ENGINEERING RESEARCH INSTITUTE
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THEORY OF FREQUENCY MODULATION NOISE

IN TUBES EMPLOYING PHASE FOCUSING

Technical Report No. 28
Electronic Defense Group
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Project 1970
TASK ORDER NO. EDG-7
CONTRACT NO. DA-36-039 sc-15358
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-04-042
SIGNAL CORPS PROJECT NO. 29-194B-0

March, 1954

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ABSTRACT

Oscillators employing phase focusing such as carcinotrons and magnetrons have fairly well defined spokes whose rotational speed is proportional to the oscillation frequency. Each spoke is composed of a finite number of electrons having random velocities and consequently is subject to random fluctuations about a mean position. This spoke "jitter" leads to fluctuations in the oscillation frequency which amounts to frequency modulation noise. It is the theoretical evaluation of the parameters of this noise as affecting the power spectrum of the oscillator output that is of interest here. The theoretical results compare favorably with the measured power spectrum of a voltage-tunable magnetron. The application of the formulae require an estimate of electron temperature - no attempt is made to evaluate this temperature. Amplitude fluctuation noise appears to be relatively unimportant in continuous-wave phase-focused oscillators - the evaluation of such noise is not undertaken here.

THEORY OF FREQUENCY MODULATION NOISE IN TUBES EMPLOYING PHASE FOCUSING

1. INTRODUCTION

It has been observed that two types of noise exist in the output of ıgnetrons and other tubes employing phase focusing (Ref. 1). As would be ‹pected, amplitude-fluctuation noise is present due to shot and flicker effects. ı addition to this, there exists frequency modulation noise, part of which is orrelated to the amplitude noise. The study here is restricted to the FM noise, ıd, in particular, to that component of FM noise introduced because the charge ı the tube[1] consists of discrete particles having random velocities. There ‹ists a second component of FM noise due to flicker and other effects at the ıthode which will not be analyzed here.

As necessary background, some results of a previous analysis will be ated (Ref. 2). If a carrier is modulated in frequency with rectangular band of ussian noise extending from a radian frequency of zero to B such that the stantaneous frequency deviation of the carrier is directly proportional to the plitude of the noise, then the power spectrum of the resulting modulated

Magnetrons will be considered specifically because more data are available.

carrier will have the asymptotic forms

$$W(\omega) = \begin{cases} \dfrac{1}{\sqrt{2\pi D^2}} \exp\left[\dfrac{(\omega - \omega_o)^2}{2D^2}\right] & \dfrac{D}{B} >> 1 \quad (1) \\ \left(\dfrac{1}{\pi B}\right) \cdot \left[\dfrac{\pi D^2/2B^2}{(\pi D^2/2B^2)^2 + (\omega - \omega_o)^2/B^2}\right] & \dfrac{D}{B} << 1 \quad (2) \end{cases}$$

where ω_o is the carrier frequency and D is the rms frequency deviation. The thr decibel half bandwidth of these power spectra are

$$B_F = \sqrt{2 \log 2}\; D = 1.18D \qquad \frac{D}{B} >> 1 \quad (3$$

$$B_F = \frac{\pi D^2}{2B} = 1.57D^2/B \qquad \frac{D}{B} << 1 \quad (4$$

For small D/B, the power spectrum falls off quite slowly with the difference frequency $\omega - \omega_o$. It should be noted that noise having such a power spectrum might have severe consequences if applicable to the local oscillator tub in a superheterodyne receiver.

2. THE MECHANISM OF FM NOISE

In a magnetron, the space charge exists in the form of spokes that rota at an angular velocity ω_r.[2] If the magnetron operates in the π mode and has N anode segments, there are N/2 spokes and

$$\omega_o = \frac{N}{2}\omega_r \quad (5)$$

where ω_o is the generated frequency in radians per second.

The current induced in the anode segments is due mainly to the rotationa

2. Data related to the analysis of magnetrons, and references to magnetrons in general, will be found in Ref. 3.

ıotion of the spokes, their radial motion having only a second-order effect which ·ill be neglected here. The electrons that comprise the spoke were at one time ·mitted from the cathode with random velocities. The probability density function ıf the velocity tangential to the cathode is Gaussian with a variance kT/m (where . is Boltzmann's constant, T is temperature, and m is the mass of the electron).[3]

The random nature of the tangential emission velocities causes the ·elocities of the electrons in the spokes to have a Gaussian probability density unction (at least approximately) with a mean related to the spoke rotational ·elocity ω_r. Since a space charge at the cathode affects only the normal omponents of velocity, the condition of the spokes with respect to the random otational velocities of the individual electrons will be the same for either pace-charge-limited or temperature-limited emission.

Although even the approximate variance of the velocity distribution of lectrons in the cathode-anode region is not known, it is believed that due to the lectron-electron interaction, growing wave amplification, and the relatively igh random velocities of the secondary electrons, the mean-square electron andom velocity between anode and cathode is very much larger than that at the athode. This random motion can be specified in terms of a temperature. Although he temperature of the electrons upon emission is only about 1000°K, in the nteraction space between cathode and anode it may be on the order of millions of egrees Kelvin.[4]

There are a certain number of electrons in the spokes at any given nstant. It is the average rotational velocity of all these electrons, suitably

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3. All units are in the M.K.S. system.
4. Guénard and Huber (Ref.4) report temperatures on the order of 10^5 degrees Kelvin in non-oscillating tubes. Although no published data seem to be available, it would appear that temperatures in the oscillating magnetron are much larger than this.

weighted in accordance with their relative effectiveness in inducing anode current, that gives the instantaneous frequency of the magnetron. New electrons a: continually being removed at the anode and entering the spokes at the cathode; from this it can be concluded that the instantaneous frequency will vary in an approximately Gaussian fashion about the average frequency. The correlation function $R(\tau)$ of these frequency variations will decrease to zero at $\tau = T$, where T is the cathode-to-anode transit time of the electrons.

Due to the averaging effect of the large number of electrons in the spokes at any instant, "jitter" of the spokes will be much smaller than that accountable to a single electron. However, the small transit time indicates tha the rate of change of instantaneous frequency is rather high. Consequently, it is apparent that the output spectrum of the magnetron will have the form of Eq (for D/B very small).

The calculation of the equivalent D and B, and especially of the half bandwidth B_F, for the type of noise described above are of primary concern here. There appears to be no reason why any correlation should be expected between this type of FM noise and amplitude-fluctuation noise.

Flicker effect at the cathode and amplitude-fluctuation noise impressed upon the various power supply voltages introduces a second kind of FM noise. Variations in the conditions of the electrons in the magnetron in this case occur rather slowly; hence, the equivalent modulator bandwidth B is quite small. On the other hand, the magnitude of these disturbances may be relatively large; therefore, it would be expected that the equivalent deviation D is large. Thus, D/B is large and the shape of the power spectrum would appear to be that of Eq 1 It is this spectrum that would be observed on the presentation of a high resolution spectrum analyzer tuned very near to the carrier frequency. This type of oise is relatively unimportant for magnetrons used as local oscillators ecause of the rapidity with which the spectrum falls off with the difference requency $\omega - \omega_o$. As contrasted to the other type of FM noise, there is every eason to expect a relatively high correlation between this type of noise and mplitude-fluctuation noise.

3. INDUCED CURRENT IN THE MAGNETRON

Consider Fig. 1 which is a diagram of spokes in a magnetron. An mpedance Z is connected between the positive and negative anode segments, and lternate segments are connected together. The induced current flows through the mpedance Z resulting in a voltage between adjacent anode segments. The magnetron ill be assumed axially symmetric and of length L.

Let it be assumed that the total charge in the spokes is constant and hat all induced current is caused by rotation of the spokes rather than by the adial motion of the electrons. Then, the space charge density function of the pokes and the function ψ_k, giving indirectly the potential of the field in the athode-anode region, can be expanded into Fourier series. ψ_k is something ike a Greens function, being unity at the electrode through which a flux alculation is desired, and zero on all other electrodes.

$$\psi_k = \sum_k A_k \cos \frac{kN\theta}{2} \tag{6}$$

$$\rho = \sum_j \left[B_j \cos \frac{jN\theta'}{2} + C_j \sin \frac{jN\theta'}{2} \right] \tag{7}$$

The angle θ is the angle in a fixed coordinate system, and θ' is the ngle in a system rotating with the spokes. For convenience, $\theta' = 0$ is assumed o correspond to the center line of a spoke. Also, it is assumed that at the

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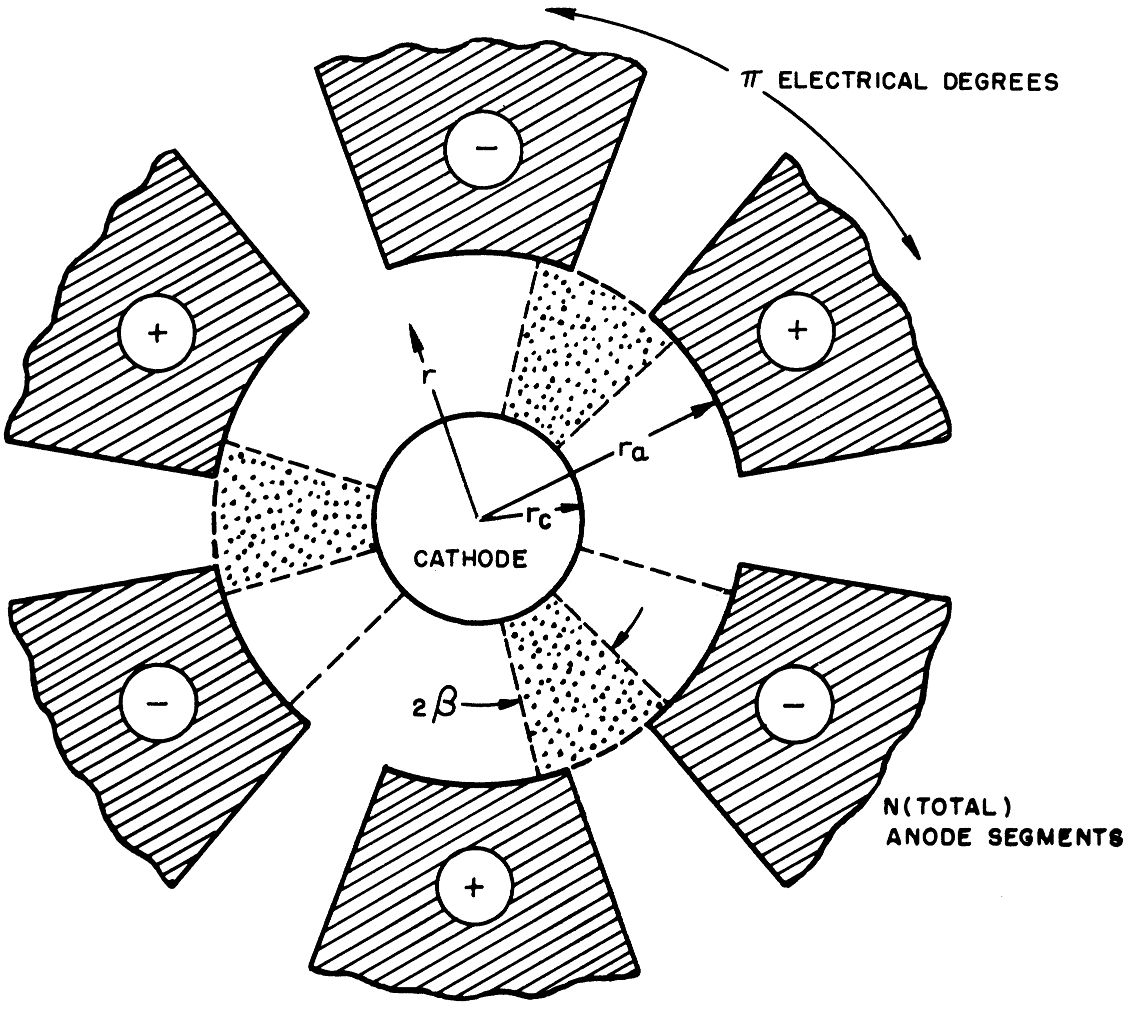

FIG. 1

CROSS SECTION OF THE MAGNETRON.

reference time t = 0, $\theta = \theta'$. Thus,

$$\theta' = \theta - \omega_r t \tag{8}$$

The displacement current induced in the anode circuit is given by Welch (Ref. 3).[5]

$$i = \frac{\partial}{\partial t} \int_{-\pi}^{\pi} \int_{r_c}^{r_a} \psi_k \, \rho \, L r dr d\theta \tag{9}$$

Substituting Equations 6 and 7 in Eq 9 (observing that because the coefficients are independent of θ', orthogonality relations exist, causing all terms for $k \neq j$ to vanish), there is obtained

$$i = L \frac{\partial}{\partial t} \sum_k \left[\int_{-\pi}^{\pi} \cos \frac{kN\theta}{2} \cos \frac{kN\theta'}{2} \int_{r_c}^{r_a} A_k B_k \, r dr + \int_{-\pi}^{\pi} \cos \frac{kN\theta}{2} \sin \frac{kN\theta'}{2} \int_{r_c}^{r_a} A_k C_k r dr \right] \tag{10}$$

Simplifying, substituting for θ', and integrating over θ results in

$$i = L\pi \frac{\partial}{\partial t} \sum_k \left[\cos \frac{kN\omega_r t}{2} \int_{r_c}^{r_a} A_k B_k r dr - \sin \frac{kN\omega_r t}{2} \int_{r_c}^{r_a} A_k C_k r dr \right] \tag{11}$$

The generated frequency is related to the rotation frequency according to Eq 5. Thus, the argument of the sine and cosine terms is $k\omega_o t$. The most important term of Eq 11 is the fundamental. Thus, for k = 1, and after differentiating with respect to time,

$$i_1 = -L\pi\omega_o \left[\sin \omega_o t \int_{r_c}^{r_a} A_1 B_1 r dr + \cos \omega_o t \int_{r_c}^{r_a} A_1 C_1 r dr \right] \tag{12}$$

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[5] The lower limit r_c can be interpreted as either the radius of the cathode or the radius of the sub-synchronous swarm, whichever is applicable.

The magnitude of i_1 is given by the square root of the sum of the squares of the quadrature components, or

$$\left|i_1\right| = \frac{2\bar{q}\,\omega_o}{\pi}\left[\left(\int_{r_c}^{r_a}\frac{\pi A_1 \pi B_1}{2q}\,Lrdr\right)^2 + \left(\int_{r_c}^{r_a}\frac{\pi A_1 \pi C_1}{2q}\,Lrdr\right)^2\right]^{1/2} \quad (13)$$

where the total charge in the spokes q has been introduced and where the equation has been written in this form because it has been proven that the maximum possible value of the square root of Eq 13 is unity (Ref. 5). Thus, $2q\,\omega_o/\pi$ is the maximum fundamental current that can be induced in the magnetron under any circumstances.

When the spokes are symmetrically located around the cathode, all the C_k are zero and Eq 13 takes the simple form

$$\left|i_1\right| = \pi L\,\omega_o \int_{r_c}^{r_a} A_1 B_1 r dr \quad (14)$$

If it is assumed that the space charge has a negligible effect upon the potential in the anode-cathode region by comparison with that caused by the voltage between anode segments, then the coefficients A_k are functions only of the radius r, and are obtainable through a solution of Laplace's equation in two dimensions. Consequently, it is permissible to assume a normalized function A(u) such that

$$A_1 = A_m A(u) \qquad A(1) = 1 \quad (15)$$

in which A_m is the value of A_1 at $r = r_a$ (which is generally the maximum value of A_1), and where the variable u is the normalized radius r/r_a.

The coefficients B_k and C_k are defined by the usual Fourier series formula **as**

$$B_k = \frac{N}{2\pi}\int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} \rho\,(r,\theta')\cos\frac{kN\theta'}{2}\,d\theta' \quad (16)$$

here $\rho(r,\theta')$ is the space-charge function. Equation 7 gives the function $\rho(r,\theta')$ n terms of the B_k and C_k (if they are not zero).

Let it be assumed that each spoke has a uniform charge density in $\beta < \theta' < \beta$ and zero charge density for $|\theta'| > \beta$; then,

$$\rho(r,\theta') = \rho(r)\ \rho(\theta') \tag{17}$$

nd

$$\int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} \rho(\theta')d\theta' = \int_{-\beta}^{\beta} \rho(\theta')d\theta' = 2\beta \tag{18}$$

The above assumption may be highly unrealistic in some cases. However, ome equivalent rectangular spoke can usually be hypothesized and some equivalent obtained.

Using Equations 15, 16, 17, and 18 in Eq 14, and integrating, there is btained

$$|i_1| = N\beta L\,\omega_o \left(\frac{\sin \frac{N\beta}{2}}{\frac{N\beta}{2}} \right) \int_{r_c}^{r_a} A_m A(u)\,\rho(r) r dr \tag{19}$$

It will be assumed that ρ is the charge density of one spoke; hence, f V is the volume of one spoke and q is the total charge in the magnetron

$$\int_V \rho(r,\theta')\,dV = q\left(\frac{2}{N}\right) \tag{20}$$

Since $dV = L\ r\ dr\ d\theta'$, Eq 20 becomes

$$\int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} \int_{r_c}^{r_a} \rho(r,\theta')\,Lrdrd\theta' = 2L\beta \int_{r_c}^{r_a} \rho(r)rdr = q\,\frac{N}{2} \tag{21}$$

from which

$$\int_{r_c}^{r_a} \rho(r) r dr = \frac{q}{L\beta N} = r_a^2 \int_{r_c/r_a}^{1} u \, \rho(u) \, du \qquad (22)$$

The function $\rho(r)$ gives the manner in which the electron density varies with r. This can be normalized to a function B(r) and $B(r/r_a)$ as

$$\rho(r) = \frac{qB(r)}{L\beta N} \qquad (23)$$

$$\rho(u) = \frac{qB(u)}{L\beta N r_a^2}$$

such that

$$\int_{r_c}^{r_a} rB(r)dr = \int_{r_c/r_a}^{1} uB(u)du = 1 \qquad (24)$$

If Eq 23 is substituted in an integral and integration is performed over B(u), only the volume of one spoke need be considered in order to include all N/2 spokes.

Using Equation 23 in Equation 19, there is obtained the final expression for induced current,

$$\left| i_1 \right| = A_m \omega_o q \left(\frac{\sin \frac{N\beta}{2}}{\frac{N\beta}{2}} \right) \int_{r_c/r_a}^{1} uA(u)B(u)du \qquad (25)$$

The maximum (but unrealizable) current is given when the spokes are very narrow and the total charge is located very close to the anode. Then A(u) = 1 and the integral over B(u) is unity giving

$$\left| i_1 \right|_{max} = A_m \omega_o q \qquad (26)$$

As a specific approximation, assume that B(u) varies as 1/u; the onstant of proportionality is found from Eq 24 as $1/(1 - r_c/r_a)$. Also, assume hat A(u) varies as u^k where k is some positive number. Then, Eq 25 becomes

$$\left| i_1 \right| = \frac{A_m}{k+1} \omega_o q \left(\frac{\sin \frac{N\beta}{2}}{\frac{N\beta}{2}} \right) \frac{1 - (r_c/r_a)^{k+1}}{1 -- r_c/r_a} \qquad (27)$$

hich is approximately

$$\left| i_1 \right| \approx \frac{A_m}{k+1} \omega_o q \qquad (28)$$

4. ELECTRONS IN VOLUME ELEMENTS

The charge dq in the volume element dV (which includes the correspond-ing element in every spoke) is found from Eq 21 as

$$\frac{N}{2} \rho(r,\theta') \, dV = dq \qquad (29)$$

But $\rho(r,\theta') = \rho(r)$ as given by Eq 23 and $dq = n'e$ where n' is the umber of electrons in dV. Thus,

$$n' = \frac{qB(u)dV}{2L\beta e r_a^2} = \frac{qB(r)dV}{2L\beta e} \qquad (30)$$

The partial current $\left| i_p \right|$ due to the charge in dV can be found via a ubstitution for A_1 and B_1 in Eq 14 as

$$\left| i_1 \right| = \pi L \omega_o \int_{r_c}^{r_a} A_m A(u) \frac{N}{2\pi} \int_{-\beta}^{\beta} \rho(r) \cos \frac{N\theta'}{2} \, d\theta' \, r dr \qquad (31)$$

Substituting for $\rho(r)$ and considering only the volume element,

$$\left| i_p \right| = \frac{A_m \omega_o q}{2L\beta} \left(\cos \frac{N\theta'}{2} \right) A(u)B(r)dV \tag{32}$$

Substituting for B(r) in Eq 32 results in

$$\left| i_p \right| = A_m \omega_o e \left(\cos \frac{N\theta'}{2} \right) A(u)n' = \left| i_p \right|_{max} A(u) \cos \frac{N\theta'}{2} \tag{33}$$

where $\left| i_p \right|_{max}$ is the value of Eq 33 at $\theta' = 0$ and $r = r_a$. Thus, the relative effectiveness of any given electron in inducing current in the anode is proportional to A(u) and to $\cos(N\theta'/2)$ as expected.

5. MEAN-SQUARE DEVIATION OF FM NOISE

The probability density function of the transverse velocity of one electron is (at least approximately) Gaussian with a variance kT/m. At a radius r, the velocity v is $\omega_r = v/r$. Thus, the variance of the density function for ω_r is $kT/(mr^2)$. In one revolution of the spoke, there are N/2 cycles of the carrier; hence, the variance of the generated frequency due to a single electron is

$$\overline{\Delta\omega_e^2} = \left(\frac{N}{2} \right)^2 \frac{kT}{mr^2} \tag{34}$$

which can be considered to refer to an electron in the most favorable position for inducing current in the anode. For any other electron, Eq 34 must be multiplied by the square of the relative effectiveness of the electron.

The variance due to the n' electrons in the volume element dV is less than that of a single electron. In order to obtain the appropriate average, q 34 must be divided by n'. Thus,

$$\overline{\Delta\omega_{n'}^{\,2}} = \left(\frac{N}{2}\right)^2 \left(\frac{kT}{mr^2}\right) \frac{\left[A(r)\cos\frac{N\theta'}{2}\right]^2}{n'} \tag{35}$$

Substituting for n' from Eq 30,

$$\frac{1}{\overline{\Delta\,\omega_{n'}^{\,2}}} = \left(\frac{2}{N}\right)^2 \left(\frac{mr^2}{kT}\right) \frac{qB(r)dV}{\left[A(r)\cos\frac{N\theta'}{2}\right]^2 2L\beta e} \tag{36}$$

The average for all electrons in the magnetron is obtained by summing q 36 for the n' electrons in each volume element dV. This integration yields he reciprocal of the variance D^2 of the FM noise as

$$\frac{1}{D^2} = \left(\frac{2}{N}\right)^2 \frac{mq}{2\beta ekT}\left(\int_{-\beta}^{\beta}\frac{d\theta'}{\cos^2\frac{N\theta'}{2}}\right)\left(\int_{r_c}^{r_a}\frac{r^3B(r)}{A^2(r)}\,dr\right) \tag{37}$$

$$= \left(\frac{2}{N}\right)^3 \frac{mq}{\beta ekT} \tan\frac{N\beta}{2}\int_{r_c}^{r_a}\frac{r^3B(r)}{A^2(r)}\,dr$$

Inverting and substituting for q from Eq 25,

$$D^2 = \left(\frac{N}{2}\right)^2 \frac{ekT\omega_o}{m|i_1|r_a^2} \cos\frac{N\beta}{2}\, F\left(\frac{r_c}{r_a}\right) \tag{38}$$

here $F(r_c/r_a)$ is a geometrical constant given by

$$F\left(\frac{r_c}{r_a}\right) = \frac{\int_{r_c/r_a}^{1} uA(u)B(u)du}{\int_{r_c/r_a}^{1}\frac{u^3B(u)}{A^2(u)}\,du} \tag{39}$$

In order to have some (albeit rough) estimate of $F(r_c/r_a)$, assume B(u) varies as 1/u as before and that $A(u) = u^k$. Then,

$$F(r_c/r_a) \cong \begin{cases} \left(\frac{3-2k}{k+1}\right) \frac{1 - (r_c/r_a)^{k+1}}{1 - (r_c/r_a)^{3-2k}} & k \neq 1.5 \\ \frac{1 - (r_c/r_a)^{2.5}}{2.5 \log (r_a/r_c)} & k = 1.5 \end{cases} \tag{40}$$

which is plotted as a function of r_c/r_a for several values of k in Fig. 2.

Except for the induced current, all quantities in Eq 39 can be specified for a given magnetron. However, if the power of the oscillations of the tube is known, the quantity $|i_1|$ can at least be estimated. Let R be the shunt resistance of the loaded cavity. Then, the power P is given approximately by

$$P = \frac{|i_1|^2 R}{2} \tag{41}$$

from which

$$|i_1| = (2P/R)^{1/2} \tag{42}$$

If the induced current is not in phase with the voltage between spokes, some inaccuracies in using the above relation will result. However, the error will not be large and can be taken into account (approximately) in many cases.

6. ESTIMATE OF NOISE BANDWIDTH

It would appear feasible to obtain an analytic expression for the bandwidth of the noise by setting up a picture of the electrons moving from cathode to anode and computing a correlation function for the instantaneous frequency. However, many of the parameters in the magnetron are known only vaguely; in particular, the transit time. Thus, rather than obtaining an

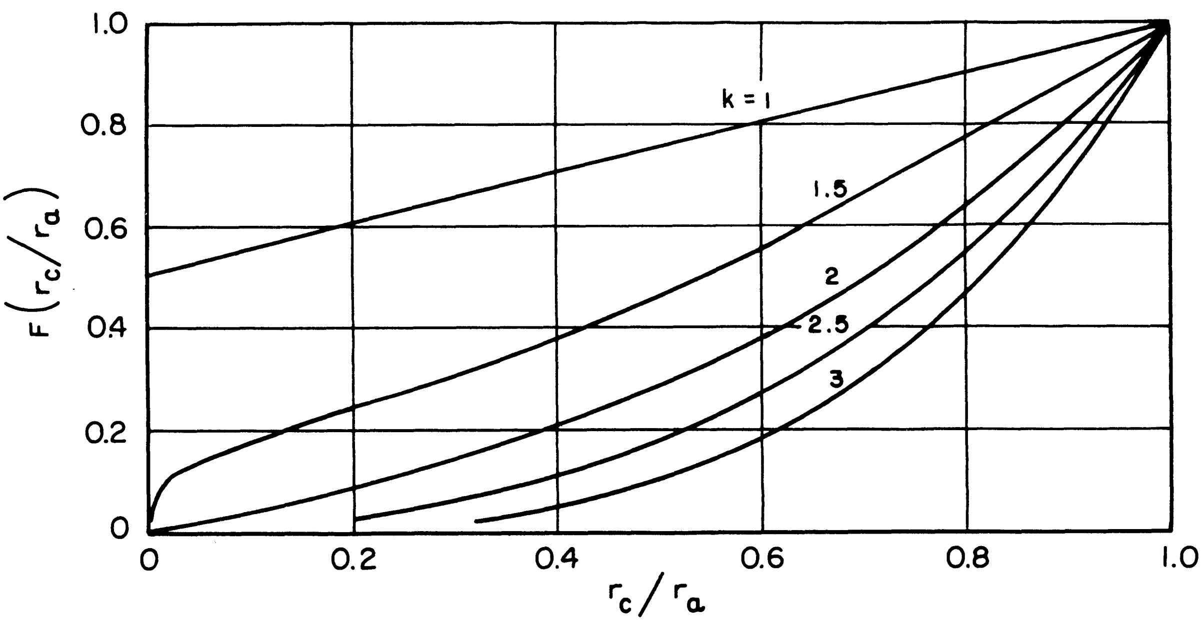

FIG. 2
GEOMETRIC CONSTANT FOR CALCULATING DEVIATION.

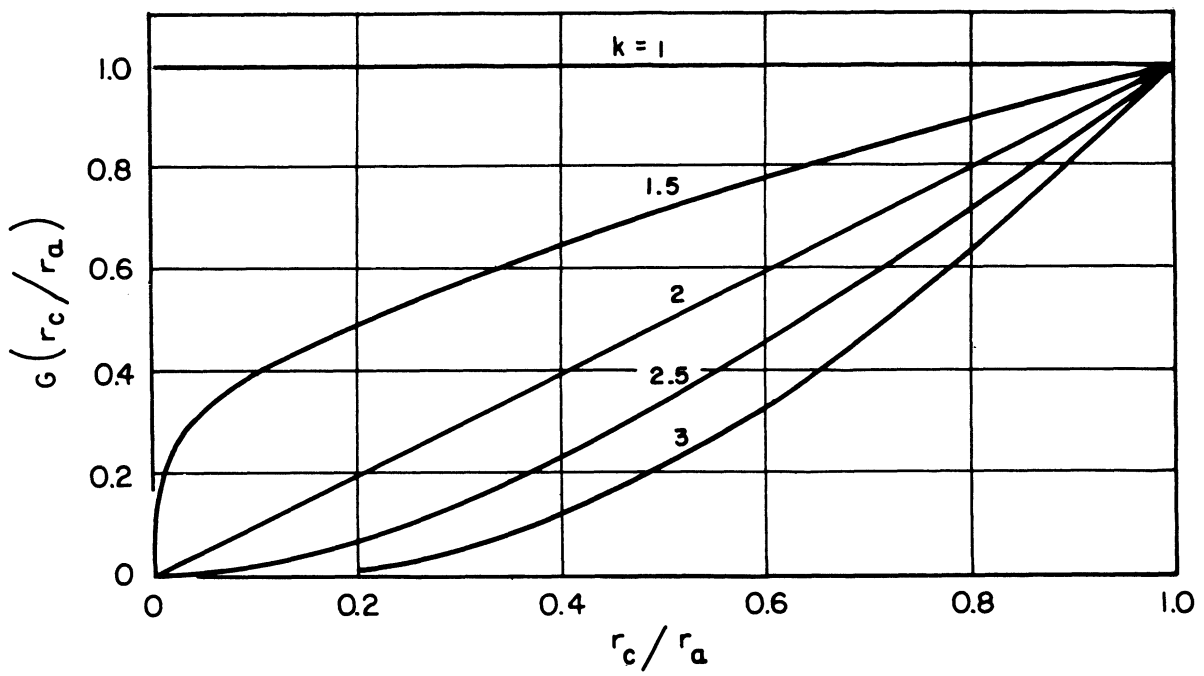

FIG. 3
GEOMETRIC CONSTANT FOR CALCULATING SPECTRUM WIDTH.

analytical solution, an estimate will be made.

If T' is the transit time, the correlation function $R(\tau)$ is zero at $\tau = T'$. In order to get a simple estimate, it will be assumed that

$$R(\tau) = \begin{cases} 1 - \left|\frac{\tau}{T'}\right| , & \left|\frac{\tau}{T'}\right| \lessgtr 1 \\ 0 , & \left|\frac{\tau}{T'}\right| > 1 \end{cases} \qquad (43)$$

The power spectrum of the equivalent low-pass modulator producing the FM noise can be calculated from this assumption as

$$W(\omega) = 4\int_0^{\infty} R(\tau) \cos \omega\tau \, d\tau = T'\left(\frac{\sin \frac{\omega T'}{2}}{\frac{\omega T'}{2}}\right)^2 \qquad (44)$$

The half-power radian bandwidth of this power spectrum is

$$B = \frac{2.74}{T'} = \frac{C_B}{T'} \qquad (45)$$

Thus, for the approximate correlation function assumed, a calculation of the bandwidth has been reduced to a calculation of the transit time.

Because the electrons most effective in inducing current are fairly close to the anode, the bandwidth given above is probably too small. If one assumes a C_B of about three or four the error should not be excessive.

Let it be assumed that the radial velocity of the electrons is constant. Then, it is possible to relate the transit time to the dc anode current I_o as

$$\text{Electrons/second} = 2\pi \times 10^{18} I_o = \frac{n}{T'} = \frac{q}{T'e} \qquad (46)$$

where n is the number of electrons in the magnetron at any one instant.

Thus

$$T' = \frac{q}{2\pi \times 10^{18} I_o e} \tag{47}$$

The value of the total charge q is substituted into this expression from Eq 25 to give

$$B = \frac{C_B}{T'} = \frac{2\pi \times 10^{18} I_o e \ \omega_o C_B}{\left| i_1 \right|} \left(\frac{\sin \frac{N\beta}{2}}{\frac{N\beta}{2}} \right) \int_{r_c/r_a}^{1} uA(u)B(u)du \tag{48}$$

Assuming that B(u) varies as 1/u and that $A(u) = u^k$, an approximate expression can be obtained as

$$B \cong \frac{2\pi \times 10^{18} I_o e \ \omega_o C_B}{(k+1) \left| i_1 \right|} \left(\frac{\sin \frac{N\beta}{2}}{\frac{N\beta}{2}} \right) \frac{1 - (r_c/r_a)^{k+1}}{1 - (r_c/r_a)} \tag{49}$$

which is very approximately

$$B \approx \frac{2\pi \times 10^{18} I_o e \ \omega_o C_B}{(k+1) \left| i_1 \right|} \tag{50}$$

7. HALF BANDWIDTH OF THE MAGNETRON SPECTRUM

Relative to unity at the center frequency, the power spectrum of the magnetron output is

$$W_M(\Delta\omega) = \frac{B_F^2}{B_F^2 + \Delta\omega^2} \tag{51}$$

where B_F is the half bandwidth given by

$$B_F = \frac{\pi D^2}{2B} \tag{52}$$

If D^2 from Eq 38 and B from Eq 48 are substituted into Eq 52, the bandwidth B_F will be neither a function of the induced current $|i_1|$ nor the generated frequency ω_o.

$$B_F = \left(\frac{N}{2}\right)^2 \frac{kT}{4 \times 10^{18} I_o m C_B r_a^2} \left(\frac{\frac{N\beta}{2}}{\tan\frac{N\beta}{2}}\right) G(r_c/r_a) \tag{53}$$

The geometric constant $G(r_c/r_a)$ is given by

$$G(r_c/r_a) = \frac{1}{\int_{r_c/r_a}^{1} \frac{u^3 B(u)}{A^2(u)}\, du} \tag{54}$$

Using the approximations often used before,

$$G(r_c/r_a) = \begin{cases} \dfrac{(1 - r_c/r_a)(3 - 2k)}{1 - (r_c/r_a)^{3-2k}} & k \neq 1.5 \\[2ex] \dfrac{1 - r_c/r_a}{\log (r_a/r_c)} & k = 1.5 \end{cases} \tag{55}$$

which is plotted in Fig. 3 as a function of r_c/r_a for various values of k. For the special value k = 2, $G(r_c/r_a) = r_c/r_a$.

The function of the angle β in Eq 53 will usually be quite close to unity and can be approximated as such. Assuming the angle factor is unity and k = 2, Eq 54 takes the particularly simple form

$$B_F \approx \frac{(N/2)^2 (r_c/r_a) kT}{4 \times 10^{18} I_o m C_B r_a^2} \tag{56}$$

8. AN APPLICATION TO A VOLTAGE TUNABLE MAGNETRON

The application of the foregoing theory can be made to a small voltage unable magnetron of the type developed over recent years at the University of ichigan Tube Laboratory. Approximated dimensions and operating conditions are aken as follows:

$N/2 = 6$

$r_a/r_c = 0.7$

$r_a = 0.005$ meters

$I_o = 0.01$ amperes

It will be assumed that

$T = 2.2\text{x}10^6$ °K (200 electron volts)

$C_B = 3$

$G(r_c/r_a) = 0.8$

Angle factor = 1.0

Using Eq 53, B_F is found to be 317 radians per second, or 50.6 cycles r second. At difference frequencies of $0.1\text{x}10^6$, 10^6, and $10\text{x}10^6$ cycles per cond, the spectrum is down about 66, 86, and 106 decibels respectively, from at at $\Delta\omega = 0$. These figures compare favorably with observed values. The ape of the spectrum has repeatedly been observed to follow the "single-tuned" sponse pattern.

It may be of interest to calculate the deviation D^2 and bandwidth B for e example. For this, Eq 38 can be used. It will be assumed that

$P = 0.2$ watts

$R = 50$ ohms

$\omega_o = 18\text{x}10^9$ radians per second

Assume $\cos (N\beta/2)F(r_c/r_a) = 0.4$

The value of induced current $|i_1|$ is found to be 0.09 amperes. Then, $D^2 = 61 \times 10^{10}$ or D = 780,000 radians per second which is 124 kilocycles per second.

Since B_F is related to D^2 and B by Eq 4, the values of B_F and D^2 calculated for the example can be used to find $B = 3020 \times 10^6$ radians per second, or 481 megacycles per second.

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