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FOR PRESCRIBED CLOSED-LOOP CHARACTERISTICS

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September, 1955

IP-132
ACKNOWLEDGEMENT

We wish to express our appreciation to John L. Stewart, Assistant Professor of Electrical Engineering, the University of Michigan, for permission to give this preprint limited distribution under the Industry Program of the College of Engineering.
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By

John L. Stewart*
Member, IRE

*University of Michigan, Ann Arbor, Michigan

ABSTRACT

Feedback amplifiers having real feedback are analyzed in general terms in order to express the open-loop transfer function uniquely in terms of a prescribed closed-loop transfer function. Then, the proper amplifier interstage compensating networks can readily be designed. It is shown that bandwidth degradation of the open-loop transfer function is required. Specific numerical functional examples are given for open- and closed-loop functions having two, three, and four poles. The four-pole function, which in the example yields a four-pole maximally-flat closed-loop characteristic, is applied to the design of a three-stage feedback amplifier with transformer output.

All-pole functions must be degraded with a pole near the origin. It is shown that the bandwidth degradation may also be accomplished with functions having both poles and zeros, with the number of poles exceeding the number of zeros by unity. A numerical example is given which shows that the open-loop bandwidth of the degraded system can be made considerably larger by using zeros as well as poles.
The design methods for all-pole functions are precise in that they result in an amplifier which, with feedback, has the transfer function specified beforehand. When zeros are added, the methods are not quite precise, although the error may be entirely negligible.

Finally, a discussion of low-frequency stabilization problems, as introduced by coupling and by-pass networks, is given. Although it does not appear practical to employ precision methods, certain optimization procedures are applicable.

From the numerical examples given, it is evident that the procedure described here not only leads to a unique solution, but is also considerably easier and quicker to apply than classical methods employing Nyquist and log-frequency plots.

INTRODUCTION


The power and facility of pole-zero design methods have for some years now been exploited (by a relatively small circle of engineers) in connection with the design of conventional low-pass and band-pass amplifiers. The full utilization of these methods has not been well documented to date, although information is becoming available. Recently, pole-zero methods have been applied to the design of servomechanisms. Similar methods


are presented here for feedback amplifiers. As will be shown, pole-zero
methods are powerful yet simple and straightforward, even for amplifiers with
several stages. Whether or not pole-zero techniques will outmode classical
methods, such as log-frequency plots and Nyquist diagrams, must await the test
of time and the more widespread dissemination of the technique.

Of necessity, the treatment here presumes a knowledge of the rela-
tionships between pole and zero positions and simple amplifier interstage cir-

cuits. Also presumed is an acquaintance with maximally-flat and similar func-
tions, the relevant polynomials, and the all-important pole-zero diagram.

The design method presented here presumes a specified closed-loop
transfer function, the development of which is a separate problem—the "ap-
proximation" problem. Accordingly, the function may be obtained from steady-
state considerations, as a result of a transient analysis, or from some other
mathematical discipline. As far as the circuit design is concerned, the par-
ticular closed-loop function is immaterial, the only effect being a relocation
of the poles of the closed-loop function, which has a surprisingly minor
effect on the locations of the poles of the open-loop transfer function.

GENERAL ANALYSIS

A single-loop feedback amplifier can be diagrammed as in Fig. 1.
It will be assumed throughout that the feedback factor β is purely real.
Accordingly, the feedback network can be represented as a resistive voltage
divider as in Fig. 2. Capacitances C_1 and C_2 are allowed if R_1C_1 = R_2C_2 so
that β remains real. The technique of realizing an R-C voltage divider in
order to make β real is useful when unavoidable stray capacitance is present.
Any external load on the amplifier, as well as any loading given by the feedback network, will be assumed accounted for in the open-loop transfer function -KG, which is

$$\frac{e_3}{e_2} = -KG = -K \frac{A}{B} = -K \frac{1 + a_1p + a_2p^2 + \ldots + a_m p^m}{1 + b_1p + b_2p^2 + \ldots + b_n p^n}$$

(1)

which will be presumed to have more poles than zeros \((n > m)\). The coefficients of \(p\) are defined so that the low-frequency gain is -K, which is real. Low-frequency effects due to interstage coupling networks, cathode and screen by-pass networks, and transformer magnetizing inductance, are neglected.

The closed-loop transfer function is determined from Fig. 1 and equation (1) as

$$\frac{e_3}{e_1} = \frac{-KG}{1 + KGB} = \frac{-KA}{B + KBA} = \frac{-K}{1 + KB} \cdot \frac{1 + a_1p + a_2p^2 + \ldots + a_m p^m}{1 + B_1p + B_2p^2 + \ldots + B_n p^n}$$

(2)

where the zeros of the open- and closed-loop transfer functions are the same and where

$$B_j = \begin{cases} \frac{b_j + KBa_j}{1 + KB}, & 1 \leq j \leq m \\ \frac{b_j}{1 + KB}, & m < j \leq n \end{cases}$$

(3)

from which

$$b_j = \begin{cases} (1 + KB) B_j - KBa_j, & 1 \leq j \leq m \\ (1 + KB) B_j, & m < j \leq n \end{cases}$$

(4)

The gain of the closed-loop system at low frequencies is \(-K/(1 + KG)\), which is approximately \(-1/\beta\) for large \(KB\).

A precision design specifies the closed-loop transfer function; that is, the coefficients \(a_j\) and \(B_j\), as well as the feedback \(KB\), are given data. From these data, the open-loop transfer function is uniquely determined. Suitable compensating networks can then be devised so that the open-loop characteristic is that required.
The most important special case is that when the open-loop transfer function, as well as the closed-loop transfer function, has only poles. (The addition of zeros will be considered later as a modification of all-pole functions.) For this, it is more convenient to write the closed-loop transfer function as

\[
\frac{e_3}{e_1} = \frac{-KC_0/(1 + KB)}{p^n + C_{n-1}p^{n-1} + C_{n-2}p^{n-2} + \ldots + C_1p + C_0}
\]  

(5)

in which \(KB\) and the coefficients \(C_j\) are presumed to be given data. The corresponding open-loop transfer function is

\[
\frac{e_3}{e_2} = \frac{-KC_0/(1 + KB)}{p^n + C_{n-1}p^{n-1} + C_{n-2}p^{n-2} + \ldots + C_2p^2 + C_1p + C_0/(1 + KB)}
\]  

(6)

Except for the coefficient of \(p^0\) in the denominator, the open- and closed-loop transfer functions are identical. For large \(KB\), one pole of the open-loop transfer function is near the origin, which requires "bandwidth degradation" of the open-loop characteristic by, for example, placing a relatively large capacitor between some signal point and ground. The approximate open-loop half-power radian bandwidth for \(KB\) large is

\[
B \cong \frac{C_0}{C_1(1 + KB)}
\]  

(7)

The closed-loop bandwidth is proportional to \(C_0\), as is also the open-loop bandwidth. The required open-loop bandwidth is essentially inversely proportional to the feedback \(KB\). The parameter \(C_1\) characterizes the type of closed-loop transfer function; that is, maximally flat, linear phase, Chebyshev, and so forth. For a given closed-loop bandwidth, the ratio \(C_0/C_1\) is somewhat larger for closed-loop functions having high-frequency peaks or other oscillatory behavior as contrasted to monotonically decreasing amplitude characteristics.
EXAMPLES FOR NO ZEROS

The following specific examples, which typify the general relations, assume a maximally-flat closed-loop transfer function with a radian low-pass bandwidth of unity. The functions have only poles.

1. Two-Pole Function

\[
\frac{e_3}{e_1} = \frac{-K/(1 + K\beta)}{p^2 + (2)^{1/2} p + 1}
\]

\[
\frac{e_3}{e_2} = \frac{-K/(1 + K\beta)}{p^2 + (2)^{1/2} p + 1/(1 + K\beta)}
\]  \hspace{1cm} (8)

For \(K\beta = 99\) (as a specific example), the open-loop transfer function has poles at

\[p_1, p_2 = -1.414, -0.00707\]  \hspace{1cm} (9)

and the bandwidth of the degraded stage is only 0.00707 times the closed-loop bandwidth.

2. Three-Pole Function

\[
\frac{e_3}{e_1} = \frac{-K/(1 + K\beta)}{p^3 + 2p^2 + 2p + 1}
\]

\[
\frac{e_3}{e_2} = \frac{-K/(1 + K\beta)}{p^3 + 2p^2 + 2p + 1/(1 + K\beta)}
\]  \hspace{1cm} (10)

For \(K\beta = 99\), the open-loop transfer function has poles at

\[p_1 = -0.00497\]

\[p_2, p_2^* = -0.998 + j 0.996.\]  \hspace{1cm} (11)

More poles go with greater lagging phase shift; consequently, more bandwidth degradation is required.

3. Four-Pole Function

\[
\frac{e_3}{e_1} = \frac{-K/(1 + K\beta)}{p^4 + 2.62p^3 + 3.42p^2 + 2.62p + 1}
\]

\[
\frac{e_3}{e_2} = \frac{-K/(1 + K\beta)}{p^4 + 2.62p^3 + 3.42p^2 + 2.62p + 1/(1 + K\beta)}
\]  \hspace{1cm} (12)
For $K_B = 99$, the open-loop transfer function has poles at
\[ p_1 = -0.00382 \]
\[ p_2 = -1.5 \]
\[ p_3, p^5 = -0.558 \pm j1.2. \] (13)

For the three- and four-pole functions, complex poles are indicated as necessary in the open-loop function. This requires that either a resonant circuit be employed as one compensating network or, if inductance is to be avoided, the open-loop system must have a minor loop (that is, an R-C feedback amplifier). The choice of the maximally-flat example was for convenience only. Had the closed-loop transfer function contained only real poles, the required open-loop function would be quite similar, the principal difference being the bandwidth of the degraded stage. Higher-order functions also lead to similar results; that is, the open-loop function has one (and only one) pole close to the origin and several complex and real poles at high frequencies.

An example can be given of the small effect on the open-loop pole positions when the closed-loop transfer function is quite different from the maximally-flat function. When the closed-loop transfer function has a fourth-order pole at $p = -1$, and when $K_B = 99$, the open-loop function is found to have poles at $p_1 = -0.00251$, $p_2 = -2$, and $p_3, p^5 = -1 \pm j1$. These values can be compared to those of equation (13). Qualitatively, the differences are insignificant, even though the closed-loop transfer functions are radically different.

**APPLICATION TO A SPECIFIC AMPLIFIER**

The four-pole function of equation (13) is characterized by a three-stage amplifier with transformer output and cathode feedback as shown in Fig. 3 (a-c connections only). It is assumed that the output represents a resistive load and the transformer secondary capacitance is neglected. Loading on the output by the feedback circuit is assumed to be purely resistive. Two
of the stages have R-C cut-off characteristics, one by virtue of stray capacitance alone, and the other padded with a larger capacitance. These two stages furnish the two real poles of equation (13). The complex-conjugate poles are given by the transformer output circuit (high-frequency equivalent). Feedback is controlled by the size of the resistor R. The capacitance C can be added to counteract any capacitance from the cathode of the first tube to ground. In amplifiers of the type shown in Fig. 3, it is important that the winding direction of the secondary of the transformer be correct—it can be changed simply by reversing connections to the secondary. If the connections are wrong, the amplifier will have positive feedback and will oscillate. Also, note that the input and feedback signals are subtracted from one another rather than added as is implied in Figs. 1 and 2. However, since subtraction is but a special case of addition, and because the sign of the output signal with respect to the input is easily controlled with the output transformer, the amplifier of Fig. 3 belongs to the general category of Figs. 1 and 2.

A typical value of K for the example amplifier is about 3000 which, for \( K\beta = 99 \), requires \( \beta = \frac{99}{3000} = 0.033 \). The full benefits of negative feedback are obtained only at frequencies where KG is large. Above the bandwidth of the degraded stage, KG decreases; therefore, the degraded stage should have a bandwidth comparable to that of the signals to be amplified, if possible. This would be about 5000 cycles for an audio-frequency example. Since the high-frequency poles of equation (13) are roughly 250 times as far from the origin of the complex plane as is the pole of the degraded stage, the bandwidth of the rest of the amplifier, including the output transformer, should be about 0.005 \( \times 250 = 1.25 \) megacycles. This implies that a very high-quality transformer be used.

Another tuning arrangement for the amplifier of Fig. 3 suggests itself. That is, the transformer output stage can be made to furnish the poles on the negative real axis by heavy overdamping. The pair of complex-conjugate poles can be furnished in the first two stages by operating them as a feedback
pair. Alternately, one interstage can employ a shunt-peaked circuit and the other a simple R-C circuit with the parameters adjusted so that the pole of the R-C circuit is cancelled by the zero of the shunt-peaked circuit.

If a transformer having the implied bandwidth is not available, the circuit can still be tuned correctly. The transformer, when overdamped, will have a pair of poles on the negative real axis. One pole should be that giving bandwidth degradation. The other will be further out on the negative real axis, but not so far as with the hypothetical high-quality unit. The one pole far out on the negative real axis can be furnished by one of the first two stages as an R-C interstage. The complex conjugate poles can be furnished with a shunt-peaked circuit (which can be made more versatile by having resistance in parallel with the capacitor as well as in series with the inductor). The zero of the shunt-peaked circuit and the high-frequency pole of the transformer can be caused to cancel. The tuning is described more thoroughly in the open-loop transfer function of Fig. 4. The shunt-peaked circuit amounts to a sophisticated kind of lead network.

Because of the possibility of saturation with large high-frequency signals, it is best not to degrade a stage near the output, which cannot always be avoided with practical output systems. Saturation problems are reduced to a minimum when degradation is accomplished at the input stage where the ratio of typical signal levels to the saturation limits of the stage is an absolute minimum. When degradation near the output is a practical necessity, saturation problems can be reduced by making the bandwidth of the degraded stage comparable to the signal frequencies of interest. The situation can further be improved with more sophisticated bandwidth degradation, which permits larger degraded bandwidths to be realized.
EFFICIENT DEGRADATION

A direct attack on closed-loop transfer functions having both poles and zeros appears cumbersome and rather uncertain. Simple checks show that such an attack does not usually result in more desirable systems than those containing only poles. A different technique, and one that results in convenient design procedures for superior systems, introduces zeros only in association with the degraded stage. Then, the closed-loop transfer function is essentially an all-pole function, although the open-loop function does contain zeros.

Consider Fig. 5, which is the Nyquist plot of a typical amplifier that is stable with feedback. The closed-loop transfer function is determined by the ratio of the two phasors in Fig. 5. If \( K \beta \gg 1 \), this ratio will be essentially the same for the dotted locus in Fig. 5 as for the locus shown with the solid line. Therefore, the exact form of the degradation is not important, as long as it constitutes a low-pass type of function, and as long as it behaves the same way as a one-pole function at frequencies where the phase shift is on the order of 90 degrees. Therefore, it is possible to use a collection of poles and zeros for the bandwidth degrading function instead of a single pole, as long as the number of poles exceeds the number of zeros by unity.

Presumably, a design as described previously results in a degraded stage obeying the transfer function

\[
\frac{k}{p + a}
\]  

(14)

where \( k \) is an arbitrary constant. The one-pole function may be replaced with

\[
\frac{k(p^r + e_{r-1}p^{r-1} + \ldots + e_1p + e_0)}{p^s + f_{s-1}p^{s-1} + \ldots + f_1p + a} \]  

(15)

which has the same behavior as equation (14) at \( \omega = 0 \) and as \( \omega \to \infty \) providing

\[
s = r + 1
\]  

(16)
and providing the high-frequency asymptotic behavior of equation (15) is reached at sufficiently low frequencies (which depends on the amount of bandwidth degradation and hence in $K\beta$),

Equation (15) is characteristic of an input impedance having shunt capacitance across the terminals, and consequently can be realized with a large variety of circuits.

With simple one-pole bandwidth degradation, the open-loop gain $K_{G}$ falls to 0.707 of its value at $\omega = 0$ at the half-power bandwidth of the degraded stage, which is $\omega = a$ in equation (14). With more sophisticated degradation, the half-power bandwidth of the degraded stage can be made considerably larger than $a$ and still behave the same way at high frequencies. The exact improvement that is possible is rather nebulous, depending on the amount of irregularity in the transfer function of the degraded stage that can be tolerated and the frequency where the behavior of the modified circuit approaches that of the one-pole function. It is interesting to note from a consideration of the Nyquist plot that for larger values of $K\beta$, more irregularity in the degrading function can be tolerated for a given amount of irregularity in the closed-loop transfer function. This implies that the bandwidth of the degraded open-loop system need not decrease as rapidly as $1/K\beta$, as is implied from a study of all-pole functions.

As an example of what can be done, a function with one zero will be employed for bandwidth degradation rather than a simple one-pole function (with a pole at $p = -a$). The new function is

$$
\frac{k}{p^2 + e_1p + ae_0}
$$

which satisfies equations (15) and (16). The zero and poles of equation (17) are

$$
z_1 = -e_0
$$

$$
p_1, p_1 = -\frac{e_1}{2} \pm j \sqrt{a e_0 - \left(\frac{e_1}{2}\right)^2}
$$
As a specific comparative numerical example, normalize to \( a = 1 \) radian and set \( e_1 = e_0 = 2 \). The zero of equation (18) is then at -2, the poles are at \(-1 + j \cdot 1\), and the bandwidth of the function is 1.8 radians. Since the bandwidth of the simple one-pole function is unity, it is evident that there has been an 80 per cent improvement in the bandwidth of the degrading function. The transfer characteristic of the new degrading function is almost flat, there being a slight peak just below the half-power frequency. Had a larger peak been allowed (which might still have a negligible effect on the closed-loop transfer function), a larger improvement could be realized.

The technique can now be described in terms of the specific example used before. The original open-loop transfer function with one-pole bandwidth degradation is shown in Fig. 6a. With the one-zero function, the modified open-loop function is as shown in Fig. 6b. Only the degrading part of the function is different; the high-frequency poles are the same and the closed-loop transfer functions are essentially the same.

The function of Fig. 6b can be mechanized in the amplifier of Fig. 3 in a very practical manner. The pair of low-frequency complex-conjugate poles can be realized with a practical transformer output stage, somewhat underdamped. If the degraded stage of Fig. 6a is 5000 cycles, then for the specific numerical example, the bandwidth of the output stage of the modified system is 5000 \((2)^{1/2}\) = 7070 cycles, which corresponds to inexpensive transformers and which is wide enough to accept most audio signal frequencies without saturation difficulties. (The increase in the transformer bandwidth is less than 80 per cent because the transformer output function does not include the low-frequency zero.)

The first two interstage networks must furnish the three high-frequency poles and the low-frequency zero. The single real high-frequency pole is easily realized as a simple R-C circuit. The pair of high-frequency complex-conjugate poles and the low-frequency zero can be realized with a modified shunt-peaked interstage network. The compensated amplifier is shown in Fig. 7. Of course, the
first and second interstage networks can be interchanged. In addition, the design can be applied to triodes rather than pentodes.

It is of interest to note that the example design worked out here has many of the characteristics found desirable when developing a design with the aid of log-frequency plots. First, the degrading function will be recognized as that establishing an early 6 db per octave slope in the log-frequency plot. Second, the resonant circuit in the second stage of Fig. 7 adds a high-frequency plateau (a step) in a log-frequency plot, which is a well-known technique.

LOW-FREQUENCY STABILITY

At very low frequencies, any feedback amplifier that is not d-c coupled may have stability problems because of interstage coupling networks, cathode and screen by-pass networks, and transformer magnetizing inductance. It does not appear feasible to extend precision design to these low frequencies, primarily because circuits that introduce poles and zeros with imaginary parts are not economical or practical. This is unfortunate because low-frequency stabilization is often a more severe problem than high-frequency stabilization because there are so many places where a leading phase shift can be introduced (which some designers circumvent with the ineconomies of d-c coupling).

If all high-frequency characteristics are ignored, which includes all high-frequency poles and zeros as well as those of the bandwidth degrading function, the open-loop transfer function reduces to a collection of poles and zeros on the negative real axis near the origin of the complex plane. The total number of poles must equal the total number of zeros; if not, the mid-band gain cannot be independent of frequency.

Each R-C interstage coupling circuit furnishes a pole and a zero as shown in Fig. 8a. The transfer function of a transformer (low-frequency equivalent) also has a pole and zero as in Fig. 8a by virtue of source and load

13
resistance and magnetizing inductance. Each cathode resistor and capacitor by-pass combination adds a pole and a zero as in Fig. 8b. Also, each screen dropping resistor and by-pass capacitor has a pole and a zero as in Fig. 8b. Figure 8c is characteristic of the low-frequency compensating circuit often used to improve the long-time response of pulse amplifiers and permits a given low-frequency response to be obtained without excessively large coupling and by-pass capacitors. It is evident that the compensating circuit is capable of entirely cancelling the effects of either a cathode or a screen by-pass circuit.

The low-frequency open-loop transfer function can be written

$$-Kg = -\frac{KA}{B} = -\frac{p^q \left(p^r + a_{r-1}p^{r-1} + \ldots + a_1p + a_0\right)}{p^{r+q} + b_{r+q-1}p^{r+q-1} + \ldots + b_1p + b_0}$$

which has \( r + q \) poles and \( r + q \) zeros, with \( q \) of the zeros at the origin.

The low-frequency characteristics of a feedback amplifier are often enhanced (although stabilization is not made easier) by a-c coupling the feedback signal at low frequencies. Accordingly, a relatively large capacitor can be placed in series with the feedback resistor. At mid-band and high frequencies, the reactance of this capacitor must be negligible. With such a capacitor, the feedback function \( \beta \) has the form

$$\beta = \beta_0 \frac{p}{p + a}$$

where \( \beta_0 \) is the high-frequency value of \( \beta \); as the capacitance value approaches infinity, \( a \to 0 \) and \( \beta \to \beta_0 \).

From equations (19) and (20), the low-frequency closed-loop transfer function can be written

$$\frac{-Kg}{1 + Kg\beta} = -\frac{KA (p + a)}{B (p + a) + Kg\beta pA}$$

In equation (19), the coefficient \( a_{r-1} \) is the negative of the sum of the real parts of the zero distances from the origin of the complex plane. Similarly, \( b_{r+q-1} \) is the negative of the sum of the real parts of the pole positions. With feedback, equation (21) yields
\[ -\sum \text{Re} \ (\text{zero positions}) = a + a_{r-1} \]  
\[ -\sum \text{Re} \ (\text{pole positions}) = \frac{a + b_{r+q-1} + K_0 a_{r-1}}{1 + K_0} \]  

(22)

A standard technique for optimizing the low-frequency response of video amplifiers is to make the two sums of equation (22) equal, which yields

\[ b_{r+q-1} = a_{r-1} + K_0 a \]  

(23)

Without considerable low-frequency compensation, the coefficient \( b_{r+q-1} \) is much larger than \( a_{r-1} \). Thus the series capacitor, which introduces the term \( K_0 a \), can be very helpful in low-frequency design.

However, equation (23) is not a guarantee of stability. In addition, the locus \( K_0 \beta \), where \( \beta \) is given by equation (21), must not encircle the -1 point, which can be viewed as a second condition necessary for low-frequency stability. This second condition can be satisfied by bandwidth degradation in a manner quite analogous to that discussed before. That is, the low-frequency cut-off points of all the coupling and by-pass circuits, with the exception of a single coupling circuit, are made as low as is feasible. The unique coupling circuit is made to have as high a cut-off frequency as is necessary to give good stability (that is, without an excessive peak in the closed-loop amplitude characteristic). After reasonable stabilization in this manner, the parameter \( a \) in equation (23) can be adjusted for an optimum.

As before, saturation problems make it desirable to obtain low-frequency degradation in an early stage in the amplifying system, or in the \( \beta \) circuit. In the audio amplifier example, this means that the magnetizing inductance of the transformer should be large so that the output stage will have a very low low-frequency half-power point; it should be as low or lower than the minimum signal frequencies expected.
FIG. 1 THE GENERAL SINGLE-LOOP FEEDBACK SYSTEM

FIG. 2 THE SINGLE-LOOP SYSTEM WITH REAL FEEDBACK
FIG. 3 A TYPICAL FEEDBACK AMPLIFIER

FIG. 4 OPEN-LOOP TRANSFER FUNCTION OF THE COMPENSATED AMPLIFIER
FIG. 5 NYQUIST PLOT OF A TYPICAL AMPLIFIER WITH REAL FEEDBACK

FIG. 6 COMPARISON OF BANDWIDTH DEGRADATION TECHNIQUES
FIG. 7 THE FULLY COMPENSATED FEEDBACK AMPLIFIER

FIG. 8 LOW-FREQUENCY POLES AND ZEROS