Retail Pricing of Substitutable Products Under Logit Demand

by

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CHAPTER I

Introduction

Retail pricing decisions are challenging in that they need to take into account a host of factors such as product values, customer demand, inventory status, and competitive landscape. This dissertation focuses on the effects of product substitution on retail pricing decisions. In the presence of substitutable products, the inventory levels and prices of different products interact in complicated ways and, therefore, insights developed in the context of single-product pricing problems do not necessarily carry over. Unfortunately, pricing problems that involve multiple products and multiple sellers tend to be analytically intractable, which has so far limited the extent of analytical results for such problems. Motivated by this gap in the literature, this dissertation studies two related problems. First, a single retailer which sells substitutable products and, second, competition among multiple sellers each offering a single product. The first problem is explored in two separate chapters, which analyze dynamic pricing in the presence of limited inventories and an extension to personalized pricing.

The remainder of this chapter is organized as follows: In §1.1, the problem of dynamically pricing two substitutable products is presented. In this problem, we assume the seller offers two products with limited inventories, sold over a predetermined, finite selling season. This setting is a good fit for seasonal products for which the sales deadline is exogenously fixed and there are limited replenishment opportunities. We consider both price-based substitution and stock-out-based substitution among products. When both types of substitution are allowed, the existing literature offers little in the way of analytical results. We add to the literature by developing a series of results about the marginal values of products, which then leads to results about the structure of the optimal pricing policy. In §1.2, the personalized dynamic pricing problem is presented. This is equivalent to group pricing practices where different prices can be charged for the same product, depending on consumer segments. An example of such pricing practices is offering discounts to students or senior citizens. We derive analytical properties of the revenue function under dynamic pricing. We conduct a numerical analysis to study the interactions between dynamic pricing and personalized pricing. In §1.3, we present a problem with multiple sellers, in which a major retailer allows smaller merchants to sell through it. In this setting, the smaller merchants gain access to the store traffic of the major retailer, in return for commission fees. An example for this practice is Amazon's third-party marketplace. We study the characteristics of the equilibrium and explore the conditions under which this business model is viable.

1.1. Dynamic Pricing of Substitutable Products with Limited Inventory under Logit Demand

We consider the problem of dynamically pricing two substitutable products over a predetermined, finite selling season. The initial inventory levels of the products are fixed exogenously and there are no replenishment opportunities during the season. We assume that each arriving customer chooses from available products based on the multinomial logit choice model, which captures the effect of prices on consumer choice. Every time a product runs out of stock, the set of choices shrinks, capturing the effect of stockouts on consumer choice. We prove a number of structural results regarding the behavior of the marginal value of a product, the price difference between the two products and the purchase probabilities. The contributions of this research are as follows:

- We show that, under the optimal pricing policy, the marginal value of an item is increasing in the remaining time and decreasing in the stock level of either product.
- We prove that the marginal value of a product is more sensitive to its own stock level than it is to the substitute's stock level.
- We show that both the optimal price difference between two products and the optimal purchase probabilities provide intuitive gauges of the optimal behavior.

1.2. Personalized Dynamic Pricing of Substitutable Products under Logit Demand

We consider the personalized dynamic pricing problem of two substitutable products over a finite selling season. We study a consumer population that consists of three segments: a segment that chooses from both products, a segment that considers only product 1, and another segment that considers only product 2. We add the possibility of another layer of price differentiation to the model described in §1.1: we assume that the seller can charge different prices to different segments for the same product (hereafter, personalized pricing). We assume inventory levels of the products are exogenous without replenishment opportunities. We describe the purchase behavior of customers by using the multinomial logit choice model which captures the effects of prices and stock-outs on consumer choices. We compare four different pricing strategies, depending on whether or not (i) dynamic pricing is used, and (ii) personalized pricing is used. The contributions of this research are as follows:

- We find that, under optimal personalized dynamic pricing policy, the marginal value of an item is increasing in the remaining time and decreasing in its own stock level and the other product's stock level.
- We identify the conditions under which personalized pricing and dynamic pricing reinforce one another (in that the benefits from one strategy increases when the other is also in place).
- We compare how different pricing strategies are affected by changes in inventory levels and customer arrival rates.

1.3. Retail Competition under Commission Contract

In this study we consider a retailer and multiple merchants who sell through it by paying a commission fee. The retailer and merchants sell products that are substitutes. A customer chooses from all the available sellers (the retailer and merchants) according to the MNL choice model. In this setting, the merchants take advantage of the store traffic at the retailer, who in return benefits from the commissions. From the retailer's point of view, the commission contract promises additional revenue. However, the contract creates direct competition between the retailer and the merchants by displaying the merchants' products to customers. A good example of such a simultaneously competitive and cooperative relationship is the third-party marketplace practiced by Amazon. We explore the conditions under which both the retailer and the merchants benefit from such a business practice. The contributions of this research are as follows:

- We prove that there exists a unique Nash equilibrium under various settings of competition and cooperation.
- We derive the conditions under which the commission contract is attractive to

both the retailer and merchants.

- We show the conditions under which the retailer refrains from stocking inventory and simply collects commission fees from the merchants.

CHAPTER II

Dynamic Pricing of Substitutable Products with Limited Inventories under Logit Demand

2.1. Introduction

ALDO, a shoe company with more than 600 retail stores in Canada, the United Kingdom and the United States, started to offer two types of sneakers early in 2007. One model was initially priced at $29 and the other at $49. ALDO wanted to clear the inventory of the sneakers by the end of June, 2007. The cheaper model turned out to be very popular and it was on track to sell out by May, while the more expensive model was off to a slow start. A merchandising manager considered reducing the price of the more expensive model, possibly all the way down to $29. However, after running a commercially-available price optimization software, the company decided to keep the price unchanged because the software suggested that, based on historical trends, the expensive model would sell out by June even at the higher price of $49 (Bergstein, 2007). Many firms selling an assortment of substitutable, short-lived products are faced with similar pricing decisions. In this paper, we consider the dynamic pricing problem faced by such a seller who has a limited inventory of substitutable products that need to be salvaged at the end of a predetermined selling season.

Presumably, ALDO's price-optimization software uses historical sales data to predict how the sales of one sneaker depends on the prices of both types of sneakers. That is, the software must have estimated price-based substitution effects. Furthermore, in coming up with its suggestion, the software must have taken into account the demand that would spill over to the more expensive model once the cheaper version ran out in May. That is, the software must have taken into account substitutions that would occur due to stockouts. Here, we use a model that allows both price-based and stockout-based substitution. In particular, we model consumer choice using the multinomial logit model (MNL). This model captures price-based substitution effects, because the attractiveness of each product depends on the prices of all products. The customer chooses from the set of products that are available at the time of her visit. Every time a product runs out, the set of choices available to the customer gets smaller, capturing the substitution effects due to stockouts.

In our model, we focus on a seller offering only two different versions of a product. This assumption is for tractability, but there are cases where a retailer's problem is to choose the prices of only two products, as the example of ALDO suggests. Likewise, Barret (2006) describes how Duane Reade, a drugstore chain in the metropolitan New York area, adjusted the prices of 50- and 24-pill bottles of a pain reliever: The retailer found out that customers choosing the 50-pill bottle over the 24-pill bottle were less price sensitive, which resulted in an increase in the price of the 50-pill bottle. Of course, in general, a retailer carries not just two, but many versions of a given product. Nonetheless, our two-product model is sophisticated enough to capture the fundamental effects of substitution, yet tractable enough to derive a number of analytical results regarding the behavior of optimal prices.

We prove a number of structural results regarding the behavior of the marginal value of a product, the price difference between the two products and the purchase probabilities. Under the optimal pricing policy, the marginal value of an item is increasing in the remaining time and decreasing in the stock level of either product.

Furthermore, the marginal value of a product is more sensitive to its own stock level than it is to the substitute's stock level. Despite such unsurprising behavior on the part of marginal values, the optimal prices themselves are not simply increasing in the remaining time or decreasing in the total stock level. We show, however, that the optimal price difference between the two products and the optimal purchase probabilities provide intuitive gauges of the optimal behavior. Under the optimal pricing policy, if a product's inventory increases or the remaining time decreases, the overall probability of making a sale increases. Furthermore, if the stock level of product i increases, the price difference between products i and j (price of i minus price of j) decreases. In addition, if the stock level of product i increases, the conditional purchase probability of product i (the probability that a customer picks product i given that she decided to purchase) will increase and the conditional purchase probability of product j will decrease.

In Section 2.2, we position our work with respect to the literature. Section 2.3 describes the model and establishes preliminary results. The monotonicity properties for the marginal value, price difference and purchase probabilities are discussed in Section 2.4. We conclude in Section 2.5.

2.2. Literature Review

There is a large and still growing literature on the dynamic pricing of limited inventories. Gallego and van Ryzin (1994) and Bitran and Mondschein (1997) were among the first to consider the dynamic pricing of an exogenously-fixed inventory of a product. Since then, this topic has received significant attention. For a detailed review of this literature, see Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003).

While substitution is of practical importance in pricing decisions, there is relatively little work on the dynamic pricing of substitutable products. Bitran, Caldentey and Vial (2007) consider this problem using as their choice model a variation of the model proposed by Hauser and Urban (1986). In this choice model, consumers differ in terms of their budget and their utility from not purchasing, but they have identical rankings of products. Each arriving customer buys the best product she can afford as long as the product provides greater utility than her no-purchase option. They solve the deterministic version of the problem (where there is a steady stream of customer arrivals), and show that this heuristic is asymptotically optimal for the original problem. Zhang and Cooper (2005) consider a revenue management problem for multiple products under consumer choice: Given a set of substitutable products, multiple classes of customers, and exogenously-fixed prices that each class is charged for each product, what is the optimal capacity allocation across classes and over time? Zhang and Cooper (2009), on the other hand, consider the dynamic pricing of substitutable products with limited inventories. They assume a non-homogenous arrival rate and allow a general choice model for an arbitrary number of products. They develop heuristic methods to solve the pricing problem and evaluate the performance of these heuristic methods. Gallego and van Ryzin (1997) study a multi-product dynamic pricing problem, where the demand for each product is a stochastic process whose intensity depends on the prices of all products. Each unit of product requires a number of resources which have fixed inventory levels. Gallego and van Ryzin (1997) examine the performance of heuristic methods based on the optimal solution to the deterministic version, which are shown to be asymptotically optimal. Maglaras and Meissner (2006) show that the problem considered by Gallego and van Ryzin (1997) and its capacity control version (in which the prices are fixed, but the demand for a product can be turned down) can be reduced to a single-product pricing problem. This equivalence is then utilized to develop new heuristic methods. In contrast to all of these papers, we focus on a specific choice model, namely the MNL consumer choice model.

Talluri and van Ryzin (2004) examine a revenue management problem where customers choose from multiple products whose prices are exogenously fixed. One of the special cases they consider is the MNL choice model, which is the choice model we use. In their model, all products draw upon one common pool of inventory. The seller's problem is to decide which products to make available at any given time. In contrast, in our model, the seller's problem is to choose the prices of products and each product has a distinct inventory level..

Dong, Kouvelis and Tian (2008) and Akcay, Natarajan and Xu (2008) are the closest to our work in that they also use the MNL consumer choice model to study dynamic pricing of substitutable products. Dong et al. (2008) demonstrate through numerical examples that the prices of products may be non-monotonic in time or inventory levels. They explain the behavior in these numerical examples through the effects of inventory scarcity and quality differences among products. In addition, Dong et al. (2008) devise heuristic methods for deciding not only the prices, but also the initial stock levels. They identify the inventory and quality conditions that favor dynamic pricing over static pricing. In Akcay et al. (2008), the MNL choice model is one of two choice models under study. In the other choice model studied by Akcay et al. (2008), there is a clear ranking of products: If the prices of all products were the same, all customers would prefer a higher ranked product over a lower ranked product. For this choice model, Akcay et al. (2008) show that prices are monotonic in time and inventory levels. However, they observe that these monotonicity properties do not hold under the MNL choice model. They use approximate dynamic programming to find near-optimal solutions for the dynamic pricing problem under the MNL choice model. In contrast to Dong et al. (2008) and Akcay et al. (2008), we restrict our attention to a two-product problem, which allows us to establish a number of analytical results about how the optimal prices behave with respect to time and inventory levels. For example, we show that even though the optimal prices themselves may not be monotonic in inventory levels, the optimal price gaps and purchase probabilities are. In summary, our work complements those of Dong et al. (2008) and Akcay et al. (2008) by proving certain properties of the optimal prices for the two-product case.

We assume that the stock levels are exogenously fixed, instead of analyzing the problem of choosing stock levels. There is a related stream of research that analyzes joint inventory and pricing decisions for substitutable products, but assumes that prices are fixed once and for all. For example, Aydin and Porteus (2008) and Maddah and Bish (2007) consider newsvendor-type firms that must make one-shot pricing and inventory decisions for substitutable products. While they model price-based substitution, they assume that if the demand for a product exceeds the stock level, the excess demand will become lost sales, ignoring stockout-based substitutions. On the other hand, Mahajan and van Ryzin (2001) study an inventory problem for substitutable products using a very general model of dynamic substitution, but the prices are exogenously fixed in their model.

In modeling consumer choice, we follow the MNL choice model. In recent years, the MNL model has been widely used in operations literature to incorporate consumer choice into operational models. See, for example, van Ryzin and Mahajan (1999), Hopp and Xu (2005), and Cachon and Kok (2007).

2.3. Model Description

We focus here on the problem with two substitutable products. This simplification is primarily for analytical convenience. By focusing on the two-product case, we are able to prove a number of results regarding the behavior of prices with respect to time and inventory levels. We assume that the seller starts with a fixed quantity of each product in inventory and there are no replenishment opportunities during the selling season. This is in keeping with our focus on products with short life cycles.

Following the approach first used by Bitran and Mondschein (1997), we assume that the selling season is divided into T periods, each of which is short enough that at most one customer arrives in a period. Let λ denote the probability that a customer arrives in a given period. Each customer, upon arrival, observes available product(s) and price(s), and chooses what product to purchase, if any. Next, we discuss the choice model, followed by the dynamic program that captures the seller's pricing problem. Throughout the paper, we use bold letters to denote vectors. In addition, we use increasing/decreasing and positive/negative in the weak sense, unless specifically qualified as strictly increasing/decreasing or non-positive/negative.

2.3.1 Choice Model

Our choice model follows the multinomial logit (MNL) model, which has been used extensively in marketing literature to model the choice behavior of an individual or a household. (See, for example, Ben-Akiva and Lerman, 1986, for a detailed discussion of the MNL model.) Many researchers in operations management have also used the MNL model to incorporate consumer choice into inventory control (e.g., van Ryzin and Mahajan, 1999).

Let $\mathbf{u} = (u_1, u_2)$ denote the vector of average utilities for the products; i.e., u_i is the average utility a customer derives from product i. Suppose that with t periods to go until the end of the season, the seller's inventory levels are given by the vector $\mathbf{y} = (y_1, y_2)$ and the prices by the vector $\mathbf{p} = (p_1, p_2)$. Let $S(\mathbf{y})$ denote the set of in-stock products; i.e., $S(\mathbf{y}) = \{i : y_i > 0, i = 1, 2\}$. A customer arriving in period t obtains a surplus of $u_i - p_i + \epsilon_{it}$ for purchasing item $i = 1, 2$, where ϵ_{it} is a random error term that captures the heterogeneity of customers. In addition, let u_0 denote the average utility obtained by a customer who chooses not to purchase. The customer's surplus for the no-purchase option is $u_0 + \epsilon_{0t}$, where ϵ_{0t} is a random error term. In accordance with the MNL model, we assume that ϵ_{it}, $i = 0, 1, 2$, $t = 1, \ldots, T$ are independent and identically distributed Gumbel error terms with shape parameter one. One implication is that the error terms are independent across products for a given customer, and independent over time (and, hence, across customers) for a given product. Hereafter, we normalize $u_0 = 0$ without loss of generality.

Let $q_i(\mathbf{p}, S(\mathbf{y}))$ denote the probability that a customer will choose option $i = 0, 1, 2$, given the vector of prices p and the set of available products $S(\mathbf{y})$ at the time of customer's arrival. We assume that if a product is out of stock at the time a customer arrives, then the customer will choose either the remaining product or the no-purchase option, ignoring the fact that there was one more product that the retailer carried earlier, which is now out-of-stock. The MNL model stipulates that the customer will choose the option that maximizes her surplus, and a standard result of the MNL model yields the following purchase probabilities:

$$
\begin{aligned}
q_i(\mathbf{p}, S(\mathbf{y})) &= 0, \ i \notin S(\mathbf{y}), \\
q_i(\mathbf{p}, S(\mathbf{y})) &= \frac{\exp(u_i - p_i)}{1 + \sum_{i \in S(\mathbf{y})} \exp(u_i - p_i)}, \ i \in S(\mathbf{y}), && (2.1) \\
q_0(\mathbf{p}, S(\mathbf{y})) &= \frac{1}{1 + \sum_{i \in S(\mathbf{y})} \exp(u_i - p_i)}. && (2.2)
\end{aligned}
$$

Notice that the price of an out-of-stock product is irrelevant to the consumer's choice. The implicit assumption is that once a product runs out of stock, it no longer affects the choice of the consumer. This assumption is reasonable in our setting where the inventories are not replenished: the customer has no incentive to wait for the out-of-stock product to become available again. Furthermore, in some cases, the customer's knowledge of the existing products may be limited to those that are available at the time the customer visits the store. Of course, there are certain customer reactions to stock-outs that this model precludes (e.g., a customer deciding to search elsewhere for the out-of-stock product). Nonetheless, our model captures the main effect of a stock-out: If the seller runs out of a product, the consumer then has fewer options available, which increases the purchase probability for the remaining product, but reduces the overall probability that the consumer will purchase from this seller.

2.3.2 Seller's Problem

Suppose the seller has $\mathbf{y} = (y_1, y_2)$ units of inventory with t periods to go. (Our convention is to count the periods backwards. Hence, the first period of the season is labeled as period T and the last period as period 1.) Let $V_t(\mathbf{y})$ be the optimal expected revenue of the seller, given the vector of inventory levels $\mathbf{y}$ at time t. Define $\mathbf{e_i}, i = 1, 2$, as the unit vector whose i^{th} component is 1 while the other is zero. In addition, define $\mathbf{e} := \mathbf{e_1} + \mathbf{e_2}$. The seller's problem is given by the following optimality equations: For $t > 0$ and $\mathbf{y} \neq \mathbf{0}$,

$$\begin{aligned} V_t(\mathbf{y}) &= \max_{\mathbf{p}} \left\{ \sum_{i \in S(\mathbf{y})} \lambda q_i(\mathbf{p}, S(\mathbf{y}))(p_i + V_{t-1}(\mathbf{y} - \mathbf{e_i})) + [1 - \lambda + \lambda q_0(\mathbf{p}, S(\mathbf{y}))] V_{t-1}(\mathbf{y}) \right\} \\ &= V_{t-1}(\mathbf{y}) + \max_{\mathbf{p}} \left\{ \sum_{i \in S(\mathbf{y})} \lambda q_i(\mathbf{p}, S(\mathbf{y}))(p_i + V_{t-1}(\mathbf{y} - \mathbf{e_i}) - V_{t-1}(\mathbf{y})) \right\} \end{aligned} \tag{2.3}$$

where $V_t(\mathbf{0}) = 0$, for any $t \in \{1, \ldots, T\}$ and $V_0(\mathbf{y}) = 0$, for any $\mathbf{y} \geq 0$. It should be noted that even though we are stating the seller's problem as choosing a vector of prices, $\mathbf{p}$, once a product runs out of stock, its price becomes irrelevant under our choice model.

Let us define the marginal value of product i, $\Delta_{it}(\mathbf{y})$ for $i \in S(\mathbf{y})$, as follows:

$$\Delta_{it}(\mathbf{y}) = V_{t-1}(\mathbf{y}) - V_{t-1}(\mathbf{y} - \mathbf{e_i}), \text{if } i \in S_t(\mathbf{y}),\ t \in \{1, \ldots, T\}. \tag{2.4}$$

Here, $\Delta_{it}(\mathbf{y})$ for $i \in S(\mathbf{y})$ is the marginal value of product i, given the vector of inventory levels $\mathbf{y} = (y_1, y_2)$ and the remaining time t (i.e., this is the potential future revenue that the seller is foregoing by selling one unit of product i in the current period instead of carrying it into the next period). Note that the marginal value of product i is not defined when product i is out of stock, since the marginal value of an out-of stock product is irrelevant to the seller's optimization problem. Let us define the vector $\mathbf{\Delta}_t(\mathbf{y})$ as the vector of Δ_{it}'s for $i \in S(y)$. Note our slight abuse of notation here: The vector $\mathbf{\Delta}_t(\mathbf{y})$ may have one or two components depending on whether one or both products are available at time t.

Using (2.4), the seller's optimization problem in period t, given by (2.3), reduces to the following: For $t > 0$ and $y \neq \mathbf{0}$,

$$V_t(\mathbf{y}) = V_{t-1}(\mathbf{y}) + \lambda \max_{\mathbf{p}} \left\{ \sum_{i \in S(\mathbf{y})} q_i(\mathbf{p}, S(\mathbf{y}))(p_i - \Delta_{it}(\mathbf{y})) \right\}. \tag{2.5}$$

In preparation for the next result, consider now a make-to-order seller offering assortment S and suppose that the seller's unit cost of procuring product i is c_i. Furthermore, suppose that the size of the consumer population is λ and the customers are choosing from the assortment according to the MNL model; i.e., a fraction $q_i(\mathbf{p}, S)$ of the population buys product $i \in S$, given the vector of prices, $\mathbf{p}$. It is well known that the optimal prices for this problem satisfy the *equal margin property*; i.e., $p_i - c_i$ is the same for all $i \in S$. (See, for example, Besanko, Gupta and Jain, 1998 or Anderson and de Palma, 1992, who prove the same property in the context of the nested logit choice model.) Now, observe that this problem is the same as problem (2.5), with assortment S replaced by assortment $S(\mathbf{y})$ and c_i replaced by $\Delta_{it}(\mathbf{y})$ in our model. Hence, the optimal prices in our model satisfy the *equal margin property*, where the marginal value of product i mimics as the unit cost of product i. This observation is formalized in the following lemma, which helps to simplify the optimization problem of the seller.

Lemma II.1. *With t periods to go until the end of the season, suppose the stock level vector $\mathbf{y} > 0$, that is, both products are available. Let $\mathbf{p}_t^*(\mathbf{y})$ denote the vector of* *optimal prices. Then:*

$$p_{1t}^*(\mathbf{y}) - \Delta_{1t}(\mathbf{y}) = p_{2t}^*(\mathbf{y}) - \Delta_{2t}(\mathbf{y}).$$

Lemma II.1 states that the seller chooses the prices so that each product's price is the marginal value plus a margin, where the margin is the same across all available products. Dong et al. (2008) and Akcay et al. (2008) also observe this equal margin property in the context of dynamic pricing of substitutable products under the MNL choice model. Given the equal margin property, the optimization problem in (2.5) can be rewritten as: For $t > 0$ and $\mathbf{y} \neq \mathbf{0}$,

$$\begin{aligned} V_t(\mathbf{y}) \quad &= \quad V_{t-1}(\mathbf{y}) + \lambda \max_m \left\{ m \cdot \sum_{i \in S_t(\mathbf{y})} q_i(\mathbf{p}, S(\mathbf{y})) \right\} \\ &\text{s.t.} \quad p_i = \Delta_{it}(\mathbf{y}) + m \text{ for } i \in S(\mathbf{y}) \end{aligned} \tag{2.6}$$

Note that the problem statement in (2.6) is valid even when one of the products is out of stock. Let $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$ denote the optimal solution to (2.6). Hereafter, we refer to this premium as the *optimal margin*. As the notation suggests, the stock level vector $\mathbf{y}$ influences the optimal margin, $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$, only through its influence on the set of available products, $S(\mathbf{y})$, and the marginal values of in-stock products, $\mathbf{\Delta}_t(\mathbf{y})$. Likewise, the remaining time t influences the optimal margin only through its effect on the marginal values.

The following lemma takes advantage of the MNL choice model to further simplify the seller's optimality equations. The same simplification has been used by Dong et al. (2008) and Akcay et al. (2008) as well.

Lemma II.2. *Given inventory vector* $\mathbf{y}$ *with* t *periods remaining in the season, the seller's optimal margin,* $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$*, the corresponding optimal price vector,*

$\mathbf{p}_t^*(\mathbf{y})$, *and the no-purchase probability,* $q_0(\mathbf{p}_t^*(\mathbf{y}), S(\mathbf{y}))$, *satisfy the following:*

$$m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = \frac{1}{q_0(\mathbf{p}_t^*(\mathbf{y}), S(\mathbf{y}))}. \tag{2.7}$$

Furthermore, given the inventory $\mathbf{y}$ *and the remaining time* t*, the seller's optimal revenue-to-go,* $V_t(\mathbf{y})$*, and the optimal margin,* $m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$*, are related as follows: For* $t > 0$ *and* $\mathbf{y} \neq \mathbf{0}$*,*

$$V_t(\mathbf{y}) = V_{t-1}(\mathbf{y}) + \lambda(m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y})) - 1). \tag{2.8}$$

As a convention, we assume that $m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = 1$ when $\mathbf{y} = \mathbf{0}$. Notice that this convention is consistent with the relationships stated in Lemma II.2 and it amounts to turning off the demand when there are no more products left to sell: Using this convention, Lemma II.2 implies that the no-purchase probability, $q_0(\mathbf{p}_t^*(\mathbf{y}), S(\mathbf{y}))$, is one when $\mathbf{y} = \mathbf{0}$.

2.4. Analysis

In the dynamic pricing of substitutable products, one would like to know not only how the price of a product depends on time (*time effect*) and the product's own stock level (*own-stock effect*), but also how the price changes with respect to the stock level of the other product (*cross-stock effect*). As Figure 2.4 shows, the cross-stock effect is not always unidirectional: a product's price may increase or decrease in the stock level of the other product. In Figure 2.4, panel (a) shows the optimal prices as a function of the stock level of product 1, when $y_2 = 4$ and $t = 10$. Panel (b) shows the optimal prices as a function of the remaining time when $y_1 = 1$ and $y_2 = 3$. It is rather surprising that even though the products are substitutable, an increase in the stock level of one product may drive up the other product's price. Furthermore, the figure shows that the time effect is not unidirectional either. The price of a product may increase as the end of the season is approaching, which is in sharp contrast to single-product dynamic pricing problems, where the price will be lower if there is less time until the end of the selling season. Such non-monotonicities that distinguish the substitutable-product problem from the single-product problem have been pointed out by Dong et al. (2008) and Akcay et al. (2008) as well. It appears from the

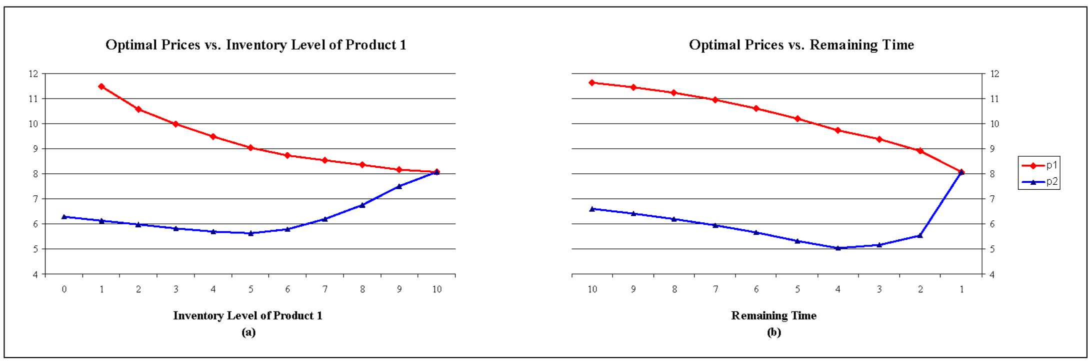

Figure 2.1: The price of product 2 is non-monotonic in the stock level of product 1 and the remaining time. In this example, $u_1 = 10$, $u_2 = 6$, $\lambda = 1$.

figure that it may be challenging to draw generalizations regarding the behavior of optimal prices. Nonetheless, we show in this section that alternative gauges of the seller's behavior, such as the price difference between the two products and purchase probabilities, exhibit more intuitive behavior with respect to time and stock levels.

2.4.1 Properties of the Marginal Values and Optimal Prices

As we observed earlier, the optimal price of a product is the marginal value of the product plus an optimal margin. Hence, in order to understand the behavior of prices, it is important to understand the behavior of the marginal values. The following proposition shows how the marginal value of an item depends on time and inventory levels.

Proposition II.3. *The marginal value of a product decreases if:*

(*a*) *its stock level increases, i.e.,* $\Delta_{it}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+2\mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1,\ldots,T\}$, *or,*

(*b*) *the stock level of the other product increases, i.e.,* $\Delta_{it}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j})$, *for* $\mathbf{y} \geq 0$, $t \in \{1,\ldots,T\}$ *and* $i \neq j$, *or,*

(*c*) *its own stock level increases by one unit and the other product's stock level decreases by one unit, i.e.,* $\Delta_{it}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j}) \geq \Delta_{it}(\mathbf{y}+2\mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1,\ldots,T\}$ *and* $i \neq j$, *or,*

(*d*) *the remaining time decreases, i.e.,* $\Delta_{i,t+1}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1,\ldots,T\}$.

The own-stock effect stated in part (a) and the time effect stated in part (d) are common properties in most dynamic pricing models: If there is less time until the end of the season or more inventory of the product, the seller is foregoing a smaller benefit by selling the product in the current period instead of carrying it into the next period. As stated in part (b), the marginal value of a product is larger if there is less inventory of the other product. This is not surprising since the products are substitutable. Akcay et al. (2008) prove properties (a), (b) and (d) for a different choice model, in which all customers would prefer a higher-ranked product to a lower-ranked product if their prices were the same. Note that, in the MNL choice model, no such ranking exists: Even when the prices of all products are the same, each and every product will be the first choice of a fraction of the population. Our results show that the marginal value has the same properties under the MNL choice model as well.

Proposition II.3(c) compares own-stock and cross-stock effects, and states that the own-stock effect dominates the cross-stock effect. This is not a trivial statement. Unlike parts (a), (b) and (d) of Proposition II.3, which are likely to hold for most dynamic pricing problems with substitutable products, part (c) is not necessarily an automatic consequence of offering substitutable products. Suppose the customer choice was fundamentally biased against one of the two products, say product 1.

Then most customers would be choosing product 2 absent a hugely discounted price for product 1. Under such a scenario, a decrease in product 1's inventory will not have a large effect on the marginal value of product 1. However, a decrease in product 2's inventory may substantially increase the chances that product 2 will eventually run out of stock, thus causing a significant improvement in the odds of a unit of product 1 being sold eventually. Hence, a decrease in product 2's inventory may cause a significant increase in product 1's marginal value. In short, when the choice model is biased against product 1, the marginal value of product 1 may be more sensitive to the inventory of product 2 than it is to the inventory of product 1. Indeed, such an example can be obtained by modifying our original choice model as follows:

Example II.4. *Suppose the purchase probability of product 1 is the original probability function, given by* (2.1)*, now weighted by* $\frac{1}{1+p_1^{3.5}}$*, whereas the choice probability of product 2 remains the same. Notice that this new model introduces a bias against product 1. Let* $u_1 = 3.2$*,* $u_2 = 1$*,* $\lambda = 1$*,* $t = 5$*. Given these parameter values,* $\Delta_{15}(2,2) = 0.038 < \Delta_{15}(3,1) = 0.073$*, yielding a counter-example to Proposition II.3(c).*

In summary, Proposition II.3(c) does not necessarily hold when the choice model is biased against one of the products, and it is the somewhat 'equitable' nature of the logit choice model that drives the result.

Proposition II.3(a) implies that the seller's revenue function $V_t(\mathbf{y})$ is separately concave in the inventory level of each product, which would make it easier to find the optimal inventory levels if those were decision variables. Consider, for example, a seller who must decide how many units of each product to order at the beginning of the selling season. Assuming linear (or, convex) ordering costs, it follows from Proposition II.3(a) that the seller's expected profit is separately concave in each product's inventory level. This suggests that the optimal stock levels could be found by rather simple search algorithms. It is interesting to note that this result is not generally true in the presence of stockout-based substitution. For example, in the context of a dynamic substitution model with fixed prices, Mahajan and van Ryzin (2001) construct counterexamples, which show that the seller's profit is not necessarily separately concave in the inventory levels.

Note that Proposition II.3(b) implies that the seller's revenue function, $V_t(\mathbf{y})$, is submodular in the stock levels, y_1 and y_2. On a technical note, this highlights the difficulty in proving the proposition. The proof of the proposition proceeds by induction. Supposing that the optimal expected revenue with t periods to go, $V_t(\mathbf{y})$, is submodular in stock levels, one needs to show that this property is preserved in period $t+1$, after the seller picks revenue-maximizing prices in period $t+1$. In general, submodularity is not preserved under maximization. However, in our model, the submodularity of the revenue function is preserved under maximization, which is what we prove to complete the induction.

As shown in Figure 2.4, the price of a product is not necessarily monotonic in the stock level of the other product or the remaining time. This non-monotonicity occurs despite the fact that a product's marginal value is monotonic in both of those, as established in Proposition II.3. In other words, there are certain properties of the marginal value that are not inherited by the optimal prices. Nevertheless, the next proposition verifies that there is one property of the marginal value that is inherited by the optimal price: As the marginal value of a product decreases in its own stock level, so does the optimal price of the product.

Proposition II.5. *The optimal price of a product is decreasing in its own stock level; i.e.,* $p^*_{it}(\mathbf{y}) \geq p^*_{it}(\mathbf{y}+\mathbf{e_i})$ *for* $y_i \geq 1$.

The same property has been observed numerically by Dong et al. (2008) and Akcay et al. (2008), and the proposition is an analytical complement to these observations. In the next section, we establish a number of results about how the optimal prices change relative to each other as time or stock levels change, and how the purchase probabilities are affected by these changes.

2.4.2 Purchase Probabilities and the Optimal Price Difference

In a dynamic pricing problem with a single product and stationary parameters (e.g., arrival rate, λ), under the optimal pricing policy, the probability of making a sale increases if there is less time until the end of the season or more inventory of the product. As we discuss next, the same behavior holds true in our model, the non-monotonicity of the prices notwithstanding. When there are t periods to go with inventory vector $\mathbf{y}$, let $q^*_{0t}(\mathbf{y})$ denote the *optimal no-purchase probability*, that is, the probability that a customer arriving in period t will not purchase any product from the seller, given the seller is charging the optimal prices. Hence, *the optimal probability of making a sale* in period t with inventory vector $\mathbf{y}$ is $1 - q^*_{0t}(\mathbf{y})$.

Proposition II.6. *The optimal probability of making a sale,* $1 - q^*_{0t}(\mathbf{y})$*, increases if:* (a) *the stock level of a product increases, or* (b) *the remaining time decreases.*

Figure 2.4.2 shows, for the same example depicted in Figure 2.4 (b), how the optimal probability of making a sale changes with respect to time. As the end of the season is approaching, the price of product 2 increases as shown in Figure 2.4(b), but the optimal probability of making a sale increases nonetheless, because the reduction in the price of product 1 more than compensates for the increase in the price of product 2. In the example of Figure 2.4(b), as the end of the season is approaching, the seller is promoting product 1 at the expense of product 2. We next examine how the stock levels and remaining time affect the seller's relative pricing of the products. To that end, consider *the price difference*; i.e., the difference between the optimal prices of the two products. The next proposition shows how this price difference depends on the stock levels.

Proposition II.7. *The difference between the prices of products* i *and* j*,* $p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y})$*, decreases if:* (a) *the stock level of product* i *increases; i.e.,* $p^*_{it}(\mathbf{y}+\mathbf{e_i}) - p^*_{jt}(\mathbf{y}+$

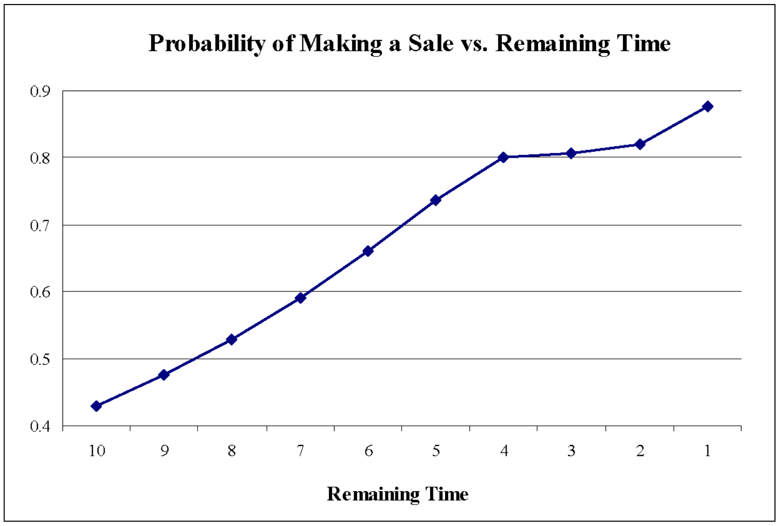

Figure 2.2: The optimal probability of making a sale increases as the end of the season is approaching. In this example, $y_1 = 1$, $y_2 = 3$, $u_1 = 10$, $u_2 = 6$, $\lambda = 1$.

$\mathbf{e_i}) \leq p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y})$ *for* $y_i, y_j \geq 1$, *or (b) the stock level of product* j *decreases; i.e.,* $p^*_{it}(\mathbf{y} + \mathbf{e_j}) - p^*_{jt}(\mathbf{y} + \mathbf{e_j}) \geq p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y})$ *for* $y_i, y_j \geq 1$.

Given a fixed number of periods to go, if the stock level of product i increases or the stock level of product j decreases, the risk of leftover inventory for product i grows relative to the same risk for product j. Hence, the seller decreases the difference between the prices of products i and j, in an effort to increase the attractiveness of product i relative to product j. Figure 2.4.2 illustrates this behavior. In this example, when the inventory of product 1 is low, it is priced well above product 2, because product 1 has higher average utility in the first place ($u_1 = 10$ versus $u_2 = 6$). However, as the inventory level of product 1 increases, the price difference shrinks and, eventually, the price of product 1 falls below that of product 2 when there are ten units of product 1 in inventory. While the behavior of the price difference described in Proposition II.7 is intuitive, it is worthwhile to note that this behavior is not trivially true for all choice models. Consider, for instance, the alternative model discussed in Example II.4, under which the choice model is biased against product 1. In that example, the difference between the prices of products 1 and 2 is -1.075 when $y_1 = 2$

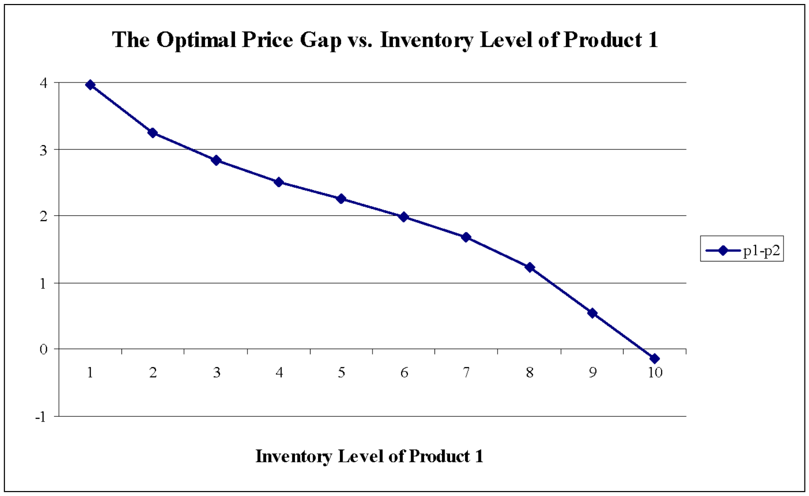

Figure 2.3: The optimal difference between the prices of products 1 and 2 decreases as the inventory of product 1 increases. In this example, $y_2 = 1$, $t = 10$, $u_1 = 10$, $u_2 = 6$, $\lambda = 1$.

and $y_2 = 1$ with five periods to go, and grows to -0.990 when y_1 increases to 3, thus yielding a counter-example to the result described in Proposition II.7.

In the example of Figure 2.4.2, as the inventory level of product 1 is increasing, the reduction in the price difference is serving to make product 1 more attractive. Hence, one would expect that a customer is becoming more likely to choose product 1 over product 2. We next formalize this intuition. Consider a seller who is charging the optimal prices with t periods to go and inventory vector $\mathbf{y}$. Let $q^*_{it}(\mathbf{y})$ denote the resulting optimal purchase probability for product i, and $\overline{q}^*_{it}(\mathbf{y})$ the probability that a customer arriving in this period will choose product i, given that the customer chooses to purchase:

$$\overline{q}^*_{it}(\mathbf{y}) = \frac{q^*_{it}(\mathbf{y})}{1 - q^*_{0t}(\mathbf{y})}$$

The following proposition states the effect of inventory levels on the conditional purchase probabilities, $\overline{q}^*_{it}(\mathbf{y})$, $i = 1, 2$.

Proposition II.8. *Under the optimal pricing policy, if the stock level of product i increases, then the conditional purchase probability of product i, $\overline{q}^*_{it}(\mathbf{y})$, increases and* *the conditional purchase probability of product j, $\overline{q}^*_{jt}(\mathbf{y})$, decreases.*

Figure 2.4.2 is an illustration of the conditional purchase probabilities as a function of the stock level of product 1.. In connection with Proposition II.8, it is important

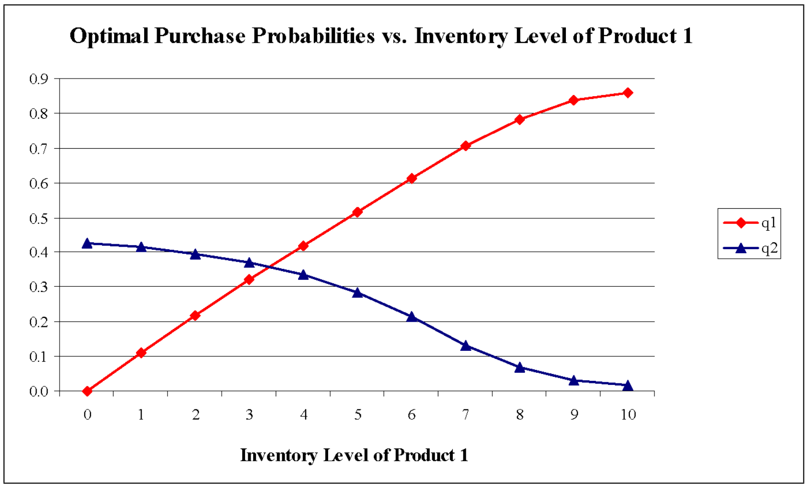

Figure 2.4: As the inventory level of product 1 increases, the purchase probability of product 1 increases and that of product 2 decreases. In this example, $y_2 = 4$, $t = 10$, $u_1 = 10$, $u_2 = 6$, $\lambda = 1$.

to note that while the conditional purchase probability, $\overline{q}^*_{jt}(\mathbf{y})$, decreases in the stock level of product i, the same is not necessarily true for the unconditional purchase probability, $q^*_{jt}(\mathbf{y})$. When the stock level of product i increases, the prices change in such a way that the overall probability of making a sale increases (as was shown in Proposition II.6). This increase in the overall probability of making a sale may translate to an increase in the unconditional purchase probability of product j. Hence, an increase in the stock level of one product may actually increase the chances of sale for the other product as well (even though a customer's chances of picking the other product decreases, given that the customer purchases). The following is a numerical example that demonstrates such behavior.

Example II.9. *Suppose that $u_1 = 2$, $u_2 = 1$, $\lambda = 1$ and $t = 3$. Given these parameter values, $q^*_{23}(4, 1) = 0.113$ and $q^*_{23}(5, 1) = 0.145$. Hence, increasing the stock level of* *product 1 from 4 to 5, results in an increase in the probability of purchase for product 2.*

The preceding discussion shows how the optimal price difference and purchase probabilities change as a function of a product's stock level. One may wonder if there is a similar structure to the effect of time on the optimal price difference and purchase probabilities. Consider, for example, Figure 2.4(b). In that example, where product 1 is the product with higher average utility (the *premium* product), the difference between the price of product 1 and the price of product 2 (the *basic* product) gets steadily smaller as the end of the season is approaching. Hence, one may conjecture that, given inventory levels, the seller will charge a larger price for the premium product when there are many periods to go, but the price difference will steadily shrink as the end of the season is approaching. It turns out, however, that the behavior of the optimal price difference may be more complicated than that. Consider the example shown in Figure 2.4.2. In this example, product 1 is the premium product but it has

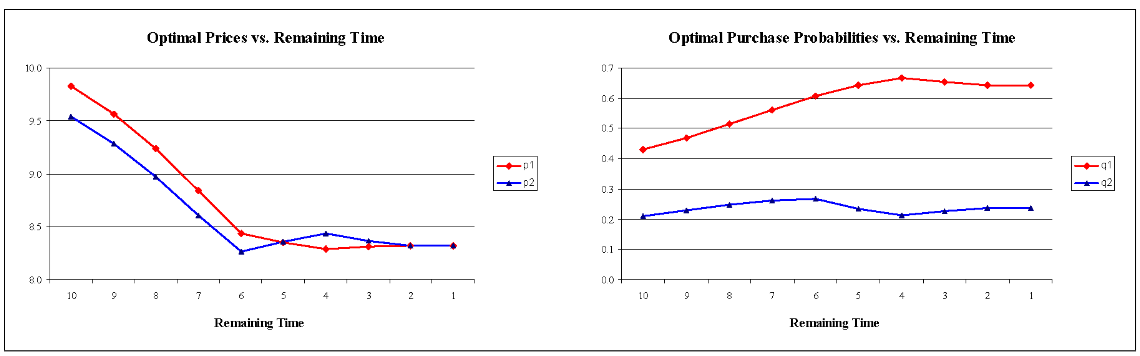

Figure 2.5: The price difference is not monotonically decreasing in time. In this example, $y_1 = 4$, $y_2 = 2$, $u_1 = 10$, $u_2 = 9$, $\lambda = 1$.

larger inventory. Product 2 is the basic one since $u_2 < u_1$. With ten periods to go, the seller charges a higher price for product 1. From period ten until period six, the price difference between products 1 and 2 keeps on decreasing (eventually becoming negative). However, the behavior of the price difference changes once period four is over. Beyond period four, the price difference between products 1 and 2 starts to increase once again, eventually becoming zero once we hit the last period.

The price difference shows this particular type of non-monotonicity because, in this example, the basic product has less inventory than the premium product. To explain this behavior, we first note that the price difference between the two products is equal to the difference between the marginal values (because the price of each product is the marginal value plus a margin that is the same across products). Now, when there are enough many periods to go, the premium product's marginal value starts out higher than that of the basic product. Hence, the price difference is positive. As the number of remaining periods decreases, the difference between the marginal values shrinks. In fact, once the number of remaining periods is at or below the inventory level of the premium product, but above the inventory level of the basic product (for example, when there are three or four remaining periods in Figure 2.4.2), the marginal value of the premium product becomes zero, while the basic product still has positive marginal value. When this happens, the price difference becomes negative. However, as the number of remaining periods decreases even further, the basic product's marginal value also decreases and eventually falls to zero, thus causing the price difference to increase from negative to zero. All in all, the price difference first shrinks (eventually becoming negative) and increases again to become zero in the last period.

2.5. Conclusion

We have considered the dynamic pricing problem of a seller who is offering two substitutable products with exogenously fixed inventory levels, over a predetermined, finite selling horizon. We show that, under the optimal pricing policy, the marginal value of an item is increasing in the remaining time and decreasing in the stock level of either product. Despite such unsurprising behavior on the part of marginal values, the optimal prices themselves are not simply increasing in the remaining time or decreasing in the total stock level. We show, however, that the optimal price difference between the two products and the optimal purchase probabilities behave in predictable ways with respect to stock levels. Under the optimal pricing policy, if the stock level of product i increases, the price difference between products i and j (price of i minus price of j) decreases. Furthermore, the purchase probability of product i will increase and the purchase probability of product j will decrease. As for the effect of time on the optimal price difference and optimal purchase probabilities, we illustrate the range of complicated behavior that may arise.

We allow the seller to increase the price during the season. One extension would be to consider a seller who can use markdowns, but can never increase a price. In such a case, interesting questions arise about the assortment. When only markdowns are allowed, there may arise cases where the seller finds it optimal to withdraw a product from the assortment. This would never happen if the seller has the ability to charge whatever price it wishes, because an additional product allows the seller to exercise price discrimination. However, if the seller's pricing flexibility is limited to markdowns only, price discrimination may lose its appeal and the seller may be better off by removing a cheap alternative that cannibalizes the demand of a more profitable product. This is likely to occur especially when the total remaining inventory is large relative to the remaining time.

We assume that the inventory levels of the products are exogenously fixed at the beginning. However, our results on the marginal value (Proposition II.3) may help characterize the optimal ordering policy in a multiple-period inventory and pricing problem, where our selling season is embedded as one period. In such a problem, our results suggest that the optimal ordering policy may be of the following type: In each period, provided that the starting inventory of product j is below a threshold, order enough of product i to bring its inventory to a base-stock level, which itself depends on the starting inventory of product j.

We focused in this paper on a two-product problem. In a numerical study, we observed that Proposition 1 holds for the case with $n > 2$ products. Unfortunately, however, our proofs do not readily lend themselves to the analysis of the n-product problem. This difficulty calls for a different approach to prove the n-product version of the results.

Acknowledgments

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CHAPTER III

Personalized Dynamic Pricing of Substitutable Products under Logit Demand

3.1. Introduction and Related Literature

Customers who visit a store may show different behavioral patterns. One customer may visit a store to look for a specific item while others may visit the store with the intent of making a choice after seeing the assortment. These differences can be attributed to personal characteristics such as variety-seeking tendency, impulsiveness, brand consciousness and so forth. Since such behavioral differences often correspond to differences in willingness to pay for the same product, firms have been making efforts to learn about individual customers. A recent survey from the Federal Trade Commission(FTC) shows that 99% of on-line companies collect personal information from the individuals visiting their websites(Seligman and Tylor, 2000). Access to such information can lead to higher profit since sellers may offer deals and prices that are tailored to specific customer groups. Annenberg Center at the University of Pennsylvania recently reported on the practices of online price discrimination. For example, an online photography service company is reported to have been charging different prices for the same service, depending on the shoppers' browsing histories. Although price discrimination is neither new nor limited to online retailers (for example, AT&T

once offered discounted rates for long-distance service to price-sensitive groups of customers), technological improvements led to a recent surge in the number of firms that experiment with innovative pricing techniques. For example, it was reported that Coca-Cola tested vending machines, which adjust the prices of soft drinks according to the outside temperature(Fracassini, 2000).

Even though price discrimination has been widely used by corporations, customers are not aware of such practices. According to the Annenberg study, 64% of American adults who have recently used the internet do not know that it is legal to charge different prices for the same item to different customers at the same time of the day. It is therefore not too surprising that consumers feel the need for legal protection against price discrimination. In 1996, Victoria's Secret was sued for distributing different versions of a catalog that offered different prices to different customer groups for the same items(Weiss and Mehrotra, 2001). Even though such price discrimination strategies may be legal, perceptions of uneven pricing can easily lead to negative consequences for the sellers(Campbell, 1999).

Nonetheless, from an economic point of view, price discrimination is often regarded as desirable, as long as it increases the efficiency of the economy. For example, companies may want to retain loyal customers by rewarding them with lower prices. Also, by identifying customers who are more demanding than others in terms of customer service, companies can charge such customers higher prices in return for better service options.

In this study, we consider two-layered price differentiation by adding consumer segments to dynamic pricing. We consider a firm that sells limited inventories of two substitutable products within an exogenously fixed sales horizon. We assume that the consumer population comprises of three segments. Segment-1 and segment-2 customers visit the seller with the intention of purchasing products 1 and 2, respectively. If their desired product is not available, they do not consider the alternative product. In contrast, segment-3 customers choose from all available products. We assume that the seller is able to observe the segment affiliation of each arriving customer. Consequently, the retailer can charge prices that depend on the customer's segment affiliation. We refer to this practice as personalized pricing. Under the presence of different customer types, we show the analytical results in Chapter II continue to hold. For example, own-stock effect and cross-stock effect as well as time effect on marginal value are preserved under optimal pricing policy of the new model.

We extend our scope to compare different pricing strategies. In particular, in addition to personalized dynamic pricing, we consider non-personalized dynamic pricing, personalized static pricing and non-personalized static pricing. We conduct a numerical study to analyze how the profit changes as the seller switches from one strategy to another. We find that the inventory level has a substantial effect on the profit gaps between these strategies.

3.2. Model Description

We keep the structure of the dynamic programming model in Chapter II. We modify the model to include three types of customer segments: segment-1, segment-2 and segment-3 customers. Segment-1 customers choose between product 1 and the no-purchase option. If product 1 is not available, segment-1 customers leave the seller without making a purchase (they do not consider product 2). In other words, segment-1 customers are dedicated to product 1 and behave as if the seller has a single item, which is product 1. Segment-2 customers can be defined similarly as dedicated to product 2. On the other hand, segment-3 customers are willing to pick from the assortment available at the time of their visit; if both products 1 and 2 are available, a segment-3 customer can either purchase product 1, product 2 or no product. We use $\lambda^j, j = 1, 2, 3$ to denote the probability that a segment-j customer arrives in a given period.

3.2.1 Choice Model with Customer Segments

In our personalized dynamic pricing model, we assume the seller is able to price discriminate among the customers; the seller can charge different prices to different segments for the same product. For example, segment-1 and segment-3 customers can be charged different prices for product 1. Similarly, different prices can be charged to segment-2 and segment-3 customers for product 2. Define the price of product i for segment-j customers as p_i^j, $i = 1, 2, j = 1, 2, 3$. Here, we use the convention that the price of a product which does not affect the choice of customers is $p_{null} = \infty$. For example, $p_2^1 = p_{null}$ since the price of product 2 does not affect the choice of segment-1 customers. Similarly, if product 1 is out of stock then $p_1^3 = p_{null}$ since unavailable product does not affect the choice of type 3 customers. Therefore, the price vector for segment-j customers ($j = 1, 2$) can be defined as $\mathbf{p}^j = (p_1^j, p_2^j)$ where $p_i^j = p_{null}$ if $i \neq j$ or if $i \notin S(\mathbf{y})$, $i = 1, 2$. The price vector for segment-3 customers is $\mathbf{p}^3 = (p_1^3, p_2^3)$ where $p_i^3 = p_{null}$ if $i \notin S(\mathbf{y})$, $i = 1, 2$. Let $q_i^j(\mathbf{p}^j, S(\mathbf{y}))$ be the purchase probability of product i given an arrival of segment-j customer, $i = 0, 1, 2, j = 1, 2, 3$, where the case of $i = 0$ is when the customer leaves the seller without making a purchase. We use the multinomial logit (MNL) choice model to consider price based substitution and stock-out based substitution. Under the MNL model, the probability that a segment-j customer will choose to buy product i is:

$$q_i^j(\mathbf{p}^j, S(\mathbf{y})) = \frac{\exp(u_i - p_i^j)}{1 + \sum_{i \in S(\mathbf{y})} \exp(u_i - p_i^j)}, \tag{3.1}$$

$$q_0^j(\mathbf{p}^j, S(\mathbf{y})) = \frac{1}{1 + \sum_{i \in S(\mathbf{y})} \exp(u_i - p_i^j)}. \tag{3.2}$$

3.2.2 Seller's Problem

The seller's expected revenue to go in period t can be expressed as the following dynamic programming equation:

$$\begin{aligned} V_t^{PD}(\mathbf{y}) &= \max_{\mathbf{p^1},\mathbf{p^2},\mathbf{p^3}} \sum_{j\in\{1,2,3\}} \sum_{i\in S(\mathbf{y})} \left\{\lambda^j q_i^j \left(\mathbf{p}^j, S(\mathbf{y})\right) \left[p_i^j + V_{t-1}^{PD}(\mathbf{y}-\mathbf{e_i})\right]\right. \\ &\quad \left. +\lambda^j q_0^j(\mathbf{p}^j) V_{t-1}^{PD}(\mathbf{y})\right\} + (1-\lambda^1-\lambda^2-\lambda^3)V_{t-1}^{PD}(\mathbf{y}) \end{aligned}$$

where $V_t^{PD}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{PD}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$. We use the superscript of PD to specify personalized dynamic pricing. We simplify the equation by using the marginal value of a product as follows:

$$V_t^{PD}(\mathbf{y}) = V_{t-1}^{PD}(\mathbf{y}) + \max_{\mathbf{p^1},\mathbf{p^2},\mathbf{p^3}} \sum_{j\in\{1,2,3\}} \sum_{i\in S(\mathbf{y})} \lambda^j q_i^j \left(\mathbf{p}^j, S(\mathbf{y})\right) \left[p_i^j - \Delta_{it}^{PD}(\mathbf{y})\right] \quad (3.3)$$

where $\Delta_{it}^{PD}(\mathbf{y}) = V_t^{PD}(\mathbf{y}) - V_t^{PD}(\mathbf{y}-\mathbf{e_i}), i \in S(\mathbf{y})$ is the marginal value of product $i = 1, 2$, given the vector of inventory levels $\mathbf{y} = (y_1, y_2)$ and the remaining time t. Using the result in Lemma II.1, we get the following:

$$\begin{aligned} V_t^{PD}(\mathbf{y}) &= V_{t-1}^{PD}(\mathbf{y}) + \sum_{j\in\{1,2,3\}} \sum_{i\in S_t(\mathbf{y})} \max_{m^j} \lambda^j m^j q_i^j(\mathbf{p}^j, S(\mathbf{y})) \quad (3.4) \\ &\text{s.t.} \quad p_i^j = \Delta_{it}^{PD}(\mathbf{y}) + m^j \text{ for } i \in S(\mathbf{y}), j = 1, 2, 3 \end{aligned}$$

where $V_t^{PD}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{PD}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$. (3.4) is valid even when one of the products is out of stock. Let $m^{j*}(\mathbf{\Delta}_t^{PD}(\mathbf{y}), S(\mathbf{y}))$ denote the optimal solution to (3.4), $j = 1, 2, 3$. Hereafter, we refer to $m^{j*}(\mathbf{\Delta}_t^{PD}(\mathbf{y}), S(\mathbf{y}))$ as the *optimal margin* for segment-j customers. As the notation suggests, the stock level vector $\mathbf{y}$ influences the optimal margins, $m^{j*}(\mathbf{\Delta}_t^{PD}(\mathbf{y}), S(\mathbf{y}))$, only through its influence on the set of available products, $S(\mathbf{y})$, and the marginal values of in-stock products, $\mathbf{\Delta}_t^{PD}(\mathbf{y})$. Likewise, the remaining time t influences the optimal margin only through its effect on the marginal values. We can apply the result in Lemma II.2 for segment-1, segment-2 and segment-3 customers to obtain the following form of dynamic programming equation. The seller's optimal revenue-to-go, $V_t^{PD}(\mathbf{y})$, and the optimal margins, $m^{j*}(\mathbf{\Delta}_t^{PD}(\mathbf{y}), S(\mathbf{y})), j = 1, 2, 3$, are related as follows: For $t > 0$ and $\mathbf{y} \neq \mathbf{0}$,

$$V_t^{PD}(\mathbf{y}) = V_{t-1}^{PD}(\mathbf{y}) + \sum_{j\in\{1,2,3\}} \sum_{i\in S(\mathbf{y})} \lambda^j \left(m^{j*}(\mathbf{\Delta}_t^{PD}(\mathbf{y}), S(\mathbf{y})) - 1 \right). \tag{3.5}$$

where $V_t^{ND}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{ND}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$.

3.2.3 Properties of the Marginal Values and Optimal Prices

As we observed earlier, the optimal price of a product is the marginal value of the product plus an optimal margin. Hence, in order to understand the behavior of prices, it is important to understand the behavior of the marginal values. The following proposition shows how the marginal value of an item depends on time and inventory levels.

Proposition III.1. *The marginal value of a product decreases if:*

(a) its stock level increases, i.e., $\Delta_{it}^{PD}(\mathbf{y} + \mathbf{e_i}) \geq \Delta_{it}^{PD}(\mathbf{y} + 2\mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1, \ldots, T\}$, *or,*

(b) the stock level of the other product increases, i.e., $\Delta_{it}^{PD}(\mathbf{y} + \mathbf{e_i}) \geq \Delta_{it}^{PD}(\mathbf{y} + \mathbf{e_i} + \mathbf{e_j})$, *for* $\mathbf{y} \geq 0$, $t \in \{1, \ldots, T\}$ *and* $i \neq j$, *or,*

(c) its own stock level increases by one unit and the other product's stock level decreases by one unit, i.e., $\Delta_{it}^{PD}(\mathbf{y} + \mathbf{e_i} + \mathbf{e_j}) \geq \Delta_{it}^{PD}(\mathbf{y} + 2\mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1, \ldots, T\}$ *and* $i \neq j$, *or,*

(d) the remaining time decreases, i.e., $\Delta_{i,t+1}^{PD}(\mathbf{y} + \mathbf{e_i}) \geq \Delta_{it}^{PD}(\mathbf{y} + \mathbf{e_i})$ *for* $\mathbf{y} \geq 0$, $t \in \{1, \ldots, T\}$.

These properties are extensions of the analogous results in Chapter II. Note that

Proposition III.1(b) implies that the seller's revenue function, $V_t^{PD}(\mathbf{y})$, continues to be submodular in the stock levels, y_1 and y_2. Also, Proposition III.1(a) implies that the seller's revenue function $V_t^{PD}(\mathbf{y})$ is separately concave in the inventory level of each product, which suggests that the optimal stock levels could be found by rather simple search algorithms.

Proposition III.2. *The optimal price of a product charged to each segment of customers is decreasing in its own stock level; i.e.,* $p_{it}^{i*}(\mathbf{y}) \geq p_{it}^{i*}(\mathbf{y}+\mathbf{e_i})$ *for* $y_i \geq 1$, $i = 1, 2$, *and* $p_{it}^{3*}(\mathbf{y}) \geq p_{it}^{3*}(\mathbf{y} + \mathbf{e_i})$ *for* $y_i \geq 1$, $i = 1, 2$.

Proposition III.3. *For a given product* $i = 1, 2$, *the price charged to a segment-*3 *customer is higher than the price charged to a segment-i customer, i.e.,* $p_{it}^{3*}(\mathbf{y}) \geq p_{it}^{i*}(\mathbf{y})$.

In contrast to customers in segment 1 and 2, who are considering only one product for purchase, customers in segment 3 are considering both products. Segment-3 customers' willingness to purchase either product implies that, everything else being equal, these customers are more likely to make a purchase compared to segment-1 or segment-2 customers. The retailer takes advantage of segment-3 customers' higher propensity to purchase by increasing the prices charged to these customers.

The proposition also suggests a way in which group pricing can be implemented in this context: The retailer may announce the price $p_{it}^{3*}(\mathbf{y})$ as the price of product i, but offer a discount in the amount of $p_{it}^{3*}(\mathbf{y}) - p_{it}^{i*}(\mathbf{y})$ to segment-i customers.

3.2.4 Purchase Probabilities and the Optimal Price Difference

In a dynamic pricing of single product and stationary parameters (e.g., constant arrival rate) the probability of making a sale increases if there is less time until the end of the season or more inventory of the product under the optimal pricing policy. As we discuss next, similar behavior holds in our model, despite the non-monotonicity of the prices. When there are t periods to go with inventory vector

$\mathbf{y}$, let $q_{0t}^{j*}(\mathbf{y}), i = 1, 2, 3$ denote the *optimal no-purchase probability for segment-j customers*, that is, the probability that a segment-j customer arriving in period t will not purchase any product from the seller, given the seller is charging the optimal prices. Hence, *the optimal probability of making a sale* in period t with inventory vector $\mathbf{y}$ is $\sum_{j=1}^{3} \lambda^j \left(1 - q_{0t}^{j*}(\mathbf{y})\right)$.

Proposition III.4. *The optimal probability of making a sale,* $\sum_{j=1}^{3} \lambda^j \left(1 - q_{0t}^{j*}(\mathbf{y})\right)$, *increases if:*

(*a*) *the stock level of a product increases, or*

(*b*) *the remaining time decreases.*

Proposition III.5. *For segment-*3 *customers, the difference between the prices of products i and j,* $p_{it}^{3*}(\mathbf{y}) - p_{jt}^{3*}(\mathbf{y})$, *decreases if*

(*a*) *the stock level of product i increases; i.e.,* $p_{it}^{3*}(\mathbf{y}+\mathbf{e_i}) - p_{jt}^{3*}(\mathbf{y}+\mathbf{e_i}) \leq p_{it}^{3*}(\mathbf{y}) - p_{jt}^{3*}(\mathbf{y})$ *for* $y_i \geq 1, y_j \geq 1, i = 1, 2, j = 1, 2,$ *or*

(*b*) *the stock level of product j decreases; i.e.,* $p_{it}^{3*}(\mathbf{y}+\mathbf{e_j}) - p_{jt}^{3*}(\mathbf{y}+\mathbf{e_j}) \geq p_{it}^{3*}(\mathbf{y}) - p_{jt}^{3*}(\mathbf{y})$ *for* $y_i \geq 1, y_j \geq 1, i = 1, 2, j = 1, 2.$

3.3. Numerical Studies

In addition to personalized dynamic pricing, we consider other pricing strategies and conduct numerical studies to compare the optimal expected revenue under alternative pricing strategies. We also study the effect of parameters such as inventory levels or customer arrival rates on the optimal prices under given a pricing strategy.

3.3.1 Alternative Pricing Strategies

We denote pricing strategies by using the indices D, S, P and N to represent dynamic pricing, static pricing, personalized pricing and non-personalized pricing, respectively. Now we combine D, S, P and N to consider the following:

- ND: Non-personalized Dynamic Pricing

Suppose the company cannot charge different prices for the same product despite the existence of customer segments. However, if the company is allowed to change prices each period, the problem can be considered as a version of the original PD problem with the constraint that the price of product i should be the same for both segment-i and segment-3 customers, $i = 1, 2$. The optimality equations for ND are as follows:

$$\begin{aligned} V_t^{ND}(\mathbf{y}) &= V_{t-1}^{ND}(\mathbf{y}) + \max_{\mathbf{p}^3} \sum_{j \in \{1,2,3\}} \sum_{i \in S(\mathbf{y})} \lambda^j q_i^j \left(\mathbf{p}^j, S(\mathbf{y})\right) \left[p_i^j - \Delta_{it}^{ND}(\mathbf{y})\right] \\ &\text{s.t.} \quad p_i^i = p_i^3 \text{ for } i \in S(\mathbf{y}) \end{aligned}$$

where $V_t^{ND}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{ND}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$.

- PS: Personalized Static Pricing

The company can charge different prices for the same product, depending on customer segments. However, the prices cannot be changed during the selling season.

$$\begin{aligned} V_T^{PS}(\mathbf{y}) &= \max_{\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3} \sum_{t=1}^{T} V_t^{PS}(\mathbf{y}) \\ &\text{s.t.} \quad V_t^{PS}(\mathbf{y}) = V_{t-1}^{PS}(\mathbf{y}) + \sum_{j \in \{1,2,3\}} \sum_{i \in S(\mathbf{y})} \lambda^j q_i^j \left(\mathbf{p}^j, S(\mathbf{y})\right) \left[p_i^j - \Delta_{it}^{PS}(\mathbf{y})\right] \end{aligned}$$

where $V_t^{PS}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{PS}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$.

- NS: Non-personalized Static Pricing

The company can choose only one set of prices regardless of customer segments and the prices are fixed at the beginning of the selling season.

$$\begin{aligned} V_T^{NS}(\mathbf{y}) &= \max_{\mathbf{p}^3} \sum_{t=1}^{T} V_t^{NS}(\mathbf{y}) \\ &\text{s.t.} \quad V_t^{NS}(\mathbf{y}) = V_{t-1}^{NS}(\mathbf{y}) + \sum_{j \in \{1,2,3\}} \sum_{i \in S(\mathbf{y})} \lambda^j q_i^j \left(\mathbf{p}^j, S(\mathbf{y})\right) \left[p_i^j - \Delta_{it}^{NS}(\mathbf{y})\right] \\ &\qquad\; p_i^i = p_i^3 \text{ for } i \in S(\mathbf{y}) \end{aligned}$$

where $V_t^{NS}(0) = 0$ for any $t \in \{1, ..., T\}$ and $V_0^{NS}(\mathbf{y}) = 0$ for any $\mathbf{y} \geq 0$.

3.3.2 Relative Benefits of Switching Pricing Strategies

First we consider relative benefit the company can expect by switching pricing strategies. We generate 2916 instances by the combinations of following parameters: $u_1 = \{1.1, 1.3, 1.5\}$, $u_2 = 1.0, T = 4$, $y_1 = \{1, 2, 3, 4, 5, 6\}$, $y_2 = \{1, 2, 3, 4, 5, 6\}$, $\lambda^1 = \{0.1, 0.2, 0.3\}$, $\lambda^2 = \{0.1, 0.2, 0.3\}$ and $\lambda^3 = \{0.1, 0.2, 0.3\}$. The optimal expected revenue is calculated for each instance under pricing strategies of *PD*, *ND*, *PS*, *NS* across the given ranges of parameters. The results are processed using Minitab, a software application for statistical analysis, to calculate the Pearson correlation coefficients.

- **D-S**: Switching from static pricing to dynamic pricing

We first consider the benefit the firm obtains by switching to dynamic pricing from static pricing. The optimal expected revenue under personalized and non-personalized pricing cases are calculated separately at time T, across all the parameters in the given ranges. Let B_P^{D-S} be the average relative benefit of switching from static pricing to dynamic pricing under personalized pricing, and let B_N^{D-S} be the average relative benefit under non-personalized pricing.

	y_1	$y_1 + y_2$	$\lambda^1 + \lambda^2 + \lambda^3$	B_P^{D-S}	B_N^{D-S}
B_P^{D-S}	-0.472	-0.633	0.224	-	0.969
B_N^{D-S}	-0.543	-0.621	0.218	0.969	-

Table 3.1: Pearson correlation coefficients show close correlation between the benefits from dynamic pricing under personalized and non-personalized pricing strategies at 99% confidence level.

Table 3.1 shows Pearson correlation coefficient between B_P^{D-S} and B_N^{D-S} is 0.969, which implies strong positive association between them. The correlation indicates that a given factor has similar effects on both B_P^{D-S} and B_N^{D-S}. From the Pearson coefficients in Table 3.1, inventory levels and the relative benefits have negative correlations. This implies that the relative benefits of switching from static pricing to dynamic pricing decrease in the inventory levels, which applies similarly to both personalized pricing and non-personalized pricing. This corresponds with established results, where the improvement brought about by dynamic pricing increases as inventory gets scarce. However, customer arrival rates and the relative benefits show positive correlations. It is shown in Table 3.1 that benefits increase in $\lambda^1+\lambda^2+\lambda^3$. In other words, implementing dynamic pricing is more beneficial if the firm is expected to face higher customer traffic. The explanation for this goes back to the scarcity of inventory: increased traffic implies higher likelihood of situations where inventory will become scarce, thus rendering dynamic pricing more beneficial.

- **P**-**N**: Switching from non-personalized pricing to personalized pricing

We consider the benefit the firm obtains by switching to personalized pricing from non-personalized pricing. The optimal expected revenues under dynamic pricing and static pricing cases are calculated separately at time T, across all the parameters in the given ranges. Let B_D^{P-N} be the average relative benefit of switching from non-personalized pricing to personalized pricing under dynamic pricing, and let B_S^{P-N} be the average relative benefit under static pricing.

	y_1	y_1+y_2	λ^1	λ^3	B_D^{P-N}	B_S^{P-N}
B_D^{P-N}	0.355	0.415	-0.407	0.399	-	0.867
B_S^{P-N}	0.348	0.559	-0.298	0.211	0.867	-

Table 3.2: Pearson correlation coefficients show close correlation between the benefits from personalized pricing under dynamic and static pricing strategies at 99% confidence level.

Table 3.2 shows Pearson correlation coefficient between B_D^{P-N} and B_S^{P-N} is 0.867, which implies strong positive association between them. The correlation indicates that a given factor has similar effects on both B_D^{P-N} and B_S^{P-N}. The Pearson coefficients are positive between the inventory levels and the relative benefit. Therefore, the improvement brought about by personalized pricing shrinks as inventory gets scarce. This follows from the properties of marginal values discussed in the previous section. If the inventory levels are low, the products have high marginal values, and customers are charged high prices regardless of their segments. Hence, personalized pricing has little effect on prices. On the other hand, when the inventory levels are high, the firm must choose low prices, but the need to charge low prices is tempered by the ability to charge higher prices to less sensitive segments, thus resulting in significant benefits from personalized pricing. λ^3 and the relative benefits show positive correlation. This is intuitive because personalized pricing should become more powerful as the segment of customers to which the company can charge a higher price gets larger. The growth of other segments would have the opposite effect, which can be explained by similar reasoning.

3.3.3 Initial Stock Levels

y_1	1	2	3	4	5	6	7
y_2	7	6	6	5	5	4	3

Table 3.3: The change in optimal stock level of product 2 with respect to stock level of product 1. In this example, $u_1 = 5, u_2 = 3, c_1 = 2, c_2 = 1.1, \lambda^1 = 0, \lambda^2 = 0, \lambda^3 = 1, T = 15$.

In proposition III.1 (a) and (b), we have shown that the marginal value of a product decreases in stock levels of both products. These properties imply the revenue function $V_t^{PD}(\mathbf{y})$ is separately concave in stock levels of product 1 and product 2. Therefore, the initial stock levels of products can be decided by relatively simple search algorithms. Furthermore, the optimal initial inventory of a product is nonincreasing in the initial inventory of the other product. An example is provided in Table 3.3, which shows that the optimal initial stock of product2 is decreasing in the inventory level of product 1.

3.3.4 Customer Arrival and Optimal Prices

y_1	λ^1	p_1	p_2
1	0.1	2.31	1.73
1	0.5	3.01	1.67
2	0.1	1.94	1.76
2	0.5	2.31	1.71
6	0.1	1.83	2.07
6	0.5	1.78	2.06

Table 3.4: Optimal prices in non-personalized dynamic pricing. In this example, $u_1 = 1.5, u_2 = 1, y_2 = 2, \lambda^2 = 0.1, \lambda^3 = 0.1, T = 8$.

Given the marginal value of a product, the optimal price for the period does not depend on the arrival rates in personalized dynamic pricing (PD). However, the PD prices would be affected if the changes in arrival rates are effective for the rest of the selling horizon. On the other hand, the changes in marginal values affect the current prices for the given period in non-personalized dynamic pricing (ND). We focus on the price changes in ND cases in this section. Suppose there is a change in customer arrival rate, as illustrated in Table 3.4. In the example with low inventory level ($y_1 = 1$), the optimal price of product 1 goes up if λ^1 increases. However, at high inventory level ($y_1 = 6$), the optimal price of product 1 decreases in λ^1. Such conflicting results point to the existence of two competing effects. First, an increase in λ^1 implies higher chances of an eventual stock-out for product 1, which pushes the retailer to increase the price of product 1. Second, an increase in λ^1 implies a relative decrease in the portion of segment-3 customers in the arrival stream. As segment-3 is the one that calls for higher prices, a reduction in their relative size leads to a decrease in the prices. At low inventory levels, the first effect dominates, thus resulting in an increase in the price of product 1. At high inventory levels, the second effect dominates, thus resulting in a decrease in the price of product 1.

3.4. Conclusion

In this study, we consider the personalized dynamic pricing problem of a seller who is offering two substitutable products with exogenously fixed inventory levels over a predetermined finite selling horizon. We assume that the consumer population consists of three segments. We show that many of the structural results from the previous chapter carry over. For example, the marginal value of an item is increasing in the remaining time and decreasing in its own stock level and the other product's stock level, under optimal personalized dynamic pricing policy. In addition, we find that customers who are choosing from the entire assortment pay higher prices than those who are considering only one product in the assortment. We provide results from our numerical studies on personalized dynamic pricing, non-personalized dynamic pricing, personalized static pricing and non-personalized static pricing strategies. We discuss the seller's relative benefits from switching pricing strategies. In the case of switching from static pricing to dynamic pricing, the relative benefit decreases in inventory levels. In other words, switching from static pricing to dynamic pricing is more beneficial when the inventory levels are getting scarce. On the other hand, in the case of switching from non-personalized pricing to personalized pricing, the relative benefit increases in inventory levels. In other words, it is better to switch to personalized pricing when the inventory is higher. We also discuss the effect of customer arrival rates on optimal prices. In the case of personalized static pricing, the optimal price can either increase or decrease due to a change in customer arrival rate, depending on inventory level.

CHAPTER IV

Retail Competition under Commission Contract

4.1. Introduction

"Marketplace is the marriage of Amazon's e-tail experience and mass capabilities with the strength of niche players"[1]. Buying online has become a significant part of daily life. The percentage of U.S. consumers who have purchased online was 49% in September 2007, compared to 22% in June 2000. In keeping with the growing number of online shoppers, online retail sales have shown growth rates of double digits, year after year. Much of online sales is dominated by top players. In fact, top 500 online retailers account for 61% of entire e-commerce sales in 2007 (PR Log, 2008). According to the estimates from Forrester's research, Amazon and ebay together account for about "$1 out of every $5 spent online" (The Wall Street Journal, 2008). The financial reports show that Amazon.com was responsible for about 18.7% of U.S. e-commerce in the fourth quarter of 2008.

The third-party marketplace program has been accounted as the strongest financial support for the success of Amazon. In their efforts to build market share, Amazon executives realized that they could offer many products without actually owning the inventory (Los Angeles Times, 2004). This was accomplished by allowing third-party sellers to use Amazon as a gateway to reach potential customers. This arrangement,

[1]Thomas Lot(President of Amazon.fr), Internet Retailer 2003

known as "the third-party marketplace" is considered a key enabler of Amazon's success. When a customer searches for an item, Amazon's product page shows many alternatives, including third-party sellers. Sometimes the offerings of third-party sellers will be shown side-by-side with Amazon's own inventory of the same item. When a third party makes a sale through Amazon, Amazon charges them a fraction of the sale amount. Essentially, this is an arrangement where third-party sellers sellers "rent" virtual shelves from Amazon and pay commission to Amazon as compensation for sharing in its customer traffic. Amazon specifies the commission rates for several product categories.

Since the launch of the program in 2000, the third-party marketplace, powered by Amazon's over 66 million active customers, has been an attractive place for merchants. The merchants include such established brands as Toyota Motor Sales Inc. and Levi Strauss & Co. According to Amazon's report, the third-party marketplace accounted for 32% of all units sold through Amazon in the first quarter of 2009, with more than 1.6 million active sellers' participation. Other retailers such as Buy.com, impressed by Amazon Marketplace's smash hit, started similar programs. As a remarkable milestone, Walmart also launched Walmart Marketplace on August 31, 2009.

For merchants who agree to pay commission fees, the contract created new business opportunities by giving them access to increased customer traffic. For the retailer who offers the contract and accommodates the marketplace, the effect of the commission contract has two opposing effects. The commission contract promises the retailer additional profits from items or entire product categories that it does not want to stock itself. On the other hand, the contract creates direct competition between the retailer and the merchants by displaying the merchants' products to customers alongside the retailer's own inventory.

In this study, we consider retail competition models under the commission contract. We model various types of retail competition using a retailer and multiple merchants. (In our terminology, the retailer corresponds to the provider of the marketplace, e.g., Amazon, and the merchants are the independent, third-party sellers.) In each market model, a customer chooses which seller to buy from, depending on the prices. We use MNL choice model to describe customers' price-based substitution among the sellers. Under several models of competition, we show a unique Nash equilibrium exists in the pricing decisions. We find the conditions which make the commission contract attractive to both the retailer and merchants. Especially, the growth of market size is found to be playing a cardinal role in driving both parties to the agreement on the contract. We exclude the possibility of lost sales; therefore, the order from a customer is assumed to be always fulfilled. However, we consider one model where the retailer may choose not to carry an item. We show that both the retailer and merchants can be better off in equilibrium by the retailer's decision to omit the item from its offering. The practice and benefits of stockless (i.e., zero-inventory) operation under duopoly competition are studied by Sun and Ryan et al. (2008).

This chapter is organized as follows: In Section 4.2, we position our work with respect to the literature, and describe the models in Section 4.3. We establish structural results for the equilibrium of these four models and numerically analyze them in Section 4.4, where we discuss the overall effect of commission fees under several competition scenarios. We conclude in Section 4.5.

4.2. Literature Review

Online retailers' third-part marketplace practices and the accompanying commission contracts have been phenomenally successful; however, the research on this practice is scant. Such research would extend the existing literature on the channel coordination, where contracts are designed to drive the decisions of individual players toward better channel outcomes. An extensive summary of supply chain coordination is presented in Cachon (2002). We focus on the commission contracts offered by online retailers to third-party merchants. One distinction of this type of contract is that the parties to it - the online retailer and the merchants - remain both competitors and collaborators.

Our structural results focus on the existence and uniqueness of the pricing game that arises under the commission contract. To obtain such results, we conduct a game-theoretic analysis. Game theoretic models have been used with increasing frequency in the supply chain literature. See Cachon and Netessine (2003) for a review of game theory techniques that found use in supply chain models. Our proofs utilize the so-called index theory approach to establish the uniqueness of the equilibrium. Netessine and Rudi (2006) use the same approach when analyzing online retailers' drop-shipping problem. Cachon and Zipkin (1999) do so in a stocking game within a two-stage supply chain.

In our models, a customer chooses one of many sellers, all of whom offer more or less the same product. We use MNL choice model to describe customers' price-based discrete choice behavior. MNL model has been widely used in the marketing literature to model consumers' choice behavior. See van Ryzin and Mahajan (1999) and Cachon and Kok (2007) for several examples. In addition, Ben Akiva and Lerman (1985) provide a more broad discussion on discrete choice models.

4.3. Model Description

We consider single period problems of retail competition under the commission contract. We model the competition among a retailer and n identical merchants. We categorize the retail environment models into three types. *Model 1* describes a market where the retailer is the only seller. *Model 2* represents a market where the retailer and the merchants can cooperate and compete simultaneously. In this model, the retailer and the merchants cooperate in that the retailer allows the merchants to access its customer base by displaying their products on the retailer's website. They compete in that the retailer and the merchants sell substitutable products. *Model 3* stands for a cooperative market where only the merchants are selling. In this model, there is no competition because the retailer does not carry the product sold by the merchants. In each model, customers in the market observe their options and choose one. Depending on the market model, the option can be one of the following: buy from the retailer, buy from one of the merchants or buy nothing. We use the MNL choice model to describe customer choice behavior. For that purpose, let the indices r and m stand for the retailer and a merchant, respectively. We assume that the sellers incur fixed costs of c_r and c_m, respectively, for carrying the product.

4.3.1 *Model 1*: Retailer only

Let p_r denote the price of the product sold by retailer under monopoly. The probability that a customer chooses to buy from the retailer can be derived using MNL choice model as follows:

$$q_r(p_r) = \frac{\exp[u_r - p_r]}{1 + \exp[u_r - p_r]}$$

Let N_1 be the size of the market. Then retailer's profit as the monopolist is the following:

$$F_r^1(p_r) = N_1 p_r q_r(p_r) - c_r$$

4.3.2 *Model 2*: Retailer-merchant coopetition

Let n be the number of merchant(s) competing against retailer in the market. An average customer values the utility of buying from the merchant as u_m (all merchants are assumed to be identical). Let $\mathbf{p} = (p_r, p_m)$ denote the pair of the retailer's and the merchant's price. The probabilities that the customer chooses to buy from the retailer, $q_r(\mathbf{p})$, and from the merchant, $q_r(\mathbf{p})$, are:

$$\begin{aligned} q_r(\mathbf{p}) &= \frac{\exp[u_r - p_r]}{1 + \exp[u_r - p_r] + n\exp[u_m - p_m]} \\ q_m(\mathbf{p}) &= \frac{\exp[u_m - p_m]}{1 + \exp[u_r - p_r] + n\exp[u_m - p_m]} \end{aligned}$$

Under coopetition, each merchant pays the retailer a fraction $b \in (0,1)$ of its revenue from every sale. As an extreme case, pure competition can be recovered by setting $b = 0$. Let N_2 be the market size under the merchants' participation. The profit functions can be expressed as follows:

$$F_r^2(\mathbf{p}) = N_2\left[p_r q_r(\mathbf{p}) + bnp_m q_m(\mathbf{p})\right] - c_r \tag{4.1}$$

$$F_m^2(\mathbf{p}) = N_2(1-b)p_m q_m(\mathbf{p}) - c_m \tag{4.2}$$

Proposition IV.1. *For $b \in [0,1)$, there exists a Nash equilibrium, and it is unique.*

4.3.3 *Model 3*: Retailer-merchant cooperation

In this model, the retailer does not carry the product and limits itself to displaying the merchants' products. Therefore, the retailer does not compete in this game but only collects commissions. The unique Nash Equilibrium in this pricing game occurs at a symmetrical price point among the identical merchants. The purchase probability of a merchant and the profit functions under the contract are the following:

$$q_m(p_m) = \frac{\exp[u_m - p_m]}{1 + n\exp[u_m - p_m]}$$

$$F_r^3(p_m) = N_3 bnp_m q_m(p_m) \tag{4.3}$$

$$F_m^3(p_m) = N_3(1-b)p_m q_m(p_m) - c_m \tag{4.4}$$

4.4. Analysis

4.4.1 Case 1: Monopolist Vs Marketplace under same market size

Consider a retailer, who is currently a monopolist but is considering launching a third-party marketplace. By launching the marketplace, the retailer enables competition from the merchants, which will reduce its market share. On the other hand, as the compensation, the retailer collects commissions from the merchants. It is not necessarily clear whether the retailer prefers to launch a marketplace or not, as the retailer's preference depends on whether the gains from commissions outweigh the losses due to competition. To study this question, we use *model 1* and *model 2* at fixed market sizes $N_1 = N_2$. We compare the profits from *model 1* and *model 2* with $N_1 = N_2$ to study numerically the effect of launching the marketplace. According to the results, the loss due to competition turns out to outweigh the gain from commission fees, even at the highest commission rate, $b = 1$. Table 4.1 shows retailer's loss due to the launch of the marketplace and the percentage contribution of merchants' commission fees to retailer's revenue. Increased number of merchants(n) creates more competition among the merchants, thus makes the retailer better off, but never enough to match the profit the retailer makes as a monopolist.

n:number of merchant(s)	1	2	3	4	5
Retailer's Loss(%)	14.89	24.35	31.01	36.07	40.10
Merchants' Contribution (%)	0.62	1.03	1.35	1.62	1.88

Table 4.1: Retailer's relative loss due to competition and merchants' relative contribution. In this example, $N_1 = N_2 = 1$, $u_r = 5$, $u_m = 1$, $b = 0.1$.

Figure 4.1 shows changes in equilibrium prices and profits using *model 2* with respect to commission rate b. The graphs show that both the retailer's price and the merchants' prices increase in commission rate. Considered from the retailer's point of view, the increase in commission rate motivates the retailer to raise the price since

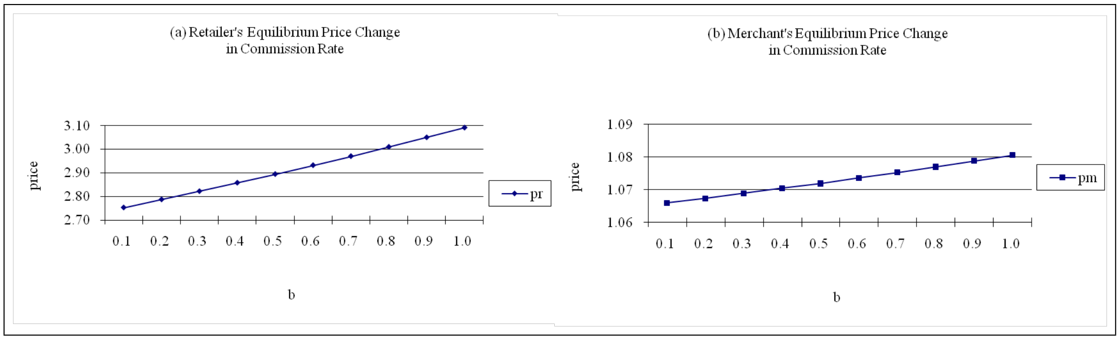

Figure 4.1: Equilibrium prices increase in commission rate b. In this example, $N_1 = N_2 = 1$, $u_r = 5$, $u_m = 1$ and $n = 5$.

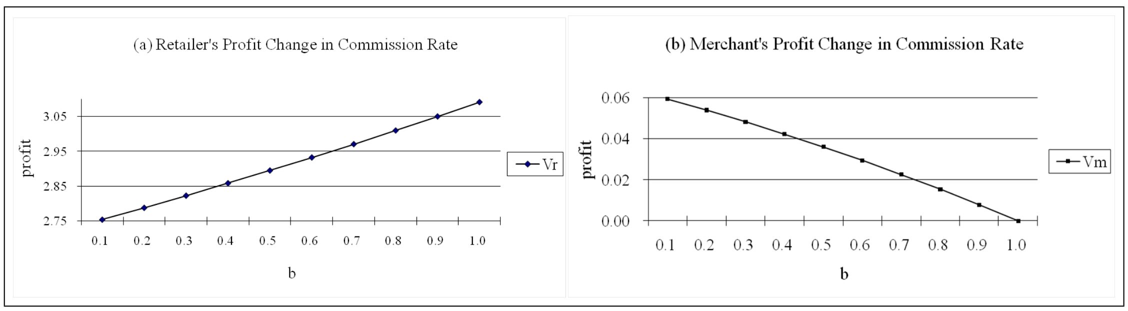

Figure 4.2: Retailer's profit increases and merchant's profit decreases in commission rate b. In this example, $N_1 = N_2 = 1$, $u_r = 5$, $u_m = 1$ and $n = 5$.

the retailer makes money even when it is not the one chosen by the customer. As a consequence of retailer's price increase, Nash equilibrium leads to an increase in the merchants' price as well. As it is implied in figure 4.1, the retailer benefits from increased price and commission fees to make a profit which increases in commission rate b. However, the merchants' gain by increased price is not enough to cover the burden of the increased commission fee. As the results show, only the retailer benefits by raising the commission rate. All in all, the conclusion is that if the market size is going to remain the same, then a monopolist retailer has no incentive to launch a marketplace as the commission revenues will always be dominated by the adverse effects of competition from the merchants.

4.4.2 Case 2: Bringing the existing merchants into a marketplace under fixed market size

Consider now a retailer that is competing with merchants in status quo. Suppose the retailer offers to launch a marketplace. We next study how the marketplace would effect the profits of the retailer and the merchants. In particular, is there any party (the retailer and/or the merchant) that will be better off as a result of the launch of the marketplace? For that purpose, we use *model 2* to compare the sellers' profits at $b = 0$ versus at $b > 0$. Figure 4.3 shows the effect of launching the marketplace business, under fixed market size N_2.

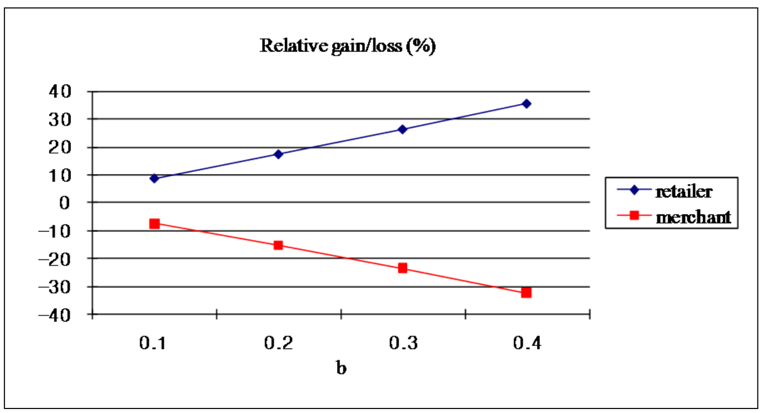

Figure 4.3: Retailer's relative gain and merchant's relative loss from launching the marketplace increases in commission rate b. In this example, $N_2 = 1$, $u_r = 5$, $u_m = 3$ and $n = 5$.

Given the fixed market size, the retailer is in favor of the business because of the benefit from commission fees. On the other hand, merchants are against it due to the loss. Therefore, marketplace business will not emerge in this case. If the market size is going to remain the same, the merchants would rather remain independent than join in a third-party marketplace.

4.4.3 Case 3: Merchants' Outlet VS Marketplace under market growth

If the launch of a third-party marketplace has no effect on the market size, Cases 1 and 2 together confirm that a third-party marketplace will not arise. In particular, as Case 1 shows, if the retailer is a monopolist in status quo, it would not gain anything by starting the the third-party marketplace. Likewise, as Case 2 shows, if the merchants are already in business, they would not want to join a third-party marketplace.

We conjecture that the existence of the third-party marketplace can be explained by increased market size. To study the effect of market size growth, we compare the equilibrium outcome of *model 2* at $b > 0$ (which corresponds to the case with third-party marketplace) versus $b = 0$ (which corresponds to the case of pure competition among the retailer and the merchants with no third-party marketplace). Now suppose the market size for the product is greater in the presence of the marketplace, i.e., letting N_2 and N_2' denote the market sizes, respectively, with and without the marketplace, assume that $N_2 > N_2'$. Figure 4.4 shows the retailer's and the merchants' preferences as a function of b, the commission rate, and $S := N_2/N_2'$, the relative growth in market size due to the marketplace.

In Figure 4.4, region I captures the area where both the retailer and the merchants are better off due to the launch of the marketplace. In contrast, in region II, only the retailer is better off. The threshold of S above which the merchants are benefit from the marketplace increases in commission rate b. Therefore, the merchants would take into account both the commission rate and prospect of the business growth when deciding whether or not to participate in the marketplace. Notice from the figure that the merchants would not join the marketplace at high values of b, even if the market size is expected to triple.

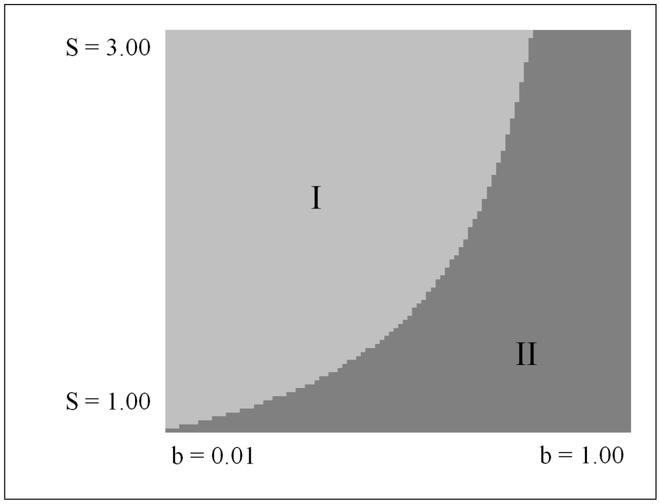

Figure 4.4: In area I, the commission contract is attractive to both retailer and merchants. In area II, only the retailer benefits from the contract. In this example, $u_r = 5$, $u_m = 2$ and $n = 5$.

4.4.4 Case 4: Stocking decision

In this scenario, we compare two alternative forms of the marketplace: one in which the retailer carries the product itself and another in which the retailer does not stock the product and makes money from commissions only.

Consider now a retailer who is carrying a product in status quo and is offering a marketplace for merchants as well. Suppose that this retailer would like to stop carrying the product so that the product will be carried only by the merchants who will sell through the marketplace. Under what conditions will the retailer benefit from such an initiative? To answer this question, we compare the equilibrium outcome of *model 2* with $b > 0$ (which corresponds to a marketplace scenario with the retailer also carrying the product) and the equilibrium outcome of *model 3* with $b > 0$ (which corresponds to a marketplace scenario with the retailer not carrying the product). To control for the effects of the market size, we normalize the market sizes under both models to 1, i.e., $N_2 = N_3 = 1$. Our numerical results are illustrated in Figures 4.5 through 4.7.

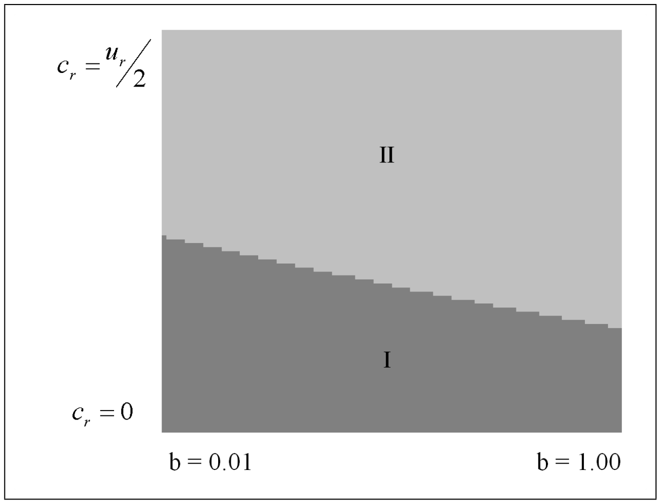

Figure 4.5: Retailer is better off by non-stocking in area II. In area I, the retailer would stock and compete with merchants. In this example, $N_2 = N_3 = 1$, $u_r = 5$, $u_m = 2$ and $n = 5$.

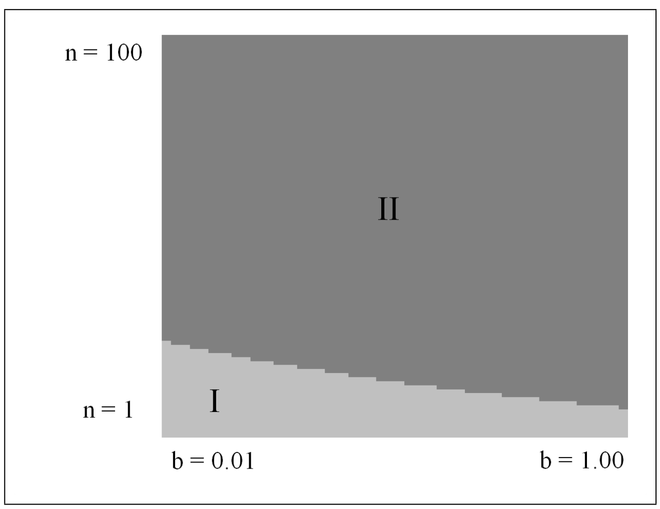

Figure 4.6: Under the fixed cost, retailer is better off by not stocking only in area II. In this example, $N_2 = N_3 = 1$, $u_r = 5$, $u_m = 2$ and $c_r = 0.5$.

When the retailer stops carrying the product, it saves the fixed cost c_r. On the flipside, however, the retailer loses revenues from its own sales and must rely on commission revenues. Figure 4.5 therefore illustrates an intuitive behavior: the retailer prefers not to carry the product when the fixed cost of carrying the product is high (in region II). The threshold cost above which the retailer stops carrying the product gets smaller as the commission rate increases.

Figure 4.6 explores the effect of the number of merchants on the retailer's preference. As the number of merchants grows, the retailer's preference changes from carrying the product (region I) to not carrying the product (region II). The number of merchants needed to flip the retailer's preference gets smaller as the commission rate increases.

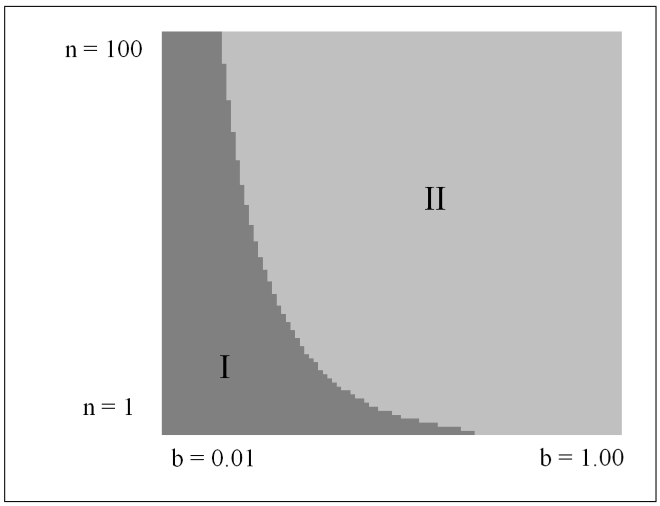

Figure 4.7: A merchant benefits from the retailer's non-stocking decision only in area I. In this example, $N_2 = N_3 = 1$, $u_r = 5$ and $u_m = 2$.

Figure 4.7 illustrates the effect of the retailer's decision on the merchants. Suppose the retailer decides not to stock. Intuitively, this decision should be a good news for the merchants since it removes the retailer as a competitor. However, this is not necessarily the case, as indicated in Figure 4.7. In region II, the merchants would prefer the retailer to carry the product. The explanation for this rather surprising result seems to be the following: When the retailer carries the product, the retailer's price plays a regulatory role, since its revenues come from both its own sales and the commissions. In an effort to keep the commissions high, the retailer would price its own product so that the merchants make a decent profit. Once the retailer stops carrying the product, there is no such regulation that can be achieved through the retailer's price and the merchants settle in a more aggressive competition with one another, thus driving each other's prices and profits down.

4.5. Conclusion

We have considered a retail competition problem which consists of a single retailer and multiple merchants. In particular, we considered marketplace business model, in which the merchants can share the retailer's system and customer traffic in exchange for a fraction of their sales revenue. Since the contract implies that both the retailer and merchants would be competing in sales, the business model would be meaningful to the retailer if the additional revenue transferred from the merchants outweighs the loss from the competition. Also, the merchants would weigh the commission fee and profitability of the business. Therefore, the marketplace can emerge only under the condition where both parties benefit.

We present three types of retail market models and show the uniqueness of the Nash equilibrium in competition between the retailer and the merchants. We numerically compare the equilibrium outcome under several different scenarios to understand the conditions under which the marketplace business model can be realized. We find that if the marketplace leads to a larger market size, then both the retailer and merchants will want to launch it. In addition, we explore how the marketplace is affected when the retailer chooses not to stock the product itself. Interestingly, we find that the merchants may prefer the retailer to stock the product.

One possible extension to this work is to add inventory decisions. At the very least, one could add a first stage of decision-making where the merchants and the retailer are making decisions about whether or not to carry the product. Another extension is to study the multi-period problem. This might require efficient implementation of the suggested first extension to include the inventory effect which transfers to the next periods. One other extension would be to consider sellers, each of which offers multiple products.

CHAPTER V

Conclusions

In this dissertation, we discuss the problems related to retail pricing under customer substitution. We provide the topics on a single retailer's dynamic pricing of substitutable products and multiple sellers in competition under a commission contract. The first topic is divided into two separate chapters to include the extension to the case of personalized pricing.

In Chapter II, *Dynamic Pricing of Substitutable Products with Limited Inventory under Logit Demand*, we consider the dynamic pricing problem of a seller who is offering two substitutable products with exogenously fixed inventory levels, over a predetermined, finite selling horizon. We summarize our contribution as the following:

- We show the marginal value of an item is increasing in the remaining time and decreasing in the stock level of either product, under the optimal dynamic pricing policy.
- We prove the marginal value of a product is more sensitive to its own stock level than it is to the substitute's stock level.
- We show that the optimal price difference between the two products and the optimal purchase probabilities provide intuitive gauges of the optimal behavior.

In Chapter III, *Personalized Dynamic Pricing of Substitutable Products under*

Logit Demand, we consider the personalized dynamic pricing problem of a seller who is offering two substitutable products with exogenously fixed inventory levels over a predetermined finite selling horizon. We assume that the customers are categorized into three segments by the buying patterns. Our contribution is summarized as follows:

- In the existence of consumer segments, we find that the marginal value of an item is increasing in the remaining time and decreasing in its own stock level and the other product's stock level under optimal personalized dynamic pricing policy.
- We perform numerical studies on personalized dynamic pricing, non-personalized dynamic pricing, personalized static pricing and non-personalized static pricing strategies. We show the relevances among the pricing strategies by considering the relative benefits of switching from one strategy to the other.
- We show the seller's benefits from using personalized dynamic pricing and personalized static pricing increase in inventory.

In Chapter IV, *Retail Competition under Commission Contract*, we have considered retail competition / coopetition problems which consist of a single retailer and multiple merchants. The models we consider reflect the aspects of online retail practices where sellers are engaged in cooperation by paying commission fees, as well as in competition at the same time. Our contributions are summarized as follows:

- We prove that there exists a unique Nash equilibrium under various settings of competition and coopetition.
- We find the conditions under which the marketplace business is attractive to both retailer and merchants, which is the growth of market size.

- We show the condition under which the retailer refrains from stocking inventory and simply collects commission fees from the merchants. From the results, we interpret the retailer's role as the profit balancer during the coopetition.

APPENDICES

APPENDIX A

Proofs for Chapter 2

In preparation for the proofs, we note that parts (a) through (d) of Proposition II.3 correspond to the following inequalities $I_1(\mathbf{y},t)$ through $I_4(\mathbf{y},t)$. For $\mathbf{y} \geq \mathbf{0}$ and $t \geq 1$:

$I_1(\mathbf{y},t) : \Delta_{it}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+2\mathbf{e_i})$, $i = 1,2$,

$I_2(\mathbf{y},t) : \Delta_{it}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j})$, $i \neq j$, $i = 1,2$, $j = 1,2$,

$I_3(\mathbf{y},t) : \Delta_{it}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j}) \geq \Delta_{it}(\mathbf{y}+2\mathbf{e_i})$, $i \neq j$, $i = 1,2$, $j = 1,2$,

$I_4(\mathbf{y},t) : \Delta_{i,t+1}(\mathbf{y}+\mathbf{e_i}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i})$, $i = 1,2$.

In order to prove Proposition II.3, we will prove inequalities $I_1(\mathbf{y},t)$ through $I_4(\mathbf{y},t)$ by induction on t. To help with the induction process, we first state four useful lemmas.

Lemma A.1. *For a given t, suppose $I_1(\mathbf{y},t)$ through $I_4(\mathbf{y},t)$ hold for $\mathbf{y} \geq 0$. Then, for $y_i \geq 0$ and $y_j \geq 1, j \neq i$:*

$$\Delta_{it}(\mathbf{y}+\mathbf{e_i}) - \Delta_{it}(\mathbf{y}+2\mathbf{e_i}) \geq \Delta_{jt}(\mathbf{y}+\mathbf{e_i}) - \Delta_{jt}(\mathbf{y}+2\mathbf{e_i}).$$

Lemma A.2. *Given a set of available products, S, let Δ_i denote the marginal value of product i for $i \in S$. Then:*

(a) If both products are available, i.e., S= {1,2}, then the optimal margin, $m^*(\Delta_1, \Delta_2, \{1, 2\})$*, decreases in* $\Delta_i, i = 1, 2$*.*

(b) If only product i is available, i.e., S= $\{i\}$*, then the optimal margin,* $m^*(\Delta_i, \{i\})$*, decreases in* Δ_i*.*

(c) Given two constants Δ_1 *and* Δ_2*,* $m^*(\Delta_1, \Delta_2, \{1, 2\}) \geq m^*(\Delta_i, \{i\})$*,* $i = 1, 2$*.*

(d) Given two constants Δ_1 *and* Δ_2*,* $m^*(\Delta_1, \{1\}) = m^*(\Delta_1, \infty, \{1, 2\})$ *and* $m^*(\Delta_2, \{2\}) = m^*(\infty, \Delta_2, \{1, 2\})$*. In addition, if* $\Delta_1 = \Delta_2 = \infty$*, then* $m^*(\Delta_1, \Delta_2, \{1, 2\}) = 1$*.*

(e) Suppose both products are available. Given a positive constant x*, the optimal margin satisfies the following:*

$$m^*(\Delta_1, \Delta_2, \{1, 2\}) - m^*(\Delta_1 + x, \Delta_2 + x, \{1, 2\}) = x - \int_0^x \frac{1}{m^*(\Delta_1 + z, \Delta_2 + z, \{1, 2\})} dz \leq x$$

(f) Suppose only product i is available. Given a positive constant x*, the optimal margin satisfies the following:*

$$m^*(\Delta_i, \{i\}) - m^*(\Delta_i + x, \{i\}) = x - \int_0^x \frac{1}{m^*(\Delta_i + z, \{i\})} dz \leq x$$

(g) Suppose both products are available. Define

$$q_i^*(\Delta_1, \Delta_2, \{1, 2\}) := q_i(\Delta_1 + m^*(\Delta_1, \Delta_2, \{1, 2\}), \Delta_2 + m^*(\Delta_1, \Delta_2, \{1, 2\}), \{1, 2\}),$$

i.e., $q_i^*(\Delta_1, \Delta_2, \{1, 2\})$ *is the optimal purchase probability for product i given that both products are available and their marginal values are* Δ_1 *and* Δ_2*. Then,* $q_i^*(\Delta_1, \Delta_2, \{1, 2\})$ *increases in* $\Delta_j, j \neq i$*.*

(h) Suppose only product 1 is available. Define $q_1^*(\Delta_1, \{1\}) := q_1(\Delta_1 + m^*(\Delta_1, \{1\}), \{1\})$*, i.e.,* $q_1^*(\Delta_1, \{1\})$ *is the optimal purchase probability for product* 1 *given that only product* 1 *is available and its marginal value is* Δ_1*. Given some* $\Delta_2 > 0$*,* $q_1^*(\Delta_1, \{1\}) \geq q_1^*(\Delta_1, \Delta_2, \{1, 2\})$*.*

(i) Suppose both products are available. Given a positive constant x, the optimal margin satisfies the following:

$$m^*(\Delta_1, \Delta_2, \{1,2\}) - m^*(\Delta_1 + x, \Delta_2, \{1,2\}) = \int_0^x q_1^*(\Delta_1 + z, \Delta_2, \{1,2\}) dz$$

(j) Suppose only product 1 is available. Given a positive constant x, the optimal margin satisfies the following:

$$m^*(\Delta_1, \{1\}) - m^*(\Delta_1 + x, \{1\}) = \int_0^x q_1^*(\Delta_1 + z, \{1\}) dz$$

Lemma A.3. *For a given t, suppose $I_1(\mathbf{y}, t)$ through $I_4(\mathbf{y}, t)$ hold for $\mathbf{y} \geq 0$. Then, for $\mathbf{y} \geq 0$:*
(a) $m^*(\boldsymbol{\Delta}_t(\mathbf{y} + \mathbf{e_i}), S(\mathbf{y} + \mathbf{e_i})) \geq m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$,
(b) $m^*(\boldsymbol{\Delta}_{t+1}(\mathbf{y}), S(\mathbf{y})) \leq m^*(\boldsymbol{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$.

Lemma A.4. *For a given t, suppose $I_1(\mathbf{y}, t)$ through $I_4(\mathbf{y}, t)$ hold for $\mathbf{y} \geq 0$. Then, for $\mathbf{y} \geq 0$:*
$m^*(\boldsymbol{\Delta}_t(\mathbf{y} + 2\mathbf{e_i}), S(\mathbf{y} + 2\mathbf{e_i})) - m^*(\boldsymbol{\Delta}_t(\mathbf{y} + \mathbf{e_i}), S(\mathbf{y} + \mathbf{e_i})) \leq \Delta_{it}(\mathbf{y} + \mathbf{e}_i) - \Delta_{it}(\mathbf{y} + 2\mathbf{e}_i)$.

Proof of Proposition II.3: In order to prove the proposition, we will prove inequalities $I_1(\mathbf{y}, t)$ through $I_4(\mathbf{y}, t)$ by induction on t. Since $\Delta_{it}(\mathbf{y}) = 0$ for $t = 1$ and $\mathbf{y} \geq \mathbf{0}$, inequalities $I_1(\mathbf{y}, t)$ through $I_4(\mathbf{y}, t)$ hold trivially when $t = 1$. Suppose that the inequalities hold for $t = k$. We will show that they hold for $t = k + 1$.

Proof of $I_1(\mathbf{y}, k+1)$: Note that

$$\Delta_{i,k+1}(\mathbf{y} + \mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y} + 2\mathbf{e_i}) = (V_k(\mathbf{y} + \mathbf{e_i}) - V_k(\mathbf{y})) - (V_k(\mathbf{y} + 2\mathbf{e_i}) - V_k(\mathbf{y} + \mathbf{e_i}))$$

Now, using (2.8) in Lemma II.2, we can rewrite the right-hand side (RHS) of the above equality:

$$\begin{aligned}
&\Delta_{i,k+1}(\mathbf{y}+\mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y}+2\mathbf{e_i}) \\
&= \Delta_{ik}(\mathbf{y}+\mathbf{e_i}) - \Delta_{ik}(\mathbf{y}+2\mathbf{e_i}) \\
&+ \lambda\left[m^*(\mathbf{\Delta}_k(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) - m^*(\mathbf{\Delta}_k(\mathbf{y}), S(\mathbf{y}))\right] \\
&- \lambda\left[m^*(\mathbf{\Delta}_k(\mathbf{y}+2\mathbf{e_i}), S(\mathbf{y}+2\mathbf{e_i})) - m^*(\mathbf{\Delta}_k(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i}))\right]
\end{aligned} \tag{A.1}$$

Since $I_1(\mathbf{y}, t)$ through $I_4(\mathbf{y}, t)$ hold for $t = k$, notice from Lemma A.4 and $\lambda \leq 1$ that

$$\Delta_{ik}(\mathbf{y}+\mathbf{e_i}) - \Delta_{ik}(\mathbf{y}+2\mathbf{e_i}) \geq \lambda \begin{bmatrix} m^*(\mathbf{\Delta}_k(\mathbf{y}+2\mathbf{e_i}), S(\mathbf{y}+2\mathbf{e_i})) \\ -m^*(\mathbf{\Delta}_k(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) \end{bmatrix} \tag{A.2}$$

In addition, it follows from Lemma A.3(a) that

$$m^*(\mathbf{\Delta}_k(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) \geq m^*(\mathbf{\Delta}_k(\mathbf{y}), S(\mathbf{y})) \tag{A.3}$$

Now, (A.1), (A.2) and (A.3) together allow us to conclude that $\Delta_{i,k+1}(\mathbf{y}+\mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y}+2\mathbf{e_i}) \geq 0$, which concludes the proof of $I_1(\mathbf{y}, k+1)$.

Proof of $I_2(\mathbf{y}, k+1)$: Note that

$$\begin{aligned}
&\Delta_{i,k+1}(\mathbf{y}+\mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j}) \\
&= (V_k(\mathbf{y}+\mathbf{e_i}) - V_k(\mathbf{y})) - (V_k(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j}) - V_k(\mathbf{y}+\mathbf{e_j})) \\
&= \Delta_{j,k+1}(\mathbf{y}+\mathbf{e_j}) - \Delta_{j,k+1}(\mathbf{y}+\mathbf{e_i}+\mathbf{e_j})
\end{aligned}$$

Therefore, showing that $\Delta_{1,k+1}(\mathbf{y}+\mathbf{e_1}) \geq \Delta_{1,k+1}(\mathbf{y}+\mathbf{e_1}+\mathbf{e_2})$ is equivalent to showing $\Delta_{2,k+1}(\mathbf{y}+\mathbf{e_2}) \geq \Delta_{2,k+1}(\mathbf{y}+\mathbf{e_1}+\mathbf{e_2})$. In other words, it suffices to show the result for $i = 1$.

In this proof we will sometimes write the vector $\mathbf{\Delta}_k(\mathbf{y})$ explicitly as a list of its components. (Recall that this vector captures the marginal values of available products and may have one or two components depending on which product(s) are available.) Using (2.8), we can write:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1) \\
&= \Delta_{1k}(y_1+1,y_2) - \Delta_{1k}(y_1+1,y_2+1) \\
&+ \lambda\left[m^*(\mathbf{\Delta}_k(y_1+1,y_2), S(y_1+1,y_2)) - m^*(\mathbf{\Delta}_k(y_1,y_2), S(y_1,y_2))\right] \\
&- \lambda\left[m^*(\mathbf{\Delta}_k(y_1+1,y_2+1), S(y_1+1,y_2+1)) - m^*(\mathbf{\Delta}_k(y_1,y_2+1), S(y_1,y_2+1))\right]
\end{aligned} \tag{A.4}$$

In addition, note from Lemma A.2(e) that

$$\begin{aligned}
&\Delta_{1k}(y_1+1,y_2) - \Delta_{1k}(y_1+1,y_2+1) \\
&= m^*(\Delta_{1k}(y_1+1,y_2+1), \Delta_{2k}(y_1+1,y_2+1), \{1,2\}) \\
&- m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1+1,y_2+1) + \Delta_{1k}(y_1+1,y_2) - \Delta_{1k}(y_1+1,y_2+1), \{1,2\}) \\
&+ \int_0^{\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)} \frac{dz}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z, \Delta_{2k}(y_1+1,y_2+1)+z, \{1,2\})}
\end{aligned}$$

We now substitute the above expression for $\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)$ in (A.4). Noting that $\Delta_{2k}(y_1+1,y_2+1)+\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1) = \Delta_{2k}(y_1,y_2+1)$ and $S(y_1+1,y_2+1) = \{1,2\}$ for any pair of stock levels $y_1, y_2 \geq 0$, this substitution helps us rewrite (A.4) as follows:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1) \\
&= \lambda\left[m^*(\mathbf{\Delta}_k(y_1+1,y_2), S(y_1+1,y_2) - m^*(\mathbf{\Delta}_k(y_1,y_2), S(y_1,y_2))\right] \\
&- \lambda\left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\}) - m^*(\mathbf{\Delta}_k(y_1,y_2+1), S(y_1,y_2+1)\right] \\
&+ (1-\lambda)\begin{bmatrix} m^*(\Delta_{1k}(y_1+1,y_2+1), \Delta_{2k}(y_1+1,y_2+1), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\}) \end{bmatrix} \\
&+ \int_0^{\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)} \frac{dz}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z, \Delta_{2k}(y_1+1,y_2+1)+z, \{1,2\})}
\end{aligned}$$

Since $I_1(\mathbf{y},k)$ through $I_4(\mathbf{y},k)$ hold by induction, we have $\Delta_{1k}(y_1+1,y_2+1) \leq \Delta_{1k}(y_1+1,y_2)$ and $\Delta_{2k}(y_1+1,y_2+1) \leq \Delta_{2k}(y_1,y_2+1)$. This observation and the fact that $m^*(\Delta_1,\Delta_2,\{1,2\})$ is decreasing in Δ_i (by Lemma A.2(a)), imply that $m^*(\Delta_{1k}(y_1+1,y_2+1), \Delta_{2k}(y_1+1,y_2+1), \{1,2\}) \geq m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\})$. Hence, the third term on the RHS of the above equality is positive. Notice that the fourth term is also positive. Therefore:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1) \\
&\geq \lambda\left[m^*(\mathbf{\Delta}_k(y_1+1,y_2), S(y_1+1,y_2) - m^*(\mathbf{\Delta}_k(y_1,y_2), S(y_1,y_2))\right] \\
&-\lambda\left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\}) - m^*(\mathbf{\Delta}_k(y_1,y_2+1), S(y_1,y_2+1)\right]
\end{aligned} \tag{A.5}$$

We will divide the rest of the proof into four cases, depending on whether $y_1 = 0$ and/or $y_2 = 0$.

Case 1 — $y_1 > 0$ and $y_2 > 0$: In this case $S(y_1,y_2) = S(y_1+1,y_2) = S(y_1,y_2+1) =$

$\{1,2\}$. Hence, we can write (A.5) as follows:

$$\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1)$$
$$\geq \lambda \left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1+1,y_2), \{1,2\})\right.$$
$$\left. - m^*(\Delta_{1k}(y_1,y_2), \Delta_{2k}(y_1,y_2), \{1,2\})\right]$$
$$-\lambda \left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\})\right.$$
$$\left. - m^*(\Delta_{1k}(y_1,y_2+1), \Delta_{2k}(y_1,y_2+1), \{1,2\})\right]$$

Because $\Delta_{2k}(y_1+1,y_2) \leq \Delta_{2k}(y_1,y_2)$, it follows from Lemma A.2(a) that

$$m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1+1,y_2), \{1,2\}) \geq m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2), \{1,2\})$$

The last two inequalities together imply:

$$\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1)$$
$$\geq \lambda \left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2), \{1,2\})\right.$$
$$\left. - m^*(\Delta_{1k}(y_1,y_2), \Delta_{2k}(y_1,y_2), \{1,2\})\right]$$
$$-\lambda \left[m^*(\Delta_{1k}(y_1+1,y_2), \Delta_{2k}(y_1,y_2+1), \{1,2\})\right.$$
$$\left. - m^*(\Delta_{1k}(y_1,y_2+1), \Delta_{2k}(y_1,y_2+1), \{1,2\})\right]$$

We now apply Lemma A.2(i) to each of the terms on the RHS of the above inequality and we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,y_2) - \Delta_{1,k+1}(y_1+1,y_2+1) \\
&\geq \lambda \int_0^{\Delta_{1k}(y_1,y_2)-\Delta_{1k}(y_1+1,y_2)} q_1^*\left(\Delta_{1k}(y_1+1,y_2)+z, \Delta_{2k}(y_1,y_2), \{1,2\}\right) dz \\
&- \lambda \int_0^{\Delta_{1k}(y_1,y_2+1)-\Delta_{1k}(y_1+1,y_2)} q_1^*\left(\Delta_{1k}(y_1+1,y_2)+z, \Delta_{2k}(y_1,y_2+1), \{1,2\}\right) dz
\end{aligned}$$

Now, on the RHS of the above inequality, observe that the range of the first integral is larger (since $\Delta_{1k}(y_1,y_2) \geq \Delta_{1k}(y_1,y_2+1)$ due to inequality $I_2(\mathbf{y},t)$, which holds for $t = k$ by the induction assumption) and the first integrand is also larger (this follows from Lemma A.2(g) and $\Delta_{2k}(y_1,y_2) \geq \Delta_{2k}(y_1,y_2+1)$). It now follows that the RHS of the above inequality is non-negative, concluding the proof for Case 1.

Case 2 — $y_1 > 0$ and $y_2 = 0$: In this case $S(y_1,y_2) = S(y_1+1,y_2) = \{1\}$ and $S(y_1,y_2+1) = \{1,2\}$. Hence, we can write (A.5) as follows:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,0) - \Delta_{1,k+1}(y_1+1,1) \\
&\geq \lambda \left[m^*(\Delta_{1k}(y_1+1,0),\{1\}) - m^*(\Delta_{1k}(y_1,0),\{1\})\right] \\
&- \lambda \left[m^*(\Delta_{1k}(y_1+1,0),\Delta_{2k}(y_1,1),\{1,2\}) - m^*(\Delta_{1k}(y_1,1),\Delta_{2k}(y_1,1),\{1,2\})\right]
\end{aligned}$$

We now apply Lemma A.2(j) and (i) to the first and second terms, respectively, on the RHS of the above inequality and we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1,0) - \Delta_{1,k+1}(y_1+1,1) \\
&\geq \lambda \int_0^{\Delta_{1k}(y_1,0)-\Delta_{1k}(y_1+1,0)} q_1^* \left(\Delta_{1k}(y_1+1,0)+z, \{1\}\right) dz \\
&- \lambda \int_0^{\Delta_{1k}(y_1,1)-\Delta_{1k}(y_1+1,0)} q_1^* \left(\Delta_{1k}(y_1+1,0)+z, \Delta_{2k}(y_1,1), \{1,2\}\right) dz
\end{aligned}$$

Now, on the RHS of the above inequality, observe that the range of the first integral is larger (because $\Delta_{1k}(y_1,0) \geq \Delta_{1k}(y_1,1)$ due to inequality $I_2(\mathbf{y},t)$, which holds for $t=k$ by the induction assumption) and the first integrand is also larger (this follows from Lemma A.2(h)). It now follows that the RHS of the above inequality is non-negative, concluding the proof for Case 2.

Case 3 — $y_1 = 0$ and $y_2 > 0$: In this case $S(y_1,y_2) = S(y_1,y_2+1) = \{2\}$ and $S(y_1+1,y_2) = \{1,2\}$. Hence, we can write (A.5) as follows:

$$\begin{aligned}
&\Delta_{1,k+1}(1,y_2) - \Delta_{1,k+1}(1,y_2+1) \\
&\geq \lambda \left[m^*(\Delta_{1k}(1,y_2), \Delta_{2k}(1,y_2), \{1,2\}) - m^*(\Delta_{2k}(0,y_2), \{2\})\right] \\
&- \lambda \left[m^*(\Delta_{1k}(1,y_2), \Delta_{2k}(0,y_2+1), \{1,2\}) - m^*(\Delta_{2k}(0,y_2+1), \{2\})\right] \qquad \text{(A.6)}
\end{aligned}$$

Because $\Delta_{2k}(1,y_2) \leq \Delta_{2k}(0,y_2)$, it follows from Lemma A.2(a) that

$$m^*(\Delta_{1k}(1,y_2), \Delta_{2k}(1,y_2), \{1,2\}) \geq m^*(\Delta_{1k}(1,y_2), \Delta_{2k}(0,y_2), \{1,2\}). \qquad \text{(A.7)}$$

In addition, observe from Lemma A.2(d) that

$$m^*(\Delta_{2k}(0,y_2),\{2\}) = m^*(\infty,\Delta_{2k}(0,y_2),\{1,2\}) \text{ and} \tag{A.8}$$

$$m^*(\Delta_{2k}(0,y_2+1),\{2\}) = m^*(\infty,\Delta_{2k}(0,y_2+1),\{1,2\}). \tag{A.9}$$

Utilizing (A.6), (A.7), (A.8) and (A.9), we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(1,y_2) - \Delta_{1,k+1}(1,y_2+1)\\
&\geq \lambda\left[m^*(\Delta_{1k}(1,y_2),\Delta_{2k}(0,y_2),\{1,2\}) - m^*(\infty,\Delta_{2k}(0,y_2),\{1,2\})\right]\\
&-\lambda\left[m^*(\Delta_{1k}(1,y_2),\Delta_{2k}(0,y_2+1),\{1,2\}) - m^*(\infty,\Delta_{2k}(0,y_2+1),\{1,2\})\right]
\end{aligned}$$

We now apply Lemma A.2(i) to each of the terms on the RHS of the above inequality and we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(1,y_2) - \Delta_{1,k+1}(1,y_2+1)\\
&\geq \lambda\int_0^\infty q_1^*\left(\Delta_{1k}(1,y_2)+z,\Delta_{2k}(0,y_2),\{1,2\}\right)dz\\
&-\lambda\int_0^\infty q_1^*\left(\Delta_{1k}(1,y_2)+z,\Delta_{2k}(0,y_2+1),\{1,2\}\right)dz
\end{aligned}$$

Now, on the RHS of the above inequality, observe that the first integrand is larger than the second integrand (this follows from Lemma A.2(g) and $\Delta_{2k}(0,y_2) \geq \Delta_{2k}(0,y_2+1)$). It now follows that the RHS of the above inequality is non-negative, concluding the proof for Case 3.

Case 4 — $y_1 = 0$ and $y_2 = 0$: In this case $S(y_1,y_2) = \emptyset$, $S(y_1,y_2+1) = \{2\}$ and $S(y_1+1,y_2) = \{1\}$. Furthermore, $m^*(\Delta_1(y_1,y_2),\Delta_2(y_1,y_2),S(y_1,y_2)) = 1$ when

$y_1 = y_2 = 0$.[1] Therefore, we can write (A.5) as follows:

$$\begin{aligned}
&\Delta_{1,k+1}(1,0) - \Delta_{1,k+1}(1,1) \\
&\geq \lambda \left[m^*(\Delta_{1k}(1,0), \{1\}) - 1)\right] \\
&- \lambda \left[m^*(\Delta_{1k}(1,0), \Delta_{2k}(0,1), \{1,2\}) - m^*(\Delta_{2k}(0,1), \{2\})\right]
\end{aligned} \tag{A.10}$$

From Lemma A.2(d), we have that

$$\begin{aligned}
m^*(\infty, \infty, \{1,2\}) &= 1, \\
m^*(\Delta_{1k}(1,0), \{1\}) &= m^*(\Delta_{1k}(1,0), \infty, \{1,2\}), \\
m^*(\Delta_{2k}(0,1), \{2\}) &= m^*(\infty, \Delta_{2k}(0,1), \{1,2\}).
\end{aligned}$$

Thus, we can rewrite the RHS of (A.10) as follows:

$$\begin{aligned}
&\Delta_{1,k+1}(1,0) - \Delta_{1,k+1}(1,1) \\
&\geq \lambda \left[m^*(\Delta_{1k}(1,0), \infty, \{1,2\}) - m^*(\infty, \infty, \{1,2\}))\right] \\
&- \lambda \left[m^*(\Delta_{1k}(1,0), \Delta_{2k}(0,1), \{1,2\}) - m^*(\infty, \Delta_{2k}(0,1), \{1,2\})\right]
\end{aligned}$$

We now apply Lemma A.2(i) to each of the terms on the RHS of the above inequality and we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(1,0) - \Delta_{1,k+1}(1,1) \\
&\geq \lambda \int_0^\infty q_1^* \left(\Delta_{1k}(1,0) + z, \infty, \{1,2\}\right) dz \\
&- \lambda \int_0^\infty q_1^* \left(\Delta_{1k}(1,0) + z, \Delta_{2k}(0,1), \{1,2\}\right) dz
\end{aligned}$$

[1] To see why, observe from (2.8) that if $m^* \neq 1$ when $y_1 = y_2 = 0$, then we would have $V_t(\mathbf{y}) \neq V_{t-1}(\mathbf{y})$ for $\mathbf{y} = 0$. However, that would be a contradiction because $V_t(\mathbf{y}) = V_{t-1}(\mathbf{y}) = 0$ for $\mathbf{y} = 0$.

The first integrand is greater than the second integrand due to Lemma A.2(g). Thus, the RHS is non-negative, which completes the proof for Case 4.

Proof of $I_3(\mathbf{y}, k+1)$: To facilitate the exposition, we will prove the result for $i = 1$. Symmetric arguments prove the result for $i = 2$. First, note that

$$\begin{aligned} &\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) \\ &= (V_k(y_1+1, y_2+1) - V_k(y_1, y_2+1)) - (V_k(y_1+2, y_2) - V_k(y_1+1, y_2)) \end{aligned} \tag{A.11}$$

Using (2.8), we can rewrite (A.11) as

$$\begin{aligned} &\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) = \Delta_{1k}(y_1+1, y_2+1) - \Delta_{1k}(y_1+2, y_2) \\ &+ \lambda\,[m^*(\mathbf{\Delta}_k(y_1+1, y_2+1), S(y_1+1, y_2+1)) - m^*(\mathbf{\Delta}_k(y_1, y_2+1), S(y_1, y_2+1))] \\ &- \lambda\,[m^*(\mathbf{\Delta}_k(y_1+2, y_2), S(y_1+2, y_2)) - m^*(\mathbf{\Delta}_k(y_1+1, y_2), S(y_1+1, y_2))] \end{aligned} \tag{A.12}$$

We will divide the proof into four cases, depending on whether $y_1 = 0$ and/or $y_2 = 0$. In this proof, we will write the vector $\mathbf{\Delta}_k(\mathbf{y})$ explicitly as a list of its components. (Recall that this vector may have one or two components depending on which product(s) are available.)

Case 1 — $y_1 > 0$ and $y_2 > 0$: In this case $S(y_1+1, y_2+1) = S(y_1, y_2+1) = S(y_1 +$

$1, y_2) = S(y_1 + 2, y_2) = \{1, 2\}$. Hence, we can rewrite (A.12) as

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) = \Delta_{1k}(y_1+1, y_2+1) - \Delta_{1k}(y_1+2, y_2) \\
&+ \lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+1, y_2+1), \Delta_{2k}(y_1+1, y_2+1), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1, y_2+1), \Delta_{2k}(y_1, y_2+1), \{1,2\}) \end{array} \right] \\
&- \lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+2, y_2), \Delta_{2k}(y_1+2, y_2), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1+1, y_2), \Delta_{2k}(y_1+1, y_2), \{1,2\}) \end{array} \right]
\end{aligned} \tag{A.13}$$

By Lemma A.2(e) and $\lambda \le 1$, we have:

$$\begin{aligned}
&\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+2, y_2), \Delta_{2k}(y_1+2, y_2), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1+1, y_2+1), D1_{2k}(y_1, y_2), \{1,2\}) \end{array} \right] \\
&\le \Delta_{1k}(y_1+1, y_2+1) - \Delta_{1k}(y_1+2, y_2)
\end{aligned}$$

where $D1_{2k}(y_1, y_2) := \Delta_{2k}(y_1+2, y_2) + \Delta_{1k}(y_1+1, y_2+1) - \Delta_{1k}(y_1+2, y_2)$. Applying the above inequality to (A.13) yields:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) \ge \\
&\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+1, y_2+1), \Delta_{2k}(y_1+1, y_2+1), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1, y_2+1), \Delta_{2k}(y_1, y_2+1), \{1,2\}) \end{array} \right] \\
-&\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+1, y_2+1), D1_{2k}(y_1, y_2), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1+1, y_2), \Delta_{2k}(y_1+1, y_2), \{1,2\}) \end{array} \right]
\end{aligned} \tag{A.14}$$

As an aside, note from Lemma A.1 that:

$$\Delta_{1k}(y_1, y_2+1) - \Delta_{1k}(y_1+1, y_2+1) \ge \Delta_{2k}(y_1, y_2+1) - \Delta_{2k}(y_1+1, y_2+1), \text{ and}$$

$$\Delta_{1k}(y_1+1, y_2) - \Delta_{1k}(y_1+2, y_2) \ge \Delta_{2k}(y_1+1, y_2) - \Delta_{2k}(y_1+2, y_2).$$

The following inequalities are obtained, respectively, from the last two inequalities, combined with the fact that $m^*(\Delta_1, \Delta_2, \{1,2\})$ is decreasing in Δ_i, $i = 1,2$ (see Lemma A.2(a)):

$$\begin{aligned}
&m^*(D2_{1k}(y_1,y_2), \Delta_{2k}(y_1, y_2+1), \{1,2\}) \\
&\geq m^*(\Delta_{1k}(y_1, y_2+1), \Delta_{2k}(y_1, y_2+1), \{1,2\}), \text{ and} \\
&m^*(\Delta_{1k}(y_1+1, y_2+1), D1_{2k}(y_1,y_2), \{1,2\}) \\
&\leq m^*(\Delta_{1k}(y_1+1, y_2+1), D3_{2k}(y_1,y_2), \{1,2\}).
\end{aligned}$$

where $D2_{1k}(y_1,y_2) := \Delta_{1k}(y_1+1, y_2+1) + \Delta_{2k}(y_1, y_2+1) - \Delta_{2k}(y_1+1, y_2+1)$ and $D3_{2k}(y_1,y_2) := \Delta_{2k}(y_1+1, y_2) + \Delta_{1k}(y_1+1, y_2+1) - \Delta_{1k}(y_1+1, y_2)$ By applying the last two inequalities to (A.14), we obtain:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) \geq \\
&\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+1, y_2+1), \Delta_{2k}(y_1+1, y_2+1), \{1,2\}) \\ -m^*(D2_{1k}(y_1,y_2), \Delta_{2k}(y_1, y_2+1), \{1,2\}) \end{array} \right] \\
&-\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1+1, y_2+1), D3_{2k}(y_1,y_2), \{1,2\}) \\ -m^*(\Delta_{1k}(y_1+1, y_2), \Delta_{2k}(y_1+1, y_2), \{1,2\}) \end{array} \right]
\end{aligned}$$

We now use Lemma A.2(e) to rewrite each of the two terms in brackets on the RHS of the above inequality:

$$\begin{aligned}
&\Delta_{1,k+1}(y_1+1, y_2+1) - \Delta_{1,k+1}(y_1+2, y_2) \geq \\
&\lambda \left[\begin{array}{l} \Delta_{2k}(y_1, y_2+1) - \Delta_{2k}(y_1+1, y_2+1) \\ -\int_0^{\Delta_{2k}(y_1,y_2+1)-\Delta_{2k}(y_1+1,y_2+1)} \frac{1}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z, \Delta_{2k}(y_1+1,y_2+1)+z, \{1,2\})} dz \end{array} \right] \\
&-\lambda \left[\begin{array}{l} \Delta_{1k}(y_1+1, y_2) - \Delta_{1k}(y_1+1, y_2+1) \\ -\int_0^{\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)} \frac{1}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z, D3_{2k}(y_1,y_2)+z, \{1,2\})} dz \end{array} \right]
\end{aligned}$$

Noting that $\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)=\Delta_{2k}(y_1,y_2+1)-\Delta_{2k}(y_1+1,y_2+1)$, we can simplify the above inequality:

$$\Delta_{1,k+1}(y_1+1,y_2+1)-\Delta_{1,k+1}(y_1+2,y_2)\geq$$
$$\lambda\left[\begin{array}{l} -\int_0^{\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)}\frac{1}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z,\Delta_{2k}(y_1+1,y_2+1)+z,\{1,2\})}dz \\ +\int_0^{\Delta_{1k}(y_1+1,y_2)-\Delta_{1k}(y_1+1,y_2+1)}\frac{1}{m^*(\Delta_{1k}(y_1+1,y_2+1)+z,D3_{2k}(y_1,y_2)+z,\{1,2\})}dz \end{array}\right]$$

Finally, the term in brackets is non-negative since $\Delta_{2k}(y_1+1,y_2+1)\leq\Delta_{2k}(y_1+1,y_2)+\Delta_{1k}(y_1+1,y_2+1)-\Delta_{1k}(y_1+1,y_2)$ (by Lemma A.1) and $m^*(\Delta_1,\Delta_2,\{1,2\})$ is decreasing in Δ_i, $i=1,2$ (by Lemma A.2(a)). This completes the proof for Case 1.

<u>*Case 2 — $y_1>0$ and $y_2=0$*:</u> In this case $S(y_1+1,y_2+1)=S(y_1,y_2+1)=\{1,2\}$ and $S(y_1+2,y_2)=S(y_1+1,y_2)=\{1\}$. Hence, We can rewrite (A.12) as

$$\begin{aligned}&\Delta_{1,k+1}(y_1+1,1)-\Delta_{1,k+1}(y_1+2,0)=\Delta_{1k}(y_1+1,1)-\Delta_{1k}(y_1+2,0)\\&+\lambda\left[m^*(\Delta_{1k}(y_1+1,1),\Delta_{2k}(y_1+1,1),\{1,2\})-m^*(\Delta_{1k}(y_1,1),\Delta_{2k}(y_1,1),\{1,2\})\right]\\&-\lambda\left[m^*(\Delta_{1k}(y_1+2,0),\{1\})-m^*(\Delta_{1k}(y_1+1,0),\{1\})\right]\end{aligned}\qquad\text{(A.15)}$$

By Lemma A.2(f) and $\lambda\leq 1$, we have:

$$\lambda\left[m^*(\Delta_{1k}(y_1+2,0),\{1\})-m^*(\Delta_{1k}(y_1+1,1),\{1\})\right]\leq\Delta_{1k}(y_1+1,1)-\Delta_{1k}(y_1+2,0)$$

Applying the above inequality to (A.15) yields:

$$\begin{aligned}&\Delta_{1,k+1}(y_1+1,1)-\Delta_{1,k+1}(y_1+2,0)\geq\\&\lambda\left[m^*(\Delta_{1k}(y_1+1,1),\Delta_{2k}(y_1+1,1),\{1,2\})-m^*(\Delta_{1k}(y_1,1),\Delta_{2k}(y_1,1),\{1,2\})\right]\\&-\lambda\left[m^*(\Delta_{1k}(y_1+1,1),\{1\})-m^*(\Delta_{1k}(y_1+1,0),\{1\})\right]\end{aligned}\qquad\text{(A.16)}$$

As an aside, note from Lemma A.1 that:

$$\Delta_{1k}(y_1, 1) - \Delta_{1k}(y_1 + 1, 1) \geq \Delta_{2k}(y_1, 1) - \Delta_{2k}(y_1 + 1, 1)$$

Hence, it follows from Lemma A.2(a) that:

$$\begin{aligned} &m^*(\Delta_{1k}(y_1 + 1, 1) + \Delta_{2k}(y_1, 1) - \Delta_{2k}(y_1 + 1, 1), \Delta_{2k}(y_1, 1), \{1, 2\}) \\ &\geq m^*(\Delta_{1k}(y_1, 1), \Delta_{2k}(y_1, 1), \{1, 2\}) \end{aligned}$$

Substituting from the last inequality in (A.16), we obtain:

$$\begin{aligned} &\Delta_{1,k+1}(y_1 + 1, 1) - \Delta_{1,k+1}(y_1 + 2, 0) \\ &\geq \lambda \left[\begin{array}{l} m^*(\Delta_{1k}(y_1 + 1, 1), \Delta_{2k}(y_1 + 1, 1), \{1, 2\}) \\ -m^*(\Delta_{1k}(y_1 + 1, 1) + \Delta_{2k}(y_1, 1) - \Delta_{2k}(y_1 + 1, 1), \Delta_{2k}(y_1, 1), \{1, 2\}) \end{array} \right] \\ &- \lambda \left[m^*(\Delta_{1k}(y_1 + 1, 1), \{1\}) - m^*(\Delta_{1k}(y_1 + 1, 0), \{1\}) \right] \end{aligned}$$

We use Lemma A.2(e) and (f) to rewrite, respectively, the first and second terms in brackets on the RHS of the above inequality. We then note that $\Delta_{1k}(y_1 + 1, 0) - \Delta_{1k}(y_1 + 1, 1) = \Delta_{2k}(y_1, 1) - \Delta_{2k}(y_1 + 1, 1)$. These steps simplify the above inequality as follows:

$$\begin{aligned} &\Delta_{1,k+1}(y_1 + 1, 1) - \Delta_{1,k+1}(y_1 + 2, 0) \\ &\geq \lambda \left[\begin{array}{l} -\int_0^{\Delta_{1k}(y_1+1,0)-\Delta_{1k}(y_1+1,1)} \frac{1}{m^*(\Delta_{1k}(y_1+1,1)+z, \Delta_{2k}(y_1+1,1)+z, \{1,2\})} dz \\ +\int_0^{\Delta_{1k}(y_1+1,0)-\Delta_{1k}(y_1+1,1)} \frac{1}{m^*(\Delta_{1k}(y_1+1,1)+z, \{1\})} dz \end{array} \right] \end{aligned}$$

The first integrand above is smaller than the second integrand, because $m^*(\Delta_{1k}(y_1 + 1, 1) + z, \Delta_{2k}(y_1 + 1, 1) + z, \{1, 2\}) \geq m^*(\Delta_{1k}(y_1 + 1, 1) + z, \{1\})$ by Lemma A.2(c). Thus, the term in brackets on the RHS above is non-negative, and this completes the proof for Case 2.

<u>*Case 3:* $y_1 = 0$ *and* $y_2 > 0$:</u> In this case $S(y_1+1, y_2+1) = S(y_1+2, y_2) = S(y_1+1, y_2) = \{1, 2\}$ and $S(y_1, y_2+1) = \{2\}$. Hence, we can rewrite (A.12) as

$$\begin{aligned}
&\Delta_{1,k+1}(1, y_2+1) - \Delta_{1,k+1}(2, y_2) = \Delta_{1k}(1, y_2+1) - \Delta_{1k}(2, y_2) \\
&+ \lambda \left[m^*(\Delta_{1k}(1, y_2+1), \Delta_{2k}(1, y_2+1), \{1, 2\}) - m^*(\Delta_{2k}(0, y_2+1), \{2\})\right] \\
&- \lambda \left[m^*(\Delta_{1k}(2, y_2), \Delta_{2k}(2, y_2), \{1, 2\}) - m^*(\Delta_{1k}(1, y_2), \Delta_{2k}(1, y_2), \{1, 2\})\right] \qquad \text{(A.17)}
\end{aligned}$$

By Lemma A.2(d) and $\lambda \leq 1$, we have:

$$\begin{aligned}
&\lambda \begin{bmatrix} m^*(\Delta_{1k}(2, y_2), \Delta_{2k}(2, y_2), \{1, 2\}) \\ -m^*(\Delta_{1k}(1, y_2+1), \Delta_{2k}(2, y_2) + \Delta_{1k}(1, y_2+1) - \Delta_{1k}(2, y_2), \{1, 2\}) \end{bmatrix} \\
&\leq \Delta_{1k}(1, y_2+1) - \Delta_{1k}(2, y_2)
\end{aligned}$$

Applying the above inequality to (A.17) yields:

$$\begin{aligned}
&\Delta_{1,k+1}(1, y_2+1) - \Delta_{1,k+1}(2, y_2) \\
&\geq \lambda \left[m^*(\Delta_{1k}(1, y_2+1), \Delta_{2k}(1, y_2+1), \{1, 2\}) - m^*(\Delta_{2k}(0, y_2+1), \{2\})\right] \\
&-\lambda \begin{bmatrix} m^*(\Delta_{1k}(1, y_2+1), \Delta_{2k}(2, y_2) + \Delta_{1k}(1, y_2+1) - \Delta_{1k}(2, y_2), \{1, 2\}) \\ -m^*(\Delta_{1k}(1, y_2), \Delta_{2k}(1, y_2), \{1, 2\}) \end{bmatrix}
\end{aligned}$$

As an aside, note from Lemma A.1 that:

$$\Delta_{1k}(1, y_2) - \Delta_{1k}(2, y_2) \geq \Delta_{2k}(1, y_2) - \Delta_{2k}(2, y_2),$$

Hence, it follows from Lemma A.2(a) that:

$$
\begin{aligned}
&m^*(\Delta_{1k}(1,y_2+1),\Delta_{2k}(2,y_2)+\Delta_{1k}(1,y_2+1)-\Delta_{1k}(2,y_2),\{1,2\}) \\
&\le m^*(\Delta_{1k}(1,y_2+1),\Delta_{2k}(1,y_2)+\Delta_{1k}(1,y_2+1)-\Delta_{1k}(1,y_2),\{1,2\}) \qquad \text{(A.18)}
\end{aligned}
$$

Furthermore, from Lemma A.2(c), we have:

$$
\begin{aligned}
&m^*(\Delta_{1k}(1,y_2+1)+\Delta_{2k}(0,y_2+1)-\Delta_{2k}(1,y_2+1),\Delta_{2k}(0,y_2+1),\{1,2\}) \\
&\ge m^*(\Delta_{2k}(0,y_2+1),\{2\}) \qquad \text{(A.19)}
\end{aligned}
$$

(A.18), (A.18) and (A.19) together yield:

$$
\begin{aligned}
&\Delta_{1,k+1}(1,y_2+1)-\Delta_{1,k+1}(2,y_2) \\
&\ge \lambda \left[\begin{array}{l} m^*(\Delta_{1k}(1,y_2+1),\Delta_{2k}(1,y_2+1),\{1,2\}) \\ -m^*(\Delta_{1k}(1,y_2+1)+\Delta_{2k}(0,y_2+1)-\Delta_{2k}(1,y_2+1),\Delta_{2k}(0,y_2+1),\{1,2\}) \end{array} \right] \\
&- \lambda \left[\begin{array}{l} m^*(\Delta_{1k}(1,y_2+1),\Delta_{2k}(1,y_2)+\Delta_{1k}(1,y_2+1)-\Delta_{1k}(1,y_2),\{1,2\}) \\ -m^*(\Delta_{1k}(1,y_2),\Delta_{2k}(1,y_2),\{1,2\}) \end{array} \right]
\end{aligned}
$$

We use Lemma A.2(e) to rewrite each term in brackets on the RHS of the above inequality and we note that $\Delta_{1k}(1,y_2)-\Delta_{1k}(1,y_2+1)=\Delta_{2k}(0,y_2+1)-\Delta_{2k}(1,y_2+1)$. The above inequality simplifies as follows:

$$
\begin{aligned}
&\Delta_{1,k+1}(1,y_2+1)-\Delta_{1,k+1}(2,y_2) \\
&\ge \lambda \left[\begin{array}{l} -\int_0^{\Delta_{1k}(1,y_2)-\Delta_{1k}(1,y_2+1)} \frac{dz}{m^*(\Delta_{1k}(1,y_2+1)+z,\Delta_{2k}(1,y_2+1)+z,\{1,2\})} \\ +\int_0^{\Delta_{1k}(1,y_2)-\Delta_{1k}(1,y_2+1)} \frac{dz}{m^*(\Delta_{1k}(1,y_2+1)+z,\Delta_{2k}(1,y_2)+\Delta_{1k}(1,y_2+1)-\Delta_{1k}(1,y_2)+z,\{1,2\})} \end{array} \right]
\end{aligned}
$$

Now, the first integrand above is smaller than the second integrand, because $\Delta_{2k}(1,y_2+1)\le\Delta_{2k}(1,y_2)+\Delta_{1k}(1,y_2+1)-\Delta_{1k}(1,y_2)$ (by Lemma A.1) and $m^*(\Delta_1,\Delta_2,\{1,2\})$

is decreasing in Δ_i, $i = 1, 2$ (see Lemma A.2(a)). Thus, the term in brackets on the RHS is non-negative, which completes the proof of this case.

<u>*Case 4* — $y_1 = y_2 = 0$:</u> In this case $S(y_1+1, y_2+1) = \{1, 2\}$, $S(y_1+2, y_2) = S(y_1+1, y_2) = \{1\}$ and $S(y_1, y_2+1) = \{2\}$. Hence, we can rewrite (A.12) as

$$\begin{aligned}&\Delta_{1,k+1}(1,1) - \Delta_{1,k+1}(2,0) = \Delta_{1k}(1,1) - \Delta_{1k}(2,0)\\&+ \lambda\left[m^*(\Delta_{1k}(1,1), \Delta_{2k}(1,1), \{1,2\}) - m^*(\Delta_{2k}(0,1), \{2\})\right]\\&- \lambda\left[m^*(\Delta_{1k}(2,0), \{1\}) - m^*(\Delta_{1k}(1,0), \{1\})\right] \qquad \text{(A.20)}\end{aligned}$$

By Lemma A.2(f) and $\lambda \leq 1$, we have:

$$\lambda\left[m^*(\Delta_{1k}(2,0), \{1\}) - m^*(\Delta_{1k}(1,1), \{1\})\right] \leq \Delta_{1k}(1,1) - \Delta_{1k}(2,0)$$

Applying the above inequality to (A.20) yields:

$$\begin{aligned}&\Delta_{1,k+1}(1,1) - \Delta_{1,k+1}(2,0)\\&\geq \lambda\left[m^*(\Delta_{1k}(1,1), \Delta_{2k}(1,1), \{1,2\}) - m^*(\Delta_{2k}(0,1), \{2\})\right]\\&- \lambda\left[m^*(\Delta_{1k}(1,1), \{1\}) - m^*(\Delta_{1k}(1,0), \{1\})\right] \qquad \text{(A.21)}\end{aligned}$$

Note from Lemma A.2(c) that

$$m^*(\Delta_{2k}(0,1), \{2\}) \leq m^*(\Delta_{1k}(1,1) + \Delta_{2k}(0,1) - \Delta_{2k}(1,1), \Delta_{2k}(0,1), \{1,2\}). \qquad \text{(A.22)}$$

(A.21) and (A.22) together yield:

$$\begin{aligned}
&\Delta_{1,k+1}(1,1) - \Delta_{1,k+1}(2,0) \geq \\
&\lambda \left[\begin{array}{l} m^*(\Delta_{1k}(1,1), \Delta_{2k}(1,1), \{1,2\}) \\ -m^*(\Delta_{1k}(1,1) + \Delta_{2k}(0,1) - \Delta_{2k}(1,1), \Delta_{2k}(0,1), \{1,2\}) \end{array} \right] \\
&-\lambda \left[m^*(\Delta_{1k}(1,1), \{1\}) - m^*(\Delta_{1k}(1,0), \{1\}) \right]
\end{aligned}$$

We use Lemma A.2(e) and (f) to rewrite, respectively, the first and second terms in brackets on the RHS of the above inequality. In addition, we note that $\Delta_{1k}(1,0) - \Delta_{1k}(1,1) = \Delta_{2k}(0,1) - \Delta_{2k}(1,1)$. These steps allow us to simplify the above inequality as follows:

$$\Delta_{1,k+1}(1,1) - \Delta_{1,k+1}(2,0) \geq \lambda \left[\begin{array}{l} -\int_0^{\Delta_{1k}(1,0)-\Delta_{1k}(1,1)} \frac{1}{m^*(\Delta_{1k}(1,1)+z, \Delta_{2k}(1,1)+z, \{1,2\})} dz \\ +\int_0^{\Delta_{1k}(1,0)-\Delta_{1k}(1,1)} \frac{1}{m^*(\Delta_{1k}(1,1)+z, \{1\})} dz \end{array} \right]$$

The first integrand is smaller than the second, because $m^*(\Delta_{1k}(1,1) + z, \Delta_{2k}(1,1) + z, \{1,2\}) \geq m^*(\Delta_{1k}(1,1) + z, \{1\})$ by Lemma A.2(c). Thus, the term in brackets on the RHS of the above inequality is non-negative, which completes the proof of $I_3(\mathbf{y}, k+1)$.

Proof of $I_4(\mathbf{y}, k+1)$: Note that

$$\Delta_{i,k+2}(\mathbf{y} + \mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y} + \mathbf{e_i}) = (V_{k+1}(\mathbf{y} + \mathbf{e_i}) - V_{k+1}(\mathbf{y})) - (V_k(\mathbf{y} + \mathbf{e_i}) - V_k(\mathbf{y}))$$

Now, using (2.8) in Lemma II.2, the equation above simplifies to:

$$\begin{aligned}
&\Delta_{i,k+2}(\mathbf{y} + \mathbf{e_i}) - \Delta_{i,k+1}(\mathbf{y} + \mathbf{e_i}) \\
&= \lambda m^*(\boldsymbol{\Delta}_{k+1}(\mathbf{y} + \mathbf{e_i}), S(\mathbf{y} + \mathbf{e_i})) - \lambda m^*(\boldsymbol{\Delta}_{k+1}(\mathbf{y}), S(\mathbf{y}))
\end{aligned} \tag{A.23}$$

Given that we already proved $I_1(\mathbf{y}, k+1)$ and $I_2(\mathbf{y}, k+1)$, we have $\mathbf{\Delta}_{k+1}(\mathbf{y}+\mathbf{e_i}) \leq \mathbf{\Delta}_{k+1}(\mathbf{y})$. Therefore, $m^*(\mathbf{\Delta}_{k+1}(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) \geq m^*(\mathbf{\Delta}_{k+1}(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i}))$ (because $m^*(\Delta_1, \Delta_2, \{1,2\})$ is decreasing in Δ_i, $i = 1, 2$ by Lemma A.2(a)). Hence, (A.23) is non-negative, which in turn implies that $I_4(\mathbf{y}, k+1)$ holds.

Proofs of Propositions II.5 through II.8

Proof of Proposition II.5: From Lemma II.1, $p^*_{it}(\mathbf{y}) = m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) + \Delta_{it}(\mathbf{y})$. Therefore, for $y_i \geq 1$:

$$
\begin{aligned}
& p^*_{it}(\mathbf{y}) - p^*_{it}(\mathbf{y}+\mathbf{e_i}) \\
&= m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) + \Delta_{it}(\mathbf{y}) - m^*(\mathbf{\Delta}_t(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) - \Delta_{it}(\mathbf{y}+\mathbf{e_i}) \\
&= (\Delta_{it}(\mathbf{y}) - \Delta_{it}(\mathbf{y}+\mathbf{e_i})) - [m^*(\mathbf{\Delta}_t(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) - m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))] \\
&\geq 0
\end{aligned}
$$

where the last inequality follows from Lemma A.4.

Proof of Proposition II.6: Since $q^*_{0t}(\mathbf{y}) = \frac{1}{m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))}$ by (2.7) in Lemma II.2, parts (a) and (b) of the result follow from Lemma A.3(a) and (b), respectively.

Proof of Proposition II.7: From Lemma II.1, we have $p^*_{it}(\mathbf{y}+\mathbf{e_i}) - p^*_{jt}(\mathbf{y}+\mathbf{e_i}) =$

$\Delta_{it}(\mathbf{y}+\mathbf{e_i}) - \Delta_{jt}(\mathbf{y}+\mathbf{e_i})$ and $p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y}) = \Delta_{it}(\mathbf{y}) - \Delta_{jt}(\mathbf{y})$. Therefore:

$$\begin{aligned}
&(p^*_{it}(\mathbf{y}+\mathbf{e_i}) - p^*_{jt}(\mathbf{y}+\mathbf{e_i})) - (p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y})) \\
&= (\Delta_{it}(\mathbf{y}+\mathbf{e_i}) - \Delta_{jt}(\mathbf{y}+\mathbf{e_i})) - (\Delta_{it}(\mathbf{y}) - \Delta_{jt}(\mathbf{y})) \\
&= -(\Delta_{it}(\mathbf{y}) - \Delta_{it}(\mathbf{y}+\mathbf{e_i})) + (\Delta_{jt}(\mathbf{y}) - \Delta_{jt}(\mathbf{y}+\mathbf{e_i})) \\
&\leq 0
\end{aligned}$$

where the last inequality follows from Lemma A.1. This proves part (a) of the proposition. Similarly, to prove part (b) of the proposition, note from Lemma A.1 that:

$$\begin{aligned}
&(p^*_{it}(\mathbf{y}+\mathbf{e_j}) - p^*_{jt}(\mathbf{y}+\mathbf{e_j})) - (p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y})) \\
&= (\Delta_{it}(\mathbf{y}+\mathbf{e_j}) - \Delta_{jt}(\mathbf{y}+\mathbf{e_j})) - (\Delta_{it}(\mathbf{y}) - \Delta_{jt}(\mathbf{y})) \\
&= -(\Delta_{it}(\mathbf{y}) - \Delta_{it}(\mathbf{y}+\mathbf{e_j})) + (\Delta_{jt}(\mathbf{y}) - \Delta_{jt}(\mathbf{y}+\mathbf{e_j})) \\
&\geq 0
\end{aligned}$$

Proof of Proposition II.8: Recall that $\overline{q}^*_{it}(\mathbf{y}) = \frac{q^*_{it}(\mathbf{y})}{1-q^*_{0t}(\mathbf{y})} = \frac{q^*_{it}(\mathbf{y})}{q^*_{1t}(\mathbf{y})+q^*_{2t}(\mathbf{y})}$. Substituting for $q^*_{it}(\mathbf{y})$ and $q^*_{0t}(\mathbf{y})$ from (2.1) and (2.2), and rearranging the terms, we obtain:

$$\overline{q}^*_{it}(\mathbf{y}) = \frac{1}{1+\exp(p^*_{it}(\mathbf{y}) - p^*_{jt}(\mathbf{y}) - u_i + u_j)}$$

The result now follows from Proposition II.7.

Proofs of Lemmas A.1 through A.4

Proof of Lemma A.1: One can check that:

$$\begin{aligned}
&(\Delta_{it}(\mathbf{y}+\mathbf{e_i}) - \Delta_{it}(\mathbf{y}+2\mathbf{e_i})) - (\Delta_{jt}(\mathbf{y}+\mathbf{e_i}) - \Delta_{jt}(\mathbf{y}+2\mathbf{e_i})) \\
&= \Delta_{it}(\mathbf{y}+\mathbf{e_i}) - \Delta_{it}(\mathbf{y}+2\mathbf{e_i}-\mathbf{e_j}) \\
&\geq 0,
\end{aligned}$$

where the inequality follows from $I_3(\mathbf{y}-\mathbf{e_j}, t)$.

Proof of Lemma A.2:

Proof of (a): Note from Lemma II.2 that if both products are available and their marginal values are Δ_1, Δ_2 then the optimal margin $m^*(\Delta_1, \Delta_2, \{1,2\})$ satisfies

$$\begin{aligned}
m^*(\Delta_1, \Delta_2, \{1,2\}) &= \frac{1}{q_0(m^*(\Delta_1, \Delta_2, \{1,2\}) + \Delta_1, m^*(\Delta_1, \Delta_2, \{1,2\}) + \Delta_2, \{1,2\})} \\
&= 1 + \sum_{i=1}^{2} \exp(u_i - m^*(\Delta_1, \Delta_2, \{1,2\}) - \Delta_i)
\end{aligned}$$

Rearranging the terms, we can write:

$$(m^*(\Delta_1, \Delta_2, \{1,2\}) - 1)\exp(m^*(\Delta_1, \Delta_2, \{1,2\})) = \sum_{i=1}^{2} \exp(u_i - \Delta_i) \qquad \text{(A.24)}$$

Notice from above that the left-hand side of (A.24) is increasing in m^* and the right-hand side is decreasing in Δ_i. Hence, if Δ_i increases, then $m^*(\Delta_1, \Delta_2, \{1,2\})$ decreases.

Proof of (b): With only product i available, (A.24) now changes as follows:

$$(m^*(\Delta_i, \{i\}) - 1)\exp(m^*(\Delta_i, \{i\})) = \exp(u_i - \Delta_i) \qquad \text{(A.25)}$$

It now follows from (A.25) that $m^*(\Delta_i, \{i\})$ is decreasing in Δ_i.

Proof of (c): While $m^*(\Delta_1, \Delta_2, \{1,2\})$ satisfies (A.24), $m^*(\Delta_i, \{i\})$ satisfies (A.25). For a given pair of Δ_1 and Δ_2 values, the right-hand side of (A.24) is larger than or equal to that of (A.25). Hence, $m^*(\Delta_1, \Delta_2, \{1,2\})$ is larger than or equal to $m^*(\Delta_i, \{i\})$.

Proof of (d): The statements that $m^*(\Delta_1, \{1\}) = m^*(\Delta_1, \infty, \{1,2\})$ and $m^*(\Delta_2, \{2\}) = m^*(\infty, \Delta_2, \{1,2\})$ follow immediately from (A.24) and (A.25). To see why, $m^*(\Delta_1 = \infty, \Delta_2 = \infty, \{1,2\}) = 1$, first observe from (A.24) that when $\Delta_1 = \Delta_2 = \infty$, the RHS of (A.24) is zero. Therefore, there are only two possible solutions for $m^*(\Delta_1 = \infty, \Delta_2 = \infty, \{1,2\})$: either 1 or $-\infty$. However, the latter solution can never occur, because $m^*(\Delta_1, \Delta_2, \{1,2\})$ can never be negative. Thus, it follows that $m^*(\Delta_1 = \infty, \Delta_2 = \infty, \{1,2\}) = 1$.

Proof of (e): For any $z \geq 0$, we can use (A.24) to write:

$$(m^*(\Delta_1 + z, \Delta_2 + z, S) - 1)\exp(m^*(\Delta_1 + z, \Delta_2 + z, S)) = \sum_{i \in S} \exp(u_i - \Delta_i - z)$$

Implicit differentiation of the above identity with respect to z and some algebra yields:

$$\frac{dm^*(\Delta_1 + z, \Delta_2 + z, S)}{dz} = -1 + \frac{1}{m^*(\Delta_1 + z, \Delta_2 + z, S)}$$

Hence:

$$\begin{aligned} m^*(\Delta_1 + x, \Delta_2 + x, S) - m^*(\Delta_1, \Delta_2, S) &= \int_0^x \left(-1 + \frac{1}{m^*(\Delta_1 + z, \Delta_2 + z, S)}\right) dz \\ &= -x + \int_0^x \frac{1}{m^*(\Delta_1 + z, \Delta_2 + z, S)} dz \end{aligned}$$

Proof of (f): The proof is similar to that of part (d).

Proof of (g): For ease of notation, we use the short-hand m^* in place of $m^*(\Delta_1, \Delta_2, \{1,2\})$.

First, using the chain rule:

$$\frac{dq_i(\Delta_1+m^*,\Delta_2+m^*,\{1,2\})}{d\Delta_j} = \frac{\partial q_i(\Delta_1+m^*,\Delta_2+m^*,\{1,2\})}{\partial \Delta_j} + \left.\frac{\partial q_i(\Delta_1+m,\Delta_2+m,\{1,2\})}{\partial m}\right|_{m=m^*} \frac{dm^*}{d\Delta_j} \quad \text{(A.26)}$$

In addition, from (2.1), we have

$$q_i(\Delta_1+m,\Delta_2+m,\{1,2\}) = \frac{\exp(u_i-\Delta_i-m)}{1+\exp(u_1-\Delta_1-m)+\exp(u_2-\Delta_2-m)}.$$

Now, by implicit differentiation of the above equality, we get:

$$\begin{aligned}
\frac{\partial q_i(\Delta_1+m,\Delta_2+m,\{1,2\})}{\partial \Delta_j} &= q_i(\Delta_1+m,\Delta_2+m,\{1,2\})q_j(\Delta_1+m,\Delta_2+m,\{1,2\}) \\
&\geq 0 \\
\frac{\partial q_i(\Delta_1+m,\Delta_2+m,\{1,2\})}{\partial m} &= -q_i(\Delta_1+m,\Delta_2+m,\{1,2\}) \\
&\times (1-q_0(\Delta_1+m,\Delta_2+m,\{1,2\})) \leq 0
\end{aligned}$$

Given the signs of the partial derivatives above and the fact that m^* decreases in Δ_j, we now observe from (A.26) that $\frac{dq_i(\Delta_1+m^*,\Delta_2+m^*,\{1,2\})}{d\Delta_j} \geq 0$.

Proof of (h): From (2.1), we have

$$\begin{aligned}
q_1^*(\Delta_1,\{1\}) &= \frac{\exp[u_1-\Delta_1-m^*(\Delta_1,\{1\})]}{1+\exp[u_1-\Delta_1-m^*(\Delta_1,\{1\})]}, \\
q_1^*(\Delta_1,\Delta_2,\{1,2\}) &= \frac{\exp[u_1-\Delta_1-m^*(\Delta_1,\Delta_2,\{1,2\})]}{1+\sum_{i=1}^{2}\exp[u_i-\Delta_i-m^*(\Delta_1,\Delta_2,\{1,2\})]}
\end{aligned}$$

Regrouping the terms in the above equalities:

$$\begin{aligned}
q_1^*(\Delta_1,\{1\}) &= \frac{\exp(u_1-\Delta_1)}{\exp(m^*(\Delta_1,\{1\}))+\exp(u_1-\Delta_1)}, \\
q_1^*(\Delta_1,\Delta_2,\{1,2\}) &= \frac{\exp(u_1-\Delta_1)}{\exp(m^*(\Delta_1,\Delta_2,\{1,2\}))+\sum_{i=1}^{2}\exp[u_i-\Delta_i]}
\end{aligned}$$

The result follows by comparing the last two equalities and remembering that $m^*(\Delta_1, \Delta_2, \{1, 2\}) \geq m^*(\Delta_1, \{1\})$ (by part (c) of the lemma).

Proof of (i): For any $z \geq 0$, we can use (A.24) to write:

$$(m^*(\Delta_1 + z, \Delta_2, S) - 1) \exp(m^*(\Delta_1 + z, \Delta_2, S)) = \exp(u_1 - \Delta_1 - z) + \exp(u_2 - \Delta_2)$$

Implicit differentiation of the above identity with respect to z and some algebra yields:

$$\frac{dm^*(\Delta_1 + z, \Delta_2, S)}{dz} = -q_1(\Delta_1 + z + m^*(\Delta_1 + z, \Delta_2, S), \Delta_2 + m^*(\Delta_1 + z, \Delta_2, S))$$

Hence:

$$\begin{aligned} & m^*(\Delta_1 + x, \Delta_2, S) - m^*(\Delta_1, \Delta_2, S) \\ &= -\int_0^x q_1(\Delta_1 + z + m^*(\Delta_1 + z, \Delta_2, S), \Delta_2 + m^*(\Delta_1 + z, \Delta_2, S)) dz \\ &= -\int_0^x q_1^*(\Delta_1 + z, \Delta_2) dz \end{aligned}$$

where the last inequality follows from the definition of $q_1^*(\Delta_1 + z, \Delta_2)$ provided in part (g) of the lemma.

Proof of (j): The proof is similar to that of part (i).

Proof of Lemma A.3:

Proof of (a): We divide the proof into four cases depending on which product(s) are available.

Case 1 — $\mathbf{y} > 0$: In this case, $S(\mathbf{y}) = S(\mathbf{y} + \mathbf{e_i}) = \{1, 2\}$, that is, both products are available when the stock level vector is $\mathbf{y}$ or $\mathbf{y} + \mathbf{e_i}$. Since $I_1(y, t)$ through $I_4(y, t)$ hold, we have $\Delta_{it}(\mathbf{y}) \geq \Delta_{it}(\mathbf{y} + \mathbf{e_i})$ by $I_1(y, t)$ and $\Delta_{jt}(\mathbf{y}) \geq \Delta_{jt}(\mathbf{y} + \mathbf{e_i})$ by $I_2(y, t)$. It now follows from Lemma A.2(a) that $m^*(\mathbf{\Delta}_t(\mathbf{y}+\mathbf{e_i}), S(\mathbf{y}+\mathbf{e_i})) \geq m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$.

Case 2 — $y_i > 0, y_j = 0$: In this case, $S(\mathbf{y}) = S(\mathbf{y}+\mathbf{e_i}) = \{i\}$, that is, only product i is available when the stock level vector is $\mathbf{y}$ or $\mathbf{y}+\mathbf{e_i}$. Since $I_1(y,t)$ through $I_4(y,t)$ hold, we have $\Delta_{it}(\mathbf{y}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i})$ by $I_1(y,t)$. The result follows from Lemma A.2(b).

Case 3 — $y_i = 0, y_j > 0$: In this case, $S(\mathbf{y}) = \{j\}$, but $S(\mathbf{y}+\mathbf{e_i}) = \{1,2\}$. Since $I_1(y,t)$ through $I_4(y,t)$ hold, $\Delta_{jt}(\mathbf{y}+\mathbf{e_i}) \leq \Delta_{jt}(\mathbf{y})$ by $I_2(y,t)$. The result now follows from Lemma A.2(c).

Case 4 — $\mathbf{y} = 0$: In this case, $S(\mathbf{y}) = \emptyset$, but $S(\mathbf{y}+\mathbf{e_i}) = \{i\}$. According to our convention, $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(y)) = 1$ when there is no product left to sell. In contrast, $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(y)) > 1$ whenever $y_1 > 0$ or $y_2 > 0$. (To see why, recall that $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(y))$ is the inverse of no-purchase probability, which yields a quantity larger than 1 whenever there is at least one unit of a product to sell.) The result follows.

Proof of (b): Suppose both products are available, i.e., $S = \{1,2\}$. Since $I_1(y,t)$ through $I_4(y,t)$ hold, we have $\Delta_{i,t+1}(\mathbf{y}) \geq \Delta_{it}(\mathbf{y}+\mathbf{e_i})$ for $i = 1,2$ by $I_4(y,t)$. The result now follows from Lemma A.2(a). The result goes through similarly when only one product is available, using Lemma A.2(b). If no product is available, i.e., $S = \emptyset$, the result holds trivially because $m^*(\mathbf{\Delta}_{t+1}(\mathbf{y}), S(\mathbf{y})) = m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = 1$ when $\mathbf{y} = 0$.

Proof of Lemma A.4: To facilitate the exposition, we will prove the result for $i = 1$. Symmetric arguments work for $i = 2$. There are two cases to consider, depending on whether or not product 2 is available: $y_2 > 0$ and $y_2 = 0$. We will prove the result for the first case, $y_2 > 0$. The proof follows similarly for the case where $y_2 = 0$. Note that, by Lemma A.1, we have $\Delta_{2t}(y_1+1, y_2) - \Delta_{2t}(y_1+2, y_2) \leq \Delta_{1t}(y_1+1, y_2) - \Delta_{1t}(y_1+2, y_2)$.

Hence, it follows from Lemma A.2(a) that:

$$
\begin{aligned}
& m^*(\Delta_{1t}(y_1+2,y_2),\Delta_{2t}(y_1+2,y_2),S(y_1+2,y_2)) \\
& - m^*(\Delta_{1t}(y_1+1,y_2),\Delta_{2t}(y_1+1,y_2),S(y_1+1,y_2)) \\
& \leq m^*(\Delta_{1t}(y_1+2,y_2),\Delta_{2t}(y_1+1,y_2)+\Delta_{1t}(y_1+2,y_2)-\Delta_{1t}(y_1+1,y_2),S(y_1+2,y_2)) \\
& - m^*(\Delta_{1t}(y_1+1,y_2),\Delta_{2t}(y_1+1,y_2),S(y_1+1,y_2)) \qquad \text{(A.27)}
\end{aligned}
$$

Now, by applying Lemma A.2(d) to the right-hand side of the above inequality, we obtain:

$$
\begin{aligned}
& m^*(\Delta_{1t}(y_1+2,y_2),\Delta_{2t}(y_1+1,y_2)+\Delta_{1t}(y_1+2,y_2)-\Delta_{1t}(y_1+1,y_2),S(y_1+2,y_2)) \\
& - m^*(\Delta_{1t}(y_1+1,y_2),\Delta_{2t}(y_1+1,y_2),S(y_1+1,y_2)) \\
& \leq \Delta_{1t}(y_1+1,y_2)-\Delta_{1t}(y_1+2,y_2) \qquad \text{(A.28)}
\end{aligned}
$$

The result follows from (A.27) and (A.28).

Proofs of Lemmas II.1 and II.2

Proof of Lemma II.1: Besanko, Gupta and Jain (1998) state virtually the same result. (See equation (5) on page 1536.) Here, we provide an outline of the proof for the sake of completeness. Note from (2.5) that the seller's problem is to pick the vector $\mathbf{p}$ to maximize

$$J(\mathbf{p}) := \sum_{i \in S_t(\mathbf{y})} (p_i - \Delta_{it}(\mathbf{y}))q_i(\mathbf{p}, S_t(\mathbf{y})) \qquad \text{(A.29)}$$

The prices for out-of-stock products are irrelevant. Hence, we will focus on products that are in stock. For notational convenience, we drop $S_t(\mathbf{y})$ from the arguments of the functions in this proof. Furthermore, since $\mathbf{y}$ is fixed in this proof, we use the shorthand notation Δ_{it} instead of $\Delta_{it}(\mathbf{y})$.

One can check that it is never optimal to set $p_i \leq \Delta_{it}$. Furthermore, one can check that $J(\mathbf{p})$ decreases as $p_i \to \infty$. Hence, any optimal solution must be an interior solution and satisfy the first-order conditions. The first-order conditions for $J(\mathbf{p})$ are given by:

$$\frac{\partial J(\mathbf{p})}{\partial p_i} = q_i(\mathbf{p}) + (p_i - \Delta_{it})\frac{\partial q_i(\mathbf{p})}{\partial p_i} + \sum_{j \in S_t(\mathbf{y}), j \neq i} (p_j - \Delta_{jt})\frac{\partial q_j(\mathbf{p})}{\partial p_i} = 0. \tag{A.30}$$

Furthermore, from the definition of $q_i(\mathbf{p})$ given by (2.1), one can check that

$$\frac{\partial q_i(\mathbf{p})}{\partial p_i} = -q_i(\mathbf{p})(1 - q_i(\mathbf{p})), \; i \in S_t(\mathbf{y}) \tag{A.31}$$

$$\frac{\partial q_j(\mathbf{p})}{\partial p_i} = q_j(\mathbf{p})q_i(\mathbf{p}), \; i, j \in S_t(\mathbf{y}), \; j \neq i \tag{A.32}$$

Substituting (A.31) and (A.32) in (A.30), we obtain:

$$\begin{aligned}\frac{\partial J(\mathbf{p})}{\partial p_i} &= q_i(\mathbf{p})\left[1 - (p_i - \Delta_{it})(1 - q_i(\mathbf{p})) + (p_j - \Delta_{jt})q_j(\mathbf{p})\right] \\ &= q_i(\mathbf{p})\left[1 - (p_i - \Delta_{it}) + J(\mathbf{p})\right] \\ &= 0 \end{aligned} \tag{A.33}$$

Since $q_i(\mathbf{p}) > 0$, equation (A.33) implies that any optimal price vector must satisfy $1 - (p_i - \Delta_{it}) + J(\mathbf{p}) = 0$ for all $i \in S_t(\mathbf{y})$. Hence, at an optimal price vector $\mathbf{p}$, we have:

$$p_i - \Delta_{it} = p_j - \Delta_{jt}, \; i, j \in S_t(\mathbf{y}), \; j \neq i. \tag{A.34}$$

We next show that the optimal price vector is unique. Define $m := p_i - \Delta_{it}(\mathbf{y})$ for $i \in S_t(\mathbf{y})$. Let $\mathbf{\Delta}_t$ denote the vector of Δ_{it} values. By virtue of (A.34), the problem of maximizing $J(\mathbf{p})$, given by (A.29), can be reduced to the problem of maximizing the following objective function, $J(m)$ (recall that $\mathbf{e}$ denotes the vector (1,1)):

$$J(m) := m \sum_{i \in S_t(\mathbf{y})} q_i(\mathbf{\Delta}_t + m\mathbf{e}) = m(1 - q_0(\mathbf{\Delta}_t + m\mathbf{e})) \tag{A.35}$$

It can be checked that

$$\frac{\partial q_0(\mathbf{\Delta}_t + m\mathbf{e})}{\partial m} = q_0(\mathbf{\Delta}_t + m\mathbf{e})(1 - q_0(\mathbf{\Delta}_t + m\mathbf{e})) \tag{A.36}$$

Using (A.36), one can show

$$\frac{\partial J(m)}{\partial m} = (1 - q_0(\mathbf{\Delta}_t + m\mathbf{e})) - q_0(\mathbf{\Delta}_t + m\mathbf{e})J(m) \tag{A.37}$$

$$\frac{\partial^2 J(m)}{\partial m^2} = -(1 - q_0(\mathbf{\Delta}_t + m\mathbf{e}))(1 + J(m)) - q_0(\mathbf{\Delta}_t + m\mathbf{e})\frac{\partial J(m)}{\partial m} \tag{A.38}$$

It now follows from (A.38) that whenever $\frac{\partial J(m)}{\partial m} = 0$, we have $\frac{\partial^2 J(m)}{\partial m^2} < 0$. Hence, any stationary point of $J(m)$ is a maximizer. Furthermore, there can be only one stationary point. (To see why, suppose there are two stationary points. Both stationary points must be maximizers. However, in order to have two stationary points, both of which are maximizers, there must be a stationary point between them, which is a minimizer. This yields a contradiction to any stationary point being a maximizer.) In addition, observe that if $m = 0$ then $J(m) = 0$, and $J(m) \to 0$ as $m \to \infty$. Therefore, the optimizer of $J(m)$ must be the unique stationary point in $(0, \infty)$. Since there is a unique m that optimizes $J(m)$, it now follows that there is a unique $\mathbf{p}$ that optimizes $J(\mathbf{p})$.

Proof of Lemma II.2: An equivalent result is also stated in Besanko, Gupta and Jain (1998). Once again, we include the proof for the sake of completeness. As shown in Lemma II.1, the optimal margin, $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$, is the unique maximizer of $J(m)$, given by (A.35), and the first-order condition $\frac{\partial J(m)}{\partial m} = 0$ is satisfied at $m =$

$m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$. This first-order condition can be written as follows, by substituting for $J(m)$ from (A.35) in (A.37):

$$\frac{\partial J(m)}{\partial m} = (1 - q_0(\mathbf{\Delta}_t(\mathbf{y}) + m\mathbf{e}))\,(1 - m \cdot q_0(\mathbf{\Delta}_t(\mathbf{y}) + m\mathbf{e})) = 0.$$

Hence, $m \cdot q_0(\mathbf{\Delta}_t(\mathbf{y}) + m\mathbf{e}) = 1$ for $m = m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))$, which yields the conclusion that

$$m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = \frac{1}{q_0(\mathbf{p}_t^*(\mathbf{y}), S_t(\mathbf{y}))}.$$

Observe that, by Lemma II.1, (2.5) can be further simplified as follows:

$$\begin{aligned} V_t(\mathbf{y}) &= V_{t-1}(\mathbf{y}) + \lambda m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) \sum_{i \in S_t(\mathbf{y})} q_i(\mathbf{p}_t^*(\mathbf{y}), S_t(\mathbf{y})) \\ &= V_{t-1}(\mathbf{y}) + \lambda m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y}))\,(1 - q_0(\mathbf{p}_t^*(\mathbf{y}), S_t(\mathbf{y}))) \end{aligned} \tag{A.39}$$

Now, using the fact that $m^*(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = \frac{1}{q_0(\mathbf{p}_t^*(\mathbf{y}), S_t(\mathbf{y}))}$, we conclude from (A.39) that:

$$V_t(\mathbf{y}) = V_{t-1}(\mathbf{y}) + \lambda(m_t(\mathbf{y}) - 1)$$

APPENDIX B

Proofs for Chapter 3

We provide the proof for Proposition III.3. The remaining analytic results can be proven by using Proposition II.3 through Proposition II.8 and their related Lemmas, using the fact that $\sum_{j=1}^{3} \lambda^j \leq 1$.

Proof of Proposition III.3: From Lemma A.2 (c), we have $m^{3*}(\Delta_1, \Delta_2, \{1,2\}) \geq m^{i*}(\Delta_i, \{i\})$, $i = 1, 2$. Suppose both product 1 and product 2 are available, and consider the following from the definitions of optimal margins of product 1:

$$m^{3*}(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = p_{1t}^{3*}(\mathbf{y}) - \Delta_{1t}(\mathbf{y})$$
$$m^{1*}(\mathbf{\Delta}_t(\mathbf{y}), S(\mathbf{y})) = p_{1t}^{1*}(\mathbf{y}) - \Delta_{1t}(\mathbf{y})$$

We get $p_{1t}^{3*}(\mathbf{y}) \geq p_{1t}^{1*}(\mathbf{y})$ from $m^{3*}(\Delta_1, \Delta_2, \{1,2\}) \geq m^{1*}(\Delta_1, \{1\})$. The proof for product 2 is similar. Also, $m^{3*}(\Delta_i, \{i\}) = m^{i*}(\Delta_i, \{i\})$ holds if a single product $i = 1, 2$ is available. Therefore the proposition holds trivially, which completes the proof.

APPENDIX C

Proofs for Chapter 4

Proof of Lemma IV.1:

We only consider *model 2* and the model number index is omitted for simplicity.

$$F_r(\mathbf{p}) = N\left[p_r q_r(\mathbf{p}) + bnp_m q_m(\mathbf{p})\right] - c_r \tag{C.1}$$

$$F_m(\mathbf{p}) = N(1-b)p_m q_m(\mathbf{p}) - c_m \tag{C.2}$$

Define $\mathbf{p}_r^*(p_m) = (p_r^*(p_m), p_m)$ as the pair of the merchant's price and the best response of retailer. Similarly, define $\mathbf{p}_m^*(p_r) = (p_r, p_m^*(p_r))$ as the pair of retailer's price and the best response of the merchant. The price vector at Nash equilibrium is defined as $\mathbf{p}^{NE} = \left(p_r^{NE}, p_m^{NE}\right)$. Consider the following derivations:

$$\begin{aligned}
\frac{\partial F_r(\mathbf{p})}{\partial p_r} &= N\left[q_r(\mathbf{p}) - p_r q_r(\mathbf{p})(1 - q_r(\mathbf{p})) + bnp_m q_m(\mathbf{p}) q_r(\mathbf{p})\right] \\
&= Np_r q_r(\mathbf{p})\left[(q_r(\mathbf{p}) - 1) + 1 + bnp_m q_m(\mathbf{p})\right] \\
&= Nq_r(\mathbf{p})\left[p_r q_r(\mathbf{p}) + bnp_m q_m(\mathbf{p}) - (p_r - 1)\right] \\
&= q_r(\mathbf{p})\left\{F_r(\mathbf{p}) + c_r - N(p_r - 1)\right\} \qquad \text{(C.3)}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial F_m(\mathbf{p})}{\partial p_m} &= N(1-b)\left[q_m(\mathbf{p}) - p_m q_m(\mathbf{p})\left(1-q_m(\mathbf{p})\right)\right] \\
&= N(1-b)q_m(\mathbf{p})\left[1 - p_m\left(1-q_m(\mathbf{p})\right)\right] \\
&= N(1-b)q_m(\mathbf{p})\left[p_m\left(q_m(\mathbf{p})-1\right)+1\right] \\
&= N(1-b)q_m(\mathbf{p})\left[p_m q_m(\mathbf{p}) - (p_m - 1)\right] \\
&= q_m(\mathbf{p})\left[F_m(\mathbf{p}) + c_m - N(1-b)(p_m-1)\right] \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_r(\mathbf{p})}{\partial p_m} &= N\left[p_r q_r(\mathbf{p})\, q_m(\mathbf{p}) - bnp_m q_m(\mathbf{p})\left(1-q_m(\mathbf{p})\right) + bnq_m(\mathbf{p})\right] \\
&= Nq_m(\mathbf{p})\left[p_r q_r(\mathbf{p}) + bnp_m q_m(\mathbf{p}) - bn(p_m-1)\right] \qquad \text{(C.5)}
\end{aligned}
$$

$$
\frac{\partial F_m(\mathbf{p})}{\partial p_r} = N(1-b)p_m q_r(\mathbf{p})\, q_m(\mathbf{p}) \qquad \text{(C.6)}
$$

$$
\begin{aligned}
\frac{\partial^2 F_r(\mathbf{p})}{\partial {p_r}^2} &= -q_r(\mathbf{p})\left(1-q_r(\mathbf{p})\right)\left[F_r(\mathbf{p}) + c_r - N(p_r-1)\right] + q_r(\mathbf{p})\left(\frac{\partial F_r(\mathbf{p})}{\partial p_r} - N\right) \\
&= \left(q_r(\mathbf{p})-1\right)\frac{\partial F_r(\mathbf{p})}{\partial p_r} + q_r(\mathbf{p})\left(\frac{\partial F_r(\mathbf{p})}{\partial p_r} - N\right) \\
&= \left(2q_r(\mathbf{p})-1\right)\frac{\partial F_r(\mathbf{p})}{\partial p_r} - Nq_r(\mathbf{p}) \qquad \text{(C.7)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F_m(\mathbf{p})}{\partial {p_m}^2} &= -q_m(\mathbf{p})\left(1-q_m(\mathbf{p})\right)\left[F_m(\mathbf{p}) + c_m - N(1-b)(p_m-1)\right] \\
&\quad + q_m(\mathbf{p})\left[\frac{\partial F_m(\mathbf{p})}{\partial p_m} - N(1-b)\right] \\
&= \left(q_m(\mathbf{p})-1\right)\frac{\partial F_m(\mathbf{p})}{\partial p_m} + q_m(\mathbf{p})\left[\frac{\partial F_m(\mathbf{p})}{\partial p_m} - N(1-b)\right] \\
&= \left(2q_m(\mathbf{p})-1\right)\frac{\partial F_m(\mathbf{p})}{\partial p_m} - N(1-b)q_m(\mathbf{p}) \qquad \text{(C.8)}
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^2 F_r(\mathbf{p})}{\partial p_r \partial p_m} &= \frac{\partial}{\partial p_m}\left(\frac{\partial F_r(\mathbf{p})}{\partial p_r}\right) \\ &= q_m(\mathbf{p})\, q_r(\mathbf{p})\left\{F_r(\mathbf{p}) + c_r - N(p_r - 1)\right\} + q_r(\mathbf{p})\frac{\partial F_r(\mathbf{p})}{\partial p_m} \\ &= q_m(\mathbf{p})\frac{\partial F_r(\mathbf{p})}{\partial p_r} + q_r(\mathbf{p})\frac{\partial F_r(\mathbf{p})}{\partial p_m} \qquad \text{(C.9)}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 F_m(\mathbf{p})}{\partial p_m \partial p_r} &= \frac{\partial}{\partial p_r}\left(\frac{\partial F_m(\mathbf{p})}{\partial p_m}\right) \\ &= q_r(\mathbf{p})\, q_m(\mathbf{p})\left\{F_m(\mathbf{p}) + c_m - N(1-b)(p_m - 1)\right\} + q_m(\mathbf{p})\frac{\partial F_m(\mathbf{p})}{\partial p_r} \\ &= q_r(\mathbf{p})\frac{\partial F_m(\mathbf{p})}{\partial p_m} + q_m(\mathbf{p})\frac{\partial F_m(\mathbf{p})}{\partial p_r} \qquad \text{(C.10)}\end{aligned}$$

(a) **The existence of *NE*.**

From (C.3), one can easily show that $\lim_{p_r\to\infty}\frac{\partial F_r(\mathbf{p})}{\partial p_r} < 0$ and $\lim_{p_m\to+0}\frac{\partial F_r(\mathbf{p})}{\partial p_r} > 0$. Similarly, it can be shown that $\lim_{p_r\to\infty}\frac{\partial F_m(\mathbf{p})}{\partial p_m} < 0$ and $\lim_{p_m\to+0}\frac{\partial F_m(\mathbf{p})}{\partial p_m} > 0$ from (C.4). Therefore we are assured that there exist $p_r > 0$ and $p_m > 0$ which satisfy (C.3) and (C.4) simultaneously. Second order conditions derived in (C.7) and (C.8) reduce to the following at $\mathbf{p}_r^*(p_m)$ and $\mathbf{p}_m^*(p_r)$:

$$\begin{aligned}\left.\frac{\partial^2 F_r(\mathbf{p})}{\partial {p_r}^2}\right|_{\mathbf{p}_r^*(p_m)} &= \left(2q_r(\mathbf{p}_r^*(p_m)) - 1\right)\left.\frac{\partial F_r(\mathbf{p})}{\partial p_r}\right|_{\mathbf{p}_r^*(p_m)} - Nq_r(\mathbf{p}_r^*(p_m)) \\ &= -Nq_r(\mathbf{p}_r^*(p_m)) < 0 \\ \left.\frac{\partial^2 F_m(\mathbf{p})}{\partial {p_m}^2}\right|_{\mathbf{p}_m^*(p_r)} &= \left(2q_m(\mathbf{p}_m^*(p_r)) - 1\right)\left.\frac{\partial F_m(\mathbf{p})}{\partial p_m}\right|_{\mathbf{p}_m^*(p_r)} - N(1-b)q_m(\mathbf{p}_m^*(p_r)) \\ &= -N(1-b)q_m(\mathbf{p}_m^*(p_r)) < 0\end{aligned}$$

The second order conditions confirm that the profit functions (4.1) and (4.2) are unimodal. Therefore we conclude that the Nash equilibrium exists in the price decisions for $0 \le b < 1$.

(b) **The uniqueness of *NE*.**

For the uniqueness of the Nash equilibrium, it is enough to show that $\left|\frac{dp_m^*(p_r)}{dp_r}\right|_{NE} < 1$

and $\left|\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} < 1$.

(b-i) Proof of $\left|\frac{dp_m^*(p_r)}{dp_r}\right|_{NE} < 1$.

The first order condition from (C.4) leads to

$$q_m\left(\mathbf{p}_m^*\right) = \frac{p_m^*(p_r) - 1}{p_m^*(p_r)} = 1 - \frac{1}{p_m^*(p_r)} \tag{C.11}$$

Using (C.11), the derivative of $q_m\left(\mathbf{p}_m^*\right)$ with respect to p_r is the following:

$$\frac{dq_m\left(\mathbf{p}_m^*\right)}{dp_r} = \frac{1}{\left(p_m^*(p_r)\right)^2} \cdot \frac{dp_m^*(p_r)}{dp_r} \tag{C.12}$$

Using the property of purchase probabilities from MNL choice model, we get the following:

$$\begin{aligned} \frac{dq_m\left(\mathbf{p}_m^*\right)}{dp_r} &= \frac{\partial q_m\left(\mathbf{p}_m^*\right)}{\partial p_r} + \frac{\partial q_m\left(\mathbf{p}_m^*\right)}{\partial p_m} \cdot \frac{dp_m^*(p_r)}{dp_r}\frac{dq_m\left(\mathbf{p}_m^*\right)}{dp_r} \\ &= q_m\left(\mathbf{p}_m^*\right) q_r\left(\mathbf{p}_m^*\right) - q_m\left(\mathbf{p}_m^*\right)\left(1 - q_m\left(\mathbf{p}_m^*\right)\right) \cdot \frac{dp_m^*(p_r)}{dp_r} \end{aligned} \tag{C.13}$$

Notice that the best response of the retailer to a merchant's price $p_m^*(p_r)$ should simultaneously satisfy both (C.12) and (C.13). Suppose $\frac{dp_m^*(p_r)}{dp_r} \leq 0$. Then we get $\frac{dq_m(\mathbf{p}_m^*)}{dp_r} > 0$ from (C.12), which is a contradiction to (C.13). Therefore, we must have $\frac{dp_m^*(p_r)}{dp_r} > 0$ and it follows from (C.12) that $\frac{dq_m(\mathbf{p}_m^*)}{dp_r} > 0$. Further, it follows from $\frac{dq_m(\mathbf{p}_m^*)}{dp_r} > 0$ and (C.13) that $\frac{dp_m^*(p_r)}{dp_r} < \frac{q_r(\mathbf{p}_m^*)}{1-q_m(\mathbf{p}_m^*)}$. Therefore $\frac{dp_m^*(p_r)}{dp_r}$ must satisfy $0 < \frac{dp_m^*(p_r)}{dp_r} < \frac{q_r(\mathbf{p}_m^*)}{1-q_m(\mathbf{p}_m^*)}$. We conclude that $0 < \frac{dp_m^*(p_r)}{dp_r} < 1$, since $q_r\left(\mathbf{p}_m^*\right) < 1 - q_m\left(\mathbf{p}_m^*\right)$.

(b-ii) Proof of $\left|\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} < 1$.

Now consider $\frac{dp_r^*(p_m)}{dp_m}$ using $\frac{dq_r(\mathbf{p}_r^*)}{dp_m}$. We break down the proof into two cases:

Pure competition ($b = 0$)

The first order conditions are the following:

$$\begin{aligned}
\frac{\partial F_r(\mathbf{p})}{\partial p_r} &= N q_r(\mathbf{p})\left[p_r q_r(\mathbf{p}) - (p_r - 1)\right] \\
\frac{\partial F_m(\mathbf{p})}{\partial p_m} &= N q_m(\mathbf{p})\left[p_m q_m(\mathbf{p}) - (p_m - 1)\right]
\end{aligned} \tag{C.14}$$

(C.14) takes a similar form as (C.4). Therefore, one can show $0 < \frac{dp_r^*(p_m)}{dp_m} < 1$ simply by using the same reasoning as in part (a).

<u>Coopetition ($b > 0$)</u>

The first order conditions are the following:

$$\begin{aligned}
\frac{\partial F_r(\mathbf{p})}{\partial p_r} &= N q_r(\mathbf{p})\left[p_r q_r(\mathbf{p}) + bnp_m q_m(\mathbf{p}) - (p_r - 1)\right] \\
\frac{\partial F_m(\mathbf{p})}{\partial p_m} &= N q_m(\mathbf{p})\left[p_m q_m(\mathbf{p}) - (p_m - 1)\right]
\end{aligned}$$

Consider the following using chain rule:

$$\begin{aligned}
\frac{dq_r(\mathbf{p}_r^*)}{dp_m} &= \frac{\partial q_r(\mathbf{p}_r^*)}{\partial p_m} + \frac{\partial q_r(\mathbf{p}_r^*)}{\partial p_r} \cdot \frac{dp_r^*(p_m)}{dp_m} \\
&= q_m(\mathbf{p}_r^*)\, q_r(\mathbf{p}_r^*) - q_r(\mathbf{p}_r^*)\left(1 - q_r(\mathbf{p}_r^*)\right) \cdot \frac{dp_r^*(p_m)}{dp_m}
\end{aligned} \tag{C.15}$$

From (C.15), we find $\frac{dq_r(\mathbf{p}_r^*)}{dp_m} \leq 0$ if $\frac{dp_r^*(p_m)}{dp_m} \geq \frac{q_m(\mathbf{p}_r^*)}{1-q_r(\mathbf{p}_r^*)}$ and $\frac{dq_r(\mathbf{p}_r^*)}{dp_m} > 0$ if $\frac{dp_r^*(p_m)}{dp_m} < \frac{q_m(\mathbf{p}_r^*)}{1-q_r(\mathbf{p}_r^*)}$. The other expression for $q_r(\mathbf{p}_r^*)$ can be obtained from the first order condition (C.3) as follows:

$$q_r(\mathbf{p}_r^*) = \frac{p_r - 1 - bnp_m q_m(\mathbf{p}_r^*)}{p_r} = 1 - \frac{1 + bnp_m q_m(\mathbf{p}_r^*)}{p_r} \tag{C.16}$$

Using (C.16), consider the following:

$$\frac{dq_r(\mathbf{p}_r^*)}{dp_m} = -\frac{\left(bnp_m \frac{dq_m(\mathbf{p}_r^*)}{dp_m} + bnq_m\left(\mathbf{p}_r^*\right)\right) p_r^*(p_m) - \left(1 + bnp_m q_m\left(\mathbf{p}_r^*\right)\right) \frac{dp_r^*(p_m)}{dp_m}}{\left(p_r^*(p_m)\right)^2}$$

Using $\frac{dq_m(\mathbf{p}_r^*)}{dp_m} = \frac{\partial q_m(\mathbf{p}_r^*)}{\partial p_m} + \frac{\partial q_m(\mathbf{p}_r^*)}{\partial p_r} \cdot \frac{dp_r^*(p_m)}{dp_m}$, we get the following after algebra and rearranging the terms:

$$\begin{aligned} \left(p_r^*(p_m)\right)^2 \frac{q_r(\mathbf{p}_r^*)}{dp_m} &= \left[1 + bnp_m q_m\left(\mathbf{p}_r^*\right) - bnp_r^*(p_m) p_m q_r\left(\mathbf{p}_r^*\right) q_m\left(\mathbf{p}_r^*\right)\right] \cdot \frac{dp_r^*(p_m)}{dp_m} \\ &\quad + bnp_r^*(p_m) q_m\left(\mathbf{p}_r^*\right) \left\{p_m\left[1 - q_m\left(\mathbf{p}_r^*\right)\right] - 1\right\} \end{aligned} \tag{C.17}$$

We only consider $\frac{q_r(\mathbf{p}_r^*)}{dp_m}$ at Nash equilibrium, therefore, we use $p_m^{NE}\left[1 - q_m\left(\mathbf{p}^{NE}\right)\right] = 1$ from (C.4) to get the following simplification:

$$bnp_r^{NE} q_m\left(\mathbf{p}_r^{NE}\right)\left\{p_m^{NE}\left[1 - q_m\left(\mathbf{p}_r^{NE}\right)\right] - 1\right\} = 0$$

Therefore the expression for $\frac{q_r(\mathbf{p}_r^*)}{dp_m}$ at $\mathbf{p}^{NE}$ boils down to the following:

$$\left.\frac{dq_r\left(\mathbf{p}\right)}{dp_m}\right|_{NE} = \frac{1 + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right) - bnp_r^{NE} p_m^{NE} q_r\left(\mathbf{p}^{NE}\right) q_m\left(\mathbf{p}^{NE}\right)}{\left(p_r^{NE}\right)^2} \cdot \left.\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} \tag{C.18}$$

First we show $\left.\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} > 0$. Consider the following:

$$\begin{aligned} \left.\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} = -\left.\frac{\frac{\partial^2 F_r(\mathbf{p})}{\partial p_r \partial p_m}}{\frac{\partial^2 F_r(\mathbf{p})}{\partial {p_r}^2}}\right|_{NE} &= q_m\left(\mathbf{p}^{NE}\right)\left[p_r^{NE} q_r\left(\mathbf{p}^{NE}\right) + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right) - bn(p_m^{NE} - 1)\right] \\ &= q_m\left(\mathbf{p}^{NE}\right)\left[\left(p_r^{NE} - 1\right) - bn(p_m^{NE} - 1)\right] \end{aligned} \tag{C.19}$$

From (C.11) and (C.16), we have $q_r\left(\mathbf{p}^{NE}\right) p_r^{NE} = \left(p_r^{NE} - 1\right) - bn(p_m^{NE} - 1)$ which implies that $p_r^{NE} - 1 > bn(p_m^{NE} - 1)$. Therefore (C.19) leads to $\left.\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} > 0$.

Next, we show $1 + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right) - bnp_r^{NE} p_m^{NE} q_r\left(\mathbf{p}^{NE}\right) q_m\left(\mathbf{p}^{NE}\right) > 0$. For simplicity, we define $A := 1 + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right)$ and $\hat{A} := -bnp_r^{NE} p_m^{NE} q_m\left(\mathbf{p}^{NE}\right) q_r\left(\mathbf{p}^{NE}\right)$ to prove $A + \hat{A} > 0$. By (C.3), we have $p_r^{NE} q_r\left(\mathbf{p}^{NE}\right) + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right) = p_r^{NE} - 1$ from which we get $A = 1 + bnp_m^{NE} q_m\left(\mathbf{p}^{NE}\right) = p_r^{NE}\left[1 - q_r\left(\mathbf{p}^{NE}\right)\right]$. Also, we define $B := bnp_r^{NE} p_m^{NE} q_m\left(\mathbf{p}^{NE}\right)\left[1 - q_m\left(\mathbf{p}^{NE}\right)\right]$ and $\hat{B} := -bnp_r^{NE} q_m\left(\mathbf{p}^{NE}\right)$. Notice that $B + \hat{B} = bnp_r^{NE} q_m\left(\mathbf{p}^{NE}\right)\left\{p_m^{NE}\left[1 - q_m\left(\mathbf{p}^{NE}\right)\right] - 1\right\} = 0$, from (C.4). Consider the following:

$$A + \hat{B} = \left(1 - p_r^{NE} q_r\left(\mathbf{p}^{NE}\right)\right) - bnp_r^{NE} q_m\left(\mathbf{p}^{NE}\right) = p_r^{NE}\left[1 - q_r\left(\mathbf{p}^{NE}\right) - bnq_m\left(\mathbf{p}^{NE}\right)\right]$$

Since $1 - q_r\left(\mathbf{p}^{NE}\right) - bnq_m\left(\mathbf{p}^{NE}\right) > 1 - q_r\left(\mathbf{p}^{NE}\right) - nq_m\left(\mathbf{p}^{NE}\right) > 0$, we conclude $A + \hat{B} > 0$.

$$\begin{aligned} B + \hat{A} &= bnp_r^{NE} p_m^{NE} q_m\left(\mathbf{p}^{NE}\right)\left[1 - q_m\left(\mathbf{p}^{NE}\right)\right] - bnp_r^{NE} p_m^{NE} q_r\left(\mathbf{p}^{NE}\right) q_m\left(\mathbf{p}^{NE}\right) \\ &= bnp_r^{NE} p_m^{NE} q_m\left(\mathbf{p}^{NE}\right)\left[1 - q_r\left(\mathbf{p}^{NE}\right) - q_m\left(\mathbf{p}^{NE}\right)\right] \end{aligned}$$

Again, $B + \hat{A} > 0$ since $1 - q_r\left(\mathbf{p}^{NE}\right) - q_m\left(\mathbf{p}^{NE}\right) > 1 - q_r\left(\mathbf{p}^{NE}\right) - nq_m\left(\mathbf{p}^{NE}\right) > 0$. Since $B + \hat{B} = 0$, $A + \hat{B} > 0$ and $B + \hat{A} > 0$ lead to $A + \hat{A} + B + \hat{B} = A + \hat{A} > 0$. Therefore, $\frac{dq_r(\mathbf{p}_r^*)}{dp_m} \leq 0$ leads to a contradiction in (C.18), and we conclude that $0 < \left.\frac{dp_r^*(p_m)}{dp_m}\right|_{NE} < 1$ for $b > 0$.

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