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SUBJECT: Maximally-Flat Transmission-Line Type L-C Filter Design

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ABSTRACT

A method is presented with which transmission-line type L-C low-pass ladder filters having maximally-flat transfer characteristics can be designed. The method is applicable to an arbitrary source-to-load resistance ratio and an arbitrary number of inductors and capacitors. All quantities are calculated from closed-form expressions.

Also given are general closed-form expressions for the element values of L-C ladder networks of the transmission-line type calculable from the numerical input admittance expression.

Finally, relations giving the optimum mismatch required to realize maximum gain-bandwidth type products are derived.

1. INTRODUCTION

Constant-k L-C ladder filters are widely used today, even though they are not too satisfactory for many applications, primarily because they are simple to design. A networks specialist is not likely to employ constant-k methods because it is within his ability to design filters having more desirable transfer characteristics, e.g., maximally flat, Tchebycheff, linear phase, etc. However, because only a small percentage of electronics engineers have the background needed to design other than classical filters, more modern and efficient structures are not likely to be utilized on a widespread scale until simple and direct design methods are developed.

The purpose of this paper is to set down (in the next section) a simple and direct method for designing transmission-line type L-C low-pass ladder filters of arbitrary complexity (e.g., number of elements) and with any desired source-to-load resistance ratio.

The procedure developed here is not applicable to m-derived filters where there exist zeros of transmission in the stop band.

By restricting interest to the maximally-flat function, considerable mathematical simplifications result which, in several respects, give rise to a more practical design procedure, because all calculations can be made from closed-form equations. In addition, the maximally-flat function is perhaps the most representive single function to consider because it has characteristics (and pole locations) midway between those applicable to high-efficiency Tchebycheff functions and low transient-distortion linear-phase functions. In any event, should the numerical form of the input admittance applicable to a Tchebycheff or linear-phase function be available, and should the admittance be realizable with a simple L-C ladder, formulas derived here can be used to find the required network.

A set of design relations applicable to the transmission-line type of filter has been worked out by Green (Ref. 1). His method is essentially that of matching coefficients, which is not the same as that presented here.

2. DESIGN PROCEDURE

The lossless filters shown in Figs. 1 (with all element values finite) have transfer functions that can be made maximally flat with a relative half-power bandwidth of unity such that

$$\frac{e_n}{e_o} = \left[\frac{R_2/(R_1 + R_2)}{p^n + B_{n-1}\, p^{n-1} + B_{n-2}\, p^{n-2} + \ldots + B_1\, p+1} \right]_{p\,=\,j\omega} \tag{1}$$

$$\left|\frac{e_n}{e_o}\right| = \frac{R_2/(R_1 + R_2)}{\sqrt{1 + \omega^{2n}}} \tag{2}$$

where n is the total number of inductors and capacitors. The coefficients B_j are given by

$$B_{n-1} = \frac{1}{\sin \Theta}$$

$$B_{n-2} = \frac{\cos \Theta}{\sin 2\Theta} B_{n-1} \tag{3}$$

$$B_{n-3} = \frac{\cos 2\Theta}{\sin 3\Theta} B_{n-2}$$

$$\vdots$$

$$B_{n-s} = \frac{\cos(s-1)\Theta}{\sin s\Theta} B_{n-s+1}$$

where $\Theta = \pi/2n$ and where the polynomial in the denominator of Eq 1 is symmetric such that $B_{n-1} = B_1$, $B_{n-2} = B_2$, and so forth.

The input admittance $Y_1(p)$ of the networks of Figs. 1 is given in general by

$$Y_1(p) = \frac{a_n p^n + a_{n-1}p^{n-1} + \ldots + a_2p^2 + a_1p + a_o}{b_{n-1} p^{n-1} + b_{n-2} p^{n-2} + \ldots + b_2p^2 + b_1p + b_o} \tag{4}$$

Specifically for the maximally-flat function,

$$Y_1(p) = G_1 \frac{2p^n + B_{n-1}(1 \pm r^{1/n}) p^{n-1} + B_{n-2}(1 \pm r^{2/n}) p^{n-2} + \ldots + B_1\left(1 \pm r^{\frac{n-1}{n}}\right)p + (1 \pm r)}{B_{n-1} (1 \mp r^{1/n})p^{n-1} + B_{n-2} (1 \mp r^{2/n})p^{n-2} + \ldots + B_1\left(1 \mp r^{\frac{n-1}{n}}\right)p + (1 \mp r)} \tag{5}$$

where

$$r = \left|\frac{R_1 - R_2}{R_1 + R_2}\right| \tag{6}$$

and where the positive sign in the numerator and the negative sign in the denominator of Eq. 5 are used for $R_1 > R_2$, and conversely for $R_1 < R_2$.

A special case of Eq. 5 is the matched one where $R_1 = R_2 = R$. Then $r = 0$ and

$$Y_1(p) = G \frac{2p^n + B_{n-1}p^{n-1} + \ldots + B_1p + 1}{B_{n-1}p^{n-1} + B_{n-2}p^{n-2} + \ldots B_1p + 1} \qquad (7)$$

Another special case is that resulting when $R_1 \to \infty$ (and $R_2 = R$) such that the source is an ideal current source with a current proportional to the voltage e_1. Then

$$Y_1(p) = G \frac{p^n + B_{n-1}p^{n-1} + \ldots + B_1p + 1}{\frac{B_{n-1}}{n}p^{n-1} + \frac{2B_{n-2}}{n}p^{n-2} + \frac{3B_{n-3}}{n}p^{n-3} + \ldots + \frac{n-1}{n}B_1p+1} \qquad (8)$$

The element values C_1, L_2, C_3, ... are given by

$$C_1 = \frac{a_n}{b_{n-1}}$$

$$L_2 = \frac{b_{n-1}}{a_{n-2} - C_1b_{n-3}}$$

$$C_3 = \frac{a_{n-2} - C_1b_{n-3}}{b_{n-3} - L_2(a_{n-4} - C_1b_{n-5})} \qquad (9)$$

$$L_4 = \frac{b_{n-3} - L_2\left(a_{n-4} - C_1 b_{n-5}\right)}{(a_{n-4} - C_1b_{n-5}) - C_3\left[b_{n-5} - L_2(a_{n-6} - C_1b_{n-7})\right]}$$

$$\vdots$$

$$C_k = \frac{\text{Denominator of } L_{k-1}}{\left[\begin{array}{l}\text{Denominator of } C_{k-2}\\ \text{with the subscript}\\ \text{numbers decreased by 2}\end{array}\right] - L_{k-1}\left[\begin{array}{l}\text{Denominator of } L_{k-1}\\ \text{with the subscript}\\ \text{numbers decreased by 2}\end{array}\right]}$$

$$L_j = \frac{\text{Denominator of } C_{j-1}}{\left[\begin{array}{l}\text{Denominator of } L_{j-2}\\ \text{with the subscript}\\ \text{numbers decreased}\\ \text{by 2}\end{array}\right] - C_{j-1}\left[\begin{array}{l}\text{Denominator of } C_{j-1}\\ \text{with the subscript}\\ \text{numbers decreased}\\ \text{by 2}\end{array}\right]}$$

It should be noted that the coefficients a_{n-k} and b_{n-k} are zero for $k > n$.

The design of a filter is carried out as follows:

(a) After selecting a suitable filter complexity n, calculate the B_j from Eqs. 3. (This could be done in tabular form once and for all).

(b) Select some desired r from Eq. 6.

(c) Calculate the coefficients a_j and b_j using the results of steps (a) and (b) from Eqs. 5, 7, or 8, whichever is applicable.

(d) Calculate element values from Eqs. 9 (When $R_1 = R_2$ and for n odd, only half the element values need be determined because the network is physically symmetric).

If a radian relative half-power bandwidth B is desired rather than a radian bandwidth of unity, all the L's and C's resulting from step (d) must be divided by B.

Reciprocity may be employed in the usual manner by turning the network end for end. Also, frequency transformations may be employed to convert the low-pass filter to a high-pass filter, a band-pass filter, or a band-eliminating filter. Of course, the dual of a filter may be obtained after its design is complete. The networks of Figs. 1, plus their duals, cover all the usual structural possibilities.

It must be pointed out that when a low-pass filter is designed for $R_1 \neq R_2$, the source and load are not matched for maximum power transfer. When a conjugate match at band center is desired, then the filter should always be designed with $R_1 = R_2$. Ideal transformers (which can only be approximated physically) can be introduced to transform the impedance level as desired. Determinant manipulation for effecting an impedance transformation is applicable to equivalent band-pass and band-elimination filters, but not to the low-pass structure.

One sometimes important advantage of maximally-flat filters over other types is that the input impedance of a low-pass filter can be made complimentary to its high-pass equivalent, the cross-over frequency being the relative half-power frequency. However, only filters designed for $R_1 \to \infty$ and $R_2 = R$ can be made complimentary in this manner. For example, $Z_1 + Z_2$ can be made equal to R for all frequencies if Y_1 is given by Eq. 8 and Y_2 by Eq. 8 with p replaced by 1/p. The dual low-pass and high-pass networks have $Y_1 + Y_2$ equal to the conductance G rather than $Z_1 + Z_2$ equal to the resistance R.

A brief discussion relating to the origins of the various relations given in this section follows:

Equations 1 and 2 are standard relations yielding the maximally-flat n-pole transfer function where all the zeros are at infinity. Equations 3, which give the coefficients of the powers of p in Eq. 1, can be worked out by studying the properties of the poles of the transfer function (which lie with equal angular spacing on the unit circle). These relations will be found in Ref. 1.

The derivation of the various input admittance expressions, Eqs. 5, 7, and 8, will be found in Ref. 2. The capacitance C_1 is a maximum: in deriving the general input admittance expression, the zeros of the input reflection coefficient, like the poles, are all taken in the left half-plane.

The closed-form expressions for the element values will be derived in the following section. It should be pointed out that these particular relations are general. As long as an L-C ladder network exists and as long as all the element values are finite, Eqs. 9 can be employed to determine the element values from the coefficients of the powers of p of the input admittance expression.

3. ELEMENT VALUES

It will be assumed that networks such as those of Figs. 1 as described by Eq. 4 exist with all element values positive and greater than zero. Then, the physically significant continued-fraction expression must exist, and we may form the following:

$$Y_1\left(\frac{n}{n-1}\right) = \frac{A}{B}$$

$$Z_2\left(\frac{n-1}{n-2}\right) = \frac{1}{Y_1 - pC_1} = \frac{B}{A-pC_1B}$$

$$Y_3\left(\frac{n-2}{n-3}\right) = \frac{1}{Z_2 - pL_2} = \frac{A-pC_1B}{B-pL_2(A-pC_1B)} \qquad (10)$$

$$Z_4\left(\frac{n-3}{n-4}\right) = \frac{1}{Y_3 - pC_3} = \frac{B-pL_2(A-pC_1B)}{A-pC_1B - pC_3\left[B-pL_2(A-pC_1B)\right]}$$

$$Y_5\left(\frac{n-4}{n-5}\right) = \frac{1}{Z_4 - pL_4} = \frac{A-pC_1B - pC_3\left[B-pL_2(A-pC_1B)\right]}{B-pL_2(A-pC_1B) - pL_4\left\{A-pC_1B - pC_3\left[B-pL_2(A-pC_1B)\right]\right\}}$$

and so forth.

In Eqs. 10, the functional dependence denoted by $\left(\frac{n-i}{n-j}\right)$ signifies that the numerator of the function is of highest degree n-i in p and that of the denominator is of highest degree n-j. This must be true if the continued-fraction expansion is to result in a network having the structural form assumed.

Specifically, consider the expression for Z_2. The numerator is of highest degree n-1 because the coefficient b_{n-1} of Eq. 4 is finite. The denominator, $A-pC_1B = (a_np^n + \dots) - pC_1(b_{n-1}p^{n-1} + \dots)$, appears to be of highest degree n. This cannot be if the leading term of the continued-fraction expansion of Z_2 is to give the element L_2. We therefore conclude that the coefficient of p^n in the denominator must be zero. Similarly, the coefficient of p^{n-1} must be zero. It is not possible for similar factors in numerator and denominator to cancel else the highest power of p in the numerator will be too small.

The above argument applies step by step to the immittances Y_3, Z_4, Y_5, ... , leading to the conclusion that all of the coefficients of p in the numerator and denominator of an expression for an immittance that are of higher degree than indicated in Eqs. 10 must be zero.

The first few relations of Eqs. 9 have been obtained by collecting the coefficients of the highest permissible powers of p in a given immittance expression.

The denominator of C_1 is the same as the numerator of L_2. Similarly, the denominator of L_2 is the same as the numerator of C_3. The general rule for forming the various expressions is then evident and the general expression for L_j and C_k can be written.

One calculates each higher numbered element value making use of the known coefficients a_j and b_j and the previously determined element values. The process is not at all difficult or tedious compared to the usual continued-fraction expansion (although it is in some respects equivalent).

It should be mentioned that since a number of the coefficients of the powers of p are zero, one can obtain alternate equations for the element values. These in turn yield a set of relations connecting the coefficients a_j and b_k. That such relations exist is not surprising because the poles and zeros of an input immittance are not independent but must be related such that the real part of the input immittance is positive. In addition, these extra relations must be satisfied in order that the input immittance be given by a simple L-C network terminated in a resistance.

Finally, Eqs. 9 simplify considerably for any practical n-pole filter because all the coefficients a_j and b_j for $j < 0$ are zero. The most complicated expressions will be those giving the elements near the physical center of the network.

4. BANDWIDTH OPTIMIZATION

When the bandwidth of a filter is limited by the capacitance C_1, an intentional mismatch may yield appreciable improvements in factors analogous to a gain-bandwidth product. The development of equations from which the optimum mismatch can be determined is the subject of this section (see Ref. 2).

The capacitance C_1 is given by

$$C_1 = \frac{2G_1}{B_{n-1}(1 \mp \rho^{1/n})} \tag{11}$$

where the negative sign applies for $R_1 > R_2$ and the positive sign for $R_1 < R_2$. This value of capacitance applies for a relative half-power bandwidth of unity. For a general radian bandwidth B, the product R_1C_1B can be formed. Multiplying this product by T^k and expressing ρ in terms of T, we get

$$R_1C_1BT^k = \frac{2T^k}{B_{n-1}\left[1 \mp (1-T)^{1/2n}\right]} \tag{12}$$

where

$$T = \frac{4R_1R_2}{(R_1 + R_2)^2} \tag{13}$$

is the ratio of the maximum possible to actual power dissipation in the load R_2 at $\omega = 0$.

For finite T, Eq. 12 is larger using the negative sign than the positive sign. Thus, the maximum possible value of the product will occur for $R_1 > R_2$. However, unless the weighting k on T (which specifies the relative importance of having a high transmission efficiency at $\omega = 0$) is made larger than unity, no unique maximum value for finite T exists; rather, the ideal current-generator type of network for which $R_1 \to \infty$ will be indicated as best.

The derivative of Eq. 12 can be taken in the usual manner to give the optimum value of T, T_o, from

$$(1 - T_o)^{1/2n} = \frac{1}{1 + \dfrac{T_o}{2nk\,(1-T_o)}}, \quad \begin{cases} k > 1 \\ R_1 > R_2 \end{cases} \tag{14}$$

For k = 2 and n = 2, 4, 10, and 40, values of T_o are 0.8, 0.76, 0.73, and 0.70. In all cases, the maximum of Eq. 12 appears to be quite broad; hence, precision in determining T_o is not required.

Other optimization relations using the relative half-power bandwidth can be derived in an analogous manner, for example, the products $R_2C_1BT^k$ and $(R_1C_1 + R_2C_n)BT^k$ where C_n is the shunt capacitance at the output of the filter for n odd (which is given by Eq. 11 with the positive sign).

Another class of optimization relations can be derived on the basis of a tolerance bandwidth B'. The equation

$$\frac{P_2}{P_{20}} = \frac{T}{1 + \omega^{2n}} \tag{15}$$

gives the ratio of the power dissipated in the load R_2 at $\omega = 0$ to that at a frequency ω. A tolerance bandwidth B' can be defined as the frequency where the value of Eq. 15 falls below a prescribed number β. Then

$$B' = (T/\beta - 1)^{1/2n} , \quad \beta < T \tag{16}$$

When the bandwidth B' is used in Eq. 12 in place of the relative half-power bandwidth B, the relevent optimization relation becomes

$$(1 - T_o)^{1/2n} = \frac{1}{1 + \dfrac{T_o - \beta}{(1 - T_o)\left[2nk(1 - \beta/T_o) + 1\right]}} , \begin{cases} \beta < T \\ k \geq 0 \\ R_1 > R_2 \end{cases} \tag{17}$$

One final optimization, similar in some respects to the preceding one, is that where the area $B'\beta$ under the curve of Eq. 15 is maximized. B' is the tolerance bandwidth and β is the tolerable value of p_2/p_{20}. We thus have

$$B'\beta = \frac{B'T}{1+(B')^{2n}} \tag{18}$$

Differentiating this expression with respect to B' (where β is assumed to be a prescribed number) there is obtained

$$B' = \frac{1}{2n-1} \tag{19}$$

Using this value for the tolerance bandwidth in Eq. 18, the optimum value of T is found to be

$$T_o = \beta\left(\frac{2n}{2n-1}\right) \tag{20}$$

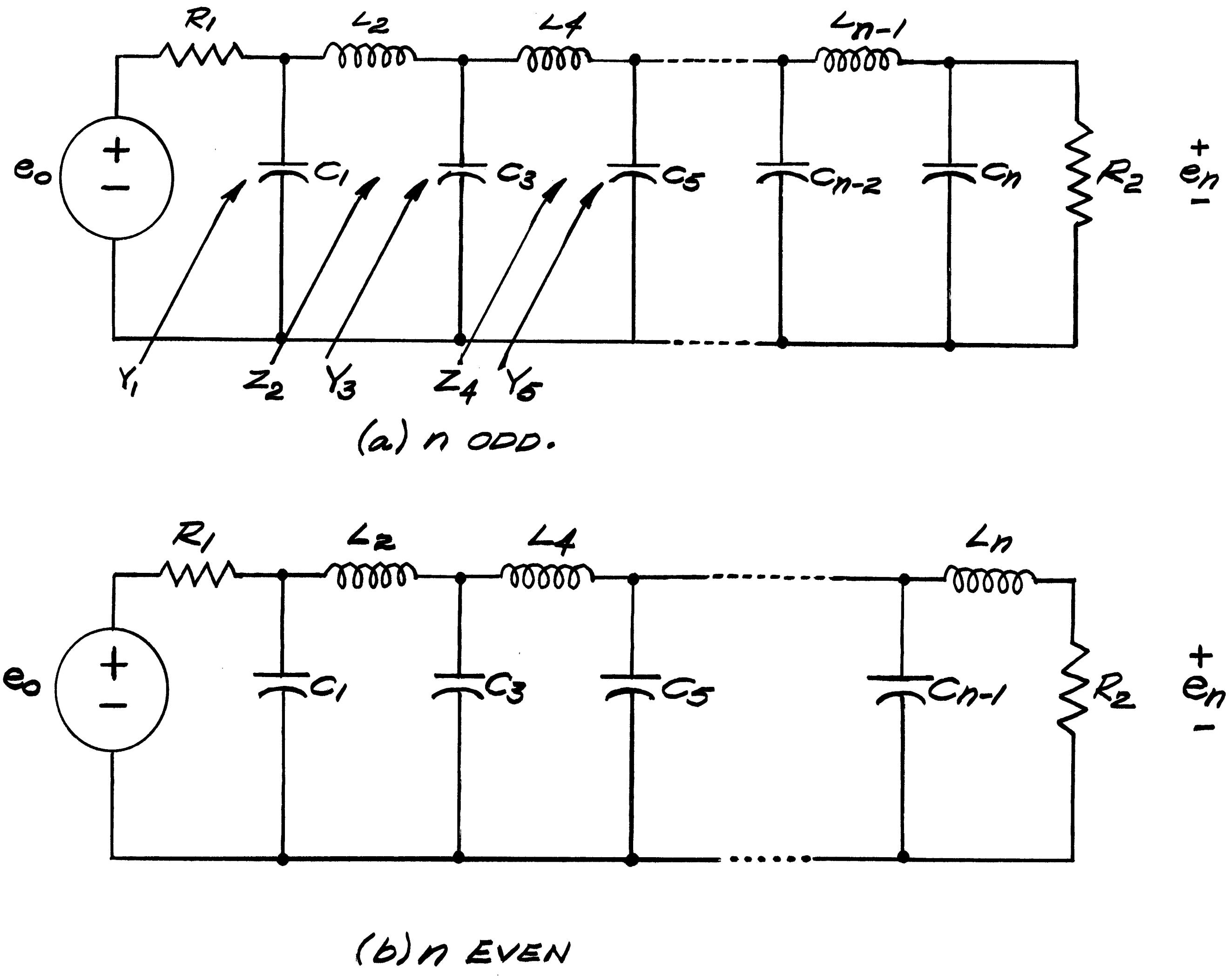

FIG. 1. TRANSMISSION-LINE TYPE LADDER FILTER.

REFERENCES

1. E. Green, "Synthesis of Ladder Networks to Give Butterworth or Chebychev Response in the Pass Band," Proc. IEE, Vol. 101, No. 7, Part 4, Aug., 1954, pp 192-203.

2. J. L. Stewart, "Shunt Capacitance and Maximally-Flat Filter Design," 1955, IRE National Convention Record, Part 2.