Noncommutative Schur P-functions and the shifted plactic monoid

by

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To Kamilah Omolara Neighbors (1976–2010)

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CHAPTER 1

Introduction

This thesis is focused on the theory of shifted Young tableaux and Schur P- and Q-functions. More specifically, we construct a product structure on the set of shifted Young tableaux, the *shifted plactic monoid*. This theory is analogous to that of the plactic monoid of Lascoux and Schützenberger for ordinary Young tableaux. The main applications are new formulations (and new proofs) of the shifted Littlewood-Richarsdon rule and the rule for the Schur expansion of Schur P-functions, and a shifted counterpart of the Lascoux-Schützenberger theory of noncommutative Schur functions in plactic variables. We also present an application of the shifted plactic monoid to prove a conjecture of Reiner, Stanton, and White regarding cyclic sieving in the hyperoctahedral group.

In **Chapter 2** we survey the classical theory of Young tableaux and symmetric functions. The celebrated *Robinson-Schensted-Knuth correspondence* [20] is a bijection between words in a linearly ordered alphabet $X = \{1 < 2 < 3 < \cdots\}$ and pairs of *Young tableaux* with entries in X. More precisely, each word corresponds to a pair consisting of a semistandard *insertion tableau* and a standard *recording tableau*. The words producing a given insertion tableau form a *plactic class*. A. Lascoux and M. P. Schützenberger [13] made a crucial observation based on a result by D. E.

Knuth [8]: the plactic classes $[u]$ and $[v]$ of two words u and v uniquely determine the plactic class $[uv]$ of their concatenation. This gives the set of all plactic classes (equivalently, the set of all semistandard Young tableaux) the structure of a *plactic monoid* $\mathbf{P} = \mathbf{P}(X)$. This monoid has important applications in representation theory and the theory of symmetric functions; see, e.g., [12].

The main results in this thesis are developed in **Chapter 3**, where we construct and study a proper analog of the plactic monoid for (semistandard) *shifted Young tableaux*, with similar properties and similar applications. The problem of developing such a theory was already posed more than 20 years ago by B. Sagan [18]. Shifted Young tableaux are certain fillings of a *shifted shape* (a shifted Young diagram associated with a strict partition) with letters in an alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$; see, e.g., [19]. M. Haiman [6] defined the (shifted) *mixed insertion correspondence*, a beautiful bijection between permutations and pairs of standard shifted Young tableaux; each pair consists of the mixed insertion tableau and the mixed recording tableau. Haiman's correspondence is easily generalized (see Section 3.4) to a bijection between words in the alphabet X and pairs consisting of a semistandard shifted mixed insertion tableau and a standard shifted mixed recording tableau. (We emphasize that this bijection deals with words in the original alphabet X rather than the extended alphabet X'.) We define a *shifted plactic class* as the set of all words which have a given mixed insertion tableau. Thus, shifted plactic classes are in bijection with shifted semistandard Young tableaux. The following key property, analogous to that of Lascoux and Schützenberger's in the ordinary case, holds (Theorem 3.8): the shifted plactic class of the concatenation of two words u and v depends only on the shifted plactic classes of u and v. Consequently, one can define the *shifted plactic monoid* $\mathbf{S} = \mathbf{S}(X)$ in which the product is, again, given by concatenation. In analogy with the classical case, we obtain a presentation of $\mathbf{S}$ by the quartic *shifted Knuth (or shifted plactic) relations*. So two words are shifted Knuth-equivalent if and only if they have the same mixed insertion tableau.

Sagan [18] and Worley [28] have introduced the *Sagan-Worley correspondence*, another analog of Robinson-Schensted-Knuth correspondence for shifted tableaux. In the case of permutations, Haiman [6] proved that the mixed insertion correspondence is dual to Sagan-Worley's. In Section 4.1, we use a semistandard version of this duality to describe shifted plactic equivalence in yet another way, namely: two words u and v are shifted plactic equivalent if and only if the recording tableaux of their inverses (as biwords) are the same.

The *plactic algebra* $\mathbb{Q}\mathbf{P}$ is the semigroup algebra of the plactic monoid. The shape of a plactic class is the shape of the corresponding tableau. A *plactic Schur function* $\mathcal{S}_\lambda \in \mathbb{Q}\mathbf{P}$ is the sum of all plactic classes of shape λ; it can be viewed as a noncommutative version of the ordinary Schur function s_λ. This notion was used by Schützenberger [21] to obtain a proof of the Littlewood-Richardson rule along the following lines. It can be shown that the plactic Schur functions span the ring they generate. Furthermore, this ring is canonically isomorphic to the ordinary ring of symmetric functions: the isomorphism simply sends each Schur function s_λ to its plactic counterpart $\mathcal{S}_\lambda$. It follows that each Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ is equal to the coefficient of a fixed plactic class T_λ of shape λ in the product of plactic Schur functions $\mathcal{S}_\mu \mathcal{S}_\nu$. In other words, $c^\lambda_{\mu,\nu}$ is equal to the number of pairs (T_μ, T_ν) of plactic classes of shapes μ and ν such that $T_\mu T_\nu = T_\lambda$.

We develop a shifted counterpart of this classical theory. The *shifted plactic algebra* $\mathbb{Q}\mathbf{S}$ is the semigroup algebra of the shifted plactic monoid, and a (shifted) *plactic Schur P-function* $\mathcal{P}_\lambda \in \mathbb{Q}\mathbf{S}$ is the sum of all shifted plactic classes of a given shifted shape. We prove that the plactic Schur P-functions span the ring they generate, and this ring is canonically isomorphic to the ring spanned/generated by the ordinary Schur P-functions. Again, the isomorphism sends each Schur P-function P_λ to its plactic counterpart $\mathcal{P}_\lambda$. This leads to a proof of the shifted Littlewood-Richardson rule (Corollary 3.22). Our version of the rule states that the coefficient $b^\lambda_{\mu,\nu}$ of P_λ in the product $P_\mu P_\nu$ is equal to the number of pairs (T_μ, T_ν) of shifted plactic classes of shapes μ and ν such that $T_\mu T_\nu = T_\lambda$, where T_λ is a fixed shifted plactic class of shape λ. The first version of the shifted Littlewood-Richardson rule was given by Stembridge [26]. In Lemma 3.25 we relate our rule to Stembridge's by a simple bijection.

It turns out that the shifted plactic relations are a "relaxation" of the ordinary Knuth (= plactic) relations. More precisely, the tautological map $u \mapsto u$ that sends each word in the alphabet X to itself descends to a monoid homomorphism $\mathbf{S} \to \mathbf{P}$. By extending this map linearly, we obtain the following theorem (Corollary 3.26): For a shifted shape θ, the coefficient g^θ_μ of s_μ in the Schur expansion of P_θ is equal to the number of shifted plactic classes of shifted shape θ contained in a fixed plactic class of shape μ. A simple bijection (Theorem 3.29) recovers a theorem of Stembridge [26]: g^θ_μ is equal to the number of standard Young tableaux of shape μ which rectify to a fixed standard shifted Young tableau of shape θ.

In the classical setting, an approach developed by Schützenberger and his school begins with the plactic monoid as the original fundamental object, and identifies each tableau T with a distinguished canonical representative of the corresponding plactic class, the *reading word* $\mathrm{read}(T)$. This word is obtained by reading the rows of T from left to right, starting from the bottom row and moving up. A word w such that $w = \mathrm{read}(T)$ for some tableau T is called a *tableau word*. By construction, tableau words are characterized by the following property. Each of them is a concatenation of weakly increasing words $w = u_l u_{l-1} \cdots u_1$, such that

(A) for $1 \leq i \leq l-1$, the longest weakly increasing subword of $u_{i+1}u_i$ is u_i.

For a tableau word w, the lengths of the segments u_i are precisely the row lengths of the Young tableau corresponding to w.

In **Chapter 4** of this thesis, we develop an analog of this approach in the shifted setting by taking the shifted plactic monoid as the fundamental object, and constructing a canonical representative for each shifted plactic class. Since shifted Young tableaux have primed entries while the words in their respective shifted plactic classes do not, the reading of a shifted Young tableau cannot be defined in as simple a manner as in the classical case. Instead, we define the *mixed reading word* $\mathrm{mread}(T)$ of a shifted tableau T as the unique word in the corresponding shifted plactic class that has a distinguished *special recording tableau*. The latter notion is a shifted counterpart of P. Edelman and C. Greene's *dual reading tableau* [1].

A word w such that $w = \mathrm{mread}(T)$ for some shifted Young tableau T is called a *shifted tableau word*. Such words have a characterizing property similar to (A), with weakly increasing words replaced by *hook words* (a hook word consists of a strictly decreasing segment followed by a weakly increasing one). In Theorem 4.9 and Proposition 4.10, we prove that w is a shifted tableau word if and only if

(B) for $1 \leq i \leq l-1$, the longest hook subword of $u_{i+1}u_i$ is u_i.

For a shifted tableau word w, the lengths of the segments u_i are precisely the row lengths of the shifted Young tableau corresponding to w.

Also in Chapter 4, we lay down the machinery needed for the proofs of the main results of Chapter 3. Building on the concept of *standard decomposition tableaux* introduced by W. Kraśkiewicz [9] and further developed by T. K. Lam [11], we define a (shifted) *semistandard decomposition tableau* (SSDT) R of shifted shape λ as a filling of λ by entries in X such that the rows $u_1, u_2, \ldots, u_l$ of R are hook words satisfying (B). We define the *reading word* of R by $\mathrm{read}(R) = u_l u_{l-1} \cdots u_1$, that is, by reading the rows of R from left to right, starting with the bottom row and moving up.

As a semistandard analog of Kraśkiewicz's correspondence [9], we develop the *SK correspondence* (see Definition 4.18). This is a bijection between words in the alphabet X and pairs of tableaux with entries in X. Every word corresponds to a pair consisting of an SSDT called the *SK insertion tableau* and a standard shifted Young tableau called the *SK recording tableau*. We prove (Theorem 4.23) that the mixed recording tableau and the SK recording tableau of a word w are the same. Furthemore, we construct (see Theorem 4.17) a bijection Φ between SSDT and shifted Young tableaux of the same shape that preserves the reading word: $\mathrm{read}(R) = \mathrm{mread}(\Phi(R))$. In light of the conditions (A) and (B) above, one can see that the counterpart of an SSDT in the ordinary case is nothing but a semistandard Young tableau.

In **Chapter 5** we present a joint project with T. K. Petersen [15]. Here we use the relationship between plactic equivalence and shifted plactic equivalence to prove a conjecture of Reiner, Stanton, and White regarding the cyclic sieving phenomenon for the set $R(w_0)$ of reduced expressions for the longest element in the hyperoctahedral group. More specifically, $R(w_0)$ possesses a natural cyclic action given by moving the first letter of a word to the end, and we show that the orbit structure of this action is encoded by the generating function for the major index on $R(w_0)$.

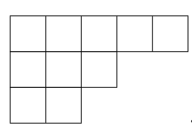
CHAPTER 2

Young tableaux and Schur functions

2.1 Partitions and Ferrers shapes

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{Z}^l$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$. Throughout this thesis we will assume that all the parts of a partition are non zero, by virtue of the identification $(\lambda_1, \ldots, \lambda_l) = (\lambda_1, \ldots, \lambda_l, 0)$. The *(Ferrers) shape* of λ is a left-justified array of square cells in which the i-th row has λ_i cells. We identify a partition λ with the Ferrers shape corresponding to λ. The *size* of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We denote $\ell(\lambda) = l$, the number of rows. The *conjugate partition*, denoted λ', is the one corresponding to the transpose of the Ferrers shape of λ. Equivalently, λ'_i is the number of parts of λ that are greater than or equal to i.

To illustrate, the Ferrers shape of $\lambda = (5, 3, 2)$ is shown below. We have $|\lambda| = 10$, $\ell(\lambda) = 3$, and $\lambda' = (3, 3, 2, 1, 1)$.



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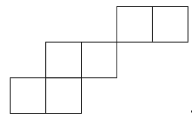
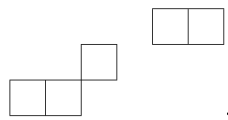
We say that a shape λ contains a shape μ if when overimposing their upper left corners, the shape μ is fully contained in λ. The *skew shape* λ/μ is then obtained by removing the shape μ from the upper left corner of λ. Below is an example of the skew shape λ/μ, for $\lambda = (5, 3, 2)$ and $\mu = (3, 1)$:



A *horizontal strip* is a (not necessarily connected) skew shape in which every column contains at most one box. The following is an example of a horizontal strip:



2.2 Young tableaux

A *(semistandard) Young tableau (SSYT)* T of shape λ is a filling of a Ferrers shape λ with letters from the alphabet $X = \{1 < 2 < \cdots\}$ such that:

- rows of T are weakly increasing;
- columns of T are strictly increasing.

If T is a filling of a shape λ, we write $\mathrm{shape}(T) = \lambda$. A *skew semistandard Young tableau* of shape λ/μ is defined analogously. When necessary, we will refer to the former as a *straight–shape SSYT*, and to the latter as a *skew SSYT*. The *content* of a tableau T is the vector $(a_1, a_2, \ldots)$, where a_i is the number of appearances of the letter i in T. The *reading word* $\mathrm{read}(T)$ of a tableau T is the word obtained by reading each row from left to right, starting from the bottom row, and moving up.

Example 2.1. The semistandard Young tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}$$

has shape $\lambda = (5, 3, 2)$, content $(3, 2, 3, 0, 2)$, and reading word $\mathrm{read}(T) = 35\,233\,11125$.

A Young tableau of shape λ is called *standard* if it contains each of the entries $1, 2, \cdots, |\lambda|$ exactly once. In other words, a standard Young tableau has content $(1, 1, \ldots, 1)$.

2.3 Robinson-Schensted-Knuth insertion

Let X be the alphabet $\{1, 2, 3, \ldots\}$.The *Robinson-Schensted-Knuth (RSK) insertion* (see, e.g., [25]) is a remarkable algorithm which gives rise to a correspondence between words in X and pairs of Young tableaux of the same shape, one of them semistandard and one standard.

Definition 2.2 (RSK insertion)**.** Let $w = w_1 \ldots w_n$ be a word in the alphabet X. We recursively construct a sequence $(P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$ of pairs of tableaux, where P_i is a semistandard Young tableau, and Q_i is a standard Young tableau, as follows. Set $(P_0, Q_0) = (\emptyset, \emptyset)$. For $i = 1, \ldots, n$, insert w_i into P_{i-1} in the following manner:

Insert w_i into the first row, bumping out the smallest element a_1 that is strictly greater than w_i. Now insert a_1 into the next row, and continue, row by row. The procedure terminates when the element a_k bumped from row k is greater than or equal to every element in the (possibly empty) row $k + 1$. In this case, a_k is placed at the right end of row $k + 1$, and the algorithm stops. The resulting tableau is denoted P_i.

The shapes of P_{i-1} and P_i differ by one box. Add that box to Q_{i-1}, and write i into it to obtain Q_i.

We call P the *insertion tableau* and Q the *recording tableau*, and denote them $P_{\mathrm{RSK}}(w)$ and $Q_{\mathrm{RSK}}(w)$, respectively.

Example 2.3. The word $w = 3152133125$ has the following insertion and recording tableaux:

$$P_{\mathrm{RSK}}(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 & 5 \\ \cline{1-2}\end{array}, \qquad Q_{\mathrm{RSK}}(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 9 & 10 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 & 7 \\ \cline{1-2}\end{array}.$$

Remark 2.4. If the word w is a permutation, then the insertion tableau P is also a standard Young tableau. This is the original *Robinson-Schensted insertion*, which was generalized by Knuth to arbitrary words. In fact, Knuth's generalization, involving biwords, is stronger than the definition we need to use herein.

arbitrary words. In fact, Knuth's generalization is stronger, involving biwords, but

Theorem 2.5 (see e.g. [25])**.** *Let w be a permutation. Then $P_{\mathrm{RSK}}(w^{-1}) = Q_{\mathrm{RSK}}(w)$.*

Theorem 2.6 (see e.g. [25])**.** *Let T be an SSYT. The insertion tableau of* $\mathrm{read}(T)$ *is precisely T.*

Edelman and Greene [1] define a *dual reading tableau* of shape λ in the following way. First, label the cells of a shape λ with the integers $1, 2, \ldots, n$ from left to right in each row, beginning with the bottom row and moving upwards. Next, sort the columns into increasing order. The steps are illustrated in the example below.

$$\begin{array}{|c|c|c|c|c|}\hline 6 & 7 & 8 & 9 & 10 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 1 & 2 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 9 & 10 \\ \hline 3 & 4 & 8 \\ \cline{1-3} 6 & 7 \\ \cline{1-2}\end{array}$$

Dual reading tableaux provide an alternative way of finding the reading word of an SSYT, by the following lemma.

Lemma 2.7 (Edelman, Greene [1])**.** *Let T be an SSYT of shape λ. The recording tableau of* $\mathrm{read}(T)$ *is precisely the dual reading tableau of shape λ.*

Thus, $\mathrm{read}(T)$ is the word corresponding to the pair (T, Q) under the RSK correspondence, where Q is the dual reading tableau of the corresponding shape.

Knuth [8] has determined when two words have the same insertion tableau, as follows.

Theorem 2.8. *Two words u and v have the same RSK insertion tableau if and only if they are equivalent modulo the following relations:*

$$acb \equiv cab \quad \text{for} \quad a \leq b < c; \tag{2.1}$$

$$bac \equiv bca \quad \text{for} \quad a < b \leq c; \tag{2.2}$$

One refers to equations (2.1) and (2.2) as the *Knuth relations*, or the *plactic relations*. Thus, Theorem 2.8 says that two words are *plactic equivalent* if and only if they have the same RSK insertion tableau.

2.4 Jeu de taquin

Jeu de taquin [20] is an important operation on skew tableaux introduced by M.–P. Schützenberger. Two skew tableaux T and U are jeu de taquin equivalent if and only if U can be obtained from T by a sequence of *jeu de taquin slides* of the form

$$\begin{ytableau} {} & b \\ a \end{ytableau} \leftrightarrow \begin{ytableau} a & b \\ {} \end{ytableau} \; (a \leq b) \qquad \begin{ytableau} {} & b \\ a \end{ytableau} \leftrightarrow \begin{ytableau} a & {} \\ b \end{ytableau} \; (a < b)$$

Example 2.9. The skew tableaux

$$T = \begin{ytableau} 1 & 1 & 1 & 2 & 5 \\ 2 & 3 & 3 \\ 3 & 5 \end{ytableau} \text{ and } U = \begin{ytableau} \none & \none & 1 & 1 & 2 \\ \none & 2 & 3 & 3 & 5 \\ 1 & 3 & 5 \end{ytableau}$$

are jeu de taquin equivalent, as can be seen from the following sequence of slides:

$$\begin{ytableau} \none & \none & 1 & 1 & 2 \\ \none & 2 & 3 & 3 & 5 \\ 1 & 3 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 1 & 2 & 5 \\ \none & 2 & 3 & 3 \\ 1 & 3 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 1 & 2 & 5 \\ 1 & 2 & 3 & 3 \\ 3 & 5 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 1 & 1 & 2 & 5 \\ 2 & 3 & 3 \\ 3 & 5 \end{ytableau} .$$

Theorem 2.10 ([13])**.** *Each jeu de taquin equivalence class contains exactly one straight–shape SSYT.*

The *jeu de taquin rectification* of a skew SSYT T, denoted $\operatorname{rect}(T)$, is the unique straight–shape SSYT in the jeu de taquin equivalence class of T.

The following two theorems link jeu de taquin and Robinson-Schensted insertion.

Theorem 2.11. *Two skew SSYT are jeu de taquin equivalent if and only if their reading words are plactic equivalent.*

Theorem 2.12. *Let T be a skew SSYT. Then* $\mathrm{rect}(T) = P_{\mathrm{RSK}}(\mathrm{read}(T))$.

2.5 Greene's Invariants

Greene [5] has described the words whose insertion (hence also recording) tableau have a given shape λ. For a word w, define a *subword* of w as a word formed by a subset of the letters of w (not necessarily consecutive), respecting the order of the letters in w. Denote by I_k (resp., D_k) the maximum length of the union of k weakly increasing (resp., strictly decreasing) disjoint subwords of w. The following example illustrates this definition.

Let $w = 3152133125$. One can see that $I_1 = 5$, using the subword 11125, and that $I_2 = 8$, using the pair of subwords 11125 and 233. Similarily, $I_3 = 10$, $D_1 = 3, D_2 = 6$, $D_3 = 8$, $D_4 = 9$, and $D_5 = 10$. Clearly, $I_k = D_l = 10$ for all $k \geq 3$, and $l \geq 5$.

Theorem 2.13 ([5])**.** *Let w be a word, and let λ be the shape of $P_{\mathrm{RSK}}(w)$. Then for all $k \geq 0$, we have*

$$I_k = \lambda_1 + \cdots + \lambda_k,$$

and

$$D_k = \lambda'_1 + \cdots + \lambda'_k,$$

In the running example $w = 3152133125$, one can see that the shape of $P_{\mathrm{RSK}}(w)$ is $\lambda = (5, 3, 2)$, agreeing with Theorem 2.13.

2.6 The plactic monoid

Using the Knuth relations (2.1) and (2.2), Lascoux and Schützenberger [13] introduced a remarkable product structure for the set of SSYT, the *plactic monoid*.

Definition 2.14. Two words u and v in the alphabet $X = \{1, 2, \ldots\}$ are *plactic equivalent* (denoted $u \sim v$) if they have the same RSK insertion tableau. By Theorem 3.8, u and v are plactic equivalent if they can be transformed into each other using the plactic relations (2.1) and (2.2).

A *plactic class* is an equivalence class under the relation $\sim$. The plactic class containing a word w is denoted by $\langle w \rangle$. We can identify a plactic class with the corresponding SSYT $T = P_{\mathrm{RSK}}(w)$, and write $\langle T \rangle = \langle w \rangle$.

Definition 2.15. The *plactic monoid* $\mathbf{P} = \mathbf{P}(X)$ is the set of plactic classes where the product is given by concatenation, namely, $\langle u \rangle \langle v \rangle = \langle uv \rangle$. Equivalently, the plactic monoid is generated by the symbols in X subject to the relations (2.1) and (2.2).

Thus, the plactic monoid is precisely the quotient of the free monoid on the letters $1, 2, \ldots$ modulo the plactic relations. From this observation, one can see that the product is well defined.

Alternatively, identifying each plactic class with the corresponding SSYT, we obtain a notion of a (plactic) associative product on the set of SSYT.

Theorem 2.16 ([13]). *Let T and U be SSYTs. The product TU in $\mathbf{P}$ can be determined as follows. Construct a skew SSYT $T \sqcup U$ by identifying the upper right corner of T with the lower left corner of U. Then TU is the rectification of $T \sqcup U$.*

Example 2.17. Let

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}.$$

Then

$$TU = \mathrm{rect}\left(\begin{array}{cccccc} & & 1 & 1 & 2 & 5 \\ & & 3 & 3 & & \\ 1 & 2 & & & & \\ 3 & 5 & & & & \end{array} \right) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}.$$

One can see that this is consistent with the definition of the plactic product, since the plactic classes corresponding to T and U are, respectively, $\langle 3512 \rangle$ and $\langle 331125 \rangle$ (their reading words). The product of these two plactic classes is the plactic class $\langle 351233125 \rangle$ of their concatenation. By Example 2.3, the RSK insertion tableau of this word is precisely TU.

The *shape* of a plactic class is defined as the shape of the corresponding SSYT.

Remark 2.18. We normally consider an infinite alphabet X, but a totally analogous theory holds for any finite alphabet $X_n = \{1 < 2 < \cdots < n\}$.

2.7 The ring of symmetric functions

Consider the ring $\mathbb{Q}[x_1, x_2, \ldots,]$ of polynomials in a (finite or infinite) set of indeterminates $x_1, x_2, \ldots$. A *symmetric function* is a formal power series in $\mathbb{Q}[x_1, x_2, \ldots]$ of bounded degree that remains invariant under any permutation of the variables. The symmetric functions form a ring, denoted Λ. Some examples of symmetric functions are the following.

The *complete homogeneous symmetric function* h_k is defined, for any integer $k \geq 1$, by

$$h_k = \sum_{a_1 \leq a_2 \leq \cdots \leq a_k} x_{a_1} x_{a_2} \cdots x_{a_k}.$$

The *elementary symmetric function* e_k is defined, for any integer $k \geq 1$, by

$$e_k = \sum_{a_1 < a_2 < \cdots < a_k} x_{a_1} x_{a_2} \cdots x_{a_k}.$$

The *power sum symmetric function* p_k is defined, for any integer $k \geq 1$, by

$$p_k = \sum_{i \geq 1} x_i^k.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$. Similarily, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$.

Theorem 2.19 (See, e.g. [25])**.** *As a ring, Λ is freely generated by the h_k, the e_k, or the p_k. In other words,*

$$\Lambda = \mathbb{Q}[h_1, h_2, \cdots] = \mathbb{Q}[e_1, e_2, \cdots] = \mathbb{Q}[p_1, p_2, \cdots].$$

The h_λ (similarily, the e_λ or the p_λ) form a basis for Λ as a vector space, as λ runs over all partitions.

2.8 Schur functions and the Littlewood-Richardson rule

For an SSYT T with content $(a_1, a_2, \ldots)$, we denote the corresponding monomial by

$$x^T = x_1^{a_1} x_2^{a_2} \cdots .$$

For example, the monomial of the SSYT in Example 2.1 is $x_1^3 x_2^2 x_3^3 x_4^2$.

For each partition λ, the *Schur function* is defined as the sum of the monomials corresponding all SSYT of shape λ, namely,

$$s_\lambda = s_\lambda(x_1, x_2, \ldots) = \sum_{\mathrm{shape}(T)=\lambda} x^T.$$

Notice that $s_{(k)} = h_k$, and $s_{1^k} = e_k$.

The *skew Schur functions* $s_{\lambda/\mu}$ are defined similarly, for a skew shape λ/μ.

The following is an example of a Schur function in two variables:

Example 2.20. For $\lambda = (3, 1)$,

$$\begin{array}{ccccccc} s_\lambda(x_1, x_2) & = & x_1^3 x_2 & + & x_1^2 x_2^2 & + & x_1 x_2^3. \\ & & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \end{array}$$

The Schur functions form a basis for the ring Λ of symmetric functions.

The *Littlewood-Richardson coefficients* $c_{\mu,\nu}^{\lambda}$ are of great importance in combinatorics, algebraic geometry, and representation theory. They appear in the expansion of the product of two Schur functions,

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

and also in the Schur expansion of a skew Schur function (see e.g. [25, 7.15])

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu.$$

There are many combinatorial interpretations for these coefficients, such as the following.

Theorem 2.21. *Let λ, μ, and ν, be partitions, where λ contains both μ and ν. Fix a standard Young tableau T of shape μ. The Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ is equal to the number of skew standard tableaux of shape λ/ν whose rectification is T.*

2.9 Plactic Schur functions and their applications

Lascoux and Schützenberger [13] have used the plactic monoid to define a non-commutative analog of Schur functions. As an application, they gave a combinatorial interpretation of the Littlewood-Richardson coefficients.

Definition 2.22. A *plactic Schur function* $\mathcal{S}_\lambda \in \mathbb{Q}\mathbf{P}$ is defined as the sum of all plactic classes of shape λ. More specifically,

$$\mathcal{S}_\lambda = \sum_{\text{shape}(T)=\lambda} \langle T \rangle.$$

Example 2.23. For $\lambda = (3,1)$ and a two–letter alphabet $X_2 = \{1,2\}$, we have:

$$\begin{array}{ccccccc} \mathcal{S}_{(3,1)} & = & [2111] & + & [2112] & + & [2122]. \\ & & \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array} & & \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array} & & \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array} \end{array} \tag{2.3}$$

The plactic Schur functions $\mathcal{S}_\lambda$ are noncommutative analogs of the ordinary Schur functions. For example, (2.3) is a noncommutative analog of

$$s_{(3,1)}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3.$$

Theorem 2.24 ([13]). *The map $s_\lambda \mapsto \mathcal{S}_\lambda$ extends to a canonical isomorphism between the algebras generated by the ordinary and plactic Schur functions, respectively. As a result, the $\mathcal{S}_\lambda$ commute pairwise, span the ring they generate, and have the same structure constants as the s_λ. In particular,*

$$\mathcal{S}_\mu \mathcal{S}_\nu = \sum_\lambda c^\lambda_{\mu,\nu} \mathcal{S}_\lambda. \tag{2.4}$$

The first application of Theorem 2.24 is (a proof of) the Littlewood-Richardson rule. By taking the coefficient of the plactic class $\langle T \rangle$ corresponding to a fixed SSYT T of shape λ on both sides of (2.4), one obtains the following:

Corollary 2.25 (Littlewood-Richardson rule). *Fix a plactic class $\langle T \rangle$ of shape λ. The Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ is equal to the number of pairs of plactic classes $\langle U \rangle$ and $\langle V \rangle$ of shapes μ and ν, respectively, such that $\langle U \rangle \langle V \rangle = \langle T \rangle$.*

Example 2.26. Let us compute $c^{31}_{3,1}$. For this, we fix the tableau word $w = 4123$ associated with the SSYT $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$. The three words in $\langle T \rangle$ are 4123, 1423, and 1243. Note that the only one that can be written as uv where u and v are reading words of shapes (3) and (1), respectively, is 1243, with $u = 124$ (associated to the tableau $U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline\end{array}$) and $v = 3$ (associated to the tableau $V = \boxed{3}$). We conclude that $c^{31}_{3,1} = 1$.

A word w is called a *tableau word* if $w = \mathrm{read}(T)$ for some SSYT T. The *shape* of a tableau word is, by definition, the shape of the corresponding SSYT. With this terminology, the Littlewood-Richardson rule can be restated in the language of words as follows.

Corollary 2.27. *Fix a tableau word w of shape λ. The Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is equal to the number of pairs of tableau words u, v of shapes μ, ν, respectively, such that $w \sim uv$.*

2.10 Noncommutative Schur functions and box-adding operators

Fomin and Greene [3] have developed the theory of noncommutative Schur functions which gives a general approach for finding Schur expansions of functions such as skew Schur functions, stable Schubert and Grothendieck polynomials, and more.

Definition 2.28 (Noncommutative Schur function)**.** Let $u_1, u_2, \ldots$ be a finite or infinite sequence of elements of some associative algebra $\mathcal{A}$ over $\mathbb{Q}$. For a partition λ, define the *noncommutative Schur function* as

$$s_\lambda(\mathbf{u}) = s_\lambda(u_1, u_2, \ldots) = \sum_T u^T,$$

where the sum runs over all SSYT T, and the noncommutative monomial u^T is determined by $\mathrm{read}(T)$, or by any other representative of the plactic class corresponding to T. (In the case of an infinite alphabet, $s_\lambda(\mathbf{u})$ is an element of the appropriate *completion* of the algebra $\mathcal{A}$.)

The following theorem was obtained in [3].

Theorem 2.29. *Assume that the elements $u_1, u_2, \ldots$ of some associative algebra satisfy the relations*

$$\begin{aligned} u_i u_k u_j &= u_k u_i u_j \quad \text{for} \quad i < j < k \\ u_j u_i u_k &= u_j u_k u_i \quad \text{for} \quad i < j < k \\ u_{i+1} u_i (u_i + u_{i+1}) &= (u_i + u_{i+1}) u_{i+1} u_i \quad \text{for} \quad i \geq 1. \end{aligned}$$

Then the $s_\lambda(\mathbf{u})$ *commute pairwise, and satisfy the Littlewood-Richardson rule:*

$$s_\mu(\mathbf{u})s_\nu(\mathbf{u}) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(\mathbf{u}).$$

Remark 2.30. The plactic relations (2.1)–(2.2) imply the relations in Theorem 2.29.

Corollary 2.31 (Noncommutative Cauchy identity [3]). *Let* $u_1, u_2, \ldots, u_n$ *be as in Theorem 2.29, and let* $x_1, x_2, \ldots, x_m$ *be a family of commuting indeterminates, also commuting with each of the* u_j. *Then*

$$\sum_\lambda s_{\lambda'}(\mathbf{u})s_\lambda(\mathbf{x}) = \prod_{i=1}^m \left(\prod_{j=n}^1 (1 + x_i u_j) \right). \tag{2.5}$$

The analogous statement also holds when the x_i, *or* u_j, *or both, are an infinite family.*

Proof. We have

$$\begin{aligned} \prod_{i=1}^m \left(\prod_{j=n}^1 (1 + x_i u_j) \right) &= \prod_{i=1}^m \left(\sum_{k \geq 0} x_i^k \sum_{a_1 > \ldots > a_k} u_{a_1} \ldots u_{a_k} \right) \\ &= \prod_{i=1}^m \sum_{k \geq 0} x_i^k e_k(\mathbf{u}). \end{aligned}$$

The last step follows from the classical Cauchy identity (see e.g. [25, Theorem 7.12.1]) together with Theorem 2.29, since now the x_i and the $e_k(\mathbf{u})$ form a commuting family of indeterminates. Note that for this reason, the ordering in the outer products in equation (2.5) does not matter. □

Definition 2.32 (Partial maps, cf. [3]). Let $\mathbf{Y}$ be a finite or countable set, and let $\mathbb{R}\mathbf{Y}$ be the vector space formally spanned over $\mathbb{R}$ by the elements of $\mathbf{Y}$. A linear map $u \in \mathrm{End}(\mathbf{Y})$ is called a *partial map* in $\mathbf{Y}$ if the image $u(p)$ of each element $p \in \mathbf{Y}$ is either another element of $\mathbf{Y}$ or zero.

Definition 2.33 (Generalized skew Schur functions, cf. [3]). Let $u_1, u_2, \ldots, u_n$ be partial maps in $\mathbf{Y}$. For any $g, h \in \mathbf{Y}$, define

$$F_{h/g}(x_1, \ldots, x_m) = \left\langle \prod_{i=1}^m \left(\prod_{j=n}^1 (1 + x_i u_j) \right) g, h \right\rangle, \tag{2.6}$$

where the formal variables x_i commute with each other and with the u_j, and the noncommuting factors of the double product are multiplied in the specified order. Here, $\langle *, * \rangle$ denotes the inner product on $\mathbb{R}\mathbf{Y}$ for which the elements of $\mathbf{Y}$ form an orthonormal basis.

Theorem 2.34 (*Generalized Littlewood-Richardson Rule*)**.** *Let the u_i be partial maps in $\mathbf{Y}$ satisfying the relations in Theorem 2.29. Then for any $g, h \in \mathbf{Y}$, the polynomial $F_{h/g}$ defined by (2.6) is a nonnegative integer combination of Schur functions. More specifically,*

$$F_{h/g}(x_1, \ldots, x_m) = \sum c_{g,\lambda}^h \, s_\lambda(x_1, \ldots, x_m),$$

where $c_{g,\lambda}^h$ is equal to the number of semistandard Young tableaux T of shape λ such that $u^T g = h$.

Proof. By the noncommutative Cauchy identity (Corollary 2.31),

$$F_{h/g}(\mathbf{x}) = \left\langle \sum_\lambda s_{\lambda'}(\mathbf{u}) s_\lambda(\mathbf{x}) g, h \right\rangle = \sum_\lambda \langle s_{\lambda'}(\mathbf{u}) g, h \rangle s_\lambda(\mathbf{x}).$$

Consequently,

$$c_{g\nu}^h = \langle s_{\lambda'}(\mathbf{u}) g, h \rangle,$$

which is precisely the number of semistandard Young tableaux T of shape λ such that $u^T g = h$. $\square$

As an application of this theory, one obtains another version of the Littlewood-Richardson Rule.

Definition 2.35 (cf. [3])**.** The *box-adding operators* u_j act on Ferrers shapes according to the following rule:

$$u_j(\lambda) = \begin{cases} \lambda \cup \{\text{box in the } j\text{-th column}\} & \text{if this gives a valid shape;} \\ 0 & \text{otherwise.} \end{cases}$$

Here the columns are numbered from left to right, starting with $j = 1$ for the leftmost column.

Example 2.36. We have

$$u_2\left(\ydiagram{3,1}\right) = \ydiagram{3,2} \qquad \text{and} \qquad u_3\left(\ydiagram{3,1}\right) = 0.$$

The maps u_i are partial maps on the vector space formally spanned by Ferrers shapes, as the image of each u_i is either a Ferrers shape or 0.

The product

$$\mathcal{A}(x) = \prod_{j=n}^{1}(1 + xu_j)$$

can be viewed as an operator that adds a horizontal strip to a fixed shape, each time introducing a power of x that is determined by the length of the strip. Setting $g = \mu$ we get

$$\prod_{i\geq 0}\left(\prod_{j=n}^{1}(1 + x_iu_j)\right)\mu = \prod_{i\geq 0}\mathcal{A}(x_i)\mu = \sum_{\lambda}\sum_{T} x^T\lambda,$$

where the last sum is over all skew SSYT T of shape λ/μ. Therefore

$$F_{\lambda/\mu}(\mathbf{x}) = \left\langle \prod_{i\geq 1}\mathcal{A}(x_i)\mu, \lambda \right\rangle = \sum_{\text{shape}(T)=\lambda/\mu} x^T = s_{\lambda/\mu}(\mathbf{x}),$$

the skew Schur function.

One can see that the box-adding operators u_i satisfy the *nil-Temperley-Lieb relations* (cf. [3, Example 2.4]):

$$\begin{aligned} u_iu_j &= u_ju_i \qquad |i-j| \geq 2, \\ u_iu_{i+1}u_i &= u_{i+1}u_iu_{i+1} = 0 \qquad i \geq 1, \end{aligned}$$

which implies that they also satisfy (2.1) and (2.2). Consequently, we can use Theorem 2.34 to obtain an expansion of a skew Schur function in terms of Schur functions. This leads to the following version of the Littlewood-Richardson rule.

Corollary 2.37. *The Littlewood-Richardson number* $c^{\lambda}_{\mu,\nu}$ *is equal to the number of semistandard tableaux* T *of shape* ν *such that* $u^T(\mu) = \lambda$*, where, as before,* u^T *is the noncommutative monomial in* $u_1, u_2, \ldots$ *defined by any representative of the plactic class associated with* T*, and each* u_i *is interpreted as a box-adding operator.*

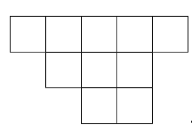
CHAPTER 3

Shifted Young tableaux and Schur P-functions

3.1 Strict partitions and shifted diagrams

A *strict partition* is a finite sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. The *shifted diagram*, or *shifted shape* of λ is an array of square cells in which the i-th row has λ_i cells, and is shifted $i-1$ units to the right with respect to the top row. Throughout this thesis, we identify the shifted shape corresponding to a strict partition λ with λ itself. The *size* of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We denote $\ell(\lambda) = l$, the number of rows.

To illustrate, we show the shifted shape $\lambda = (5, 3, 2)$, with $|\lambda| = 10$ and $\ell(\lambda) = 3$, is shown below:



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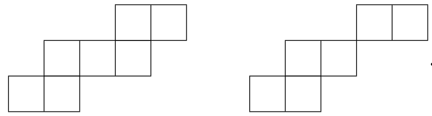
A shifted shape λ is said to contain a smaller shape μ, if after overimposing the corresponding shifted diagrams by identifying their upper left corners, the shape of μ is fully contained inside the shape λ. A *skew shifted diagram* (or shape) λ/μ is obtained by removing a shifted shape μ from a larger shape λ containing μ.

A *border strip* is a skew shape which does not contain a 2×2 square. Unless otherwise specified, a border strip is connected. The following are examples of a connected border strip and a disconnected one:



3.2 Shifted Young tableaux

A *(semistandard) shifted Young tableau* T of shape λ is a filling of a shifted shape λ with letters from the alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$ such that:

- rows and columns of T are weakly increasing;
- each k appears at most once in every column;
- each k' appears at most once in every row;
- there are no primed entries on the main diagonal.

If T is a filling of a shape λ, we write $\mathrm{shape}(T) = \lambda$.

A *skew shifted Young tableau* is defined analogously.

The *content* of a tableau T is the vector $(a_1, a_2, \ldots)$, where a_i is the number of times the letters i and i' appear in T.

Example 3.1. The shifted Young tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3' & 4 \\ \hline & 4 & 5 & 5 & \\ \cline{2-4} & & 6 & 9' & \\ \cline{3-4} \end{array}$$

has shape $\lambda = (5, 3, 2)$ and content $(2, 1, 1, 2, 2, 1, 0, 0, 1)$.

A tableau T of shape λ is called *standard* if it contains each of the entries $1, 2, \ldots, |\lambda|$ exactly once. In particular, standard shifted Young tableaux have no primed entries. A standard shifted tableau has content $(1, 1, \ldots, 1)$.

3.3 Sagan-Worley insertion

There are two main correspondences which can be viewed as shifted analogs of the Robinson-Schensted-Knuth correspondence. They are the Sagan-Worley insertion and the (Haiman) mixed insertion.

Sagan [18] and Worley [28] introduced the *Sagan-Worley insertion*, a remarkable correspondence between permutations and pairs of shifted Young tableaux. The first tableau is a shifted standard Young tableau and the second tableau is a semistandard shifted Young tableau of content $(1, 1, \ldots, 1)$. (In other words, both tableaux are "standard", except that the second tableau is allowed to have primed entries.)

Definition 3.2 (Sagan-Worley insertion)**.** Let $w = w_1 \ldots w_n$ be a permutation. We recursively construct a sequence $(T_0, U_0), \ldots, (T_n, U_n) = (T, U)$ of tableaux, where T_i is a shifted semistandard Young tableau, and U_i is a shifted standard Young tableau, similarly to the Robinson-Schensted insertion. Set $(T_0, U_0) = (\emptyset, \emptyset)$. For $i = 1, \ldots, n$, insert w_i into T_{i-1} in the following manner:

Insert w_i into the first row of T_{i-1} by bumping the smallest element a that is greater than w_i.

1. If a was not located in the main diagonal, insert a in the next row, using the same procedure as above, and repeat.

2. If a is located in the main diagonal, insert it in the next column using the same procedure as for inserting it in a row. Continue column-inserting the remaining letters.

The insertion process terminates once a letter is placed at the end of a row or column, bumping no new element. The resulting tableau is T_i.

The shapes of T_{i-1} and T_i differ by one box. Add that box to U_{i-1}, and write i or

i' into it, depending on wether the last letter was row-inserted or column-inserted, to obtain U_i.

We call T the *Sagan-Worley insertion tableau* and U the *Sagan-Worley recording tableau*, and denote them $P_{\mathrm{SW}}(w)$ and $Q_{\mathrm{SW}}(w)$, respectively.

Sagan-Worley insertion has been defined more generally as a correspondence between words in the alphabet X and pairs of tableaux, the first one being a shifted semistandard Young tableau with no primed entries, and the second one being a shifted semistandard Young tableau of content $(1, 1, \dots, 1)$. We will not make use of this in this thesis, but we refer the reader to the definition in [18].

3.4 Shifted mixed insertion

M. Haiman [6] has introduced *(shifted) mixed insertion*, a remarkable correspondence between permutations and pairs of shifted Young tableaux.

The following is a semistandard generalization of shifted mixed insertion, which we call *semistandard shifted mixed insertion*. It is a correspondence between words in the alphabet X and pairs of shifted semistandard Young tableaux, one of them standard. Throughout this paper we refer to semistandard shifted mixed insertion simply as *mixed insertion*.

Definition 3.3 (Mixed insertion)**.** Let $w = w_1 \dots w_n$ be a word in the alphabet X. We recursively construct a sequence $(T_0, U_0), \dots, (T_n, U_n) = (T, U)$ of tableaux, where T_i is a shifted Young tableau, and U_i is a standard shifted Young tableau, as follows. Set $(T_0, U_0) = (\emptyset, \emptyset)$. For $i = 1, \dots, n$, insert w_i into T_{i-1} in the following manner:

Insert w_i into the first row, bumping out the smallest element a that is strictly greater than w_i (in the order given by the alphabet X').

1. if a is not on the main diagonal, do as follows:

   (a) if a is unprimed, then insert it in the next row, as explained above;

   (b) if a is primed, insert it into the next column to the right, bumping out the smallest element that is strictly greater than a;

2. if a is on the main diagonal, then it must be unprimed. Prime it, and insert it into the next column to the right.

The insertion process terminates once a letter is placed at the end of a row or column, bumping no new element. The resulting tableau is T_i.

The shapes of T_{i-1} and T_i differ by one box. Add that box to U_{i-1}, and write i into it to obtain U_i.

We call T the *mixed insertion tableau* and U the *mixed recording tableau*, and denote them $P_{\text{mix}}(w)$ and $Q_{\text{mix}}(w)$, respectively.

Example 3.4. The word $w = 3415961254$ has the following mixed insertion and recording tableaux:

$$P_{\text{mix}}(w) = \begin{array}{ccccc} 1 & 1 & 2 & 3' & 4 \\ & 4 & 5 & 5 & \\ & & 6 & 9' & \end{array} \qquad Q_{\text{mix}}(w) = \begin{array}{ccccc} 1 & 2 & 4 & 5 & 9 \\ & 3 & 6 & 8 & \\ & & 7 & 10 & \end{array}.$$

Both mixed insertion and Sagan-Worley insertion can be generalized to *biwords*, namely two-rowed arrays in which the columns are arranged lexicographically, with priority given to the element in the top row. In this setting, one inserts the elements from the bottom row, left to right, while recording the corresponding elements from the top row. The ordinary mixed insertion is obtained if the top row is $(1, 2, 3, \ldots)$. Thus, one can identify a word u with a biword where u is in the bottom row, and the numbers $1, 2, 3, \ldots$ form the top row. The inverse of u, denoted u^{-1}, is the biword obtained by switching the two rows in u, and then sorting the columns lexicographically with priority given to the element on the top row. Haiman has proved that mixed insertion is dual to Sagan-Worley insertion in the following sense.

Theorem 3.5 (Haiman). *Let w be a permutation. Then*

$$P_{\mathrm{mix}}(w) = Q_{\mathrm{SW}}(w^{-1}), \quad \textit{and} \quad Q_{\mathrm{mix}}(w) = P_{\mathrm{SW}}(w^{-1})$$

This was extended to biwords by Fomin [2, p. 291], as follows.

Theorem 3.6 (Fomin). *Let w be a word in X. Then*

$$P_{\mathrm{mix}}(w) = Q_{\mathrm{SW}}(w^{-1}), \quad \textit{and} \quad Q_{\mathrm{mix}}(w) = P_{\mathrm{SW}}(w^{-1})$$

3.5 Shifted Greene's Invariants

In this section we introduce a shifted analog of Greene's invariants which determines precisely what conditions a word must satisfy for its mixed insertion tableau to have shape λ.

Define a *hook word* as a word $w = w_1 \cdots w_l$ such that for some $1 \leq k \leq l$, we have

$$w_1 > w_2 > \cdots > w_k \leq w_{k+1} \leq \cdots \leq w_l. \tag{3.1}$$

Now, define a k-hook subword of w as a union of k hook subwords of w such that:

- any number can appear in at most 2 of the words, and
- any pair of hook words can only have at most 1 number in common.

The length of a k-hook subword of w is the sum of the lengths of the hook subwords contained in it.

Note that if a letter i appears in w more than once, then each instance of i is treated separately in the k-hook subword.

The following theorem can be considered a dual of [18, Corollary 5.2], or a semistandard version of [11, Corollary 3.39].

Theorem 3.7. *Let w be a word, and let λ be the shape of $P_{\mathrm{mix}}(w)$. Denote by $I_k(w)$ the maximum length of a hook subword of w. Then for all $k \leq \ell(\lambda)$,*

$$I_k(w) = \lambda_1 + \cdots + \lambda_k + \binom{k}{2}.$$

The proof of Theorem 3.7 is given in Chapter 4.

As an example, let $w = 3415961254$, where the insertion tableau has shape $(5, 3, 2)$ (see Example 3.1). One can see that a maximal 1-hook subword is 96125, a maximal 2-hook subword is $96125, 4159$, and a maximal 3-hook subword is $96125, 4159, 3124$. Thus $I_1(w) = 5$, $I_2(w) = 9$, and $I_3(w) = 13$, agreeing with Theorem 3.7.

3.6 The shifted plactic monoid

Theorem 3.8 below is one of our main results. It is a shifted analog of the plactic relations (2.1)–(2.2) [8]. It can be considered a semistandard generalization of results by Haiman [6] and Kraśkiewicz [9].

Theorem 3.8. *Two words have the same mixed insertion tableau if and only if they are equivalent modulo the following relations:*

$$abdc \equiv adbc \quad \textit{for} \quad a \leq b \leq c < d \quad \textit{in } X; \tag{3.2}$$

$$acdb \equiv acbd \quad \textit{for} \quad a \leq b < c \leq d \quad \textit{in } X; \tag{3.3}$$

$$dacb \equiv adcb \quad \textit{for} \quad a \leq b < c < d \quad \textit{in } X; \tag{3.4}$$

$$badc \equiv bdac \quad \textit{for} \quad a < b \leq c < d \quad \textit{in } X; \tag{3.5}$$

$$cbda \equiv cdba \quad \textit{for} \quad a < b < c \leq d \quad \textit{in } X; \tag{3.6}$$

$$(3.7) \qquad dbca \equiv bdca \quad \textit{for} \quad a < b \leq c < d \quad \textit{in } X;$$

$$(3.8) \qquad bcda \equiv bcad \quad \textit{for} \quad a < b \leq c \leq d \quad \textit{in } X;$$

$$(3.9) \qquad cadb \equiv cdab \quad \textit{for} \quad a \leq b < c \leq d \quad \textit{in } X.$$

Consequently, the mixed insertion tableau of a concatenation of two words is uniquely determined by their mixed insertion tableaux.

The proof of Theorem 3.8 is given in Chapter 4.

A concise alternative description of relations (3.2)–(3.9) is the following.

Remark 3.9. As noted in [22], the plactic relations (2.1)–(2.2) can be understood in the following way. Let us call $w = w_1 \cdots w_n$ a *line word* if

$$w_1 > w_2 > \cdots > w_n$$

or

$$w_1 \leq w_2 \leq \cdots \leq w_n.$$

Line words are precisely those words w for which the shape of $P_{\mathrm{RSK}}(w)$ is a single row or a single column.

In this language, the plactic relations (2.1)–(2.2) can be stated as follows. Two 3-letter words w and w' in the alphabet X are plactic equivalent if and only if:

- w and w' differ by an adjacent transposition, and
- neither w nor w' is a line word.

The relations (3.2)–(3.9) are called the *shifted plactic relations*, and can be described in a similar way.

Observe that w is a hook word if and only if $P_{\text{mix}}(w)$ consists of a single row. Now, the shifted plactic relations (3.2)–(3.9) are precisely all the relations $w \equiv w'$ in which:

- w and w' are plactic equivalent 4-letter words, and
- neither w nor w' is a hook word.

Definition 3.10. Two words u and v in the alphabet X are *shifted plactic equivalent* (denoted $u \equiv v$) if they can be transformed into each other using the shifted plactic relations (3.2)–(3.9). By Theorem 3.8, u and v are shifted plactic equivalent if and only if they have the same mixed insertion tableau.

A *shifted plactic class* is an equivalence class under the relation $\equiv$. The shifted plactic class containing a word w is denoted by $[w]$. We can identify a shifted plactic class with the corresponding shifted Young tableau $T = P_{\text{mix}}(w)$, and write $[T] = [w]$.

Example 3.11. Figure 3.1 shows the shifted plactic classes of 5-letter words of content $(3, 2)$, while Figure 3.2 shows the plactic classes of the same.

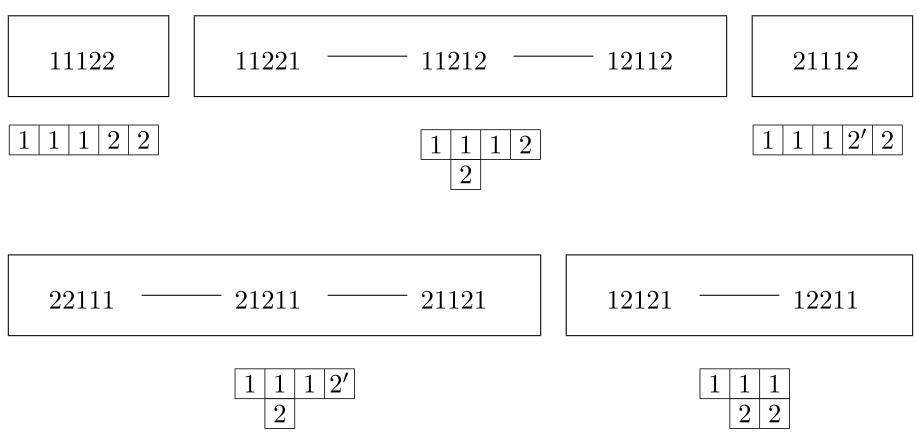


Figure 3.1: Shifted plactic classes of words of content $(3, 2)$. Each box contains a shifted plactic class, with the edges representing shifted plactic relations; the corresponding shifted tableau is shown underneath.

The following proposition can be verified by direct inspection.

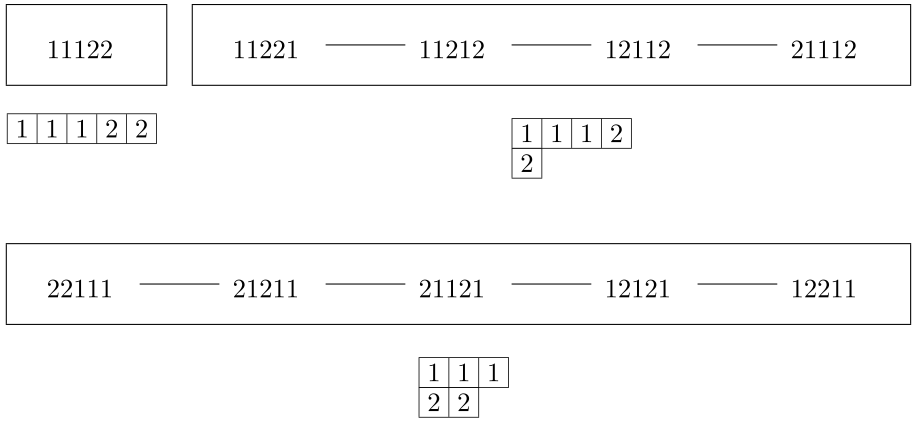


Figure 3.2: Plactic classes of words of content $(3, 2)$. Each box contains a plactic class, with the edges representing plactic relations; the corresponding tableau is shown underneath.

Proposition 3.12. *Shifted plactic equivalence is a refinement of the plactic equivalence. That is, each plactic class is a disjoint union of shifted plactic classes. To put it yet another way: if two words are shifted plactic equivalent, then they are plactic equivalent.*

Proposition 3.12 can be illustrated by comparing Figures 3.1 and 3.2.

Definition 3.13. The *shifted plactic monoid* $\mathbf{S} = \mathbf{S}(X)$ is the set of shifted plactic classes with multiplication given by $[u][v] = [uv]$. (This multiplication is well defined by Theorem 3.8.) Equivalently, the monoid is generated by the symbols in X subject to the relations (3.2)–(3.9).

Alternatively, identifying each shifted plactic class with the corresponding shifted Young tableau, we obtain the notion of a (shifted plactic) associative product on the set of shifted tableaux.

The *shape* of a shifted plactic class is defined as the shape of the corresponding shifted Young tableau.

The *shifted plactic algebra* $\mathbb{Q}\mathbf{S}$ is the semigroup algebra of the plactic monoid.

Remark 3.14. We normally consider X as an infinite alphabet, but a totally analogous theory holds for any finite alphabet $X_n = \{1 < 2 < \cdots < n\}$.

3.7 The ring of Schur P- and Q-functions and Stembridge's shifted Littlewood-Richardson rule

Recall that the ring of symmetric functions Λ is generated by the power symmetric functions; in other words,

$$\Lambda = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

An important subring $\Omega \subset \Lambda$ is defined by

$$\Omega = \mathbb{Q}[p_1, p_3, p_5, \ldots].$$

The ring Ω plays a key role in the Schubert calculus of isotropic Grassmannians, and in the theory of projective representations of the symmetric groups. In this section, we recall two important bases for this ring: the Schur P- and Schur Q-functions.

For a shifted Young tableau T with content $(a_1, a_2, \ldots)$, we denote the corresponding monomial by

$$x^T = x_1^{a_1} x_2^{a_2} \cdots .$$

For example, for the shifted Young tableau T in Example 3.1, the monomial is $x^T = x_1^2 x_2 x_3 x_4^2 x_5^2 x_6 x_9$.

For each strict partition λ, the *Schur P-function* is defined as the generating function for the shifted Young tableaux of shape λ, namely

$$P_\lambda = P_\lambda(x_1, x_2, \ldots) = \sum_{\text{shape}(T)=\lambda} x^T.$$

The *Schur Q-function* is defined by

$$Q_\lambda = Q_\lambda(x_1, x_2, \ldots) = 2^{\ell(\lambda)} P_\lambda.$$

Thus, Q_λ is the generating function for a different kind of shifted Young tableaux, namely those in which the elements on the main diagonal are allowed to be primed. For partitions with only one part, it is common to denote $Q_{(k)}$ by q_k.

The *skew Schur P- and Q-functions* $P_{\lambda/\mu}$ and $Q_{\lambda/\mu} = 2^{\ell(\lambda)-\ell(\mu)}P_{\lambda/\mu}$ are defined similarly, for a skew shifted shape λ/μ.

The following is an example of a Schur P-function in two variables:

Example 3.15. For $\lambda = (3, 1)$,

$$\begin{array}{ccccccccc} P_\lambda(x_1, x_2) & = & x_1^3x_2 & + & x_1^2x_2^2 & + & x_1^2x_2^2 & + & x_1x_2^3. \\ & & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 2' & 2 \\ \hline \end{array} \\ & & \;\;\begin{array}{|c|}\hline 2 \\ \hline \end{array} & & \;\;\begin{array}{|c|}\hline 2 \\ \hline \end{array} & & \;\;\begin{array}{|c|}\hline 2 \\ \hline \end{array} & & \;\;\begin{array}{|c|}\hline 2 \\ \hline \end{array} \end{array}$$

Theorem 3.16 (see e.g. [14])**.** *The Schur P- and Q-Schur functions form linear bases for the ring Ω.*

The *shifted Littlewood-Richardson coefficients*, $b^\lambda_{\mu,\nu}$ are of great importance in combinatorics, algebraic geometry, and representation theory. They appear in the expansion of the product of two Schur P-functions

$$P_\mu P_\nu = \sum_\lambda b^\lambda_{\mu,\nu} P_\lambda$$

and also in the expansion of a skew Schur Q-function

$$Q_{\lambda/\mu} = \sum_\nu b^\lambda_{\mu,\nu} Q_\nu.$$

The latter can be rewritten as

$$P_{\lambda/\mu} = \sum_\nu 2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} b^\lambda_{\mu,\nu} P_\nu.$$

The first combinatorial interpretation for the shifted Littlewood-Richardson coefficients has been given by Stembridge [26]. Thomas and Yong [27] have recently given a similar one involving (co)minuscule roots.

Theorem 3.17 (Stembridge)**.** *Fix a standard shifted tableau Q of shape ν. The shifted Littlewood-Richardson coefficient $b^\lambda_{\mu,\nu}$ is equal to the number of standard shifted skew tableaux of shape λ/μ which rectify to Q.*

It is well known that the Schur P-functions are Schur positive, i.e., the coefficients g^λ_μ in the expansion

$$P_\lambda = \sum_\mu g^\lambda_\mu s_\mu$$

are positive. Stembridge [26] gave a combinatorial description of these coefficients:

Theorem 3.18 (Stembridge)**.** *The coefficient g^λ_μ is equal to the number of standard Young tableaux of shape μ which rectify to a fixed standard shifted Young tableau of shape λ.*

3.8 Plactic Schur P-functions and their applications

In this section we use the theory of the shifted plactic monoid to give a new proof (and a new version of) the shifted Littlewood-Richardson rule.

Definition 3.19. A *shifted plactic Schur P-function* $\mathcal{P}_\lambda \in \mathbb{Q}\mathbf{S}$ is defined as the sum of all shifted plactic classes of shape λ. More precisely,

$$\mathcal{P}_\lambda = \sum_{\mathrm{shape}(T)=\lambda} [T].$$

Example 3.20. Representing each shifted plactic class as $[w]$, for some representative w, we have:

$$\begin{array}{ccccccccc} \mathcal{P}_{(3,1)} & = & [1211] & + & [2211] & + & [1212] & + & [2212]. \\ & & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} \end{array}$$

The reader can check that each word gets mixed inserted into the tableau underneath, making it a valid representative of the corresponding plactic class.

The $\mathcal{P}_\lambda$ are noncommutative analogs of the Schur P-functions. For example, $\mathcal{P}_{(3,1)}$ is a noncommutative analog of

$$P_{(3,1)}(x_1, x_2) = x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3.$$

Theorem 3.21. *The map $P_\lambda \mapsto \mathcal{P}_\lambda$ extends to a canonical isomorphism between the algebra generated by the ordinary and shifted plactic Schur P-functions, respectively. As a result, the $\mathcal{P}_\lambda$ commute pairwise, span the ring they generate, and multiply according to the* shifted Littlewood-Richardson rule*:*

$$\mathcal{P}_\mu \mathcal{P}_\nu = \sum_\lambda b_{\mu,\nu}^\lambda \mathcal{P}_\lambda. \tag{3.10}$$

The proof of Theorem 3.21 is given in Chapter 4.

The concept of *jeu de taquin* [25, A1] has a shifted analog; both are special instances of a general construction due to Schützenberger [23]. For both the ordinary and the shifted cases, a *rectification* of a skew standard tableau T is a non-skew standard tableau U obtained from T by applying a sequence of jeu de taquin moves. The rectification is unique as proven by Schützenberger [23] in the ordinary case, and by Sagan [18] in the shifted case. [1]

Our first application of Theorem 3.21 is a new proof (and a new version of) the shifted Littlewood-Richardson rule. Stembridge [26] proved that the shifted Littlewood-Richardson number $b_{\mu,\nu}^\lambda$ is equal to the number of standard shifted skew tableaux of shape λ/μ which rectify to a fixed standard shifted tableau of shape ν.

By taking the coefficient of the shifted plactic class $[T]$ corresponding to a fixed tableau T of shape λ on both sides of (3.10), one obtains the following:

[1] Sagan [18] has extended the concept of jeu de taquin to shifted semistandard skew tableaux. Unfortunately, his version does not fit our purposes nor have we been able to develop an alternative that does. Nevertheless, in Chapter 4 we develop the concept of rectification.

Corollary 3.22 (Shifted Littlewood-Richardson rule)**.** *Fix a shifted plactic class* $[T]$ *of shape* λ*. The shifted Littlewood-Richardson coefficient* $b^{\lambda}_{\mu,\nu}$ *is equal to the number of pairs of shifted plactic classes* $[U]$ *and* $[V]$ *of shapes* μ *and* ν*, respectively, such that* $[U][V] = [T]$*.*

Example 3.23. Let us compute $b^{31}_{3,1}$. For this, we fix the shifted tableau word $w = 1423$ associated with the shifted Young tableau $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$. The two words in $[T]$ are 1423 and 1243. Among there, the only one that can be written as uv where u and v are reading words of shapes (3) and (1), respectively, is 1243, with $u = 124$ (associated to the tableau $U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline\end{array}$) and $v = 3$ (associated to the tableau $V = \boxed{3}$). We conclude that $b^{31}_{3,1} = 1$.

Similarly, let us compute $b^{31}_{2,2}$ using the same w and T. Note that both words in $[T]$ can be written as uv, where u and v are both reading words of shape (2). The word 1243, with $u = 12$ (associated to the tableau $U = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$) and $v = 43$ (associated to the tableau $V = \begin{array}{|c|c|}\hline 3 & 4' \\ \hline\end{array}$). The word 1423, with $u = 14$ (associated to the tableau $U = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}$) and $v = 23$ (associated to the tableau $V = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$). We conclude that $b^{31}_{2,2} = 2$.

Corollary 3.22 can be restated in the language of words as follows. In Section 4.1 we introduce a canonical representative of the shifted plactic class $[T]$ corresponding to the tableau T. This representative is called the *mixed reading word* of T, and denoted by $\mathrm{mread}(T)$. (See Definition 4.4 for precise details.) A word w is called a *shifted tableau word* if $w = \mathrm{mread}(T)$ for some shifted Young tableau T. The *shape* of a shifted tableau word is, by definition, the shape of the corresponding tableau. With this terminology, the shifted Littlewood-Richardson rule can be restated as follows:

Corollary 3.24. *Fix a shifted tableau word w of shape λ. The shifted Littlewood-Richardson coefficient $b^{\lambda}_{\mu,\nu}$ is equal to the number of pairs of shifted tableau words u, v of shapes μ, ν, respectively, such that $w \equiv uv$.*

The representatives we have picked in Example 3.20 are precisely the mixed reading words of the corresponding tableaux; c.f. Example 4.6.

The following result is proved in Chapter 4

Lemma 3.25. *Fix a shifted tableau word w of shape λ and a standard shifted tableau Q of shape ν. The number of pairs of shifted tableau words u, v of shapes μ and ν, respectively, such that $uv = w$ is equal to the number of standard shifted skew tableaux R of shape λ/μ which rectify to Q.*

As a corollary, we recover Theorem 3.17.

The second application of the shifted plactic monoid is a new proof (and a new version of) the *Schur expansion of a Schur P-function.* Stembridge [26] has found a combinatorial interpretation for the coefficients g^{λ}_{μ} appearing in the sum

$$P_{\lambda} = \sum_{\mu} g^{\lambda}_{\mu} s_{\mu}. \tag{3.11}$$

Below we give a different description of the numbers g^{λ}_{μ} in terms of shifted plactic classes.

By Proposition 3.12, any two shifted plactic equivalent words are plactic equivalent; in other words, relations (3.2)–(3.9) are valid instances of (2.1)–(2.2). This yields the natural projection

$$\pi : \mathbf{S} \to \mathbf{P}$$

which maps the shifted plactic class $[u]$ to the plactic class $\langle u \rangle$.

We next consider the image of a plactic Schur P-function under π. We get:

Theorem 3.26. *The plactic Schur P-function $\mathcal{P}_\lambda$ is sent by the projection π to a sum of plactic Schur functions $\mathcal{S}_\mu$. Specifically (cf. 3.11),*

$$\pi(\mathcal{P}_\lambda) = \sum_{\mu} g_\mu^\lambda \mathcal{S}_\mu.$$

Since the span of the $\mathcal{P}_\lambda$ and the span of the $\mathcal{S}_\lambda$ are isomorphic to Ω and Λ, respectively, the following statement holds.

Corollary 3.27. *The coefficient g_μ^λ is equal to the number of shifted plactic classes $[u]$ of shifted shape λ such that $\pi([u]) = \langle v \rangle$ for some fixed plactic class $\langle v \rangle$ of shape μ.*

Example 3.28. Let μ be the ordinary shape $(3,1)$, and λ be the shifted shape (4). Let us compute g_μ^λ, the coefficient of s_μ in P_λ. For this, we fix $\langle u \rangle = \langle 2134 \rangle$, the plactic class corresponding to the Young tableau $U = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$. Note that the words in $\langle u \rangle$ are $2134, 2314$, and 2341. These get split into two shifted plactic classes, namely $[2134]$ corresponding to the shifted Young tableau $\begin{array}{|c|c|c|c|}\hline 1 & 2' & 3 & 4 \\ \hline\end{array}$, and $[2314] = [2341]$ corresponding to the shifted Young tableau $\begin{array}{|c|c|c|}\hline 1 & 2' & 4 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$. Since only one of these shifted plactic classes has shape λ, namely $[2134]$, we conclude that $g_\mu^\lambda = 1$.

Theorem 3.29. *Let λ be a shifted shape and U_λ a fixed standard shifted tableau of shape λ. Fix a plactic class $\langle T_\mu \rangle$ of (ordinary) shape μ. Then the number of shifted plactic classes $[T_\lambda]$ of shape λ for which $\pi([T_\lambda]) = \langle T_\mu \rangle$ is equal to the number of standard Young tableaux of shape μ which rectify to U_λ.*

Theorems 3.26 and 3.29 are proved in Section 4.3. As a corollary, we recover Theorem 3.18.

3.9 Noncommutative Schur P-functions and box-adding operators

Definition 3.30 (Noncommutative Schur P-function)**.** Let $u_1, u_2, \ldots$ be a finite or infinite sequence of elements of some associative algebra $\mathcal{A}$ over $\mathbb{Q}$. (We will always assume that these elements satisfy the shifted plactic relations.) For a shifted shape λ, define

$$P_\lambda(\mathbf{u}) = P_\lambda(u_1, u_2, \ldots) = \sum_T u^T,$$

where T runs over all shifted Young tableaux, and the monomial u^T is determined by $\mathrm{mread}(T)$, or by any other representative of the shifted plactic class corresponding to T. (In the infinite case, $P_\lambda(\mathbf{u})$ is an element of the appropriate completion of the algebra $\mathcal{A}$.)

Theorem 3.21 implies the following result.

Corollary 3.31. *Assume that the elements $u_1, u_2, \ldots$ of some associative algebra satisfy the shifted plactic relations (3.2)–(3.9) (the element u_i is represented by the letter i in the alphabet X). Then the noncommutative Schur P-functions $P_\lambda(\mathbf{u})$ introduced in Definition 3.30 commute pairwise, and satisfy the shifted Littlewood-Richardson rule:*

$$P_\mu(\mathbf{u})P_\nu(\mathbf{u}) = \sum_\lambda b_{\mu,\nu}^\lambda P_\lambda(\mathbf{u}).$$

Corollary 3.32 (Noncommutative Cauchy identity)**.** *Let $u_1, u_2, \ldots, u_n$ be as in Corollary 3.31, and let $x_1, x_2, \ldots, x_m$ be a family of commuting indeterminates, also commuting with each of the u_j. Then*

$$\sum_\lambda P_\lambda(\mathbf{u})Q_\lambda(\mathbf{x}) = \prod_{i=1}^m \left(\prod_{j=n}^1 (1 + x_i u_j) \prod_{j=1}^n (1 - x_i u_j)^{-1} \right). \tag{3.12}$$

The analogous statement also holds when the x_i, or u_j, or both, are an infinite family.

Proof. We have

$$\begin{aligned}
\prod_{i=1}^m \left(\prod_{j=\infty}^1 (1 + x_i u_j) \prod_{j=1}^\infty (1 - x_i u_j)^{-1} \right) &= \prod_{i=1}^m \left(\sum_{k\geq 0} x_i^k \sum_{a_1 > \ldots > a_i \leq a_{i+1} \leq \cdots \leq a_k} u_{a_1} \ldots u_{a_k} \right) \\
&= \prod_{i=1}^m \sum_{k\geq 0} x_i^k q_k(\mathbf{u}).
\end{aligned}$$

The last step follows from the classical shifted Cauchy identity (see e.g. [18, Corollary 8.3]) together with Corollary 3.31, since now the x_i and the $q_k(\mathbf{u})$ form a commuting family of indeterminates. Note that for this reason, the ordering in the interior products in 3.12 is as indicated, whereas the ordering in the outer products does not matter. $\square$

Recall the notion of *partial maps* from Definition 2.32.

Definition 3.33 (Generalized skew Schur Q-functions, cf. [3]). Let $u_1, u_2, \ldots, u_n$ be partial maps in $\mathbf{Y}$. For any $g, h \in \mathbf{Y}$, define

$$G_{h/g}(x_1, \ldots, x_m) = \left\langle \prod_{i=1}^{m} \left(\prod_{j=n}^{1} (1 + x_i u_j) \prod_{j=1}^{n} (1 - x_i u_j)^{-1} \right) g, h \right\rangle, \tag{3.13}$$

where the variables x_i commute with each other and with the u_j, and the noncommuting factors of the double product are multiplied in the specified order. Here, $\langle *, * \rangle$ denotes the inner product on $\mathbb{R}\mathbf{Y}$ for which the elements of $\mathbf{Y}$ form an orthonormal basis. By the argument at the end of the proof of Corollary 3.32, the order of factors in the outer product doesn't matter, which implies that the $G_{h/g}$ are (ordinary) symmetric polynomials in $x_1, \ldots, x_m$.

Theorem 3.34 (*Generalized shifted Littlewood-Richardson Rule*). *Let the u_i be partial maps in $\mathbf{Y}$ satisfying the shifted plactic relations (3.2)–(3.9). Then for any $g, h \in \mathbf{Y}$, the polynomial $G_{h/g}$ defined by (3.13) is a nonnegative integer combination of Schur Q-functions. More specifically,*

$$G_{h/g}(x_1, \ldots, x_m) = \sum b^h_{g,\lambda} \, Q_\lambda(x_1, \ldots, x_m),$$

where $b^h_{g,\lambda}$ is equal to the number of shifted semistandard Young tableaux T of shape λ such that $u^T g = h$.

Proof. By the noncommutative Cauchy identity (Corollary 3.32),

$$G_{h/g}(\mathbf{x}) = \left\langle \sum_{\lambda} P_\lambda(\mathbf{u}) Q_\lambda(\mathbf{x}) g, h \right\rangle = \sum_{\lambda} \langle P_\lambda(\mathbf{u}) g, h \rangle Q_\lambda(\mathbf{x}).$$

Consequently,

$$b^h_{g\nu} = \langle P_\lambda(\mathbf{u}) g, h \rangle,$$

which is precisely the number of shifted semistandard Young tableaux T of shape λ such that $u^T g = h$. $\square$

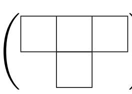
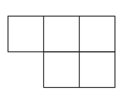
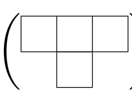
As an application of this theory, we obtain yet another version of the shifted Littlewood-Richardson Rule.

Definition 3.35 (cf. [3])**.** The *diagonal box-adding operators* u_j act on shifted shapes according to the following rule:

$$u_j(\lambda) = \begin{cases} \lambda \cup \{\text{box in the } j\text{-th diagonal}\} & \text{if this gives a valid shape;} \\ 0 & \text{otherwise.} \end{cases}$$

Here the diagonals are numbered from left to right, starting with $j = 1$ for the main diagonal.

Example 3.36. We have

u_2 () =  and u_1 () $= 0$.

The maps u_i are partial maps on the vector space formally spanned by the shifted shapes.

The product

$$\mathcal{B}(x) = \prod_{j=n}^{1} (1 + x u_j) \prod_{j=1}^{n} (1 - x u_j)^{-1}$$

can be viewed as an operator that adds a (possibly disconnected) border strip to a fixed shifted shape (the first product will add a horizontal strip, and the second one a vertical strip), each time introducing a power of x that is determined by the length of the strip. Setting $g=\mu$ and $h=\lambda$, we get

$$\prod_{i\geq 0}\left(\prod_{j=n}^{1}(1+x_iu_j)\prod_{j=1}^{n}(1-x_iu_j)^{-1}\right)\mu=\prod_{i\geq 0}\mathcal{B}(x_i)\mu=\sum_T x^T\lambda,$$

where the sum is over all semistandard skew shifted Young tableaux T of shape λ/μ. Therefore

$$G_{\lambda/\mu}(\mathbf{x})=\left\langle\prod_{i\geq 1}\mathcal{B}(x_i)\mu,\lambda\right\rangle=\sum_T x^T=Q_{\lambda/\mu}(\mathbf{x}),$$

the skew Schur Q-function.

One can see that the box-adding operators u_i satisfy the *nil-Temperley-Lieb relations of type B* (cf. [3, Example 2.4] [4]):

$$\begin{aligned} u_iu_j &= u_ju_i && |i-j|\geq 2,\\ u_i^2 &= 0 && i\geq 1,\\ u_iu_{i+1}u_i &= 0 && i\geq 2,\\ u_{i+1}u_iu_{i+1} &= 0 && i\geq 1, \end{aligned}$$

which implies that they also satisfy (3.2)–(3.9). Consequently, we can use Theorem 3.34 to obtain an expansion of the skew Schur Q-functions in terms of Schur Q-functions. This leads to the following new version of the shifted Littlewood-Richardson rule.

Corollary 3.37. *The shifted Littlewood-Richardson number $b^{\lambda}_{\mu,\nu}$ is equal to the number of shifted semistandard tableaux T of shape ν such that $u^T(\mu)=\lambda$, where, as before, u^T is the noncommutative monomial in $u_1,u_2,\ldots$ defined by any representative of the shifted plactic class associated with T, and each u_i is interpreted as a diagonal box-adding operator.*

The following is a direct connection between the versions of the shifted Littlewood-Richardson rule given in Corollary 3.37 and in Corollary 3.17.

Let T be a tableau of shape ν with the property that $u^T\mu = \lambda$. We wish to match it to a standard filling of a shape λ/μ that rectifies to a fixed standard tableau of shape ν. This filling will be obtained as follows: Note that $\mathrm{mread}(T)$ adds boxes to a shape μ one by one, until a shape λ is obtained. If we add a label i to the i-th box added in this procedure, we obtain a filling of a shape λ/μ that will rectify to the tableau of shape ν in which the first box in the i-th row will contain the number $\lambda_1 + \ldots + \lambda_{i-1} + 1$, and the numbers will be increasing by one in every row.

Example 3.38. For the computation the coefficient $b^{542}_{31,43}$, one has the following standard shifted Young tableau of shape $(542)/(31)$ which rectifies to a fixed tableau (say, the one below) of shape (43):

$$\begin{array}{|c|c|c|c|c|}\hline & & & 1 & 4 \\ \hline \multicolumn{1}{c|}{} & & 2 & 3 & 5 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 \\ \cline{2-4} \end{array}.$$

Recording the diagonals in which the numbers $7, 6, \ldots, 1$ are located within the skew shifted tableau on the left, we obtain the sequence $w = 2145324$. Note that w is the mixed reading word of the shifted semistandard Young tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2' & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 4' & 5 \\ \cline{2-4} \end{array}$$

which means that its corresponding monomial $u_2u_1u_4u_5u_3u_2u_4$ appears in $P_{(4,3)}(\mathbf{u})$. Note that this monomial, when applied to a shape $(3, 1)$ yields a shape $(5, 3, 2)$, as claimed.

CHAPTER 4

Shifted tableau words, semistandard decomposition tableaux, and proofs

In this Chapter, we develop the machinery for the proofs of the main theorems in the thesis. We define a canonical *mixed reading word* for a shifted semistandard Young tableau, and state several of its important properties. We define *semistandard decomposition tableaux*, which are related to shifted semistandard Young tableaux, and are sometimes easier to manipulate. Most of the proofs rely on the key *standardization* lemmas which show that many theorems about permutations and shifted standard tableaux can be extended to the generality of words and shifted semistandard tableaux.

4.1 Shifted tableau words

In this section, we begin to systematically develop the theory of the shifted plactic monoid by defining a canonical representative of each shifted plactic class and stating its characterizing properties.

Recall that the mixed insertion correspondence associates each word with its insertion and recording tableau. Associating each word in a shifted plactic class $[T]$ with its recording tableau gives a natural correspondence between representatives of $[T]$ and standard tableaux of the same shape as T. Thus, by constructing a canonical standard shifted Young tableau of each shape, we specify a canonical representative of each shifted plactic class.

A skew shifted tableau T is called a *vee* if its shape is a (possibly disconnected) border strip, and the entries $i, i+1, \ldots, k$ appear in T in the following manner. There exists some $i \leq j \leq k$ such that:

- the entries $i, i+1, \ldots, j$ form a vertical strip;
- these entries are increasing down the vertical strip;
- the entries $j, j+1, \ldots, k$ form a horizontal strip;
- these entries are increasing from left to right;
- each box in the vertical strip is left of those boxes in the horizontal strip that are on the same row.

The size of a vee is the number k of its entries. A vee is *connected* if the corresponding skew shape is connected.

Example 4.1. A vee of shape $(5,3,2)/(3,2)$ and size 5 is shown below:

	6	10
	7	
8	9	

Definition 4.2. A standard shifted Young tableau of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a *special recording tableau* if it is standard, and for every i such that $1 \leq i \leq l$, the entries $\lambda_l + \ldots + \lambda_{i+1} + 1, \lambda_l + \ldots + \lambda_{i+1} + 2, \ldots, \lambda_l + \ldots + \lambda_{i+1} + \lambda_i$ form a connected vee.

Note that for every shape, the special recording tableau is unique.

This concept is a shifted analog of the *dual reading tableau* defined by P. Edelman and C. Greene [1].

Example 4.3. The steps for building a special recording tableau of shape $(5,3,2)$, by adding connected vees formed by the following sets of numbers: $\{1,2\}, \{3,4,5\}$, and $\{6,7,8,9,10\}$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \end{array}\ , \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 10 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 8 & 9 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array}\ .$$

Definition 4.4. The *mixed reading word* of a shifted Young tableau T is the word corresponding to the pair (T,U) under the mixed insertion correspondence, where U is the shifted dual reading tableau of the same shape as T. The mixed reading word of T is denoted $\mathrm{mread}(T)$.

Example 4.5. We find $\mathrm{mread}(T)$ for the tableau T in Example 3.1. Recall that the special recording tableau of shape $(5,3,2)$ is U, given in Example 4.3.

To obtain the last letter of the mixed reading word, one first removes (using the mixed insertion algorithm backwards) the element in T corresponding to the largest entry in U, namely the 10. In T, this is precisely the 4 in the top right corner. Thus, the last letter of $\mathrm{mread}(T)$ is 4. Continuing in this fashion, we obtain $\mathrm{mread}(T) = 3451196524$.

Example 4.6. The mixed reading words of all tableaux of shape $\lambda = (3,1)$ in the alphabet $\{1,2\}$ are shown below:

$$\begin{array}{cccc}
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} &
\begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} &
\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} &
\begin{array}{|c|c|c|}\hline 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \\
\\
1211 & 2211 & 1212 & 2212.
\end{array}$$

Definition 4.7. A word w in the alphabet X is a *shifted tableau word* if there exists a shifted Young tableau T such that $w = \mathrm{mread}(T)$. The shape of a shifted tableau word is given by the shape of the corresponding tableau.

Theorem 4.8. *Every shifted plactic class* $[T]$ *contains exactly one shifted tableau word, namely* $\mathrm{mread}(T)$*.*

Proof. The fact that $P_{\mathrm{mix}}(\mathrm{mread}(T)) = T$ is direct from the definition. The uniqueness follows from the fact that each shifted plactic class corresponds to a unique shifted Young tableau. $\square$

We proceed to characterize shifted tableau words by certain properties.

Recall that a hook word is a word $w = w_1 \cdots w_l$ such that for some $1 \leq k \leq l$, the inequalities (3.1) hold. It is formed by the *decreasing part* $w_{\searrow} = w_1 \cdots w_k$, and the *increasing part* $w_{\nearrow} = w_{k+1} \cdots w_l$. Note that the decreasing part of a hook word is always nonempty.

Theorem 4.9. *A word* w *is a shifted tableau word if and only if it is of the form* $w = u_l u_{l-1} \cdots u_1$*, and:*

(1) each u_i *is a hook word,*

(2) u_i *is a hook subword of maximum length in* $u_l u_{l-1} \cdots u_i$*, for* $1 \leq i \leq l-1$*.*

Furthermore, for $1 \leq i \leq l$*, the shape of the tableau* $P_{\mathrm{mix}}(u_l \cdots u_{i+1} u_i)$ *is* $(\lambda_i, \lambda_{i+1}, \ldots, \lambda_l)$*, where* $|u_i| = \lambda_i$*. In particular,* $\mathrm{shape}(P_{\mathrm{mix}}(u)) = \lambda$*.*

This theorem is proved in Section 4.2.

Proposition 4.10. *An equivalent characterization of a shifted tableau word is obtained when one replaces condition* (2) *in Theorem 4.9 by the following:*

(2′) u_i *is a hook subword of maximum length in* $u_{i+1} u_i$*, for* $1 \leq i \leq l-1$*.*

This proposition is proved at the end of Chapter 4.

Consider the alphabets $-X = \{\cdots < -3 < -2 < -1\}$ and $-X' = \{\cdots - 2 < -2' < -1 < -1'\}$, where by convention, any letter of $-X$ $(-X')$ is smaller than any letter in X (X'). The mixed reading word of a skew shifted Young tableau T of shape λ/μ is defined as follows: Fill the shape μ with any shifted Young tableau with letters in $-X'$. In this way, one obtains a shifted Young tableau U with letters in $X' \cup -X'$. Note that $\mathrm{mread}(U)$ has letters in the alphabet $X \cup -X$. Let $\mathrm{mread}(T)$ be the restriction of $\mathrm{mread}(U)$ to the alphabet X, and let $\mathrm{rect}(T)$, the *rectification* of T be $P_{\mathrm{mix}}(\mathrm{mread}(T))$. The following lemma confirms that this reading is well defined.

Lemma 4.11. *Let T be a shifted skew Young tableau of shape λ/μ. Both* $\mathrm{mread}(T)$ *and* $\mathrm{rect}(T)$ *are independent of the filling of the shape μ with letters in $-X$.*

Proof. Let a be a letter in $X \cup -X$, and U a shifted Young tableau with letters in $X' \cup -X'$. The mixed insertion of a into U gives rise to a sequence of letters $a = a_1, a_2, \ldots, a_k = b$, each getting bumped at every stage, until b gets added to a new row or column. By definition, $a_1 < a_2 < \cdots < a_k$. Thus, when one removes b using the inverse process, the letters bumped are $b = a_k > \cdots > a_1 = a$. Thus, if $a \in -X'$, there is some $1 \leq i \leq k$ such that $a_i, a_{i-1}, \ldots, a_1 \in -X'$. This implies that once the reverse bumping sequence enters $-X'$, it will stay in $-X'$. Since we are restricting $\mathrm{mread}(U)$ to X', the result follows. □

We define the *shifted plactic skew Schur P-function* of shape λ/μ to be the following element of $\mathbb{Q}\mathbf{S}$:

$$\mathcal{P}_{\lambda/\nu} = \sum_{\mathrm{shape}(T)=\lambda/\mu} [\mathrm{rect}(T)].$$

Conjecture 4.12. *$\mathcal{P}_{\lambda/\mu}$ belongs to the ring generated by the shifted plactic Schur P-functions.*

Corollary 4.13 (of Conjecture 4.12)**.** *Fix a shifted Young tableau U of shape ν. The coefficient of P_ν in $P_{\lambda/\mu}$ is equal to the number of skew shifted Young tableaux T with* $P_{\mathrm{mix}}(\mathrm{mread}(T)) = U$.

4.2 Semistandard decomposition tableaux

The fact that the mixed reading word of T can be decomposed into hook subwords, precisely of the same lengths as the rows of T, hints that arranging these words as rows of a shifted diagram would yield an interesting object. Based on the concept of a standard decomposition tableau introduced by W. Kraśkiewicz [9] and further developed by T. K. Lam [11], we introduce the following notion.

Definition 4.14. A *semistandard decomposition tableau* (SSDT) is a filling R of a shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ with elements of X, such that:

(1) the word u_i formed by reading the i-th row from left to right is a hook word of length λ_i, and

(2) u_i is a hook subword of maximum length in $u_l u_{l-1} \cdots u_i$, for $1 \leq i \leq l-1$.

The *reading word* of R is $\mathrm{read}(R) = u_l u_{l-1} \cdots u_1$. The *content* of an SSDT is the content of $\mathrm{read}(R)$.

By Proposition 4.10, a filling R of a shape λ is an SSDT if and only if each of the fillings formed by two consecutive rows is an SSDT. By the definition, a filling R is an SSDT of shape λ if and only if $\mathrm{read}(R)$ is a shifted tableau word of shape λ.

Example 4.15. An SSDT, with its corresponding reading word:

$$R = \begin{array}{|c|c|c|c|c|} \hline 9 & 6 & 5 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} \qquad \mathrm{read}(R) = 34\ 511\ 96524.$$

The content of R is $(2, 1, 1, 2, 2, 1, 0, 0, 1)$.

Remark 4.16. An SSDT can be viewed as a shifted analog of a (ordinary) Young tableau for the following reason. A word w in the alphabet X is the reading word of a Young tableau T of shape λ if and only if it is of the form $u_l u_{l-1} \cdots u_1$, where:

(1) each word u_i is weakly increasing,

(2) the length of u_i is λ_i, for $1 \leq i \leq l$, and

(3) u_i is a weakly increasing subword of maximal length in $u_l u_{l-1} \cdots u_i$, for $1 \leq i \leq l-1$.

In this case the u_i are precisely the rows of T.

Shifted Young tableaux and SSDT share many properties, and many theorems about shifted Young tableaux can be proved more easily in the language of SSDT. To translate from one language to another, one can use the following bijection.

Theorem 4.17. *Let $\mathcal{Y}(\lambda)$ be the set of shifted Young tableaux of shape λ. Let $\mathcal{D}(\lambda)$ be the set of SSDT of shape λ. The map*

$$\begin{array}{rccc} \Phi: & \mathcal{D}(\lambda) & \to & \mathcal{Y}(\lambda) \\ & R & \mapsto & P_{\text{mix}}(\text{read}(R)) \end{array}$$

is a bijection. Furthermore, $\text{read}(R) = \text{mread}(\Phi(R))$, *i.e.,* Φ *is a word preserving bijection.*

Theorem 4.17 is proved below in this section.

For a more informal definition of Φ, see Remark 4.24

As an example, the image under Φ of the SSDT in Example 4.15 is the shifted Young tableau of Example 3.1.

In the special case of tableaux with only one row, the image of a shifted Young tableau is the SSDT formed by reading the primed entries from right to left, and then the unprimed entries from left to right. For example,

$$\boxed{6}\boxed{3}\boxed{2}\boxed{1}\boxed{2}\boxed{2}\boxed{4}\boxed{5} \overset{\Phi}{\mapsto} \boxed{1}\boxed{2'}\boxed{2}\boxed{2}\boxed{3'}\boxed{4}\boxed{5}\boxed{6'}.$$

In order to find the inverse of Φ directly, we define a semistandard version of Kraśkiewicz insertion [9].

Definition 4.18 (Semistandard Kraśkiewicz (SK) insertion)**.** Given a hook word $u = y_1 \cdots y_k \cdots y_s$, where $u_{\searrow} = y_1 \cdots y_k$ and $u_{\nearrow} = y_{k+1} \cdots y_s$, and a letter x, the insertion of x into u is the word ux if ux is a hook word, or the word u' with an element y that gets bumped out, as follows:

(1) let y_j be the leftmost element in $u_{\nearrow}$ which is strictly greater than x;

(2) replace y_j by x;

(3) let y_i be the leftmost element in $u_{\searrow}$ which is less than or equal to y_j;

(4) replace y_i by y_j, to obtain u', bumping $y = y_i$ out of x.

To insert a letter x into an SSDT T with rows $u_1, u_2, \ldots, u_l$, one first inserts $x = x_1$ into the top row u_1. If an element x_2 gets bumped, it will get inserted into the second row u_2, and so on. The process terminates when an element x_i gets placed at the end of row u_i.

The *SK insertion tableau* of the word $w = w_1 \cdots w_n$, denoted $P_{\mathrm{SK}}(w)$, is obtained by starting with an empty shape and inserting the letters $w_1, \ldots, w_n$ from left to right, forming an SSDT.

The *SK recording tableau* of w, denoted $Q_{\mathrm{SK}}(w)$, is the standard shifted Young tableau that records the order in which the elements have been inserted into $P_{\mathrm{SK}}(w)$. In other words, the shapes of $P_{\mathrm{SK}}(w_1 \cdots w_{i-1})$ and $P_{\mathrm{SK}}(w_1 \cdots w_i)$ differ by one box; $Q_{\mathrm{SK}}(w)$ has a letter i on that box.

Example 4.19. The following are the steps for inserting 3 into the tableau

$$\begin{array}{ccccccc} 6 & 5 & 4 & 2 & 1 & 1 & 4 \\ & 6 & 3 & 2 & 1 & 5 & \\ & & 5 & 2 & 2 & & \end{array}\ .$$

At every step, notation $u \leftarrow a \equiv b \leftarrow u'$ indicates that the insertion of the letter a into row u produces a new row u', with the letter b bumped out and inserted into the next row. The vertical bar separates $u_{\nearrow}$ and $u_{\searrow}$. We have:

$$\begin{aligned} 65421\,|\,14 \leftarrow 3 &\equiv 4 \leftarrow 65\mathbf{4}21\,|1\mathbf{3}, \\ 6321|5 \leftarrow 4 &\equiv 3 \leftarrow 6\mathbf{5}21|\mathbf{4}, \\ 52|2 \leftarrow 3 &\equiv 52|2\mathbf{3}, \end{aligned}$$

and the resulting tableau is

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ \hline \multicolumn{1}{c|}{} & 6 & 5 & 2 & 1 & 4 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 & 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{3-6} \end{array}\ .$$

Example 4.20. The word $w = 3415961254$ has the following SK insertion and recording tableau

$$P_{\mathrm{SK}}(w) = \begin{array}{|c|c|c|c|c|} \hline 9 & 6 & 5 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} \qquad Q_{\mathrm{SK}}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & 10 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array}\ .$$

In Section 4.3, we define the *standardization* of a word, of a shifted Young tableau, and of an SSDT, based on techniques introduced by Sagan [18] and Haiman [6]. Standardization enables us to translate results on permutations to results on words with repeated letters. We prove (Lemmas 4.28, 4.30 and 4.31) that standardization commutes with the shifted plactic equivalence, the SK insertion, and the mixed insertion. Thus, we obtain the following results which can be viewed as semistandard counterparts of theorems by Kraśkiewicz [9], Lam [10] [11], and Sagan [18]. Specifically, Theorem 4.21 is the semistandard extension of [9, Theorem 5.2], while Theorem 4.22, Proposition 4.23, and Theorem 4.25 are semistandard extensions of [11, Lemma 3.5, Corollary 3.6, Lemma 4.8], [10, Theorem 3.34], and [10, Theorem 3.25], respectively. Their proofs follow directly from the result in for permutations and the standardization lemmas.

Theorem 4.21. *SK insertion is a bijection between words in the alphabet X and pairs of tableaux (P, Q), where P is an SSDT and Q is a standard shifted Young tableau of the same shape as P.*

Theorem 4.22. *Two words are shifted plactic equivalent if and only if they have the same SK insertion tableau. In particular, for a word w, $w \equiv \mathrm{read}(P_{\mathrm{SK}}(w))$ (or equivalently, $P_{\mathrm{SK}}(\mathrm{read}(P_{\mathrm{SK}}(w))) = P_{\mathrm{SK}}(w)$). Furthermore, $Q_{\mathrm{SK}}(\mathrm{read}(P_{\mathrm{SK}}(w)))$ is the special recording tableau of the same shape as $P_{\mathrm{SK}}(w)$.*

Proposition 4.23. *The recording tableau of a word is the same under mixed insertion and SK insertion. Namely, $Q_{\mathrm{mix}}(w) = Q_{\mathrm{SK}}(w)$ for any word w in the alphabet X.*

Remark 4.24. A more informal way to view Φ is as follows. Let w be a word in the alphabet X. Then Φ sends $P_{\mathrm{SK}}(w)$ to $P_{\mathrm{mix}}(w)$.

Proof of Theorem 4.17. We will prove that the inverse map is given by

$$\begin{array}{rccc} \Psi: & \mathcal{Y}(\lambda) & \rightarrow & \mathcal{D}(\lambda) \\ & T & \mapsto & P_{\mathrm{SK}}(\mathrm{mread}(T)). \end{array}$$

Let $T \in \mathcal{Y}(\lambda)$. Then by Theorem 4.22, $Q_{\mathrm{SK}}(\mathrm{mread}(T))$ is the special recording tableau of shape λ. Therefore, $\Psi(T) = P_{\mathrm{SK}}(\mathrm{mread}(T))$ has shape λ, i.e., $\Psi(T) \in \mathcal{D}(\lambda)$. Similarly, if $R \in \mathcal{D}(\lambda)$, then $\Phi(R) \in \mathcal{Y}(\lambda)$.

Now, note that for $T \in \mathcal{Y}(\lambda)$,

$$\begin{aligned} \Phi(\Psi(T)) &= P_{\mathrm{mix}}(\mathrm{read}(P_{\mathrm{SK}}(\mathrm{mread}(T)))) \\ &= P_{\mathrm{mix}}(\mathrm{mread}(T)) \\ &= T. \end{aligned}$$

And similarly, for $R \in \mathcal{D}(\lambda)$, $\Psi(\Phi(R)) = R$. Therefore, Φ is a bijection. □

Proof of Theorem 4.9. By Theorem 4.17, w is a shifted tableau word if and only if $w = \mathrm{read}(R)$ for some SSDT R. Therefore, (1) follows from the definition of an SSDT.

(2) from the definition of an SSDT, it follows that if $u_l, \ldots, u_1$ are rows of an SSDT, then so are $u_l, \ldots, u_i$ for all $1 \leq i \leq l$. The result follows from this observation. $\square$

Note that an ordinary shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ can be considered a skew shifted shape $(\lambda + \delta)/\delta$, where $\delta = (l, l-1, \ldots, 1)$. The following result, which can be viewed as the semistandard version of [10, Theorem 3.25], relates the mixed (or SK) recording tableau of a word to its Robinson-Schensted-Knuth recording tableau.

Theorem 4.25. *Let w be a word in the alphabet X. Then the tableau $Q_{\mathrm{mix}}(w)$ (which is the same as $Q_{\mathrm{SK}}(w)$) can be obtained by treating $Q_{\mathrm{RSK}}(w)$ as a skew shifted Young tableau and applying shifted jeu de taquin slides to it to get a standard shifted Young tableau.*

4.3 Semistandardization and Proofs

Definition 4.26. Recall that a word of length n is a permutation if it contains each of the letters $1, 2, \ldots, n$ each exactly once. A shifted Young tableau of content $(1, 1, \ldots, 1)$ will be called a *Haiman tableau*. In other words, a Haiman tableau is a standard shifted tableau, possibly with some primed off-diagonal entries.

Let $w = w_1 w_2 \ldots w_m$ be a word. Let $(\alpha_1, \ldots, \alpha_m)$ be the *content* of w, i.e., α_i letters in w are equal to i. The *standardization* of w, denoted $\mathrm{stan}(w)$ is the permutation obtained by relabelling the elements labelled i by $\alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_i$, from left to right.

The *standardization* of an SSDT R, denoted $\mathrm{stan}(R)$ is the filling obtained by applying the same procedure used in the standardization of a word, namely, taking the order of the elements from $\mathrm{read}(R)$. Note that in general, any filling of a shifted shape with letters in X can be standardized, using this method.

The standardization of a shifted Young tableau T, denoted $\mathrm{stan}(T)$ is the Haiman tableau of the same shape, obtained as follows: Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be the content of T. For every i, the entries labelled i' or i are relabelled $\alpha_1 + \ldots + \alpha_{i-1} + 1, \alpha_1 + \ldots + \alpha_{i-1} + 2, \ldots, \alpha_1 + \ldots + \alpha_{i-1} + \alpha_i$. One assigns these values in increasing order, starting from the boxes labelled i', from top to bottom, and then moving to the boxes labelled i, from left to right. If the old element in a box was primed, so is the new one. This procedure has been suggested by Haiman [6].

Example 4.27. Standardizations of a word w, a shifted Young tableau T, and an SSDT R:

$$w = 23314211, \qquad \mathrm{stan}(w) = 46718523,$$

$$R = \begin{array}{|c|c|c|c|} \hline 4 & 2 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 3 & 1 & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}, \qquad \mathrm{stan}(R) = \begin{array}{|c|c|c|c|} \hline 8 & 5 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 6 & 1 & 7 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array},$$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 4 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}, \qquad \mathrm{stan}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4' \\ \hline \multicolumn{1}{c|}{} & 5 & 6' & 8 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}.$$

In his study of decompositions of reduced words in the hyperoctahedral group B_N, Kraśkiewicz [9] has introduced *standard decomposition tableaux* (SDT), which are defined in the same way as SSDT, with the extra condition that the reading word must be a reduced word in B_N. He has also introduced the *Kraśkiewicz correspondence*, which assigns to every reduced word in B_N, a pair of tableaux consisting of the Kraśkiewicz insertion tableau (an SDT), and the Kraśkiewicz recording tableau (a standard Young tableau).

We do not use the full power of Kraśkiewicz insertion, but we use it for permutations, since a permutation is always a reduced word in B_N, for some N. (Here the letter i stands for the i-th generator of B_N as a Coxeter group.) We abuse notation and call $P_{\mathrm{SK}}(w)$ and $Q_{\mathrm{SK}}(w)$ the *Kraśkiewicz insertion* and *recording tableaux*, respectively, and use the term SDT to refer to a semistandard decomposition tableau whose reading word is a permutation. We also note that for permutations, the shifted plactic relations are equivalent to the *B-Coxeter-Knuth relations* in [11], and to the dual equivalence relations given in Corollary 3.2 in [7].

Lemma 4.28. *Two words u and v in the alphabet X are shifted plactic equivalent if and only if they have the same content, and* $\mathrm{stan}(u) \equiv \mathrm{stan}(v)$*.*

Proof. It is enough to check this for relations (3.2)–(3.9). For example, for (3.2), there are four possible type of words that will have $abdc$ as their standardization, with $a < b < c < d$, namely $aaba, aacb, abcb, abdc$. Similarly for $adbc$, we have $abaa, abac, acab, adbc$. These eight words are paired by relation (3.2) as follows:

$$\begin{aligned} aaba &\equiv abaa; \\ aacb &\equiv acab; \\ abcb &\equiv acbb; \\ abdc &\equiv adbc. \end{aligned}$$

Checking the other seven relations is routine, and left for the reader. □

Lemma 4.29. *Standardization respects the property of being a shifted tableau word, an SSDT, or a semistandard shifted Young tableau. More precisely:*

- *a word w is a shifted tableau word if and only if* $\mathrm{stan}(w)$ *is;*
- *a filling of a shifted shape R with letters in X is an SSDT if and only if* $\mathrm{stan}(R)$ *is an SDT;*
- *a filling of a shifted shape T with letters in X' is a semistandard shifted Young tableau if and only if* $\mathrm{stan}(T)$ *is a Haiman tableau.*

Proof. The first two points follow from the observation that a word w is a hook word if and only if its standardization $\mathrm{stan}(w)$ is a hook word. The third one is a routine check of the properties that define a shifted Young tableau. □

Lemma 4.30. *Let w be a word and R an SSDT. Then*

$$\begin{aligned} P_{\mathrm{SK}}(\mathrm{stan}(w)) &= \mathrm{stan}(P_{\mathrm{SK}}(w)); \\ Q_{\mathrm{SK}}(\mathrm{stan}(w)) &= Q_{\mathrm{SK}}(w); \\ \mathrm{read}(\mathrm{stan}(R)) &= \mathrm{stan}(\mathrm{read}(R)). \end{aligned}$$

Proof. Cases (1) and (2) of the SK insertion algorithm treat the two following scenarios in the same fashion:

- when a letter i gets bumped by a letter $j > i$;
- when a letter i gets bumped by another letter i.

Therefore, if two letters i are in an SSDT R, the algorithm treats them as if the rightmost one in the reading word was larger. Furthermore, the i at the right will always remain at the right of the i at the left, i.e., they will always maintain their relative position with respect to each other. The third equation is a rewording of the definition of standardization of an SSDT. □

Lemma 4.31. *Let w be a word and T a shifted Young tableau. Then*

$$\begin{aligned} P_{\mathrm{mix}}(\mathrm{stan}(w)) &= \mathrm{stan}(P_{\mathrm{mix}}(w)); \\ Q_{\mathrm{mix}}(\mathrm{stan}(w)) &= Q_{\mathrm{mix}}(w); \\ \mathrm{mread}(\mathrm{stan}(T)) &= \mathrm{stan}(\mathrm{mread}(T)). \end{aligned}$$

Proof. Let $w = w_1 \cdots w_l$, and $T = P_{\mathrm{mix}}(w)$. The result follows from the next claim. Say w has two letters equal to i, say, $w_a = w_b = i$ for $a < b$. We say that a letter is primed or unprimed depending of the state of its corresponding entry in T. Let r_a and c_a be the row and column where w_a gets located in T, and r_b, c_b the row and column where w_b gets located. Then either

- w_a and w_b are unprimed and $c_a < c_b$,
- w_a and w_b are primed and $r_a < r_b$, or
- w_a is primed and w_b is unprimed.

The claim is proved as follows. Assume that both w_a and w_b are unprimed. As they both have the label i, then whenever w_b reaches the row where w_a is, it must be at its right. No number smaller than i can bump w_b before bumping w_a. Therefore, w_a always remains strictly at the left of w_b, i.e., $c_a < c_b$.

If both w_a and w_b are primed, namely, they have the label i', it means they both reached the diagonal and later got bumped out of the diagonal. An analogous argument to the above one, replacing columns for rows, will show that w_a must be in a row above w_b, i.e., $r_a < r_b$.

It remains to show that the case where w_a is unprimed and w_b is primed can not occur. For this to happen, w_b must reach the diagonal, but w_a must not. But this is impossible since w_b must remain at the right of w_a when they are both unprimed. □

Proof of Theorem 3.7. The result follows either as a dual of [18, Corollary 5.2], or by [11, Corollary 3.39] and Lemmas 4.30 and 4.31. □

Proof of Theorem 3.8. Let u, v be words in the alphabet X. By Lemma 4.28, $u \equiv v$ if and only if they have the same content and $\mathrm{stan}(u) \equiv \mathrm{stan}(v)$.

By Corollary 3.2 in [7], $\mathrm{stan}(u) \equiv \mathrm{stan}(v)$ if and only if $P_{\mathrm{mix}}(\mathrm{stan}(u)) = P_{\mathrm{mix}}(\mathrm{stan}(v))$. By Lemma 4.31, the latter is equivalent to $\mathrm{stan}(P_{\mathrm{mix}}(u)) = \mathrm{stan}(P_{\mathrm{mix}}(v))$.

By the definition of standardization of shifted Young tableaux, it is clear that $P_{\mathrm{mix}}(u) = P_{\mathrm{mix}}(v)$ if and only if u and v have the same content and $\mathrm{stan}(P_{\mathrm{mix}}(u)) = \mathrm{stan}(P_{\mathrm{mix}}(v))$. The result then follows. □

The following result is the semistandard extension of [11, Lemma 3.11].

Lemma 4.32. *Let R be an SSDT and $w = w_1 \cdots w_l$ a word in the alphabet X. Let S be the SSDT obtained by SK inserting $w_1, \ldots, w_l$ into R. Let $\mu = \text{shape}(R)$ and $\lambda = \text{shape}(S)$. Then w is a hook word if and only if the entries in the standard skew tableau of shape λ/μ that records the insertion of $w_1, \ldots, w_l$ in R is a vee.*

Proof of Theorem 3.21. It suffices to prove that the shifted plactic Schur P-functions satisfy the Pieri rule, namely

$$\mathcal{P}_\mu \mathcal{P}_{(k)} = \sum_\lambda 2^{c(\lambda/\mu)-1} \mathcal{P}_\lambda,$$

where λ runs over all strict partitions such that λ/μ is a (possibly disconnected) border strip with $c(\lambda/\mu)$ connected components.

It is well known that a skew tableau $U_{\lambda/\mu}$ of shape λ/μ is a vee if and only if it rectifies to the single row with entries $i, i+1, \ldots, k$ under jeu de taquin, and that the number of vees of this shape with a fixed content is exactly $2^{c(\lambda/\mu)-1}$, where $c(\lambda/\mu)$ is the number of connected components of λ/μ.

Recall that $\mathcal{D}(\lambda)$ is the set of SSDT of shape λ, where we denote $\mathcal{D}((k))$ by $\mathcal{D}(k)$. Let $\mu \oplus k$ be the set of shifted shapes λ such that λ/μ is a (possibly disconnected) border strip of size k. Let $\mathcal{V}(\lambda/\mu)$ be the set of vees of shape λ/μ filled with the entries $|\mu|+1, |\mu|+2, \cdots .|\lambda|$.

We will prove that $\mathcal{D}(\mu) \times \mathcal{D}(k)$ and $\bigcup_{\lambda \in \mu \oplus k} (\mathcal{D}(\lambda) \times \mathcal{V}(\lambda/\mu))$ are in bijection, and moreover, that the content of the elements in $\mathcal{D}(\mu)$ and $\mathcal{D}(k)$ adds up to the content of the element in $\mathcal{D}(\lambda)$ in their image. Thus, the theorem will follow since $\mathcal{P}_\lambda$ is the formal sum of all elements in $\mathcal{D}(\lambda)$.

Consider the map

$$\Phi : \mathcal{D}(\mu) \times \mathcal{D}(k) \to \bigcup_{\lambda \in \mu \oplus k} (\mathcal{D}(\lambda) \times \mathcal{V}(\lambda/\mu))$$

defined as follows. Given $S_\mu \in \mathcal{D}(\mu)$ and $T_k \in \mathcal{D}(k)$, insert the elements of $\mathrm{read}(T_k)$ into S_μ from left to right to obtain an SSDT R_λ of some shape λ. Since $\mathrm{read}(T_k)$ is a hook word then by Lemma 4.32, $\lambda \in \mu \oplus k$, and the standard skew tableau $V_{\lambda/\mu}$ that records this insertion is in $\mathcal{V}(\lambda/\mu)$. We define $\Phi(S_\mu, T_k) = (R_\lambda, V_{\lambda/\mu})$. Clearly, the content of S_μ and T_k add up to the content of R_λ.

The inverse map

$$\Psi : \bigcup_{\lambda \in \mu \oplus k} (\mathcal{D}(\lambda) \times \mathcal{V}(\lambda/\mu)) \to \mathcal{D}(\mu) \times \mathcal{D}(k)$$

is defined in a very similar way, by removing the elements of R_λ in the order given by the vee $V_{\lambda/\mu}$, to obtain $S_\nu \in \mathcal{D}(\mu)$ and $T_k \in \mathcal{D}(k)$. Since this map removes the elements that were inserted in the definition of Φ, it is clear that the composition of these two maps is the identity.

The following example illustrates the bijection:

$$\Phi\left(\begin{array}{cccc} 4&2&1&1\\ &3&1&3\\ &&2& \end{array}, \begin{array}{ccccc} 5&3&1&2&2 \end{array}\right) = \left(\begin{array}{ccccccc} 5&3&1&1&1&2&2\\ &3&1&2&4&&\\ &&2&3&&& \end{array}, \begin{array}{ccccccc} \square&\square&\square&\square&1&4&5\\ &\square&\square&\square&2&&\\ &&\square&3&&& \end{array}\right). \qquad \square$$

The following definition appears in [11] and [18].

Definition 4.33. Let U be a standard ordinary or shifted Young tableau. Define $\Delta(U)$ to be the tableau obtained by applying the following operations:

1. remove the entry 1 from U;

2. apply a jeu de taquin slide into this box;

3. deduct 1 from the remaining boxes.

The first two parts of the following lemma follow straight from [11, Theorem 4.14], the standardization Lemmas 4.28 and 4.30, and Theorem 4.17. The last part is a straightforward application of the theory of jeu de taquin in [25].

Lemma 4.34. *Let $w_1 w_2 \cdots w_l$ be a word. Then,*

$$Q_{\mathrm{SK}}(w_2 \cdots w_l) = \Delta Q_{\mathrm{SK}}(w_1 w_2 \cdots w_l),$$

$$Q_{\mathrm{mix}}(w_2 \cdots w_l) = \Delta Q_{\mathrm{mix}}(w_1 w_2 \cdots w_l),$$

and

$$Q_{\mathrm{RSK}}(w_2 \cdots w_l) = \Delta Q_{\mathrm{RSK}}(w_1 w_2 \cdots w_l),$$

Proof of Lemma 3.25. The bijection Φ in the proof of Theorem 3.21 is a special case of the following bijection:

$$\Phi : \mathcal{D}(\mu) \times \mathcal{D}(\nu) \to \bigcup_{\lambda} \left(\mathcal{D}(\lambda) \times \mathcal{V}(\lambda/\mu)\right),$$

where $\mathcal{V}(\lambda/\mu)$ is the set of standard shifted skew tableau which rectify to the special recording tableau of shape ν. This bijection is described in the exact same way, so we will not go into much detail. However, it is necessary to check that $\mathcal{V}(\lambda/\mu)$ is indeed the set we claim.

For a skew tableau T and an integer k, let $T + k$ be the skew tableau of the same shape, where all the entries are raised by k. Let $S_\mu \in \mathcal{D}(\mu)$, $T_\nu \in \mathcal{D}(\nu)$. Let λ be the shape of $P_{\mathrm{mix}}(\mathrm{mread}(S_\mu)\,\mathrm{mread}(T_\nu))$. Let $U_{\lambda/\mu} \in \mathcal{V}(\lambda/\mu)$ be the standard shifted skew tableau of shape λ/μ which records the order in which the entries of $\mathrm{mread}(T_\nu)$ get inserted into S_μ. Note that $U_{\lambda/\mu} + |\mu|$ is precisely the subtableau of $Q_{\mathrm{mix}}(\mathrm{mread}(S_\mu)\,\mathrm{mread}(T_\nu))$ corresponding to the shape λ/μ. By Lemma 4.34 applied repeatedly, one can see that $Q_{\mathrm{mix}}(\mathrm{mread}(T_\nu) = \Delta^{|\mu|}(U_{\lambda/\mu} + |\mu|)$. But $\Delta^{|\mu|}(U_{\lambda/\mu} + |\mu|)$ is nothing more than the restriction of $U_{\lambda/\mu}$. Since $Q_{\mathrm{mix}}(\mathrm{mread}(T_\nu))$ is the standard recording tableau of shape ν, by definition, then $U_{\lambda/\mu} \in \mathcal{V}(\lambda/\mu)$.

Lemma 3.25 then follows, if one lets $u = \mathrm{mread}(S_\mu)$, $v = \mathrm{mread}(T_\nu)$, and $Q = Q_{\mathrm{mix}}(\mathrm{mread}(T_\nu))$, i.e., the special recording tableau of shape ν. $\square$

Proof of Theorems 3.26 and 3.29. Let U_λ be the special recording tableau of shifted shape λ. Let $\mathcal{H}(\lambda, \mu)$ be the set of (ordinary) standard Young tableaux of shape μ which rectify to U_λ (the proof is the same if U_λ is any other standard shifted Young tableau of shape λ).

Let $\langle P_\mu \rangle$ be a plactic class of shape μ, and $\mathcal{G}(\lambda, P_\mu)$ be the set of shifted plactic classes $[T_\lambda]$ of shifted shape λ such that $\pi([T_\lambda]) = \langle P_\mu \rangle$. We will prove that the size of $\mathcal{G}(\lambda, P_\mu)$ does not depend on P_μ (only on λ and μ), by finding a bijection Θ between $\mathcal{G}(\lambda, P_\mu)$ and $\mathcal{H}(\lambda, \mu)$.

Define the maps

$$\begin{array}{rccc} \Theta: & \mathcal{G}(\lambda, P_\mu) & \to & \mathcal{H}(\lambda, \mu) \\ & [T_\lambda] & \mapsto & Q_{\mathrm{RSK}}(\mathrm{mread}(T_\lambda)) \end{array}$$

and

$$\begin{array}{rccc} \Gamma: & \mathcal{H}(\lambda, \mu) & \to & \mathcal{G}(\lambda, P_\mu) \\ & Q & \mapsto & [P_{\mathrm{mix}}(w)] \end{array}$$

where w is the word in the alphabet X such that $P_{\mathrm{RSK}}(w) = P_\mu$ and $Q_{\mathrm{RSK}}(w) = Q$.

First assume that $[T_\lambda] \in \mathcal{G}(\lambda, P_\mu)$. Let $w = \mathrm{mread}(T_\lambda)$ (i.e., the canonical representative of $[T_\lambda]$), so by definition, $Q_{\mathrm{mix}}(w) = U_\lambda$. By Theorem 4.25, $Q_{\mathrm{RSK}}(w)$ rectifies to U_λ, so $\Theta([T_\lambda]) = Q_{\mathrm{RSK}}(\mathrm{mread}(T_\lambda)) = Q_{\mathrm{RSK}}(w) \in \mathcal{H}(\lambda, \mu)$.

Now assume that $Q \in \mathcal{H}(\lambda, \mu)$. Let w be such that $P_{\mathrm{RSK}}(w) = P_\mu$ and $Q_{\mathrm{RSK}}(w) = Q$. Then, again by Theorem 4.25, $Q = Q_{\mathrm{RSK}}(w)$ rectifies to $Q_{\mathrm{mix}}(w)$, but as $Q \in \mathcal{H}(\lambda, \mu)$, then $Q_{\mathrm{mix}}(w)$ has shape λ. Furthermore, as $P_{\mathrm{RSK}}(w) = P_\mu$, then $\Gamma(Q) = [P_{\mathrm{mix}}(w)] \in \mathcal{G}(\lambda, P_\mu)$.

To prove that Θ and Γ are inverse maps, again let $Q \in \mathcal{H}(\lambda/\mu)$. Thus, $\Gamma(Q) = [P_{\mathrm{mix}}(w)]$ where w is the word such that $P_{\mathrm{RSK}}(w) = P_\mu$ and $Q_{\mathrm{RSK}}(w) = Q$. Since Q rectifies to U_λ, $Q_{\mathrm{RSK}}(w)$ rectifies to $Q_{\mathrm{mix}}(w)$ (by Theorem 4.25), and $Q_{\mathrm{mix}}(w) = Q$, then $Q_{\text{mix}}(w) = U_\lambda$. Thus, w is the canonical representative of $[P_{\text{mix}}(w)]$, which implies that $\text{mread}(P_{\text{mix}}(w)) = w$. Therefore,

$$\begin{aligned}
\Theta(\Gamma(Q)) &= \Theta([P_{\text{mix}}(w)]) \\
&= Q_{\text{RSK}}(\text{mread}(P_{\text{mix}}(w))) \\
&= Q_{\text{RSK}}(w) \\
&= Q.
\end{aligned}$$

The proof that $\Gamma(\Theta([T_\lambda])) = [T_\lambda]$ is similar.

Since $\mathcal{H}(\lambda, \mu)$ clearly does not depend on the choice of P_μ, but only on the shape μ, then neither does the number of shifted plactic classes $[T_\lambda]$ of shifted shape λ such that $\pi([T_\lambda]) = \langle P_\mu \rangle$. This number is precisely g^λ_μ. Furthermore, it is also equal to the size of $\mathcal{H}(\lambda, \mu)$, which proves Theorem 3.29. $\square$

Proof of Proposition 4.10. Let $u_1, \ldots, u_l$ be hook words. We use the fact that $w = u_1 \cdots u_l$ is a shifted tableau word if and only if the tableau formed by the rows $u_1, \ldots, u_l$ from top to bottom is an SSDT.

Clearly, (2) implies (2′). We will prove the converse.

By an inductive argument, it suffices to prove the following claim: If $u_n \cdots u_2$ and $u_{n-1} \cdots u_1$ are both shifted tableau words, then so is $u_n \cdots u_1$.

Equivalently, we will prove that if the filling of the shape $(a_1, a_2, \ldots, a_{n-1})$ with rows $u_1, \cdots, u_{n-1}$, and the filling of the shape $(a_2, \ldots, a_n)$ with rows $u_2, \ldots, u_n$ are both SSDT, then the filling of the shape $(a_1, \ldots, a_n)$ with rows $u_1, \ldots, u_n$ is an SSDT as well.

By Theorem 4.22, $Q_{\text{SK}}(u_n \cdots u_2)$ is the special recording tableau of shape $(a_2, \ldots, a_n)$. Let us assume that the tableaux formed by the rows $u_1, \ldots, u_n$ is not an SSDT. This means that the longest hook subword in the word $u_n \ldots u_1$ is of length d, for some

$d > a_1$. Therefore, by Theorem 3.7, the top row of $Q_{\text{SK}}(u_n \ldots u_1)$ is of length d. By Lemma 4.34 applied repeatedly, $Q_{\text{SK}}(u_{n-1} \cdots u_1) = \Delta^{a_n} Q_{\text{SK}}(u_n \ldots u_1)$. But note that since $Q_{\text{SK}}(u_n \cdots u_2)$ is a subtableau of $Q_{\text{SK}}(u_n \cdots u_1)$, and $\Delta^{a_n} Q_{\text{SK}}(u_n \cdots u_2) = Q_{\text{SK}}(u_{n-1} \cdots u_2)$ which is a special recording tableau of shape $(a_2, \ldots, a_{n-1})$, then the top row of $\Delta^{a_n}(Q_{\text{SK}}(u_{n-1} \ldots u_1))$ has length a_1 (because applying Δ^{a_1} to $Q_{\text{SK}}(u_n \cdots u_1)$ will not alter the top length of the top row, since it doesn't alter the top row of $Q_{\text{SK}}(u_n \cdots u_2)$). This contradicts the assumption that $u_{n-1} \cdots u_1$ is a shifted tableau word, completing the proof. $\square$

CHAPTER 5

Cyclic sieving for longest reduced words in the hyperoctahedral group

In this Chapter, we apply the relations between plactic equivalence and shifted plactic equivalence to show that the set $R(w_0)$ of reduced expressions for the longest element in the hyperoctahedral group exhibits the cyclic sieving phenomenon. More specifically, $R(w_0)$ possesses a natural cyclic action given by moving the first letter of a word to the end, and we show that the orbit structure of this action is encoded by the generating function for the major index on $R(w_0)$.

5.1 The hyperoctahedral group

The *hyperoctahedral* group B_n is the group of symmetries of an n-dimensional cube. As an abstract group, B_n is a Coxeter group generated by the simple reflections $s_1, \ldots, s_n$ subject to the relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i \text{ for } |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } 2 \leq i \leq n \\ s_1 s_2 s_1 s_2 &= s_2 s_1 s_2 s_1. \end{aligned}$$

The smallest number of generators s_i whose product is equal to a given element

$w \in B_n$ is called the *length* of w, and denoted by $\ell(w)$. Such a shortest factorization is called a reduced word for w. The longest element in B_n is $w_0 = w_0^{(B_n)} = (s_1 s_2 \cdots s_n)^n$, with the length $\ell(w_0) = n^2$.

5.2 Promotion on standard Young tableaux

For a partition λ, let $SYT(\lambda)$ denote the set of standard Young tableaux of shape λ. If λ is a strict partition, then let $SYT'(\lambda)$ denote the set of standard Young tableaux of shifted shape λ. We now describe the action of *jeu de taquin promotion*, first defined by Schützenberger [20].

We will consider promotion as a permutation of the set of tableaux of a fixed shape (resp. shifted shape), $p : SYT(\lambda) \to SYT(\lambda)$ (resp. $p : SYT'(\lambda) \to SYT'(\lambda)$). Given a standard Young tableau T of shape $\lambda \vdash n$, we form $p(T)$ by the following algorithm. (We denote the entry in row a, column b of a tableau T, by $T_{a,b}$.)

1. Remove the entry 1 in the upper left corner and decrease every other entry by 1. The empty box is initialized in position $(a, b) = (1, 1)$.

2. Perform jeu de taquin:

   (a) If there is no box to the right of the empty box and no box below the empty box, then go to 3).

   (b) If there is a box to the right or below the empty box, then swap the empty box with the box containing the smaller entry, i.e., $p(T)_{a,b} := \min\{T_{a,b+1}, T_{a+1,b}\}$. Set $(a, b) := (a', b')$, where (a', b') are the coordinates of box swapped, and go to 2a).

3. Fill the empty box with n.

Here is an example:

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 6 & 7 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} = p(T).$$

5.3 Cyclic sieving

Suppose we are given a finite set X, a finite cyclic group $C = \langle \omega \rangle$ acting on X, and a polynomial $X(q) \in \mathbb{Z}[q]$ with integer coefficients. Following Reiner, Stanton, and White [16], we say that the triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* (CSP) if for every integer $d \geq 0$, we have that $|X^{\omega^d}| = X(\zeta^d)$ where $\zeta \in \mathbb{C}$ is a root of unity of multiplicitive order $|C|$ and X^{ω^d} is the fixed point set of the action of the power ω^d. In particular, since the identity element fixes everything in any group action, we have that $|X| = X(1)$ whenever $(X, C, X(q))$ exhibits the CSP.

If the triple $(X, C, X(q))$ exhibits the CSP and ζ is a primitive $|C|^{th}$ root of unity, we can determine the cardinalities of the fixed point sets $X^1 = X$, X^{ω}, $X^{\omega^2}, \ldots, X^{\omega^{|C|-1}}$ via the polynomial evaluations $X(1), X(\zeta), X(\zeta^2), \ldots, X(\zeta^{|C|-1})$. These fixed point set sizes determine the cycle structure of the canonical image of ω in the group of permutations of X, S_X. Therefore, to find the cycle structure of the image of any bijection $\omega : X \to X$, it is enough to determine the order of the action of ω on X and find a polynomial $X(q)$ such that $(X, \langle \omega \rangle, X(q))$ exhibits the CSP.

The cyclic sieving phenomenon has been demonstrated in a variety of contexts. The paper of Reiner, Stanton, and White [16] itself includes examples involving noncrossing partitions, triangulations of polygons, and cosets of parabolic subgroups of Coxeter groups. An example of the CSP with standard Young tableaux is due to Rhoades [17] and will be discussed further in Section 5.5.

5.4 CSP for words in $R(w_0)$.

Now we turn to the CSP of interest to this chapter.

Abbreviate a reduced word $w = w_{i_1}w_{i_2}\cdots w_{i_l}$ in B_n as $i_1i_2\cdots i_l$. For example, the reduced word $s_1s_2s_1s_3s_2s_3s_1s_2s_3$ for $w_0^{(B_3)}$ will be abbreviated by 121323123. It turns out that if we cyclically permute these letters, we always get another reduced expression for w_0. Said another way, $s_iw_0s_i = w_0$ for $i = 1, \ldots, n$. The reason for this is that in the standard reflection representation of the type B_n Coxeter group, the longest element w_0 is the scalar transformation -1, and thus commutes with all simple reflections s_i. The same is not true for longest elements of other classical types. In type A, we have $s_iw_0^{(A_n)}s_{n+1-i} = w_0^{(A_n)}$, and for type D,

$$w_0^{(D_n)} = \begin{cases} s_iw_0^{(D_n)}s_i & \text{if } n \text{ even or } i > 2, \\ s_iw_0^{(D_n)}s_{3-i} & \text{if } n \text{ odd and } i = 1, 2. \end{cases}$$

Recall that $R(w_0)$ is the set of reduced expressions for w_0 in type B_n and let $c : R(w_0) \to R(w_0)$ denote the action of placing the first letter of a word at the end. Then the orbit in $w_0(B_3)$ of the word above is:

$$\{121323123 \to 213231231 \to 132312312 \to 323123121 \to 231231213$$

$$\to 312312132 \to 123121323 \to 231213231 \to 312132312\}.$$

As the length of w_0 is n^2, we clearly have $c^{n^2} = 1$, and the size of any orbit divides n^2. For an example of a smaller orbit, notice that the word 213213213 has cyclic order 3.

For any word $w = w_1 \ldots w_l$, (e.g., a reduced expression for w_0), a *descent* of w is defined to be a position i in which $w_i > w_{i+1}$.The *major index* of w, $\mathrm{maj}(w)$, is defined as the sum of the descent positions. For example, the word $w = 121323123$

has descents in positions $2, 4$, and 6, so its major index is $\mathrm{maj}(w) = 2 + 4 + 6 = 12$. Let $f_n(q)$ denote the generating function for this statistic on words in $R(w_0)$:

$$f_n(q) = \sum_{w \in R(w_0)} q^{\mathrm{maj}(w)}.$$

The following is the main result in this chapter.

Theorem 5.1 ([15]). *The triple $(R(w_0), \langle c \rangle, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = q^{-n\binom{n}{2}} f_n(q).$$

For example, let us consider the case $n = 3$. We have

$$\begin{aligned} X(q) &= q^{-9} \sum_{w \in w_0(B_3)} q^{\mathrm{maj}(w)} \\ &= 1 + q^2 + 2q^3 + 2q^4 + 2q^5 + 4q^6 + 3q^7 + 4q^8 + 4q^9 \\ &\quad + 4q^{10} + 3q^{11} + 4q^{12} + 2q^{13} + 2q^{14} + 2q^{15} + q^{16} + q^{18}. \end{aligned}$$

Let $\zeta = e^{\frac{2\pi i}{9}}$. Then we compute:

$$\begin{array}{lll} X(1) = 42 & X(\zeta^3) = 6 & X(\zeta^6) = 6 \\ X(\zeta) = 0 & X(\zeta^4) = 0 & X(\zeta^7) = 0 \\ X(\zeta^2) = 0 & X(\zeta^5) = 0 & X(\zeta^8) = 0 \end{array}$$

Thus, the 42 reduced expressions for $w_0^{(B_3)}$ split into two orbits of size three (the orbits of 123123123 and 132132132) and four orbits of size nine.

To prove Theorem 5.1 we rely on a pair of remarkable bijections due to Haiman [6, 7], and a recent result of Rhoades [17, Thm 3.9]. The composition of Haiman's bijections relates $R(w_0)$ with the set $SYT(n^n)$ of standard Young tableaux of square shape. Rhoades' result is that there is a CSP for $SYT(n^n)$ with respect to the action of *promotion* (defined in Section 5.2). Our main strategy is to show that Haiman's bijections carry the orbit structure of promotion on $SYT(n^n)$ to the orbit structure of c on $R(w_0)$.

5.5 Rhoades's Theorem

Rhoades [17] proved an instance of the CSP related to the action of promotion on rectangular tableaux. His result is quite deep, and his proof involves the theory of Kahzdan-Lusztig cell representations.

Recall that for any partition $\lambda \vdash n$, the standard tableaux of shape λ are enumerated by the Frame-Robinson-Thrall *hook-length formula*:

$$f^\lambda = |SYT(\lambda)| = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}},$$

where the product is over the boxes (i,j) in λ and h_{ij} is the *hook-length* at the box (i,j), i.e., the number of boxes directly east or south of the box (i,j) in λ, counting itself exactly once. To obtain the polynomial used for cyclic sieving, we replace the hook-length formula with a natural q-analogue. First, recall that for any $n \in \mathbb{N}$, $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$.

Theorem 5.2 ([17], Theorem 3.9)**.** *Let $\lambda \vdash N$ be a rectangular shape and let $X = SYT(\lambda)$. Let $C := \mathbb{Z}/N\mathbb{Z}$ act on X via promotion. Then, the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \frac{[N]_q!}{\Pi_{(i,j)\in\lambda}[h_{ij}]_q}$$

is the q-analogue of the hook-length formula.

5.6 Haiman's bijections

We first describe the bijection between reduced expressions and shifted standard tableaux of doubled staircase shape. This bijection is described in Section 5 of [7].

Let T be in $SYT'(2n-1, 2n-3, \ldots, 1)$. Notice the largest entry in T, (i.e., n^2), occupies one of the outer corners. Let $r(T)$ denote the row containing this largest entry, numbering the rows from the bottom up. The *promotion sequence* of T is defined to be $\Phi(T) = r_1 \cdots r_{n^2}$, where $r_i = r(p^i(T))$. Using the example above of

$$T = \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \boxed{8} \\ & \boxed{3} & \boxed{6} & \boxed{9} & \\ & & \boxed{7} & & \end{array},$$

we see $r(T) = 2, r(p(T)) = 1$, $r(p^2(T)) = 3$, and since $p^3(T) = T$, we have

$$\Phi(T) = 132132132.$$

Haiman's result is the following.

Theorem 5.3 ([7], Theorem 5.12)**.** *The map $T \mapsto \Phi(T)$ is a bijection $SYT'(2n-1, 2n-3, \ldots, 1) \to R(w_0)$.*

By construction, then, we have

$$\Phi(p(T)) = c(\Phi(T)),$$

i.e., Φ is an orbit-preserving bijection

$$(SYT'(2n-1, 2n-3, \ldots, 1), p) \longleftrightarrow (R(w_0), c).$$

Next, we will describe the bijection

$$H : SYT(n^n) \to SYT'(2n-1, 2n-3, \ldots, 1)$$

between squares and doubled staircases. Though not obvious from the definition below, we will demonstrate that H commutes with promotion.

Recall that by Theorem 4.25, if we view $Q_{\mathrm{RSK}}(w)$ as a skew shifted standard Young tableau and apply jeu de taquin to obtain a standard shifted Young tableau, the result is $Q_{\mathrm{mix}}(w)$ (independent of any choices in applying jeu de taquin).

For example, if $w = 332132121$, then

$$(P,Q) = \left(\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6&8\\\hline 4&7&9\\\hline\end{array}\right),$$

$$(P',Q') = \left(\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2'&3'\\\hline \multicolumn{1}{c|}{}&2&2&3'&\multicolumn{1}{c}{}\\\cline{2-4} \multicolumn{2}{c|}{}&3&\multicolumn{2}{c}{}\\\cline{3-3}\end{array}, \begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&8\\\hline \multicolumn{1}{c|}{}&3&6&9&\multicolumn{1}{c}{}\\\cline{2-4} \multicolumn{2}{c|}{}&7&\multicolumn{2}{c}{}\\\cline{3-3}\end{array}\right).$$

Performing jeu de taquin we see:

$$\begin{array}{|c|c|c|c|c|}\hline &&1&2&5\\\hline \multicolumn{1}{c|}{}&&3&6&8\\\cline{2-5} \multicolumn{2}{c|}{}&4&7&9\\\cline{3-5}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline &&1&2&5\\\hline \multicolumn{1}{c|}{}&3&4&6&8\\\cline{2-5} \multicolumn{2}{c|}{}&7&9&\multicolumn{1}{c}{}\\\cline{3-4}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline &1&2&5&8\\\hline \multicolumn{1}{c|}{}&3&4&6&\multicolumn{1}{c}{}\\\cline{2-4} \multicolumn{2}{c|}{}&7&9&\multicolumn{1}{c}{}\\\cline{3-4}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&8\\\hline \multicolumn{1}{c|}{}&3&6&9&\multicolumn{1}{c}{}\\\cline{2-4} \multicolumn{2}{c|}{}&7&\multicolumn{2}{c}{}\\\cline{3-3}\end{array}.$$

Haiman's bijection is precisely $H(Q) = Q'$. That is, given a standard square tableau Q, we embed it in a shifted shape and apply jeu de taquin to create a standard shifted tableau. That this is indeed a bijection follows from Theorem 4.25, but is originally found in [6, Proposition 8.11].

Remark 5.4. Haiman's bijection applies more generally between rectangles and "shifted trapezoids", i.e., for $m \leq n$, we have $H : SYT(n^m) \to SYT'(n+m-1, n+m-3, \ldots, n-m+1)$. All the results presented here extend to this generality, with similar proofs. We restict to squares and doubled staircases for clarity of exposition.

We will now fix the tableaux P and P' to ensure that the insertion word w has particularly nice properties. Recall that Proposition 3.12 states that the set of words that mixed insert into $P' = P_{\text{mix}}(w)$ is contained in the set of words that RSK insert into $P = P_{\text{RSK}}(w)$. We apply this proposition to the word

$$w = \underbrace{n \cdots n}_{n} \cdots \underbrace{2 \cdots 2}_{n}\underbrace{1 \cdots 1}_{n}.$$

If we use RSK insertion, we find P is an $n \times n$ square tableau with all 1s in row first row, all 2s in the second row, and so on. With such a choice of P it is not difficult to show that any other word u inserting to P has the property that for all indices j

and all initial subwords $u_1 \cdots u_i$, there are at least as many letters $(j+1)$ as letters j. Such words are sometimes called (reverse) *lattice words* or (reverse) *Yamanouchi words*. Notice also that any such u has n copies of each letter i, $i = 1, \ldots, n$. We call the words inserting to this choice of P *square words*.

On the other hand, if we use mixed insertion on w, we find P' as follows (with $n = 4$):

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 2' & 3' & 4' \\
\hline
\multicolumn{1}{c|}{} & 2 & 2 & 2 & 3' & 4' & \multicolumn{1}{c}{} \\
\cline{2-6}
\multicolumn{2}{c|}{} & 3 & 3 & 4' & \multicolumn{2}{c}{} \\
\cline{3-5}
\multicolumn{3}{c|}{} & 4 & \multicolumn{3}{c}{} \\
\cline{4-4}
\end{array}\,.$$

In general, on the "shifted half" of the tableau we see all 1s in the first row, all 2s in the second row, and so on. In the "straight half" we see only primed numbers, with $2'$ on the first diagonal, $3'$ on the second diagonal, and so on. Proposition 3.12 tells us that every u that mixed inserts to P' is a square word. But since the sets of recording tableaux for P and for P' are equinumerous, we see that the set of words mixed inserting to P' is precisely the set of all square words.

Remark 5.5. Yamanouchi words give a bijection with standard Young tableaux that circumvents insertion completely. In reading the word from left to right, if $w_i = j$, we put letter i in the leftmost unoccupied position of row $n+1-j$. (See [25, Proposition 7.10.3(d)].)

5.7 Proof of the theorem

The operator e_j acting on words $w = w_1 \cdots w_l$ is defined in the following way. Consider the subword of w formed only by the letters j and $j+1$. Consider every $j+1$ as an opening bracket and every j as a closing bracket, and pair them up accordingly. The remaining word is of the form $j^r(j+1)^s$. The operator e_j leaves all of w invariant, except for this subword, which it changes to $j^{r-1}(j+1)^{s+1}$. (This operator is widely used in the theory of *crystal graphs*.)

As an example, we calculate $e_2(w)$ for the word $w = 3121221332$. The subword formed from the letters 3 and 2 is

$$3 \cdot 2 \cdot 22 \cdot 332,$$

which corresponds to the bracket sequence ()))((). Removing paired brackets, one obtains))(, corresponding to the subword

$$\cdot \cdot \cdot \cdot 22 \cdot 3 \cdot \cdot.$$

We change the last 2 to a 3 and keep the rest of the word unchanged, obtaining $e_2(w) = 31212\mathbf{3}1332$.

The following lemma shows that this operator leaves the recording tableau unchanged. The unshifted case is found in work of Lascoux, Leclerc, and Thibon [12]; the shifted case follows from the unshifted case, and the fact that the mixed recording tableau of a word is uniquely determined by its RSK recording tableau (Proposition 3.12).

Lemma 5.6 ([12] Theorem 5.5.1)**.** *Recording tableaux are invariant under the operators e_i. That is,*

$$Q_{\mathrm{RSK}}(e_i(w)) = Q_{\mathrm{RSK}}(w),$$

and

$$Q_{\mathrm{mix}}(e_i(w)) = Q_{\mathrm{mix}}(w).$$

Let $\overline{e} = e_1 \cdots e_{n-1}$ denote the composite operator given by applying first e_{n-1}, then e_{n-2} and so on. It is clear that if $w = w_1 \cdots w_{n^2}$ is a square word, then $\overline{e}(\widehat{w})1$ is again a square word.

Theorem 5.7. *Let $w = w_1 \cdots w_{n^2}$ be a square word. Then,*

$$p(Q_{\mathrm{RSK}}(w)) = Q_{\mathrm{RSK}}(\overline{e}(\widehat{w})1),$$

and

$$p(Q_{\mathrm{mix}}(w)) = Q_{\mathrm{mix}}(\overline{e}(\widehat{w})1).$$

In other words, Haiman's bijection commutes with promotion:

$$p(H(Q_{\mathrm{RSK}})) = H(p(Q_{\mathrm{RSK}})).$$

Proof. By Lemma 4.34, we see that $Q_{\mathrm{RSK}}(\widehat{w})$ is only one box away from $p(Q_{\mathrm{RSK}}(w))$. Further, repeated application of Lemma 5.6 shows that

$$Q_{\mathrm{RSK}}(\widehat{w}) = Q_{\mathrm{RSK}}(e_{n-1}(\widehat{w})) = Q_{\mathrm{RSK}}(e_{n-2}(e_{n-1}(\widehat{w}))) = \cdots = Q_{\mathrm{RSK}}(\overline{e}(\widehat{w})).$$

The same lemmas apply show $Q_{\mathrm{mix}}(\overline{e}(\widehat{w}))$ is one box away from $p(Q_{\mathrm{mix}}(w))$.

All that remains is to check that the box added by inserting 1 into $P(\overline{e}(\widehat{w}))$ (resp. $P'(\overline{e}(\widehat{w}))$) is in the correct position. But this follows from the observation that $\overline{e}(\widehat{w})1$ is a square word, and square words insert (resp. mixed insert) to squares (resp. doubled staircases). $\square$

Now thanks to Theorem 5.7 we know that H preserves orbits of promotion, and as a consequence we see the CSP for doubled staircases.

Corollary 5.8. *Let $X = SYT'(2n-1, 2n-3, \ldots, 1)$, and let $C := \mathbb{Z}/n^2\mathbb{Z}$ act on X via promotion. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \frac{[n^2]_q!}{[n]_q^n \prod_{i=1}^{n-1} ([i]_q \cdot [2n-i]_q)^i}$$

is the q-analogue of the hook-length formula for an $n \times n$ square Young diagram.

Because of Theorem 5.3 the set $R(w_0)$ also exhibits the CSP.

Corollary 5.9 ([17], Theorem 8.1)**.** *Let $X = R(w_0)$ and let $X(q)$ as in Corollary 5.8. Let $C := \mathbb{Z}/n^2\mathbb{Z}$ act on X by cyclic rotation of words. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon.*

Corollary 5.9 is the CSP for $R(w_0)$ as stated by Rhoades. This is nearly our main result (Theorem 5.1), but for the definition of $X(q)$.

In spirit, for a CSP $(X, C, X(q))$, the polynomial $X(q)$ should be some q-enumerator for the set X. That is, it should be expressible as

$$X(q) = \sum_{x \in X} q^{s(x)},$$

where s is an intrinsically defined statistic for the elements of X. Indeed, nearly all known instances of the cyclic sieving phenomenon have this property. For example, it is known ([25, Cor 7.21.5]) that the q-analogue of the hook-length formula can be expressed as follows:

$$f^\lambda(q) = q^{-\kappa(\lambda)} \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}, \tag{5.1}$$

where $\kappa(\lambda_1, \ldots, \lambda_l) = \sum_{1 \le i \le l} (i-1)\lambda_i$ and for a tableau T, $\mathrm{maj}(T)$ is the sum of all i such that i appears in a row above $i+1$. Thus $X(q)$ in Theorem 5.2 can be described in terms a statistic on Young tableaux.

With this point of view, Corollaries 5.8 and 5.9 are aesthetically unsatisfying. Section 5.8 is given to showing that $X(q)$ can be defined as the generating function for the major index on words in $R(w_0)$. It would be interesting to find a combinatorial description for $X(q)$ in terms of a statistic on $SYT'(2n-1, 2n-3, \ldots, 1)$ as well, though we have no such description at present.

5.8 Combinatorial description of $X(q)$

As stated in the introduction, we will show that

$$X(q) = q^{-n\binom{n}{2}} \sum_{w \in R(w_0)} q^{\mathrm{maj}(w)}.$$

If we specialize (5.1) to square shapes, we see that $\kappa(n^n) = n\binom{n}{2}$ and

$$X(q) = q^{-n\binom{n}{2}} \sum_{T \in SYT(n^n)} q^{\mathrm{maj}(T)}.$$

Thus it suffices to exhibit a bijection between square tableaux and words in $R(w_0)$ that preserves major index. In fact, the composition $\Psi := \Phi H$ has a stronger feature.

Define the *cyclic descent set* of a word $w = w_1 \cdots w_l$ to be the set

$$D(w) = \{i : w_i > w_{i+1}\} \pmod{l}$$

That is, we have descents in the usual way, but also a descent in position 0 if $w_l > w_1$. Then $\mathrm{maj}(w) = \sum_{i \in D(w)} i$. For example with $w = 132132132$, $D(w) = \{0, 2, 3, 5, 6, 8\}$ and $\mathrm{maj}(w) = 0 + 2 + 3 + 5 + 6 + 8 = 24$.

Similarly, we follow [17] in defining the cyclic descent set of a square (in general, rectangular) Young tableau. For T in $SYT(n^n)$, define $D(T)$ to be the set of all i such that i appears in a row above $i + 1$, along with 0 if $n^2 - 1$ is above n^2 in $p(T)$. Major index is $\mathrm{maj}(T) = \sum_{i \in D(T)} i$. We will see that Ψ preserves cyclic descent sets, and hence, major index. Using our earlier example of $w = 132132132$, one can check that

$$T = \Psi^{-1}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 8 \\ \hline 4 & 7 & 9 \\ \hline \end{array}$$

has $D(T) = D(w)$, and so $\mathrm{maj}(T) = \mathrm{maj}(w)$.

Lemma 5.10. *Let $T \in SYT(n^n)$, and let $w = \Psi(T)$ in $R(w_0)$. Then $D(T) = D(w)$.*

Proof. First, we observe that both types of descent sets shift cyclically under their respective actions:

$$D(p(T)) = \{i - 1 \pmod{n^2} : i \in D(T)\},$$

and

$$D(c(w)) = \{i - 1 \pmod{n^2} : i \in D(w)\}.$$

For words under cyclic rotation, this is obvious. For tableaux under promotion, this is a lemma of Rhoades [17, Lemma 3.3].

Because of this cyclic shifting, we see that $i \in D(T)$ if and only if $0 \in D(p^i(T))$. Thus, it suffices to show that $0 \in D(T)$ if and only if $0 \in D(w)$. (Actually, it is easier to determine if $n^2 - 1$ is a descent.)

Let $S = \Phi^{-1}(w)$ be the shifted doubled staircase tableau corresponding to w. We have $n^2 - 1 \in D(w)$ if and only if n^2 is in a higher row in $p^{-1}(S)$ than in S. But since n^2 occupies the same place in $p^{-1}(S)$ as $n^2 - 1$ occupies in S, this is to say $n^2 - 1$ is above n^2 in S. On the other hand, $n^2 - 1 \in D(T)$ if and only if $n^2 - 1$ is above n^2 in T. It is straightforward to check that since S is obtained from T by jeu de taquin into the upper corner, the relative heights of n^2 and $n^2 - 1$ (i.e., whether n^2 is below or not) are the same in S as in T. This completes the proof. □

This lemma yields the desired result for $X(q)$.

Theorem 5.11. *The q-analogue of the hook-length formula for an $n \times n$ square Young diagram is, up to a shift, the major index generating function for reduced expressions of the longest element in B_n:*

$$\sum_{w \in R(w_0)} q^{\mathrm{maj}(w)} = q^{n\binom{n}{2}} \cdot \frac{[n^2]_q!}{[n]_q^n \prod_{i=1}^{n-1} ([i]_q \cdot [2n-i]_q)^i}.$$

Theorem 5.11, along with Corollary 5.9, complete the proof of Theorem 5.1. Because this result can be stated purely in terms of the set $R(w_0)$ and a natural statistic on this set, it would be interesting to obtain a self-contained proof, i.e., one that does not appeal to Haiman's or Rhoades' work. More specifically, a proof that doesn't rely on promotion on Young tableaux.

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