A Study of Skills, Problem Solving, and Collaboration Networks

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Economics)
in The University of Michigan
2010

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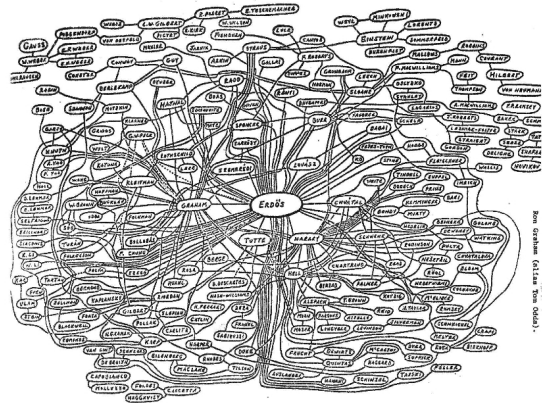   Assistant Professor Stephan Lauermann

PLAYERS

PROBLEM!

NETWORK!

COLLABORATION

ACKNOWLEDGEMENTS

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ABSTRACT

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Problem solving plays an important role in many contexts, including scientific innovation and economic production. Individual problem solvers work together, combining their skills to innovate and solve problems that none of them could solve alone. The collaborative links between individuals form a network, the structure of which affects behavior and outcomes. This thesis focuses on the formation of collaboration networks, specialization in problem solving populations, and the effect of network structure on group formation.

In the first chapter, I present a formal model of collaborative problem solving. I show that the number of collaborators an individual has is a highly non-linear function of her set of skills. I show that the degree distribution of the network as a whole will be fat-tailed–that is, a small number of players solve the vast majority of the problems, while most players solve relatively few. This result holds, even when skills are distributed independently across the problem solvers. The degree distribution becomes more skewed when problems are difficult for the population, and when skills are arranged into disciplines.

In the second chapter, I examine the equilibrium population of specialists and gen-

eralists in problem solving communities. I show that if problems are one-dimensional, a population of generalists can only be sustained if there are significant barriers between disciplines. I then evaluate the social optimality of this equilibrium. I find that because generalists internalize the costs of diversifying their skills, some populations suffer from an undersupply of generalists, suggesting that more problems may be solved by subsidizing the costs of skill diversification.

In the final chapter, I model how individuals form problem solving teams when constrained by an exogenous social network. I show that without network constraints, the equilibrium of a sequential group formation game is highly suboptimal–groups tend to be much too large. I then introduce an exogenous social network constraint, and show that this constraint mitigates the tendency for groups to get too large. The efficiency of the equilibrium depends on the topology of the underlying social network; as the social network becomes more sparse, social welfare increases.

CHAPTER I

Introduction

Innovation is largely the result of collaboration between individuals. Despite the picture given to us by history and popular culture, the lone innovator is the exception, rather than the rule; there is seldom a Watson without a Crick, and even Thomas Edison relied on the contributions of an army of fellow researchers when producing his patents. There are many advantages to collaboration in problem solving and innovation. On a practical level, difficult problems often require talents that are beyond the capacity of any single individual. Collaboration allows people with diverse talents to pool their skills and solve problems that no individual could solve alone. For example, the projects funded by the X-prize Foundation (a group that funds large cash prizes for particularly difficult but ground-breaking problems) are so complex that they could hardly be accomplished by even the most ambitious individual. Individuals with different backgrounds also bring new perspectives to old problems, which can allow for large leaps in thinking where only incremental progress had been made before. Finally, on a practical level, collaboration can allow individuals with great talents to spread those talents across multiple projects, and thus have an even greater impact than they would have while working alone. The Hungarian mathematician Paul Erdos, for instance, who had over 500 collaborators in his lifetime, could not have had nearly the same impact if he had worked alone. The importance of collaboration

in problem solving and innovation makes the study of collaborative relationships a valuable area of study. This thesis will be devoted to exploring several aspects of collaborative problem solving, including the structure and function of collaboration networks.

Collaborative relationships embed individuals in a vast network of connections. Problem solvers use these connections for a wide range of other activities, and thus the structure and function of these collaboration networks can have a huge effect on both individual outcomes and the overall progress of innovation in the collaborative community. In particular, because ideas are built on other ideas, and individuals talk to their collaborators about new and exciting advancements in their fields, an individual's innovative potential will be affected by the connections she has in the collaboration network. The structure of a collaboration network as a whole can help or hinder the diffusion of information, and thus affect the efficacy of the community in solving problems. Moreover, outcomes for individual innovators may be strongly affected by their positions on the collaboration network–individuals who have many connections may have better access to information and resources, and may be more influential in their collaborative communities. Thus, there is enormous value in understanding how that structure is formed, and the effects of social network structure on innovation.

In this thesis, I will model collaborative problem solving and innovation in the context of collaboration networks. In particular, I will look at the formation of collaboration networks, the effects of network structure on the assembly of problem solving teams, and the acquisition of skills in problem solving communities. In this work, I draw upon and make contributions to two distinct literatures: the collaborative problem solving literature, and the social networks literature. Here, I will briefly discuss these two literatures, and outline the contributions of this thesis to both.

1.1 Collaborative Problem Solving and Innovation

Collaborative problem solving is important in a wide range of contexts. By collaborating, individual problem solvers are able to pool their resources, and solve problems that none of them could solve alone. As problems become more difficult, collaboration becomes increasingly important, driving basic science research and allowing individuals to solve ever-more-difficult problems and extending the frontiers of knowledge through innovation.

In an economic context, collaborative problem solving has become increasingly important as our economy moves away from manufacturing towards knowledge-based industries. This transition has been framed by Hagel et al (2009) as a movement from knowledge exploitation (the use of existing stores of knowledge to create value) towards knowledge creation (the development of value through innovation). They call this transition "The Big Shift". This shift in the nature of production has brought a change in the nature of work. Increasingly, knowledge-based firms rely on team-based production–small groups of specialists, who work together on problems (Lipnack and Stamps (1993)). There has also been an increased emphasis on interfirm collaboration (Powell et al. (1996)), which, in some industries, has led to "open innovation" projects, which allow firms to share their knowledge more widely.

These changes in production are associated with a few patterns in labor and organization, including an increasingly skewed distribution of labor demand (Rosen (1981)) and income (Juhn et al. (1991), Machin (2008)), and a flattening of organizational structures (Bresnahan et al. (2002), Rajan and Wulf (2006)). However, there are very few models of these new kinds of production. Better models of collaborative problem solving may provide insights into the origins of the patterns that we are now observing and improve the explanatory power of empirical models of labor outcomes. One major thrust of this thesis is developing a model of collaborative problem solving that is both detailed and flexible enough to address relevant questions, and also

tractable enough to provide clear answers. In the rest of this section, I will look at three different aspects of collaborative problem solving that will be relevant in the coming chapters: team assembly, agent heterogeneity, and skill acquisition.

1.1.1   Team Assembly

Given the importance of team-based production in modern, knowledge-based industries, the literature on team-assembly is relevant to the problem of collaboration in problem solving.[1] In this literature, individuals make group membership decisions. In the context of collaborative problem solving, team assembly means forming groups to work on projects. Traditionally, these models are static, with all individuals making their membership decisions simultaneously (Hart and Kurz (1983), Nitzen (1991), Yi and Shin (2000)). However, these games have multiple equilibria, not all of which are efficient. Dynamic models, in which individuals make their group membership decisions sequentially, provide a form of equilibrium refinement (Cooley and Smith (1989), Bloch (1996), Arnold and Schwalbe (2002), Konishi and Ray (2003), Macho-Stadler et al (2004)). However, none of these models have explored the effects of social network structure on group formation. All of these models assume that players can form groups with anyone. But in practice, group membership is constrained by various social, spatial, and institutional barriers–for example, it is more difficult to enter a collaboration with a stranger at a different institution than it is to collaborate with a person you already know. In Chapter 4, I present a model of group formation in which individual group membership decisions are constrained by an exogenous social network. In particular, individual players can join a group only if they already know one of the members of that group. I use this model to look at how the structure of this exogenous social network affects the efficiency of the groups formed.

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[1]Note that there is considerably variation in the terms used to describe this literature. Relevant work appears under the terms "group formation," "club formation," and "coalition formation" as well as "team assembly".

1.1.2   Agent Heterogeneity

The importance of diversity in collaborative problem solving has been widely recognized in the business and economics literatures. On a theoretical level, Hong and Page (2001) and (2004) show that diverse teams of intelligent problem solvers will outperform teams of experts when performing difficult tasks. On an empirical level, Guimera et al (2005) show that less diverse teams of academic collaborators tend to have lower impact than those that are more diverse. However, there have been few models that explicitly incorporate diverse problem solvers.

Traditionally, in models of economic production, workers are allowed to differ along a single dimension–what we might usually think of as "ability". While this might be a reasonable measure of the worth of a worker in manufacturing industries, where workers largely perform a single task, it is more problematic when thinking about problem solving production, where individuals bring a variety of useful skills to the table. We might be tempted to give each individual a "type" or speciality. However, this method of dealing with agent heterogeneity is not as general as it might be–in many cases it is difficult to categorize an individual's talents in this way–and it is not clear that such an assumption is without empirical consequences.

Chapter 2 of this thesis presents a more general model of skills and collaborative problem solving, which subsumes both a model with one-dimensional ability and a model with types or specialities. Moreover, I show that the way that skills are modeled has implications for the amount of variation in outcomes that can be explained empirically. In particular, the predictions of a model in which individuals have a type or specialty are considerably different from the predictions of a model where individuals can have overlapping sets of skills. This illustrates the importance of a more fine-grained approach to modeling skills in a problem solving context.

1.1.3 Skill Acquisition

Given this more detailed treatment of skills, we may then take a step back and look at the acquisition of those skills. How do individuals choose the kinds of skills that they should acquire? Presumably, the optimal decision about which skills to acquire will depend on the skills that others have, and perhaps the skills that one already possesses. The division of skills into disciplines further complicates matters. We value individuals with skills in a wide range of areas, because they provide vital bridges between otherwise distinct collaborative communities. However, if there are costs associated with acquiring diverse sets of skills, it is not at all clear that people will do it. Unfortunately, given the somewhat coarse treatment of skills in the literature, it has not been possible to consider these questions in the kind of detail they deserve. The finer treatment outlined in Chapter 2 allows for a more careful consideration. Near the end of Chapter 2, I consider some basic elements of the skill acquisition problem. In Chapter 2, I look at the problem of diverse skills in more detail, and consider under what conditions it is rational for an individual to obtain skills in more than one discipline.

1.2 Network Structure and Behavior

In the past decade, there has been an increasing interest in understanding the role of network structure in governing individual behavior. This has, in turn, sparked interest in the origins of social network structure itself. In this section, I will give a brief introduction to the terminology of social networks and then describe the growing literature in the area, including where this thesis fits into this literature.

1.2.1 Introduction to some relevant network concepts

A network has two components: *nodes* and *links* (also called *edges*). In a social network, the nodes represent agents, such as individuals or firms. The links between nodes represent a relationship between those agents. A link may represent anything from friendship to trading relationships, to professional association. In a collaboration network, two individuals are connected by a link if they have collaborated on a project.

The *degree* of a node is the number of links that that node has to other nodes.

A *path* between nodes i and j is a series of links starting at node i and ending at node j. The *distance* between two nodes is the length of the shortest path between those two nodes. The *diameter* of a network is the longest distance between two nodes.

The *clustering coefficient* of a node is the probability that two of the node's neighbors are connected. The *clustering* of a network is the average clustering coefficient over all nodes in the network.

A node's *betweenness* counts the average fraction of the shortest paths between points that go through a given node. Betweenness is a measure of how central a node is to the network, and nodes with high betweenness tend to connect otherwise disconnected communities within a network.

1.2.2 The structure and function of social networks

Empirically, collaboration networks display some remarkable structural similarities. First, the average distance between two nodes in a collaboration network is very low, meaning that any two nodes in the network are connected by a relatively small number of hops. They also display a clustering coefficient that is much higher than what would be expected in a random network. Finally, the degree distribution of the networks tends to be fat tailed–that is, a few of the nodes in the network have a large number of links while the majority of the nodes have very few (see Figure 1.1
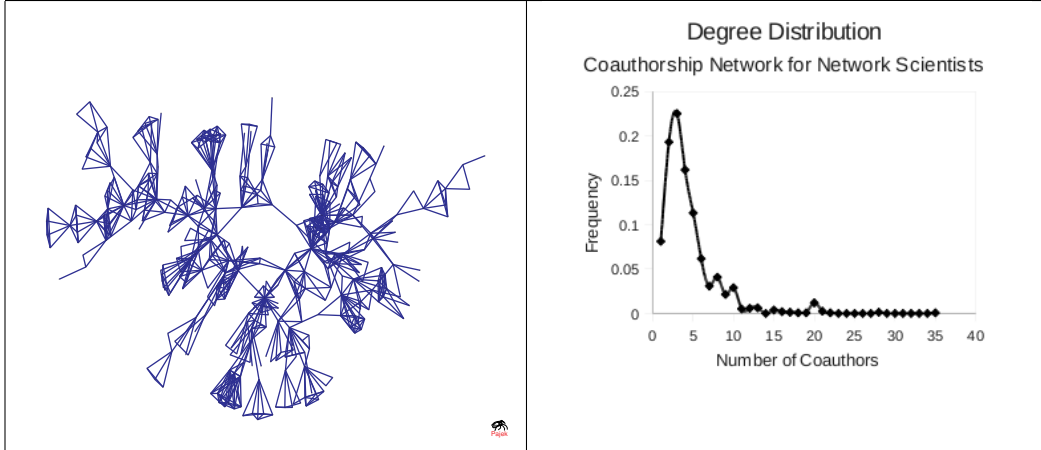
Figure 1.1: A typical coauthorship network. The network on the left is derived from data on coauthorship between network scientists. Two nodes in this network are connected if they have coauthored a paper together. The degree distribution of this network is on the right. This fat-tailed distribution, in which a few individuals have a large number of coauthors, is typical of all empirically-observed collaboration networks. Data from Newman (2006).

for an example). These characteristics are shared by collaboration networks across contexts, including in academic coauthorship networks in a wide range of disciplines (Newman (2001), Moody (2004), Goyal et al (2006), Acedo et al (2006)), networks of broadway artists (Uzzi and Spiro (2005)), film actors (Barabasi and Albert (1999)), jazz musicians (Gleiser and Danon (2003)), and interfirm collaboration (Powell et al (1996), Iyer et al (2006)).

The structure of social networks is important because it governs a wide range of behaviors, such as the flow of information and ideas (Jackson and Rogers (2007), Newman (2003)), the adoption of new technologies (Ryan and Gross (1943), Hagerstrand (1967)), and opinion formation (DeGroot (1974)). By channeling these activities, social networks affect both individual welfare and equilibrium outcomes. An individual's position on the network affects her access to information and the degree of influence she has over others, which in turn affects her outcomes. For example, it has been shown that an individual's position in a social network affects her access to

information about jobs, and thus her eventual job outcomes (Granovetter (1973) and (1995)). Individuals with high betweenness tend to control the flow of information and are thus likely to have greater power or influence (Burt (2001)). Individuals with higher degree have more input when opinions are forming and may affect the time to consensus (DeMarzo et al (2003), Golub and Jackson (2007)). On a more network-wide level, network structure affects the timing of communication and the flow of information in the network. Networks with a long average distance between nodes (a large diameter) are likely to have slow or noisy communication when compared with networks with a short average distance. Networks that consist of many tightly-knit communities with few links between are likely to suffer from impeded information flows, leading to a kind of "echo chamber" effect.

Because of the importance of the structure of social networks on determining outcomes, the networks community has placed a premium on understanding the origins of social network structure. The models of social network formation can roughly be divided into two types: statistical models and behavioral models. In statistical models, linking decisions are made through some kind of stochastic process. Many of these models are variations on *preferential attachment*, a model proposed by Barabasi and Albert (1999). In preferential attachment models, new nodes connect to older nodes at random, but they connect to high-degree nodes with greater probability. This creates a statistical "rich get richer" phenomenon, and the resulting network has a power law degree distribution ($f(k) \propto \alpha^k$), which resembles that found in empirical collaboration networks. Several variations on the preferential attachment model produce degree distributions that are an even better fit to the observed distributions–see, for example, Jackson and Rodgers (2007) and Ramansco et al (2007). Another statistical model of network structure is the *Watts-Strogatz small world network*. In this model, a fraction of links are made to nearest neighbors and a fraction are made at random. This creates a network with high clustering and low network diameter, much as is

observed in empirical social networks. A more recent model, introduced by Guimera et al (2005), uses two parameters to balance incumbency and diversity, producing networks that have a similar degree distribution to observed collaboration networks, as well as some other, secondary structures.

While statistical models do a good job of replicating the observed structure of social networks, including collaboration networks, their stochastic nature makes it difficult to draw conclusions about the connection between network structure and incentives or behavior. Thus, there has been a move among social scientists towards models of network formation in which individuals make their linking decisions based on payoff maximization. For example, Jackson and Woolinsky (1996) present a model of coauthorship networks in which individuals divide their attention across a number of different projects. They use this model to show that collaboration networks tend to be more connected than is efficient. Goyal and Moranga-Gonzalez (2001) construct a model of inter-firm collaboration, in which firms enjoy spillover effects from their neighbors' R&D efforts. They use this model to look at the efficiency of networks with high inter-firm rivalry and low inter-firm rivalry. These models give us a much better understanding than statistical models of how the nature of social interaction, institutional structures, and individual incentives affect the structure of networks that form. However, in all of these models, individual agents are homogeneous, and thus the network structures that are produced are highly symmetrical and do not resemble network structures that we observe empirically. Additionally, with homogeneous agents, it can be difficult to address the determinants of an individual's position in the social network.

This thesis makes progress on several questions that are central to the literature on social networks. Network structure both affects and is affected by individual behavior. In order to simplify our analysis of the complex feedback between the two, we often divide analysis into two areas: first is the effect of social network structure

on behavior and outcomes and second is the effect of behavior on social network structure. Chapter 4 of this thesis looks at how social network structure affects the ability of individuals to form teams. In particular, in that chapter, I present a of model team assembly in which individuals can only form teams with people to whom they are connected on an existing social network. I then use that model to examine the effect of social network structure on the efficiency of the teams formed. Chapter 2 turns this question around to look at how behavior affects social network structure. On an individual level, I look at how an individual's characteristics affect her position in the social network. This allows me to look at what distinguishes an individual with many links from one with few links, and identify individuals who are likely to be central to the collaborative community. On a more global level, I look at how overall network structure is affected by the composition of the problem solving community. This allows me to look at things like the effect of problem difficulty on network structure, and the role of specialization.

1.3   Overview of this thesis

In the second chapter of this thesis, I model the formation of collaboration networks. I examine how the structure of a collaboration network is affected by the mixture of skills in a problem-solving population and how an individual's position in this network is affected by her individual skills and those of other problem solvers. In the third chapter, I look at the skill acquisition decision and the equilibrium distribution of skills in a problem-solving population. In the fourth chapter, I examine the other side of the relationship between network structure and behavior–namely, how the formation of groups, including problem-solving groups, is affected by the structure of an exogenous social network. The fifth chapter concludes by presenting some possible extensions to the current work.

CHAPTER II

Collaboration Network Formation and the Demand for Knowledge Workers with Heterogeneous Skills

2.1 Introduction

Collaborative problem solving is important in a wide range of contexts, including economic production, product development, policy making, and academic research. In all of these, individual problem solvers work together to solve problems that none of them could solve alone. For example, research groups in a pharmaceutical firm search for new and better molecules; architectural firms design new buildings; teams of programmers create more efficient algorithms; and academic collaborations answer open scientific questions. Collaboration is widely recognized as a vital part of problem solving, because it allows diverse teams of individuals to pool their skills towards a common goal.[1] As problems become more difficult, few individuals have all of the pieces required, and collaboration becomes even more important.[2]

By linking two players who work together on a problem, we create a collaboration network. An individual's position in the network reflects her prominence in the community of collaborators and her value as a problem solver. Players with more

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[1]Philips et al. (2004), Polzer et al. (2002), Thomas-Hunt et al. (2003)

[2]Hong and Page (2001) and (2004) show the importance of collaboration in problem solving. They show that under a wide range of conditions, diverse teams of problem solvers will outperform teams of experts.

connections are presumably more important to the community because their skills are in higher demand. The overall structure of this network reflects the nature of the problem-solving community. In particular, the degree distribution of this network shows how output is distributed across the problem solvers. In networks where the degree distribution is skewed, a few individuals solve most of the problems, while the majority solve relatively few. These network structures are important because they, in turn, govern a wide range of other interactions, including the information and ideas (Jackson and Rogers (2007), Newman (2003)), individual reputation (Golub and Jackson (2007)), and opinion formation (DeGroot (1974)).

In this chapter, I present a formal model of collaborative problem solving and collaboration network formation, in which individual problem solvers have heterogeneous skills and collaborate in order to solve difficult problems. In this model, skills are pieces of knowledge useful for solving problems. For example, a skill might be familiarity with a complex tool or technique, ability as a programmer, or knowledge of a particular field.[3] Each problem solver has a subset of the total set of skills, representing her human capital. Problems, in this model, are activities requiring certain sets of skills.[4] Although individual problem solvers may have some of the skills required to solve a given problem, most problems are too difficult to be solved by an individual working alone. Thus, problem solvers in my model collaborate with others to gain access to the skills they lack. The number of problems they help solve is the demand for their skills, and proxy for their value to the collaborative community.

In the first part of this chapter, I look at the relationship between an individual's

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[3]Note that skills (such as the ability to program in java, or familiarity with the field of combinatorics) are distinguished from information (such as an observation about local weather conditions, or the availability of employment at a firm) by the fact that skills they are non-transferable in the short run. Whereas information can be passed easily from individual to individual, and may even be aggregated, skills cannot.

[4]For example, if the problem solvers are biologists, they may face an open research question requiring experience working with a particular organism, familiarity with a difficult lab technique, C-programming skills, knowledge of an unusual statistical tool, and familiarity with a the literature in a particular sub-field.

skills and the demand for her as a collaborator. I show that the number of problems a player solves is a supermodular function of her set of skills, and cannot be determined by pricing her skills individually. This is because in collaborative problem solving, collaborators bring all of their skills to the problem at hand. Thus *combinations of skills* are important. In particular, an individual with a useful combination of skills can outperform one with many rare skills, bringing into question the utility of one-dimensional ability measures in models of problem solving.

By linking players who collaborate together on a problem, we form a collaboration network. An individual's degree on this network is the number of problems that she helps to solve and the number of collaborators she has. In the second part of the chapter, I make a connection between overall structure of this collaboration network and the distribution of skills in the problem solving population. I find that even when skills are distributed independently across players, the degree distribution of the collaboration network is highly skewed–that is, a few players solve the majority of the problems, while most players solve very few. This creates a network with a distinctive, "hub and spoke" structure, similar to that observed in empirical collaboration networks. The inequality in the distribution of degree holds even when the skills are independently distributed in the population (the Bernoulli Skills Model), and becomes even more pronounced as problems become more difficult. When skills are arranged into disciplines (the Ladder Model), the degree distribution becomes even more skewed.

This chapter makes contributions to several distinct literatures. First, the results of this chapter have important implications for labor and industrial organization, as our economy shifts away from manufacturing towards more knowledge-based production. It is widely recognized that the US economy has undergone a transition from production based in knowledge exploitation to one based in knowledge creation (Hagel

et al. (2009)). This transition[5] is associated with a wide range of effects in labor and industrial organization, including an attenuation of the distribution of output Rosen (1981) and income (see Juhn et al. (1991) and Machin (2008)) and a flattening of organizational structures (Bresnahan et al. (2002) and Rajan and Wulf (2006)). In addition, because problem solving production is an intensely cooperative effort, we observe an increasing number of collaborative connections between firms (Powell et al. (1996)). Unfortunately, most current models of production are still based in existing models of manufacturing and trade. Labor productivity in these models is denoted either by a one dimensional ability measure (eg: speed) or a labor type (eg: speciality). This makes it very difficult to answer questions particular to collaborative problem solving.

The detailed treatment of skills in this model adds considerable value, when compared with these more traditional models of labor production. Players in this model have multiple skills, and their skill sets can overlap in any of a number of ways. This treatment of labor is advantageous because it is much broader than the traditional treatment, encompassing both ability-based and type-based models. It also allows us to ask questions about the value of skill *combinations*, which would not be relevant in a model with individual skills or one-dimensional ability levels. Moreover, the relationships revealed by this treatment of skills are not what we would naively expect, given our understanding of more coarse-grained models. In particular, I find that the distribution of labor demand will be skewed towards a few, highly productive individuals, similar to what is observed in empirical labor markets. Moreover, this model provides a framework for creating a more general model of organization within knowledge-based firms. I am also able to answer questions about the value of a particular skill to a particular problem solver. I show that the value of a skill depends on both the supply and demand for that skill in the population, and the set of skills the

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[5]which Hagel et al. call "The Big Shift"

problem solver already has, indicating that optimal training decisions will be highly individualized.

This chapter also contributes to the network formation literature. A growing literature has demonstrated the importance of social networks in social, political, and economic interactions.[6] The structure of collaboration networks shapes a wide range of other interactions, affecting the spread of information and ideas (Jackson and Rogers (2007), Newman (2003)), individual reputation (Golub and Jackson (2007)), and opinion formation (DeGroot (1974)). The position of an individual in the collaboration network governs her access to knowledge, tools, and information (Coleman et al (1966)), and thus it may shape the kinds of questions she addresses. The structure of collaboration networks also affects job search and hiring (Granovetter (1973) and (1995)), the adoption of new technologies (Ryan and Gross (1943), Hagerstrand (1967)), and the influence of individual researchers (DeMarzo et al (2003), Golub and Jackson (2007)). In the case of academic research, network structure may even affect the course of scientific inquiry.

Empirically, we observe that collaboration networks have some common structural characteristics, which transcend context. In particular, the degree distribution of these networks is fat-tailed.[7] This means that a small number of individuals participate in the vast majority of the collaborations, while most individuals participate in relatively few. This skewed degree distribution has been observed in a wide range of collaboration networks, including interfirm collaboration (Powell et al (1996), Iyer et al (2006)), creative artists in broadway plays (Uzzi and Spiro (2005)), film actors (Barabasi and Albert (1999)), jazz musicians (Gleiser and Danon (2003)), and coauthorship networks in a variety of fields.[8]

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[6]See Jackson (2008) for a good survey of the existing literature.

[7]That is, there are more players with very high degree and low degree, as compared to a random network with the same average degree. Exponential and scale-free distributions are two examples of fat-tailed distributions.

[8]Newman (2001) examines coauthorship networks for several subdisciplines of physics, biomedical fields, and computer science. Moody (2004) does the same for sociology. Goyal et al (2006) looks

Because the structure of collaboration networks affects other behaviors, there is a premium attached to understanding the origins and determinants of that structure. Statistical models of network formation, such as preferential attachment,[9] and models based on incumbency[10] do a good job of recreating the fat-tailed network structure in empirical collaboration networks. However, these models rely on stochastic processes to drive link formation. Players do not make choices about which links to make, so they cannot answer questions about the relationship between behavior and network structure.

There have been several attempts to model network formation behaviorally. In these models, players choose their links strategically, in order to maximize their payoffs. Jackson and Wolinsky (1996) present a model of coauthorship networks, in which each link represents a single paper. Players in their model must allocate effort across various projects, and thus the payoff from a paper is inversely related to the number of links the two coauthors have.[11] Goyal and Moranga-Gonzalez (2001) construct a model of collaboration among firms, rather than individuals. Firms in their model choose a set of links and an effort level to put into research and development. The firm's immediate neighbors experience perfect spillover effects from the firm's efforts, whereas unconnected firms receive imperfect spillover effects. [12] Both of these mod-

at economists. Acedo et al (2006) present data on researchers in management and organizational studies, and while they do not directly address the degree distribution, their data includes more high-degree nodes than would be expected in a random network, suggesting a fat-tailed distribution.

[9]First introduced by Barabasi and Albert (1999). In preferential attachment models, new nodes connect to older nodes at random, but they connect to high-degree nodes with greater probability. This creates a statistical "rich get richer" phenomenon, and the resulting network has a power law degree distribution (eg: $f(k) \propto \alpha^k$). Several variations on the preferential attachment model produce degree distributions that are an even better fit to the observed distributions–see, for example, Jackson and Rodgers (2007) and Ramansco et al (2007)

[10]Guimera et al (2005) presents a model sequential team assembly based on the balance between experience and diversity. The model is statistical, and has two parameters, representing the probability that a newcomer enters the field and the probability that an incumbent player works with the same team twice.

[11]They show that in the equilibrium of this game, players form into a collection of fully-connected groups, each of a different size and the efficient configuration arranges all of the players into partnerships, indicating that collaboration networks tend to be more connected than is efficient.

[12]They show that the complete network is the unique pairwise stable network structure. They also compare the efficiency of equilibrium networks under different levels of firm rivalry. When firm

els provide insights into the relationship between incentives and network structure. However, because players are homogeneous in these models, the network structure obtained is very symmetrical. Moreover, it is impossible to answer questions about the value of particular skill sets in a model with homogeneous players.

The model I present here is behavioral–players receive payoffs for solving problems, and choose a set of links that maximizes that payoff. However, it differs from existing behavioral models in its treatment of skills and problem solving. Here, I allow the problem solvers to be heterogeneous. This creates a rich collaborative environment, in which players seek out others with complementary skills, and allows me to ask questions about how a particular individual's skill set is valued in the community. This heterogeneity in players also breaks the symmetry of the resulting collaboration network, resulting in a network with a degree distribution similar to that observed empirically. This allows me to look at how the degree distribution is affected by the population of problem solvers.

Note that this model combines two distinct lines of research. Existing models of collaboration network formation do not consider the impact of skills on individual degree and overall network structure, and existing models of problem solving and collaboration do not consider the network structures that result from the interactions of individuals. The model I present in this chapter bridges that gap, and provides a framework which can be used to address a wide array of new questions.

The rest of the chapter is organized as follows. Section 2 presents the model, and offers a brief discussion of it's characteristics. Section 3 examines the relationship between a player's degree on this network and her set of skills. Sections 5 and 6 take a step back and look at how the overall structure of the collaboration network depends on the distribution of skills in the population. Section 7 discusses the implications of these results in the labor market and industrial organization. Section 8 presents

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rivalry is low, the equilibrium configuration is also efficient. When firm rivalry is high, the complete network is inefficient when compared with a network with fewer links.

some possible extensions and Section 9 concludes.

2.2 A General Model of Skills, Problem Solving, and Collaboration Networks

2.2.1 Inputs: Problem Solving Population and Problems

The inputs to this model are a single problem and a population of problem solvers.

Let $I = \{1, 2, ...N\}$ be the set of problem solvers.

Let $S = \{a_1...a_M\}$ denote the set of all skills.

$A_i \subseteq S$ is the subset of those skills possessed by player i, which I will call her *skill set*.[13] The players' skill sets are distributed according to Ψ, a probability measure with support $\Sigma(\Psi) \subseteq 2^S$–that is, $\Psi(A)$ is the fraction of the players in I who have the skill set $A \subseteq S$.[14]

Each player is endowed with a single problem, $\omega_i \subseteq S$, which requires a subset of the skills in the population.

A *collaboration* is a subset of the players, $C \subset I$. A player and her collaborators can *solve* a problem if together they possess all of the required skills–that is, if $\omega_i \subseteq \bigcup_{j \in C_i} A_j$ (see Figure 2.1 for an illustration).

The problem yields a payoff of 1 if solved. If the player can solve her problem alone (that is, if $A_i = \omega_i$), then she keeps the entire payoff. If she solves it with the help of other players, then she splits the payoff evenly with them, giving each a share of $\frac{1}{|C_i|}$ and retaining a similar share for herself. Each player faces a problem, and thus player i's payoff is the sum the payoff she gets from solving her own problem, plus

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[13]We could think of A_i as player i's human capital.

[14]Formally, Ψ is a *frequency* distribution–that is, Ψ is a *realized* distribution of skill sets across players, rather than a statistical one. The distinction between frequency and probability distributions disappears when N is large, but using a frequency distribution allows me to also make statements about small N as well.
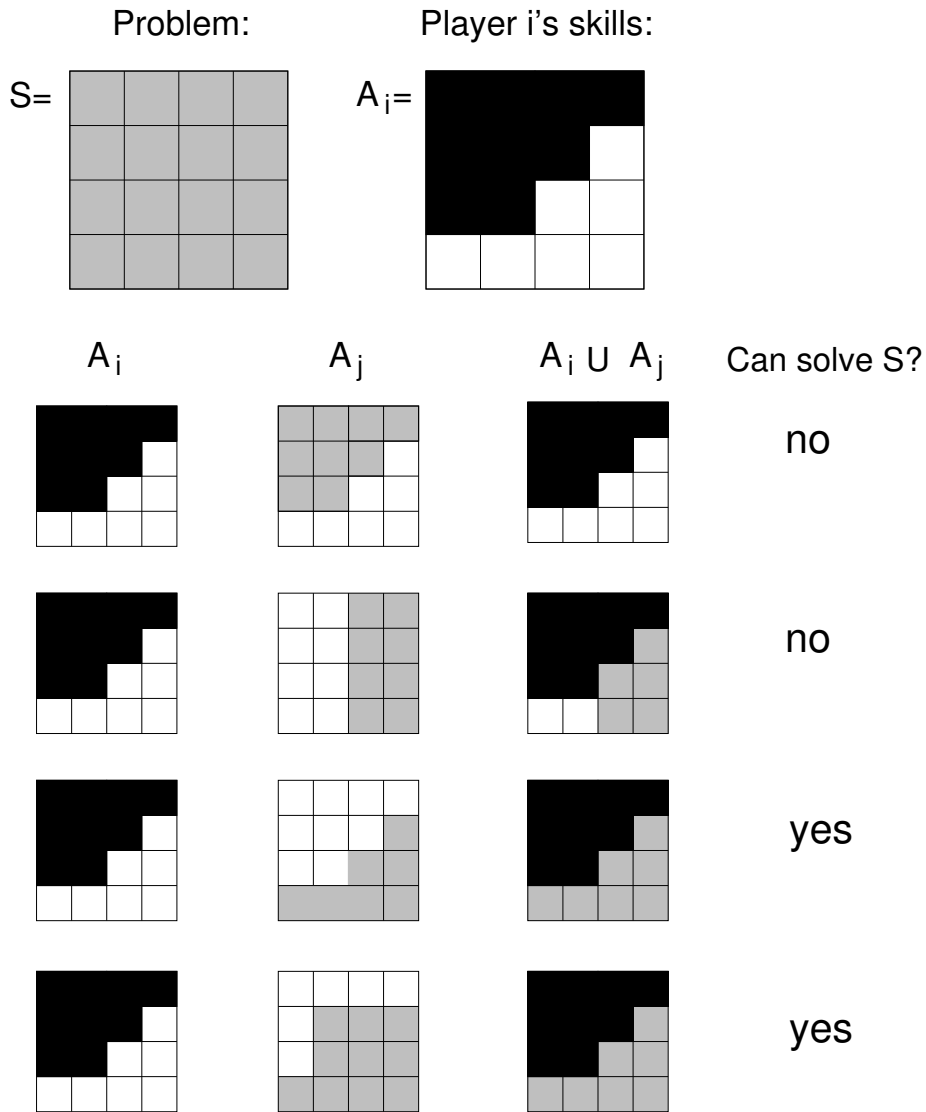
Figure 2.1: A graphical example of collaboration and problem solving in this model. The problem to be solved requires 16 skills, represented by the boxes. Player i has 9 of the required skills, represented by the filled boxes. Player i can solve the problem only by collaborating with someone who has the skills she lacks.
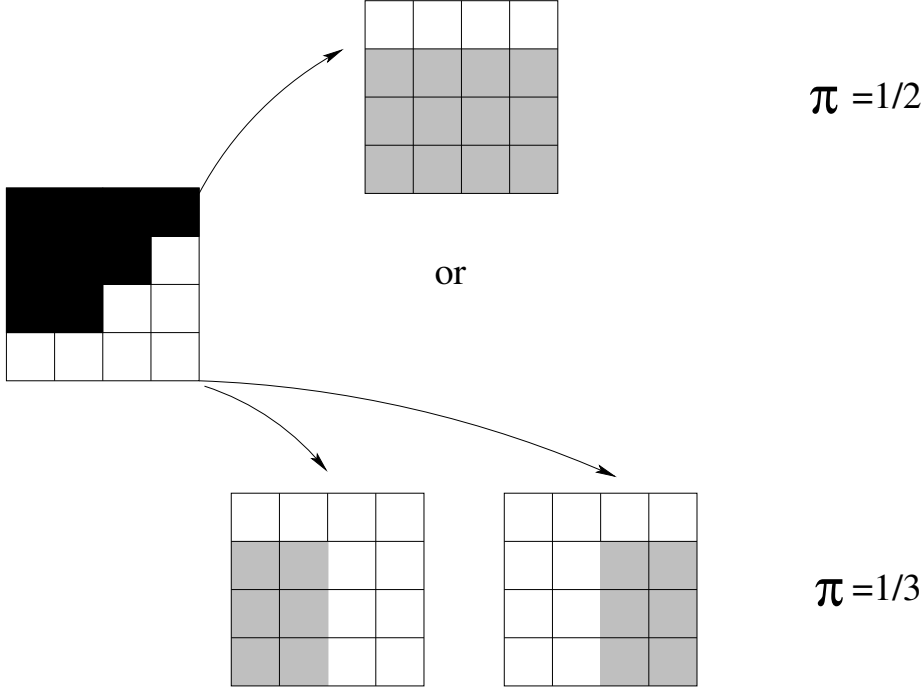
Figure 2.2: A graphical illustration of the player's optimization decision. Solving a problem yields a payoff of 1. Because the player splits this payoff equally with her collaborators, she optimizes by choosing the minimum number required to solve the problem.

any payoffs she gets from collaborating with others on their problems:

$$u_i = \frac{1}{|C_i|} + \sum_{j \neq i \ st \ i \in C_j} \frac{1}{|C_j|}$$

A player chooses her set of collaborators (C_i) to maximize her utility. Note that player i's payoff to solving her own problem is always positive, and thus it is always incentive compatible for her to find a solution to the problem. Since the player controls only her own collaborative decisions, a utility-maximizing player chooses C_i to minimize the number of connections she must make–in other words, she chooses a minimal subcover of the set of skills she lacks–$A_i^c = \omega_i \backslash A_i$ (see Figure 2.2 for an illustration). Let $\mathbb{C}_i$ denote the set of all minimal subcovers of A_i^c. I assume that if there exist multiple minimal subcovers (ie: if $|\mathbb{C}_i| > 1$) then the player will choose a

random minimal subcover, $C_i^* \in \mathbb{C}_i$.[15]

2.2.2 Cost-minimizing Collaboration Networks

For a given a set of collaborations, $C = \{C_1...C_N\}$, the collaboration network is represented by an adjacency matrix, $g(C)$, where $g_{ij}(C) = 1$ if $j \in C_i$. Note that the network is directed–since $j \in C_i$ does not necessarily imply $i \in C_j$, it may be that $g_{ij}(C) \neq g_{ji}(C)$. However, the links are mutual, in the sense that neither player wants to terminate a link (see Section 2.2.5 for further discussion). When all collaborators are chosen optimally (that is, when $C_i \in \mathbb{C}_i \, \forall i$), I will call the result a *cost-minimizing collaboration network.*

Definition. A network, $g(C)$, is a *cost minimizing network* if each player in the network chooses a minimal set of collaborators required to solve her problem–eg: if $C_i \in \mathbb{C}_i \, \forall i$.

Since the set of minimal subcovers for each player ($\mathbb{C}_i$), depends on the distribution of skills in the population, I use $\Gamma(\Psi)$ to denote the set of cost-minimizing collaboration networks for a particular distribution of skills, Ψ.

Before continuing, a brief word about network notation is in order. First, for ease of reading I will usually drop the argument of $g(C)$. I will denote a link from player i to player j by ij. Using a slight abuse of notation, I will use g to refer to both the adjacency matrix (as above) *and* the set of links in the network–that is, $ij \in g$ if i is connected to j in the network g. In a similar abuse of notation, I will use $g - ij$ to represent the network that results when the link ij is removed from an existing network, g, and $g + ij$ to represent the network that results when the link ij is added to the existing network, g.

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[15]Since players are indifferent between minimal subcovers, this choice at random follows convention. The results are not sensitive to this assumption.

2.2.3 Example

An example will help clarify the structure of this model. Suppose all of the players face the same problem requiring three skills: $\omega_i = \omega = \{a, b, c\}$ $\forall i$. Suppose the distribution of skills is such that every player has at least one skill, but no player has all of the skills required. In other words, the support of Ψ is the set $\{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$. In this particular case, each player needs to make only one link in order to solve the problem–a player with skill set $\{a\}$ must link to one of the players with the skill set $\{b, c\}$, a player with skill set $\{a, b\}$ may choose from those with skill sets $\{c\}, \{a, c\}$, and $\{b, c\}$, and so on. Figure 2.3 shows a schematic of the model. The inputs are the problem, and a particular skill distribution (in this case, $\Psi(A) = \frac{1}{6}$ for all $A \in \{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$ and $\Psi(A) = 0$ otherwise). The players optimize their choice of collaborators, and the result is a collaboration network. The figure shows an example network for a particular set of optimal choices. The set $\Gamma(\Psi)$ is composed of many networks with the same skill distribution, but different choices of minimal subcovers.

2.2.4 Discussion

Speaking generally, there are two inputs to the model: a problem (ω) and a distribution of skills (Ψ). The output of the model is a set of cost minimizing collaboration networks, in which each player has chosen a minimal set of links in order to solve her problem.

This model produces outcomes consistent with several empirical facts about collaboration and problem solving. First, the model predicts that as problems become increasingly difficult–that is, as each individual has a smaller fraction of the skills required to solve the problems she faces–collaboration networks will become more densely connected. This prediction is born out in data from a variety of aca-
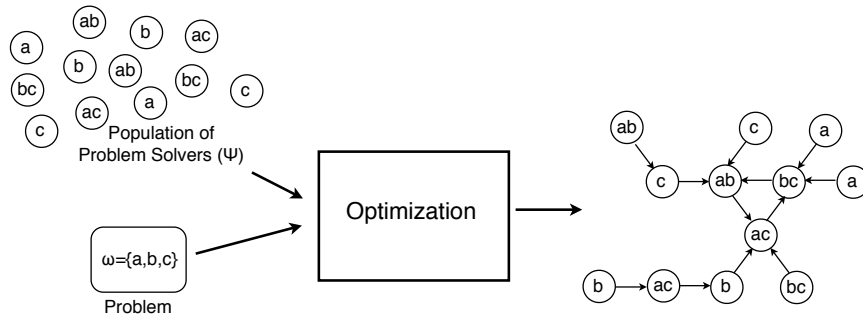
Figure 2.3: An example, illustrating the model. The inputs to the model are a problem, ω, and a population of problem solvers with a distribution of skills. In this case, there are $N = 12$ players. The players face a problem requiring three skills: $\omega = \{a, b, c\}$. Each player has either one or two skills, and they have an equal probability of having any combination. That is, $\Psi(A) = \frac{1}{6}$ for $A \in \{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$ and $\Psi(A) = 0$ otherwise. Players optimize their set of collaborators, and the result is a cost minimizing collaboration network. The pictured network is one example of a cost minimizing collaboration network for this problem and population.

demic fields–collaborative work has become increasingly common in mathematics,[16] physics,[17] sociology,[18] management science,[19] and economics.[20] Moreover, the literature supports a connection between increased collaboration, the difficulty of problems faced, and the increasing complexity of required methodologies.[21]

The model also predicts that problem solvers will seek out collaborators that are unlike themselves–that is, players who have complementary skills.[22] Diversity

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[16]Grossman and Ion (2002)

[17]Barabasi et al (2002)

[18]Moody (2004)

[19]Acedo et al (2006)

[20]Laband and Tollison (2000) look at papers published in three prominant economic journals (*American Economic Review, Journal of Political Economy,* and *The Quarterly Journal of* Economics*)* from 1930-1995. The percentage of economics papers that were coauthored is around 10% in the period from 1930-1960, but rises to over 50% by 1990. The number of authors per paper also rises, from essentially 1 to 1.5 by the mid-1990s. Goyal et al (2006) notes that the average number of coauthors per individual in economics nearly doubled in the period from 1970-1999.

[21]Laband and Tollison (2000) suggest that coauthorship is more common in fields where intellectual advances are difficult or costly, and that the rise in coauthorship in economics and biology over the past 50 years could be attributed to increasingly complex methodologies, which are more costly to learn. Moody (2004)

[22]This is because in this case, players do not benefit from redundant skills. However, the result

is widely recognized as contributing to the success of collaborative problem-solving groups,[23] and theoretical work indicates that diversity may even be more important to collaborative success than raw ability.[24] Empirical data backs up these assertions, indicating that collaborators are more likely to collaborate if they have dissimilar backgrounds.[25]

2.2.5 A Note on Stability and Efficiency of the Cost Minimizing Social Network

Before considering specific questions about the cost-minimizing collaboration network, it is worth considering the stability and efficiency of that network. Jackson and Wolinsky (1996) introduce an equilibrium concept of network stability, called *pairwise stability.* Briefly, a network is pairwise stable if no individual would prefer to terminate an existing link, and if no pair of individuals would prefer to add a link (see Appendix A for a more formal definition). Together, these two conditions ensure that links are mutual. That is, if a network is pairwise stable, then both players agree to maintain the link.

Theorem 1 states that any cost minimizing collaboration network is pairwise stable,[26] and thus all links in the network are mutual. Moreover, it states that any cost-minimizing collaboration network is strongly efficient–that is, the players extract the maximum possible value from the network.

Theorem 1. *Any cost minimizing collaboration network, $g \in \Gamma(\Psi)$, is pairwise stable and strongly efficient.*

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still holds for alternative production functions–we just need for the returns to a single skill to be decreasing in the number of copies of that skill obtained.

[23]Using a longitudinal study of work groups, Polzer *et al.* (2002) find that diversity improves group performance.

[24]Hong and Page (2004) shows that under a broad range of conditions, a randomly-selected group of diverse problem solvers will out-perform a group of non-diverse experts.

[25]Fafchamps et al (2006) show that economics researchers are more likely to cooperate if they have dissimilar experience and ability levels.

[26]It is actually *more* than pairwise stable, because linking players choose an optimal *set* of links from all possible sets.

Proof. See Appendix A □

This means that in any cost-minimizing collaboration network, all collaborative links are mutually beneficial, and the problem solvers choose an efficient network structure. This result is in contrast with other models of network formation–for example, Jackson and Wolinsky (1996) and Goyal and Moranga-Gonzalez (2001)–in which pairwise stability and efficiency do not coexist.[27]

2.3 Skills and Degree: Skill Sets and Collaborative Success

A player's *in-degree* in the network–which I will denote d_i– is the number of links that are directed towards that player. In the context of collaboration networks, it represents the number of problems that the player helps to solve. In this section, I consider how a player's degree in the collaboration network depends on her set of skills. I show that when players can have multiple skills, a player's degree[28] in the network is a highly non-linear function of her set of skills, meaning that the value of a *combination* of skills may be greater than the sum of its parts. This result suggests that as an input to production, skills should be valued much differently than either raw materials or man hours.

2.3.1 Skills and Degree: An Example

Before presenting the main results of this section, it is useful to see an example. Suppose players face a problem requiring three skills, $S = \{a, b, c\}$. Further, suppose each player has one or two of those skills, so that Ψ has support $\{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$. The number of problems a player will help solve,

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[27]In both of these papers, the pairwise stable network has too many links, when compared with the efficient network.

[28]Here, and in most of the following, I will drop the modifier and refer to in-degree simply as "degree". I use in-degree because it has a clear, empirical interpretation, but the results qualitatively similar if we consider a player's degree to be the sum of his in-degree and out-degree, or use the degree of the player in a network where directed links are projected into undirected links.

and thus her in-degree on the network, will depend on the number of players who need her skills and the number of other players who have those same skills. For example, consider a player with the skill set $\{a\}$. She can help any player who has the complementary set of skills, $\{b, c\}$. A player with $\{b, c\}$ may ask anyone with skill a for help, including those with skill sets $\{a\}, \{a, b\},$ or $\{a, c\}$. So the expected degree of a player with skill set $\{a\}$ is

$$E\left[d\left(\{a\}\right)\right] = \frac{\Psi\left(\{b, c\}\right)}{\Psi\left(\{a\}\right) + \Psi\left(\{a, b\}\right) + \Psi\left(\{a, c\}\right)}$$

Similarly, a player with the skill set $\{a, b\}$ can help any player who needs skill a or skill b, yielding expected degree

$$E\left[d\left(\{a, b\}\right)\right] = \frac{\Psi\left(\{b, c\}\right)}{\Psi\left(\{a\}\right) + \Psi\left(\{a, b\}\right) + \Psi\left(\{a, c\}\right)} + \frac{\Psi\left(\{a, c\}\right)}{\Psi\left(\{b\}\right) + \Psi\left(\{a, b\}\right) + \Psi\left(\{b, c\}\right)} + \frac{\Psi\left(\{c\}\right)}{\Psi\left(\{a, b\}\right)}$$

Note that the expected degree of a player with both skills a and b is greater than $E\left[d\left(a\right)\right] + E\left[d\left(b\right)\right]$. This is because a player with both skills can help players who need skill a, players who need skill b, *and players who need both*.

2.3.2   Skills and Degree: General Results

Theorem 2 states that degree is a supermodular function of a player's set of skills. That is, regardless of the skill set required for the problem, or the distribution of skills, a player with skill set $A \cup B$ can solve at least as many problems as players with A and B put together. The sketch of the proof is similar to the above example–the set of all problems that can be solved by a player who has the skill set $A \cup B$ includes those that can be solved by a player with skill set A, and those that can be solved by a player with skill set B, and those requiring some skills from *both* sets.

Theorem 2. *For any set of skills, S, and distribution of those skills, Ψ, a player's expected degree over the networks in $\Gamma\left(\Psi\right)$ is a supermodular function of her set of*

skills. That is, $Ed(A \cup B) + Ed(A \cap B) \geq Ed(A) + Ed(B)$.

Proof. Here, I will prove the result for the case where players need only one collaborator to solve their problem. The proof for the general result is similar, and can be found in Appendix B. For the sake of clarity, I consider the case where $A \cap B = \emptyset$ (again, the more general result appears in Appendix B). Since $d(A \cap B) = d(\emptyset) = 0$, we need to show that $Ed(A \cup B) \geq Ed(A) + Ed(B)$. Consider $d(A \cup B)$. Recall that a player can help anyone in the population who requires a subset of her skills. The fraction of players who need a particular subset of $A \cup B$ is the fraction who have exactly the complementary set of skills, so the fraction needing $C \subseteq A \cup B$ is $\delta(C) = \Psi(S \backslash C)$. The fraction who can supply the set C is $\sigma(C) = \sum_{D \subseteq S \backslash C} \Psi(C \cup D)$. Thus, a player with the skill set $A \cup B$ has expected degree

$$E[d(A \cup B)] = \sum_{C \subseteq A \cup B} \frac{\Psi(S \backslash C)}{\sum_{D \subseteq S \backslash C} \Psi(C \cup D)} = \sum_{C \subseteq A \cup B} \frac{\delta(C)}{\sigma(C)}$$

We can divide the subsets of $A \cup B$ into one of three categories according to the skills required:

1. Problems requiring only skills in A

2. Problems requiring only skills in B

3. Problems requiring some skills from A and some from B

which gives us the following:

$$
\begin{aligned}
E[d(A \cup B)] &= \sum_{C \subseteq A} \frac{\delta(C)}{\sigma(C)} + \sum_{C \subseteq B} \frac{\delta(C)}{\sigma(C)} + \sum_{C \subseteq A \cup B \text{ and } C \cap A, C \cap B \neq \emptyset} \frac{\delta(C)}{\sigma(C)} \\
&= E[d(A)] + E[d(B)] + \phi \\
&\geq E[d(A)] + E[d(B)]
\end{aligned}
$$

$\square$

This theorem suggests an immediate corollary.

Corollary 3. *Adding skills to a player's skill set will never decrease her degree in a cost minimizing collaboration network.*

Proof. Suppose a player has a skill set, A, and we add a new skill that he did not have $a \notin A$. From Theorem 2, $E\left[d\left(A \cup a\right)\right] + E\left[d\left(A \cap a\right)\right] = E\left[d\left(A \cup a\right)\right] \geq E\left[d\left(A\right)\right] + E\left[d\left(a\right)\right]$, and so $E\left[d\left(A \cup a\right)\right] - E\left[d\left(A\right)\right] \geq E\left[d\left(a\right)\right] \geq 0$. $\qquad\square$

2.3.3 Bundled Skills and the Importance of Skill Combinations

Theorem 2 indicates that an individual's importance in a community of collaborators reflects not only the supply and demand of her individual skills, but also the supply and demand of her *combination* of skills. This means that a player with a useful combination of skills might be more important to the community than a player with many skills or rare skills.

This result is best illustrated using an example–consider a problem requiring five skills, $S = \{a, b, c, d, e\}$, which are distributed across $5N$ players, as shown in Table 2.1. Each skill is held by exactly 2 players, and thus no skill is rarer than the others. Traditionally, we might condense the information contained in this table into a single measure of ability. Player 1 and 2 have the most skills, and therefore, we would expect them to have the most value in the community. However, despite having fewer skills than players 1 and 2, player 3 will receive more links, in expectation. Note that player 3's skills are not rare, individually. However, her combination of skills *is* rare and valuable to many different people.[29] Therefore, she receives more links than a tally of her individual skills might predict.[30]

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[29]Note that a player's skill combination must be both rare and *useful*–that is, complementary to the skills of other players. In Table 2.1 players 4 and 5 both have rare combinations of skills, but the combination they have is not useful for any player, and thus they receive few links.

[30]Examples of this phenomenon are not difficult to find. For example, consider a group conducting a field experiment in a remote mountain area. This type of problem requires a wide range of skills, including the ability to pose questions, design experiments, acquire funding, collect data, and (given

	a	b	c	d	e	in-degree
1	X	X	X			1.5
2	X	X	X			1.5
3				X	X	3
4					X	.5
5				X		.5

Table 2.1: An example where degree is not monotone in the size of the skill set. 5 skills are distributed across $5N$ players as shown. In this population, all skills occur with equal frequency, and therefore there are no rare skills. Players 1 and 2 have the largest skill sets. However, player 3 has more links. This demonstrates that a player with a useful combination of skills may receive more links than one with many skills.

This example also highlights another implication of Theorem 2–because a player's degree is a supermodular function of her set of skills, it is not generically possible to assign prices to individual skills in a way that captures a player's degree. This means that examining the supply and demand of single skills in isolation does not capture an individual's value to a community of problem solvers.

Corollary 4. *There need exist no vector of prices for individual skills, μ, such that* $\sum_{a \in A} \mu_a = d(A)$.

To further emphasize this point, consider the skill distribution shown in Table 2.2. This distribution is identical to that in Table 2.1, except that players 4 and 5 have both gained two skills. However, gaining these skills does not influence the degree of either player. In fact, endowing players 4 and 5 with those skills does not change any part of the degree distribution. Skills a and b have value to players 1 and 2, but not

---

the remote location) mountaineering skills. In this context, player 3 represents a lab assistant who also has mountaineering experience.

players 3 or 4–clearly, no linear weighting of the individual skills could produce that pattern.

	a	b	c	d	e	in-degree
1	X	X	X			1.5
2	X	X	X			1.5
3				X	X	3
4	X	X			X	.5
5		X	X	X		.5

Table 2.2: Consider the previous example, pictured in Table 2.1. Now, suppose players 4 and 5 were endowed with two extra skills, as shown here. Neither player's degree is affected by this change, because their combinations of skills are not useful to any of the players in the game. This example illustrates that we cannot value a player's skill set by determining the value of her skills individually.

Two characteristics of problem solving production contribute to this result. First, skills are bundled within a person. Thus, in evaluating a collaborator, that person's combination of skills must be considered as a unit. Second, players share their payoffs from solving the problem among their collaborators, and thus have an incentive to minimize the number of collaborators they work with. Together, these two factors mean that a player's value as a collaborator may be more than the sum of her individual skills. This non-linearity has implications for the distribution of degree in the population as a whole. Sections 2.4 and 2.5 consider those implications in further detail.

2.4 Skill Distributions and the Distribution of Prominence: The Bernoulli Skills Model

Collaboration network structure governs a number of important interactions, including the spread of information, control of reputation, and even the process of finding jobs. Therefore, we would like to obtain a better understanding of the topology of these networks. In the previous section, I looked at a local measure of network topology–degree. In this section, I take a step back, and consider how the distribution of skills in the population affects the structure of the collaboration network as a whole. I use a special case where skills are independently and identically distributed to show that even when skills are distributed evenly across the population, the degree distribution is highly skewed–that is, a very small number of players help to solve the vast majority of the population's problems, while a large number of players solve very few. I then examine how the degree distribution is affected by changes in the problems faced and the distribution of skills in the population. I show that as problems become more difficult for a population, the degree distribution becomes increasingly unequal–in other words, superstars emerge.

2.4.1 The Bernoulli Skills Model

In this section, I consider a special case in which skills are distributed independently, with equal probability–that is, $Prob\left(a_i \in A | a_j \in A\right) = Prob\left(a_i \in A\right) = p \, \forall \, i \neq j \in S$. I call this the Bernoulli Skills model because each player's skill set can be thought of as the result of a set of M Bernoulli trials, each with probability p of success. This means that the distribution of skill set sizes in the population is binomial, implying that the fraction of the players who have a *particular* set of k skills is $\Psi\left(A\right) = p^k \left(1 - p\right)^{M-k}$, and the fraction having *any* k skills is $\binom{M}{k} p^k \left(1 - p\right)^{M-k}$.

This special case has several characteristics which make it interesting. First, be-

cause skills are completely uncorrelated and occur with the same frequency, all players with the same number of skills will have the same degree, in expectation. This construction enables a clear picture of the effects of supermodularity on the degree distribution of the network. Second, because this model has only 2 parameters–M and p–I can use this model to illustrate how the distribution of skills in the population affects the structure of the collaboration network.

2.4.2 Degree Distribution of the Bernoulli Skills Model

Let Δ denote the distribution of expected degree. That is, $\Delta(d)$ is the fraction of players who have expected degree d, where the expectation is taken over all $g \in \Gamma(\Psi)$.[31] In this particular case, I will use a convenient shorthand: $\Delta_{M,p}$ represents the distribution of expected degree when M skills are independently distributed with probability p.

Theorem 5 states the closed form expression for the degree of a player in the Bernoulli Skills Model.

Theorem 5. *Suppose players face a problem, $\omega(S)$, requiring M skills. If the skills are distributed independently with $Prob(a) = p \, \forall \, a \in S$, then the expected degree of a player with k skills is*

$$E[d(k)] = p^M \left[\left(\frac{1 - p + p^2}{p^2} \right)^k - 1 \right]$$

Proof. Since $\Sigma(\Psi) = 2^S$, every player needs to make only one link. Thus, we can write $E[d(A)] = \sum_{C \subseteq A} \frac{\delta(C)}{\sigma(C)}$, where $\delta(C)$ is the fraction of players who need skill set C and $\sigma(C)$ is the fraction of players who can provide skill set C. Since the skills are independent, we can separate this sum according to the size of the skill set required.

---

[31] Alternatively, we might plot the distribution of degree across all networks $g \in \Gamma(\Psi)$. That is, we could set $\Delta(d) = \sum_{g \in \Gamma(\Psi)} \delta_g(d)$ where $\delta_g(d)$ is the fraction of players in network g with degree d. This choice does not affect the results.

If we start with the players who are lacking exactly one skill in A_i and end with players needing all of the skills, we obtain the following sum:

$$
\begin{aligned}
E\left[d\left(A\right)\right] &= \sum_{i \in A} \frac{p^{M-1}\left(1-p\right)}{p} + \sum_{i,j \in A} \frac{p^{M-2}\left(1-p\right)}{p^2} + ... + \frac{p^{M-k}\left(1-p\right)^k}{p^k} \\
&= \sum_{i=1}^{k} \binom{k}{i} \frac{p^{M-i}\left(1-p\right)^i}{p^i} \\
&= p^M \left[\left(\frac{1-p+p^2}{p^2}\right)^k - 1 \right]
\end{aligned}
$$

$\square$

Note that when skills are independent, a player's degree depends only on the size of the player's skill set, k.[32] Therefore, in this particular case, it is appropriate to interpret the size of a player's skill set as her "ability"–something that we cannot do in the more general case (recall Table 2.1 in the previous section). This suggests a corollary to Theorem 5.

Corollary 6. *Suppose players face a problem, $\omega\left(S\right)$, requiring M skills. If the skills are distributed independently with $Prob\left(a\right) = p \, \forall \, a \in S$, then expected degree is strictly increasing in the size of the player's skill set.*

However, we still cannot price the skills individually in such a way that we capture degree, despite the fact that skills are independently distributed. Theorem 7 formalizes this statement.

Theorem 7. *Suppose the players face a problem, $\omega\left(S\right)$, requiring M skills . If the skills are distributed independently with $Prob\left(a\right) = p \, \forall \, a \in S$, then there exists no vector of prices, μ, such that $\sum_{a \in A} \mu_a = d\left(A\right)$ for all $A \subseteq S$.*

---

[32]In the Bernoulli skills model, skills are uncorrelated and occur with equal frequency. Therefore, there is no statistical difference between two sets of skills of the same size, and degree depends only on the number of skills the player has, rather than the exact set of skills.

Proof. Any such vector would be required to set $\mu_a = d(a) = p^{M-2}(1-p)$ for all $a \in S$. But that would imply that $d(A) = kp^{M-2}(1-p)$ for $|A| = k$. This is clearly not true for $k > 1$. $\qquad\square$

This also means that there is no way to price the individual skills such that a player's utility is the sum of the prices of her individual skills (Theorem 8)

Theorem 8. *Suppose the players face a problem, $\omega(S)$, requiring M skills . If the skills are distributed independently with $Prob(a) = p \forall a \in S$, then there exists no vector of prices, μ, such that a player can recover his utility, that is, there exists no price vector, μ such that $\sum_{a \in A_i} \mu_a = u_i(A_i)$ for all $A_i \subseteq S$.*

Theorem 5 implies that despite the fact that ability is binomial, the degree distribution for the Bernoulli skills model is highly skewed–a few players have a disproportionately large number of links, while the majority of players receive no links at all. As an illustration, consider a Bernoulli skills model with $M = 3$ and $p = \frac{1}{3}$. Table 2.3 lists the expected degree of every type of player in this case. Although the distribution of ability is binomial, the distribution of links is highly skewed towards those with more skills. For example, although the players with $\{a, b, c\}$ comprise less than 4% of the population and hold only 11% of the total skills, they help to solve nearly 50% of the problems. The vast majority of the players solve only their own problem, helping no other players at all.
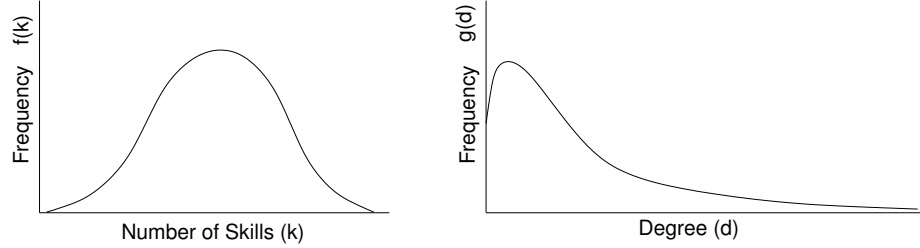
Figure 2.4:
In the Bernoulli Skills Model, a player's degree is increasing in the size of her set of skills. The supermodularity of degree means that players with more skills receive many more links. This exaggerates any initial inequalities in the size of the skill sets. The resulting network has a very distinctive structure–a small number of players participate in a majority of the collaborations.

k	fraction of players	fraction of skills	links per player	fraction of links
3	1/27	1/9	≈ 12.7	$\approx .49$
2	6/27	4/9	≈ 1.8	$\approx .41$
1	12/27	4/9	$\approx .2$	$\approx .10$
0	8/27	0	0	0

Table 2.3:
An illustration of the non-linearity of degree in a Bernoulli Skills model with $M = 3$ and $p = \frac{1}{3}$.

The fact that the degree distribution is skewed is surprising, since the distribution of skill set size (ability) is symmetrical in the Bernoulli Skills model. Figure 2.4 shows both distributions for a representative set of parameters. This example illustrates that even when the distribution of skills across players is symmetrical, the distribution of links is not–because degree is supermodular, small differences in skill set size are magnified in the degree distribution, making the distribution of links highly uneven. This structure is similar to that which we observe in empirical collaboration networks. Figure 2.5 illustrates an example: a network science coauthorship network, in which two scientists are linked if they have coauthored a paper together. The degree distribution of this network is remarkably similar to that of a cost-minimizing collaboration

Degree Distribution
Coauthorship Network for Network Scientists
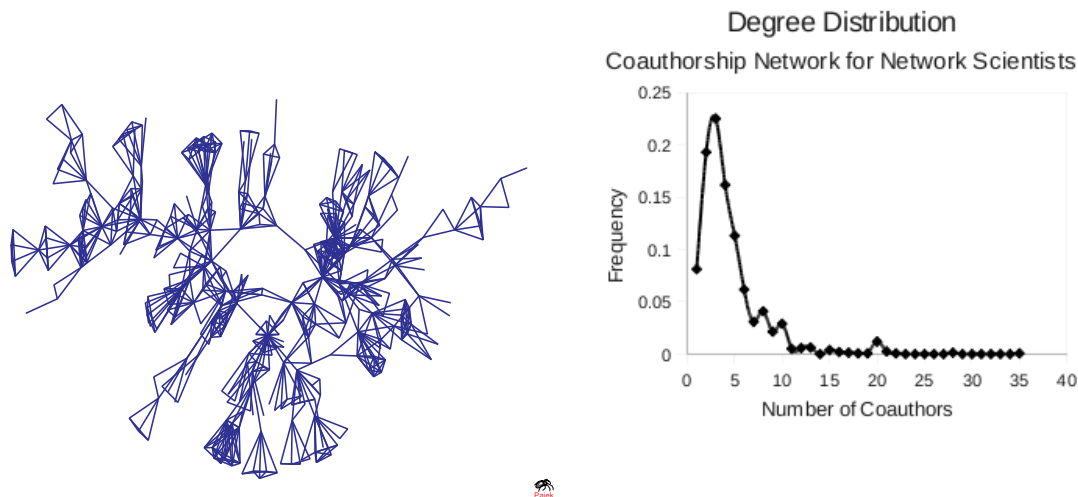
Figure 2.5: Empirically, most collaboration networks have a skewed degree distribution. The left panel depicts a coauthorship network for network scientists– two nodes in this network are connected if the scientists coauthored a paper together (only the largest connected component is shown here). The right panel depicts the degree distribution for this network.

network such as the one pictured in Figure 2.4.

This highly centralized network structure, in which a small number of players participate in most of the collaborations, has implications for other behaviors that take place over collaborative ties. For example, suppose players use collaborative links to share information about jobs, new technologies, and open areas of study. In a network with a skewed degree distribution, most players are connected to a few, high-degree hubs. Thus, the average distance between two nodes in a collaboration network is shorter than we would find in a random network.

A player's degree in the collaboration network reflects their importance to the collaborative community, and therefore the degree distribution also reflects the distribution of influence. In networks with a highly skewed distribution, the high degree players have a more significant impact on opinion formation, reputation, and the decisions over new technology than they would in a network with a more equitable distribution of links. In the following section, I look at the determinants of network

structure more carefully–by looking at the comparative statics on the Bernoulli skills model, I examine how changing the distribution of skills in the population affects the degree distribution of the resulting collaboration network.

2.4.3 Problem Difficulty and the Distribution of Degree

In the following, will use the Gini coefficient as my measure of distributional equality. The Gini coefficient measures the area between the Lorenz curve of a distribution (in this case, the distribution of expected degree), and the line of equality. In the case of a discrete distribution with values $y_0...y_N$ where $y_i < y_{i+1}$, the Lorenz curve is a piecewise function connecting points (F_i, D_i) where $F_i = \sum_{k=0}^{i} \Delta(y_k)$ is the fraction of players with strictly less than y_i links, and $D_i = \frac{\sum_{k=0}^{i} \Delta(y_k) y_k}{\sum_{k=0}^{N} \Delta(y_k) y_k}$ is the fraction of the total number of links held by those players. See Figure 2.6 for an example. The Gini coefficient for a discrete distribution is given by $G = 1 - \sum_{i=1}^{N} D_i (F_i - F_{i-1})$. Lower values of the gini coefficient indicate a more equal distribution of links across players, and higher values indicate a more skewed distribution of links. The coefficient is 0 when the distribution is perfectly equal (ie: the bottom $x\%$ of the population holds exactly $x\%$ of the links) and 1 when all of the links are held by a single player.

Theorem 9 presents a comparative static on the number of skills required to solve the problem. It shows that the degree distribution becomes more uneven when problems require more skills ($M \uparrow$)or the individual skills are less common ($p \downarrow$).

Theorem 9. *Suppose the players face a problem, $\omega(S)$, requiring M skills. If the skills are distributed independently with $Prob(a) = p \, \forall \, a \in S$, then the Gini coefficient of $\Delta_{M,p}$ (the degree distribution of the resulting network) is increasing in M. That is, the distribution of links in the collaboration network is more uneven when the problem being solved requires more skills.*

Proof. Using the fact that $\Delta_{M,p}(d(k)) = \binom{M}{k} p^k (1-p)^{M-k}$ for $k \in \{0, 1, 2, ...M\}$,
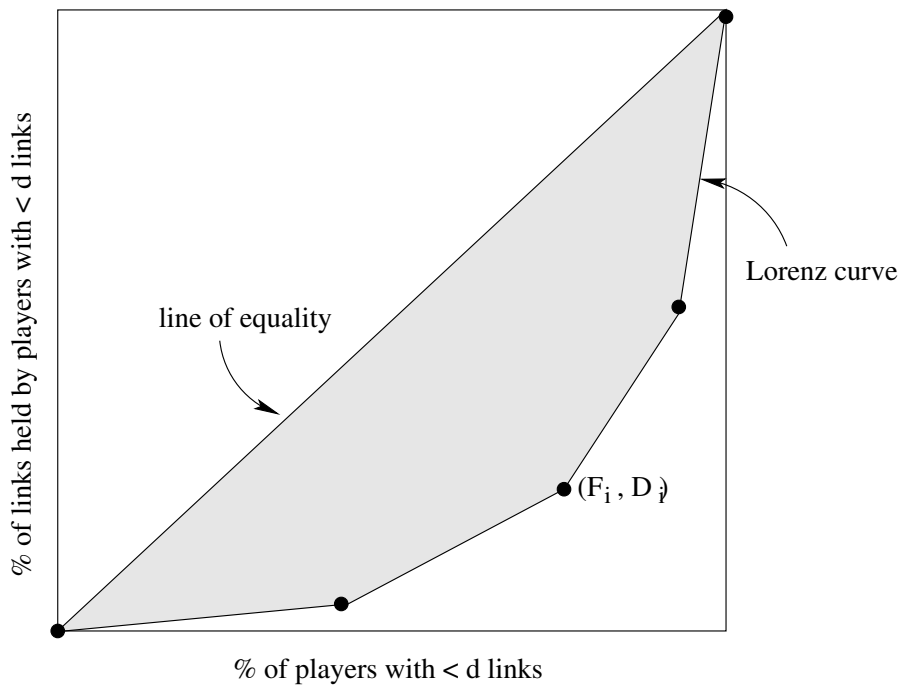
Figure 2.6:
An example of the Gini coefficient for a discrete distribution, $\Delta(y)$. In this case, the random variable y takes on one of five values, $y_0...y_4$. The Gini coefficient is the area of the shaded region between the line of equality and the Lorenz curve.

the gini coefficient of the degree distribution $\Delta_{M,p}$ is

$$G = 1 - \frac{\left(1 - p^M\right)\left(1 - p\right)}{p^M\left[\left(1 - p + p^2\right)\left[\left(\frac{1-p+p^2}{p^2}\right)^M - 1\right] - M\left(1 - p\right)\right]}$$

It can be shown that $\frac{\delta G}{\delta M} > 0$, meaning that the degree distribution becomes more un-even as the number of skills required for the problem increases (holding the probability of having a skill constant). □

Figure 2.7 illustrates this comparative static graphically.
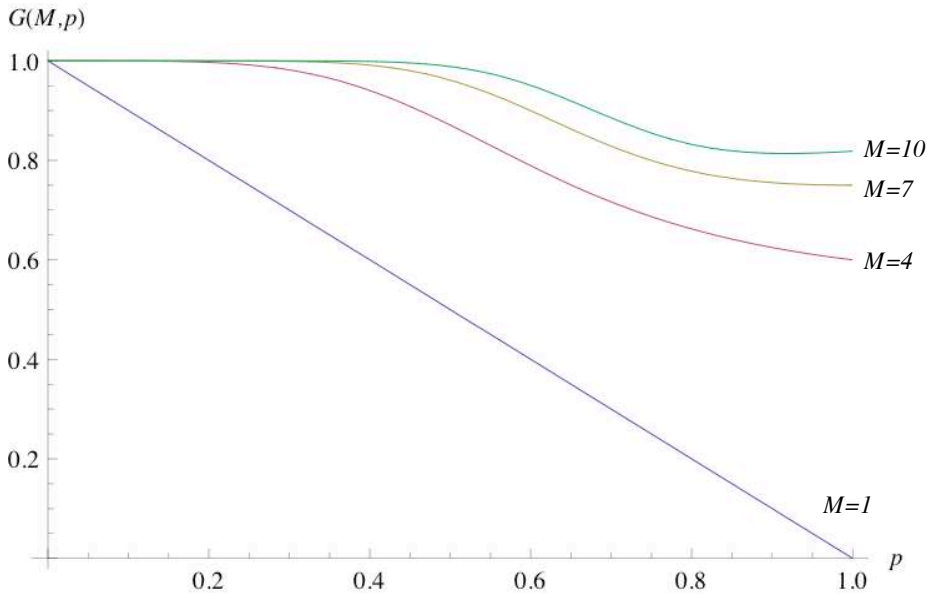
Figure 2.7: A graph of the Gini coefficient, $G(p, M)$, for different populations. The four curves pictured represent different values of M, and the x axis represents the value of p. The difficulty of a problem rises as the number of skills required increases or the probability of having a particular skill falls.

Together, the two parameters of the Bernoulli skills model reflect the difficulty of the problem the population faces–a problem is difficult if it requires many skills, or if the average individual has only a few of them. Theorem 9 can therefore be interpreted as a comparative static on problem difficulty–the model predicts that as

problems become increasingly difficult, a few "superstars" emerge, who participate in most of the collaborations and help solve a disproportionate number of problems.

2.5   Skill Ladders: Specialization and Degree

In the previous section, I considered a special case in which skills are entirely uncorrelated. Although there are cases where problem-solving skills are essentially uncorrelated, we would also like to understand the impact of correlations between skills. In this section, I consider a case where skills are divided into disciplines, and the skills within a discipline build on one another, much as calculus builds on algebra and algebra builds on arithmetic. I show that when skills are correlated in this way, the degree distribution of the collaboration network becomes even more unequal than when skills are uncorrelated. This suggests that as fields become increasingly specialized, a very small number of players will tend to dominate the collaboration network.

2.5.1   Notation and Definitions

Some additional notation is needed to formalize this concept of specialization. I will define a *ladder* to be an ordered set of skills, $L = \{a_1, a_2, a_3...a_l\} \subseteq S$, such that any player who has the ith skill in the set must have all of the skills that precede it in the set.[33]

Definition 10. A *ladder* is an ordered set of skills, $L = \{a_1, a_2, a_3...a_l\} \subseteq S$, such that $Prob$ (have a_i|have a_{i+1}) $= 1$. An example of a single ladder with 6 skills is shown in Figure 2.8.

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[33]Page (2007) introduces this concept of skill ladders, where each skill builds on the one before it.
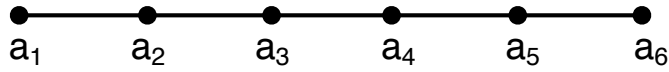
Figure 2.8: A ladder of 6 skills–a player with skill a_i in this set must have all of the skills that precede it: $a_1...a_{i-1}$.

Here, I consider a special case where the skills in S are partitioned into m ladders of equal length.[34] The set of all ladders is denoted $\hat{S} = \{L_1...L_m\}$. Figure 2.9 shows an example with 12 skills arranged into four ladders.
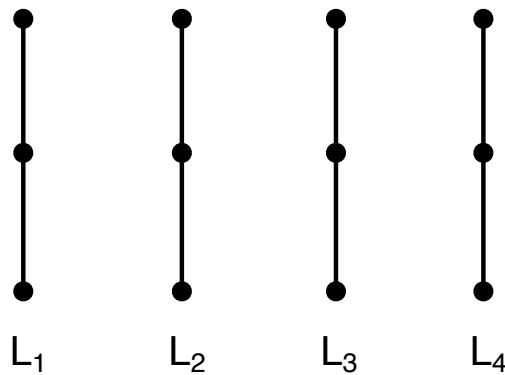
Figure 2.9: An example of 12 skills arranged into four ladders of equal length.

I will call a player who has all of the skills in a single ladder an "expert" in that ladder, and I will call the set of ladders that player i is an expert in $\hat{A}_i \subseteq \hat{S}$.

Definition 11. A player is an *expert* in a ladder L_k if she possesses all of the skills in that ladder.

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[34]Obviously, since the length of a ladder is an integer, there will only be equal-length ladders if m divides M evenly. To simplify the exposition, I have written the following as if this is true. However, all of the the following results hold if the ladders are equal length up to integer constraints, which allows for cases where m does not divide M evenly.

One additional assumption will allow us to compare the results in this section to the results of the Bernoulli skills model. Assume that the conditional probability of having the next skill in a ladder is the same for all skills–that is, $Prob\,(\text{have } a_i|\text{ have } a_{i-1}) = p$ for all a_i.[35] The probability of being an expert in a ladder of l skills is then p^l.

The number of ladders, m, will be our measure of how specialized the problem-solving skills are. Thanks to the above assumption, the case where $m = M$ corresponds to the Bernoulli skills model. On the other extreme, $m = 1$, and all of the skills are arranged in a single ladder. Before considering ladders of arbitrary length, I will first look at this case, where $m = 1$.

2.5.2 Example: a single ladder of skills

Suppose the skills in the set S comprise a single ladder of length M. Because $Prob\,(\text{have } a_i|\text{have } a_{i+1}) = 1 \ \forall i \in S$, a player's skill set can be represented by the number of skills she has ($|A_i| = k$ implies $A_i = \{a_1, a_2...a_k\}$) .

The linking behavior in this case is very simple. The only players who have skill a_M are those who also have skills $a_1...a_{M-1}$. All of the players who don't have all M skills link to one of the players who does. The resulting collaboration network is a set of isolated stars, each with $\frac{1-p^M}{p^M}$ links, on average.

Figure 2.10 compares the network structure in the case with one skill ladder ($m = 1$) to the network structure in the Bernoulli skills model, where skills are in-dependent ($m = M$). The two networks have the same number of skills and players, and players have the same probability of having an additional skill. This means that the probability of having all of the skills required to solve the problem is the same in both networks. Moreover, in both, exactly one player has all of the skills required.

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[35]In other words, I assume that putting a skill at the end of a ladder doesn't change the essential difficulty of obtaining that skill. We could imagine cases where putting a skill at the top of a ladder would make the skill easier to obtain (eg: because it builds on previous experience). We could also imagine a case where skills at the top of the ladder are more difficult to obtain (eg: because they are more demanding than the skills that came before). These would both make interesting extensions.
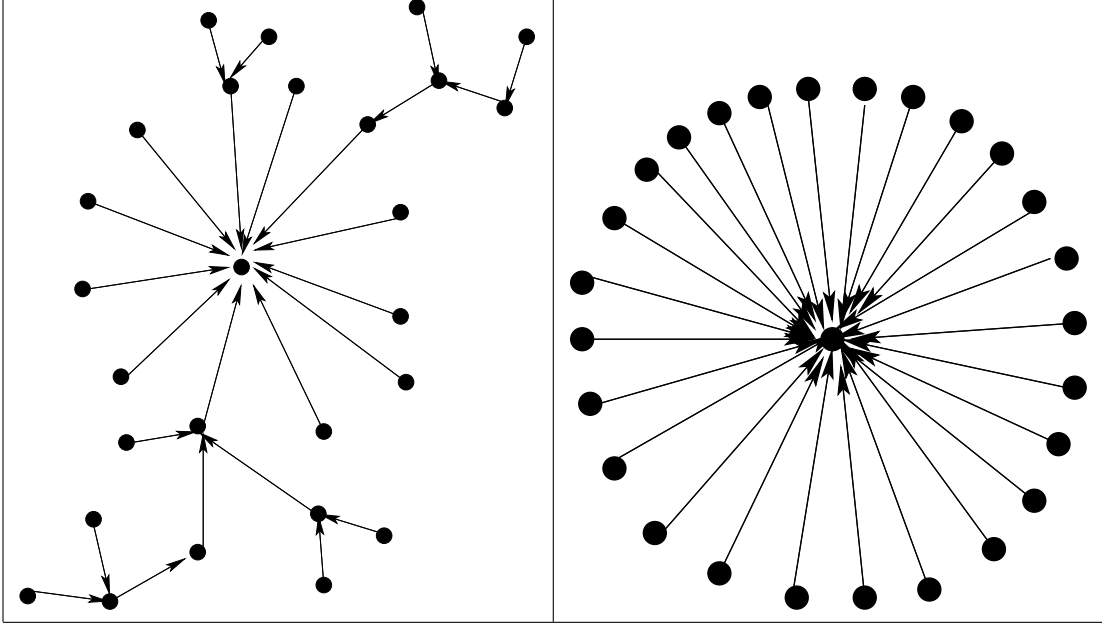
Figure 2.10: Two collaboration networks with 27 players. In both cases, the players are solving a problem requiring $M = 3$ skills. These networks represent the two ends of a spectrum of skill specialization, with $m = M$ ladders on the left and $m = 1$ ladders on the right.

However, the two networks have a much different structure.

2.5.3 Results for m ladders

Using insights gained from this example, I can derive a more general result. Suppose the skills are arranged in m equal-length ladders.[36] As in the previous example, the only players with all of the skills in a ladder are those who are experts in that field. Thus, a model with m ladders reduces to a Bernoulli skills model with m independent skills. Theorem 12 presents a closed-form expression for a player's degree in the case with m skill ladders.

Theorem 12. *If Ψ is a distribution of skills such that $\hat{S} = \{L_1...L_m\}$ is a partition of S into m equal-length ladders with $Prob\left(\text{have } a_i^j \mid \text{have } a_{(i-1)}^j\right) = p$, then a player with the skill set A will have expected degree $Ed(A) = p^M \left[\left(\frac{1 - p^{\frac{M}{m}} + p^{2\frac{M}{m}}}{p^{2\frac{M}{m}}}\right)^k - 1\right]$,*

---

[36]Again, the results are the same if the ladders are equal length to integer constraints.

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where k is the number of disciplines the player is an expert in.

Proof. The ladders are of equal length, so the length of a single ladder is $\frac{M}{m}$, and the probability that a player is an expert in any one ladder is $p^{\frac{M}{m}}$. A player receives a link only if she is an expert in a field. Define a new set of skills that correspond to the set of ladders: $\hat{S} = \{L_1...L_m\}$. The player's new skill set is $\hat{A}_i$, where $L_k \in \hat{A}_i$ if she is an expert in ladder L_k. Each of these new skills has a probability equal to the probability of being an expert in that field, so define $\hat{p} = p^{M/m}$. The probability of being an expert in a particular ladder is independent of the probability of being an expert in any other ladder, so this problem reduces to one with m independent skills, with probability $p^{\frac{M}{m}}$. The result then is a simple extension of Theorem 5. $\square$

We can now do a comparative static on the number of skill ladders, to see how the number of "disciplines" affects the structure of the collaboration network. Theorem 13 indicates that as the skills are concentrated into fewer and fewer disciplines, the collaboration network becomes increasingly skewed.

Theorem 13. *Suppose S skills are arranged in m ladders of equal length, with constant conditional probability $Prob\left(have\ a_i^j\mid have\ a_{i-1}^j\right) = p$ and $Prob\left(have\ a_1^j\right) = p$ $\forall j = 1...m$. The gini coefficient of the resulting network is decreasing in the number of ladders, m. That is, when there are fewer skill ladders, the degree distribution becomes increasingly uneven.*

Proof. Recall the gini coefficient for the Bernoulli Skills model, presented in the proof for Theorem 9. A model with m skill ladders is equivalent to a Bernoulli Skills model with m skills and $p = p^{M/m}$. Substituting into the previous equation, we obtain the following:

$$G = 1 - \frac{(1 - p^m)\left(1 - p^{\frac{M}{m}}\right)}{p^m\left[\left(1 - p^{\frac{M}{m}} + p^{2\frac{M}{m}}\right)\left[\left(\frac{1 - p^{\frac{M}{m}} + p^{2\frac{M}{m}}}{p^{2\frac{M}{m}}}\right)^m - 1\right] - m\left(1 - p^{\frac{M}{m}}\right)\right]}$$

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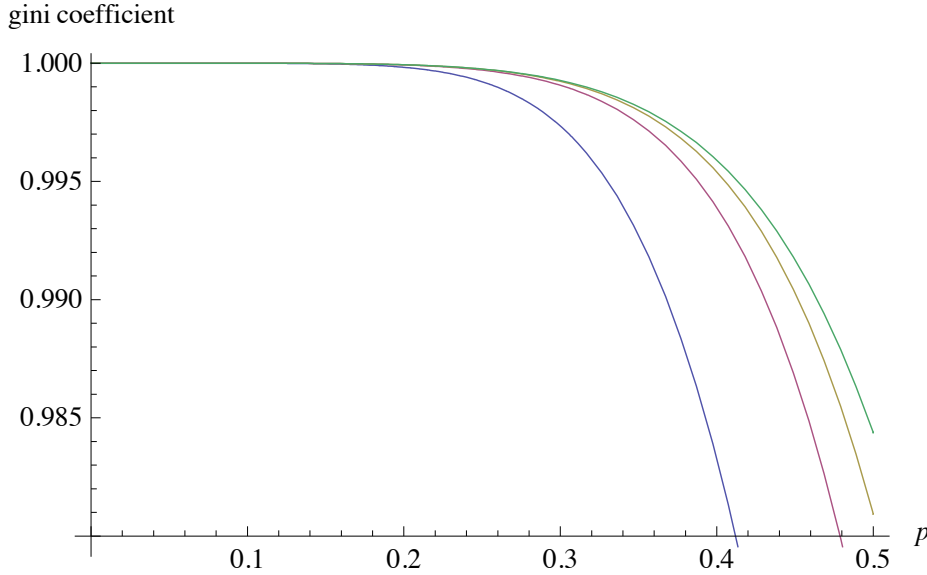
gini coefficient

Figure 2.11: The gini coefficient for $M = 10$, divided into different sets of ladders. The bottom curve pictures the case where $m = 10$, the second curve pictures the case where $m = 5$, the third shows $m = 2$, and the fourth shows $m = 1$.

which is decreasing in the number of ladders, m. Figure 2.11 shows how the gini coefficient depends on the number of ladders for the case where $M = 10$.

□

Skills often build on one another because fields are specialized. The result in Theorem 13 can thus be interpreted in terms of specialization and reliance on experts– as skills become increasingly specialized, we would predict that the degree distribution would become increasingly unequal. This is because when skills are specialized, most players are not useful collaborators. As a result, most problems are solved by a few, high-degree experts. On the other hand, when skills are distributed independently, most players are capable of being useful to someone, and thus the networks will tend to have a much more even distribution of links.

2.6 A Comparison to a Model with a Single Skill

One of the major contributions of the model presented above is its detailed treatment of problem-solving skills. Rather than having a single specialty or "type", players in this model have sets of skills, which may overlap. These two models allow for different levels of analysis. In a "type-based" model, a player's skills are represented by a single unit–her type. In contrast, a "skill-based" model represents a player's skills individually, allowing players' skill sets to overlap and interact in complex ways. In this section, I examine how this more general representation of skills impacts network topology by directly comparing a type-based model of problem-solving to a skill-based model.

In the previous sections, I showed that when players have sets of skills, the relationship between a player's skills and her value as a problem solver is often highly non-linear. In particular, an individual's combination of skills may be valuable, even if her skills are not valuable individually. Since this nonlinearity of degree is driven by the desirability of combinations of skills, we would expect to regain the linear relationship between skills and degree in the special case where players' skill sets do not overlap–that is, where they have a type or specialty.

Consider the following pair of examples. First, suppose a problem requires 3 skills, $S = \{a, b, c\}$, and all three skills are distributed independently to each player–that is, $Prob\,(have\,skill\,i\,|\,have\,skill\,j) = Prob\,(have\,skill\,i)\,\forall i \neq j \in S$. This means that the probability of having skill set A is $\Psi\,(A) = \prod_{i \in A} p_i$. Let $p_a = Prob\,(have\,skill\,a) = \frac{1}{2}$, $p_b = Prob\,(have\,skill\,b) = \frac{1}{3}$, and $p_c = Prob\,(have\,skill\,c) = \frac{1}{6}$. A player in this population will have, on average, exactly one skill. Table 2.4 shows the expected degree for all 8 possible combinations of skills. Note that we cannot price individual skills in such a way that it characterizes a player's degree in the network. To see this, suppose that such a pricing scheme existed–that is, suppose $\exists \mu = [\mu_a, \mu_b, \mu_c]$ such that $\sum_{i \in A} \mu_i = Ed\,(A)$. Then clearly $\mu\,(i) = Ed\,(i)$ for $i \in S$. But that price scheme

A	$\Psi(A)$	$E[d(A)]$
$\emptyset$	$\frac{10}{36}$	0
a	$\frac{10}{36}$	$\frac{1}{18}$
b	$\frac{5}{36}$	$\frac{1}{6}$
c	$\frac{2}{36}$	$\frac{5}{6}$
ab	$\frac{5}{36}$	$\frac{5}{9}$
ac	$\frac{2}{36}$	$2\frac{5}{9}$
bc	$\frac{1}{36}$	6
abc	$\frac{1}{36}$	$18\frac{1}{18}$

Table 2.4:
In this example, the problem requires 3 skills: $S = \{a, b, c\}$. The skills are distributed independently with $Prob\left(have\,a\right) = \frac{1}{2}$, $Prob\left(have\,b\right) = \frac{1}{3}$, and $Prob\left(have\,c\right) = \frac{1}{6}$. This table shows the frequency of each skill set, and the expected degree of an individual with those skills.

would predict that $Ed\left(\{b, c\}\right) = 1$, whereas the actual degree of a player with those skills is $Ed\left(\{b, c\}\right) = 6$.

Now, consider a modification of this example–suppose that instead of skills being distributed independently, we assume that each player has exactly one skill, with $\Psi\left(a\right) = \frac{1}{2}$, $\Psi\left(b\right) = \frac{1}{3}$, $\Psi\left(c\right) = \frac{1}{6}$. The population in this example shares many characteristics with the previous population–the frequency of each skill in the population remains the same, and in both cases, the average player holds one skill. However, in contrast with the previous example, skills in this population can be priced individually. If a player has skill i, her expected degree in the cost minimizing collaboration network is $\frac{1 - \Psi(i)}{\Psi(i)}$. If we set the prices of skills a, b, and c to be $\mu_a = 1$, $\mu_b = 2$, and $\mu_c = 5$, then each player's degree is a linear function of her endowments, weighted by the prices.

The structure of the collaboration network is also considerably different when players' skill sets can overlap. For example, consider the two networks and associated degree distributions pictured in Figure 2.12. In both cases, the players are solving a problem requiring three skills: $S = \{a, b, c\}$. In the network on the left, each player has exactly one skill, and all three skills occur with equal probability ($\Psi\left(a\right) = \Psi\left(b\right) = $
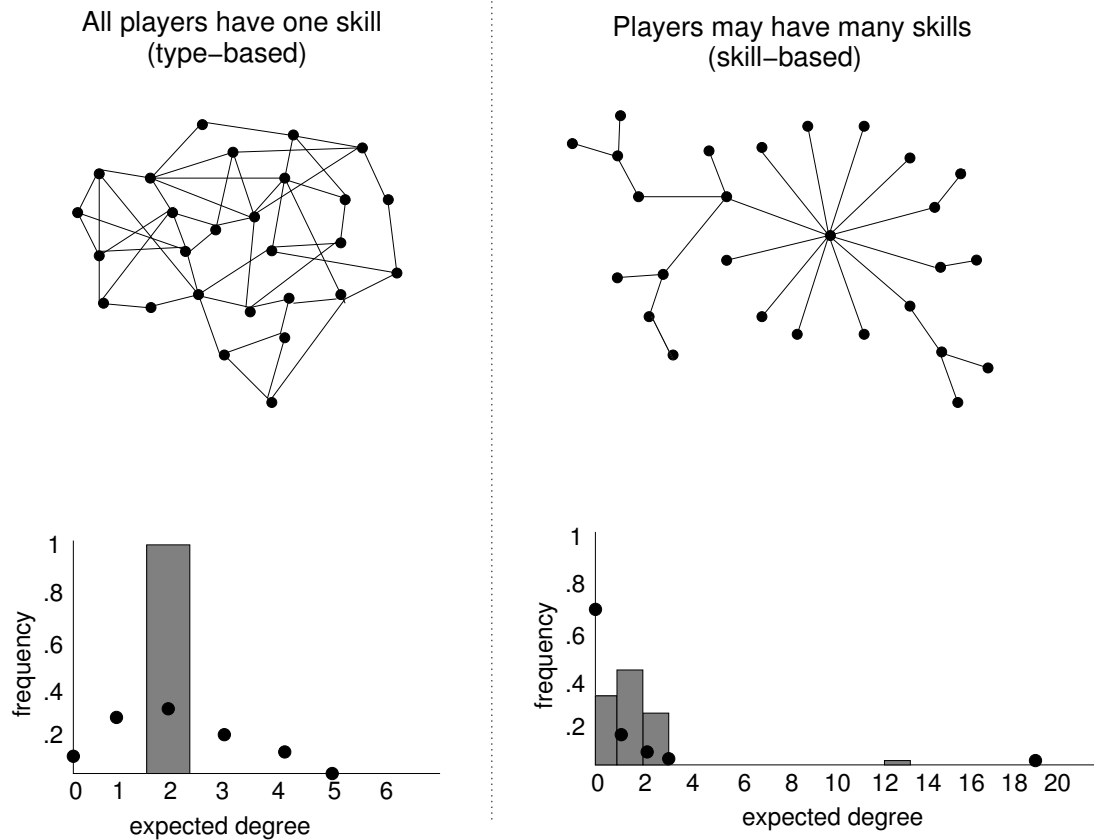
Figure 2.12: Contrasting the structure of collaboration networks resulting from type-based and skill-based models of collaboration. In both cases, the problem faced requires three skills. In the network on the left, each player has a single skill. In the network on the right, the three skills are distributed independently with probability $p = \frac{1}{3}$. The bottom panels show both the distribution of degree in the pictured networks (black dots) and the distribution of expected degree in the set of cost minimizing networks (grey bars).

$\Psi(c) = \frac{1}{3}$). In the network on the right, the skills are distributed independently, as in the Bernoulli skills model, with $Prob(have\ skill\ i) = \frac{1}{3}$ for $i = a, b, c$. In both of these networks, the average player has one skill. However, the degree distribution of the collaboration network is much different in the two cases. Every player in the left hand network has, on average, $\frac{1 - \frac{1}{3}}{\frac{1}{3}} = 2$ links, and the distribution of links in a typical cost-minimizing collaboration network is symmetric around that value. In the left hand network, the network structure is much different. Because combinations of skills are valuable, players with more skills help to solve a disproportionate number of problems. The initial inequalities in the distribution of skills in the population are magnified, resulting in a network of interconnected stars.

This pair of examples illustrates two points. First, the type-based model is a special case of the skill-based model–in particular, it is the case where each player has exactly one skill. Second, it highlights the value of a more detailed treatment of skills in modeling problem solving. Although in some contexts it is appropriate to assume that each player has a type or specialty, this assumption is not necessarily benign. That modeling choice impacts our predictions about the value of certain individuals in the community, as well as impacting the structure of the collaboration network.

In particular, this comparison highlights the difference between modeling the value of labor in manufacturing production and modeling the value of labor in problem-solving. In manufacturing, players can plausibly be given a type or specialty, and the wages of an individual are a function of their type. As production transitions away from manufacturing towards problem-solving, it becomes more difficult to classify workers according to a type or specialty. Thus, the shift from manufacturing production to problem-solving can be modeled as a shift from a type-based model to a skill-based model. This comparison is particularly valuable because it allows us to draw conclusions and make predictions about the labor market as we transition from manufacturing to knowledge-based industries. The following section explores this in

greater detail.

2.7   Implications: Labor Markets and Industrial Organization

2.7.1   Extending the Model to Employment by Firms

Before going further it is worth taking a detour to look at the applicability of this model outside of collaboration. Although the results of the previous section were framed in terms of collaboration between individuals, it is relatively simple to extend this model to firm/employer labor market interactions. Suppose there are two types of agents in an economy: firms and problem solvers. Firms face problems, which have value if solved. Let δ be a probability measure on the skill sets required to solve the firms' problems–that is, $\delta(A)$ is the probability that a firm faces a problem requiring the skills $A \subseteq S$, with $\sum_{A \subseteq S} \delta(A) = 1$. The firms hire problem-solvers to work on projects, giving them each a share of the proceeds from solving the problem.[37] The expected number of projects a player contributes to can be calculated much the same way as before:

$$E\left[d\left(A\right)\right] = \sum_{C \subseteq A} \frac{\delta\left(C\right)}{\sigma\left(C\right)}$$

The sole difference between this case and that in the previous sections is that now the demand for a particular set of skills is decoupled from the supply of those skills.

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[37]I will assume that problem solvers don't bundle themselves together and offer their services jointly for a single share of the problem-solving proceeds. This is consistant with the previous interpretation of the model.

Since the proofs of the results in this chapter do not depend on such a coupling, we can obtain a supermodularity result analogous to Theorem 2 for this firm-based case as well. As a result, all of the results from previous sections apply to a case where individuals work for firms, as well as cases where individuals collaborate.

2.7.2 Returns to Skills and Optimal Training Decisions

Although individuals in this model have static skill sets, we can use it to look at the returns to skills, and thus optimal training decisions. Using the result from Theorem 2, Theorem 14 shows that in a static population, returns to acquiring new skills are non-decreasing.

Theorem 14. *If a player's utility is increasing in her degree in the collaboration network, then players experience non-decreasing returns to obtaining additional skills, holding the rest of the population fixed.*

Proof. This is a simple application of supermodularity.

Theorem 2 states that $d(A_i \cup A_j) + d(A_i \cap A_j) \geq d(A_i) + d(A_j)$. Rearranging, this tells us that $d(A_i \cup A_j) - d(A_i) \geq d(A_j) - d(A_i \cap A_j)$. Both sides of this inequality represent a gain in degree from obtaining the skills in the set $A_j \backslash A_i$. The fact that this gain is greater when the player already has A_i indicates that returns are non-decreasing. $\square$

The implication of this result is that every player will obtain the maximum number of skills possible. Thus, this model predicts that a shift in the economy from manufacturing to knowledge-based production might be accompanied by increasing returns to skills, and thus increased skill acquisition.

However, that is not the whole story–individuals also have to choose which skills to obtain in which order. In a world with many possible skills, and costs to obtaining new skills, how will a worker determine the optimal skills to acquire? I can

calculate the contribution of each skill in a player's skill set to the total demand for her skills using a Shapely value decomposition. The demand for player i's skills can generically be written as $d(A_i) = \sum_{C \subseteq A_i} \frac{\delta(C)}{\sigma(C)}$ where $\delta(C) = \Psi(S \backslash C)$[38] and $\sigma(C) = \sum_{D \subseteq S \backslash C} \Psi(C \cup D)$. Using this demand as a value function,[39] we can obtain an expression for the Shapely value of a skill, a, to a player, i.

Theorem 15. *The Shapley value for a skill, a, to a player, i, is*

$$\phi_{a,i}(d) = \sum_{B \subseteq A_i \backslash \{a\}} \frac{1}{\binom{|A_i|}{|B|}} \left(\sum_{C \subseteq B} \frac{\Psi(S \backslash (C \cup a))}{\sum_{D \subseteq S \backslash (C \cup a)} \Psi((C \cup a) \cup D)} \right)$$

Theorem 15 shows that the Shapley value of a skill to an individual depends on 1) the existing skill distribution in the population and 2) the set skills the individual already has. This theorem suggests that optimal training decisions should be highly individualized, because the value of a skill depends on problem solver's existing skill set and the skills of others in the population.

2.7.3 Variation in Labor Demand

The number of collaborators an individual has (her degree in the collaboration network) can be interpreted as the demand for her skills as a collaborator. Empirically, it has been observed that output is highly concentrated among a small number of people, creating an extremely skewed distribution of demand. For example, Rosen (1981) observes that in certain creative fields (eg: music, film, textbook writing),

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[38]Note that this is the expression when each player needs exactly one partner to solve their problem. The results are similar for the more general case.

[39]Note that $d(.)$ satisfies both requirements for a value function: $d(\emptyset) = 0$ and according to Theorem 2, $d(A \cup B) + d(A \cap B) \geq d(A) + d(B)$, which implies superadditivity.

a large fraction of demand goes to an extremely small number of producers. Uzi and Spiro (2005) observe a similar patter among the directors, producers, and other creative artists on Broadway. Similarly, data on academic collaborations suggests that a small fraction of researchers are responsible for the majority of output (see Newman (2001), Moody (2004), Acedo et al (2006), and Goyal et al (2006)).

This long-tailed distribution has implications for the distribution of wages and welfare (see the next section for further discussion) and thus there has been considerable interest in understanding why such a concentration in labor demand occurs. Some existing models (for example, Rosen (1981)) can induce a long-tailed distribution when there is a high premium on quality, and production technology decouples effort from output quantity (eg: in creative industries, where a single performance or album can be enjoyed by many consumers). However, such technologies are not relevant in knowledge-based industries, where effort is not decoupled from output volume.

The model presented in this chapter induces a long-tailed distribution of demand in cases where collaboration is important. As noted above, the search for complementarities and the bundling of skills within individuals exaggerates existing inequalities in skill distributions, creating a long-tailed distribution of demand. Thus, this model can explain uneven output demand when effort is not fixed, but skills are varied. Moreover, the model makes predictions about the comparative statics of the distribution of output. Specifically, the distribution of demand will become more skewed as problems become more difficult. We can observe this trend in longitudinal studies of coauthorship, such as Moody (2004) and Grossman and Ion (2002). These studies observe the number of collaborations maintained by authors in the fields of sociology and mathematics, respectively. It is widely believed that the problems faced in these fields have become more difficult over time. The authors show that the upper tail of the collaboration distribution extends over time, indicating that a small number of in-

dividuals capture an ever-increasing fraction of the collaborative demand. This model connects those two trends, attributing the attenuation of the demand distribution to the increasing difficulty of the problems being faced.

2.7.4 Industrial Organization

The results that precede also have implications of the organization of knowledge-based firms. Traditional organizational structures were hierarchical, with several layers of supervisors. The theoretical underpinnings of these hierarchical structures are explored in a wide range of models, including Rosen (1982). However, evidence indicates that organizational structure within firms is changing–hierarchical structures are flattening, and workplaces are becoming more decentralized (see, for example, Bresnahan *et al* (2002) and Rajan and Wulf (2006)).

The model presented in this chapter provides a more general model of organizational structure, one that allows for these more complex interrelationships between individuals. In particular, this model allows me to explore how a shift towards knowledge-based industries and team-based production affects organizational structures. In knowledge-based firms, value is created through the creation of new knowledge, rather than the exploitation of existing knowledge.

My model produces strictly hierarchical structures in one special case–that in which skills are arranged in a single ladder (see section 2.5.2 for the details of this case). This case corresponds to a model in which ability is measured on a one-dimensional scale. One dimensional measures of ability make sense in industries that create value by exploiting an existing bank of knowledge. However, as demand for workers shifts towards industries that create value by creating new knowledge, we expect skills to be arranged in ladders less often. My model predicts that as such a shift occurs, organizational structure should move away from hierarchies, towards flatter, more distributed structures.

2.8   Extensions

The model I present in this chapter suggests a wealth of extensions. In the following, I will briefly discuss a few of the more promising of these extensions.

2.8.1   Bargaining Over Surplus

As mentioned earlier, there are several reasons to consider an even split of payoffs between collaborators–indeed, when payoffs to problem solving are non-monetary, it may be difficult to split returns any other way.[40] The results in this chapter also hold for a variety of other payoff-splitting methods–in particular, the results hold for any payoff split which induces an individual to minimize the number of collaborators she has.[41]

However, one might also want to consider what happens when individuals can bargain over the surplus from solving a problem. This bargaining process gives each player a "wage" from the collaboration,[42] which I can then use to produce an income distribution, much as we produce an output distribution in this chapter.

By producing an income distribution, I can attempt to explain some empirical trends in labor markets. In particular, income inequality has grown dramatically over the past 50 years, a trend that holds even if one controls for years of education and tenure (Juhn et al (1991)). The labor literature draws a connection between this trend towards greater wage inequality and the shift towards knowledge-based production in our economy, telling a story that is potentially consistent with the model presented in this chapter: knowledge-based production creates increasing returns to skills, a trend which benefits high-skilled workers over low-skilled workers, widening the wage

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[40]It is, for example, difficult to split the authorship of a publication into arbitrarily-sized shares.

[41]Note that this is the case whenever an individual's payoff to solving her problem is decreasing in the number of collaborators she has. So, for example, if links are costly, the results will be the same.

[42]It is possible to show two necessary conditions for non-zero wages: first, firms must face some friction in hiring (such as search costs) and second, problem solvers must face constraints on their time.

gap.[43] More recent work notes a second-order effect of the shift–the wage distribution widens most significantly in the upper tail of the distribution. According to Machin (2008), the wage difference between the 90th percentile and the 50th percentile rose throughout the 80s, 90s, and early 2000s. On the other hand, the wage difference between the 50th percentile and wages in the 10th percentile increased significantly in the 80s, only slightly in the 90s, and actually contracted in the 2000s.

These empirical observations are not inconsistent with the predictions of this model. As noted in Section 2.6, we can model the shift in production as a shift from a type-based model to a skill-based model. As production becomes more problem-solving based, the upper tail of the output distribution stretches dramatically, while the lower tail contracts, increasing income inequality overall. By explicitly connecting income to the output, I will be able to determine whether these results on the output distribution can be extended to the income distribution.

2.8.2   Long Run Skill Acquisition

Thus far, I have assumed that the skill distribution in the population is fixed. However, in the long run, we would expect problem-solvers to acquire new skills, based on what will optimize their expected outcomes. I have already shown that in a static world, the individual's skill acquisition decisions will depend on the current state of the world (distribution of skills in the population) and the set of skills the individual possesses. It would be interesting to look at the long-run steady state population of problem solvers that results from this dynamic skill acquisition process.

One might ask a number of question about this steady state population. Is the equilibrium distribution of skills in the population efficient? Do individual players

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[43]See, for example, Juhn et al (1991), who find a trend towards increased wage inequality, and attribute that gap to increasing returns to skills (which they specifically differentiate from years of education or tenure). Other, related papers include Kruger (1993) and Berman *et al* (1998). More recently, Machin (2008) refines this notion, pointing out that technological advances in the 80s eliminated many routine tasks, accelerating the increasing returns to skills.

acquire too many skills, from a societal perspective? I can also use this model to ask questions about the equilibrium levels of specialization and generalization in the problem solving population. Under what conditions should we see individuals specializing in a set of related skills? In a companion paper, I explore whether specialists and generalists can coexist in the same equilibrium population. It would also be interesting to see whether, in the extreme long run, disciplines (groups of individuals with related, specialized skills) emerge. Finally, if the distribution of problems faced changes over time, then the steady state population may be subject to shocks in the types of problems faced. A long run model with skill acquisition can explore questions about the robustness of the populations to these shocks.

2.9 Conclusion

In this chapter, I present a model of problem-solving and collaboration. I use this model to look at the demand for a problem-solver's skills as a collaborator. I show that because a problem solver's skills are bundled, the number of problem she solves is a supermodular function of her set of skills–in other words, an individual's value as a problem solver is more than the sum of her individual skills. Each additional skill multiplies the number of skill combinations that the player can use, unlocking many potential synergies with other problem solvers. Moreover, a player who has a particularly useful combination of skills may participate in more collaborations than a player who has many rare skills, but does not fill such a hole in the organization. The fact that players collaborate with those who have complementary skills also has an effect on the structure of the collaboration network as a whole–the model predicts that the degree distribution of a collaboration network will be skewed, even if skills are distributed independently across players. In other words, the model predicts that a few players will solve the majority of the problems a population faces, even if the distribution of skills in the population is symmetrical. Finally, this model connects

the nature of problems and problem solvers to the structure of the collaboration network–as problems become more difficult for a population, the model predicts that the collaboration network become more centralized around a few, high degree hubs. In sum, this framework, in which problem solvers have skill sets and collaborate to solve problems, appears to be sufficiently flexible to address many questions about the returns to skills in knowledge production.

CHAPTER III

Foxes and Hedgehogs: Equilibrium Skill Acquisition Decisions in Problem-solving Populations

3.1 Introduction

In this chapter, I consider the decision to be a generalist or a specialist. Being a generalist is costly. Each of us has a limited capacity for learning new things. By focusing on a narrow field of study, specialists are able to concentrate their efforts and maximize the use of that limited capacity. Generalists, on the other hand, are forced to spread themselves more thinly in the pursuit of a wider range of knowledge. They pay a penalty for each new field of study they pursue, in the form of effort expended learning new jargon, establishing new social contacts in a field, and becoming familiar with new literatures. This tradeoff between focus and depth has been observed empirically in Adamic et al (2010).

In the economics and sociology literatures, it has become common to refer to this distinction between specialists and generalists via a metaphor, used by Isaiah Berlin in an essay on Leo Tolstoy: "The fox knows many things, but the hedgehog knows one big thing" (Berlin (1953)).[1] In other words, generalists are foxes, who know

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[1]See, in particular, Tetlock (2005), which uses the metaphor to examine whether specialists or

many things, and specialists are hedgehogs, who know one big thing. The question I will address in this chapter is as follows: given the significant penalty paid by foxes in acquiring their skills, is there ever a reason that a rational actor would choose to become one? And is the decision made by individual problem solvers socially optimal?

As I will show in this chapter, it can be difficult to justify the decision to become a fox. In fact, when problems are simple and disciplinary boundaries fluid, I will show that it is impossible to rationalize being a fox–obtaining skills in multiple disciplines is costly, and there are no benefits to doing so. In order to justify becoming a fox, there must be some advantage to having skills in multiple areas.

Here, I look at two conditions under which foxes can have an advantage over hedgehogs. First, by virtue of having background in multiple areas, foxes may be exposed to more ideas and have opportunities to address a wider variety of questions. Second, if problems have multiple parts, then it may be the case that some elements of a given problem are best addressed using skills in one discipline and other elements are best addressed using skills in another discipline. In other words, as problems become more difficult,[2] there may exist problems that are truly interdisciplinary, and can best be addressed by individuals with skills in many different areas.

The rest of this chapter is as follows. In the first section, I model an individual problem-solver's skill acquisition decision–in particular, her decision whether to specialize in one discipline, or generalize over multiple disciplines. Foxes (generalists) pay a fixed cost for obtaining skills in a new discipline. In the following two sections, I use this model to characterize the conditions under which there is a population of foxes in equilibrium. I show that when problems are simple (consist of only one part), and there are no barriers to communication between fields, foxes receive no benefit from diversifying their skills, and in equilibrium all problem solvers will choose to be

generalists make better predictions.

[2] I use the term "difficult" in a similar sense to Page (2008)–difficult problems have many possible solutions, none of which is optimal, *a priori.*

hedgehogs. When problems are simple, a population of foxes can only be sustained if there are significant communication or institutional barriers between disciplines, which prevent specialists in one field from working on problems in another field. In the presence of such barriers, being a fox is rationalizable as long as problems are distributed relatively evenly across the different disciplines. I then show that from a societal perspective, this equilibrium population may be suboptimal. In particular, there are parameter regions where society would prefer to have some foxes, but the equilibrium population consists of only hedgehogs. In other words, because foxes internalize the costs of diversifying their skills, some populations suffer from an undersupply of foxes. This suggests that more problems may be solved by subsidizing the costs of skill diversification. Finally, I present an extension of the model, which looks at problems with multiple parts. I show that when problems have more than one part, a population of foxes can be sustained, even in the absence of institutional barriers between disciplines.

3.2 Model

I construct a two period model. In period 1, players choose their skills. In period 2, they, as individuals, attempt to solve a problem using those skills. Note that in this model, there is no collaboration. The problem in period 2 is drawn from a distribution of problems. Although players know the distribution of problems when making their skill acquisition decisions, they do not know the particular problem they will face. Each individual chooses her skills in period 1 to maximize her individual probability of solving the problem in period 2. I will solve for the equilibrium choice of skills.

Let S be the set of all possible skills.[3] The skills are arranged into 2 disciplines, d_1 and d_2, each with K skills, $s_{1d}...s_{Kd}$. An example with six skills arranged into

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[3]Skills are defined as bits of knowledge, tools, and techniques useful for solving problems and not easily acquired in the short run. See Anderson (2010) for a model with a similar treatment of skills.
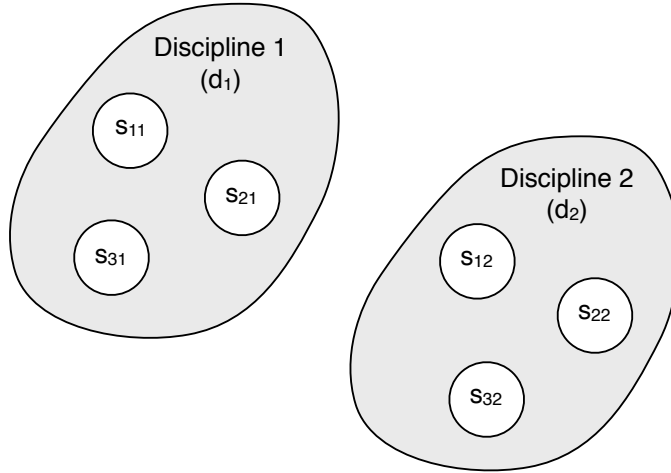
Figure 3.1: Two disciplines, each with three skills.

two disciplines is shown in Figure 3.1. A *specialist* is a person who chooses skills within a single discipline. A *generalist* is a person who chooses some skills from both disciplines.

A problem, y, is a task faced by the individuals in the model. A *skill* is a piece of knowledge that can be applied to the problem in an attempt to solve it. Each skill $s_{kd} \in S$ has either a high probability (H) or a low probability (L) of solving the problem. I will define a problem by the matrix of probabilities that each skill will solve the problem. That is, $y = \begin{bmatrix} y_{11} & y_{12} \\ \vdots & \vdots \\ y_{K1} & y_{K2} \end{bmatrix}$ where $y_{kd} = H$ if skill k in discipline d has a high probability of solving the problem and L if it has a low probability of solving the problem. So, for example, if there are two disciplines, each with three skills, a problem might be

$$y = \begin{bmatrix} L & H \\ H & H \\ H & L \end{bmatrix}$$

meaning that two of the skills in each discipline have a high probability of solving the problem, and one skill in each discipline has a low probability of solving the problem.

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Define $h \equiv 1 - H$ and $l \equiv 1 - L$.

The mechanics of the model are as follows. In period 1, the players, $i_1...i_N$, each choose a set of skills $A_i \subset S$. In period 2, the players attempt to solve a problem using those skills. I will assume that players have a *capacity* for learning skills, which limits the number of skills they can obtain. For the moment, I will assume that all players have the same capacity for learning new skills. Let $M \in \mathbb{Z}^+$ represent the players' capacities for new skills. For the moment, I will assume that all skills are equally costly to obtain, and in particular, if q_{dk} is the cost of obtaining skill k in discipline d, then $q_{dk} = q = 1$. I will also assume that player pay a fixed cost, c, for learning skills in a new discipline. That is, players pay $1 + c$ to obtain the first skill in a discipline, and $q = 1$ for every additional skill in that discipline. For simplicity, I will assume that $M = K + c$. This assumption means that a specialist can obtain all K skills in one discipline, and a generalist can obtain a total of $K - c$ skills spread over the two disciplines.

Although players in period 1 do not know the particular problem they will face, they do know the distribution from which those problems are drawn. In particular, they know the probability that each skill will be an H skill or an L skill. I will assume that the probability that skill s_{kd} is an H skill is independent of the probability that skill $s_{k'd'}$ is an H skill.[4] For the moment, I will also assume that skills are *symmetric within disciplines*. That is, I will assume that every skill in a discipline has an equal probability of being an H skill. This assumption simplifies one decision made by the players–namely which skills a generalist will choose within each discipline. When skills are symmetric within a discipline, a generalist's skill acquisition decision is simply a division of her skills across the two disciplines–within a discipline, she can choose her skills at random.

Let δ_d be the probability that a skill in discipline d is an H skill–that is, $\delta_d =$

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[4]This assumption means that skills must work more or less independently. That is, it cannot be the case that skills are used in combination to solve problems, or that skills build on one another.

$E\left[Prob\left(y_{kd}=H\right)\right]$ where the expectation is taken over the distribution of problems, which I will call Δ. I will assume that the vector of probabilities in the two disciplines, $\delta=[\delta_1,\delta_2]$, is known *ex ante.* This means that while the players do not know the particular problem they will face, they do know the *distribution* of problems.

Players choose their skills to maximize their expected probability of solving the problem in period 2. A Nash equilibrium of this game is a choice of skill set for each player in the population, $A=\{A_1...A_N\}$, such that no player has an incentive to unilaterally change her skill set, given the distribution of problems.

3.3 Results: Specialization and Barriers Between Disciplines

In this section, I consider two questions. The first question concerns individual decision-making–what is the equilibrium skill acquisition decision of the problem solvers? Under what conditions do individuals decide to generalize? The second question concerns the optimality of that population from a societal perspective. Is the equilibrium population optimal?

3.3.1 A Special Case: Symmetry Across Disciplines

Before considering the general model, I first look at a special case where $\delta_1=\delta_2=\delta$. In this case, all disciplines are expected to be equally useful in expectation. This case is useful because it produces particularly clear results. The following subsection generalizes the results from this section to a case where $\delta_1\neq\delta_2$.

Given that generalists pay a significant penalty for diversifying their skills, it is difficult to explain the existence of generalists in the population. Theorem 16 states that if there are no barriers to problem solvers working on problems in other fields, then there will be no generalists in equilibrium.

Theorem 16. *If skills are symmetric, and players can work on any available problem,*

then no player will ever want to be a generalist. In other words, the equilibrium population will contain only specialists.

Proof. The *ex ante* probability that a specialist in discipline i will be able to solve a problem from a given distribution, Δ, is

$$
\begin{aligned}
E\left[P\left(S_i\right)\right] &= \sum_y Prob\left(\text{one of skills solves } y\right) * \Delta\left(y\right) \\
&= \sum_y \left(1 - Prob\left(\text{none do}\right)\right) * \Delta\left(y\right) \\
&= 1 - \sum_{n_i=0}^K h^{n_i} l^{K-n_i} \binom{K}{n_i} \delta^{n_i} \left(1-\delta\right)^{K-n_i} \\
&= 1 - \left(\delta h + \left(1 - \delta\right) l\right)^K
\end{aligned}
$$

where n_i is the number of H skills in discipline i in a particular problem, y.

Now, consider a generalist who is spreading his skills across both disciplines. The *ex ante* probability that a generalist with x skills in discipline 1 and $K - c - x$ skills in discipline 2 will solve a problem from a given distribution, Δ, is

$$
\begin{aligned}
E\left[P\left(G\right)\right] &= 1 - \sum_y Prob\left(\text{none of skills solve } y\right) * \Delta\left(y\right) \\
&= 1 - \left(\delta h + \left(1 - \delta\right) l\right)^{K-c}
\end{aligned}
$$

Clearly, $1 - \left(\delta h + \left(1 - \delta\right) l\right)^K > 1 - \left(\delta h + \left(1 - \delta\right) l\right)^{K-c}$, and thus no individual will ever be a generalist in two disciplines.[5] □

Theorem 16 clearly indicates that when there are no barriers to solving problems in other fields, there is no advantage to being a generalist. However, in practice, there may be several barriers between disciplines that prevent individuals in one discipline from solving problems in another. An individual in one field may simply

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[5]Note that this result generalizes to a case with more than two disciplines. Generalists do worse as they add skills in additional disciplines, so this result holds regardless of the number of disciplines a generalist spreads himself across.

be unaware of problems that exist in other areas, even if her skills would be useful in solving them. Cultural or institutional barriers may prevent her from working on the questions she knows about, either because resources are not forthcoming, or because publication is difficult. Communication barriers are also a significant impediment to interdisciplinary work–although a physicist may have skills that would be useful in solving a biology problem, field-specific jargon may make it difficult for her to communicate her insights, and if communication barriers are severe enough, she may even have difficulty understanding what open questions exist.

Barriers to working on problems outside ones discipline give specialists a disadvantage, because they are only able to work on problems in their own fields. Theorem 32 states that if there are barriers to working on problems outside of ones field, then a population of all generalists may be sustained in equilibrium. In particular, if there are two disciplines, and a fraction ϕ of all problems occur in discipline 1 , then generalists dominate the equilibrium population for values of ϕ in a particular range. As shorthand, let $\pi(\delta, h, l) \equiv (\delta h + (1 - \delta) l)$ be the expected probability that a skill won't be able to solve a problem drawn from Δ. Figure 3.2 shows the ranges of ϕ in which individuals will specialize and generalize, as a function of π. If a sufficient fraction of the problems are assigned to one or the other discipline, then all players will specialize in that discipline. If problems tend to be evenly distributed across the two disciplines, then players will tend to be generalists and acquire skills in both disciplines.

Theorem 17. *If skills are symmetric, and problems are assigned to one of two disciplines, then players will generalize (obtain $K - c$ skills spread across the two disciplines) if $1 - \left(\frac{1 - \pi^{K-c}}{1 - \pi^K} \right) \leq \phi \leq \frac{1 - \pi^{K-c}}{1 - \pi^K}$ where ϕ is the fraction of problems assigned to discipline 1. If $\phi > \frac{1 - \pi^{K-c}}{1 - \pi^K}$, then the player will specialize in discipline 1 and if $\phi < 1 - \left(\frac{1 - \pi^{K-c}}{1 - \pi^K} \right)$ then the player will specialize in discipline 2.*

Proof. In this case, the *ex ante* probability that a problem is solved by a specialist is

All specialists in discipline 2

All generalists

All specialists in discipline 1

0 $\dfrac{\pi^K - \pi^{K-c}}{1 - \pi^K}$ $\dfrac{1 - \pi^{K-c}}{1 - \pi^K}$ 1 ϕ
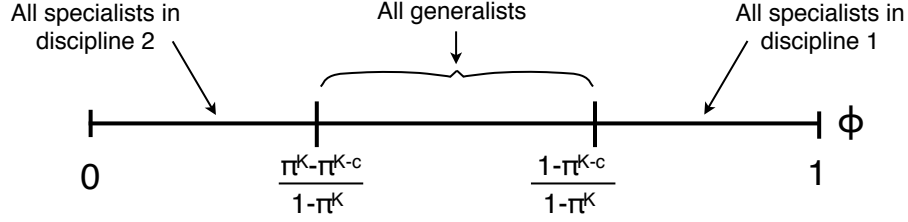
Figure 3.2: The equilibrium skill acquisition decisions as a function of the fraction of problems assigned to discipline 1 in the case where skills are symmetric across disciplines (that is, where $\delta_1 = \delta_2 = \delta$). If enough problems are assigned to one of the disciplines, then all players will specialize in that discipline. If problems are more evenly distributed across disciplines, than players will tend to generalize. The point at which individuals will start to generalize depends on $\pi = \delta h + (1 - \delta) l$, the expected probability that a single skill will not solve the problem.

$\phi \left(1 - (\delta h + (1 - \delta) l)^K\right)$ for a specialist in discipline 1 and $(1 - \phi) \left(1 - (\delta h + (1 - \delta) l)^K\right)$ for a specialist in discipline 2. Since generalists can work on problems in both disciplines, their expected probability of solving the problem is $1 - (\delta h + (1 - \delta) l)^{K-c}$. A player will generalize if $E[P(S_1)] < E[P(G)]$ and $E[P(S_2)] < E[P(G)]$. The result follows immediately. $\square$

3.3.2 Optimality of the Equilibrium

The next question that we can ask is whether the distribution of specialists and generalists in the population is socially optimal. There is reason to believe that it would not be. From a societal standpoint, we would like to maximize the probability that someone in the problem solving population manages to solve the problem. Individuals pay the price for diversifying their skills, which makes it individually rational to specialize. However, on a societal level, it is optimal to have problem solvers apply a wide range of skills to the problems faced. Theorem 18 states that for some values of ϕ (the fraction of problems that occur in discipline 1) the equilibrium population
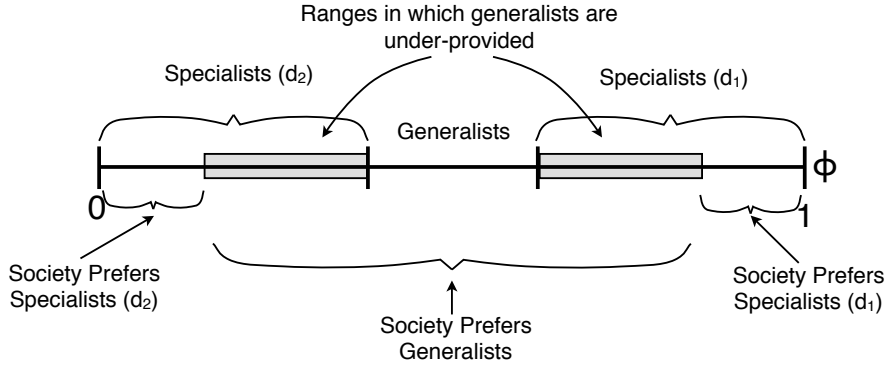
Figure 3.3: Regions of social suboptimality. The line shows values of the parameter ϕ–the fraction of problems assigned to discipline 1. The top part of the figure shows the parameter regions where individuals decide to specialize. The bottom part of the figure shows where specialization is optimal from a societal perspective. The shaded areas indicate the parameter values where individuals choose to specialize, but society would prefer to have at least a few generalists.

is suboptimal. In particular, there will be an underprovision of generalists. Figure 3.3 illustrates these regions of social suboptimality. Note that these regions will exist for any value of $\pi \in (0, 1)$.

Theorem 18. *If skills are symmetric and problems are assigned to one of two disciplines, then there is a range of values for ϕ (the fraction of problems assigned to discipline 1) such that generalists are underprovided in the equilibrium population of problem solvers.*

In particular, generalists are underprovided when $\frac{1-\pi^{K-c}}{1-\pi^K} < \phi < \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}}$ or $1 - \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}} < \phi < 1 - \frac{1-\pi^{K-c}}{1-\pi^K}$.

Proof. The probability that at least one of the N problem-solvers in the population solves the problem is $1 - Prob$ (none of them do). If all of the individuals in the population are specialists in discipline 1, then with probability ϕ, each specialist has a probability $1 - \pi^K$ of solving the problem and π^K of not solving it. With probability $1 - \phi$, the problem is assigned to the other discipline, and no specialist

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solves it. Thus, the probability of someone in a population of discipline 1 specialists solving the problem is

$$
\begin{aligned}
Prob\,(\text{one of N solve it}) \; &= \; 1 - Prob\,(\text{none of N solve it}) \\
&= \; 1 - [\phi Prob\,(\text{none solve problem in } d_1) \\
&\quad\; + (1 - \phi)\,Prob\,(\text{none solve problem in } d_2)] \\
&= \; 1 - \Big[\phi Prob\,(\text{one fails})^N + (1 - \phi) * 1\Big] \\
&= \; 1 - \Big[\phi\left(\pi^K\right)^N + (1 - \phi) * 1\Big] \\
&= \; \phi\left(1 - \pi^{KN}\right)
\end{aligned}
$$

On the other hand, if they are all generalists, then the probability of at least one solving the problem is

$$
\begin{aligned}
Prob\,(\text{one of N solve it}) \; &= \; 1 - Prob\,(\text{none of N solve it}) \\
&= \; 1 - \left(\pi^{K-c}\right)^N \\
&= \; 1 - \pi^{N(K-c)}
\end{aligned}
$$

Society is better off with a population of generalists when $1 - \pi^{N(K-c)} > \phi\left(1 - \pi^{KN}\right)$, which is true when $\phi < \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}}$. However, there is a population of generalists when $\phi \leq \frac{1-\pi^{K-c}}{1-\pi^{K}}$. It is always the case that $\frac{1-\pi^{K-c}}{1-\pi^{K}} \leq \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}}$. So if $\frac{1-\pi^{K-c}}{1-\pi^{K}} < \phi < \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}}$, then society is better off with a population of generalists, but has a population of specialists.

We can make a similar argument for specialists in discipline 2. Society is better off with a population of generalists when $1 - \pi^{N(K-c)} > (1 - \phi)\left(1 - \pi^{KN}\right)$, which is true when $\phi > 1 - \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}}$. However, there is a population of generalists when $\phi > 1 - \frac{1-\pi^{K-c}}{1-\pi^{K}}$. It is always the case that $1 - \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}} \leq 1 - \frac{1-\pi^{K-c}}{1-\pi^{K}}$. So if $1 - \frac{1-\pi^{N(K-c)}}{1-\pi^{NK}} < \phi < 1 - \frac{1-\pi^{K-c}}{1-\pi^{K}}$, then society is better off with a population of

generalists, but has a population of specialists. □

Note that although in this section I have assumed that disciplines are symmetric–that is, that they are all equally useful for solving problems, in expectation–the results easily extend to a more general case, where one discipline is more useful, in expectation, than the other. The equivalent results are stated and proved in Appendix D.

3.3.3 Discussion

In this section, I have shown that because diversifying ones skills is costly, it can be difficult to justify the decision to be a generalist. The costs associated with learning a new field, with its new jargon, literature, and social patterns, must be outweighed by some kind of benefit–in this case, barriers to addressing problems in other fields. In the real world, there are many reasons that individuals may have trouble working on problems outside their home discipline. On a purely practical level, communication barriers make it difficult to understand the content of questions in unfamiliar fields, and those lacking familiarity with the field may also be unable to place questions in context. If it is possible to understand the questions, techniques in another field may be foreign enough to make communicating results extremely difficult. And culturally, it may be impossible to be taken seriously without credentials in the field a problem originates in. By possessing some skills in multiple areas, generalists are able to overcome these barriers and work on problems in many different fields.

However, the final lesson of this section is that the equilibrium level of generalization is not always socially optimal. From a societal standpoint, we would like to maximize the amount of progress made on problems we face. This means applying skills to all problems that arise. However, when there are barriers to working on problems outside of ones own field, there is a danger that not all all skills will be applied to all problems. Generalists are able to apply skills from one discipline to

problems in another discipline. However, since they bear the costs of gaining skills in a new field individually, it is possible for generalists to be underprovided.

3.4 An Extension: Problems with Multiple Parts

In the previous section, I showed that barriers to addressing problems in other disciplines can induce problem solvers to diversify their skills. In this section, I consider an extension of the previous model, which highlights a second scenario in which individuals can be incentivized to acquire skills in multiple disciplines: problems with multiple parts. As problems become more difficult, they may be broken down into many different sub-problems. Although in some cases, these subproblems may all be best addressed within a single discipline, in others, different subproblems will be best addressed using different skills. In this section, I show that when different parts of a problem are best addressed using different disciplines, then a population of generalists can be sustained.

3.4.1 Problems With Multiple Parts

Here, I consider an extension of the previous model. Many of the model elements are the same as before. Once again, the skills in the set S are divided into two disciplines, d_1 and d_2. Players use their skills to address a problem, the nature of which is not known *ex ante.* They will choose to be a specialist or generalist in period 1 to maximize their chances of solving the problem in period 2.

But now, suppose each problem consists of two parts, y^1 and y^2. Each part of the problem is addressed independently by the skills in each of the disciplines. Thus, much like the problems in the previous section, we can define the parts of the problem by a matrix of probabilities that each skill will solve the problem. That is,

$$y^i = \begin{bmatrix} y^i_{11} & y^i_{12} \\ \vdots & \vdots \\ y^i_{K1} & y^i_{K2} \end{bmatrix}$$ where $y^i_{kd} = H$ if skill k in discipline d has a high probability

of solving part i and L if it has a low probability of solving part i.

Again, the probability that a given skill is an H skill is not known *ex ante*. However, the players know the expected probability that a skill is an H skill. I will allow the expected probabilities to vary across parts of the problem–in other words, it is possible that a discipline will be more useful in solving one of the parts of the problem than in solving the other part of the problem. Let δ_{id} be the probability that a skill from discipline d is an H skill for part i of the problem. That is, $\delta_{id} = E\left[Prob\left(y^i_{kd} = H\right)\right]$. The matrix $\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$ describes a distribution of problems, Δ, and is known *ex ante*.

Suppose, that there are just two possibilities for δ_{id}: δ_1 and δ_0, where $\delta_0 < \delta_1$. Then we can categorize the possible distributions of problems according to which discipline is useful for which part of the problem. Roughly, these fall into three categories, which are outlined in Figure 3.4.

In the first category, both disciplines are equally likely to be useful on both parts of the problem. When this is the case, the results are similar to those obtained in Section 3.3.1, where skills were symmetric across disciplines. In particular, if individuals can work on any problem they like, then there will be no generalists in equilibrium.

In the second category of problem distributions, one discipline is more likely to be useful on at least one of the parts. In this case, the results are similar to those obtained in Section D, where on average one discipline is more useful in solving the problems faced than the other discipline. Again, if all problems are accessible to all individuals, no individual will ever chose to be a generalist.

The third category of problems is most interesting. In this category, one of the disciplines is useful on one of the parts of the problem and the other discipline is

Category 1: Symmetric across disciplines

$$\begin{bmatrix} X & X \\ X & X \end{bmatrix} \quad \begin{bmatrix} X & X \\ O & O \end{bmatrix} \quad \text{etc...}$$

Category 2: One discipline more useful in both parts

$$\begin{bmatrix} X & O \\ X & O \end{bmatrix} \quad \begin{bmatrix} X & O \\ X & X \end{bmatrix} \quad \begin{bmatrix} X & X \\ X & O \end{bmatrix} \quad \text{etc...}$$

Category 3: Each discipline useful for a different part

$$\begin{bmatrix} X & O \\ O & X \end{bmatrix} \quad \begin{bmatrix} O & X \\ X & O \end{bmatrix}$$

Figure 3.4: A taxonomy of problem distributions. In the above, each problem has two parts. The matrix $\begin{bmatrix} \delta_{11} & \delta_{21} \\ \delta_{12} & \delta_{22} \end{bmatrix}$ represents the distribution of problems, where δ_{id} is the probability of a skill in discipline d being an H skill for part i of the problem. In this case, I have simplified the problem distributions by assuming that either $\delta_{id} = X$ (high probability that the discipline will be useful on part i) or $\delta_{id} = O$ (low probability that the discipline will be useful on part i). The problem distributions can be divided into three categories according to which disciplines are more useful on which parts of the problem.

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useful on the other part of the problem. This case does not resemble any of those already explored, and it is the one I will focus on here. This case, where problems have multiple parts, each of which is best addressed within the context of a different discipline, presents a different kind of opportunity for generalists than before. Here, generalists benefit by having diverse skills, useful in different contexts. Note that unlike in the case where problems are simple (have only one part), when problems are difficult, there need be no communication barriers in order for individuals to generalize their skills. Difficult problems can be inherently interdisciplinary, giving generalists the advantage they need without barriers to working in other fields.

Claim 19. When problems have multiple parts, each of which is best addressed using a different discipline (eg: $\delta = \begin{bmatrix} \delta_1 & \delta_0 \\ \delta_0 & \delta_1 \end{bmatrix}$ with $\delta_1 > \delta_0$) then there is a set of values of $\pi_1 = \delta_1 h + (1 - \delta_1) l$ and $\pi_0 = \delta_0 h + (1 - \delta_0) l$ such that it is individually optimal for problem solvers to be generalists, even when problems are open.

In order to see this, consider the case mentioned above, where $\delta = \begin{bmatrix} \delta_1 & \delta_0 \\ \delta_0 & \delta_1 \end{bmatrix}$ with $\delta_1 > \delta_0$. This means that a specialist in discipline 1 will have K skills, all in discipline 1. Each of her skills has a probability δ_1 of being an H skill for part 1 of the problem, and a probability $\delta_0 < \delta_1$ of being an H skill for part 2 of the problem. In contrast, a specialist in discipline 2 will have K skills, each of which has a probability δ_0 of being an H skill for part 1 of the problem and a probability δ_1 of being an H skill for part 2 of the problem. A generalist will have skills in both disciplines. When solving part 1 of the problem, her skills in discipline 1 will have a higher probability of being H skills. When solving part 2, her skills in discipline 2 will have a higher probability of being H skills. In this particular case, it will be optimal for a generalist to split her skills evenly between the two disciplines, and she will obtain $\frac{K-c}{2}$ skills in each.

For a specialist in either discipline, the expected probability that her skills will
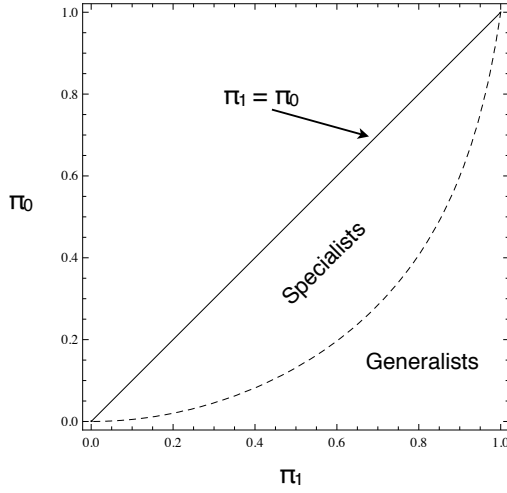
Figure 3.5: Equilibrium decisions to specialize and generalize when problems have two parts. The boundry between the regions where individuals specialize and generalize is defined by the equation $\pi_1^K + \pi_0^K = (\pi_1\pi_0)^{\frac{K-c}{2}} - (\pi_1\pi_0)^{K-c} + (\pi_1\pi_0)^K$. This graph illustrates these regions in the case where $K = 3$ and $c = 1$. This boundry moves upwards as costs (c) rise, relative to a problem solver's capacity $(M = K + c)$.

solve *both* parts of the problem is $E\left[P\left(\text{success on part 1}\right)\right] * E\left[P\left(\text{success on part 2}\right)\right]$, which is $\left(1 - \pi_1^K\right)\left(1 - \pi_0\right)^K$ where $\pi_1 = \delta_1 h + (1 - \delta_1)\,l$ and $\pi_0 = \delta_0 h + (1 - \delta_0)\,l$.

For a generalist, the expected probability that she will solve both parts of the problem is $\left(1 - \pi_1^{\frac{K-c}{2}}\pi^{\frac{K-c}{2}}\right)^2$.

In order to find the parameter region where individuals choose to generalize, we look for the region where $\left(1 - \pi_1^{\frac{K-c}{2}}\pi_0^{\frac{K-c}{2}}\right)^2 > \left(1 - \pi_1^K\right)\left(1 - \pi_0\right)^K$.[6] This region, as a function of π_1 and π_0, is illustrated in Figure 3.5 for $K = 3$ and $c = 1$. As would be expected, the region where individuals specialize shrinks as the costs to generalizing (c) become smaller, relative to the individual's total capacity for learning new skills $(M = K + c)$.

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[6]Note that when $\delta_1 = \delta_0$, we have a case that fits into the first category in the taxonomy of problem distributions in Figure 3.4. Since $\delta_1 = \delta_0 \implies \pi_1 = \pi_0$, we can use this calculation to verify the claim made above that in the case where skills are symmetric, the results are the same as in Section 3.3.1.

3.5 Conclusion

Given the cost of obtaining skills in new areas, it may seem difficult to rationalize the decision to diversify ones skills. However, there are clearly a large (and growing) number of individuals in research communities who choose to do so. In addition, national funding organizations have put considerable effort into promoting interdisciplinary effort. This chapter indicates that both of those actions can be rationalized under particular conditions. In particular, it is possible to rationalize the decision to obtain skills in multiple disciplines if there are significant barriers to working on questions in fields with which one is unfamiliar. Those who pay the initial price of learning the jargon and literature of a new field reap the benefits in the form of a larger pool of problems to solve. These benefits may be even larger if problems are solved in a collaborative context, making adding collaboration an interesting possible extension to the current work. In particular, in a collaborative context, people with skills in multiple disciplines may be able to link specialists in one discipline with specialists in another discipline.

It is also possible to show that being a generalist is optimal if problems are difficult, because different parts of a difficult problem may be best addressed using skills in different disciplines. Again, it would be interesting to allow individuals in this model to collaborate on problems, because in a collaborative context, we might imagine that the advantage to generalists would be enhanced.

Finally, the results of this chapter indicate that the equilibrium population of problem solvers need not be socially optimal. In particular, because generalists internalize the costs of diversifying their skills, it is possible for generalists to be undersupplied in the population. If this were the case, a more socially optimal outcome could be obtained by subsidizing the costs to diversifying ones skills, much as national funding organizations are now doing. However, it is unclear whether our current situation is one in which such funding is required. More careful consideration of this question is

a good candidate for further work.

CHAPTER IV

Team Assembly with Network Constraints

4.1 Introduction

In a group formation game, each player can be a member of one and only one group, and individual payoffs depend, directly or indirectly, on group structure. Many difficult and pressing economic problems fall into this category, including rent-seeking, resource management, contract bidding, volunteer organization, problem solving, and political lobbying. Depending on the application, the collection of individuals may be called a group, club, or coalition. However, apart from this basic semantic difference, all of these problems share the same basic characteristics–they all create incentives, such as economies of scale, risk-sharing, skill aggregation, and social capital accumulation, that make working within a group (coalition or club) more attractive than working alone.

Consider, for example, a simple rent-seeking game, in which players compete for a single rent. There are many incentives for individuals to pool their efforts and compete as a group: risk averse individuals may be willing to trade some rent in expectation for a more consistent income stream; a complicated rent seeking task may require a range of skills; and economies of scale may make larger groups more likely to win. However, individuals must weigh these advantages against the disadvantages of higher maintenance costs, free-rider problems, and the division of rents. Many of these

incentives come down to a decision over different size groups. Thus, we might think of this example as one in which individuals have determined that a rent-seeking group of size 10 maximizes their expected utilities by mitigating risk and taking advantage of economies of scale. Beyond that ideal size, the losses from managing a larger size group, free-riding off of others efforts, and division of rents between a larger number of people reduce the expected utility of the individuals in those groups. However, they still prefer that larger group to pursuing the rents alone. When a group has more than 17 members, the costs of maintaining that group make pursuing rents alone more attractive than staying in the group.

In this chapter, I combine two recent areas of interest–dynamic coalition formation games and social network constraints–to explore how groups form when individuals move dynamically and face social, spatial, and institutional constraints on group membership.

Traditionally, group formation has been modeled statically–that is, players make their group membership decisions simultaneously–see, for instance, Hart and Kurz (1983), Nitzen (1991), Yi and Shin (1997), Konishi and Weber (1997), and Heintzel-man *et al* (2006). These models are advantageous because of their analytical simplicity and clarity. However, as I will show in Section 4.2.1, static group formation games often have multiple equilibria, suggesting that there are potentially large gains to clarifying the process by which they form. The presence of multiple equilibria raises a new, more complicated set of questions. Which of these equilibria can we realistically expect to reach given a dynamic group formation process? Will the equilibrium outcome reached be efficient? What characteristics of the problem affect that outcome? These questions cannot be addressed using a static model, and thus researchers have increasingly turned towards dynamic models of the coalition formation process. Recent work spans may different subfields, including industrial organization, political economy, rent-seeking, and local public good provision and includes (among

many others) Bloch (1996), Yi (2000), Arnold and Schwalbe (2002), Konishi and Ray (2003), Macho-Stadler *et al* (2004), Arnold and Wooders (2005), and Page and Wooders (2007).

These dynamic group formation models assume that players are unencumbered by social, spatial, or institutional barriers to group membership. That is, the players interact freely with all other individuals in the game and can join any of the groups in the game without regard to the composition of the group. This is a reasonable assumption in some contexts, such as political parties, unions, and other large organizations–however, in many other cases, individuals face substantial barriers in making their group membership decisions. The nature of these barriers will differ depending on the context of the specific problem being considered. Barriers to group membership may be social (eg: an individual can only join a group that contains someone he knows) or spatial (eg: an individual can only join a group with close neighbors). Some of these barriers are explicit (eg: a requirement that a current member "vouch" for the applicant) but others are implicit (eg: a social norm against attending a party composed only of strangers). The barriers may either limit actions (an individual is unable to join a particular group) or information (an individual does not know about the group). However, by modeling these constraints explicitly, we can look beyond the more superficial of these differences and ask a whole new set of questions. How do characteristics of the underlying constraint affect the eventual group structure? Are individuals better or worse off when they are constrained more heavily? How do constraints of different types affect social welfare?

In this chapter, I model the constraints faced by individuals via a network of connections–a player can only join a group if she is connected to a current member. This method allows me to use machinery from the burgeoning networks literature, which explores how social, spatial and institutional networks affect individual behavior. This literature encompasses a wide range of subfields that (as noted in Jackson

(2005)) have only recently started to interact. One branch of the literature has developed tools used to identify communities within existing social networks (see Fortunato (2010) for a survey of the literature); another branch examines how limiting interactions between individuals can affect strategic behavior (see, for instance, Galeotti *et al* (2007) and Charness and Jackson (2006)); and a third branch explores the dynamics of how social networks form (see Jackson (2005) for a survey of this work).

By combining elements of these two emerging literatures, I demonstrate the importance of both dynamics and network constraints in the group formation process. As a baseline for comparison, I start with a static game in which individuals are completely unconstrained in their choice of groups and show that this game has multiple equilibria. I then allow individuals to move sequentially, and solve explicitly for the set of Nash Equilibria of this game. I show that the dynamics act as an equilibrium refinement. However, the equilibrium reached in the dynamic game is highly suboptimal–the negative externality imposed by entering individuals drives groups to be much too large, relative to the social optimum.

I then compare the grouping behavior of the unconstrained individuals to the behavior of individuals constrained by an exogenous network of connections. The network limits a player's action set to those groups containing individuals she is connected to. I show that the network constraint mitigates the tendency for groups to get too large. The efficiency of the outcome depends on the topological characteristics of the network constraint–social welfare is higher when the network is sparse and highly ordered. This result has the surprising implication that informational, institutional, and geographic barriers to group membership may actually improve social welfare by restricting groups from becoming too large.

Finally, I consider optimal institutional design and show that the optimal membership rule also depends on network topology–when a network is dense or random, the *exclusive membership rule* (which allows a group to reject members who do not im-

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prove the group's welfare) is always optimal. However, when the network is sparse or highly ordered, the exclusive membership rule can lead to highly suboptimal results.

The structure of the chapter is as follows. In Section 4.2, I introduce a static coalition formation model, in which individuals choose their group membership simultaneously. I characterize the set of Nash Equilibria of that game, and show that only one is optimal. In Section 4.3, I transform the static model by allowing individuals to make their group membership decisions sequentially over time. This defines a dynamic game similar to that of Arnold and Wooders (2005). I characterize the set of Nash Equilibria of this game, and show that a single, highly suboptimal equilibrium survives. In Section 4.4, I introduce the network constraint. I first characterize the set of Nash Equilibria of the constrained static game. I then move to the dynamic game and show how network topology affects social welfare. In Section 4.5, I consider optimal institutional design and show that the optimal membership rule depends on the topology of the network constraint. In Section 4.6, I conclude and discuss extensions to the model.

4.2 Basic Model Elements and the Static Game

Before considering the behavior of individuals who face a constraint, I will first consider a game in which individuals are unconstrained. This game is actually a special case of the constrained game (ie: one in which all individuals are connected) and thus provides a good baseline for comparison between this game and the existing unconstrained literature.

Consider a group formation game with N homogeneous individuals, $I = \{1, 2, ...N\}$. An individual can be a member of one and only one group–thus, the group structure at time t is a partition of I, $\pi(t) = \{G_1 G_2 ... G_{J(t)}\}$, where G_j denotes the set of individuals in group j. Note that the number of groups is determined endogenously, and thus $J(t)$ may vary from one period to the next. The set of all such partitions of

the players into groups is denoted Π.

Although all of the games defined in this chapter could be played using a generalized payoff function, in the following I will assume that the players have identical payoff functions that depend only on the size of the player's own group. Thus, individual $i's$ payoffs are given by $f(g_j)$ where $i \in G_j$ and $g_j = |G_j|$ is the size of group j. I also assume that $f(g)$ is single-peaked with maximum value g^*. The assumption that payoffs depend only on own group size obviously does not allow for externalities between groups. Nor does it allow players to have preferences over group composition. However,this is an appropriately simple starting point for dynamic analysis–to the extent that inter-coalition externalities muddy behavior, they are best left to future extensions.

I will also assume that payoffs are single-peaked in group size. This assumption is useful because individual and social preferences are aligned–the individuals all want to be in groups of size g^*,[1] and social welfare is highest when this occurs. I will show that the equilibrium reached is suboptimal, *despite this alignment.* The assumption that payoffs are single-peaked also covers nearly all cases that we might encounter–generalizing further would add considerable complication without yielding much useful insight. However, extensions to more general payoff forms are obviously important, and are of interest for future studies. Section 4.6 includes a discussion of these generalizations.

Define $\bar{g}$ to be the smallest g such that $f(g+1) < f(2)$. That is, $\bar{g}$ is the largest group that will form before an individual forms a new group of size 2. If $f(N) > f(2)$, then a new group will never form, and for convenience, I will define $\bar{g} = N$ in these cases. Figure 4.1 illustrates an example of $\bar{g}$ with $\bar{g} < N$.

Note that since individuals in this game are homogeneous, the exact arrangement of the players in the groups is not as important as the sizes of the groups. Thus, I will

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[1]The results that follow can be extended to cases where individuals have different ideal group sizes, but the results are not particularly illuminating.
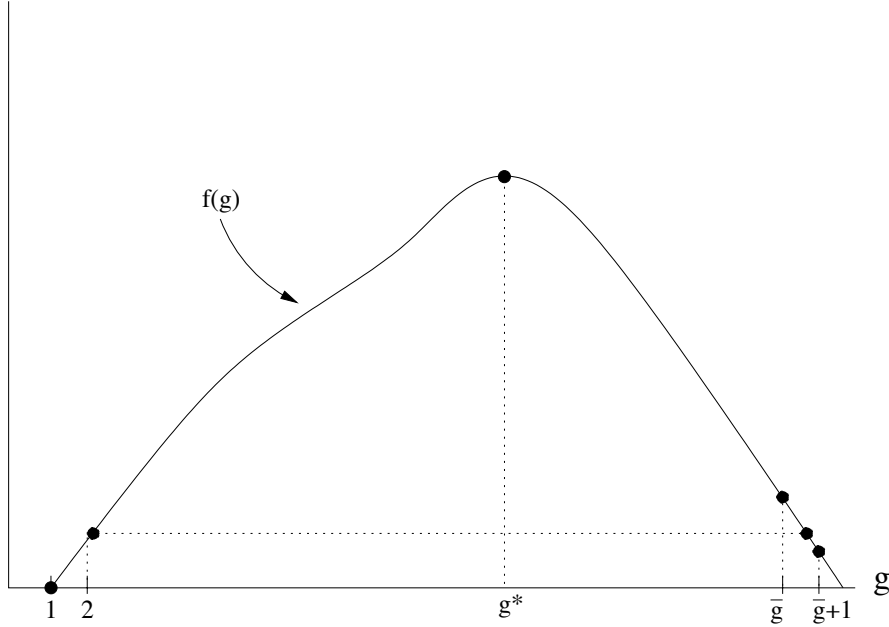
Figure 4.1: An illustration of $\bar{g}$ –the smallest g such that $f(g+1) < f(2)$.

often find it convenient to refer to the vector of group sizes resulting from a particular partition of the individuals into coalitions, rather than referring to the partition itself. To that end, define the *group size vector* of a partition $\pi(t) = \{G_1...G_J\}$ by $\langle g_1...g_J \rangle$. Note that the mapping from partitions to group size vectors is many-to-one, and thus the mapping from equilibrium partitions to equilibrium group size vectors will be as well.

4.2.1 Static Group Formation Game

Ultimately, the process of group formation is a dynamic one–individuals join, leave, and form new groups over time. However, the assumption that moves are made simultaneously may be accurate in some instances and since dynamics add a good deal of analytical complication to the model, it is reasonable to ask whether making the model dynamic adds to our understanding of the problem. To that end, I will first examine a static group formation model. I show that when payoffs are single peaked in group size, there are often multiple Nash equilibria. In Section 4.3,

I allow individuals to choose their group membership sequentially, and show that the dynamics of the game refine the set of equilibria, leaving a single equilibrium group size vector. This demonstrates that adding model dynamics can yield insights beyond those gained from static models.

Consider a static group formation game with N players and payoffs, $f(g)$, single-peaked in group size. Individuals choose their group membership simultaneously. We can think of the individuals as choosing a "location", and all of the individuals who jump to the same location are then members of the same group–thus, an individual's behavior strategy consists of a choice of coalition: $\beta \in \{1, 2...N\}$. The pair $(N, f(g))$ defines the static coalition formation game. A Nash equilibrium of this game is a partition of the players into coalitions, such that no individual wishes to deviate unilaterally.[2] Let $\Omega(N, f(g))$ be the set of partitions of the individuals into coalitions such that no individuals wishes to move unilaterally–that is, $\Omega(N, f(g)) = \{\pi = \{G_1...G_j\} \in \Pi | f(|G_j|) \geq f(|G_k| + 1) \forall G_j \, and \, G_k \in \pi\}$. Let $\varepsilon(N, f(g))$ denote the set of Nash Equilibrium coalition size vectors induced by those equilibrium partitions.

In the following, I characterize $\varepsilon(N, f(g))$. This characterization reveals several interesting aspects of group formation with single-peaked utility and also establishes the need for equilibrium refinement. 20-22 establish several characteristics that an equilibrium of the static game will have: 1) the coalitions will mostly be larger than the social optimum (at most one will be smaller) and 2) all of the groups larger than the optimum will be approximately the same size. Theorem 23 assembles these conditions into a complete characterization of $\varepsilon(N, f(g))$. Finally, Theorem 24 puts a lower bound on the number of equilibria in the set $\varepsilon(N, f(g))$, showing that this static game will often have multiple equilibria.

Lemma 20 states that in equilibrium, most groups will be larger than the social

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[2]This is equivalent to both an equilibrium in the Open Membership Game from Yi and Shin (1997) and the free mobility equilibrium in Konishi et al (1997)

optimum–at most one group can be too small.

Lemma 20. Let $(N, f(g))$ be a static group formation game with $f(g)$ single-peaked. If players are unconstrained in their choice of group, then there $\exists$ no equilibrium $\langle g_1...g_J \rangle \in \varepsilon(N, f(g))$ such that $g_i \leq g_j < g^*$, $i \neq j$. That is, in equilibrium at most one group will be smaller than the social optimum.

Proof. Towards a contradiction, suppose $\exists \langle g_1...g_J \rangle \in \varepsilon(N, f(g))$ such that $g_1 \leq g_2 < g^*$. $f(.)$ is strictly increasing in that range, so $f(g_1) < f(g_2 + 1)$. But then players in group 1 have an incentive to move to group 2, so $\langle g_1...g_J \rangle$ cannot be an equilibrium $\quad \square$

Lemma 20 implies that in characterizing $\varepsilon(N, f(g))$, we need consider only two cases: either all of the groups are larger than the socially optimal size $(g_1...g_k \geq g)$, or exactly one group is small $(g_1 < g^*$ and $g_2...g_k \geq g^*)$. The following two Lemmas address the sizes of the groups in these two different cases. Lemma 21 shows that in any equilibrium where all groups are larger than the social optimum, the groups must be approximately the same size. Lemma 22 sets a more restrictive condition in the case where one group is smaller than the social optimum.

Lemma 21. Let $(N, f(g))$ be a static group formation game with $f(g)$ single-peaked. If players are unconstrained in their choice of group, then for all $\langle g_1...g_J \rangle \in \varepsilon(N, f(g))$, $|g_i - g_j| \leq 1 \, \forall g_i, g_j \geq g^*$. That is, in equilibrium, all groups larger than the social optimum must be the same size, up to integer constraints.

Proof. Towards a contradiction, suppose $\exists \langle g_1...g_J \rangle \in \varepsilon(N, f(g))$ such that $g_1 > g_2 \geq g^*$ and $g_1 - g_2 > 1$. $f(.)$ is strictly decreasing in this range, so $f(g_1) < f(g_2 + 1)$ But then players in group 1 have an incentive to move to group 2, so $\langle g_1...g_J \rangle$ cannot be an equilibrium. $\quad \square$

Note that this result extends a result in Nitzen (1991) to the case of single-peaked utility. Arnold and Wooders (2005) prove a similar result for a sequential game. The

following lemma extends that result to the case where one group is smaller than the social optimum. The Nash Equilibrium requires a slightly stronger restriction on the size of the groups.

Lemma 22. Let $(N, f(g))$ be a static group formation game with $f(g)$ single-peaked. If players are unconstrained in their choice of group, then for all $\langle g_1...g_J \rangle \in \varepsilon\,(N, f(g))$ such that $g_1 < g^{*3}$, both of the following must be true:

1. $f(g_j) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_j + 1)\,\forall\,j > 1$

2. $g_j = g_k\,\forall\,j, k \neq 1$

Proof. Let $\langle g_1...g_J \rangle \in \varepsilon\,(N, f(g))$ such that $g_1 < g^*$.

Part 1: Consider group 1 (the small coalition) and an arbitrary group j, such that $g_j \geq g^*$ Note that $f(g_1) < f(g_1 + 1)$ and $f(g_j) > f(g_j + 1)$. If $f(g_1) < f(g_j + 1)$, then players in group 1 would move to group k. Similarly, if $f(g_j) < f(g_1 + 1)$, then players in group k would move to group 1. Together, these three inequalities imply $f(g_j) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_j + 1)\,\forall\,j > 1$

Part 2: consider two arbitrary groups, j and k, such that $g_k \geq g_j \geq g^*$. Lemma 21 indicates that $g_k - g_j \leq 1$. Towards a contradiction, suppose $g_k - g_j = 1$, so that $g_k = g_j + 1$. By Part 1, $f(g_j + 1) \leq f(g_1)$. Since we assumed $g_k = g_j + 1$, this implies that $f(g_k) \leq f(g_1)$. But since $f(g_1) < f(g_1 + 1)$ to the left of the optimum, $f(g_k) < f(g_1 + 1)$, meaning that players in coalition j would move to coalition 1. Thus, it must be that $g_j = g_k$ exactly.

$\square$

Theorem 23 combines the insights from Lemmas 20-22 to fully characterize $\varepsilon\,(N, f(g))$.

Theorem 23. *Let $(N, f(g))$ be a static group formation game with single-peaked payoff function $f(g)$. If the individuals are unconstrained in their choice of group, the set of Nash Equilibria of that game, $\varepsilon\,(N, f(g))$, is the union of two sets:*

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[3]By Lemma 20, this implies $g_l \geq g^*\forall l \neq k$

1. $\{\langle g_1...g_J \rangle | \bar{g} \geq g_j \geq g^* \, \forall j \, and \, |g_j - g_k| \leq 1 \, \forall j, k \}$

2. $\{\langle g_1...g_J \rangle | g_1 < g*, \, g_j \geq g^* \, \forall j \neq 1, \, g_j = g_k \, \forall j, k \neq 1,$

   $and \, f(g_k) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_k + 1) \}$

Example IV.1. A Static Group Formation Game with Logistic Utility

The implications of Lemmas 20-22 and Theorem 23 can be illustrated through a specific example. Consider a static group formation game with 100 players and a logistic payoff function $f(g) = g(20-g)$. This function is single-peaked with maximum $g^* = 10$ and $\bar{g} = 17$. It is illustrated in Figure 4.2.

u=f(g)

f(g)=g(20–g)

2   g*=10 $\bar{g}$=17 18   g
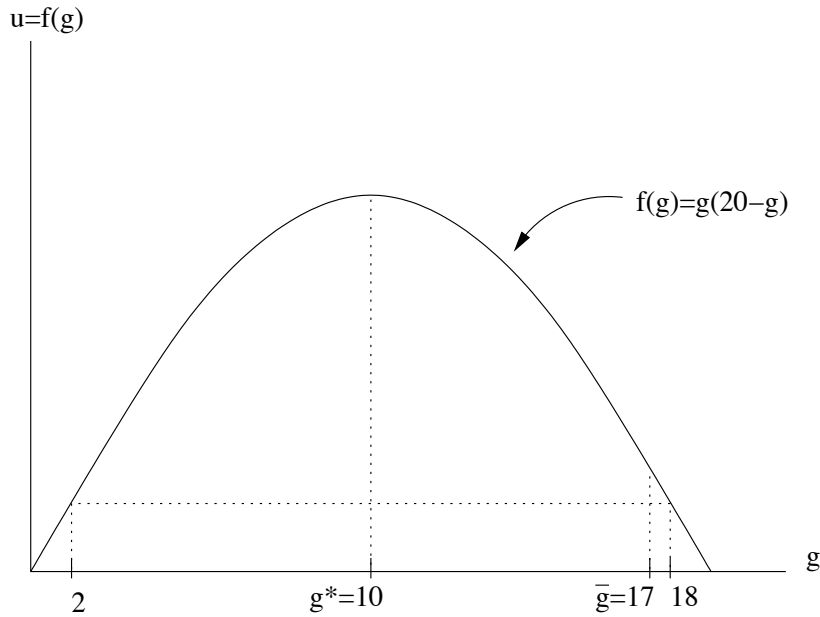
Figure 4.2:
Individual payoff function for Example IV.1. Note that the players enjoy the highest payoff in a coalition of size 10.

Lemma 20 indicates that in any Nash equilibrium of this game, at most one coalition will be smaller than the socially optimal group size, $g^* = 10$. Lemma 21 indicates that all of the groups larger than the social optimum will be approximately the same size. Using these two facts, one can show that there are 5 Nash Equilibria of this static game: $\langle 10, 10, 10, 10, 10, 10, 10, 10, 10, 10 \rangle$, $\langle 11, 11, 11, 11, 11, 11, 11, 11, 12 \rangle$,

$\langle 12, 12, 12, 12, 13, 13, 13, 13 \rangle$, $\langle 14, 14, 14, 14, 14, 15, 15 \rangle$, and $\langle 16, 16, 17, 17, 17, 17 \rangle$. Note that $\langle 20, 20, 20, 20, 20 \rangle$ is *not* an equilibrium, because an individual in a group of size 20 is better off striking out as an individual.

This example illustrates several difficulties with the static game. First of all, the set of stable coalition configurations is highly sensitive to the particular parameters used. For example, with $N = 100$ individuals, the game illustrated above does not have an equilibrium with a small group. However, if we change the game slightly, so that there are $N = 101$ individuals, there will be an "odd-sized" equilibrium: $\langle 16, 16, 16, 16, 16, 16, 5 \rangle$.

Secondly, most games will have multiple stable group size configurations. In fact, it is possible to put a lower bound on the number of equilibria for a given game. Theorem 24 does just that.

Theorem 24. *Let $(N, f(g))$ be a static group formation game. Then $|\varepsilon(N, f(g))| \geq \frac{N}{g^*} - \frac{N}{\bar{g}} - 1$.*

Proof. I will set the lower bound by enumerating the equilibria in which all groups are larger than the social optimum (ie: the first set in Theorem 23). Note that since all groups are approximately the same size, each equilibrium with all large groups is entirely characterized by the *number* of groups. The largest possible group is $\bar{g}$ and the smallest possible group is g^*. Thus, there should be one equilibrium for each integer in the interval $\left[\frac{N}{\bar{g}}, \frac{N}{g^*} \right]$,[4] or $\frac{N}{g^*} - \frac{N}{\bar{g}} - 1$. $\square$

Since the lower bound in Theorem 24 is usually greater than 1, the static game will usually have multiple equilibria. However, the static game provides no insight into which of those equilibria is most likely to occur. Are they all equally likely, or is there a distribution of equilibria? Does that distribution depend in a predictable way

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[4]This is actually also a lower bound on the number of equilibria with all large groups. There could be more, depending on whether g^* and $\bar{g}$ divide N evenly, but including that complication only adds more equilibria, keeping the lower bound accurate (albeit a bit lower than is strictly necessary).

on the observable elements of a particular problem? These questions are particularly important because some of the equilibria lead to considerably higher social welfare than others.

The following section shows that a more dynamic model of the group formation process, in which individuals make their decisions sequentially over time, provides an equilibrium refinement. As I will show, not all of the equilibria characterized in Theorem 23 are attainable when players start the game as individuals.[5] Surprisingly, the surviving equilibrium group size vector is the worst possible of the static equilibria.

4.3 Sequential Group Formation Game–The Unconstrained Case

While a static group formation game $(N, f(g))$ will typically have more than one equilibrium, only one is efficient. This obviously begs the question–will individuals moving sequentially reach the efficient coalition arrangement, or will they reach an inefficient outcome? Will that outcome be unique or are several outcomes possible? In this section, I show that allowing the players to move sequentially refines the set of equilibria from the static game. When the players start the game as individuals, they will always reach an equilibrium group structure with the same coalition size vector. Furthermore, it is not the socially optimal one–when individuals make their group formation decisions dynamically, they end up in groups that are much too large, despite a clear alignment between individual and social welfare.

4.3.1 Sequential Coalition Formation Game

In the unconstrained sequential group formation game, individuals are able to join any currently existing group $(G_j \in \pi(t))$, or alternatively they can strike off as

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[5]Note that this equilibrium configuration is *not* a "Garden of Eden" configuration (Epstein and Hammond (2002)) because it can be obtained from some initial configurations.

an individual, forming a group of size 1. Thus, individual i's action set at time t can be denoted by $A_i(t) = \pi(t) \cup \emptyset$, where $\emptyset$ denotes the action of striking out as an individual. In the following, I will assume that individuals make their group membership decisions myopically–that is, they decide which group will maximize their return, given only the *current* group structure. This defines a behavior strategy that simply maps the current group partition, $\pi(t)$, to the individual's action set as defined above: $\beta(\pi(t)) \in A_i(t)$.

This myopic behavior strategy is identical to that used in Arnold and Schwalbe (2002) and Arnold and Wooders (2005). The myopia assumption is convenient because it makes the analysis more tractable. However, in the case of sequential coalition formation games, it is also behaviorally more reasonable than perfect foresight. The sequential nature of this game induces an explosion in number of possible states, making the sequential coalition formation game more like Chess or Go than Tic-tac-toe. Moreover, as I will mention later, one can show that myopia is not the sole cause of the observed behavior, making the assumption relatively innocuous.

$(N, f(g), \phi)$ defines an unconstrained sequential coalition formation game, where ϕ is a particular order of motion for the players.[6] An equilibrium of this dynamic game is a partition of the players into groups, $\pi = \{G_1...G_J\}$ such that $f(g_j) \geq f(g_k + 1) \forall G_j, G_k \in \pi$–that is, a group configuration is an equilibrium if no individual wishes to deviate unilaterally. Let $\varepsilon(N, f(g), \phi)$ represent the set of equilibrium coalition size vectors resulting from those partitions.

Note that any Nash Equilibrium of the dynamic game must be a stable group configuration in the static game, and therefore $\varepsilon(N, f(g), \phi) \subseteq \varepsilon(N, f(g))$. Theorem

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[6]This game was first introduced by Arnold and Wooders (2005). However, whereas Arnold and Wooders consider a Nash Club Equilibrium (a group structure which is stable to deviations by coalitions of individuals within a particular group) and a k-remainder Nash Club Equilibrium (which is stable to deviations when k individuals are dropped from the system) I use a Nash Equilibrium. See the later text for a discussion of deviations by groups of n individuals. It is worth noting that I obtain dramatically different results using the Nash Equilibrium than Arnold and Wooders do using the Nash Club and k-remainder Equilibria.

25 shows that the sequential game has a unique equilibrium up to the symmetry of the players. Furthermore, Theorem 26 shows that when players start the game as individuals,[7] this equilibrium is always the worst possible stable group configuration from the standpoint of social welfare, despite the alignment between social and individual preferences.

Theorem 25. *Let $(N, f(g), \phi)$ be an unconstrained sequential group formation game with $f(g)$ single-peaked. Then there is a unique Nash Equilibrium group size vector, $\gamma(N, f(g)) = \langle g_1...g_J \rangle$, which is a function of the number of players and the payoff function alone.*

Theorem 26. *Let $(N, f(g), \phi)$ be an unconstrained sequential group formation game with $f(g)$ single-peaked. Let $(N, f(g))$ be the static coalition formation game with the same number of players and payoff function. If $\langle g_1...g_J \rangle$ is the (unique) Nash Equilbrium of $(N, f(g), \phi)$ then $\langle g_1...g_J \rangle = \arg\min_{\varepsilon(N,f(g))} \sum_{i \in I} f(g_i)$. That is, the Nash Equilibrium of the sequential game is the element of $\varepsilon(N, f(g))$ that minimizes social welfare.*

Proof. Let $(N, f(g), \phi)$ be an arbitrary sequential group formation game. Let $(N, f(g))$ be the corresponding static group formation game. The equilibrium of the static game $(N, f(g))$ that yields the lowest social welfare is the equilibrium with groups of the largest size–or conversely, the equilibrium with the smallest number of groups. I will show that regardless of the order of motion, ϕ, the players always reach a configuration with the minimum number of groups, and thus the lowest possible social welfare value.[8]

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[7]Obviously the equilibrium reached will depend on the initial condition. Starting the game with the individuals acting alone seems very natural, and also mimics the spirit of the static game. A full characterization of the basins of attraction for the different equilibria of this game is beyond the scope of the current work. However, I will note that the results that follow are unchanged if the individuals start the game in a grand coalition.

[8]Note that an "odd-size" equilibrium (one with a single small group) will always have higher social

If $f(g)$ is strictly increasing or decreasing, then the result follows trivially. So suppose $f(g)$ is unimodal. $f(g)$ unimodal implies $f(1) < f(2)$. Thus, the first individual will always want to start a new group. Since $f(g+1) > f(2) \forall g < \bar{g}$, subsequent individuals will prefer to join the existing group to forming a new group of size 2. In fact, it is only worthwhile to create a second group of size 2 when the existing group is size $\bar{g}$. If $\bar{g} \geq N$, then a second group never forms–the individuals ultimately form one large group of size N and the result follows trivially. So suppose $\bar{g} < N$.

For the sake of clarity, let $\bar{r} = N \bmod \bar{g} > 0$.[9] Thus, the equilibrium in $\varepsilon(N, f(g))$ with the lowest social welfare is the equilibrium with $\frac{N-\bar{r}}{\bar{g}} + 1$ groups. Regardless of the order of motion, a new group forms only if all existing groups have reached size $\bar{g}$. Thus, the final group forms only once there are $\frac{N-\bar{r}}{\bar{g}}$ groups of size $\bar{g}$. This implies that the unique equilibrium of the sequential game will have $\frac{N-\bar{r}}{\bar{g}} + 1$ groups. $\qquad\square$

The basic insights of the proof can best be appreciated via a specific example.

Example IV.2. An Unconstrained Sequential Group Formation Game with Logistic Utility

Note that while a static group formation game, $(N, f(g))$, often has multiple equilibrium group size vectors, only one is efficient. Recall from Example IV.1 that while the game $(100, g(20 - g))$ has 5 Nash equilibria– $\langle 10, 10, 10, 10, 10, 10, 10, 10, 10, 10 \rangle$, $\langle 11, 11, 11, 11, 11, 11, 11, 11, 12 \rangle$, $\langle 12, 12, 12, 12, 13, 13, 13, 13 \rangle$, $\langle 14, 14, 14, 14, 14, 15, 15 \rangle$, and $\langle 16, 16, 17, 17, 17, 17 \rangle$– only the equilibrium with groups of size 10 is efficient.

Now, consider the sequential group formation game with the same parameters and an arbitrary order of play: $(100, g(20 - g), \phi)$. According to Theorem 25, this sequential game has a unique equilibrium. Moreover, Theorem 26 indicates that the equilibrium will be the stable group structure with the lowest possible social

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welfare than the equilibrium with the smallest possible number of groups. However, it should become clear in the following that such an equilibrium could never arise through sequential movement with the given initial condition. Therefore, I will not address it explicitly here.

[9]The same result holds for $\bar{r} = 0$, but this assumption simplifies the exposition.

welfare–in this case, the configuration with coalitions of size 16 and 17. Note that this equilibrium is inefficient, because all players are better off in groups of size 10. The following analysis shows how players wind up in this suboptimal group structure.

The players start the game as individuals, so the first player to move faces a choice between remaining as an individual and forming a group of size 2. The player is myopic, so she chooses the group of size 2 because it gives her higher utility in the next period (Figure 4.3). The second player to move faces a similar choice–she must

u=f(g)
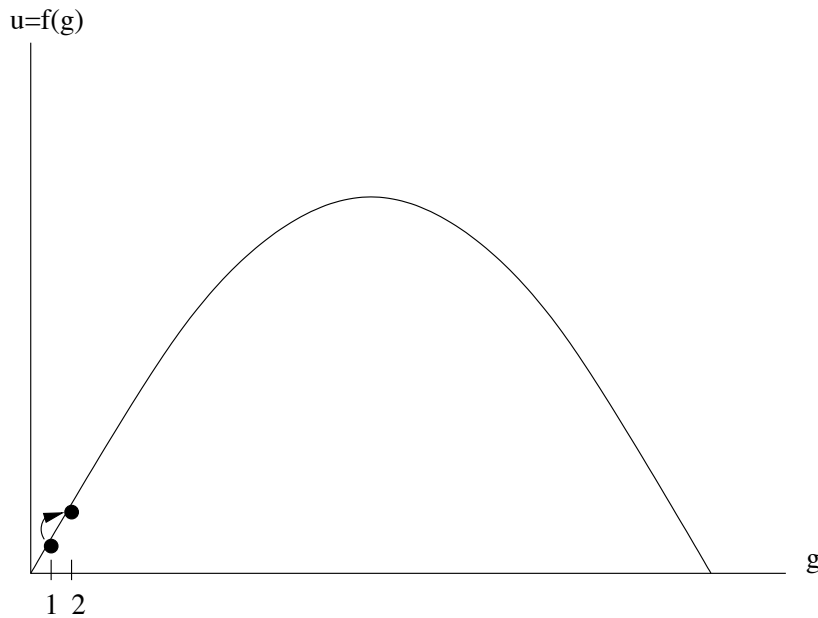
Figure 4.3: The first individual to move joins another individual to form a group of size 2.

decide whether to join the existing large group to form a group of 3, or join another individual to form a second group of 2. The group of 3 gives her higher utility, so she joins that group (Figure 4.4).

A new group only forms when $f(2) \geq f(g+1)$ where g is the size of the existing large group. The smallest such g is obviously $\bar{g}$– in this case, a group of 17 (Figure 4.5). This is true regardless of how many "large" groups (groups with more than one individual) there are. Thus, the second group forms when there are 83 individuals and one group of 17, the third group forms when there are 69 individuals and two
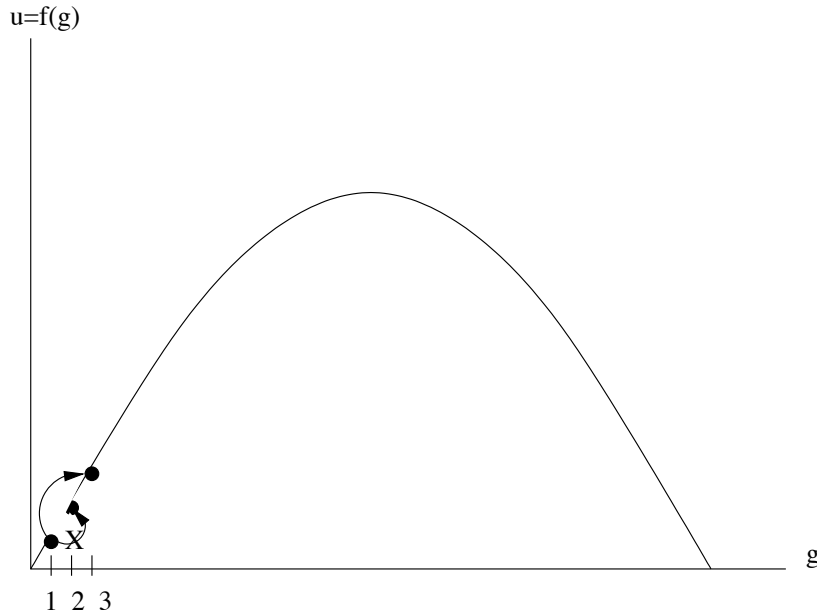
Figure 4.4: The second individual to move must choose between forming a new group of size 2 or joining the existing group of 3. He will choose the group of 3, since it gives him higher utility than the group of 2.

groups of size 17, and so on. The last group forms when there are 15 individuals and five groups of size 17.

This sixth group is the final group that will ever form. Individuals may (and indeed, will) move between the existing groups, but no new group will ever form. The individuals will stop moving when all six groups are approximately the same size–namely, in the configuration with two groups of size 16 and four groups of size 17: $\langle 16, 16, 17, 17, 17, 17 \rangle$. As predicted by Theorem 26, this is the stable group arrangement with the lowest possible social welfare value. Note also that at no point did we specify the order of play–thus, the players will reach the arrangement $\langle 16, 16, 17, 17, 17, 17 \rangle$ regardless of their order of motion.

4.3.2   Discussion–Externalities in Coalition Membership

One might be tempted to attribute the behavior detailed in Theorems 25 and 26 to the players' myopia. However, it is possible to show that even perfectly forward-
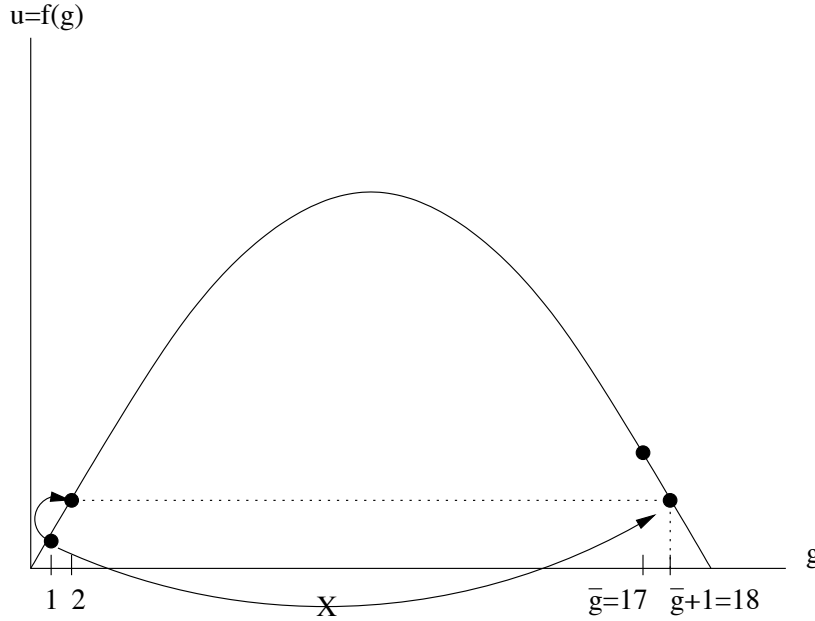
Figure 4.5: A new group forms when the large group is size $\bar{g} = 17$ because the individual is better off in a new group of size 2.

looking players will form groups that are larger than the socially optimal size.[10] Since even forward-looking agents reach a suboptimal equilibrium, it is clear that myopia is not all that is at work in this result.

The cause of the observed behavior is the externality that joining a group imposes on existing group members. When the group is smaller than the social optimum, that externality is positive. However, when the group is at the optimal size, the externality is a negative one. The entering member is obviously made better off by the change (otherwise, he would not move), but the rest of the group is made worse off. The negative externality causes individuals to enter a group that does not benefit from the extra member, which then drives groups to become too large.

Obviously, in the real world, groups of individuals will sometimes make their membership decisions together. If we allow a subgroup of up to n individuals to move as a group, then any equilibrium that exists will necessarily have smaller groups. However, the set of equilibria that are stable to such coalitional deviations are largely

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[10]An example with six players is available from the author upon request.

empty (see Arnold and Wooders (2005) for a discussion of this problem). More importantly, when we move on to games with a network constraint, as in the following section, it becomes less clear what is meant by a configuration that is stable to "coalitional deviations". Analysis of more complicated, network-specific equilibrium concepts are obviously venues for future work.

Additionally, there is empirical evidence that groups tend to be too large. Many institutions exist that constrain the size of coalitions, a measure that would be unnecessary if individuals found themselves in groups of ideal size. In the following section, I consider the effects of social and spatial constraints on individual behavior and show that such constraints can improve total social welfare. Section 4.5 continues this discussion by exploring the effect of the network constraint on optimal institutional design.

4.4   Sequential Group Formation with a Network Constraint

The analysis of the previous section (as well as much of the existing literature) assumes that individuals are free to join any existing group, regardless of its current composition. However, the cases where individuals are completely unconstrained are relatively few–in most instances, individuals face social, spatial, and information constraints when making their membership decisions. Consider, for example, a set of farmers forming water management groups along the banks of a river. Although it is conceivable that the farmers would organize into groups at random, they are more likely to join farmers who are adjacent to them on the river than those in a distant location. Similarly, research groups are more likely to be composed of colleagues than strangers, and an individual is unlikely to attend a party unless he already knows someone who is attending.

The most natural way of modeling these constraints is via a network of connections. I give each individual an exogenous network of connections to other peo-

ple. An individual can only join a group if it contains a person she is connected to on the network. More formally, $(N, f(g), \phi, C)$ defines a particular sequential group formation game with a network constraint, where C is an exogenous, unchanging[11] matrix of connections between individuals–that is, $C_{ij} = 1$ if i is connected to j on the network and 0 otherwise.[12] In the constrained game, an individual's action set is restricted to include only those groups she is connected to: $A_i(t) = \{G \mid C_{ij} = 1 \, for \, some \, j \in G\} \cup \emptyset \subseteq \pi(t) \cup \emptyset$. Clearly such a matrix of connections can model any set of constraints faced by individual agents, making the network formulation of this problem extremely general.

However, there is an additional advantage of using a network constraint–namely, it allows us to draw conclusions about general "classes" of constraints that seem similar, without getting caught up in the details of a particular case. For instance, we might want to determine how individuals on a spatial network behave differently than individuals on a social network, without getting tied up in the details of a particular network structure. Fortunately, by varying only a few parameters, we can obtain a natural spectrum of network structures that correspond nicely to the types of networks that we would observe in the real world. For my analysis of network topology I will use a Watts-Strogatz small world network. This particular network has only two parameters. The first is average degree, d, which enumerates the average number of connections each individual in the network has. The second is the Watts-Strogatz parameter, $p \in (0, 1)$, which allows us to examine a spectrum of different network types–when $p = 0$, the network is regular and approximates a spatial network; when

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[11]One obvious extension is to allow the network structure to evolve over time. This could provide insights into network formation. Jackson (2005) provides some background and a survey of the network formation literature.

[12]Note that this differs significantly from the use of networks in Page and Wooders (2007), which uses a bipartite network to illustrate the partition of individuals into groups–ie: each individual is linked to the group to which it is a member. Thus, this network contains no information about how individuals are connected. In this chapter, the network links individuals to one another, restricting an individual's choice of groups. Although we could use a second, bipartite network to denote the division of players into groups, it adds no insights in the current context.

$p = 1$, the individuals are connected at random; for values of p between 0 and 1, the network has a "small world" structure, which approximates that of a social network. [13] A pair (d, p) describes a family of networks with similar topological characteristics.

Note that when the network is fully connected, every player knows someone in every group and therefore $A_i(t) = \pi(t) \cup \emptyset$. Thus, the unconstrained static and sequential games considered previously are a special case of the constrained game– namely one where the average degree is at a maximum: $d = N - 1$. I will first use the static game to illustrate the effects of the network constraints on individual behavior given a particular network and then I will show how social welfare is affected by the network constraint.

4.4.1 The Static Game–Network Constraints and Variable Group Size

As in the unconstrained case, I will start by looking at individual behavior when players make their group membership decisions simultaneously. This section generalizes the results of Section 4.2.1 to the constrained case.

An equilibrium of the static game with a network constraint is a partition of the players into groups that is both feasible and individually rational.

Definition 27. A Nash equilibrium of the game $(N, f(g), C)$ is a partition of the players into groups, $\{G_1...G_J\}$, such that $\forall i$, $i \in G_j$ implies:

1. $f(g_j) \geq f(g_k + 1) \, \forall G_k \in \{G_1...G_J\}$

2. $C_{ij} = 1$ for some $j \in G_j$

One result of adding a network constraint is that the analogue to Lemma 21 need not be true. That is, when players are constrained, there may exist stable group structures in which groups are different sizes.

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[13] Watts and Strogatz (1998)

Claim 28. For a given static group formation game $(N, f(g), C)$, there may exist a Nash equilibrium group structure, $\{G_1...G_J\}$, such that $|g_j - g_k| > 1$ for some $g_j, g_k > g^*$.

As an illustration of this claim, consider a game with 12 players on a ring, as pictured in Figure 4.6. Note that the constraint of the ring could represent either a constraint on actions (the players would move if they could) or information (the players would move if they knew). It could also equally well represent an explicit constraint (a legal constraint), an implicit constraint (a social norm), or a functional constraint (a geographic coincidence). Further suppose $g^* = 2$ and $\bar{g} = 6$, so that all
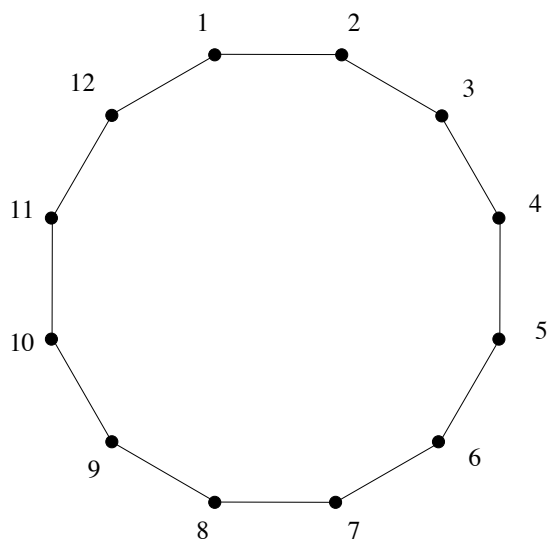
Figure 4.6: A game with 12 players arranged on a ring.

individuals want to be in a group of size 2, and will never form a group larger than size 6. Figure 4.7 illustrates a stable coalition structure of the static game $(N, f(g), C)$ with uneven group sizes.

It is obvious from Figure 4.7 how the ring affects the stability of this configuration. The individuals in group C would like to join group A, but they are unable to because they are not connected to that group on the social network. If the network were fully connected, the individuals in group C would like to move to group A, and the configuration would not be stable.
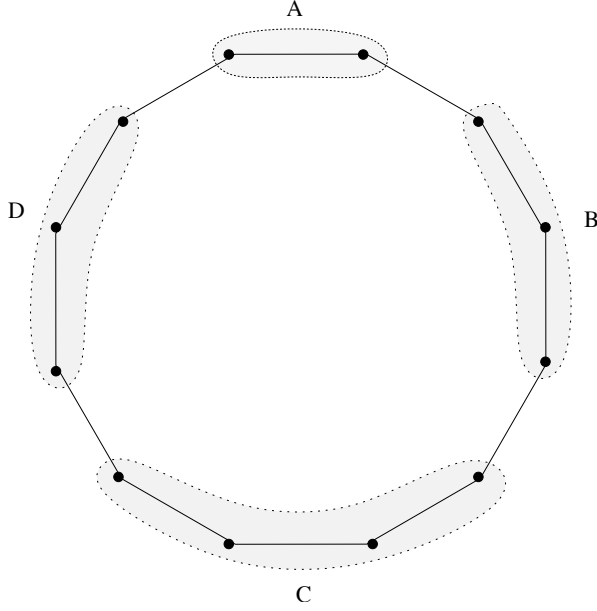
Figure 4.7: An example of an equilibrium group configuration on a ring. Note that this would not be an equilibrium on the fully-connected network because the players in group C would move group A.

This result is significant because while existing models predict that groups will be the same size in equilibrium, real-world groups are seldom identical in size. This analysis indicates that if individuals are constrained, group sizes need not be the same. By exploiting the fact that any two connected individuals form a fully connected subgraph, we can extend the results in Theorem 23 to the current case. To do so, we need one final definition: given a network constraint C, I call two groups, G_j and G_k *connected* if $\exists h \in G_j$ and $i \in G_k$ such that $C_{hi} = 1$.

Theorem 29. *Let $(N, f(g), C)$ be a static group formation game with single-peaked payoff function $f(g)$ and network constraint C. $\{G_1...G_J\} \in \varepsilon(N, f(g), C)$ if for all connected groups, G_j and G_k, either*

1. $g_j, g_k \geq g^*$ and $|g_j - g_k| \leq 1$

   or

2. $g_j < g^* \leq g_k$ and $f(g_k) \geq f(g_j + 1) \geq f(g_j) \geq f(g_k + 1)$

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Proof. Simply note that any pair of connected groups contains a pair of connected agents, who form a fully connected subgraph of the original graph. The result above follows immediately from Theorem 23. □

4.4.2   The Sequential Game–Network Topology and Efficiency

As in the special case of a fully connected network, the set of equilibria of the dynamic game is a subset of the equilibria of the static game: $\varepsilon\left(N, f(g), \phi, C\right) \subseteq \varepsilon\left(N, f(g), C\right)$. However, claim 30 indicates that Theorem 25 need not be true–there will often be multiple equilibrium group size configurations and the set of equilibria may depends on the order of play.

Claim 30. A sequential coalition formation game $(N, f(g), \phi, C)$ need not have a unique equilibrium. Furthermore, the set of equilibria of this game may depend on the order of play, ϕ.

Examples illustrating this claim can be found in Appendix E.

Since the outcome of the constrained group formation game may depend on the order of play and random moves and there is no strong theoretical foundation for a particular order of play or set of random choices, I must somehow deal with this multiplicity of equilibria. One method would be to determine the distribution of outcomes combinatorially and calculate the expected social welfare exactly. However, this method would yield results that are overly narrow, applying only to the specific networks considered. As discussed earlier, I would like to draw conclusions about a "class" of networks with similar topologies.

To that end, I will rely on the computational version of the combinatorial argument above. I will average social welfare over a large number of similar games. In this case, I use a Watts-Strogatz network, which is constructed as follows. Start with a regular network of degree d–this is a network in which every individual is connected to her $\frac{d}{2}$ nearest neighbors on each side. Then, rewire each of the links in the regular

graph with probability p. A link is rewired by disconnecting one end and reconnecting it to a different, random node in the network. Thus, the pair (d, p) describes a family of networks with a similar topology.

As an illustration of the effect of varying these two parameters, consider a coalition formation game with 12 players. Figure 4.8 depicts four networks with different degree. In the first panel, every player is connected to every other player–$d = 11$.
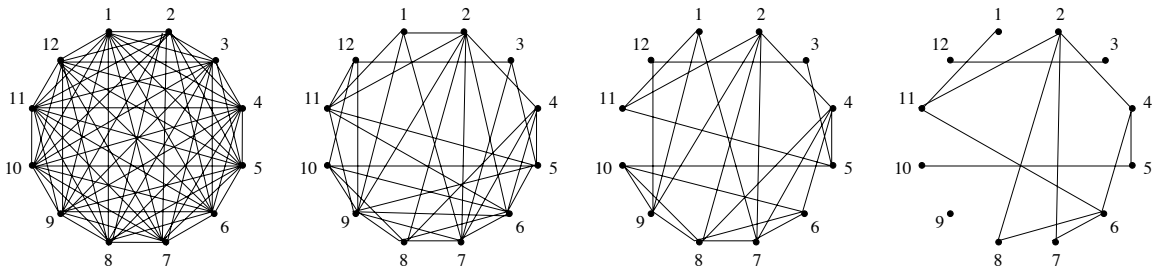
Figure 4.8: Four different networks connecting 12 players. Degree of the network decreases from left to right. The first panel depicts a fully connected network. As edges are removed at random, average degree declines and players potentially become more constrained in their choice of groups.

This is called a fully-connected graph, and represents the special case examined in Sections 4.2.1 and 4.3. The subsequent panels depict the same network with random links removed. Obviously, as the degree of the network decreases, the individuals within that network have fewer choices of groups to join (the size of an individual's action set is bounded above by the number of neighbors she has). This parameter potentially has different meaning in different types of networks–in a spatial network, the average degree specifies how far an individuals can "see" in all directions[14], whereas in a social network, the average degree varies inversely with the "familiarity" required for membership.[15]

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[14]For instance, in the case of the farmers on the river, it might indicate how many upstream and downstream neighbors an individual can interact with.

[15]Consider, for example, an individual deciding to join a club. Membership in a college activity may simply require having an acquaintance in the club. This low threshold of familiarity implies a densely connected network constraint. Membership in a secret society, on the other hand, may require an applicant to have a very strong social tie to a current member. This network constraint would be much more sparse.
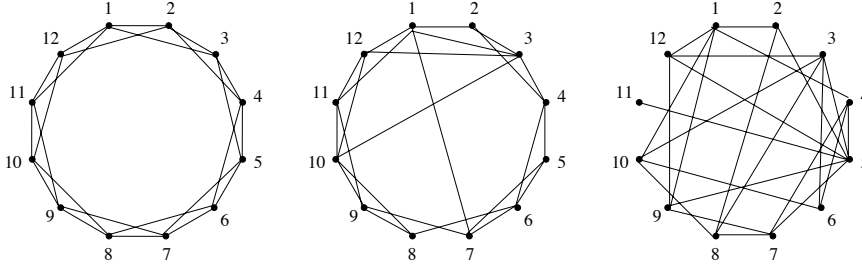
Figure 4.9: Three different networks connecting 12 players. In the first panel, the players are connected to their two nearest neighbors on each side. This is called a regular network, and is often used to represent arrangements of individuals in space. In the second panel, a small number of the links in the regular network are rewired at random. The result is called a small world network, and is a simple model of a social network. In the last panel, all of the links in the original network are rewired at random. The result is a random network, similar to those depicted in Figure 4.8. Random networks are easily analyzed, but a poor approximation of social connections.

As noted earlier, the parameter p allows us to examine networks with different topologies. When $p = 0$, none of the links are rewired, and we have a regular network such as that pictured in the first panel of Figure 4.9. These networks have a high clustering coefficient (average probability that two of a node's neighbors are connected) and a high network diameter (largest minimum path length between two nodes), and are a good model for spatial networks. On the other hand, when $p = 1$, all of the links in the network are rewired at random. The result is a completely random network, pictured in the last panel of Figure 4.9. These networks have a low clustering coefficient and low diameter. Although easy to work with statistically, random networks are unfortunately relatively rare empirically. By rewiring a small, but non-zero fraction of the links, we obtain a small world network, pictured in the second panel of Figure 4.9. A small world network has a high clustering coefficient but low diameter, and is a reasonable first-order approximation of a social network.

In the following analysis, I average social welfare over 100 games with random order of play and networks with the same pair (d, p). Note that with the exception

of the regular networks ($p = 0$), the network structure will differ from one run to the next, even as the parameters remain the same. This allows me to average over a number of networks with the same parameters, which gives the results greater generality. Since I hope to isolate the effects of network topology on outcomes, the non-network elements of the game remain the same. All results in this section use a game with 100 players and logistic utility function $f(g) = g(20 - g)$.

Figure 4.10 shows that holding the Watts-Strogatz parameter constant, social welfare declines in the degree of the network constraint.[16] Since the size of an individual's action set is bounded above by her degree on the network, degree provides a rough measure of how constraining the network is on individual behavior. As the degree of the network decreases, the individuals are more constrained in their choice of groups, which mitigates the tendency for groups to get too large. The fact that social welfare increases as individuals are more constrained is consistent with the hypothesis that groups are too large because of a negative externality.

Figure 4.11 shows the effects of the Watts-Strogatz parameter on social welfare.[17] As the graph moves from regular, to small world, to random, social welfare declines. One possible reason for this trend is that as the Watts-Strogatz parameter increases, the clustering coefficient decreases. The clustering coefficient is the probability that two of a node's neighbors are connected. As the clustering coefficient decreases, the probability that an individual knows more than one person in a group decreases, and the expected size of the action set increases. Thus, as the clustering coefficient decreases, the network becomes less constraining and average social welfare declines.

---

[16]For Figure 4.10, I used a random graph ($p = 1$). The results are qualitatively similar for other values of p.

[17]For Figure 4.11, I used networks of degree 2, 4, and 6. The results are qualitatively similar for networks of different degree. Obviously, as degree increases, the drop in social welfare from the regular graph to the random graph becomes less dramatic.

4.5 The Effects of Network Topology on Optimal Institutional Design

The most compelling evidence that real-world groups tend to be too large is that groups have developed institutions to artificially restrict membership–a measure that I argue would not be required if individuals self-organized optimally. In this section, I examine how network topology affects the optimal choice of membership rule.

When individuals are homogeneous, there are only two possible membership rules:[18] the "open membership rule" (no restriction on group membership) and the "exclusive membership rule" (groups can reject a member).[19] When individuals are unconstrained, the exclusive membership rule is always preferable to the open membership rule. In other words, the coalition members should never allow the group to get larger than g^*.

One might think that the exclusive membership rule would *always* be preferable. However, Figure 4.12 illustrates that when individuals are restricted in their choice of groups via a social network, the exclusive membership rule can sometimes result in group configurations with extremely low social welfare values. An even simpler example uses a ring network. Suppose 20 individuals are arranged on a ring as shown in Figure 4.13. If $g^* = 3$, then the exclusive membership rule may cause individuals to be "isolated" between groups of the ideal size. Both of these examples highlight why the exclusive membership rule is less beneficial when individuals are constrained in their choice of groups. In the unconstrained case, all of the individuals who were excluded from other groups could band together. When individuals are restricted,

---

[18]Because players are homogeneous, all current members will have the same opinion on whether or not to admit a new member. Therefore, we need only consider two rules: one where the group and the individual need to agree, and one where the individual can act as a dictator. Charness and Jackson (2006) examines how the game play of entire groups depends on the voting rule used to make decisions within groups. In his case, the individuals may have heterogenous opinions on the strategic decision, and thus he must consider different types of voting rules. Any extension of this work to include heterogeneous players would have to make more careful consideration of how a group decides to admit or reject a member.

[19]This terminology was introduced in Yi (2000)

they no longer have that option, and there is a much greater chance of individuals being forced into low utility outcomes.

Once again, more general results can be obtained by averaging over many runs on topologically similar networks. Figure 4.14 shows that when degree is low, the exclusive membership rule is no longer the clear optimal choice. Similarly, Figure 4.15 shows that when the graph is very ordered, the exclusive membership rule is, on average, less beneficial. Given the possibility of extremely poor outcomes, such as those pictured in Figures 4.12 and 4.13, the open membership rule may be more desirable when the network constraint is low degree or highly ordered.

4.6 Extensions and conclusions

The model I have introduced in this chapter lends itself to numerous interesting extensions. Although I have chosen a very simple payoff function for this initial work, the basic game structure of the model is easily generalized to model any problem in which payoffs depend on coalition structure. For instance, by making payoffs a function of the entire coalition size vector ($f(g_1g_2...g_J)$), we can include inter-coalitional externalities. This would allow us to model (among other things) a sequential version of the traditional 2-period rent-seeking game (such as that in Nitzen (1991)) or resource-allocation game (such as that in Heintzelman *et al* (2006)) and ask whether inter-coalitional externalities are affected by the structure of the underlying network constraint. For example, we might ask whether social welfare is higher when resource management groups are formed according to geography (eg: water management groups on a river) or social/family ties (eg: fisheries on a bay).

If we allow heterogeneity of players, then the payoff function might depend on the *composition* of the groups, as well as their size ($f(G_1G_2...G_J)$), which would allow us to explore another set of problems. If individuals are heterogeneous in ability or tool sets, then we can ask where self-organized teams contain an optimal

level of diversity for problem-solving. If individuals are heterogeneous in ideology, we can explore the formation of lobbyist organizations and political parties. More abstractly, if individuals are heterogeneous in an arbitrary characteristic, we can model discriminatory behavior.

Finally, making the network structure endogenous may add some insight into network formation (See Jackson (2005) for an excellent survey of this growing literature). This might be accomplished either by making links weighted, or allowing links to be added an lost over time.

The process of group formation is one that has attracted increasing interest in the past decade. In this chapter, I use a simple extension of a static coalition formation game to illustrate the importance of dynamics and membership constraints in the coalition formation process. I show that in the sequential coalition formation game, individuals tend to form groups that are too large, especially when players are unconstrained. Real world groups often implement membership restrictions, indicating that without such restrictions, the groups would tend to be too large. I also show that if individuals are constrained by a network, they tend to form groups that are closer to the ideal size, without the addition of membership restrictions. In fact, constraining group membership by requiring a social or spatial connection can effectively substitute for the institutional constraint of a membership rule.

Average Efficiency--Random Graphs (...)
(ideal group size = 10)
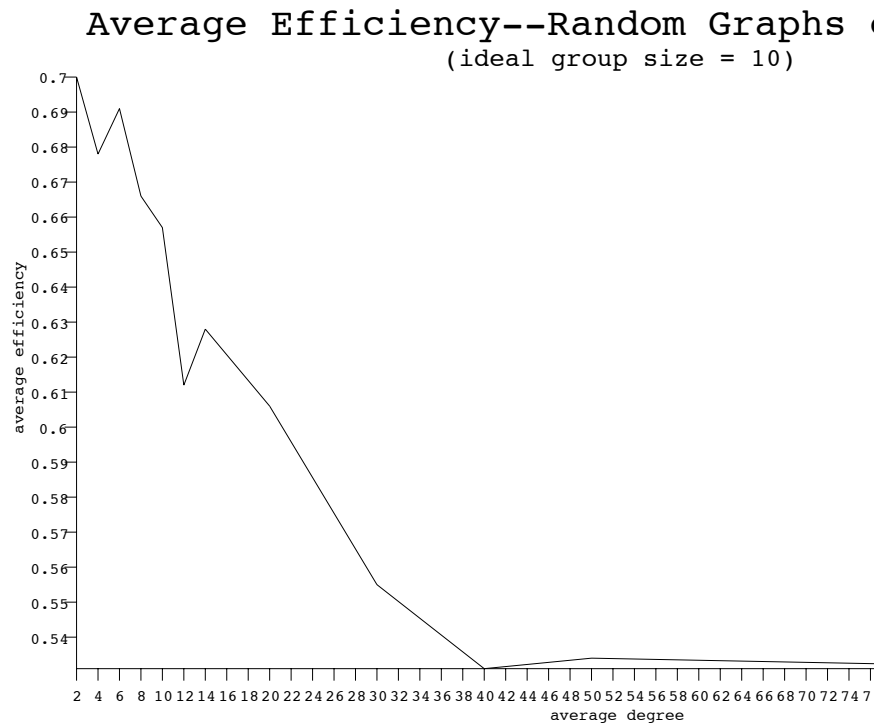
Figure 4.10: Define efficiency to be ratio of actual social welfare to the maximum possible social welfare. This plot shows average efficiency over 100 runs of a sequential coalition formation game with $N = 100$ and $f(g) = g(20 - g)$. The network constraints are random ($p = 1$). As the degree of the network constraint decreases, social welfare increases. Social welfare increases because the constraint binds more heavily, mitigating the tendency for groups to get too large.

Average Efficiency--Watts Strc
(ideal group size 10)
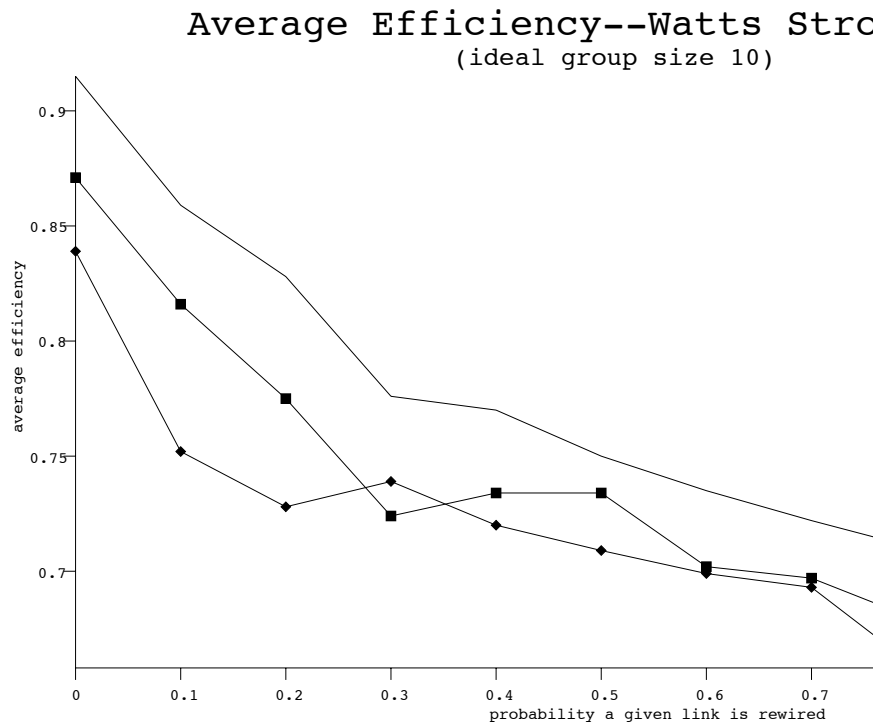
Figure 4.11: Define efficiency to be ratio of actual social welfare to the maximum possible social welfare. This plot shows average efficiency for 100 runs of a sequential coalition formation game. For all runs, $N = 100$ and $f(g) = g(20 - g)$. Holding degree constant (at 2, 4, 6) average social welfare declines in the Watts–Strogatz parameter–that is, social welfare is higher when the network is ordered than when it is random.

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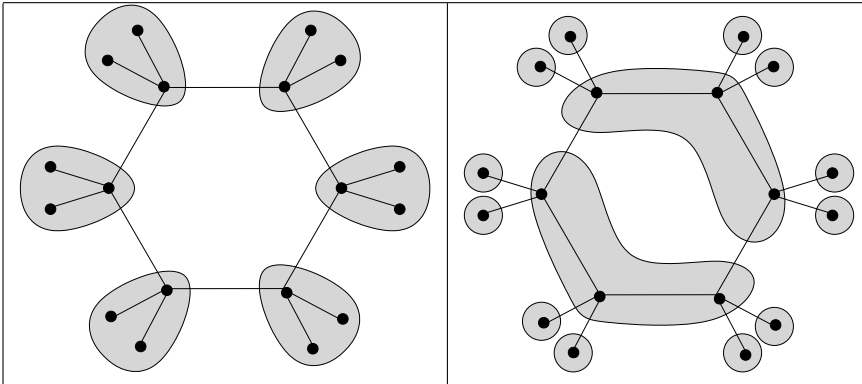
Figure 4.12: An example in which the Exclusive Membership Rule induces a poor outcome. In this game, $g^* = 3$. If groups are able to deny membership, there is a possibility that more central individuals will form one group, leaving the outliers to a poor payoff. This degree distribution of this network is that of a hierarchical social structure.
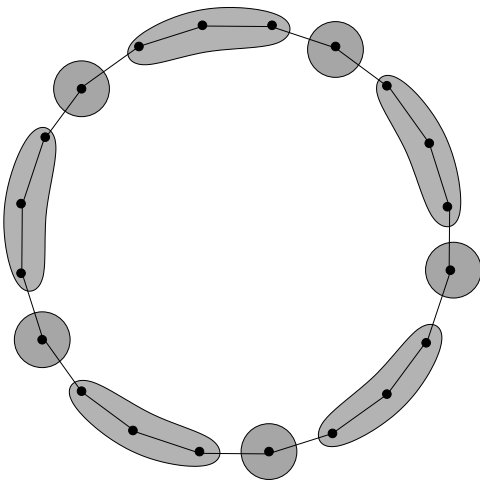
Figure 4.13: An example illustrating how the Exclusive Membership Rule can be detrimental on a ring. This is a game with 20 individuals arranged on a ring and $g^* = 3$. Because the large groups can prevent them from joining, the isolated individuals must accept a lower payoff.
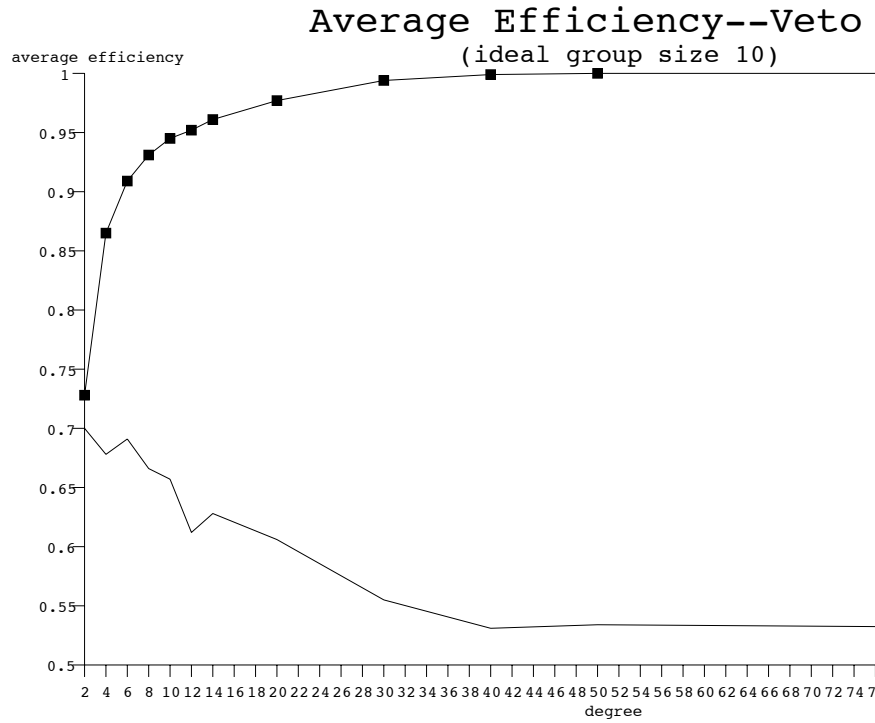
Figure 4.14: Comparing the exclusive membership rule to the open membership rule for network constraints of different degree.

Average Efficiency--Watts Stro

(ideal group size 10)
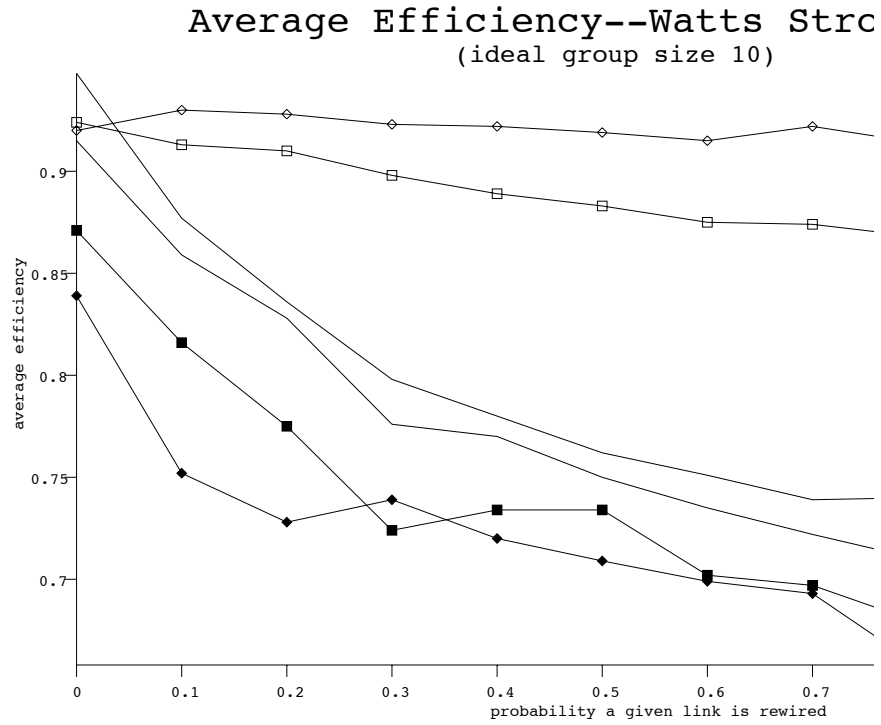
Figure 4.15:

Comparing the exclusive membership rule to the open membership rule for network constraints with different Watts-Strogatz parameters.

CHAPTER V

Conclusion and Extensions

The research agenda started by the work in this thesis is fairly ambitious. The thesis itself is a small step in the direction of understanding the interplay between skills, problem solving, and collaboration networks and the work contained here suggests many pathways for future research. In the chapters themselves, I have mentioned a few of the more modest extensions, some of which represent works in progress. In this concluding section, I will look at a few of the more ambitious possibilities suggested by this work.

Empirical connection between skills and degree

In Chapter 2, I make a connection between the set of skills an individual has and her position on the collaboration network. This connection is valuable, because it has the potential of providing a clearer explanation of the variation in outcomes among problem solvers. If one assumes that skills contribute to an individual's success as a problem solver in a linear manner, when in fact skills interact with one another, we will be able to explain a relatively small amount of the variation in outcomes based on skills. Moreover, the problem becomes worse if we, as researchers, are unable to observe skills that are known to the agents themselves. If we assume that skills contribute to outcomes in a linear fashion, then unobserved skills reduce the

amount of variation we can explain in a linear fashion. On the other hand, if skills contribute to outcomes in combination, there will be a nonlinear drop-off in explained variation with every unobserved skill. Thus, it would be valuable to be able to test the predictions of this model in a real world collaborative setting.

Unfortunately, data that links skills and collaborative success is relatively rare. However, with the increase in freely-available online data, there are several new and exciting data sources that might be used. One particularly good source of data on skills and problem solving comes from the world of large-scale online gaming. In large-scale games, such as World of Warcraft, individual players work together in groups to solve puzzles and complete tasks. The advantage of this data is that it contains very explicit information about individuals' skills and the tasks that they complete. Moreover, because skills are measured within the context of the game, and easily observed by both other players and the researcher, empirical work is quite clean. Other possible sources of data include patent and grant-writing networks. In both of these, there is usually a sense of which individuals bring which skills to the table. In particular fields, the number of patents or grants written is large enough to produce a network that is reasonably well-connected. The observability of skills in this data is less clear than in large-scale gaming, which makes this data more difficult to work with. On the other hand, the context of that data is closer to the kinds of problems that are relevant to businesses and universities, making it a worthwhile avenue to pursue.

Collaboration and skill acquisition

In Chapter 3, I take a first look at some questions surrounding the acquisition of skills. The results of that chapter are the first step into what is clearly a rich and valuable field of research. One particularly compelling sets of questions would consider equilibrium skill acquisition in the context of collaboration. In this thesis, I

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have mostly considered skill acquisition in the context of individual problem solving. It would certainly be valuable to consider the equilibrium decisions individuals would make if they were working in a more collaborative environment. One might suspect that specialists and generalists would play different roles in a collaborative environment, and perhaps occupy different positions in the collaboration network. Modeling these questions theoretically will require a considerable amount of work, but may provide insights into the "ecology" of collaborative communities.

Empirically, it would be interesting to study the connection between specialization of skills and outcomes. Data on this subject is becoming increasingly available as more individuals participate in work-based online social networking. In the context of academic collaboration, one might ask questions about how an individual researcher's outcomes depend on her advisor's position in the social network. Do individuals whose advisors are more central to the network do better on the job market, holding all other factors equal? One could also consider the connection between focus and outcomes. Suppose we look at the committee members for two young researchers. In one case, the researcher's committee members are all located in a single, tight-knit collaborative community within the collaboration network. In the other case, the committee members are spread across a wide range of relatively disconnected collaborative communities. All other things equal, which of these researchers will have a better outcome? Will the individual whose committee resides in a single community benefit from having greater focus? Or will the individual with a widespread committee benefit from a greater diversity of ideas and social information?

Finally, we might consider skill-acquisition in a much more long-run context. In the extreme long-run, we would suspect that the kinds of problems faced by a problem-solving community (such as a group of academic researchers) would have an effect on the skills acquired by those individuals. We might expect that over time, skills would cluster into groups, based on the combinations of skills that are useful for

solving different kinds of problems. Thus, we might expect to observe the evolution of disciplines over time. It would be interesting to look at the robustness of these equilibrium communities to shocks in the kinds of problems faced. It may be that communities become "overfitted" to the distributions of problems they face, causing failures when that distribution experiences external shocks. On the other hand, it may be that by sustaining a small population of generalists, these communities are robust to the effects of shocks.

Dynamic, long run model of networks and collaboration

As mentioned in the introduction to this thesis, there are two aspects to the relationship between social network structure and behavior. On the one hand, social networks clearly affect individual behavior–they shape interactions between individuals and a person's position in the network affects how others view her. On the other hand, social network structure is clearly a product of individual behavior. Chapters 2 and 4 of this thesis address these two aspects of network formation individually. However, there is clearly a feedback between the two. Who a person knows now affects who she is able to interact with in the future, both because she is more likely to interact with new people through your existing social network, and also because others' perceptions of your quality as a collaborator depend on your position in the network. Linking the effects of network structure on behavior into a more dynamic model of network formation is a difficult task. However, it promises to provide insights into the evolution of collaborative communities, and the networks that support them.

APPENDICES

APPENDIX A

Pairwise Stability and Efficiency

Briefly, a network is pairwise stable if no individual would prefer to terminate an existing link, and if no pair of individuals would prefer to add a link. Although this definition is usually used in undirected networks, it works equally well in the current context. Formally, a collaboration network, g, is *pairwise stable* if

1. for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$

2. for all $ij \notin g$, if $u_j(g + ij) > u_j(g)$ then $u_i(g + ij) < u_i(g)$

Together, these two conditions ensure that links are mutual. That is, if a network is pairwise stable, then both players agree to maintain the link.

Theorem. *Any cost minimizing collaboration network, $g \in \Gamma(\Psi)$, is pairwise stable. In other words, $\forall ij \in g$ $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$ and for all $ij \notin g$, if $u_j(g + ij) > u_j(g)$ then $u_i(g + ij) < u_i(g)$.*

Proof. Let g be a cost-minimizing collaboration network. First, consider whether any player wishes to unilaterally remove a link, $ij \in g$. Severing this link deprives player j of his share of the payoff from solving $i's$ problem, $\left(\frac{1}{|C_i|+1} \geq 0 \right)$, and thus he will never choose to terminate one of his incoming links. Since player i chooses a minimal

set of collaborators that allowed her to solve the problem, removing a link means that she can no longer solve the problem and no longer receives a payoff. Since her share of the payoff is greater than zero $\left(\frac{1}{|C_i|+1} \geq 0\right)$, this ensures that she will also never choose to severe a link unilaterally. Finally, note that no player will ever want to add an outgoing link to a cost-minimizing collaboration network–because every player has chosen a set of collaborators optimally, any additional link would require her to further split her prize. $\square$

Theorem. *Any cost minimizing collaboration network, $g \in \Gamma(\Psi)$, is strongly efficient. In other words, $\sum_i u_i(g) \geq \sum_i u_i(g') \; \forall \, g' \in G$.*

Proof. Because all value is generated from solving problems, the maximum possible value in the network is N. Since solving the problem is incentive compatible for every player, and the payoff from solving the problem is split evenly between collaborators, with no loss, the players always extract the maximum value from the network. $\square$

APPENDIX B

General Proof of Supermodularity

Theorem. *For any distribution of skills, Ψ, a player's expected degree over the networks in $\Gamma\left(\Psi\right)$ is a supermodular function of her set of skills. That is, $E\left[d\left(A\cup B\right)\right]+E\left[d\left(A\cap B\right)\right] \geq E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right]$.*

Proof. A player with the set $A\cup B$ will be able to help players needing any subset of those skills. Let $\delta\left(C\right)$ be the demand for a particular set of skills, C. In the general case,

$$\delta\left(C\right) = \Psi\left(S\backslash C\right) + \sum_{D\,:\,\Psi(C\cup D)=0} \Psi\left(S\backslash\left(C\cup D\right)\right)$$

. The fraction who can supply the set C is $\sigma\left(C\right) = \sum_{D\subseteq S\backslash C} \Psi\left(C\cup D\right)$. Note that $\delta\left(C\right)$ and $\sigma\left(C\right)$ depend only on the particulars of the problem (S), the distribution of skills (Ψ), and the subset of skills (C). Thus, any player with the skill set $A\cup B$ has expected degree

$$E\left[d\left(A\cup B\right)\right] = \sum_{C\subseteq A\cup B} \frac{\delta\left(C\right)}{\sigma\left(C\right)}$$

We can divide the problems that a player with $A\cup B$ can solve into three groups:

1. Requires only skills from set A: $C\subseteq A$

2. Requires only skills from set B, including at least one found only in B:

$$\{C \mid C \subseteq B \, and \, \exists \, b \in C \, st \, b \in B \backslash A\}$$

3. Requires at least one skill from each set that can only be found in that set:

$$\{C \mid C \subseteq A \cup B, \, where \, \exists \, a, b \in C \, s.t. \, a \in A \backslash B \, and \, b \in B \backslash A\}$$

Using this partition, we can write

$$
\begin{aligned}
E\left[d\left(A \cup B\right)\right] &= \sum_{C \subseteq A} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C \subseteq B \, and \, C \cap B \neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C \subseteq A \cup B \, and \, C \cap A, \, C \cap B \neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} \\
&= E\left[d\left(A\right)\right] + \sum_{C \subseteq B \, and \, C \cap B \neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \phi
\end{aligned}
$$

which implies that

$$
\begin{aligned}
E\left[d\left(A \cup B\right)\right] + E\left[d\left(A \cap B\right)\right] &= E\left[d\left(A\right)\right] + \sum_{C \subseteq B \, and \, C \cap B \neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \phi + E\left[d\left(A \cap B\right)\right] \\
&= E\left[d\left(A\right)\right] + \left(\sum_{C \subseteq B \, and \, C \cap B \neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C \subseteq A \cap B} \frac{\delta\left(C\right)}{\sigma\left(C\right)} \right) + \phi \\
&= E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right] + \phi \\
&\geq E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right]
\end{aligned}
$$

$\square$

APPENDIX C

Shapley Value of Each Skill in a Skill Set

We can calculate the contribution of each skill in a player's skill set to the total demand for her skills using a Shapely value decomposition. The demand for player i's skills can generically be written as $d(A_i) = \sum_{C \subseteq A_i} \frac{\delta(C)}{\sigma(C)}$ where $\delta(C) = \Psi(S \backslash C)$[1] and $\sigma(C) = \sum_{D \subseteq S \backslash C} \Psi(C \cup D)$. Using this demand as a value function,[2] we can obtain an expression for the Shapely value of a skill, a, to player i:

$$\phi_{a,i}(d) = \sum_{B \subseteq A_i \backslash \{a\}} \frac{1}{\binom{|A_i|}{|B|}} \left(\sum_{C \subseteq B} \frac{\Psi(S \backslash (C \cup a))}{\sum_{D \subseteq S \backslash (C \cup a)} \Psi((C \cup a) \cup D)} \right)$$

This decomposition highlights several points that have been made earlier in the paper: first, the value of a skill to a player depends on the rest of the skills that player has, and second, the value of a skill depends on the population of problem solvers.

---

[1] Note that this is the expression when each player needs exactly one partner to solve their problem. The results are similar for the more general case.

[2] Note that $d(.)$ satisfies both requirements for a value function: $d(\emptyset) = 0$ and according to Theorem 2, $d(A \cup B) + d(A \cap B) \geq d(A) + d(B)$, which implies superadditivity.

In the particular case of the Bernoulli Skills Model, we can make several other observations. In this case, the value of skill a to player i is

$$\phi_{a,i}\left(d\right) = p^M \left(1 - p\right) \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) \left(\frac{1 - p + p^2}{p^2}\right)^j$$

Skills are symmetric in the Bernoulli Skills model, and thus each skill has the same value to a particular player–that is, $\phi_{a,i} = \phi_i \forall a \in A_i$. Also, because skills are distributed independently, the value of a skill to a player is strictly increasing in the number of skills the player already has–that is, $\phi_{a,i} > \phi_{a,j}$ iff $|A_i| > |A_j|$.

APPENDIX D

A More General Case: Asymmetric Disciplines

Theorem 31 is the equivalent of Theorem 16, and states that if individuals can work on any available problem, then there is no advantage to being a generalist.

Theorem 31. *If skills are symmetric within disciplines, and players can work on any available problem, then no player will ever want to be a generalist. In other words, the equilibrium population will contain only specialists.*

Proof. As above, the *ex ante* probability that a specialist in discipline i will be able to solve a problem from a given distribution, Δ, is

$$
\begin{aligned}
E\left[P\left(S_i\right)\right] &= 1 - \left(\delta_i h + \left(1 - \delta_i\right) l\right)^K \\
&= 1 - \pi_i^K
\end{aligned}
$$

where $\pi_i = \left(\delta_i h + \left(1 - \delta_i\right) l\right)$.

WLOG, suppose $\delta_1 > \delta_2$. Since $h < l$, this means that $\pi_1 < \pi_2$ and $E\left[P\left(S_1\right)\right] > E\left[P\left(S_2\right)\right]$. Thus, to determine whether any player will generalize, I need to compare $E\left[P\left(S_1\right)\right]$ to $E\left[P\left(G\right)\right]$.

The *ex ante* probability that a generalist with x skills in discipline 1, and $K - c - x$

skills in discipline 2 solves a problem from a given distribution, Δ, is

$$
\begin{aligned}
E\left[P\left(G\right)\right] &= 1 - \left(\delta_1 h + \left(1 - \delta_1\right) l\right)^x \left(\delta_2 h + \left(1 - \delta_2\right) l\right)^{K-c-x} \\
&= 1 - \pi_1^x \pi_2^{K-c-x}
\end{aligned}
$$

Since $\pi_1 < \pi_2$, $E\left[P\left(G\right)\right]$ is strictly increasing in x. This means that a generalist will set $x = K - c - 1$ and $E\left[P\left(G\right)\right] = 1 - \pi_1^{K-c-1}\pi_2$, which is clearly less than $E\left[P\left(S_1\right)\right] = 1 - \pi_1^K$. $\qquad\square$

Theorem 32 is a generalized version of Theorem 17, and states the parameter range in which individuals will choose to diversify their skills when there are barriers to working interdisciplinarily.

Theorem 32. *If skills are symmetric within disciplines, and problems are assigned to one of two disciplines, then players will generalize for intermediate values of ϕ, the fraction of problems assigned to discipline 1.*

In particular, the ranges are as follows:

- *If $\delta_1 = \delta_2 = \delta$, players will obtain $K - c$ skills spread across the two disciplines when $1 - \frac{1-\pi^{K-c}}{1-\pi^K} \leq \phi \leq \frac{1-\pi^{K-c}}{1-\pi^K}$, K skills in discipline 1 when $\phi > \frac{1-\pi^{K-c}}{1-\pi^K}$, and K skills in discipline 2 when $\phi < 1 - \frac{1-\pi^{K-c}}{1-\pi^K}$.*

- *If $\delta_1 > \delta_2$, then players will obtain $K - c - 1$ skills in discipline 1 and one skill in discipline 2 when $1 - \left(\frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_2^K}\right) \leq \phi \leq \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_1^K}$, K skills in discipline 1 when $\phi > \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_1^K}$, and K skills in discipline 2 when $\phi < 1 - \left(\frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_2^K}\right)$.*

- *If $\delta_2 > \delta_1$, then players will obtain $K - c - 1$ skills in discipline 2 and one skill in discipline 1 when $1 - \left(\frac{1-\pi_2^{K-c-1}\pi_1}{1-\pi_1^K}\right) \leq \phi \leq \frac{1-\pi_2^{K-c-1}\pi_1}{1-\pi_1^K}$, K skills in discipline 1 when $\phi > \frac{1-\pi_2^{K-c-1}\pi_1}{1-\pi_1^K}$, and K skills in discipline 2 when $\phi < 1 - \left(\frac{1-\pi_2^{K-c-1}\pi_1}{1-\pi_2^K}\right)$.*

Proof. In this case, the *ex ante* probability that a problem is solved by a specialist is $\phi\left(1 - \pi_1^K\right)$ for a specialist in discipline 1 and $\left(1 - \phi\right)\left(1 - \pi_2^K\right)$ for a specialist

in discipline 2. Since generalists can work on problems in both disciplines, their expected probability of solving the problem is $1 - \pi_1^x \pi_2^{K-c-x}$ where x is the number of skills the generalist chooses to acquire in discipline 1. First, suppose $\delta_1 > \delta_2$. Since $h < l$, this means that $\pi_1 < \pi_2$ and $E[P(G)]$ is strictly increasing in x. Thus, a generalist will choose a minimal number of skills in the less useful discipline, and $E[P(G)] = 1 - \pi_1^{K-c-1} \pi_2$.

A player will generalize if $E[P(S_1)] < E[P(G)]$ and $E[P(S_2)] < E[P(G)]$. Setting $\phi(1 - \pi_1^K) < 1 - \pi_1^{K-c-1} \pi_2$ implies that $\phi \leq \frac{1 - \pi_1^{K-c-1} \pi_2}{1 - \pi_1^K}$. Setting $(1 - \phi)(1 - \pi_2^K) < 1 - \pi_1^{K-c-1} \pi_2$ implies that $1 - \frac{1 - \pi^{K-c}}{1 - \pi^K} \leq \phi$. We can verify that in the appropriate ranges, players choose to specialize. The result follows immediately. The proof for $\delta_2 > \delta_1$ is similar. For the proof when $\delta_1 = \delta_2$, see Theorem 17. $\qquad\square$

Finally, Theorem 33 is the generalization of Theorem 18. It states that there is a parameter region in which individuals choose to specialize, but society would prefer to have at least a few generalists.

Theorem 33. *If skills are symmetric within disciplines and problems are assigned to one of two disciplines, then there is a range of values for ϕ (the fraction of problems assigned to discipline 1) such that generalists are underprovided in the equilibrium population of problem solvers.*

In particular, generalists are underprovided in the following ranges:

- *If $\delta_1 = \delta_2$, then generalists are underprovided when $\frac{1 - \pi^{K-c}}{1 - \pi^K} < \phi < \frac{1 - \pi^{N(K-c)}}{1 - \pi^{NK}}$ or $1 - \frac{1 - \pi^{N(K-c)}}{1 - \pi^{NK}} < \phi < 1 - \frac{1 - \pi^{K-c}}{1 - \pi^K}$*

- *If $\delta_1 > \delta_2$, then generalists are underprovided when $\frac{1 - \pi_1^{K-c-1} \pi_2}{1 - \pi_1^K} < \phi < \frac{1 - \pi_1^{N(K-c-1)} \pi_2}{1 - \pi_1^{NK}}$ or $1 - \frac{1 - \pi_1^{N(K-c-1)} \pi_2}{1 - \pi_2^{NK}} < \phi < 1 - \frac{1 - \pi_1^{K-c-1} \pi_2}{1 - \pi_2^K}$*

- *If $\delta_2 > \delta_1$, then generalists are underprovided when $\frac{1 - \pi_1 \pi_2^{K-c-1}}{1 - \pi_1^K} < \phi < \frac{1 - \pi_1 \pi_2^{N(K-c-1)}}{1 - \pi_1^{NK}}$ or $1 - \frac{1 - \pi_1 \pi_2^{N(K-c-1)}}{1 - \pi_2^{NK}} < \phi < 1 - \frac{1 - \pi_1 \pi_2^{K-c-1}}{1 - \pi_2^K}$*

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Proof. First, suppose that $\delta_1 > \delta_2$. The probability that at least one of the N problem-solvers in the population solves the problem is $1 - Prob$ (none of them do). If all of the individuals in the population are specialists in discipline 1, then every individual has probability ϕ of a problem occurring in her discipline. In that case, each specialist in discipline has a probability $1 - \pi_1^K$ of solving the problem and π_1^K of not solving it. With probability $1 - \phi$, the problem is assigned to the other discipline, and no specialist solves it. Thus, the probability of someone in a population of specialists solving the problem is

$$
\begin{aligned}
Prob \,(\text{one of N solve it}) \;&=\; 1 - Prob \,(\text{none of N solve it})\\
&=\; 1 - \left[\phi Prob\,(\text{none solve problem in } d_1) + (1 - \phi)\, Prob\,(\text{none solve prob}\right.\\
&=\; 1 - \left[\phi Prob\,(\text{one fails})^N + (1 - \phi) * 1\right]\\
&=\; 1 - \left[\phi \left(\pi_1^K\right)^N + (1 - \phi) * 1\right]\\
&=\; \phi\left(1 - \pi_1^{KN}\right)
\end{aligned}
$$

Through a similar argument, if everyone in the population is a specialist in discipline 2, then the probability that someone in the population solves the problem is $(1 - \phi)\left(1 - \pi_2^{KN}\right)$.

If everyone in the population is a generalists, then the probability of at least one person in solving the problem is

$$
\begin{aligned}
Prob \,(\text{one of N solve it}) \;&=\; 1 - Prob \,(\text{none of N solve it})\\
&=\; 1 - \left(\pi_1^{K-c-1}\pi_2\right)^N\\
&=\; 1 - \pi_1^{N(K-c-1)}\pi_2^N
\end{aligned}
$$

Society is better off with a population of generalists than a population of discipline 1 specialists when $1 - \pi_1^{N(K-c-1)}\pi_2^N > \phi\left(1 - \pi_1^{KN}\right)$, which is true when

$\phi < \frac{1-\pi_1^{N(K-c-1)}\pi_2^N}{1-\pi_1^{NK}}$. However, there is a population of generalists when $\phi \leq \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_1^K}$.

It is always the case that $\frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_1^K} \leq \frac{1-\pi_1^{N(K-c-1)}\pi_2}{1-\pi_1^{NK}}$. Thus, if $\frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_1^K} < \phi < \frac{1-\pi_1^{N(K-c-1)}\pi_2}{1-\pi_1^{NK}}$, then society is better off with a population of generalists, but has a population of specialists.

Through a similar argument, society is better off with a population of generalists than a population of discipline 2 specialists when $1-\pi_1^{N(K-c-1)}\pi_2^N > (1-\phi)\left(1-\pi_2^{KN}\right)$, which is true when $\phi > 1 - \frac{1-\pi_1^{N(K-c-1)}\pi_2^N}{1-\pi_2^{NK}}$. However, there is a population of generalists when $\phi \leq 1 - \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_2^K}$. It is always the case that $1 - \frac{1-\pi_1^{N(K-c-1)}\pi_2}{1-\pi_2^{NK}} \leq 1 - \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_2^K}$. Thus, if $1 - \frac{1-\pi_1^{N(K-c-1)}\pi_2}{1-\pi_2^{NK}} < \phi < 1 - \frac{1-\pi_1^{K-c-1}\pi_2}{1-\pi_2^K}$, then society is better off with a population of generalists, but has a population of specialists.

The proof for $\delta_2 > \delta_2$ is similar. See the proof of Theorem 18 for the case where $\delta_1 = \delta_2$. $\qquad\square$

APPENDIX E

Examples Where Sequential Coalition Formation Game Have No Unique Equilibrium

I will illustrate the second half of this claim first–that the order of play can affect the set of Nash Equilibria. Consider a game with 12 players arranged in a ring, as shown in Figure 4.6. Further, suppose the payoff function is $f(g)$ such that $g^* = 2$ and $\bar{g} = 6$.

For the first case, suppose that the players proceed in order around the ring– that is, $\phi_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. Figure E.1 shows game play leading to an equilibrium coalition structure with two groups of size six. Because of the order of play, the individuals are always choosing between joining an existing large group, forming a new group of two, or remaining as an individual. This choice is much the same as the choice players face in the unconstrained game with the same payoff function–pictured Figure E.2. Thus, it should be unsurprising that the players reach the same equilibrium coalition structure as they would in the unconstrained game: $\langle 6, 6 \rangle$. In fact, this is the only equilibrium coalition size vector possible in this particular coalition formation game.

Now consider a second game with the same number of players, network constraint, and payoff function, but a different order of play $\phi_2 = (2, 3, 5, 6, 8, 9, 11, 12, 1, 7, 4, 10)$.
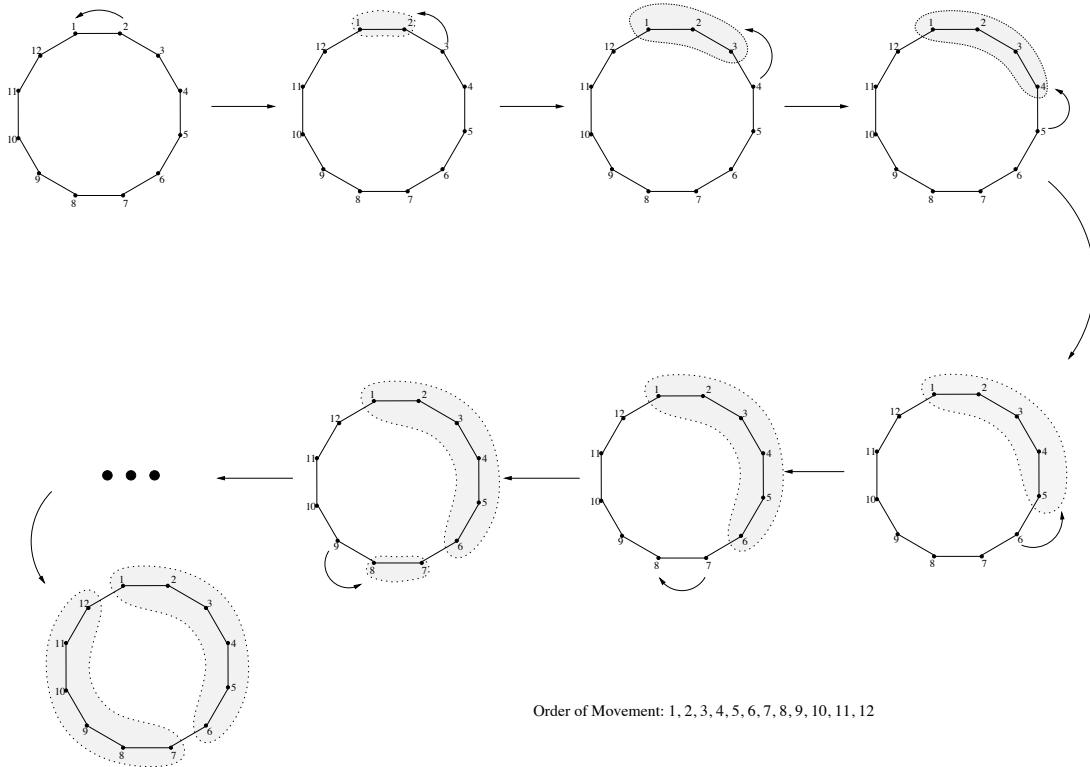
Figure E.1: In this game, 12 individuals are arranged in a ring. The payoff function, $f(g)$, has maximum $g^* = 2$ and $\bar{g} = 6$. The individuals move in order around the ring–$\phi_1 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$–and wind up in two groups of size 6. In fact, $\langle 6, 6 \rangle$ is the only equilibrium group size configuration of the game $(12, f(g), \phi_1)$. Figure E.3 shows the same game with a different order of play.

Figure E.3 shows one possible sequence of game play, given ϕ_2. Because the first few players to move are separated from the existing large groups, they are unable to impose on the groups that have already formed, as they did in the previous example. The result is an equilibrium coalition structure with four groups of the ideal size: $\langle 3, 3, 3, 3 \rangle$. Since $\langle 3, 3, 3, 3 \rangle$ is in $\varepsilon(N, f(g), \phi_2, C)$ but not in $\varepsilon(N, f(g), \phi_1, C)$, it is clear that the order of motion does affect the set of equilibria.

Of course, the outcome pictured in Figure E.3 is not the only possible equilibrium of the game with order of play ϕ_2. Many players in this game are forced to make random choices. Figure E.4 shows that if some of those players make different choices, then the players will find themselves in a different configuration–in this case, $\langle 4, 4, 4 \rangle$.
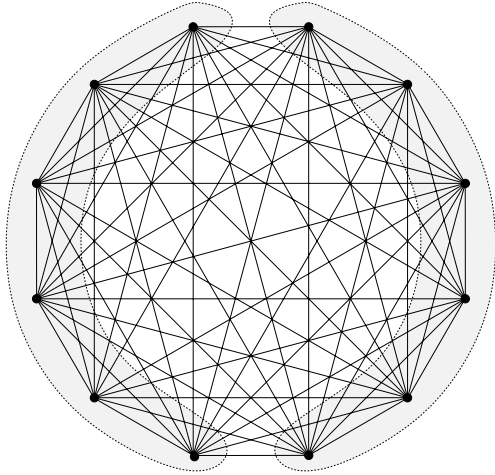
Figure E.2:
An example of an equilibrium on fully-connected network. When $g^* = 3$ and $\bar{g} = 6$, Theorem 26 suggests that the individuals will form into two groups of size 6.

This is an illustration of the first half of Claim 30, which states that a coalition formation game with a network constraint need not have a unique equilibrium.
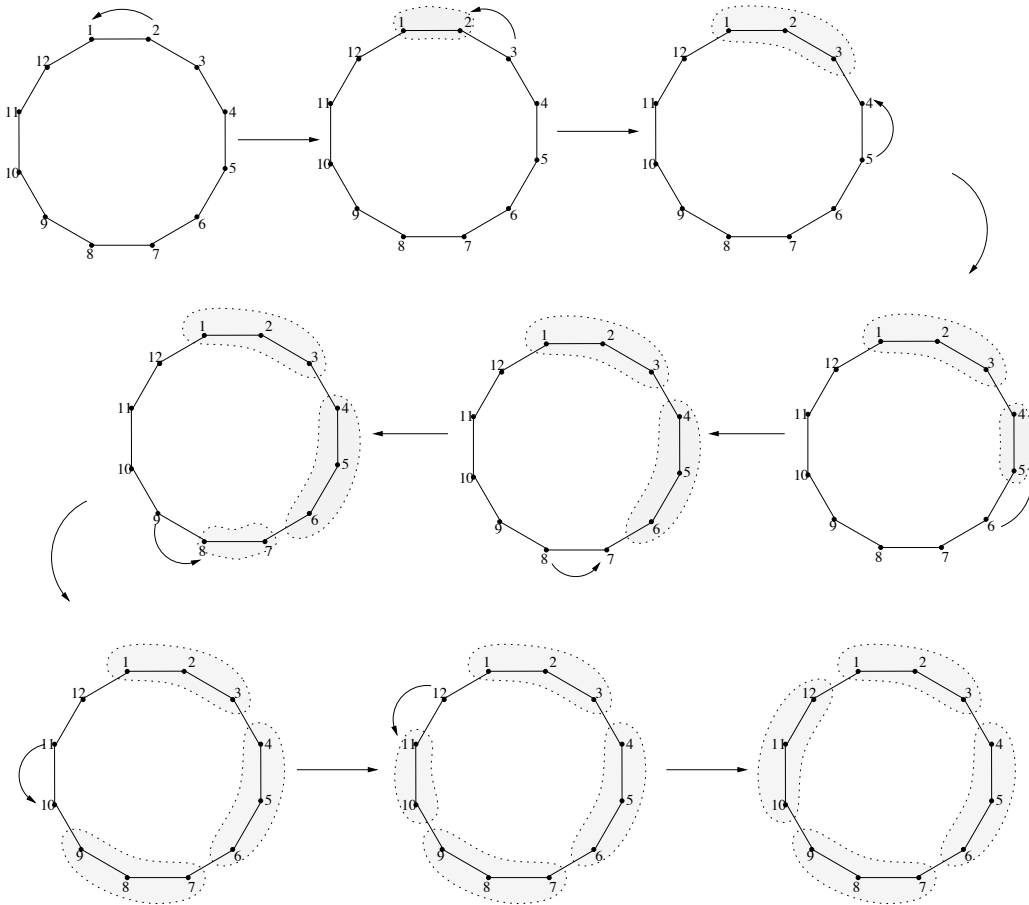
Figure E.3: This game is identical to the game presented in Figure E.3 except for the order of motion: $\phi_2 = 2, 3, 5, 6, 8, 9, 11, 12, 1, 7, 4, 10$. This figure shows a particular sequence of moves, which leads to groups of the ideal size: $\langle 3, 3, 3, 3 \rangle$. Note that $\langle 3, 3, 3, 3 \rangle$ is not an equilibrium of the game presented in Figure E.3, proving that the set of equilibria may depend on the order of play.
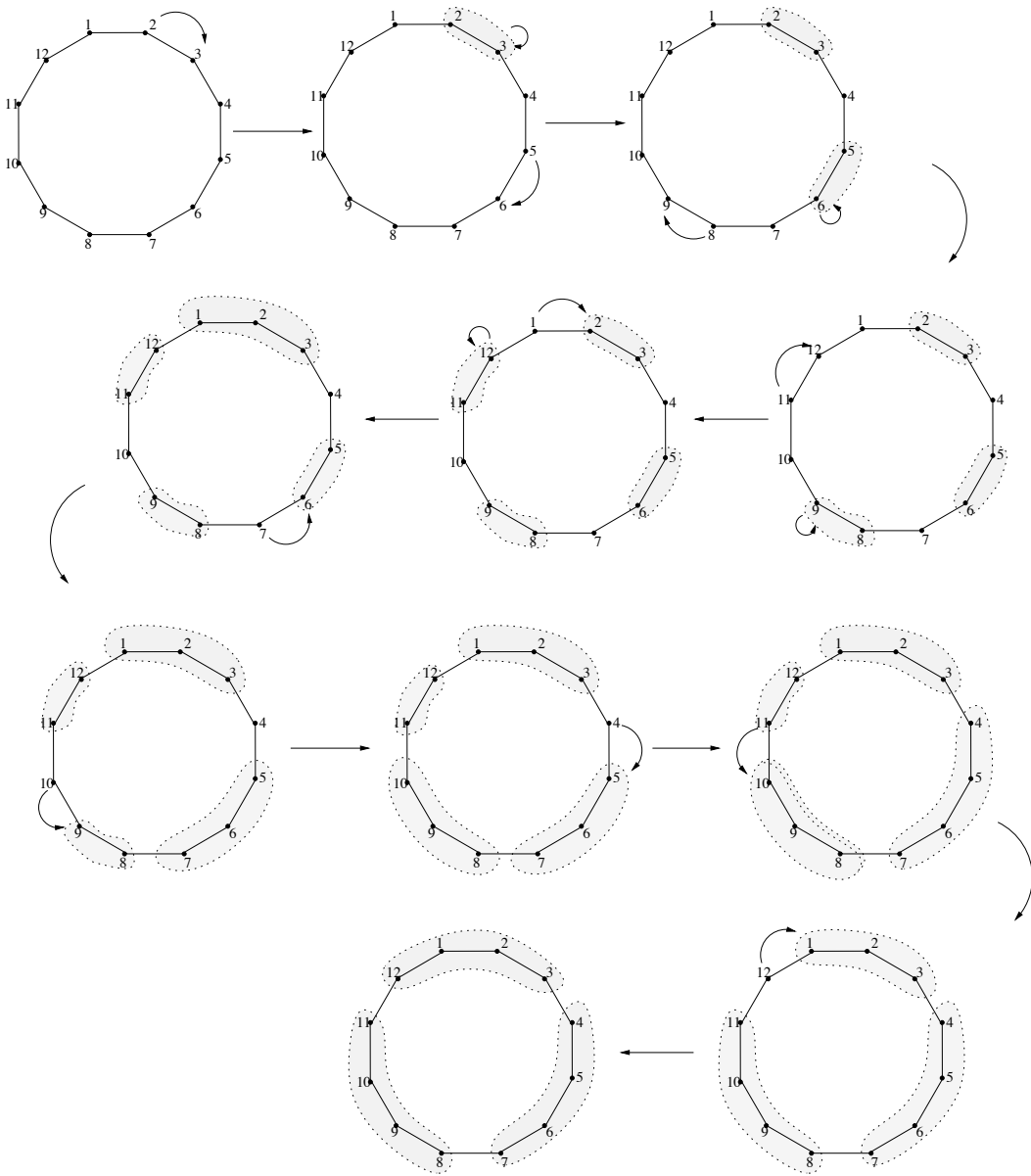
Figure E.4: This game is identical to that presented in Figure E.3. Note, in particular, that the order of play is the same: $\phi_2 = 2, 3, 5, 6, 8, 9, 11, 12, 1, 7, 4, 10$. However, the players have made different random choices, leading to a different equilibrium outcome: $\langle 4, 4, 4 \rangle$. This shows that when players are sufficiently constrained, there need not be a unique equilibrium coalition size configuration.

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