General Presentations of Algebras

by

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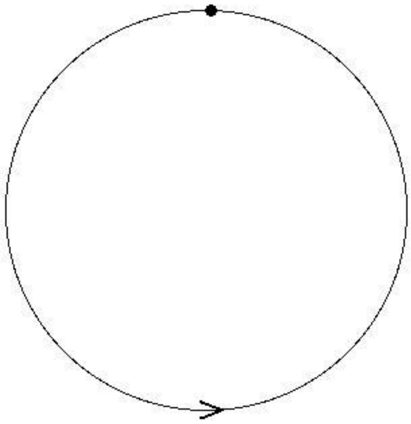

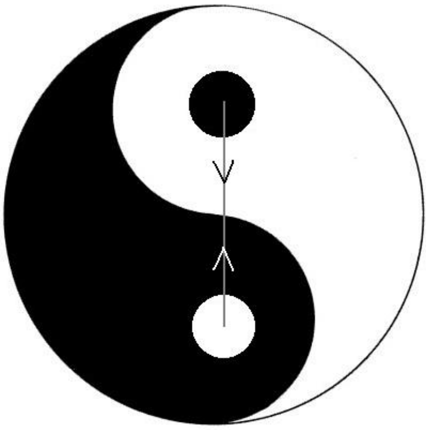

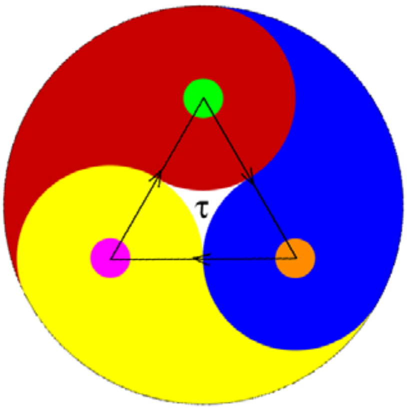

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To my papa and mama

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CHAPTER I

Introduction

A *quiver* Q is a pair $Q = (Q_0, Q_1)$ consisting of the set of vertices Q_0 and the set of arrows Q_1. If $a \in Q_1$ is an arrow, then ta and ha denote its tail and its head respectively. A path is a sequence of arrows $p = a_1 a_2 \cdots a_s$, with $ta_i = ha_{i+1}$ for all i. We define $tp = ta_s$ and $hp = ha_1$. For each vertex $v \in Q_0$ we also define the trivial path e_v of length 0, satisfying $te_v = he_v = v$. An oriented cycle is a nontrivial path satisfying $hp = tp$. In this thesis, we assume that Q_0 and Q_1 are finite but allow quivers to have oriented cycles. For any field k, the path algebra kQ is the k-vector space spanned by all paths. The multiplication in the algebra kQ is defined as follows. If p and q are paths, then their product $p \cdot q$ is the concatenation of the paths if $tp = hq$, and is defined 0 otherwise. The path algebra is bigraded: $kQ = \bigoplus_{v,w \in Q_0} e_v kQ e_w$. A relation $r = \sum_{i=1}^{s} c_i p_i$ with $c_i \in k$ and p_i a path, is called *admissible* if r is homogeneous with respect to the grading, i.e., there exist $tr, hr \in Q_0$ such that $tp_i = tr$ and $hp_i = hr$ for all i. We assume that I is an *admissible ideal*, i.e., a two sided ideal generated by admissible relations of length $\geqslant 2$. The path algebra *with relations* is the quotient algebra kQ/I. It contains a maximal semisimple subalgebra R spanned by $\{e_v\}_{v \in Q_0}$. We assume that k is algebraically closed and $A = kQ/I$ is finite dimensional henceforth.

A *dimension vector* α is a non-negative integer-valued function on Q_0. Given a dimension vector α, let M be the R-module k^α, which is a family of finite-dimensional k-vector spaces $\{M(v)\}_{v\in Q_0}$ with $\dim M(v) = \alpha(v)$. A *representation* M of kQ is a R-module M together with a family of linear maps $\{M(a) : M(ta) \to M(ha)\}_{a\in Q_1}$. Fixed a dimension vector α, the representation space $\mathrm{Rep}_\alpha(kQ)$ of all α-dimensional representations is the vector space $\bigoplus_{a\in Q_1} \mathrm{Hom}(k^{\alpha(ta)}, k^{\alpha(ha)})$. For any path $p = a_1 a_2 \cdots a_s$, we define $M(p) : M(tp) \to M(hp)$ to be the composition $M(a_1)M(a_2)\cdots M(a_s)$. So the assignment $M \mapsto M(p)$ defines a polynomial map

$$F_p : \mathrm{Rep}_\alpha(kQ) \to \mathrm{Hom}(k^{\alpha(tp)}, k^{\alpha(hp)}).$$

A representation M of $A = kQ/I$ is a representation of kQ satisfying all the relations in I, i.e., $M(r) = 0$ for all $r \in I$. Let $k[\mathrm{Rep}_\alpha(kQ)]$ denote the ring of polynomial functions on $\mathrm{Rep}_\alpha(kQ)$. Then F_p is represented by an $\alpha(hp) \times \alpha(tp)$ matrix with entries in $k[\mathrm{Rep}_\alpha(kQ)]$. Let $\tilde{I} \subseteq k[\mathrm{Rep}_\alpha(kQ)]$ be the ideal generated by the entries of all F_r for which $r \in I$. The representation space $\mathrm{Rep}_\alpha(A)$ is the scheme $\mathrm{Spec}(k[\mathrm{Rep}_\alpha(kQ)]/\tilde{I})$. The coordinate ring $k[\mathrm{Rep}_\alpha(kQ)]/\tilde{I}$ represents the following functor: $B \mapsto \mathrm{Hom}_{R-alg}(A, \mathrm{End}_B(B \otimes M))$, from the category of finitely generated commutative k-algebra to the Sets. As a set, $\mathrm{Rep}_\alpha(A)$ consists of all α-dimensional representation of A. The group $\mathrm{GL}_\alpha := \prod_{v\in Q_0} \mathrm{GL}_{\alpha(v)}$ acts on $\mathrm{Rep}_\alpha(A)$ by the natural base change. Two representations $M, N \in \mathrm{Rep}_\alpha(A)$ are isomorphic if they lie in the same GL_α-orbit.

A morphism $f : M \to N$ between two representations is a collection of linear maps $\{f(v) : M(v) \to N(v)\}_{v\in Q_0}$ such that for each $a \in Q_1$ we have $N(a)f(ta) = f(ha)M(a)$. In this way, the category $\mathrm{Rep}(A)$ of equivalent representations of A is an abelian category. Given any $M \in \mathrm{Rep}(A)$, we can make the vector space $\bigoplus_{v\in Q_0} M(v)$ into a left A-module as follows. For each vertex $v \in Q_0$, e_v acts as the projection onto $M(v)$. For each $a \in Q_1$, a acts by $a|_{M(ta)} = M(a)$ and $a|_{M(v)} = 0$ if $v \neq ta$. It is easy to see that under this correspondence the category $\mathrm{Rep}(A)$ is equivalent to the category of all finite dimensional left A-modules. For each vertex $v \in Q_0$, let S_v be the 1-dimensional simple representation with $S_v(v) = k$. The Grothendieck group $K_0(\mathrm{Rep}(A))$ of this abelian category is a free abelian group generated by all S_v [1, Theorem III.3.5]. So the dimension vector of a representation M is its class in $K_0(\mathrm{Rep}(A))$. A representation $M \in \mathrm{Rep}(A)$ is called *indecomposable* if $M = L \oplus N$ for $L, N \in \mathrm{Rep}(A)$, then one of L and N must be the zero objects. The category $\mathrm{Rep}(A)$ has the *Krull-Schmidt property*, meaning that each representation has a unique decomposition into indecomposable summands.

Kac studied in [13] properties of general representations of path algebras. We say that a *general representation* of dimension α has a certain property, if all representations in some Zariski open (and dense) subset of $\mathrm{Rep}_\alpha(kQ)$ have that property. We say that $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r$ is the *canonical decomposition* of a dimension vector α if a general representation V of dimension α has a decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where each V_i is indecomposable of dimension α_i. For such a decomposition, each α_i is a *Schur root*, which means a general representation of dimension α_i is indecomposable. Conversely, Kac showed that if $\alpha_1, \ldots, \alpha_r$ are Schur roots, and $\mathrm{ext}(\alpha_i, \alpha_j) = 0$ for all $i \neq j$, then $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_r$ is the canonical decomposition. Here, $\mathrm{ext}(\alpha, \beta)$ denotes $\dim \mathrm{Ext}^1_{kQ}(V, W)$ where V and W are general of dimension α and β respectively. In the same spirit, Schofield showed in [17] that $\mathrm{ext}(\alpha, \beta) = 0$ if and only if a general representation of dimension $\alpha + \beta$, has an α dimensional subrepresentation. Schofield also gave recursive formulas for $\mathrm{ext}(\alpha, \beta)$. An efficient algorithm for finding the canonical decomposition of a dimension vector was given in [5]. Moreover, in the same paper a simplicial complex governing the canonical decomposition was described.

A natural question is whether the results about general representations of path algebras, can be generalized to path algebras with relations. Unfortunately, the representation spaces $\mathrm{Rep}_\alpha(A)$ for path algebras with relations may not be irreducible, so one have to work with general representations in a given irreducible component as in [4]. As another approach, we can study general *presentations*. For fixed projective representations P_0, P_1 of A, a *general presentation* $f : P_1 \to P_0$ is a general element in the vector space $\mathrm{Hom}_A(P_1, P_0)$. For two presentations f, g, we will define a finite dimensional space $\mathrm{E}(f, g)$ which plays the role of Ext^1_{kQ} for path algebras. We will prove analogs of the canonical decomposition and subpresentations of general presentations of path algebras with relations in this context. In the special case of path algebras (without relations), we recover the results of Kac and Schofield. Similar questions for general presentations of path algebras (without relations) were studied in [11] through a different approach.

Recall that the category $\mathrm{Rep}(A)$ has enough projective objects. All indecomposable projective representations are of form $P_v := Ae_v$ for some vertex $v \in Q_0$. They are characterized by the property that $\mathrm{Hom}_A(P_v, N) = N(v)$ for any $N \in \mathrm{Rep}(A)$. For any $\beta \in \mathbb{N}_0^{Q_0}$, we denote $P(\beta) := \bigoplus_{v \in Q_0} P_v^{\beta(v)}$ and define the space of presentation

$$\mathrm{PHom}_A(\beta_1, \beta_0) := \mathrm{Hom}_A(P(\beta_1), P(\beta_0)).$$

The automorphism group $\mathrm{Aut}_A(M)$ of $M \in \mathrm{Rep}(A)$ consists of invertible elements in $\mathrm{Hom}_A(M, M)$. The group $\mathrm{Aut}_A(P(\beta_1)) \times \mathrm{Aut}_A(P(\beta_0))$ acts on $\mathrm{PHom}_A(\beta_1, \beta_0)$ by $(g_1, g_0)f = g_0 f g_1^{-1}$. We define the *$\delta$-vector* of a presentation $f \in \mathrm{PHom}_A(\beta_1, \beta_0)$ by $\delta(f) = \beta_0 - \beta_1 \in \mathbb{Z}^{Q_0}$. Conversely, for any $\delta \in \mathbb{Z}^{Q_0}$, there is a unique decomposition $\delta = \beta_0 - \beta_1$ such that $\beta_0, \beta_1 \in \mathbb{N}_0^{Q_0}$ have disjoint supports, i.e., $\beta_0(v) = 0$ or $\beta_1(v) = 0$

for all $v \in Q_0$. To δ we associate the *reduced* presentation space $\mathrm{PHom}_A(\delta) = \mathrm{PHom}_A(\beta_1, \beta_0)$.

The homotopy category $K^b(\text{proj-}A)$ of bounded complexes of projective representations of A is a triangulated category with the Grothendieck group $K_0(K^b(\text{proj-}A))$ isomorphic to $K_0(\mathrm{Rep}(A))$. In fact, if we embed $K^b(\text{proj-}A)$ and $\mathrm{Rep}(A)$ canonically in the bounded derived category $D^b(\mathrm{Rep}(A))$, then the Euler form

$$\langle P, M\rangle = \chi(\mathrm{RHom}_{D^b(\mathrm{Rep}(A))}(P, M)) \text{ on } K^b(\text{proj-}A) \times \mathrm{Rep}(A)$$

gives us a dual pairing with the class of P_v dual to the class of S_v. The δ-vector of a presentation is nothing but its class in $K_0(K^b(\text{proj-}A))$. Let $P^\bullet = P_l \to \cdots \to P_1 \to P_0$ be a length $l+1$ complex in $K^b(\text{proj-}A)$. The *i-th Betti vector* $\beta_i(f)$ is the minimal vector such that $P^\bullet$ is equivalent to some complex $P(\beta_l(f)) \to \cdots \to P(\beta_1(f)) \to P(\beta_0(f))$ in $K^b(\text{proj-}A)$.

A representation M of A is called *rigid* if $\mathrm{Ext}^1_A(M, M) = 0$. A rigid representation M is *partial tilting* if in addition it has projective dimension ≤ 1. Brenner-Butler's classical tilting theory [1, VI] says that any partial tilting representation M can be completed to a maximal one $\overline{M}$ in the sense that any indecomposable representation N for which $\overline{M} \oplus N$ is partial tilting is isomorphic to a direct summand of $\overline{M}$. Moreover, the number of pairwise nonisomorphic indecomposable summands of the completion $\overline{M}$ is equal to the rank of $K_0(\mathrm{Rep}(A))$ or equvalently the number of vertices of the quiver. We will prove a similar statement for *rigid presentations*, i.e., those presentation f satisfying $\mathrm{E}(f, f) = 0$. For a finite-dimensional path algebra, there are exactly two ways to complete a *sincere almost tilting* representation M, that is, M has $|Q_0| - 1$ nonisomorphic indecomposable summands and $\dim M(v) \neq 0$ for any $v \in Q_0$. We show that this is true for presentations of path algebras with relations as well. Moreover, we define a simplicial complex which governs the decomposition of rigid presentations. In the case of path algebra (without relations), this simplicial complex is the well-known *cluster complex* [11] associated to the cluster algebra of an acyclic quiver. *Cluster algebras* were introduced by Fomin and Zelevinsky in 2000 [8]. Representation theory of path algebras with relations has been used to study the combinatorics of cluster algebras. In [6, 7], Weyman, Zelevinsky and my advisor use *quivers with potentials* to prove results about cluster algebras *with coefficients*. From the *potential*, one can derive a set of relations. In this context, the space $\mathrm{E}(f, g)$ is the same as the one defined in [7]. If V is a representation corresponding to a *cluster variable*, then $\mathrm{E}(f, f) = 0$ where f is the *minimal presentation* of V. It is unknown whether the converse is true.

The rest of this thesis is organized as follows. In Section 2.1 we discuss quiver Grassmannians, introduced by Schofield in [17]. For our later purpose, we only need Corollary II.6 though. In Section 2.2, we collect some interesting results concerning the projective presentations. The Chapter III and IV is the main bulk of this thesis. In Section 3.1 we introduce the space E and prove our first main result Theorem III.8, which is an analogue of Schofield's result on general subrepresentations. In the next section we prove our second main result Theorem III.12, which is an analogue of Kac's canonical decomposition. In Section 4.1, we show how to complete a rigid presentation and prove another main result Theorem IV.4 as an analogue of the classical tilting theory. Later we study the different complements to an almost maximal rigid presentation. In Section 4.2, we introduce the simplicial complex $\mathcal{S}(A)$ and geometrically realize it on a sphere. In the last chapter, we discuss several applications and examples, from representations of path algebras to path algebras with relations. In particular, in Section 5.3 we briefly mention an application to quivers with potentials.

CHAPTER II

Quiver Grassmannians and Projective Presentations

2.1 Quiver Grassmannians

For dimension vectors α, β, we define $\mathrm{Gr}\binom{\alpha+\beta}{\alpha} := \prod_{v \in Q_0} \mathrm{Gr}\binom{(\alpha+\beta)(v)}{\alpha(v)}$, where $\mathrm{Gr}\binom{(\alpha+\beta)(v)}{\alpha(v)}$ is the usual Grassmannian variety of $\alpha(v)$-dimensional subspaces of $k^{(\alpha+\beta)(v)}$.

Let $A = kQ/I$ be a finite dimensional algebra and R be the maximal semisimple subalgebra spanned by all $e_v \in kQ$. For any $\mathrm{GL}_\alpha \times \mathrm{GL}_\beta$ (resp. $\mathrm{GL}_{\alpha+\beta}$)-stable subvariety Y (resp. Z) of $\mathrm{Rep}_\alpha(A) \times \mathrm{Rep}_\beta(A)$ (resp. $\mathrm{Rep}_{\alpha+\beta}(A)$), we define

$$Z(Y) = \{(M, L) \in Z \times \mathrm{Gr}\binom{\alpha+\beta}{\alpha} \mid L \text{ is a subrepresentation of } M \text{ with } (L, M/L) \in Y\}.$$

A point $x = (M, L) \in Z(Y)$ corresponds to an exact sequence:

$$0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0, \tag{2.1}$$

where ι and π can differ by an automorphism of L and N. Apply the bifunctor

$\mathrm{Hom}_A(-,-)$ to (2.1) and itself, then we obtain the following double complex:

$$
\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & \mathrm{Ext}^1_A(N,N) & \xrightarrow{\pi_N^e} & \mathrm{Ext}^1_A(M,N) & \xrightarrow{\iota_N^e} & \mathrm{Ext}^1_A(L,N) & \to & \mathrm{Ext}^2_A(N,N) & \to \cdots \\
 & \uparrow & & \uparrow{\scriptstyle \pi_e^M} & & \uparrow{\scriptstyle \pi_e^L} & & \uparrow & \\
\cdots \to & \mathrm{Ext}^1_A(N,M) & \xrightarrow{\pi_M^e} & \mathrm{Ext}^1_A(M,M) & \xrightarrow{\iota_M^e} & \mathrm{Ext}^1_A(L,M) & \to & \mathrm{Ext}^2_A(N,M) & \to \cdots \\
 & \uparrow{\scriptstyle \iota_e^N} & & \uparrow{\scriptstyle \iota_e^M} & & \uparrow{\scriptstyle \iota_e^L} & & \uparrow & \\
\cdots \to & \mathrm{Ext}^1_A(N,L) & \xrightarrow{\pi_L^e} & \mathrm{Ext}^1_A(M,L) & \to & \mathrm{Ext}^1_A(L,L) & \to & \mathrm{Ext}^2_A(N,L) & \to \cdots \\
 & \uparrow & & \uparrow{\scriptstyle \partial^M} & & \uparrow{\scriptstyle \partial^L} & & \uparrow & \\
0 \to & \mathrm{Hom}_A(N,N) & \to & \mathrm{Hom}_A(M,N) & \xrightarrow{\iota_N^h} & \mathrm{Hom}_A(L,N) & \xrightarrow{\partial_N} & \mathrm{Ext}^1_A(N,N) & \to \cdots \\
 & \uparrow & & \uparrow{\scriptstyle \pi_h^M} & & \uparrow{\scriptstyle \pi_h^L} & & \uparrow & \\
0 \to & \mathrm{Hom}_A(N,M) & \to & \mathrm{Hom}_A(M,M) & \xrightarrow{\iota_M^h} & \mathrm{Hom}_A(L,M) & \xrightarrow{\partial_M} & \mathrm{Ext}^1_A(N,M) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \iota_e^N} & \\
0 \to & \mathrm{Hom}_A(N,L) & \to & \mathrm{Hom}_A(M,L) & \to & \mathrm{Hom}_A(L,L) & \to & \mathrm{Ext}^1_A(N,L) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 & & \vdots &
\end{array}
$$

To begin with, we need a quick review on derivations. Recall that if B is an A-A bimodule, an R-derivation $d : A \to B$ is a linear map such that $d(aa') = ad(a') + d(a)a'$ for all $a, a' \in A$ and $d(R) = 0$. A derivation d is called *inner* if there is some $b \in B$ such that $d(a) = ab - ba$ for all $a \in A$. We denote by $\mathrm{Der}_R(A, B)$ the space of all R-derivations $A \to B$ and $\mathrm{Der}^0_R(A, B)$ the subspace of inner R-derivations. For any (left) A-modules L and N, $\mathrm{Hom}(N, L)$ has a natural A-A bimodule structure. If $d \in \mathrm{Der}_R(A, \mathrm{Hom}(N, L))$, then let $a' = e_{ta}$, we get $d(a) = d(ae_{ta}) = d(a)e_{ta}$ and similarly $d(a) = d(e_{ha}a) = e_{ha}d(a)$. So each $d(a)$ can be identified with an element in $\mathrm{Hom}(N(ta), L(ha))$ and if d is inner, then $d(a) = F(ha)N(a) - L(a)F(ta)$ for some

$F \in \mathrm{Hom}_R(N, L)$. The following fact is proved in [15, Section 3.7] for the absolute case, i.e., over k. We can slightly modify that proof to obtain

Lemma II.1. *The tangent space* $\mathrm{T}_M \mathrm{Rep}_\alpha(A)$ *of the scheme* $\mathrm{Rep}_\alpha(A)$ *at* M *is isomorphic to* $\mathrm{Der}_R(A, \mathrm{End}(M))$, *and the tangent space* $\mathrm{T}_M(\boldsymbol{O}_M)$ *of the orbit* $\boldsymbol{O}_M = \mathrm{GL}_\alpha \cdot M$ *is isomorphic to* $\mathrm{Der}_R^0(A, \mathrm{End}(M))$. *Moreover,*

$$\mathrm{Ext}^1_A(N, L) = \mathrm{Der}_R(A, \mathrm{Hom}(N, L)) / \mathrm{Der}^0_R(A, \mathrm{Hom}(N, L)).$$

Proof. Recall that the scheme $\mathrm{Rep}_\alpha(A)$ represents the functor:

$$B \mapsto \mathrm{Hom}_{R-alg}(A, \mathrm{End}_B(B \otimes V)),$$

where B is a finitely generated commutative k-algebra and V is an α-dimensional R-module. So the tangent space $\mathrm{T}_M \mathrm{Rep}_\alpha(A)$ is equal to

$$\{D : A \to \mathrm{End}(V) \mid M + D\epsilon \in \mathrm{Hom}_{R-alg}(A, \mathrm{End}_{k[\epsilon]}(k[\epsilon] \otimes V))\},$$

where $k[\epsilon]$ is the algebra of dual numbers. This requires for any $a, a' \in Q_1$,

$$\begin{aligned} M(aa') + D(aa')\epsilon &= (M(a) + D(a)\epsilon)(M(a') + D(a')\epsilon) \\ &= M(a)M(a') + (M(a)D(a') + D(a)M(a'))\epsilon. \end{aligned}$$

This is equivalent to $D(aa') = M(a)D(a') + D(a)M(a')$, so D is an R-derivation from A to $\mathrm{End}(M)$. Hence $\mathrm{T}_M \mathrm{Rep}_\alpha(A) = \mathrm{Der}_R(A, \mathrm{End}(M))$.

Consider the differential $d\mu_e : \mathfrak{gl}_\alpha \twoheadrightarrow \mathrm{T}_M(\boldsymbol{O}_M)$ of the orbit map $\mathrm{GL}_\alpha \xrightarrow{\mu} \boldsymbol{O}_M$. For any $G \in \mathfrak{gl}_\alpha = \mathrm{End}_R(M)$,

$$(\mathrm{Id} + G\epsilon)|_{ta} M(a) (\mathrm{Id} - G\epsilon)|_{ha} = M(a) + (G(ha)M(a) - M(a)G(ta))\epsilon.$$

So $d\mu_e(G)(a) = G(ha)M(a) - M(a)G(ta)$. Hence $\mathrm{T}_M(\boldsymbol{O}_M) = \mathrm{Der}^0_R(A, \mathrm{End}(M))$.

For the last statement, recall that an element $\xi \in \operatorname{Ext}^1_A(N, L)$ corresponds to an exact sequence up to an automorphism of M:

$$\begin{array}{ccccccccc} 0 & \to & L & \to & M & \to & N & \to & 0 \\ & & \downarrow\mathrm{Id} & & \downarrow\pi & & \downarrow\mathrm{Id} & & \\ 0 & \to & L & \to & M & \to & N & \to & 0 \end{array}$$

The action of $a \in Q_1$ on M can be represented by the block-form $M(a) = \left(\begin{smallmatrix} L(a) & \mu(a) \\ 0 & N(a) \end{smallmatrix}\right)$, where $\mu(a) \in \operatorname{Hom}(N(ta), L(ha))$. The condition $M(aa') = M(a)M(a')$ for $a, a' \in Q_1$ implies $\mu(aa') = L(a)\mu(a') + \mu(a)N(a')$. So $\mu \in \operatorname{Der}_R(A, \operatorname{Hom}(N, L))$. An automorphism ϕ of M can be represented by the block-form $\left(\begin{smallmatrix} \mathrm{Id}_{L(v)} & F(v) \\ 0 & \mathrm{Id}_{N(v)} \end{smallmatrix}\right)_{v \in Q_0}$, then μ and μ' are equivalent if and only if

$$\begin{pmatrix} \mathrm{Id}_{L(ha)} & F(ha) \\ 0 & \mathrm{Id}_{N(ha)} \end{pmatrix} \begin{pmatrix} L(a) & \mu(a) \\ 0 & N(a) \end{pmatrix} = \begin{pmatrix} L(a) & \mu(a) \\ 0 & N(a) \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{L(ta)} & F(ta) \\ 0 & \mathrm{Id}_{N(ta)} \end{pmatrix}$$

for all $a \in Q_1$, or equivalently $\mu'(a) - \mu(a) = F(ha)N(a) - L(a)F(ta)$. So $\mu' - \mu \in \operatorname{Der}^0_R(A, \operatorname{Hom}(N, L))$. Therefore,

$$\operatorname{Ext}^1_A(N, L) = \operatorname{Der}_R(A, \operatorname{Hom}(N, L)) / \operatorname{Der}^0_R(A, \operatorname{Hom}(N, L)).$$

$\square$

So for $y = (L, N) \in Y \subseteq \operatorname{Rep}_\alpha(A) \times \operatorname{Rep}_\beta(A)$, the normal space $\mathrm{N}_y\, Y$ of $\mathcal{O}_{(L,N)} = \mathrm{GL}_\alpha \cdot L \times \mathrm{GL}_\beta \cdot N$ in Y can be identified with a subspace of $\operatorname{Ext}^1_A(L, L) \times \operatorname{Ext}^1_A(N, N)$.

We will use matrix notation for maps from and/or to direct sums. For example, if $f : A \to B$, $g : A \to C$, $h : D \to B$ then $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ denotes a map $A \to B \oplus C$ and $(f\ h)$ represents a map $A \oplus D \to B$.

Lemma II.2. *Let $p : Z(Y) \to Z$ be the first factor projection, then there is an exact sequence:*

$$0 \to \left(\begin{smallmatrix} \partial^L \\ -\partial_N \end{smallmatrix}\right)^{-1} \mathrm{N}_y\, Y \to \mathrm{T}_x\, Z(Y) \xrightarrow{dp} \mathrm{T}_M\, Z \to \frac{\mathrm{N}_M\, Z}{\left(\begin{smallmatrix} \iota^e_M \\ \pi^M_e \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} \iota^L_e & 0 \\ 0 & \pi^e_N \end{smallmatrix}\right) \mathrm{N}_y\, Y} \to 0.$$

Proof. Choose a basis of L and N such that $M(a)$ has the matrix form $\left(\begin{smallmatrix} L(a) & \tilde{\xi}(a) \\ 0 & N(a) \end{smallmatrix}\right)$ for each $a \in Q_1$, then this determines an element $\xi \in \operatorname{Ext}^1_A(N, L)$, which is represented by $\tilde{\xi} \in \operatorname{Der}_R(A, \operatorname{Hom}(N, L))$. Let

$$(\mu, \varphi) \in \mathrm{T}_x\, Z \times \operatorname{Gr}\tbinom{\alpha+\beta}{\alpha} \subseteq \operatorname{Der}_R(A, \operatorname{End}(M)) \times \operatorname{Hom}_R(L, N),$$

and $k[\epsilon]$ be the algebra of dual numbers. Then $(\mu, \varphi) \in \mathrm{T}_x\, Z(Y)$ if and only if there is some R-linear lifting $\tilde{\varphi} = \left(\begin{smallmatrix} \varphi_1 & \varphi_3 \\ \varphi & \varphi_2 \end{smallmatrix}\right) \in \operatorname{End}_R(M)$ of φ such that

$$\begin{aligned} &\left(\begin{smallmatrix} L(a)+\epsilon\lambda(a) & \tilde{\xi}(a)+\epsilon\xi'(a) \\ 0 & N(a)+\epsilon\nu(a) \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} \mathrm{Id}+\epsilon\varphi_1(ha) & \epsilon\varphi_3(ha) \\ \epsilon\varphi(ha) & \mathrm{Id}+\epsilon\varphi_2(ha) \end{smallmatrix}\right)^{-1} [\left(\begin{smallmatrix} L(a) & \tilde{\xi}(a) \\ 0 & N(a) \end{smallmatrix}\right) + \epsilon\mu(a)] \left(\begin{smallmatrix} \mathrm{Id}+\epsilon\varphi_1(ta) & \epsilon\varphi_3(ta) \\ \epsilon\varphi(ta) & \mathrm{Id}+\epsilon\varphi_2(ta) \end{smallmatrix}\right), \end{aligned} \tag{2.2}$$

where $\{\lambda(a), \nu(a)\}_{a\in Q_1} \in \mathrm{T}_y\, Y$ but no restriction on ξ'.

To compute the kernel of dp, we set $\mu(a) = 0$, then (2.2) is equal to

$$\left(\begin{smallmatrix} L(a)+\epsilon(L(a)\varphi_1(ta)-\varphi_1(ha)L(a)+\tilde{\xi}(a)\varphi(ta)) & \tilde{\xi}(a)+\epsilon(L(a)\varphi_3(ta)-\varphi_3(ha)N(a)-\varphi_1(ha)\tilde{\xi}(a)+\tilde{\xi}(a)\varphi_2(ta)) \\ 0 & N(a)+\epsilon(N(a)\varphi_2(ta)-\varphi_2(ha)N(a)-\varphi(ha)\tilde{\xi}(a)) \end{smallmatrix}\right).$$

So $\varphi \in \operatorname{Ker} dp \iff \begin{cases} \lambda(a) = L(a)\varphi_1(ta) - \varphi_1(ha)L(a) + \tilde{\xi}(a)\varphi(ta), \\ \nu(a) = N(a)\varphi_2(ta) - \varphi_2(ha)N(a) - \varphi(ha)\tilde{\xi}(a). \end{cases}$

As in [17, Lemma 3.2], one can easily verify that φ actually lies in $\operatorname{Hom}_A(L, N)$ so $\partial^L(\varphi)$ is represented by $\{\tilde{\xi}(a)\varphi(ta)\}_{a\in Q_1} \in \operatorname{Der}_R(A, \operatorname{End}(L))$ and $\partial_N(\varphi)$ by $\{\varphi(ha)\tilde{\xi}(a)\}_{a\in Q_1} \in \operatorname{Der}_R(A, \operatorname{End}(N))$. Notice that $L(a)\varphi_1(ta) - \varphi_1(ha)L(a)$ and $N(a)\varphi_2(ta) - \varphi_2(ha)N(a)$ are inner R-derivations, then the above equation simply means that $\binom{\partial^L}{-\partial_N}\varphi \in \mathrm{N}_y\, Y$. Hence, $\operatorname{Ker} dp \cong \binom{\partial^L}{-\partial_N}^{-1} \mathrm{N}_y\, Y$.

To compute the image of dp, we get from (2.2) that $\mu(a)$ is equal to

$$\left(\begin{smallmatrix} \lambda(a)-\tilde{\xi}(a)\varphi(ta)+\varphi_1(ha)L(a)-L(a)\varphi_1(ta) & \xi'(a)-L(a)\varphi_3(ta)+\varphi_3(ha)N(a)+\varphi_1(ha)\tilde{\xi}(a)-\tilde{\xi}(a)\varphi_2(ta) \\ \varphi(ha)L(a)-N(a)\varphi(ta) & \nu(a)+\varphi(ha)\tilde{\xi}(a)+\varphi_2(ha)N(a)-N(a)\varphi_2(ta) \end{smallmatrix}\right).$$

So the image of $\mu(a)$ under the natural action of ι (resp. π) is given by the first column (resp. second row) of the above matrix, and up to an inner derivation this corresponds to the image of $\lambda(a) - \tilde{\xi}(a)\varphi(ta)$ (resp. $\nu(a) + \varphi(ha)\tilde{\xi}(a)$) under the natural action of ι (resp. π). Note that if λ, μ, ν are inner, so are their images under the natural actions. Finally a simple diagram chasing can show that we get the desired cokernel.

$\square$

Let $\xi_x \in \operatorname{Ext}^1_A(N, L)$ represents the exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$. Note that there is a natural $\operatorname{Aut}_A(L) \times \operatorname{Aut}_A(N)$-action on $\operatorname{Ext}^1_A(N, L)$ so $x = (M, L') \in Z(Y)$ determines a unique $\operatorname{Aut}_A(L) \times \operatorname{Aut}_A(N)$-orbit $\mathcal{O}_{\xi_x}$ in $\operatorname{Ext}^1_A(N, L)$. For any $\operatorname{Aut}_A(L) \times \operatorname{Aut}_A(N)$-stable subvariety Ξ of $\operatorname{Ext}^1_A(N, L)$, we define

$$Z(\Xi) = \{x' = (M, L') \in Z(\mathcal{O}_{(L,N)}) \mid \xi_{x'} \in \Xi\}.$$

We will identify the tangent space $\mathrm{T}_{\xi_x} \Xi$ with a subspace in $\operatorname{Ext}^1_A(N, L)$.

Lemma II.3. *There is an exact sequence*

$$0 \to \begin{pmatrix} \pi^L_h & \iota^h_N \end{pmatrix} \begin{pmatrix} -\partial_M & 0 \\ 0 & \partial^M \end{pmatrix}^{-1} \begin{pmatrix} \iota^N_e \\ \pi^e_L \end{pmatrix} \mathrm{T}_{\xi_x} \Xi \to \mathrm{T}_x Z(\Xi) \xrightarrow{dp} \mathrm{T}_M Z \to \frac{\operatorname{Ext}^1_A(M, M)}{\iota^M_e \pi^e_L \mathrm{T}_{\xi_x} \Xi} \to 0.$$

Proof. We use the same setting as in the previous lemma but set $\lambda(a), \nu(a)$ to 0 and require $\{\xi'(a)\}_{a \in Q_1}$ represents an element in $\mathrm{T}_{\xi_x} \Xi$. Then we get

(2.3)

$$\varphi \in \operatorname{Ker} dp \iff \begin{cases} 0 = L(a)\varphi_1(ta) - \varphi_1(ha)L(a) + \tilde{\xi}(a)\varphi(ta), \\ 0 = N(a)\varphi_2(ta) - \varphi_2(ha)N(a) - \varphi(ha)\tilde{\xi}(a), \\ \xi'(a) = L(a)\varphi_3(ta) - \varphi_3(ha)N(a) - \varphi_1(ha)\tilde{\xi}(a) + \tilde{\xi}(a)\varphi_2(ta). \end{cases}$$

Now let $\phi_1 = (\varphi_1, \varphi)^T \in \operatorname{Hom}_R(L, M)$ and $\phi_2 = (\varphi, \varphi_2) \in \operatorname{Hom}_R(M, N)$, then by the first equation of (2.3)

$$M(a)\phi_1(ta) - \phi_1(ha)L(a) = \begin{pmatrix} L(a) & \tilde{\xi}(a) \\ 0 & N(a) \end{pmatrix} \begin{pmatrix} \varphi_1(ta) \\ \varphi(ta) \end{pmatrix} - \begin{pmatrix} \varphi_1(ha) \\ \varphi(ha) \end{pmatrix} L(a) = 0.$$

So ϕ_1 actually lies in $\operatorname{Hom}_A(L,M)$ and similarly $\phi_2 \in \operatorname{Hom}_A(M,N)$. Note that $L(a)\varphi_3(ta) - \varphi_3(ha)N(a)$ is inner. Chase the diagram a bit and we find that the last equation of (2.3) can be interpret as $\left(\begin{smallmatrix} -\partial_M & 0 \\ 0 & \partial^M \end{smallmatrix}\right) \binom{\phi_1}{\phi_2} \in \binom{\iota_e^N}{\pi_L^e} \operatorname{T}_{\xi_x} \Xi$, which implies that

$$\operatorname{Ker} dp \cong \left(\begin{smallmatrix} \pi_h^L & \iota_N^h \end{smallmatrix}\right) \left(\begin{smallmatrix} -\partial_M & 0 \\ 0 & \partial^M \end{smallmatrix}\right)^{-1} \binom{\iota_e^N}{\pi_L^e} \operatorname{T}_{\xi_x} \Xi.$$

The computation of the cokernel of dp is similar to the one in Lemma II.2 but easier.

□

Definition II.4. We define the *quiver Grassmannian*

$$\operatorname{Gr}_\alpha(M) = \{L' \in \operatorname{Gr}\tbinom{\alpha+\beta}{\alpha} \mid L' \text{ is a subrepresentation of } M\}.$$

For fixed $L \in \operatorname{Rep}_\alpha(A), N \in \operatorname{Rep}_\beta(A)$, and $\xi \in \operatorname{Ext}_A^1(N,L)$, we define its subvariety

$$\operatorname{Gr}_L(M) = \{L' \in \operatorname{Gr}_\alpha(M) \mid L' \cong L\}, \operatorname{Gr}^N(M) = \{L' \in \operatorname{Gr}_\alpha(M) \mid M/L' \cong N\},$$

$$\operatorname{Gr}_L^N(M) = \operatorname{Gr}_L(M) \cap \operatorname{Gr}^N(M), \text{ and } \operatorname{Gr}(\xi, M) = \{L' \in \operatorname{Gr}_L^N(M) \mid (M,L') \in Z(\mathcal{O}_\xi)\}.$$

Corollary II.5.

(i) $\operatorname{T}_x \operatorname{Gr}_\alpha(M) \cong \operatorname{Hom}_A(L,N)$.

(ii) $\operatorname{T}_x \operatorname{Gr}_L(M) \cong \operatorname{Im}(\pi_h^L)$.

(ii*) $\operatorname{T}_x \operatorname{Gr}^N(M) \cong \operatorname{Im}(\iota_N^h)$.

(iii) $\operatorname{T}_x \operatorname{Gr}_L^N(M) = \operatorname{T}_x \operatorname{Gr}_L(M) \cap \operatorname{T}_x \operatorname{Gr}^N(M)$.

(iv) $\operatorname{T}_x \operatorname{Gr}(\xi, M) \cong \operatorname{Im}(\pi_h^L \iota_M^h) = \operatorname{Im}(\iota_N^h\ \pi_h^M)$.

Proof. We will only prove (ii) and the rest is similar. Define

$$Z_L^\beta = \{(M,L') \in \operatorname{Rep}_{\alpha+\beta}(A) \times \operatorname{Gr}\tbinom{\alpha+\beta}{\alpha} \mid L' \subseteq M, \text{ and } L' \cong L\},$$

then $\mathrm{Gr}_L(M) = p^{-1}(M)$, where $p : Z_L^\beta \to \mathrm{Rep}_{\alpha+\beta}(A)$ is the projection. The variety Y in Lemma II.2 is $\mathrm{GL}_\alpha \cdot L \times \mathrm{Rep}_\beta(A)$, so $\mathrm{N}_y Y = \mathrm{Ext}^1_A(L, L)$. Hence, $\binom{\partial^L}{-\partial_N}^{-1} \mathrm{N}_y Y = \partial_L^{-1} \mathrm{Ext}^1_A(L, L) = \mathrm{Im}(\pi_h^L)$. □

There is an open dense subset $\mathrm{Hom}_A(L, M)_\gamma$ of $\mathrm{Hom}_A(L, M)$, in which the morphisms have the *general rank* γ, so the image of the canonical map $\mathrm{Hom}_A(L, M)_\gamma \to \mathrm{Gr}_\gamma(M)$ is irreducible. In particular, $\mathrm{Gr}_L(M)$ is irreducible, and dually, so is $\mathrm{Gr}^N(M)$.

Corollary II.6. *Let L be a projective representation of A, then $\mathrm{Gr}_L(M)$ is smooth and irreducible of dimension $\dim \mathrm{Hom}_A(L, M/L)$. If M is also projective, then $\mathrm{Gr}_L^N(M)$ is a smooth irreducible subvariety of $\mathrm{Gr}_L(M)$ with codimension equal to $\dim \mathrm{Ext}^1_A(N, N)$. Moreover $\mathrm{Gr}_L^N(M) = \mathrm{Gr}(\xi, M)$ for some ξ.*

Proof. The first statement is clear from Corollary II.5(ii). The codimension and smoothness of $\mathrm{Gr}_L^N(M)$ follows from Corollary II.5(iii) and the exact sequence:

$$\mathrm{Hom}_A(M, N) \xrightarrow{\iota_N^h} \mathrm{Hom}_A(L, N) \to \mathrm{Ext}^1_A(N, N) \to \mathrm{Ext}^1_A(M, N) = 0.$$

$\mathrm{Gr}_L^N(M)$ is open in the irreducible variety $\mathrm{Gr}^N(M)$ by Corollary II.5(ii*) and hence irreducible. Finally any $M' \in \mathrm{Gr}_L^N(M)$ determines a unique $\mathrm{Aut}_A(L) \times \mathrm{Aut}_A(N)$-orbit ξ in $\mathrm{Ext}^1_A(N, L)$. By Corollary II.5(iv) $\mathrm{Gr}(\xi, M)$ is open in $\mathrm{Gr}_L^N(M)$, but $\mathrm{Gr}_L^N(M)$ is irreducible, so any two such open sets must meet. Hence $\mathrm{Gr}_L^N(M) = \mathrm{Gr}(\xi, M)$ for some ξ. □

2.2 Projective Presentations

Any projective presentation $f \in \mathrm{Hom}_A(P_1, P_0)$ induces an exact sequence

$$0 \to K \xrightarrow{\iota} P_1 \xrightarrow{f} P_0 \xrightarrow{\pi} C \to 0. \tag{2.4}$$

In this sitation, we say that C is presented by f. Apply the bifunctor $\mathrm{Hom}_A(-,-)$ to (2.4) and itself, then we obtain the following double complex:

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(C,C) \hookrightarrow & \mathrm{Hom}_A(P_0,C) \xrightarrow{f^C} & \mathrm{Hom}_A(P_1,C) \xrightarrow{\iota^C} & \mathrm{Hom}_A(K,C) \\
\uparrow & \uparrow & \uparrow_{\pi_*} & \uparrow \\
\mathrm{Hom}_A(C,P_0) \hookrightarrow & \mathrm{Hom}_A(P_0,P_0) \xrightarrow{f^*} & \mathrm{Hom}_A(P_1,P_0) \rightarrow & \mathrm{Hom}_A(K,P_0) \\
\uparrow & \uparrow & \uparrow_{f_*} & \uparrow \\
\mathrm{Hom}_A(C,P_1) \hookrightarrow & \mathrm{Hom}_A(P_0,P_1) \rightarrow & \mathrm{Hom}_A(P_1,P_1) \rightarrow & \mathrm{Hom}_A(K,P_1)
\end{array}$$

Let $\mathrm{Hom}_A(P_0,P_1) \xrightarrow{d_0} \mathrm{Hom}_A(P_1,P_1) \oplus \mathrm{Hom}_A(P_0,P_0) \xrightarrow{d_1} \mathrm{Hom}_A(P_1,P_0)$ be the induced complex. If we identify the Lie algebra of $\mathrm{Aut}_A(P_1) \times \mathrm{Aut}_A(P_0)$ with $\mathrm{End}_A(P_1) \oplus \mathrm{End}_A(P_0)$, then it is easy to see that $d_1 = (f_*, f^*)$ is the differential of the orbit map at f. So the image of d_1 gives the tangent space of the $\mathrm{Aut}_A(P_1) \times \mathrm{Aut}_A(P_0)$-orbit $\mathcal{O}_f$ of f, and thus the normal space N_f of $\mathcal{O}_f$ in $\mathrm{Hom}_A(P_1,P_0)$ is the quotient space $\mathrm{Hom}_A(P_1,P_0)/\operatorname{Im} d_1$. An easy diagram chasing can show that it can be identified with $\mathrm{Hom}_A(P_1,C)/\operatorname{Im} f^C$ under π_*. In the meanwhile, the normal space N_C of $\mathrm{GL}_\alpha \cdot C$ in $\mathrm{Rep}_\alpha(A)$ is $\mathrm{Ext}^1_A(C,C) \cong \operatorname{Ker} \iota^C/\operatorname{Im} f^C$. We denote by π_f the projection $\operatorname{Ker} \iota^C \to \operatorname{Ker} \iota^C/\operatorname{Im} f^C$.

For any $\mathrm{Aut}_A(P_1) \times \mathrm{Aut}_A(P_0)$-stable subvariety Y of $\mathrm{Hom}_A(P_1,P_0)$ and GL_α-stable subvariety X of $\mathrm{Rep}_\alpha(A)$, we define

$$\begin{aligned}
Z(Y,X) = \{(f,\pi,C) \in Y \times \mathrm{Hom}_R(P_0,k^\alpha) \times X \mid & \pi \in \mathrm{Hom}_A(P_0,C) \\
& \text{and } P_1 \xrightarrow{f} P_0 \xrightarrow{\pi} C \to 0 \text{ is exact } \}.
\end{aligned}$$

Let p (resp. q) be the projection of $Z(Y,X)$ given by $z = (f,\pi,C) \mapsto C$ (resp. $\mapsto f$).

Lemma II.7. *There are two exact sequences*

(i) $$\mathrm{T}_z\, Z(Y,X) \xrightarrow{dp} \mathrm{T}_C\, X \to \frac{\mathrm{N}_C\, X}{\mathrm{N}_f\, Y \cap \mathrm{N}_C\, X} \to 0.$$

$$\text{(ii)} \qquad \mathrm{T}_z\, Z(Y,X) \xrightarrow{dq} \mathrm{T}_f\, Y \to \frac{\pi_*^{-1}(\pi_f^{-1}(\mathrm{N}_C\, X))}{\operatorname{Im} d_1} \to 0.$$

Proof. The proof of (ii) is similar to (i) but easier. We don't need (ii) later so its proof is omitted. Using the algebra of dual numbers $k[\epsilon]$, we compute the tangent space $\mathrm{T}_z\, Z(Y,X)$ as follows. The condition $\pi \in \operatorname{Hom}_A(P_0, C)$ is defined by $C(a)\pi(ta) = \pi(ha)P_0(a)$ for all $a \in Q_1$. Since the rank function is upper-semicontinuous, the exactness of the sequence is equivalent to $\pi f = 0$ plus an open condition. So a triple $(F, G, \gamma) \in \mathrm{T}_f\, Y \times \operatorname{Hom}_R(P_0, C) \times \mathrm{T}_C\, X$ lies in $\mathrm{T}_z\, Z(Y,X)$

$$\iff \begin{cases} (\pi + \epsilon G)(f + \epsilon F) = 0, \\ (C(a) + \epsilon\gamma(a))(\pi(ta) + \epsilon G(ta)) = (\pi(ha) + \epsilon G(ha))P_0(a). \end{cases}$$

$$\text{(2.5)} \qquad \iff \begin{cases} Gf + \pi F = 0, \\ \gamma(a)\pi(ta) = G(ha)P_0(a) - C(a)G(ta). \end{cases}$$

For any $\gamma \in \mathrm{T}_C\, X \subseteq \operatorname{Der}_R(A, \operatorname{End}(C))$, let ξ_γ be the corresponding element in $\operatorname{Ext}_A^1(C, C)$, then ξ_γ has a representative $h \in \operatorname{Ker} \iota_C$. A simple diagram chasing can show that the exactness of (i) at $T_C X$ is equivalent to that $\gamma \in \operatorname{Im} dp$ if and only if h can be choosen in $\pi_*\, \mathrm{T}_f\, Y$.

Suppose that $\gamma \in \operatorname{Im} dp$, then it satisfies (2.5). We have that $Gf = -\pi F \in \pi_*\, \mathrm{T}_f\, Y \subseteq \operatorname{Hom}_A(P_1, C)$ and it lies in $\operatorname{Ker} \iota_C$. We claim that $h = Gf$ represents ξ_γ. Consider the following commutative diagram, where $\left(\begin{smallmatrix} C \\ \uparrow_\gamma \\ C \end{smallmatrix}\right)$ is the representation obtained by extension using γ as in Lemma II.1.

$$\begin{array}{ccccccccc} & & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{\pi} & C & \to & 0 \\ & & \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle \binom{G}{\pi}} & & \| & & \\ 0 & \to & C & \xrightarrow{\binom{0}{\mathrm{Id}}} & \left(\begin{smallmatrix} C \\ \uparrow_\gamma \\ C \end{smallmatrix}\right) & \xrightarrow{(0,\mathrm{Id})} & C & \to & 0 \end{array}$$

Note that the second equation of (2.5) says that the map $\binom{G}{\pi}$ in the middle is a morphism of representations. Now the claim follows easily by apply $\mathrm{Hom}_A(-, C)$ to the upper row.

Conversely, suppose that $h = \pi F$ representing ξ_γ for some $F \in \mathrm{T}_f\, Y$. Since $\mathrm{Ext}^1_A(P_0, C) = 0$, $\{\gamma(a)\pi(ta)\}_{a \in Q_1} \in \mathrm{Der}_R(A, \mathrm{Hom}(P_0, C))$ is inner, that is,

$$\gamma(a)\pi(ta) = G'(ha)P_0(a) - C(a)G'(ta) \text{ for some } G' \in \mathrm{Hom}_R(P_0, C).$$

Then

$$\begin{aligned} G'f(ha)P_1(a) - C(a)G'f(ta) &= G'(ha)P_0(a)f(ta) - C(a)G'(ta)f(ta) \\ &= \gamma(a)\pi(ta)f(ta) = 0, \end{aligned}$$

so $G'f \in \mathrm{Ker}\,\iota_C \subseteq \mathrm{Hom}_A(P_1, C)$. By the same diagram replacing Gf by $G'f$, we see that $G'f$ also represents $\xi_\gamma \in \mathrm{Ext}^1_A(C, C)$. Hence $G'f - h \in \mathrm{Im}\, f^C$. So we can write h as $h = Gf$ with $G' - G \in \mathrm{Hom}_A(P_0, C)$. Finally we get the second equation of (2.5)

$$\gamma(a)\pi(ta) = G'(ha)P_0(a) - C(a)G'(ta) = G(ha)P_0(a) - C(a)G(ta).$$

Therefore, $\gamma \in \mathrm{Im}\, dp$. ∎

Consider

$$\begin{tikzcd} & Z(\mathrm{Hom}_A(P_1, P_0), \mathrm{Rep}_\alpha(A)) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Hom}_A(P_1, P_0) & & \mathrm{Rep}_\alpha(A) \end{tikzcd}$$

Corollary II.8. *The projection p is open.*

Proof. Let $Y = \mathrm{Hom}_A(P_1, P_0)$, then for any GL_α-stable subvariety X of $\mathrm{Rep}_\alpha(A)$, $\mathrm{T}_z\, Z(Y, X) \xrightarrow{dp} \mathrm{T}_C\, X$ is surjective because $\mathrm{N}_f\, Y = \mathrm{Hom}_A(P_1, C)/\,\mathrm{Im}\, f^C \supseteq \mathrm{N}_C\, X$. Hence, the projection $p : Z(Y, X) \to X$ is dominant.

Now let $X = \mathrm{Rep}_\alpha(A)$, then by a theorem of Chevalley $\mathrm{Im}\, p$ is constructible. We need to prove $\mathrm{Im}\, p$ is open in X. This is now equivalent to show that $\mathrm{Im}\, p$ is stable under any generization, namely, if x is a point in the scheme X and $\mathrm{Im}\, p \cap \overline{\{x\}}$ is non-empty, then $x \in \mathrm{Im}\, p$ [10, Exercise III.3.18]. In fact, it is not hard to show that it is sufficient to consider the GL_α-stable points of X. By the above lemma, we know that $\mathrm{Im}\, p \cap \overline{\{x\}}$ is dense in $\overline{\{x\}}$, so $\mathrm{Im}\, p$ must contain the generic point x of $\overline{\{x\}}$. □

Lemma II.9. *The projection q is a principle GL_α-bundle over its image.*

Proof. The same proof as [16, Corollary 3.4] also works here. □

Lemma II.10. *Given any two presentations $f, f' \in \mathrm{Hom}_A(P_1, P_0)$ of C and an automorphism g of C, we can complete the diagram with $g_i \in \mathrm{Aut}_A(P_i), i = 0, 1$.*

$$\begin{tikzcd}
P_1 \arrow[r, "f"] & P_0 \arrow[r, "\pi"] & C \arrow[r] & 0 \\
P_1 \arrow[r, "f'"] \arrow[u, dashed, "g_1"] & P_0 \arrow[r, "\pi'"] \arrow[u, dashed, "g_0"] & C \arrow[r] \arrow[u, "g"] & 0
\end{tikzcd}$$

Proof. $P_0 \xrightarrow{\pi} C \to 0$ can be split into $P_0' \oplus P_0'' \xrightarrow{(\pi',0)} C \to 0$, where P_0' is the projective cover of C. Then the map $g\pi'$ induces a morphism $P_0 \xrightarrow{g_0'} P_0'$, which is surjective because P_0 is the projective cover. The kernel of g_0' must be isomorphic to P_0'', so it induces a projective $g_0'' : P_0 \to P_0''$. We set g_0 to be $\binom{g_0'}{g_0''}$, which is an isomorphism. Apply the same construction to $P_1 \xrightarrow{f} \mathrm{Ker}(\pi)$, then we get an isomorphism g_1. □

Corollary II.11. *The maps pq^{-1} and qp^{-1} give a bijection between $\mathrm{Aut}_A(P_1) \times \mathrm{Aut}_A(P_0)$-stable subvariety of $\mathrm{Im}\, q$ and GL_α-stable subvariety of $\mathrm{Im}\, p$, preserving openness, closure, and irreducibility.*

Remark II.12. One can generailize all above lemmas from projective presentations to projective resolutions, and similar results as in Corollary II.8 and II.11 hold for projective resolutions as well.

We can view any projective presentation $f \in \mathrm{Hom}_A(P_1, P_0)$ as an object in the category $\mathrm{Com}(\mathrm{Rep}(A))$ of bounded complexes of representations of A. We define a partial order $\leq_{deg}$ on $\mathrm{Hom}_A(P_1, P_0)$ by $f \leq_{deg} g$ if there is some presentation h such that $0 \to g \to f \oplus h \to h \to 0$ is exact in $\mathrm{Com}(\mathrm{Rep}(A))$.

As a special case of [12, Theorem 2], we have the following relation between the partial order and the orbit closure.

Proposition II.13. *Suppose that $f, g \in \mathrm{Hom}_A(P_1, P_0)$, then $f \leq_{deg} g$ if and only if $g \in \overline{\mathcal{O}_f}$.*

Among all presentations of $M \in \mathrm{Rep}_\alpha(A)$, the minimal projective presentation $P(\beta_1(M) \to P(\beta_0(M)) \to M \to 0$ may be the most interesting one. Let us give a concrete way to compute the Betti-vectors $\beta_0(M)$ and $\beta_1(M)$ for any $M \in \mathrm{Rep}(A)$. Suppose that $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ is the minimal resolution of M. Apply $\mathrm{Hom}_A(-, S_v)$ to the resolution and we get

$$0 \to \mathrm{Hom}_A(M, S_v) \to \mathrm{Hom}_A(P_0, S_v) \xrightarrow{d_1} \mathrm{Hom}_A(P_1, S_v) \xrightarrow{d_2} \mathrm{Hom}_A(P_2, S_v) \to \cdots .$$

The minimality of the resolution implies that d_i are all zero maps. So $\beta_i(M)_v = \dim \mathrm{Hom}_A(P_i, S_v) = \dim \mathrm{Ext}^i_A(M, S_v)$.

To proceed further, we need the canonical injective resolution of S_v. Suppose that $A = kQ/I$ and Q_2 be a basis of $\frac{I}{JI+IJ}$, then A has the following canonical (or simplicial) A-A-bimodule resolution [3, (1.4)]:

$$\cdots \to \bigoplus_{r \in Q_2} Ae_{hr} \otimes e_{tr}A \xrightarrow{d_1} \bigoplus_{a \in Q_1} Ae_{ha} \otimes e_{ta}A \xrightarrow{d_0} \bigoplus_{v \in Q_0} Ae_v \otimes e_v A \xrightarrow{\mu} A \to 0,$$

where μ is the multiplication of A, d_0 is defined by $d_0(e_{ha} \otimes e_{ta}) = a \otimes e_{ta} - e_{ha} \otimes a$, and d_1 is defined by $d_1(e_{hr} \otimes e_{tr}) = \sum_{a \in Q_1} r_2^a \otimes r_1^a$, where $r = r_2^a a r_1^a$. Tensoring the above resolution with M yields the canonical projective resolution of M:

$$\cdots \to \bigoplus_{r\in Q_2} P_{hr}\otimes M_{tr} \to \bigoplus_{a\in Q_1} P_{ha}\otimes M_{ta} \to \bigoplus_{v\in Q_0} P_v\otimes M_v \to M \to 0. \tag{2.6}$$

Applying the dual construction to $M = S_v$, we obtain the canonical injective resolution of S_v:

$$0 \to S_v \to I_v \xrightarrow{\varphi^*} \bigoplus_{ha=v} I_{ta} \xrightarrow{\psi^*} \bigoplus_{hr=v} I_{tr} \to \cdots .$$

Here φ^* is given by $\varphi^*(p^*) = p^*(a)$ and the restriction of ψ^* on I_{ta} is $\psi^*(p^*) = p^*(a^{-1}r)$, where a^{-1} is the formal inverse of a. Apply $\operatorname{Hom}_A(M,-)$ to this resolution and take the trivial dual, we obtain the complex

$$\cdots \to \bigoplus_{hr=v} M(tr) \xrightarrow{\psi} \bigoplus_{ha=v} M(ta) \xrightarrow{\varphi} M(v). \tag{2.7}$$

We conclude that $\beta_0(M)_v = \dim\operatorname{Coker}\varphi$ and $\beta_1(M)_v = \dim\operatorname{Ker}\varphi/\operatorname{Im}\psi$.

Let $K^2(\text{proj-}A)$ be the set of all presentations in $K^b(\text{proj-}A)$, then all the indecomposable objects in $K^2(\text{proj-}A)$ are in one-to-one correspondence with $P_v[1]$ and minimal projective presentations of $M \in \operatorname{Rep}(A)$. We define the *AR-transformation* τ on $K^2(\text{proj-}A)$ as follows. For any indecomposable $f \neq P_v, P_v[1]$, we define $\tau(f)$ to be the minimal projective presentation of $\tau(\operatorname{Coker}(f))$, the classical Auslander-Reiten transformation [1, Definition IV.2.3] of $\operatorname{Coker}(f)$; define $\tau(P_v) = P_v[1]$ and $\tau(P_v[1])$ to be the minimal projective presentation of I_v, the indecomposable injective representation corresponding to the vertex v. Then we can extend the definition additively to the whole $K^2(\text{proj-}A)$. It follows immediately from the classical theory [1, Proposition IV.2.10] that τ is bijective on $K^2(\text{proj-}A)$ and let τ^{-1} be its inverse. In this way, the preprojective and preinjective components of this algebra are connected pictorially by

$$\cdots \to P_v \xrightarrow{\tau} P_v[1] \xrightarrow{\tau} (\text{projective presentation of } I_v) \to \cdots .$$

CHAPTER III

General Presentations of Algebras

3.1 Subpresentations of a General Presentation

Given any projective presentation $P_1 \xrightarrow{f} P_0$, a presentation $P_1' \xrightarrow{f'} P_0'$ is called *subpresentation* of f if it is a subcomplex of f:

$$\begin{tikzcd} P_1 \arrow[r, "f"] & P_0 \\ P_1' \arrow[u, hook] \arrow[r, "f'"'] & P_0' \arrow[u, hook] \end{tikzcd}$$

We call f' *genuine* if $P_1'' = P_1/P_1'$ and $P_0'' = P_0/P_0'$ are both projectives.

Given any two projective presentations $f' \in \mathrm{Hom}_A(P_1', P_0')$ and $f'' \in \mathrm{Hom}_A(P_1'', P_0'')$ with cokernel L and N respectively, there is an induced double complex with exact rows and columns:

$$\begin{array}{ccccc} \mathrm{Hom}_A(L, N) & \hookrightarrow & \mathrm{Hom}_A(P_0', N) & \xrightarrow{f_N'} & \mathrm{Hom}_A(P_1', N) \\ \uparrow & & \twoheaduparrow & & \twoheaduparrow \\ \mathrm{Hom}_A(L, P_0'') & \hookrightarrow & \mathrm{Hom}_A(P_0', P_0'') & \xrightarrow{f_0'} & \mathrm{Hom}_A(P_1', P_0'') \\ \uparrow & & \uparrow{\scriptstyle f_0''} & & \uparrow{\scriptstyle -f_1''} \\ \mathrm{Hom}_A(L, P_1'') & \hookrightarrow & \mathrm{Hom}_A(P_0', P_1'') & \xrightarrow{f_1'} & \mathrm{Hom}_A(P_1', P_1'') \end{array} \tag{3.1}$$

Definition III.1. We define $\mathrm{E}(f', f'') := \mathrm{Hom}_{K^b(\text{proj-}A)}(f'[-1], f'') = \mathrm{Coker}(f_0', -f_1'')$. In addition, we denote $\mathrm{Ker}(f_0', -f_1'')$ by $\mathrm{H}(f', f'')$.

An easy diagram chasing can show:

Lemma III.2. $\mathrm{E}(f', f'') \cong \operatorname{Coker} f'_N$. *In particular,* $\mathrm{E}(f', f'') \supseteq \operatorname{Ext}^1_A(L, N)$ *and if* f' *is injective, then they are equal.*

So by definition $\mathrm{E}(f', f'')$ is homotopy invariant, and by the above lemma it depends on $N = \operatorname{Coker} f''$ rather than f'' itself.

For projective representations P_1, P_0, P_1^s, P_0^s, we denote $G := \operatorname{Gr}_{P_1^s}(P_1) \times \operatorname{Gr}_{P_0^s}(P_0)$ and $G_\oplus := \operatorname{Gr}_{P_1^s}^{P_1^q}(P_1) \times \operatorname{Gr}_{P_0^s}^{P_0^q}(P_0)$ if both P_1^q and P_0^q are projective. To make our notation more compact, we will always write P_i'' for P_i/P_i' when P_i' varies in $\operatorname{Gr}_{P_i^s}(P_i)$.

Lemma III.3. *There is a vector bundle* $\mathrm{T}^w\, G$ *on* G*, whose fiber over a point* (P_1', P_0') *is* $\operatorname{Hom}_A(P_1', P_0'') \oplus \operatorname{Hom}_A(P_0', P_1'')$*.*

Proof. Both G and $\operatorname{Gr}_{P_1^s \oplus P_0^s}(P_1 \oplus P_0)$ are smooth and irreducible by Corollary II.6. Let $\mathrm{T}_* G$ and $\mathrm{T}_* \operatorname{Gr}_{P_1^s \oplus P_0^s}(P_1 \oplus P_0)$ be their tangent bundles. Consider the embedding $\iota : G \hookrightarrow \operatorname{Gr}_{P_1^s \oplus P_0^s}(P_1 \oplus P_0)$ given by $\iota(P_1', P_0') = \iota_1(P_1') \oplus \iota_0(P_0')$, where ι_i is the canonical embedding $P_i \hookrightarrow P_1 \oplus P_0$. It is not hard to see that the fiber of the pullback bundle $\iota^*(\mathrm{T}_* \operatorname{Gr}_{P_1^s \oplus P_0^s}(P_1 \oplus P_0))$ over a point (P_1', P_0') is

$$\operatorname{Hom}_A(P_1', P_1'') \oplus \operatorname{Hom}_A(P_0', P_0'') \oplus \operatorname{Hom}_A(P_1', P_0'') \oplus \operatorname{Hom}_A(P_0', P_1'').$$

So there is a natural vector bundle morphism $\rho : \iota^*(\mathrm{T}_* \operatorname{Gr}_{P_1^s \oplus P_0^s}(P_1 \oplus P_0)) \to \mathrm{T}_* G$ given by the fiberwise projection from the first two direct summands. Its kernel on each fiber is of the same rank, so the kernel of this bundle morphism is our desired $\mathrm{T}^w\, G$. ∎

We define $Z = \{(f, P_1', P_0') \in \operatorname{Hom}_A(P_1, P_0) \times G_\oplus \mid f(P_1') \subseteq P_0'\}$. Let $p_1 : Z \to$

$\mathrm{Hom}_A(P_1, P_0)$ be the first factor projection and $p_2 : Z \to G_\oplus$ be the second one:

$$
\begin{tikzcd}
 & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Hom}_A(P_1, P_0) & & G_\oplus
\end{tikzcd}
$$

Any point $x = (f, P_1', P_0') \in Z$ determines the following diagram up to automorphisms of P_1', P_0', P_1'', P_0''.

$$
\begin{array}{ccccccccc}
0 & \to & P_0' & \stackrel{\iota_0}{\to} & P_0 & \stackrel{\pi_0}{\to} & P_0'' & \to & 0 \\
 & & \uparrow_{f'} & & \uparrow_{f} & & \uparrow_{f''} & & \\
0 & \to & P_1' & \stackrel{\iota_1}{\to} & P_1 & \stackrel{\pi_1}{\to} & P_1'' & \to & 0
\end{array}
$$

Lemma III.4. *The space Z is a vector bundle over $G_\oplus$ with fiber $p_2^{-1}(P_1', P_0')$ isomorphic to $\mathrm{Hom}_A(P_1', P_0') \oplus \mathrm{Hom}_A(P_1'', P_0)$. Hence Z is smooth and irreducible of dimension*

$$\dim(\mathrm{Hom}_A(P_1^s, P_1^q) \oplus \mathrm{Hom}_A(P_0^s, P_0^q)) + \dim \mathrm{Hom}_A(P_1, P_0) - \dim \mathrm{Hom}_A(P_1^s, P_0^q).$$

Proof. Define the vector bundle morphism $\rho : \mathrm{Hom}_A(P_1, P_0) \times G \to \mathrm{T}^w G$ fiberwise by the map $(\iota_1^* \pi_{0*}, 0) : \mathrm{Hom}_A(P_1, P_0) \to \mathrm{Hom}_A(P_1', P_0'') \oplus \mathrm{Hom}_A(P_0', P_1'')$. The image on the fiber is equal to $\mathrm{Im}\, \iota_1^*$ because $\pi_{0*} : \mathrm{Hom}_A(P_1, P_0) \to \mathrm{Hom}_A(P_1, P_0'')$ is surjective. Now look at the exact sequence:

$$\mathrm{Hom}_A(P_1, P_0'') \xrightarrow{\iota_1^*} \mathrm{Hom}_A(P_1', P_0'') \xrightarrow{\partial} \mathrm{Ext}_A^1(P_1'', P_0'') = 0$$

and we conclude that the bundle morphism has the constant rank $\dim \mathrm{Hom}_A(P_1', P_0'')$ over $G_\oplus$ and it is easy to see that the restriction of its kernel on $G_\oplus$ is exactly Z. Then the fiber $p_2^{-1}(P_1', P_0') = \mathrm{Ker}\, \iota_1^* \pi_{0*}$, which is isomorphic to $\mathrm{Hom}_A(P_1', P_0') \oplus \mathrm{Hom}_A(P_1'', P_0)$ and $\dim Z = \dim G_\oplus + \dim p_2^{-1}(P_1', P_0')$. $\square$

Lemma III.5. *The tangent space $\mathrm{T}_x\, p_1^{-1}(f)$ is isomorphic to $\mathrm{Ker}(f_0', -f_1'')$.*

Proof. For any $(F, \varphi_1, \varphi_0) \in \mathrm{T}_x(\mathrm{Hom}_A(P_1, P_0) \times G_\oplus)$

$$\cong \mathrm{Hom}_A(P_1, P_0) \oplus \mathrm{Hom}_A(P_1', P_1'') \oplus \mathrm{Hom}_A(P_0', P_0''),$$

let $\tilde{\varphi}_i \in \mathrm{Hom}_A(P_i', P_i)$ be some lifting of φ_i, and $\tilde{\varphi}_i$ can be further lifted to some $\bar{\varphi}_i \in \mathrm{End}_k(P_i)$. Using the algebra of dual numbers $k[\epsilon]$, we compute the tangent space $\mathrm{T}_x Z$ as follows. $(F, \varphi_1, \varphi_0) \in \mathrm{T}_x Z$

$$\begin{aligned}
&\iff (f + \epsilon F)(\mathrm{Id} + \epsilon\bar{\varphi}_1)P_1' \subseteq (\mathrm{Id} + \epsilon\bar{\varphi}_0)P_0' \\
&\iff (\mathrm{Id} - \epsilon\bar{\varphi}_0)(f + \epsilon F)(\mathrm{Id} + \epsilon\bar{\varphi}_1)P_1' \subseteq P_0' \\
&\iff (f\bar{\varphi}_1 - \bar{\varphi}_0 f + F)P_1' \subseteq P_0' \\
&\iff (f\tilde{\varphi}_1 - \tilde{\varphi}_0 f' + F)P_1' \subseteq P_0' \\
&\iff \pi_0(f\tilde{\varphi}_1 - \tilde{\varphi}_0 f' + F)P_1' = 0 \\
&\iff (f''\varphi_1 - \varphi_0 f' + \pi_0 F)P_1' = 0 \\
&\iff \pi_0 F \iota_1 = \varphi_0 f' - f''\varphi_1
\end{aligned}$$

Hence $\mathrm{T}_x\, p_1^{-1}(f) \cong \mathrm{Ker}(f_0', -f_1'')$. $\square$

Corollary III.6. *The function $Z \to \mathbb{Z}^+$ given by $(f, P_1', P_0') \mapsto \dim \mathrm{E}(f', f'')$ is upper semi-continuous.*

In the mean while, the function $\dim \mathrm{E}(-,-)$ on $\mathrm{Hom}_A(P_1', P_0') \times \mathrm{Hom}_A(P_1'', P_0'')$ is also upper semi-continuous because it is the corank function of a morphism of vector bundles on $\mathrm{Hom}_A(P_1', P_0') \times \mathrm{Hom}_A(P_1'', P_0'')$. Since for any (f', f'') there is a point $x \in Z$ corresponding to it, the minimal values of these two functions must coincide.

It is proved in [11, Theorem V.2.2] and also follows from Corollary III.10 later that for any $\beta_1, \beta_0 \in \mathbb{N}^{Q_0}$ a general presentation in $\mathrm{PHom}_A(\beta_1, \beta_0)$ is homotopy equivalent to a general presentation in the reduced space $\mathrm{PHom}_A(\beta_0 - \beta_1)$, so the following definition make sense. Let $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'')$ be the minimal value of

$\dim \mathrm{E}(-,-)$ on $\mathrm{PHom}_A(\beta_1', \beta_0') \times \mathrm{PHom}_A(\beta_1'', \beta_0'')$.

Proposition III.7. *For any $\beta_i', \beta_i'' \in \mathbb{N}^{Q_0}$, let $\beta_i = \beta_i' + \beta_i''$. The set*

$$\{f \in \mathrm{PHom}_A(\beta_1, \beta_0) \mid f \textit{ has a genuine subpresentation in } \mathrm{PHom}_A(\beta_1', \beta_0')\}$$

has codimension $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'')$ *in* $\mathrm{PHom}_A(\beta_1, \beta_0)$.

Proof. Since Z is smooth, $\dim Z = \dim \mathrm{Im}\, p_1 + \min_{x \in Z}\{\dim \mathrm{T}_x\, p_1^{-1}(f)\}$ by a theorem on the generic smoothness [10, Corollary III.10.7]. Combining the dimension formulas in Lemma III.5 and Lemma III.4, we get the codimension:

$$\begin{aligned}
&\dim \mathrm{Hom}_A(P_1, P_0) - \dim \mathrm{Im}\, p_1 \\
&= \dim \mathrm{Hom}_A(P_1^s, P_0^q) - \dim(\mathrm{Hom}_A(P_1^s, P_1^q) \oplus \mathrm{Hom}_A(P_0^s, P_0^q)) + \min_{x \in Z}\{\dim \mathrm{Ker}(f_0', -f_1'')\} \\
&= \min_{x \in Z}\{\dim \mathrm{Coker}(f_0', -f_1'')\}.
\end{aligned}$$

This minimal value is exactly $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'')$ by the proceding remarks. $\square$

We have the following easy consequence.

Theorem III.8. *The following statements are equivalent:*

(1) *A general presentation in* $\mathrm{PHom}_A(\beta_1, \beta_0)$ *has a genuine subpresentation in* $\mathrm{PHom}_A(\beta_1', \beta_0')$.

(2) $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'') = 0$.

(3) *There are* $f' \in \mathrm{Hom}_A(P_1', P_0')$, $f'' \in \mathrm{Hom}_A(P_1'', P_0'')$ *with* $N = \mathrm{Coker}\, f''$ *such that* $\mathrm{Hom}_A(P_0', N) \xrightarrow{f_*'} \mathrm{Hom}_A(P_1', N)$ *is surjective.*

3.2 Canonical Decomposition of a General Presentation

Let $\mathrm{Gr}_{\oplus^2}(P_i) = \{(P_i', P_i'') \in \mathrm{Gr}_{P_i^s}^{P_i^q}(P_i) \times \mathrm{Gr}_{P_i^q}^{P_i^s}(P_i) \mid P_i' \cap P_i'' = 0\}$. This is open in $\mathrm{Gr}_{P_i^s}^{P_i^q}(P_i) \times \mathrm{Gr}_{P_i^q}^{P_i^s}(P_i)$ and hence smooth irreducible. Let $G_{\oplus^2} = \mathrm{Gr}_{\oplus^2}(P_1) \times \mathrm{Gr}_{\oplus^2}(P_0)$

and we define

$$Z_{\oplus^2} = \{(f, P_1', P_1'', P_0', P_0'') \in \mathrm{Hom}_A(P_1, P_0) \times G_{\oplus^2} \mid f(P_1') \subseteq P_0', f(P_1'') \subseteq P_0''\}.$$

Let $p_1 : Z_{\oplus^2} \to \mathrm{Hom}_A(P_1, P_0)$ be the first factor projection and $p_2 : Z_{\oplus^2} \to G_{\oplus^2}$ be the second one. Any point $x = (f, P_1', P_1'', P_0', P_0'') \in Z_{\oplus^2}$ determines up to automorphisms of P_1', P_0', P_1'', P_0'', the following diagram with split rows.

$$\begin{array}{ccccccccc} 0 & \to & P_0' & \underset{\pi_0'}{\overset{\iota_0'}{\rightleftarrows}} & P_0 & \underset{\iota_0''}{\overset{\pi_0''}{\rightleftarrows}} & P_0'' & \to & 0 \\ & & \uparrow_{f'} & & \uparrow_{f} & & \uparrow_{f''} & & \\ 0 & \to & P_1' & \underset{\pi_1'}{\overset{\iota_1'}{\rightleftarrows}} & P_1 & \underset{\iota_1''}{\overset{\pi_1''}{\rightleftarrows}} & P_1'' & \to & 0 \end{array}$$

The proofs of the following lemma and corollary are similar to those in the last section, so we left the details for interested readers.

Lemma III.9.

(i) *$Z_{\oplus^2}$ is a vector bundle over $G_{\oplus^2}$ with fiber $p_2^{-1}(P_1', P_1'', P_0', P_0'')$ isomorphic to* $\mathrm{Hom}_A(P_1', P_0') \oplus \mathrm{Hom}_A(P_1'', P_0'')$.

(ii) *The tangent space $\mathrm{T}_x\, p_1^{-1}(f)$ is isomorphic to* $\mathrm{H}(f', f'') \oplus \mathrm{H}(f'', f')$.

Sketch of proof.

(i) A similar construction in Lemma III.3 produces a vector bundle $\mathrm{T}^w\, G_{\oplus^2}$, whose fiber over $(P_1', P_1'', P_0', P_0'')$ is

$$\mathrm{Hom}_A(P_1', P_0'') \oplus \mathrm{Hom}_A(P_0', P_1'') \oplus \mathrm{Hom}_A(P_1'', P_0') \oplus \mathrm{Hom}_A(P_0'', P_1').$$

Then one can realize $Z_{\oplus^2}$ as the kernel of the vector bundle morphism $\mathrm{Hom}_A(P_1, P_0) \times G_{\oplus^2} \to \mathrm{T}^w\, G_{\oplus^2}$ fiberwise defined by the map $(\iota_1'^* \pi_{0*}'', 0, \iota_1''^* \pi_{0*}', 0)$. It is a vector bundle because the morphism has a constant rank.

(ii) For any $(F, \varphi_1', \varphi_1'', \varphi_0', \varphi_0'') \in \mathrm{T}_x(\mathrm{Hom}_A(P_1, P_0) \times G_{\oplus^2})$

$\cong \mathrm{Hom}_A(P_1, P_0) \oplus \mathrm{Hom}_A(P_1', P_1'') \oplus \mathrm{Hom}_A(P_1', P_1'') \oplus \mathrm{Hom}_A(P_0', P_0'') \oplus \mathrm{Hom}_A(P_0'', P_0')$, as in Lemma III.5 one can show that $(F, \varphi_1', \varphi_1'', \varphi_0', \varphi_0'') \in \mathrm{T}_x Z_{\oplus^2}$ if and only if $\pi_0'' F \iota_1' = \varphi_0' f' - f'' \varphi_1'$ and $\pi_0' F \iota_1'' = \varphi_0'' f'' - f' \varphi_1''$. Hence $\mathrm{T}_x p_1^{-1}(f) \cong \mathrm{H}(f', f'') \oplus \mathrm{H}(f'', f')$. $\square$

Corollary III.10. *For any $\beta_i', \beta_i'' \in \mathbb{N}^{Q_0}$, let $\beta_i = \beta_i' + \beta_i''$. The set*

$$\{f \in \mathrm{PHom}_A(\beta_1, \beta_0) \mid f = f' \oplus f'' \text{ with } f' \in \mathrm{PHom}_A(\beta_1', \beta_0'), f'' \in \mathrm{PHom}_A(\beta_1'', \beta_0'')\}$$

has codimension $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'') + \mathrm{e}(\beta_0'' - \beta_1'', \beta_0' - \beta_1')$ *in* $\mathrm{PHom}_A(\beta_1, \beta_0)$.

Proof. $Z_{\oplus^2}$ is smooth and irreducible by the first part of the above lemma. We will perform the dimension counting using the two projections of $Z_{\oplus^2}$. From the projection p_1, we know from the second part of the above lemma and the generic smoothness that

$$\dim Z_{\oplus^2} = \dim \mathrm{Im}\, p_1 + \min_{x \in Z_{\oplus^2}} \{\dim(\mathrm{H}(f', f'') \oplus \mathrm{H}(f'', f'))\}.$$

While the projection p_2 yields that

$$\dim Z_{\oplus^2} = \dim G_{\oplus^2} + \dim(\mathrm{Hom}_A(P_1^s, P_0^s) \oplus \mathrm{Hom}_A(P_1^q, P_0^q)).$$

So the codimension:

$$\begin{aligned}
&\dim \mathrm{Hom}_A(P_1, P_0) - \dim \mathrm{Im}\, p_1 \\
&= \dim \mathrm{Hom}_A(P_1, P_0) - \dim G_{\oplus^2} - \dim(\mathrm{Hom}_A(P_1^s, P_0^s) \oplus \mathrm{Hom}_A(P_1^q, P_0^q)) \\
&\quad + \min_{x \in Z_{\oplus^2}} \{\dim(\mathrm{H}(f', f'') \oplus \mathrm{H}(f'', f'))\} \\
&= \dim \mathrm{Hom}_A(P_1^s, P_0^q) - \dim(\mathrm{Hom}_A(P_1^s, P_1^q) \oplus \mathrm{Hom}_A(P_0^s, P_0^q)) + \min_{x \in Z_{\oplus^2}} \{\dim \mathrm{H}(f', f'')\} \\
&+ \dim \mathrm{Hom}_A(P_1^q, P_0^s) - \dim(\mathrm{Hom}_A(P_1^q, P_1^s) \oplus \mathrm{Hom}_A(P_0^q, P_0^s)) + \min_{x \in Z_{\oplus^2}} \{\dim \mathrm{H}(f'', f')\} \\
&= \min_{x \in Z_{\oplus^2}} \{\dim(\mathrm{E}(f', f'') \oplus \mathrm{E}(f'', f'))\}.
\end{aligned}$$

This minimal value is exactly $\mathrm{e}(\beta_0' - \beta_1', \beta_0'' - \beta_1'') + \mathrm{e}(\beta_0'' - \beta_1'', \beta_0' - \beta_1')$. $\square$

Let l be a fixed integer greater than 1, then the abelian category $\mathrm{Com}^l(\mathrm{Rep}(A))$ of bounded complexes of length l can be viewed as the category $\mathrm{Rep}(A_l)$ for some finite-dimensional algebra A_l, which has Krull-Schmidt property [1, Corollary I.4.8]. So its full subcategory consisting of projective presentations is Krull-Schmidt as well. In particular, every presentation f has a unique decomposition $f = f_1 \oplus f_2 \oplus \cdots \oplus f_s$ with each f_i an indecomposable object in $\mathrm{Com}^l(\mathrm{Rep}(A))$. Note that each f_i is a projective presentation.

Definition III.11. $\delta \in \mathbb{Z}^{Q_0}$ is called indecomposable if a general presentation in $\mathrm{PHom}_A(\delta)$ is indecomposable. We call $\delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s$ the *canonical decomposition* of δ if a general element in $\mathrm{PHom}_A(\delta)$ decompose into (indecomposable) ones in each $\mathrm{PHom}_A(\delta_i)$.

It is easy to see that if a general presentation f decomposes as $f = f_1 \oplus f_2 \oplus \cdots \oplus f_s$, then each summand $f_i \in \mathrm{PHom}_A(\delta_i)$ can be chosen to be general too, so each δ_i in the canonical decomposition is indecomposable. Thus Corollary III.10 can be easily generalized to the following.

Theorem III.12. *$\delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s$ is the canonical decomposition of δ if and only if $\delta_1, \cdots, \delta_s$ are indecomposable, and $\mathrm{e}(\delta_i, \delta_j) = 0$ for $i \neq j$.*

Proof. $\Rightarrow$. We see from the canonical decomposition of δ that a general presentation in $\mathrm{PHom}_A(\delta)$ has a summand in $\mathrm{PHom}_A(\delta_i + \delta_j)$ $(i \neq j)$. But this summand is general in $\mathrm{PHom}_A(\delta_i + \delta_j)$ and it is a direct sum of two presentations in $\mathrm{PHom}_A(\delta_i)$ and $\mathrm{PHom}_A(\delta_j)$. By Corollary III.10, $\mathrm{e}(\delta_i, \delta_j) = 0$.

$\Leftarrow$. We prove by induction on s. $s = 1$ is the trivial case. Now assume this is true for $s - 1$, then $\delta' = \delta_1 \oplus \cdots \oplus \delta_{s-1}$ is the canonical decomposition of δ' and clearly $\mathrm{e}(\delta', \delta_s) = \mathrm{e}(\delta_s, \delta') = 0$. By Corollary III.10, a general presentation $f \in \mathrm{PHom}_A(\delta)$

can be decomposed to $f' \in \mathrm{PHom}_A(\delta')$ and $f'' \in \mathrm{PHom}_A(\delta_s)$, but f' is general in $\mathrm{PHom}_A(\delta')$, so we can finish the proof by the induction hypothesis. $\square$

Recall from Section 2.2 that $\mathrm{E}(f,f)$ can be interpreted as the normal space of f in $\mathrm{Hom}_A(P_1, P_0)$.

Definition III.13. A presentation f is called *rigid* if $\mathrm{E}(f,f) = 0$. An indecomposable $\delta \in \mathbb{Z}^{Q_0}$ is called *real* if there is a rigid $f \in \mathrm{PHom}_A(\delta)$; is called *tame* if it is not real but $\mathrm{e}(\delta,\delta) = 0$; is called *wild* if $\mathrm{e}(\delta,\delta) > 0$.

If δ is real or tame, then by Theorem III.12, $m\delta = \underbrace{\delta \oplus \cdots \oplus \delta}_{m}$ is the canonical decomposition for any $m \in \mathbb{N}$. In particular, δ is indivisible.

Question III.14. *If δ is wild, are all $m\delta$ wild (in particular indecomposable)?*

CHAPTER IV

Rigid Presentations

4.1 Maximal Rigid Presentations

Definition IV.1. A rigid presentation f is called *generating* if $K^b(\text{proj-}A)$ is generated by $\mathrm{Ind}(f)$:={nonisomorphic indecomposable direct summands of f}.

We will first show how to turn a rigid presentation f into a generating presentation. We denote the object complex with A in degree 0 simply by A, then $\mathrm{E}(A, f) = 0$. Let $e = \dim \mathrm{E}(f, A) = \dim \mathrm{Hom}_{K^b(\text{proj-}A)}(f[-1], A)$ and $f^e[-1] \xrightarrow{can} A$ be the canonical map. Using the triangulated structure of $K^b(\text{proj-}A)$, we can complete the above map to a triangle

$$f^+[-1] \to f^e[-1] \xrightarrow{can} A \to f^+. \tag{4.1}$$

It is easy to see that the mapping cone f^+ is a complex concentrated in degree $0, -1$, in other words, it is a presentation. Let us verify that $\mathrm{E}(f, f^+) = 0$. We denote $\mathrm{Hom}_{K^b(\text{proj-}A)}$ simply by Hom. Apply $\mathrm{Hom}(f, -)$ to the triangle (4.1), we get

$$\mathrm{Hom}(f, f^e) \xrightarrow{\partial} \mathrm{Hom}(f, A[1]) \to \mathrm{Hom}(f, f^+[1]) \to \mathrm{Hom}(f, f^e[1]) = 0.$$

By construction 4.1, ∂ is surjective so we have that $\mathrm{E}(f, f^+) = \mathrm{Hom}(f, f^+[1]) = 0.$

Apply $\operatorname{Hom}(-, f)$ and $\operatorname{Hom}(-, f^+)$, we get two exact sequences

$$0 = \operatorname{Hom}(f^e, f[1]) \to \operatorname{Hom}(f^+, f[1]) \to \operatorname{Hom}(A, f[1]) = 0,$$

$$0 = \operatorname{Hom}(f^e, f^+[1]) \to \operatorname{Hom}(f^+, f^+[1]) \to \operatorname{Hom}(A, f^+[1]) = 0.$$

So we can conclude $\mathrm{E}(f \oplus f^+, f \oplus f^+) = 0$, ie., $f \oplus f^+$ is rigid, and in fact generating by the above triangle. We call $\tilde{f}^+ := f \oplus f^+$ the positive completion of f. Similarly, we can construct the negative completion of f by using $A[1]$ instead of A. Namely, let $e^- = \dim \mathrm{E}(A[1], f) = \dim \operatorname{Hom}(A, f)$ and take the triangle

$$f^-[-1] \to A \xrightarrow{can} f^{e^-} \to f^-. \tag{4.2}$$

Then $\tilde{f}^- = f \oplus f^-$ is the negative completion of f.

Next, we claim that $n(f) := |\operatorname{Ind}(f)| \leqslant r(A) := \operatorname{rank}(K_0(\operatorname{Rep}(A)))$ for rigid f.

Definition IV.2. The *universal regularization* of A with respect to a projective presentation $P_1 \xrightarrow{f} P_0$ is an algebra epimorphism $\pi^f : A \twoheadrightarrow A^f$ universal with respect to the property that $A^f \otimes_A P_1 \xrightarrow{A^f \otimes_A f} A^f \otimes_A P_0$ is injective.

Here is a concrete construction of π^f. Suppose that $P_0 = \bigoplus_{i=1}^m P_{u_i}$ and $P_1 = \bigoplus_{j=1}^n P_{v_j}$, where P_{u_i}, P_{v_j} are indecomposable projective. Then P_0, P_1 are (column) vectors with entries in P_{u_i}, P_{v_j}, and f can be represented by an $n \times m$ matrix (a_{ji}) with a_{ji} a linear combination of paths from u_i to v_j. Let I_0 be the two-sided ideal in A generated by the entries of all vectors in $\operatorname{Ker} f$ and $A_1 = A/I_0$. If $A_1 \otimes_A f$ becomes injective, then we take $A^f = A_1$ and π^f to be the canonical projection. Otherwise, we repeat the previous step for the projective presentation in $\operatorname{Rep}(A_1)$: $A_1 \otimes_A P_1 \xrightarrow{A_1 \otimes_A f} A_1 \otimes_A P_0$. This procedure must terminate in finitely many, say s steps. We get a sequence of projections:

$$A_0 = A \to A_1 = A_0/I_0 \to \cdots \to A_s = A_{s-1}/I_{s-1}.$$

Our desired π^f is the composition of these projections. In particular, if $f_v := P_v \to 0$, then by construction $A^{f_v} = A/\langle e_v \rangle$, where e_v is the trivial path corresponding to the vertex v.

Lemma IV.3. *If* $\mathrm{E}(f, f') = 0$ *and* $M \in \mathrm{Rep}(A)$ *is the cokernel of* $P_1' \xrightarrow{f'} P_0'$, *then* M *is in fact a representation of* A^f *(of projective dimension one), and moreover* $\mathrm{E}(A^f \otimes_A f, A^f \otimes_A f') = 0$.

Proof. Suppose that $\cdots \to P_2 \xrightarrow{g} P_1 \xrightarrow{f} P_0$ is a projective resolution, then g is represented by elements in $\operatorname{Ker} f$. Apply $\operatorname{Hom}(-, M)$ to the resolution, we get

$$\operatorname{Hom}_A(P_0, M) \to \operatorname{Hom}_A(P_1, M) \xrightarrow{g^M} \operatorname{Hom}_A(P_2, M) \to \cdots .$$

Since $\mathrm{E}(f, f') = 0$, g^M must be a zero map by Lemma III.2. In other words, the representation M satisfies all the relations generated by the entries of all vectors in $\operatorname{Ker} f$. So $M \in \mathrm{Rep}(A_1)$, where $A_1 = A/I_0$ as in the construction above. In the mean while, $\mathrm{E}(A_1 \otimes_A f, A_1 \otimes_A f')$ still equals 0, so we can complete the proof by induction. $\square$

Now we prove the claim that $n(f) \leqslant r(A)$ for rigid f. Suppose that $n(f) > r(A)$ and in addition $P_v[1] \notin \mathrm{Ind}(f)$. Applying the above lemma to $f' = f$, we see by Lemma III.2 that $M = \mathrm{Coker}(f)$ as a classical (partial) tilting A^f-module has more than $r(A) = r(A^f)$ nonisomorphic indecomposable summands, which contradicts the classical tilting theory [1, Corollary VI.4.4]. If $\mathrm{Ind}(f)$ contains some $P_v[1]$, apply the above lemma to f_v and f, then we can easily reduce to the previous situation for another algebra A' with $r(A') < r(A)$.

Theorem IV.4. *For a rigid presentation* f, *the following are equivalent:*

(1) $|\mathrm{Ind}(f)| = \mathrm{rank}(K_0(\mathrm{Rep}(A)))$.

(2) *f is maximal rigid in the sense that* $\mathrm{E}(f \oplus f', f \oplus f') \neq 0$ *for any indecomposable* $f' \notin \mathrm{Ind}(f)$.

(3) *f is generating.*

Proof. We have already proved (1) implies (2). Now assume that f is maximal rigid, then $K^b(\text{proj-}A)$ can be generated by $\mathrm{Ind}(f)$, otherwise the (positive or negative) completion of f generates $K^b(\text{proj-}A)$ contradicting the maximality of f. Hence (2) implies (3). Finally if f is generating, then $n(f) \geqslant r(A)$ because $K_0(K^b(\text{proj-}A))$ is isomorphic to $K_0(\mathrm{Rep}(A))$. So (3) implies (1) finishing the proof. □

Remark IV.5. The above proposition may be proved entirely in the derived category setting. Our *ad hoc* approach looks simpler but depends on the classical tilting theory.

Definition IV.6. For any maximal rigid presentation f, suppose that $\mathrm{Ind}(f) = \{f_1, f_2, \cdots, f_n\}$ and let $f_{\widehat{k}} = \oplus_{i \neq k} f_i$. If $\mathrm{Ind}(f) = \mathrm{Ind}(\tilde{f}_{\widehat{k}}^{+})$ (resp. $= \mathrm{Ind}(\tilde{f}_{\widehat{k}}^{-})$), then we say f_k is the *positive* (resp. *negative*) *complement* of $f_{\widehat{k}}$, denoted by f_+ (resp. f_-). We define the *mutation* of $\mathrm{Ind}(f)$ at f_k to be $\mathrm{Ind}(\tilde{f}_{\widehat{k}}^{-})$ (resp. $\mathrm{Ind}(\tilde{f}_{\widehat{k}}^{+})$).

Now suppose that $|\,\mathrm{Ind}(f)| = n - 1$ and f_+, f_- be the positive and negative complements. First we claim that f_+ and f_- are always different. Let H be the hyperplane in $K_0(K^b(\text{proj-}A)) \cong \mathbb{Z}^{|Q_0|}$ spanned by the classes in $\mathrm{Ind}(f)$. Since the classes of A and $A[1]$ lie in two different sides of H, we see from the two triangles (4.1),(4.2) that the classes of f_+ and f_- are also separated by H.

Let f_c be any complement of f and we claim that f_c must be either f_+ or f_-. Otherwise, f_c and one of $\delta(f_+), \delta(f_-)$ must stay in the same side of H. Suppose that $\delta(f_c)$ and $\delta(f_+)$ live together, then the interiors of rational convex cones spanned by the classes in $\mathrm{Ind}(f \oplus f_+)$ and $\mathrm{Ind}(f \oplus f_c)$ must intersect, so by Theorem III.12 we can find some δ which possesses two canonical decompositions, one involving $\delta(f_+)$ and the other involving $\delta(f_c)$ but no $\delta(f_+)$. This contradicts the uniqueness of the canonical decomposition.

Applying $\mathrm{Hom}(f_+, -)$ to the triangle $f^-[-1] \to A \xrightarrow{can} f^{e^-} \to f^-$, we get

$$0 = \mathrm{Hom}(f_+, f^e[1]) \to \mathrm{Hom}(f_+, f^-[1]) \to \mathrm{Hom}(f_+, A[2]) = 0.$$

So $\mathrm{E}(f_+, f^-) = 0$ and hence $\mathrm{E}(f_+, f_-) = 0$. Then $\dim \mathrm{E}(f_-, f_+) = d \neq 0$ by Theorem IV.4. Complete $f_-^d \xrightarrow{can} f_+[1]$ to a triangle $f_+ \to f' \to f_-^d \xrightarrow{can} f_+[1]$ and apply $\mathrm{Hom}(f_-, -)$ and $\mathrm{Hom}(-, f_-)$ to it, we get

$$\mathrm{Hom}(f_-, f_-^d) \xrightarrow{can} \mathrm{Hom}(f_-, f_+[1]) \to \mathrm{Hom}(f_-, f'[1]) \to \mathrm{Hom}(f_-, f_-^d[1]) = 0,$$

$$0 = \mathrm{Hom}(f_-^d, f_-[1]) \to \mathrm{Hom}(f', f_-[1]) \to \mathrm{Hom}(f_+, f_-[1])) = 0.$$

So $\mathrm{E}(f_-, f') = \mathrm{E}(f', f_-) = 0$. Apply $\mathrm{Hom}(f_+, -)$ to the same triangle and we get

$$0 = \mathrm{Hom}(f_+, f_+[1]) \to \mathrm{Hom}(f_+, f'[1]) \to \mathrm{Hom}(f_+, f_-^d[1]) = 0,$$

so $\mathrm{E}(f_+, f') = 0$. Finally apply $\mathrm{Hom}(-, f')$ to the triangle again and we get

$$0 = \mathrm{Hom}(f_-^d, f'[1]) \to \mathrm{Hom}(f', f'[1]) \to \mathrm{Hom}(f_+, f'[1]) = 0,$$

so $\mathrm{E}(f', f') = 0$. Hence $\mathrm{E}(f' \oplus f_-, f' \oplus f_-) = 0$. Similar argument goes through the triangle $f_+^d \to f'' \to f_- \xrightarrow{can} f_+^d[1]$ as well. We summarize as follows.

Proposition IV.7. *f_+, f_- are the two and only two complements of f. They are related by the triangle $f_+ \to f' \to f_-^d \to f_+[1]$ and $f_+^d \to f'' \to f_- \to f_+^d[1]$, where $d = \dim \mathrm{E}(f_-, f_+)$. Moreover, both $f' \oplus f_-$ and $f'' \oplus f_+$ are rigid and $\mathrm{E}(f_+, f_-) = \mathrm{E}(f_+, f') = \mathrm{E}(f'', f_-) = 0$. In particular, $d = 1$ if and only if $f' = f''$ belongs to the subcategory generated by $\mathrm{Ind}(f)$.*

Question IV.8. *What is a necessary and sufficient condition for $d = 1$?*

4.2 The Simplicial Complexes

Now we can attach to the algebra A an abstract simplicial complex $\mathcal{S}(A)$ as follows. The set $\mathcal{S}(A)_p$ of p-simplexes consists of all $\{\delta_1, \ldots, \delta_p\}$ such that each δ_i is indecomposable and $\mathrm{e}(\delta_i, \delta_j) = 0$ for $i \neq j$. Let $\mathcal{S}^r(A)$ be the subcomplex of $\mathcal{S}(A)$ consisting of all the simplexes whose vertexes are all real. When $A = kQ$ is a finite-dimensional path algebra (without relation), $\mathcal{S}^r(A)$ is the well-known cluster complex by Lemma III.2.

In general, for any abstract simplical complex $\mathcal{K}$, the *geometric realization* $|\mathcal{K}|$ of $\mathcal{K}$ is defined as follows. Let $\mathcal{K}_0$ be the set of 0-simplexes, then $|\mathcal{K}|$ is the subset of the vector space $\mathbb{R}^{\mathcal{K}_0}$ consisting of all $x = \sum t_i v_i, t_i \in [0,1], v_i \in \mathcal{K}_0$ with the property that $\sum t_i = 1$ and set of all vertices v_i with nonzero coefficient is a simplex in $\mathcal{K}$. We topologize $|\mathcal{K}|$ by giving it the weakest topology with respect to the property that a map $\lambda : |\mathcal{K}| \to X$ is continuous if and only if it is continuous on every closed simplex, i.e., the set of all $x \in |\mathcal{K}|$ which are nonnegative linear combinations of the vertices of a simplex in $\mathcal{K}$.

Let $\Lambda_0 : \mathcal{S}(A)_0 \to \mathbb{R}^{|Q_0|}$ be the map assigning each vertex δ_i to itself, then Λ_0 can be piecewise-linearly extended to a continuous map $\Lambda : |\mathcal{S}(A)| \to \mathbb{R}^{|Q_0|}$. We define $\lambda : |\mathcal{S}(A)| \to S^{|Q_0|-1}$ by $\lambda(\delta) = \Lambda(\delta)/\|\delta\|$, where $\|\cdot\|$ is the usual Euclidean norm.

Proposition IV.9. *The map λ is injective. If $\mathcal{S}^r(A)$ is finite, then $\mathcal{S}(A) = \mathcal{S}^r(A)$ and λ gives a triangulation of the sphere $S^{|Q_0|-1}$.*

Proof. Suppose that λ is not injective and the fiber over ε has more than one point, then some integral multiple $k\varepsilon \in \mathbb{Z}^n$ would have two different canonical decompositions by Theorem III.12. This contradicts the uniqueness of the canonical decomposition.

When $\mathcal{S}^r(A)$ is finite, we define $\lambda^r : |\mathcal{S}^r(A)| \to S^{|Q_0|}$ exactly the same way as λ. By the construction in Section 4.1 and Proposition IV.7, the fan corresponding to the simplicial complex $\lambda^r|\mathcal{S}^r(A)|$ has the two properties of the following lemma. Our claim follows. $\square$

Lemma IV.10. *Suppose that $\mathcal{F} \subseteq \mathbb{R}^n$ is a closed fan. Let $\mathcal{C}_i$ be the set of i-dimensional simplicial cones in the fan. Assume that*

(1) *every element F in $\mathcal{C}_i$ with $i < n$ is the face of an simplicial cone in $\mathcal{C}_n$.*

(2) *every element F in $\mathcal{C}_{n-1}$ is the face of two distinct simplicial cones in $\mathcal{C}_n$.*

Then the fan covers $\mathbb{R}^n$.

Proof. Since every cone $C \in \mathcal{F}_n$ contains a rational point, $\mathcal{F}_n$ is countable. By property (1), $\mathcal{F}_i$ is countable for all i. Suppose that $p \in \mathbb{R}^n$ does not lie in the fan. Let $C \in \mathcal{C}_n$. For every element $F \in \mathcal{C}_{n-2}$, let $\overline{F}$ be the span of F and p. Let D be the union of all $\overline{F}$, with $F \in \mathcal{F}_{n-2}$. Then D is a union of simplicial cones of dimension $\leq n-1$. Since D has zero measure, it cannot contain C. Choose $q \in C \setminus D$. Define $\gamma : [0,1] \to \mathbb{R}^n$ by $\gamma(t) = (1-t)p + tq$. Then γ is a continous path from p to q. We have $\gamma^{-1}(\mathcal{F})$ is closed, hence compact. Therefore $\gamma^{-1}(\mathcal{F})$ has a smallest element, say t_0. Now $\gamma(t_0)$ lies in C for some $C \in \mathcal{C}_i$. By property (1), we know that $\gamma(t_0)$ lies in C for some $C \in \mathcal{C}_n$. Now $\gamma(t_0)$ cannot lie in the interior of C because then $\gamma(t_0 - \varepsilon) \in C$ for some small $\varepsilon > 0$. Therefore, $\gamma(t_0)$ lies in some $n-1$-dimensional facet F of C. If $\gamma(t_0)$ does not lie in the relative interior of F, then $\gamma(t_0)$ lies in F' for some $n-2$ dimensional face of F. But from $(1-t_0)p + t_0 q \in F'$ follows that $q \in \overline{F'}$, which is a contradiction. Therefore, $\gamma(t_0)$ lies in the relative interior of F. Besides C there must be another n-dimensional simplicial cone of $\mathcal{F}$ such that F is a facet of C'.But then $\gamma(t_0)$ lies in the interior of $C \cup C'$, and $\gamma(t_0 - \varepsilon) \in \mathcal{F}$ for some small $\varepsilon > 0$. Contradiction. Therefore, it is not possible to choose p outside the fan, so the fan covers $\mathbb{R}^n$. $\square$

Question IV.11. *If the algebra A is of finite type, then $\mathcal{S}^r(A)$ is finite. Is the converse true?*

A similar simplicial complex for quivers was studied in [5]. It was proved in [9, Theorem 7.1] that $\mathcal{S}^r(kQ)$ is always connected. In general, $\mathcal{S}(A)$ or $\mathcal{S}^r(A)$ may be disconnected.

Example IV.12. Consider the Yin-Yang quiver Q: $u \overset{\times 3}{\underset{\times 3}{\rightleftarrows}} v$ with three arrows in each direction. Let A be the algebra of kQ modulo the relations generated by all paths of length 2. A simplicial complex governing the canonical decomposition of general representations of the generalized Kronecker quiver $\Theta : u \xrightarrow{\times 3} v$ was shown in [5, Page 249]. It is not hard to see that that simplicial complex is the part of $\mathcal{S}(A)$ in the 1st and 4th quadrants as shown in figure 4.1 and $\mathcal{S}(A)$ is symmetric about the origin. The missing part on the circle contains the wild δ-vectors. Note that the

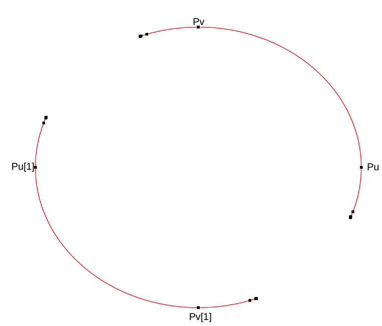

Figure 4.1: A disconnected $\mathcal{S}(A)$

algebra A has infinite global dimension.

Question IV.13. *Is $\mathcal{S}(A)$ or $\mathcal{S}^r(A)$ always connected for connected algebras of finite global dimension?*

CHAPTER V

Applications and Examples

5.1 Application to Quiver Representations

We suppose that Q is a quiver without oriented cycle and A is the path algebra kQ. Recall that any $M \in \mathrm{Rep}_\alpha(kQ)$ admits a canonical presentation (2.6):

$$0 \to P(\beta_1) = \bigoplus_{a \in Q_1} P_{ha} \otimes M(ta) \xrightarrow{f_M^{can}} P(\beta_0) = \bigoplus_{v \in Q_0} P_v \otimes M(v) \to M \to 0,$$

where f_M^{can} is given by $e_{ha} \otimes m \mapsto e_{ha} \otimes am - a \otimes m$. So $(\beta_1, \beta_0) = (D\alpha, \alpha)$, where D is the matrix whose rows and columns are labeled by Q_0, with the diagonal entries all zero and the other entries $D_{u,v}$ equal to the number of arrows from v to u. Moreover, the cokernel of a general presentation in $\mathrm{PHom}_{kQ}^{can}(\alpha) := \mathrm{PHom}_{kQ}(D\alpha, \alpha)$ corresponds to a general element in $\mathrm{Rep}_\alpha(kQ)$ by Corollary II.8. As in [17], we denote by $\mathrm{ext}(\alpha_1, \alpha_2)$ the minimal value of the upper semi-continuous function $\dim \mathrm{Ext}_{kQ}(-,-)$ on $\mathrm{Rep}_{\alpha_1}(kQ) \times \mathrm{Rep}_{\alpha_2}(kQ)$.

Corollary V.1 (Schofield). *A general representation in* $\mathrm{Rep}_{\alpha_1+\alpha_2}(kQ)$ *has a subrepresentation in* $\mathrm{Rep}_{\alpha_1}(kQ)$ *if and only if* $\mathrm{ext}(\alpha_1, \alpha_2) = 0$.

Proof. $\Rightarrow$. If $M \in \mathrm{Rep}_{\alpha_1+\alpha_2}(kQ)$ is general and let $L \in \mathrm{Rep}_{\alpha_1}(kQ)$ be its subrepresentation. We can splice the canonical presentations of L and M/L together to get a presentation of M in $\mathrm{PHom}_{kQ}^{can}(\alpha_1+\alpha_2)$. We can conclude that a general presentation in $\mathrm{PHom}_{kQ}^{can}(\alpha_1+\alpha_2)$ has a genuine subpresentation in $\mathrm{PHom}_{kQ}^{can}(\alpha_1)$. So by Theorem III.8 and Lemma III.2, $\mathrm{ext}(\alpha_1, \alpha_2) = 0$.

$\Leftarrow$. Suppose that $\mathrm{ext}(\alpha_1, \alpha_2) = 0$, then $\mathrm{e}(\alpha_1 - D\alpha_1, \alpha_2 - D\alpha_2) = 0$ by Lemma III.2, so a general presentation in $\mathrm{PHom}_{kQ}^{can}(\alpha_1 + \alpha_2)$ has a genuine subpresentation in $\mathrm{PHom}_{kQ}^{can}(\alpha_1)$. Let $(P_1, P_0) = (P(D(\alpha_1 + \alpha_2)), P(\alpha_1 + \alpha_2)), (P_1^s, P_0^s) = (P(D\alpha_1), P(\alpha_1))$, and Z be the corresponding variety constructed in Section 3.1. Since the upper semi-continuous function $(f, P_1', P_0') \mapsto \dim \mathrm{Ker}\, f''$ attains its minimun 0 on Z, we see that for a general point on Z, its induced quotient $f'' \in \mathrm{PHom}_{kQ}^{can}(\alpha_2)$ is injective. Now if $M \in \mathrm{Rep}_{\alpha_1+\alpha_2}(kQ)$ is general enough, then so is its canonical presentation $f_M^{can} \in \mathrm{PHom}_{kQ}^{can}(\alpha_1 + \alpha_2)$. Since the projection p_1 is dominant, its fiber $p_1^{-1}(f_M^{can})$ must contain a point which induces an injective quotient f''. It follows by the snake lemma that M has a subrepresentation $L = \mathrm{Coker}\, f'$ in $\mathrm{Rep}_{\alpha_1}(kQ)$. $\square$

Theorem III.12 can be specialized to the quiver case in a similar fashion as Corollary V.1.

Corollary V.2 (Kac)**.** *$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$ is the canonical decomposition of α if and only if $\mathrm{ext}(\alpha_i, \alpha_j) = 0$ for $i \neq j$ and a general representation in $\mathrm{Rep}_{\alpha_i}(kQ)$ is indecomposable for each α_i .*

Example V.3. Consider the quiver Q:

$$\begin{tikzcd} & v \arrow[dr, "b"] & \\ w \arrow[ur, "a"] \arrow[rr, "c"] & & u \end{tikzcd}$$

A similar simplicial complex as $\mathcal{S}(kQ)$ was studied in [5]. Note that $(-1, 0, 1)$ is the only tame δ-vector of kQ. The stereographic projection of $\mathcal{S}(kQ)$ looks like

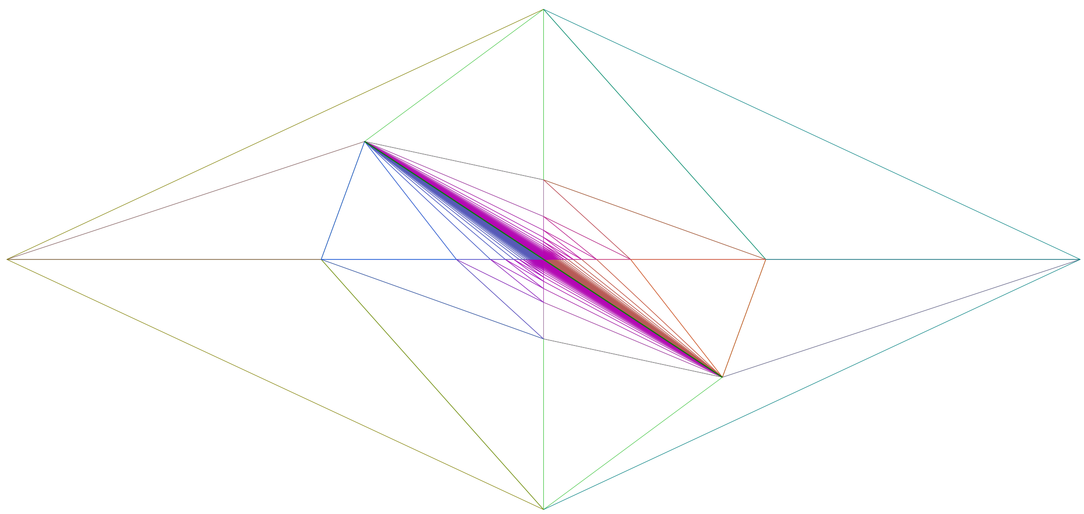

Figure 5.1: $\mathcal{S}(kQ)$ for a quiver algebra kQ

5.2 An Example from String Algebras

We quickly review some basics of unimodular bilinear forms. Let $\langle \cdot, \cdot \rangle$ be a unimodular bilinear form on $\mathbb{Z}^n$. A semiorthogonal basis $(e_1, \ldots, e_n)$ is a basis of $\mathbb{Z}^n$ such that $\langle e_i, e_i \rangle = 1, \langle e_i, e_j \rangle = 0, i < j$. Its dual semiorthogonal basis $(e_1^\vee, \ldots, e_n^\vee)$ is a basis satisfying $\langle e_i, e_j^\vee \rangle = \delta_{ij}$. Define $F : \mathbb{Z}^n \to \mathbb{Z}^{n*}$ by $Fu = \langle u, \cdot \rangle$. For any $s, t \in \mathbb{Z}^{n*}$, we define $\langle s, t \rangle_F = \langle F^{-1}s, F^{-1}t \rangle$. $\langle s, t \rangle_F$ can be computed using any semiorthogonal basis $(e_1, \ldots, e_n)$ and its dual as $\langle s, t \rangle_F = \sum_{i=1}^n s(e_i)t(e_i^\vee)$.

Let us first consider the following Diophantine problem: $M_1 = M_2 = M_3 = 1$, where M_i is the minor of the integer matrix $M = \begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ by removing the i-th column.

Let $\langle \cdot, \cdot \rangle$ be a unimodular bilinear form on $\mathbb{Z}^3$ given by $\langle u, v \rangle = uC^{\mathrm{T}}v^{\mathrm{T}}$, where $C = \left(\begin{smallmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. It is trivial to check that $\{v_1, v_2, v_3\}$ form a semiorthogonal basis, where $v_i = (m_i + 1, n_i - m_i - 1, -(n_i - 1))$ if and only if the above Diophantine equation is satisfied. In this case, $\langle v_2, v_1\rangle = \langle v_3, v_1\rangle = \langle v_3, v_2\rangle = 2$ and the dual basis $(v_3^\vee, v_2^\vee, v_1^\vee) = (v_3 C^{\mathrm{T}} C^{-1}, v_2 - 2v_1, v_1)$. Notice that for any $p, q \in \mathbb{Z}$, $\delta_0 = (p, q-p, -q)$ is isotropic, i.e., $\langle \delta_0, \delta_0\rangle = 0$. So $0 = \langle F\delta_0, F\delta_0\rangle_F = (x_1 - x_2)^2 - x_3^2$, where $x_i = \langle \delta_0, v_i\rangle = qm_i - pn_i$. An elementary calculation can show that $m_2 = m_1 + m_3$ and $n_2 = n_1 + n_3$.

Now we only interested in solutions that lie in the region $m \geqslant -1, n \geqslant 1$. Suppose that $S = \begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ is a solution with all $m_i \geqslant 1$, then it generates three other solutions. They are

$$\sigma_1(S) = \begin{pmatrix} m_2 & 2m_2 - m_1 & m_3 \\ n_2 & 2n_2 - n_1 & n_3 \end{pmatrix}, \sigma_3(S) = \begin{pmatrix} m_1 & 2m_2 - m_3 & m_2 \\ n_1 & 2n_2 - n_3 & n_2 \end{pmatrix},$$

and $\sigma_2(S)$ to be one of

$$\begin{pmatrix} 2m_1 - m_2 & m_1 & m_3 \\ 2n_1 - n_2 & n_1 & n_3 \end{pmatrix} \text{ and } \begin{pmatrix} m_1 & m_3 & 2m_3 - m_2 \\ n_1 & n_3 & 2n_3 - n_2 \end{pmatrix}$$

depending on the order of m_1 and m_3 (It is easy to see that it is the same as the order of n_1 and n_3). If we allow some $m_i < 1$, then one of $\sigma_1(S)$ and $\sigma_3(S)$ may run out of this region. We define the rank of the solution S to be $m_1 + n_1 + m_3 + n_3$. It is clear that σ_1, σ_3 increase the rank, while σ_2 reduces the rank. It is elementary to check that 0 is the least possible rank and the only rank 0 solution is $S_0 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$. Notice that the inverse of σ_2 is either σ_1 or σ_3, so we can conclude that any solution inside this region can be obtained by recursively applying σ_1 or σ_3 to S_0.

Example V.4. Consider the quiver Q

$$x \underset{a_2}{\overset{a_1}{\rightrightarrows}} y \underset{b_2}{\overset{b_1}{\rightrightarrows}} z$$

with relation $b_1a_1 = b_2a_2 = 0$. The algebra $A = kQ/I$ is a string algebra of global dimension 2 with the Cartan matrix $C = \left(\begin{smallmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, i.e., the matrix whose rows are the dimension vectors of the indecomposable projective representations. The general representations of A in irreducible components were studied in [14]. It is known that there are only three possible cases for the dimension vector α of a generally indecomposable representation.

1. $\alpha_-(m,n) = (m-1, m+n-1, n-1)$;

2. $\alpha_+(m,n) = (m+1, m+n+1, n+1)$;

3. $\alpha_0(m,n) = (m, m+n, n)$ with GCD(m,n)=1.

An easy rank argument can show that all these representations have projective dimension less than 1, so δ-vectors of theirs minimal presentation are determined by the Cartan matrix. In fact, all the indecomposable δ-vectors are the following

1. $\delta_-(m,n) = (m-1, n-m+1, -(n+1))$, for $m \geqslant 1, n \geqslant 1$ or $= -1$ and $(m,n) = (0,-1), (1,0)$;

2. $\delta_+(m,n) = (m+1, n-m-1, -(n-1))$, for $n \geqslant 1, m \geqslant 1$ or $= -1$ and $(m,n) = (-1,0), (0,1)$;

3. $\delta_0(m,n) = (m, n-m, -n)$, for $m, n \geqslant 0$ and GCD$(m,n) = 1$.

Except in case (1) when $n = -1$ or $(m,n) = (1,0)$, all the general presentations are injective and the dimensions of the cokernels match with the first list.

Then using Lemma III.2, it is ready to check that (1),(2) are real δ-vectors and (3) are tame ones. For $m \geqslant 1$ and $n \geqslant 1$, τ takes the rigid presentation in $\mathrm{PHom}_A(\delta_-(m,n))$ to the rigid presentation in $\mathrm{PHom}_A(\delta_+(m,n))$ and vice versa; and takes a general presentation in $\mathrm{PHom}_A(\delta_0(m,n))$ to a general one in the same space.

One can check that the above transformation of the δ-vectors are given by $-C^{\mathrm{T}}C^{-1}$. In case (1) when $n=-1$, τ takes the rigid presentation in $\mathrm{PHom}_A(\delta_-(m,-1))$ to the rigid one in $\mathrm{PHom}_A(\delta_-(m+2,-1))$; in case (2) when $m=-1$, τ takes the rigid presentation in $\mathrm{PHom}_A(\delta_+(-1,n))$ to the rigid one in $\mathrm{PHom}_A(\delta_+(-1,n-2))$; in case (3) when $mn=0$, τ takes a general presentation in $\mathrm{PHom}_A(\delta_0(0,n))$ or $\mathrm{PHom}_A(\delta_0(m,0))$ to a general one in the same space. There is one singular case when the rigid presentation is $P_z[1]$. $\tau P_z[1]=I_z$ has the minimal presentation $P_z\oplus P_y^2\to P_x^2$, but this presentation is not general in the corresponding space $\mathrm{PHom}_A((2,-2,-1))$. In fact, its δ-vector decomposes as $(2,-2,-1)=2\cdot(1,-1,0)\oplus(0,0,-1)$. We will see in the next section that this can never happen for the Jacobian algebra of a quiver with potential.

We are going to examine which δ-vectors can form a maximal simplex in $\mathcal{S}^r(A)$. We first make an easy observation that all δ-vectors in a maximal simplex must belong to the same class of the list with the only exceptions being $\delta_-(0,-1)=(-1,0,0)$ and $\delta_+(-1,0)=(0,0,1)$. Otherwise, that simplex will contain some $\delta_0(m,n)$ as an interior point. So we can restrict ourself to the δ-vectors of form $\delta_+(m,n)$. It is not hard to check that the situation for $\delta_-(m,n)$ is symmetric. Now if $\{\delta_+(m_i,n_i)\}_{i=1,2,3}$ form a maximal simplex and let f_i be the corresponding rigid presentation, then $f_1\oplus f_2\oplus f_3$ is a tilting complex because they are all injective. So its endomorphism algebra is derived equivalent to A, so we may assume that $\mathrm{Hom}_{K^b(\text{proj-}A)}(f_i,f_j)=0$ for $i<j$, then (f_1,f_2,f_3) is an exceptional sequence in $K^b(\text{proj-}A)$. Hence, $\left(\begin{smallmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{smallmatrix}\right)$ satisfies the Diophantine equation. Now by Proposition IV.7 and the dynamical description of the solutions, we know that this condition is also sufficient. Since we can easily mutate $\{P_x,P_y,P_z\}$ to $\{P_x[1],P_y[1],P_z[1]\}$ in three steps, we proved that $\mathcal{S}^r$ is connected.

The stereographic projection of $\lambda(\mathcal{S}(A))$ from the point $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ looks like

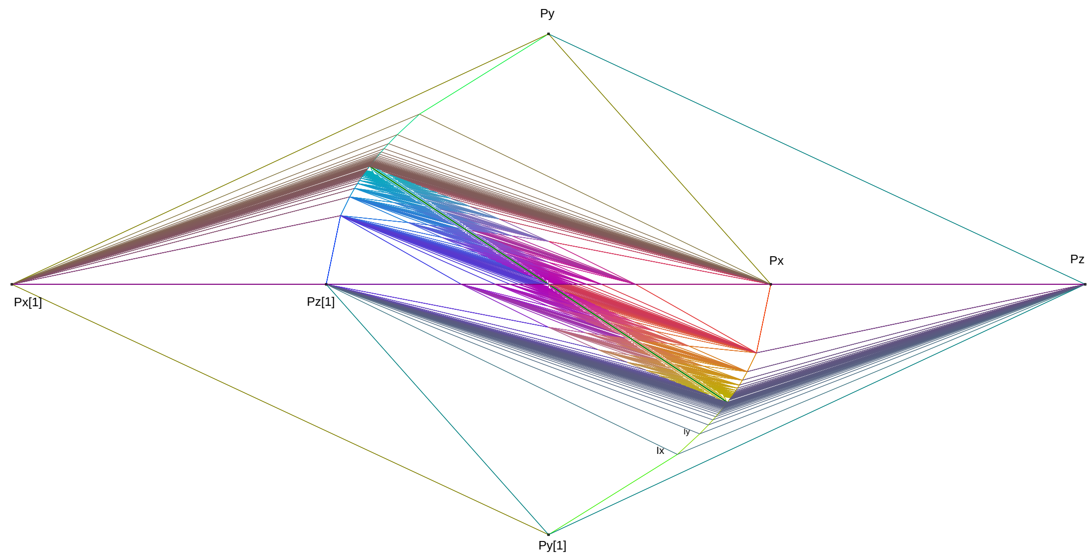

Figure 5.2: $\mathcal{S}(A)$ for a string algebra A

The dark green river in the middle consists of all the tame δ-vectors. The upper (resp. lower) light green stream is formed by real δ-vectors not involving P_x (resp. P_z). Roughly speaking, τ fixes the river, makes the stream flow, and takes the rest to the other bank. We see that both $\lambda(\mathcal{S}(A))$ and $\lambda(\mathcal{S}^r(A))$ are connected and the latter is contractible.

5.3 Application to Quivers with Potentials

Recall that a *decorated representation* of A is a pair $\mathcal{M} = (M, V)$, where $M \in \operatorname{Rep}(A)$ is the positive part of $\mathcal{M}$ and $V \in \operatorname{Rep}(R)$ is the negative part of $\mathcal{M}$. $\mathcal{M}$ is called positive (resp. negative) if $V = 0$ (resp. $M = 0$). Let $\mathcal{R}ep(A)$ be the set of decorated representations of A up to isomorphism. There is a bijection between $\mathcal{R}ep(A)$ and $K^2(\text{proj-}A)$ mapping any positive representation M to its minimal presentation in $\operatorname{Rep}(A)$, and simple negative representation $\mathcal{S}_v^-$ to $P_v \to 0$. So the AR-transformation τ and its inverse are defined in $\mathcal{R}ep(A)$. They clearly commute with the trivial dual $\mathcal{M}^* := \mathrm{Hom}(\mathcal{M}, k)$. For any decorated representation $\mathcal{M}$, we will write $f(\mathcal{M})$ for its image in $K^2(\text{proj-}A)$, and denote its δ-vector and i-th Betti vector by $\delta(\mathcal{M})$ and $\beta_i(\mathcal{M})$. Then we have that $\delta(\mathcal{M}) = \beta_0(\mathcal{M}) - \beta_1(\mathcal{M}), \beta_0(\mathcal{M}) = \beta_0(M)$, and $\beta_1(\mathcal{M}) = \beta_1(M) + \dim V$. Note that $\beta_i(M)$ agree with the classical Betti vectors for representations. Moreover, it is clear that $\beta_0(\tau\mathcal{M}^*) = \beta_1(\mathcal{M})$ and $\beta_1(\tau\mathcal{M}^*) = \beta_0(\mathcal{M})$.

We follow [7] and define

Definition V.5. $\mathrm{E}_A^{\mathrm{proj}}(\mathcal{M}, \mathcal{N}) = \mathrm{E}(f(\mathcal{M}), f(\mathcal{N}))$ and $\mathrm{E}_A^{\mathrm{inj}}(\mathcal{M}, \mathcal{N}) = \mathrm{E}_{A^{\mathrm{op}}}^{\mathrm{proj}}(\mathcal{N}^*, \mathcal{M}^*)$. $\mathcal{M}$ is called rigid if $f(\mathcal{M})$ is. We denote $\mathrm{Hom}_A^{+}(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_A(M, N)$.

From now on, we fix a quiver Q without oriented 2-cycles and a potential S on Q. Recall that a *potential* S is an element in $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$, where $\widehat{kQ}$ is the completion of the path algebra kQ. The vector space $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ has as basis the set of oriented cycles up to cyclic permutation. For each arrow $a \in Q_1$, the *cyclic derivative* ∂_a on $\widehat{kQ}$ is defined to be

$$\partial_a(a_1 \cdots a_s) = \sum_{k=1}^{d} a^*(a_k) a_{k+1} \cdots a_s a_1 \cdots a_{k-1}.$$

For each potential S, its *Jacobian ideal* ∂S is the (two-sided) ideal in $\widehat{kQ}$ generated by all $\partial_a S$. Let $A = \widehat{kQ}/\partial S$ be the *Jacobian algebra*, and assume it is finite dimensional, in which case completion is unnecessary. The key notion in [6] is the definition of mutation μ_v of the Jacobian algebra A and its decorated representations $\mathcal{M}$ at some vertex $v \in Q_0$. In fact, the mutation is defined at the level of quivers with potentials, but we do not need this. We refer the readers to that paper for details. In the general context of quivers with relations, the maps α and γ involved in the definition of the mutation are exactly φ and ψ in (2.7). So the h-vector and g-vector of $\mathcal{M}$ defined in

[7, (3.2) and (1.13)] are nothing but $-\beta_0(\mathcal{M}^*)$ and $-\delta(\mathcal{M}^*)$. One important feature of the mutation is that it is involutive. Moreover, one can check directly from the definition that it commutes with the trivial dual.

Lemma V.6. [7, Lemma 5.2, Proposition 6.1, and Theorem 7.1]

(i) $\beta_0(\mu_v\mathcal{M})_v = \beta_1(\mathcal{M})_v$, *so* $\beta_1(\mu_v\mathcal{M})_v = \beta_0(\mathcal{M})_v$ *and* $\delta(\mu_v\mathcal{M})_v = -\delta(\mathcal{M})_v$.

(ii) $\dim \mathrm{Hom}^+_{\mu_v A}(\mu_v\mathcal{M}, \mu_v\mathcal{N}) - \dim \mathrm{Hom}^+_A(\mathcal{M}, \mathcal{N}) = \beta_1(\mathcal{M})_v\beta_1(\mathcal{N}^*)_v - \beta_0(\mathcal{M})_v\beta_0(\mathcal{N}^*)_v$.

(iii) $\dim \mathrm{E}^{\mathrm{proj}}_{\mu_v A}(\mu_v\mathcal{M}, \mu_v\mathcal{N}) - \dim \mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{N}) = \beta_0(\mathcal{M})_v\beta_1(\mathcal{N})_v - \beta_1(\mathcal{M})_v\beta_0(\mathcal{N})_v$, *and* $\dim \mathrm{E}^{\mathrm{inj}}_{\mu_v A}(\mu_v\mathcal{M}, \mu_v\mathcal{N}) - \dim \mathrm{E}^{\mathrm{inj}}_A(\mathcal{M}, \mathcal{N}) = \beta_1(\mathcal{M}^*)_v\beta_0(\mathcal{N}^*)_v - \beta_0(\mathcal{M}^*)_v\beta_1(\mathcal{N}^*)_v$.

Corollary V.7. $\dim \mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{M})$ *and* $\dim \mathrm{E}^{\mathrm{inj}}_A(\mathcal{M}, \mathcal{M})$ *are mutation invariant, so the simplicial complex* $\mathcal{S}^r(A)$ *and* $\mathcal{S}^r(\mu_v A)$ *are isomorphic.*

Question V.8. *Does* μ_v *send a general presentation in* $\mathrm{PHom}_A(\delta)$ *to a general presentation in* $\mathrm{PHom}_A(\delta')$ *for some* δ'*?*

Lemma V.9. [7, *Corollary 10.8 and Proposition 7.3*]

(i) $\mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}^+_A(\mathcal{N}, \tau\mathcal{M})^*$ *and* $\mathrm{E}^{\mathrm{inj}}_A(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}^+_A(\tau^{-1}\mathcal{N}, \mathcal{M})^*$.

(ii) $\mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{M}) = \mathrm{E}^{\mathrm{inj}}_A(\mathcal{M}, \mathcal{M})$.

Corollary V.10. $\mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{M}) = \mathrm{E}^{\mathrm{proj}}_A(\tau\mathcal{M}, \tau\mathcal{M})$. *In particular,* τ *induces a simplicial automorphism on* $\mathcal{S}^r(A)$.

Proof. $\mathrm{E}^{\mathrm{proj}}_A(\mathcal{M}, \mathcal{M}) = \mathrm{Hom}^+_A(\mathcal{M}, \tau\mathcal{M})^* = \mathrm{Hom}^+_A(\tau^{-1}(\tau\mathcal{M}), \tau\mathcal{M})^* = \mathrm{E}^{\mathrm{inj}}_A(\tau\mathcal{M}, \tau\mathcal{M})$ $= \mathrm{E}^{\mathrm{proj}}_A(\tau\mathcal{M}, \tau\mathcal{M})$. Now if $\mathcal{M}$ is rigid, then so is $\tau\mathcal{M}$. Since τ is a bijection on the set of indecomposable objects, it induces a simplicial automorphism on $\mathcal{S}^r(A)$. □

Question V.11. *Is it true that* $\mathcal{S}(A)$ *and* $\mathcal{S}(\mu_v A)$ *are isomorphic and* τ *induces a simplicial automorphism on* $\mathcal{S}(A)$*?*

The following was shown in [2].

Lemma V.12. *For any finite-dimensional algebra A, two representations M, M' of A are isomorphic if and only if* $\dim \mathrm{Hom}_A(N, M) = \dim \mathrm{Hom}_A(N, M')$ *for any* $N \in \mathrm{Rep}(A)$. *There is also a dual statement.*

Corollary V.13. *The following are equivalent*

1. *Two decorated representations $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic.*

2. *For any $\mathcal{N} \in \mathcal{R}ep(A)$,* $\dim \mathrm{Hom}_A^+(\mathcal{N}, \mathcal{M}) = \dim \mathrm{Hom}_A^+(\mathcal{N}, \mathcal{M}')$
   and $\dim \mathrm{E}_A^{\mathrm{inj}}(\mathcal{N}, \mathcal{M}) = \dim \mathrm{E}_A^{\mathrm{inj}}(\mathcal{N}, \mathcal{M}')$.

(2)* *For any $\mathcal{N} \in \mathcal{R}ep(A)$,* $\dim \mathrm{Hom}_A^+(\mathcal{M}, \mathcal{N}) = \dim \mathrm{Hom}_A^+(\mathcal{M}', \mathcal{N})$
and $\dim \mathrm{E}_A^{\mathrm{proj}}(\mathcal{M}, \mathcal{N}) = \dim \mathrm{E}_A^{\mathrm{proj}}(\mathcal{M}', \mathcal{N})$.

Proof. Since (2^*) is the dual of (2), we will prove (2) implies (1) only. By the above lemma, $\mathcal{M}$ and $\mathcal{M}'$ have the same positive part. By Lemma V.9 and the dual of the above lemma, $\tau^{-1}\mathcal{M}$ and $\tau^{-1}\mathcal{M}'$ have the same positive part, then by the definition of τ, $\mathcal{M}$ and $\mathcal{M}'$ must have the same negative part. □

Proposition V.14. *The AR-transformation τ commutes with the mutation μ_v at any vertex v.*

Proof. We first verify that $\dim \mathrm{Hom}_A^+(\mathcal{N}, \tau\mu_v\mathcal{M}) = \dim \mathrm{Hom}_A^+(\mathcal{N}, \mu_v\tau\mathcal{M})$ for any $\mathcal{N} \in \mathcal{R}ep(A)$. Applying Lemma V.6 and V.9, we get the following equation

$$
\begin{aligned}
LHS &= \dim \mathrm{E}_A^{\mathrm{proj}}(\mu_v\mathcal{M}, \mathcal{N}) \\
&= \dim \mathrm{E}_{\mu_v A}^{\mathrm{proj}}(\mathcal{M}, \mu_v\mathcal{N}) + \beta_0(\mathcal{M})_v\beta_1(\mu_v\mathcal{N})_v - \beta_1(\mathcal{M})_v\beta_0(\mu_v\mathcal{N})_v \\
&= \dim \mathrm{Hom}_{\mu_v A}^+(\mu_v\mathcal{N}, \tau\mathcal{M}) + \beta_1(\tau\mathcal{M}^*)_v\beta_1(\mu_v\mathcal{N})_v - \beta_0(\tau\mathcal{M}^*)_v\beta_0(\mu_v\mathcal{N})_v \\
&= RHS.
\end{aligned}
$$

We remain to verify that $\dim \mathrm{E}^{\mathrm{inj}}_A(\mathcal{N}, \tau\mu_v\mathcal{M}) = \dim \mathrm{E}^{\mathrm{inj}}_A(\mathcal{N}, \mu_v\tau\mathcal{M})$ for any $\mathcal{N} \in \mathcal{R}ep(A)$. Applying Lemma V.6 and V.9, we get the following equation

$$\begin{aligned}
LHS &= \dim \mathrm{Hom}^+_A(\mu_v\mathcal{M}, \mathcal{N}) \\
&= \dim \mathrm{Hom}^+_{\mu_v A}(\mathcal{M}, \mu_v\mathcal{N}) + \beta_1(\mathcal{M})_v\beta_1(\mu_v\mathcal{N}^*)_v - \beta_0(\mathcal{M})_v\beta_0(\mu_v\mathcal{N}^*)_v \\
&= \dim \mathrm{E}^{\mathrm{inj}}_{\mu_v A}(\mu_v\mathcal{N}, \tau\mathcal{M}) + \beta_0(\tau\mathcal{M}^*)_v\beta_1(\mu_v\mathcal{N}^*)_v - \beta_1(\tau\mathcal{M}^*)_v\beta_1(\mu_v\mathcal{N}^*)_v \\
&= RHS.
\end{aligned}$$

$\square$

Example V.15. Consider the quiver Q'

$$\begin{tikzcd}
& v \arrow[dl, "b"'] & \\
w \arrow[rr, shift left, "c"] \arrow[rr, shift right, "d"'] & & u \arrow[ul, "a"']
\end{tikzcd}$$

with the potential $S = cba$. Note that its Jacobian algebra A is of tame type and has infinite global dimension. If we perform the mutation at the vertex v, then we get the same quiver Q as in Example V.3 with zero potential. Note that $(-1, 0, 1)$ and $(1, 0, -1)$ are the only tame δ-vectors of kQ and A respectively, and the mutation takes a general presentation in one space to the other, so $\mathcal{S}(A) = \mathcal{S}(kQ)$.

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