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THE FREQUENCY OF LINEAR OSCILLATORS

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ABSTRACT

A general method is derived for determining the oscillation frequency of linear oscillators and application is made to a wide-range reactance-modulated oscillator.

In addition, it is demonstrated that all oscillators can be treated as ordinary phase-shift oscillators.

The generalized equations derived here allow the relative advantages and values of certain types of circuits to be easily ascertained. The method actually amounts to Barkhausen's criteria aided with the concepts furnished by poles and zeros.

THE FREQUENCY OF LINEAR OSCILLATORS

1. THEORY

A conventional oscillator can be diagramed as shown in Fig. 1. Let some driving voltage e_n be applied at the input and let the feedback path be opened. Then, the ratio of output to input voltages e_o/e_n is given by $-G = -K\,A/B$ where K is a real constant and where A and B are polynomials in the complex variable p having the form

$$A = a_o + a_1 p + a_2 p^2 + \dots + a_m p^m$$

$$B = b_o + b_1 p + b_2 p^2 + \dots + b_n p^n \qquad (1)$$

The steady state prevails for $p = j\omega$. When the loop is closed $e_o/e_n = -G/(1 + G)$ and oscillation will take place at the frequency where the phase shift of G is -180 degrees provided that $|G| \geq 1$ at this frequency. In obtaining the open loop transfer function, all impedances at the input to the network are referred to the output of the network such that the input impedance can be assumed to be infinite.

More generally, when K may be either positive or negative, oscillation will take place at a frequency where the phase shift of the open loop transfer function is some multiple of 180 degrees. The tangent of such phase angles is zero.

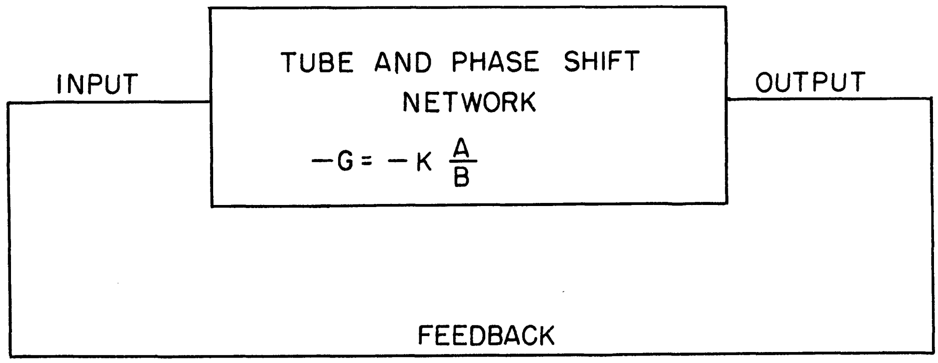


FIG. I

BLOCK DIAGRAM OF CONVENTIONAL OSCILLATOR.

The open loop transfer function can be expressed in real and imaginary parts according to

$$- K(A/B) = - K\,(AB^*/BB) = -\,(K/BB^*)\left[(\text{Ev }A\text{ Ev }B^* + \text{Od }A\text{ Od }B^*) + (\text{Ev }A\text{ Od }B^* + \text{Od }A\text{ Ev }B^*)\right] \tag{2}$$

where the first term in the brackets is the real part and the second term is the imaginary part. It is assumed that the equation is evaluated at $p = j\omega$. "Ev" and "Od" are the operators that take the even and odd parts of the polynomials upon which they operate, respectively, i. e., Ev $A = a_o + a_2p^2 + a_4p^4 + \dots$. B^* is the complex conjugate of B at $p = j\omega$.

Along the $j\omega$ axis, Ev B^* = Ev B and Od B^* = -Od B thus allowing the phase angle θ of (2) to be represented as

$$j \tan\theta = \left[\frac{\text{Od }A\text{ Ev }B - \text{Ev }A\text{ Od }B}{\text{Ev }A\text{ Ev }B - \text{Od }A\text{ Od }B}\right]_{p=j\omega} \tag{3}$$

which has poles and zeros only along the $j\omega$ axis. The numerator of (3) factors as

$$\text{Od }A\text{ Ev }B - \text{Ev }A\text{ Od }B = cp(p^2 + \omega_1^2)(p^2 + \omega_2^2)\ \dots \tag{4}$$

where the constant c is the coefficient of the highest power of p and where the ω_i are the frequencies at which occur phase shifts θ that are multiples of 180 degrees. At these frequencies, the numerator of (3) and hence tan θ becomes zero. As a side interest, it can be observed that the denominator of (3) factors as

$$\text{Ev A Ev B} - \text{Od A Od B} = d(p^2 + \omega_a^2)(p^2 + \omega_b^2) \ldots \tag{5}$$

where the frequencies are those at which occur phase shifts that are odd multiples of 90 degrees.

The loop gain at one of the frequencies ω_i is found by using the condition that (4) be zero in Eq 2 to obtain

$$- K(A/B)_{p=j\,\omega_i} = -K \left[\frac{\text{Ev A Ev B} - \text{Od A Od B}}{BB^*} \right]_{p=j\,\omega_i} \tag{6}$$

which must be equal to or greater than unity in order for oscillation to take place. If the phase shift is an odd multiple of 180 degrees at the oscillation frequency, (6) will be positive.

Of particular interest is the case when the transfer function has no zeros. Then, $A = a_o$ and the left side of (4) reduces to

$$- a_o \text{ Od B} = cp(p^2 + \omega_1^2)(p^2 + \omega_2^2) \ \ldots \tag{7}$$

and the loop gain at one of the frequencies ω_i becomes

$$- K(A/B)_{p=j\,\omega_i} = - K\, a_o/(\text{Ev B})_{p=j\omega_i} \tag{8}$$

2 FUNCTIONS HAVING ZEROS ONLY AT THE ORIGIN

Many types of oscillators have open loop transfer functions whose zeros are all at the origin of the complex frequency plane. This has the effect of giving a large leading phase angle at low frequencies which must be partially overcome with the lagging phase angle contributed by the poles in order for oscillation to take place.

If there are q zeros at the origin and q is even, Eq 4 gives the possible frequencies ω_i as

$$- p^q \text{ Od B } = cp^{q+1} (p^2 + \omega_1^2)(p^2 + \omega_2^2) \ \ldots \tag{9}$$

which reduces to the equation used when the transfer function contains only poles.

If the number of zeros r at the origin is odd, Eq 4 gives

$$p^r \text{ Ev B } = cp^{r+1}(p^2 + \omega_1^2)(p^2 + \omega_1^2) \ \ldots \tag{10}$$

which is quite similar to that used for transfer functions containing only poles.

Certain types of R-C oscillators have open loop transfer functions that contain several zeros at the origin. The tuned plate, tuned grid and Hartley oscillators also belong in this catagory.

3. APPLICABILITY*

Conventional oscillators can be treated as phase shift oscillators -- a point of view that is all to frequently misunderstood. In order to demonstrate

* For a comprehensive discussion of oscillators, see "Vacuum Tube Oscillators", W. A. Edson, John Wiley and Sons, Inc., New York, 1953

this assertion, a few one tube circuits along with their phase shift developments are shown in Fig. 2. In obtaining these equivalents, the cathode is assumed to be grounded even though it may not be grounded in the physical circuit. Then, it can be observed that the feedback quantity is always applied to the grid of the tube. In the equivalent circuits of Fig. 2 the tube is represented as a current generator driven with an external driving voltage in series with the grid--this facilitates setting up the transfer functions.

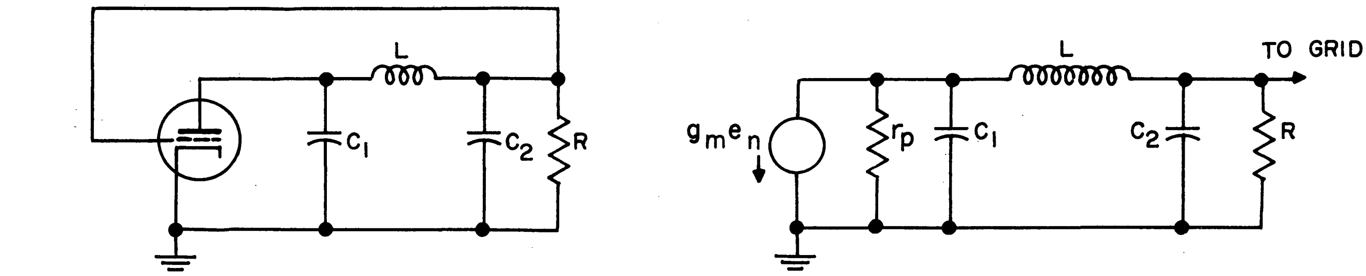
Evidently, the only essential difference between the various circuits is the magnitude of the transfer function at the oscillation frequency. The study of such circuits enables one to estimate the dependence of the frequency of oscillation upon such parameters as the tube plate resistance and inter-electrode capacitances. Capacitances in Fig. 2 include tube and miscellaneous stray and other capacitance. The resistance terminating the network represents in part the grid conduction resistance if the grid draws current. The plate resistance of the tube is r_p and the transconductance is g_m.

4. EXAMPLES

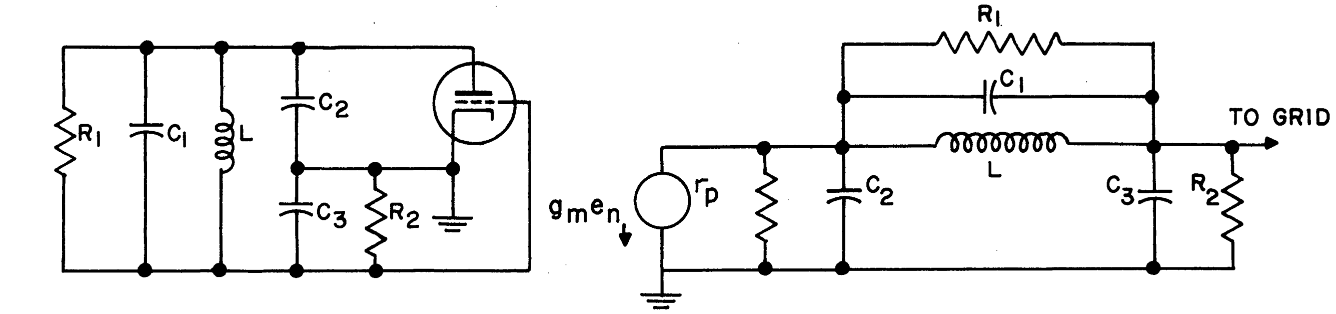
As an example, consider a network with characteristics similar to those of a three section R-C network with shunt C or a Pi section L-C network with shunt C loaded with a resistance. These networks have three poles and no zeros in their transfer functions. Then,

$$- K\,A/B \;=\; Ka_o/(b_o + b_1p + b_2p^2 + b_3p^3) \qquad (11)$$

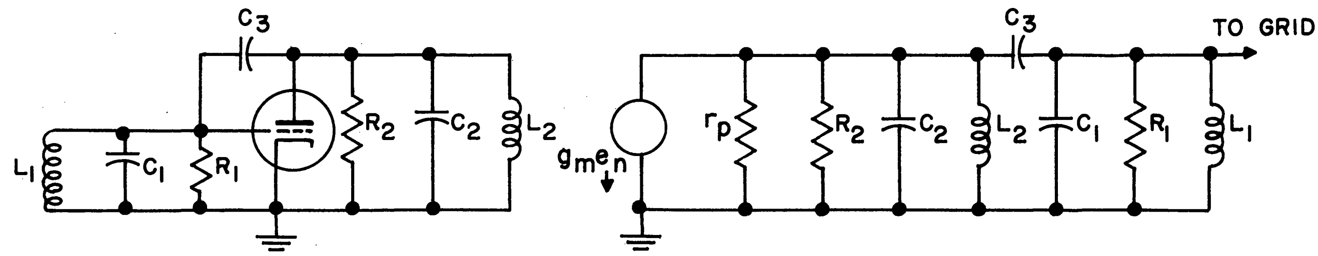
Setting the odd part of the denominator equal to the corresponding factored expression $b_3p(p^2 + \omega_1^2)$ the oscillation frequency is immediately found



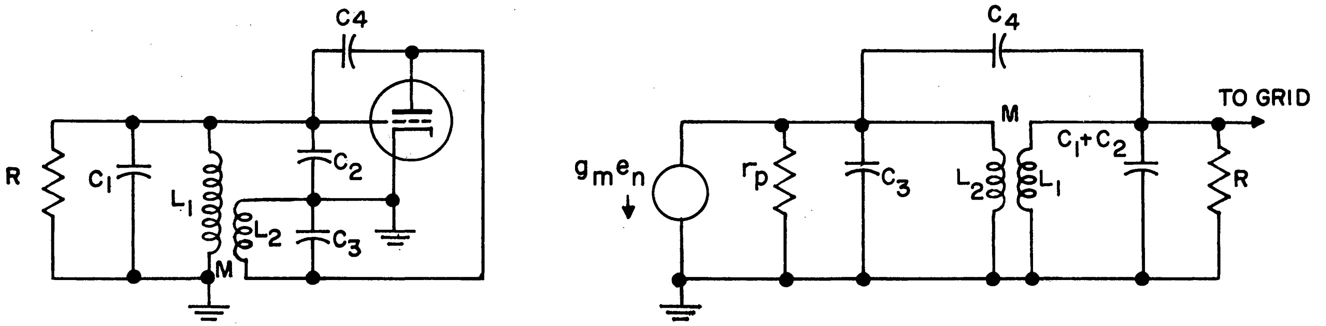
STANDARD PHASE SHIFT OSCILLATOR



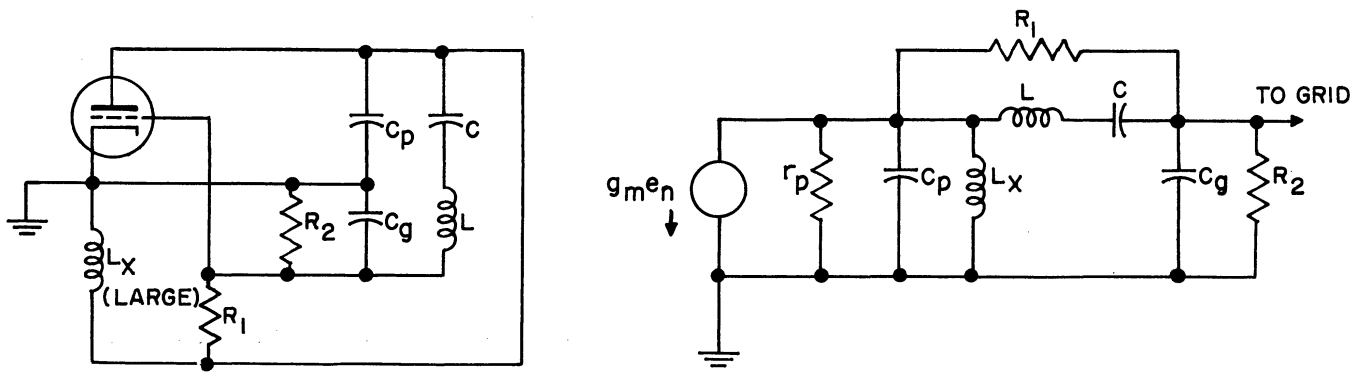
COLPITTS OSCILLATOR



TUNED PLATE TUNED GRID OSCILLATOR



HARTLEY OSCILLATOR



CLAPP'S OSCILLATOR

FIG. 2

PHASE SHIFT DEVELOPMENT OF SEVERAL CONVENTIONAL OSCILLATORS.

to be $\omega_1 = (b_1/b_3)^{1/2}$ at which the loop gain is $-Ka_o/(b_o - b_1b_2/b_3)$.

As a second example, consider the transfer function

$$- K A/B = Ka_o/(b_o + b_1p + b_2p^2 + \dots + b_6p^6) \tag{12}$$

The odd part of the denominator of Eq 12 factors as

$$p(b_1 + b_3p^2 + b_5p^4) = b_5p (p^2 + \omega_1^2)(p^2 + \omega_2^2) \tag{13}$$

Equating coefficients, the frequencies ω_1 and ω_2 are found to be

$$\omega_1^2, \omega_1^2 = b_3/2b_5 \pm \left[(b_3/2b_5)^2 - b_1/b_5\right]^{1/2} \tag{14}$$

where the negative sign applies to the frequency where the phase shift is -180 degrees and the positive sign to the frequency where the phase shift is -360 degrees.

The results of Eq 14 can be used to find the resonant frequency and range of frequencies of the reactance modulated oscillator described by Dennis and Felch[1]. The circuit is shown in Fig. 3 in which the plate conductances of the tubes are neglected. Frequency variation is achieved by changing g_{m2}, the transconductance of the modulator tube.

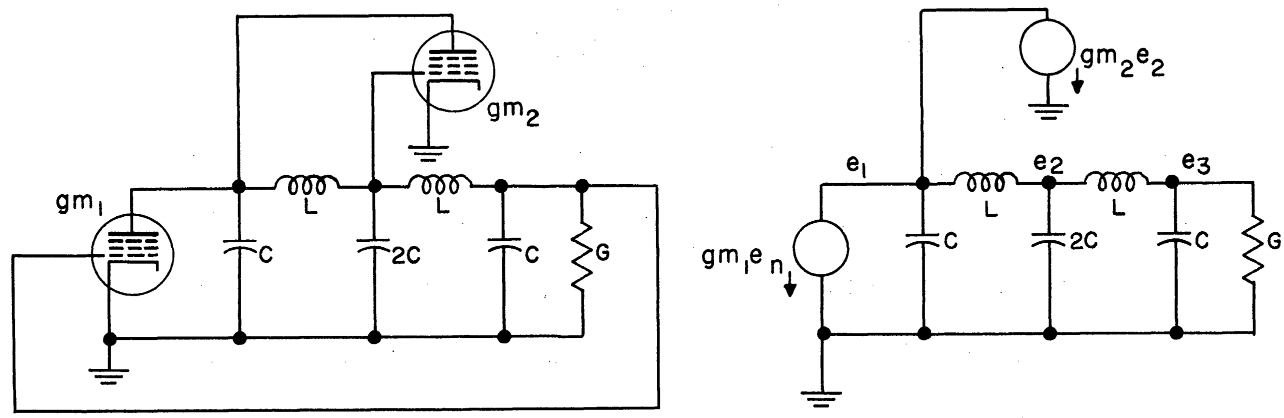


FIG. 3
REACTANCE MODULATED OSCILLATOR.

1. F. R. Dennis and E. P. Felch, "Reactance Tube Modulation of Phase Shift Oscillators", BSTJ, vol 28, No. 4, pp 601-7, October,1949.

Using a nodal analysis, the transfer function is found to be

$$e_3/e_n = -g_{m1} \Big/ \Big[p^5(2L^2C^3) + p^4(2L^2C^2G) + p^3(6LC^2) + p^2(4LCG + LCg_{m2}) + p(4C + LGg_{m2}) + (G + g_{m2}) \Big] \tag{15}$$

Applying Eq 14 for the lower of the two possible frequencies,

$$\omega_1^2 = (3/2LC) \left[1 - \frac{\sqrt{1-2LGg_{m2}/C}}{3} \right] \tag{16}$$

The minimum frequency is given when $g_{m2} = 0$ as

$$\omega_{min}^2 = 1/LC \tag{17}$$

and the maximum when the radical in (16) is zero as

$$\omega_{max}^2 = 3/2LC \tag{18}$$

which requires

$$(g_{m2})_{max} = C/2LG \tag{19}$$

The tuning ratio is found to be

$$\text{Ratio} = \frac{\omega_{max} - \omega_{min}}{(\omega_{max}\ \omega_{min})^{1/2}} = 0.203 \tag{20}$$

It should be observed that there is nothing of basic importance about the relative values of L and C in Fig. 3. It would appear that the frequency range could be materially increased by choosing different values for L and C rather than those shown in Fig. 3. To illustrate the improvement that can be obtained, assume that all the C's in Fig. 3 are equal. Then, the transfer function is

$$e_3/e_n = -g_{m1}/\left[p^5 (L^2C^3) + p^4 (L^2C^2G) + p^3 (4LC^2)\right.$$

$$\left. + p^2(3LCG + LCg_{m2}) + p(3C + LGg_{m2}) + (G + g_{m2})\right] \tag{21}$$

in which case,

$$\omega_1^2 = (2/LC) \left[1 - \frac{\sqrt{1 - LGg_{m2}/C}}{2} \right] \tag{22}$$

and

$$\text{Ratio} = \frac{\omega_{max} - \omega_{min}}{(\omega_{max}\,\omega_{min})^{1/2}} = 0.348 \tag{23}$$

A method for further extending the tuning range can be seen by examining (16); that is, g_{m2} can be made negative. To accomplish this, two reactance tubes can be used in the oscillator such that one or the other (but not both) has a finite transconductance at any given frequency. A circuit embodying such an arrangement is shown in Fig. 4.

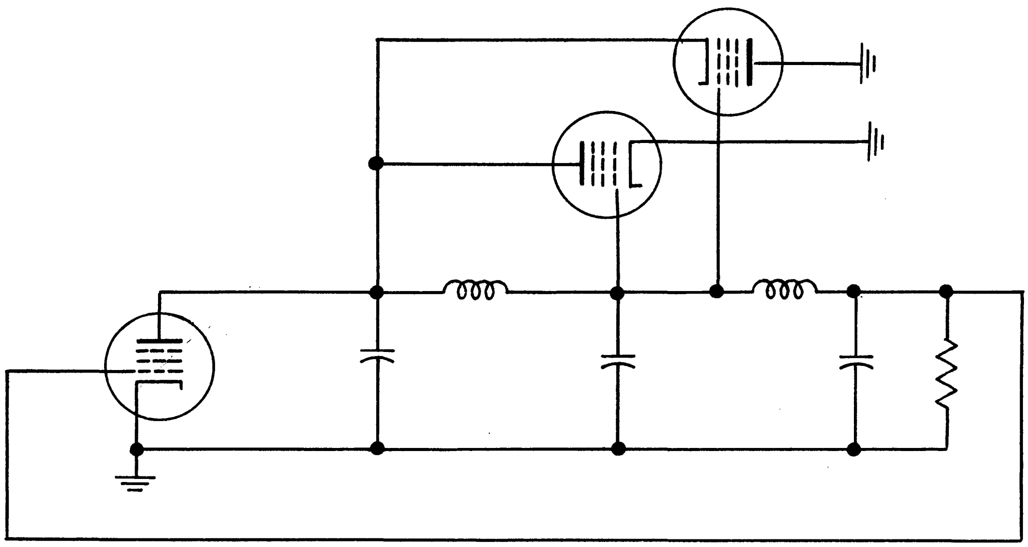


FIG. 4
EXTENDED RANGE OSCILLATOR.

It might be thought that the tuning range of the oscillator of Fig. 3 could be extended by adding a second reactance tube as shown in Fig. 5. However, an investigation of the transfer function quickly shows that only the even part of the denominator is affected; hence, no change in the oscillation frequency will result with the addition of the second reactance tube.

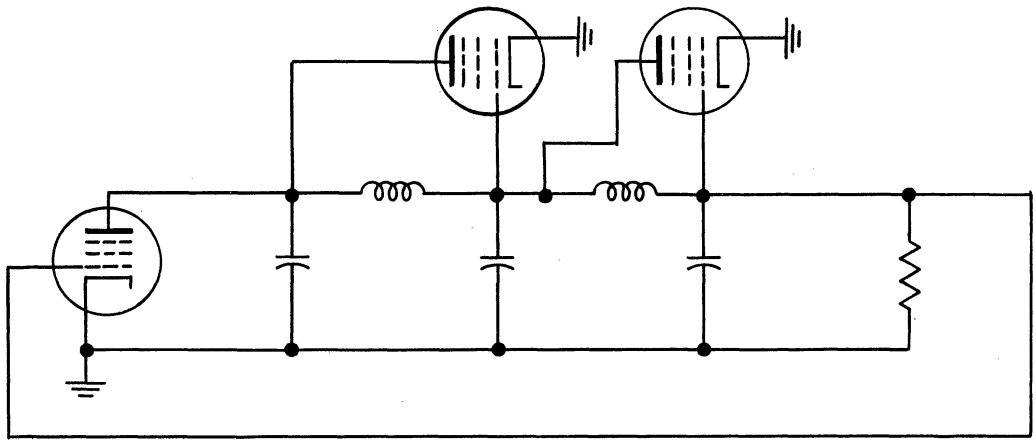


FIG. 5
AN INEFFECTIVE EXTENSION.

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