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THE POWER SPECTRA OF SIGNALS PHASE AND FREQUENCY
MODULATED BY GAUSSIAN NOISE

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ABSTRACT

General expressions for the power spectra of
carriers frequency and phase modulated by Gaussian noise
are obtained as functions of the shapes of the power
spectra of the modulating voltages. Specific asymptotic
(closed-form) expressions are obtained for the special
case when the modulating voltage has a rectangular low-
pass power spectrum. This particular case corresponds
to the use of maximally flat and similar video amplifier
tuning schemes. The half-power bandwidths of the power
spectra of the modulated carriers are obtained from these
asymptotic expressions and are plotted as functions of
the root-mean-square frequency and phase deviations. The
delta function at the carrier frequency (representing
power remaining in the carrier) is evaluated for phase
and frequency modulation where, in the latter case, the
delta function exists only if the power spectrum of the
modulating voltage does not extend to zero frequency.

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1. INTRODUCTION

A knowledge of the power spectra of phase-and frequency-modulated signals is of considerable interest in the design of networks intended to accommodate these signals - in particular, when the modulating voltage consists of random Gaussian noise. In addition to interest in such signals per se, Gaussian noise is often a reasonable "signal" to assume as a replacement for many types of very complex modulating waves such as those of voice and television.

It should not be inferred that a knowledge of the power spectrum of a frequency-or phase-modulated carrier tells a great deal. In fact, the power spectrum of FM or ∅M yields much less (relative) data than that of an amplitude-modulated wave. For example, if the frequency deviation of an FM signal is larger than the bandwidth of the modulating voltage, the shape of the spectrum is essentially independent of the bandwidth of the modulating signal. Yet, a knowledge of whether the bandwidth of the modulating signal is, say, one megacycle per second rather than one cycle per year is of obvious importance.

Nevertheless, an important piece (albeit incomplete) of information relating to frequency-or phase-modulated signals (modulated by Gaussian noise) is the power spectrum. It is the spectrum of such signals that is of concern here.

The problem solved in this report was solved (in essence) by Middleton[*]. The theoretical approach given here is a modification of Middleton's (suggested to the author by Professor Hok of this laboratory) and the resulting answers apply to somewhat different situations. Middleton did not treat the case of a rectangular modulating signal power spectrum - he solved for the Gaussian case. Actually, the rectangular spectrum more closely corresponds to the use of maximally flat, shunt-peaked, and similar amplifier tuning schemes. Middleton presented his results in the form of power spectrum curves computed from infinite series. Here, asymptotic (closed form) expressions are obtained with the half-power bandwidth of the power spectrum ultimately appearing graphically.

2. SUMMARY OF RESULTS

The power spectrum and autocorrelation function of a time variable are Fourier pairs; thus, if one is known, the other is completely specified.

In the present analysis, the time function of the frequency modulated wave is used directly to find the autocorrelation function of the modulated carrier in terms of the autocorrelation function of the modulating voltage. By Fourier inversion, there results an expression for the power spectrum of the frequency-

[*] D. Middleton, "On the Distribution of Energy in Noise-and Signal-Modulated Waves", Cruft Laboratory, Harvard University, Cambridge, Massachusetts, Technical Report No. 99, March 1, 1950.

modulated signal in terms of the power spectrum of the normally distributed modulating signal. A similar expression is obtained for phase modulation.

Once the general expressions have been derived, the special cases may be inserted and the equations solved.

The power spectrum of a frequency-modulated wave is given by

$$W_F(\Delta\omega) = \begin{cases} \dfrac{A_0^2/2}{\left[2\pi D_0^2\right]^{1/2}} \exp\left(-\Delta\omega^2/2D_0^2\right), & D_0/B \gg 1 \\[3em] \dfrac{A_0^2/2}{\pi B} \cdot \dfrac{\pi D_0^2/2B^2}{(\pi D_0^2/2B^2)^2+(\Delta\omega/B)^2}, & D_0/B \ll 1 \end{cases} \qquad (2.01)$$

where

A_0 = peak amplitude of carrier voltage (assuming a one ohm impedance level),

D_0^2 = mean-squared instantaneous radian frequency deviation (proportional to the mean-squared modulating voltage),

$\Delta\omega$ = radian difference frequency from the unmodulated carrier frequency,

B = radian bandwidth of the modulating voltage.

The power spectrum of the modulating voltage is assumed to be constant from a frequency of zero to B and zero for all higher frequencies.

The two asymptotic expressions can be set equal to $A_o^2/4$ to define a particular $\Delta\omega$ at the relative half-power point of the power spectrum of the modulated wave. This value is $\Delta\omega = B_F$ which is obtained from the two asymptotic expressions and is plotted in Fig. 1. The dotted line in this figure shows the general way that the bandwidth actually varies (the asymptotic expressions are quite approximate for some values of D_o/B).

The power spectrum for $D_o/B \gg 1$ has the familiar Gaussian shape with variance D_o^2. For $D_o/B \ll 1$ it is that of a single-tuned resonant circuit fed with white noise.

If the spectrum of the modulating voltage is not constant to zero frequency but cuts off at a frequency of B_1 (where B_1 is much smaller than the upper cut-off frequency B_2), there exists a delta-function in the power spectrum of the frequency-modulated signal as well as the continuous spectrum. This delta function singularity is given by

$$W_F'(\Delta\omega) = (A_o^2/2)\ \exp\ (-D_o^2/B_1 B_2)\ \delta(\Delta\omega) \tag{2.02}$$

where $\delta(\Delta\omega)$ is the delta function at $\Delta\omega = 0$. For $B_1 B_2 \ll B_2^2$, the power in the delta function is very small and can be neglected - it is generally too small to appreciably affect the magnitude of the continuous part of the power spectrum.

Similar asymptotic expressions can be obtained for the power spectrum of a phase-modulated wave. For large D_o, the spectrum is
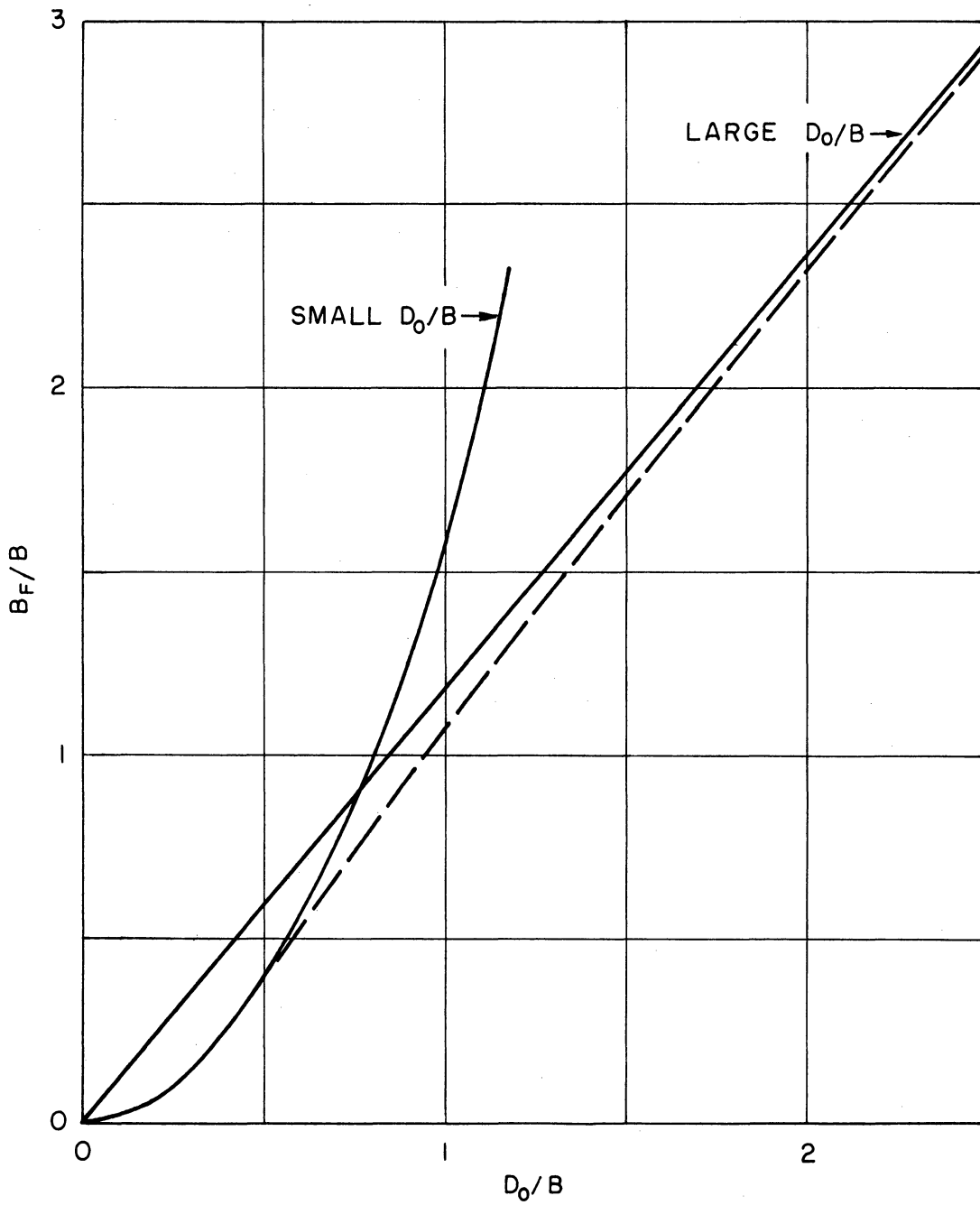
FIG. 1

HALF-POWER BANDWIDTH OF FM POWER SPECTRUM

$$W_\phi(\Delta\omega) = (A_o{}^2/2)\left[\exp(-D_o{}^2)\ \delta(\Delta\omega) + \frac{\exp\left(-\Delta\omega^2/\frac{2D_o{}^2B^2}{3}\right)}{(2\pi D_o{}^2 B^2/3)^{1/2}}\right] \qquad (2.03)$$

Here, the delta function is always present. The only difference in the terminology between this and the case of FM for large D_o/B is that $D_o{}^2$ is the mean-squared instantaneous phase deviation caused by the modulating signal. (As before, it is proportional to the mean-squared value of the modulating voltage).

For small D_o

$$W_\phi(\Delta\omega) = \begin{cases} (A_o{}^2/2)\exp(-D_o{}^2)\ \left[\delta(\Delta\omega) + D_o{}^2/2B\right], & \Delta\omega < B \\[2em] 0, & \Delta\omega > B \end{cases} \qquad (2.04)$$

which, except for the delta-function, has the same half-bandwidth as that of the modulating signal and is rectangular just as is the power spectrum of the modulating signal.

The half-power bandwidth of the continuous parts of these asymptotic expressions for the power spectrums of phase-modulated signals (by normally distributed noise) may be found as before. The bandwidth relations are plotted in Fig. 2 analogously to that for frequency modulation in Fig. 1.

3. BASIC EQUATION FOR FREQUENCY MODULATION

In this section, the general formula for the spectrum of a frequency-modulated carrier (by normally distributed noise) will be obtained. Correlation analysis will be used with some useful techniques applied to simplify the work. In the sequel, the results of this section will be extended to include a phase-modulated carrier.
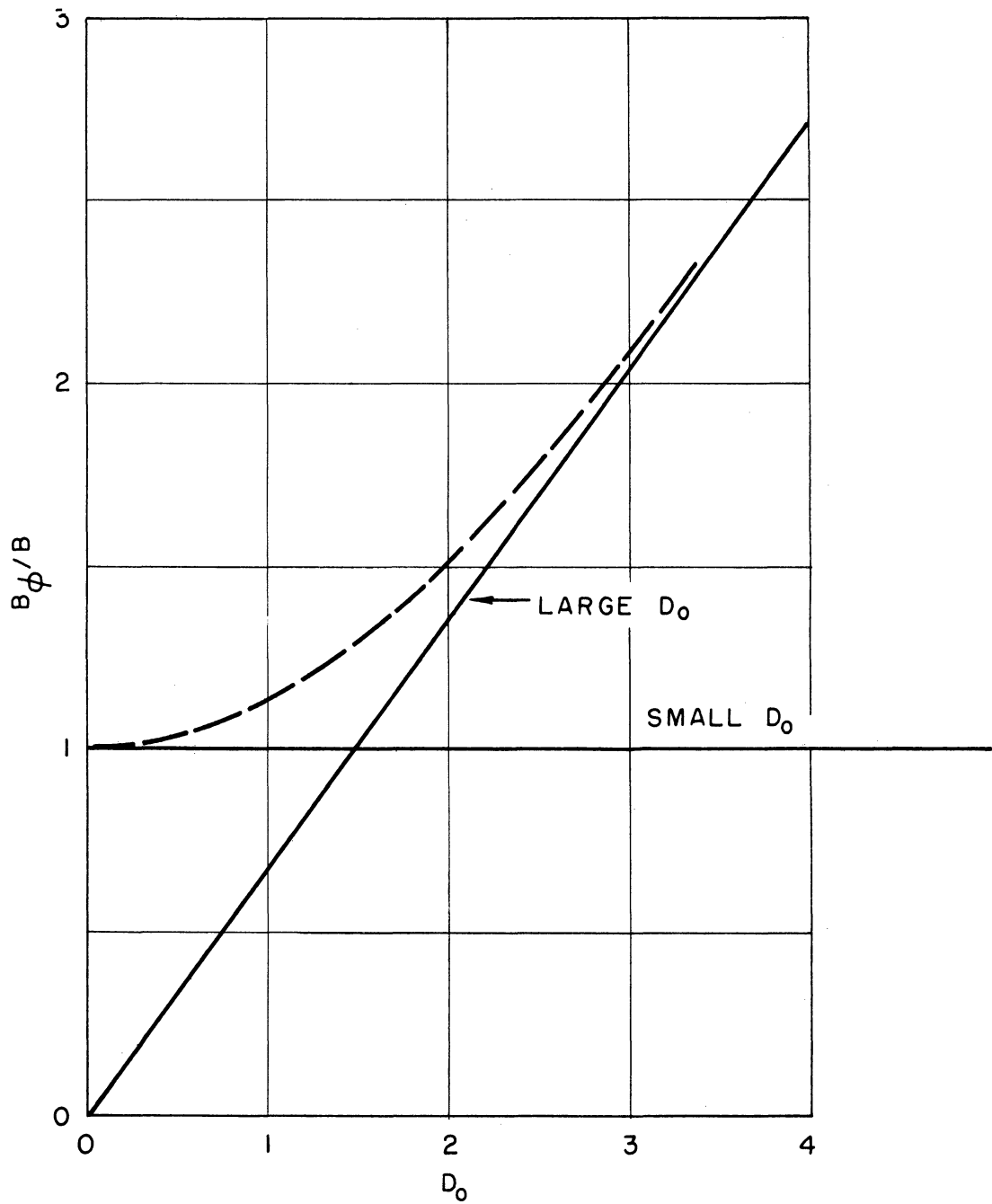
FIG. 2

HALF-POWER BANDWIDTH OF ϕM POWER SPECTRUM

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The voltage of an FM wave can be taken as

$$V(t) = A_0 \exp \left\{ j \left[\omega_0 t + \Psi(t) \right] \right\} \qquad (3.01)$$

where A_0 is the peak amplitude, ω_0 is the (radian) carrier frequency, and $\psi(t)$ is the instantaneous phase-angle shift created by the modulation. In FM,

$$\psi(t) = \int^t D(t')dt' \qquad (3.02)$$

where $D(t')$ is the modulating voltage. If a normalized modulating voltage $V_n(t)$ is defined such that

$$\overline{\left[V_n(t) \right]^2} = 1 \qquad \overline{\left[V_n(t) \right]} = 0 \qquad (3.03)$$

the total modulating voltage becomes

$$D(t) = D_0 V_n(t) \qquad (3.04)$$

where D_0^2 is the mean-squared deviation (from the carrier frequency) of the FM. It is a radian measure. It is also the mean-squared value of the modulating voltage.

The total power in the wave is the autocorrelation function $R(\tau)$ for $\tau = 0$,

$$R(0) = (1/2) \operatorname{Re} \overline{\left[V(t)V^*(t) \right]} = \frac{A_0^2}{2} \qquad (3.05)$$

where V^* is the complex conjugate of V and the answer is obvious.

The autocorrelation function is

$$R(\tau) = 1/2 \ \text{Re} \ \overline{[V(t)V^*(t+\tau)]} \tag{3.06}$$

$$= \frac{A_o^2}{2} \ \text{Re} \left\{ \exp\left[-j\omega_o\right] \ \overline{\exp\left[j\left\{\psi(t)-\psi(t+\tau)\right\}\right]} \right\}$$

The two dimensional normal probability density function $W_2(x,y)$ has a characteristic function given by [*]

$$\psi(t,\mu) = E\left\{ \exp\left[j(tx+\mu y)\right] \right\} = \overline{\exp\left[j(tx+\mu y)\right]}$$

$$= \int_{-\infty}^{+\infty} \exp\left[j(tx+\mu y)\right] W_2(x,y)dx \ dy \tag{3.07}$$

$$= \exp\left[-\frac{\mu_{11}t^2+\mu_{12}t\mu+\mu_{22}\mu^2}{2}\right]$$

where "E" signifies "expectation", where the process is assumed to be stationary (time and ensemble averages are the same), and where the moments are

$$\mu_{11} = \overline{x^2} \qquad \mu_{22} = \overline{y^2} \qquad \mu_{12} = \overline{xy} \tag{3.08}$$

Setting $t = 1$, $\mu = -1$ and $\mu_{11} = \mu_{22}$, (3.07) becomes

$$E\left\{\exp\left[j(x-y)\right]\right\} = \exp\left[-(\mu_{11}-\mu_{12})\right] \tag{3.09}$$

which amounts to particular evaluation of the integral of (3.07).

Equation (3.09) has the same form as the average of (3.06). Thus,

$$R(\tau) = \frac{A_o^2}{2} \cos \omega_o \tau \ \exp\left[-(\overline{\psi^2} - \overline{\psi_1 \psi_2})\right] \tag{3.10}$$

where

[*] H. Cramer, "Mathematical Methods of Statistics", Princeton University Press, 1946 P. 278

$$\overline{\psi^2} = \overline{[\psi(t)]^2}$$

(3.11)

$$\overline{\psi_1 \psi_2} = \overline{[\psi(t)\,\psi(t+\tau)]}$$

The second equation of (3.11) can be used to define ψ_1 and ψ_2.

The average $\overline{\psi_1 \psi_2}$ is the autocorrelation function of the angle modulation.

$$R_\psi(\tau) = D_0^2 \left[\int^t V_n(t')dt' \right] \left[\int^{t+\tau} V_n(t'')dt'' \right]$$ (3.12)

Derivatives can be taken of expected values (under the defining integrals) as follows: *

$$R_\psi(\tau) = E\left\{ \psi(t)\,\psi(t+\tau) \right\}$$

(3.13)

then

$$\frac{d}{d\tau}\left[R_\psi(\tau) \right] = E\left\{ \psi(t)\,\psi'(t+\tau) \right\}$$

(3.14)

changing the variable t to $t_1 - \tau$,

$$\frac{d}{d\tau}\left[R_\psi(t) \right] = E\left\{ \psi(t_1-\tau)\,\psi'(t_1) \right\}$$

(3.15)

then

$$\frac{d^2}{d\tau^2}\left[R_\psi(\tau) \right] = E\left\{ \psi(t)\,\psi''(t+\tau) \right\} = -E\left\{ \psi'(t_1-\tau)\,\psi'(t_1) \right\}$$

(3.16)

Changing the variable again,

$$\frac{d^2}{d\tau^2}\left[R_\psi(\tau) \right] = -E\left\{ \psi'(t)\,\psi'(t+\tau) \right\}$$

(3.17)

* Derivatives and integrals of normal functions are also normal functions.

Substituting for ψ' (observe equation (3.12),

$$\frac{d^2}{d\tau^2}\left[R_\psi(\tau)\right] = -D_o^2 E\left\{ V_n(t)V_n(t+\tau)\right\} \tag{3.18}$$

$$= -D_o^2 R_m(\tau)$$

where $R_m(\tau)$ is the autocorrelation function of the (normalized) modulating voltage.

The Wiener-Khintchine theorem can be used at this point. If $W_m(\omega)$ is the power spectrum (normalized) of a time function, the direct and inverse Fourier integral relations are

$$W_m(\omega) = \frac{2}{\pi}\int_0^\infty R_m(\tau)\cos \omega\tau\, d\tau \tag{3.19}$$

$$R_m(\tau) = \int_0^\infty W_m(\omega)\cos \omega\tau\, d\omega \tag{3.20}$$

$R_\psi(\tau)$ in (3.18) can be obtained by integrating (3.20) twice with respect to τ. Thus,

$$R_\psi(\tau) = D_o^2\int_0^\infty \frac{W_m(\omega)\cos \omega\tau}{\omega^2}d\omega \tag{3.21}$$

Equation (3.10) can also be written

$$R(\tau) = \frac{A_o^2}{2}\cos \omega_o\tau \exp\left\{ -D_o^2\left[R_\psi(0)-R_\psi(\tau)\right]\right\} \tag{3.22}$$

Again using the Wiener-Khintchine theorem and substituting for $R(0)$ and $R(\tau)$, the power spectrum of the frequency modulated wave is found to be

$$W_F(\omega) = \frac{A_o^2}{\pi} \int_0^\infty \cos \omega\tau \cos \omega_o\tau \exp\left\{-D_o^2 \int_0^\infty W_m(\omega) \frac{1-\cos \omega\tau}{\omega^2} d\omega\right\} d\tau \quad (3.23)$$

The product of cosines can be converted to sum and difference functions. The function $\cos(\omega+\omega_o)$ varies so rapidly compared to the exponential that its contribution is entirely negligible. (In particular, this is true when the narrow band approximation applies). In this case, there is obtained

$$W_F(\Delta\omega) = \frac{A_o^2}{2\pi} \int_0^\infty \cos \Delta\omega\tau \exp\left\{-D_o^2 \int_0^\infty W_m(\omega) \frac{1-\cos \omega\tau}{\omega^2} d\omega\right\} d\tau \quad (3.24)$$

where $\Delta\omega = \omega - \omega_o$ is the frequency difference (from the carrier frequency ω_o).

4. BASIC EQUATION FOR PHASE MODULATION

Equation (3.01) is also valid for the case of phase modulation. However, $\psi(t)$ is the modulating voltage directly rather than its integral,

$$\psi(t) = D(t) = D_o V_n(t) \quad (4.01)$$

where $V_n(t)$ is the normalized modulating voltage as before. D_o^2 is now the mean-squared phase deviation--it remains the mean-squared amplitude of the modulating voltage.

Equation (3.10) is also valid but ψ has a somewhat different and simpler interpretation.

$$\overline{\psi^2} = \overline{[D(t)]^2} = D_o^2 \quad (4.02)$$

$$\overline{\psi_1 \psi_2} = D_o^2 \overline{[V_n(t)V_n(t+\tau)]} = D_o^2 R_m(\tau) \quad (4.03)$$

Substituting (4.02) and (4.03) into (3.10),

$$R(\tau) = \frac{A_0{}^2}{2} \cos \omega_0 \tau \exp \left\{ -D_0{}^2 \left[R_m(0) - R_m(\tau) \right] \right\} \qquad (4.04)$$

By the Wiener-Khintchine theorem, (4.04) becomes

$$R(\tau) = \frac{A_0{}^2}{2} \cos \omega_0 \tau \exp \left\{ -D_0{}^2 \int_0^\infty W_m(\omega) \left[1 - \cos \omega \tau \right] d\omega \right\} d\tau \qquad (4.05)$$

which leads to the spectrum of a phase-modulated signal as

$$W_\phi(\Delta\omega) = \frac{A_0{}^2}{2\pi} \int_0^\infty \cos \Delta\omega \tau \exp \left\{ -D_0{}^2 \int_0^\infty W_m(\omega) \left[1 - \cos \omega \tau \right] d\omega \right\} d\tau \qquad (4.06)$$

The only observable difference between (3.24) and (4.06) is the factor $1/\omega^2$. By taking an FM wave modulated with a voltage having the power spectrum $W_m(\omega)$ and changing the spectrum by means of a linear filter to $\omega^2 W_m(\omega)$ (a differentiating network), phase modulation is obtained. The converse also applies. The close relation between FM and ϕM is notable.

5. FM SPECTRUM WITH RECTANGULAR MODULATION POWER SPECTRUM

At this point, a specific (normalized) power spectrum can be assumed for $W_m(\omega)$ in Equation (3.24). This will lead to a specific expression for the power spectrum of a frequency modulated wave. A rectangular power spectrum will be assumed for the modulating voltage. This choice will simplify some of the relations while at the same time is a fair approximation to many types of wide-band amplifier functions.

Let $W_m(\omega)$ be given by

$$W_m(\omega) = \begin{cases} \frac{1}{B}, & 0 < \omega < B \\ \\ 0, & \omega > B \end{cases} \tag{5.01}$$

where B is the radian bandwidth of the modulating-voltage power spectrum. Using (5.01) in (3.24) results in

$$W_F(\Delta\omega) = \frac{A_o^2}{2\pi} \int_0^\infty \cos \Delta\omega\tau \exp\left\{-\frac{D_o^2 \tau}{B} \int_0^{\frac{B\tau}{2}} \left[\frac{\sin \mu}{\mu}\right]^2 d\mu\right\} d\tau \tag{5.02}$$

Changing the variable $B\tau$ to x, expanding $(\sin \mu/\mu)^2$, and integrating termwise,

$$W_F(\Delta\omega) = \frac{A_o^2}{2\pi B} \int_0^\infty \cos \frac{\Delta\omega x}{B} \exp\left\{-\frac{D_o^2}{B^2}\left[\frac{x^2}{1.2!} - \frac{x^4}{3.4!} + \cdots\right]\right\} dx \tag{5.03}$$

It is to be noted that $\Delta\omega$, the frequency deviation, appears only in a normalized form (with respect to B). For large D_o/B, the exponential is appreciable only for small x; that is, the value of the integrand is negligible otherwise. Thus, the asymptotic case exists:

$$W_F(\Delta\omega) \cong \frac{A_o^2}{2\pi B} \int_0^\infty \cos \frac{\Delta\omega x}{B} \exp\left[-\frac{D_o^2 x^2}{2B^2}\right] dx \tag{5.04}$$

This integral is well known[*] with the result

$$W_F(\Delta\omega) \cong \left[\frac{A_o^2}{2}\right] \frac{\exp\left(-\Delta\omega^2/2D_o^2\right)}{\left(2\pi D_o^2\right)^{1/2}} \tag{5.05}$$

[*] H. B. Dwight , "Tables of Integrals and Other Mathematical Data", revised Edition, The Macmillan Co., New York, 1947 Integral No. 863.3.

which (fortunately) turns out to have the correct multiplying factor and which is Gaussian. To facilitate plotting, (5.05) can be normalized to B (although B does not appear) to give

$$W_F(\Delta\omega) \cong \left[\frac{A_o^2}{2}\right] \frac{\exp\left[\frac{(\Delta\omega/B)^2}{2(D_o/B)^2}\right]}{\left[2\pi(D_o/B)^2\right]^{1/2}} \tag{5.06}$$

It is to be noted that the carrier has been smeared entirely into a continuum.

The Gaussian nature of the power spectrum is a more general consequence of large D_o/B than might at first be thought. In fact, by inspecting the nature of the expansion of (5.03), almost any "ordinary" sort of spectrum shape for $W_m(\omega)$ leads to the Gaussian power spectrum for $W_F(\omega)$.

The relative half-power point of (5.06) is given when $W_F(\omega) = 1/2 \left[\frac{A_o^2}{2}\right]$. If the frequency yielding this is called B_F, there is obtained

$$B_F = D_o \sqrt{2 \ln 2} = 1.18\, D_o \tag{5.07}$$

which is one of the asymptotic expressions plotted in Fig. 1.

At the other extreme is the spectrum $W_F'(\omega)$ as $D_o/B \to 0$. In this case, inspection of (5.02) shows that only a small part of the total value of the integral in the exponential is made up of parts for which $\frac{B\tau}{2}$ is small. In the limit, the only significant contributions occur for large $\frac{B\tau}{2}$. Thus the integral of $(\sin \mu/\mu)^2$ can be approximated by its asymptotic value of $\pi/2$ to give

$$W_F'(\Delta\omega) \cong \frac{A_0^2}{2\pi B} \int_0^\infty \cos\frac{\Delta\omega x}{B} \exp\left[-\frac{\pi D_0^2 x}{2B^2}\right] dx \tag{5.08}$$

This integral is also known[*] with the result

$$W_F'(\Delta\omega) \cong \left[\frac{A_0^2}{2}\right]\left[\frac{1}{\pi B}\right]\frac{\frac{\pi D_0^2}{2B^2}}{\left(\frac{\pi D_0^2}{2B^2}\right)^2 + \left(\frac{\Delta\omega}{B}\right)^2} \tag{5.09}$$

which is that at the output of a simple resistance-capacitance filter with white noise at the input.

The half power point of (5.09) occurs for a $\Delta\omega = B_F'$ of

$$B_F' = \frac{\pi}{2} \cdot \frac{D_0^2}{B} = 1.57\frac{D_0^2}{B} \tag{5.10}$$

Equation (5.10) (divided by B) is also plotted in Fig. 1. In this figure, the dotted line shows the "estimated" true curve.

Equations 5.07 and 5.10 yield the same value at the "crossover" point for a D_0/B of

$$\left[D_0/B\right]_{c.o.} = \frac{2}{\pi}\sqrt{2\ln 2} = 0.751 \tag{5.11}$$

As a matter of classification, the usual frequency-modulated signal can be said to persist for $D_0/B > 0.751$ whereas for $D_0/B < 0.751$, the condition approaches that of narrow-band frequency modulation.

6. EFFECT OF LOW-FREQUENCY CUTOFF

An interesting effect exists with frequency modulation when the modulation power spectrum is not constant down to zero

[*] Dwight, loc. cit., Integral No. 863.2

frequency. It will be assumed that the spectrum is flat out to a radian frequency of B_2 as before but that it is zero below $\omega = B_1$. That is,

$$W_m'(\omega) = \begin{cases} \dfrac{1}{B_2 - B_1}, & B_1 < \omega < B_2 \\[2mm] 0, & \omega > B_2 \\[2mm] 0, & \omega < B_1 \end{cases} \qquad (6.01)$$

Using (6.01) in (3.24) there is obtained

$$W_F''(\Delta\omega) = \frac{A_0^2}{2\pi} \int_0^\infty \cos \Delta\omega\tau \, \exp\left\{ - \frac{D_0^2}{B_2 - B_1} \int_{B_1}^{B_2} \frac{1 - \cos \omega\tau}{\omega^2} \, d\omega \right\} d\tau \qquad (6.02)$$

The integral in the exponential can be expressed as the sum of two integrals and rearranged to give

$$W_F''(\Delta\omega) = \frac{A_0^2}{2\pi} \int_0^\infty \cos \Delta\omega\tau \, \exp\left[- \frac{D_0^2 \tau}{B_2 - B_1} \int_0^{\frac{B_2 \tau}{2}} \left(\frac{\sin \mu}{\mu} \right)^2 d\mu \right.$$

$$(6.03)$$

$$\left. \left[1 - \frac{\displaystyle\int_0^{\frac{B_1 \tau}{2}} \left(\frac{\sin \mu}{\mu} \right)^2 d\mu}{\displaystyle\int_0^{\frac{B_2 \tau}{2}} \left(\frac{\sin \mu}{\mu} \right)^2 d\mu} \right] \right] d\tau$$

For small $B_1\tau$, the ratio of the two integrals is small and can be neglected. In this case, the same spectrum is obtained as was the case when $B_1 = 0$. However, for large $B_1\tau$, the ratio is practically unity thus yielding a constant for the exponent; hence, the presence of a delta function at $\Delta\omega = 0$ is indicated. This can more clearly be seen by noting that at $\Delta\omega = 0$ in (6.03) for $B_1 > 0$, $W_F''(\Delta\omega)$ is divergent. An approximation can be utilized to obtain the magnitude of this delta function.

Using

$$\left(\frac{\sin \mu}{\mu}\right)^2 \cong \frac{C}{1+\mu^2} \tag{6.04}$$

where C is a constant and which is correct "on the average" for large μ, the ratio of the two integrals becomes

$$\frac{\displaystyle\int_0^{\frac{B_1\tau}{2}} \frac{d\mu}{1+\mu^2}}{\displaystyle\int_0^{\frac{B_2\tau}{2}} \frac{d\mu}{1+\mu^2}} = \frac{\tan^{-1} \frac{B_1\tau}{2}}{\tan^{-1} \frac{B_2\tau}{2}} \tag{6.05}$$

Expanding the arctangent in a series valid for large $\frac{B\tau}{2}$ and manipulating in such a manner that only the linear terms are retained, the ratio becomes

$$\frac{\tan^{-1} \frac{B_1\tau}{2}}{\tan^{-1} \frac{B_2\tau}{2}} \approx \frac{\frac{\pi}{2} - \frac{2}{B_1\tau}}{\frac{\pi}{2} - \frac{2}{B_2\tau}} \approx 1 - \frac{4}{\pi\tau}\left(\frac{1}{B_1} - \frac{1}{B_2}\right) \tag{6.06}$$

which, for $B_2 \gg B_1$, finally becomes

$$1 - \frac{4}{\pi B_1 \tau} \tag{6.07}$$

Using (6.07) in (6.03) (and the fact that $B_2 \gg B_1$),

$$W_F''(\Delta\omega) \cong \frac{A_o^2}{2\pi} \int_{\tau\,\text{large}} \cos \Delta\omega\tau \, \exp\left[-\frac{D_o^2}{B_2} \int_0^{\frac{B_2\tau}{2}} \left(\frac{\sin \mu}{\mu}\right)^2 d\mu \cdot \frac{4}{\pi B_1}\right] d\tau \tag{6.08}$$

Since $B_2\tau$ is considered large in (6.08), the integral of $(\sin \mu/\mu)^2$ can be replaced with its asymptotic value $\pi/2$ to give

$$W_F''(\Delta\omega) = \frac{A_o^2}{2\pi} \int_{\tau\,\text{large}} \cos \Delta\omega\tau \, \exp\left(-\frac{2D_o^2}{B_1 B_2}\right) d\tau \tag{6.09}$$

But the half delta function is defined as

$$\frac{1}{2} \, \delta(\omega) = \frac{1}{2\pi} \int_0^\infty \cos \omega\tau \, d\tau \tag{6.10}$$

which finally results in

$$W_F''(\Delta\omega) = \left(\frac{A_0^2}{2}\right) \exp\left(- \frac{2D_0^2}{B_1B_2}\right) \delta(\Delta\omega) \tag{6.11}$$

When $(D_0/B_2)^2$ is at all large, D_0^2/B_1B_2 is even larger; consequently, the delta function will generally be quite small (as long as $B_2 \gg B_1$) and usually can be neglected. The power that remains in the carrier will not be great enough to have an appreciable effect on the continuous part of the power spectrum as obtained in Section 5.

7. $\emptyset$M SPECTRUM WITH RECTANGULAR MODULATION POWER SPECTRUM

In this section, the power spectrum of a phase-modulated signal with a rectangular modulation power spectrum will be obtained in close analogy to that done for the case of frequency-modulated waves in Section 5. Substituting the modulation spectrum as given by (5.01) into (4.06),

$$W_\emptyset(\Delta\omega) = \frac{A_0^2}{2\pi} \int_0^\infty \cos \Delta\omega\tau \exp\left[- \frac{4D_0^2}{B\tau} \int_0^{\frac{B\tau}{2}} \sin^2\mu d\mu \right] d\tau \tag{7.01}$$

Using a change of variable, this expression becomes

$$W_\emptyset(\Delta\omega) = \frac{A_0^2}{2\pi B} \int_0^\infty \cos \frac{\Delta\omega x}{B} \exp\left[- \frac{4D_0^2}{x} \int_0^{X/2}\sin^2 \mu d\mu \right] dx \tag{7.02}$$

The integral in the exponent can be evaluated to give

$$W_\emptyset(\Delta\omega) = \frac{A_0^2}{2\pi B} \int_0^\infty \cos \frac{\Delta\omega x}{B} \exp\left[-D_0^2 \left(1- \frac{\sin x}{x}\right)\right] dx \tag{7.03}$$

At $\Delta\omega=0$, the integral of (7.03) does not converge. This indicates the presence of a delta function at $\Delta\omega=0$. It is necessary to remove this singularity from the integral before the continuous part of the spectrum may be obtained.

The number $e^{-D_0^2}$ may be added and subtracted in a factor containing the exponential in (7.03) as

$$W_\phi(\Delta\omega) = \frac{A_0^2}{2\pi B}\int_0^\infty \cos\frac{\Delta\omega x}{B}\left\{\exp(-D_0^2)-\exp(-D_0^2)\right.$$
$$\left. + \exp\left[-D_0^2\left(1-\frac{\sin x}{x}\right)\right]\right\}\,dx \qquad (7.04)$$

which gives

$$W_\phi(\Delta\omega) = \frac{A_0^2}{2\pi B}\exp(-D_0^2)\int_0^\infty \cos\frac{\Delta\omega x}{B}\,dx + \frac{A_0^2}{2\pi B}\int_0^\infty \cos\frac{\Delta\omega x}{B}$$
$$\left\{\exp\left[-D_0^2\left(1-\frac{\sin x}{x}\right)\right] - \exp(-D_0^2)\right\}\,dx \qquad (7.05)$$

The first of these integrals is related to the half delta function and the second is convergent at $\Delta\omega=0$ which indicates that it gives the continuous part of the power spectrum. Thus,

$$W_\phi(\Delta\omega) = \frac{A_0^2}{2}\exp(-D_0^2)\;\delta(\Delta\omega) + \frac{A_0^2}{2\pi B}\int_0^\infty \cos\frac{\Delta\omega x}{B}$$
$$\left\{\exp\left[-D_0^2\left(1-\frac{\sin x}{x}\right)\right] - \exp(-D_0^2)\right\}\,dx \qquad (7.06)$$

For large D_0, the energy remaining in the carrier becomes quite small and can often be neglected.

An interesting observation concerning FM and ϕM can be made by comparing (7.06) and (5.09). As the magnitude of the modulating voltage in FM is decreased, the spectrum becomes narrower and narrower without limit finally resulting in a delta function.

However, the spectrum is continuous at all times. In phase modulation, on the other hand, reducing the magnitude of the modulating voltage removes power from the side bands and places it in the ever present delta function at the carrier frequency causing the delta function to increase in included area.

The second term of (7.06) gives the continuous part of the spectrum. For large D_o, the integrand of (7.06) is appreciable only for small x. In this case

$$\exp\left[-D_o^2\left(1-\frac{\sin x}{x}\right)\right] -\exp(-D_o^2) = \exp\left[-D_o^2\left(\frac{x^2}{3!}-\frac{x^5}{5!}+\cdots\right)\right]$$

$$- \exp(-D_o^2) \cong \exp\left(-\frac{D_o^2 x^2}{6}\right) \tag{7.07}$$

Using this in (7.06) and performing the integration,

$$W_\phi(\Delta\omega) \cong \left[\frac{A_o^2}{2}\right] \left[\exp(-D_o^2)+ \frac{\exp\left(-\frac{\Delta\omega^2}{2D_o^2 B^2/3}\right)}{(2\pi D_o^2 B^2/3)^{1/2}}\right] \tag{7.08}$$

The half power point of the continuous part of the spectrum occurs at a $\Delta\omega = B_\phi$ of

$$B_\phi = D_o B \sqrt{\frac{2\ln 2}{3}} = 0.68\ D_o B \tag{7.09}$$

Equation (7.09) is one of the asymptotic expressions plotted in Fig. 2.

For small D_o, there exist important contributions for large values of x in Equation (7.04) as well as for small values. However, $D_o^2 \frac{\sin x}{x}$ is always small; hence, the series expansion for the exponential may be used to advantage to give

$$W_\phi(\Delta\omega) = \frac{A_o^2}{2} \exp(-D_o^2) \; \delta(\Delta\omega) + \frac{A_o^2 D_o^2}{2\pi B} \exp(-D_o^2) \int_0^\infty \cos \frac{\Delta\omega x}{B} \left[\frac{\sin x}{x} + \right.$$

$$\left. \frac{D_o^2}{2!} \left(\frac{\sin x}{x} \right)^2 + \frac{D_o^4}{3!} \left(\frac{\sin x}{x} \right)^3 + .. \right] dx \qquad (7.10)$$

The integral of (7.10) becomes a sum of integrals. The first order term is a known integral,

$$\int_0^\infty \frac{\sin x \cos \frac{\Delta\omega x}{B}}{x} \, dx = \begin{cases} 0, & \left(\frac{\Delta\omega}{B} \right)^2 > 1 \\\\ \frac{\pi}{4}, & \left(\frac{\Delta\omega}{B} \right)^2 = 1 \\\\ \frac{\pi}{2}, & \left(\frac{\Delta\omega}{B} \right)^2 < 1 \end{cases} \qquad (7.11)$$

To a first order, the spectrum consists of a delta function at $\Delta\omega = 0$ and a continuous and flat spectrum over the range of frequencies occupied by the modulating signal. For this asymptotic case, the bandwidth of the continuous part of the spectrum is constant at

$$B_\phi' = B \qquad (7.12)$$

which is plotted as one of the asymptotic expressions in Fig. 2.

The spectrum for small D_o can thus be written

$$W_\phi(\Delta\omega) \cong \begin{cases} \frac{A_o^2}{2} \left[\exp(-D_o^2) \; \delta(\Delta\omega) + \frac{D_o^2}{2B} \exp(-D_o^2) \right], & \Delta\omega < B \\\\ 0, & \Delta\omega > B \end{cases} \qquad (7.13)$$

It is instructive to sketch the shape of the spectrum for phase modulation for large and small D_o as is done in Fig. 3.

A second-order approximation for the continuous part of the spectrum for small D_o can be obtained by including the second integral of the series of (7.10). Only the general effect will be

noted here. The function $\left(\frac{\sin x}{x}\right)^2$ can be approximated very roughly with the Gaussian function $e^{-a^2x^2}$ in which case, the correction term becomes a Gaussian function which is to be added to the rectangular spectrum of Fig. 3. The "corrected" spectrum might appear as in Fig. 4. This means, simply, that there is a continuous transition from the rectangular to Gaussian spectrum as D_o increases.
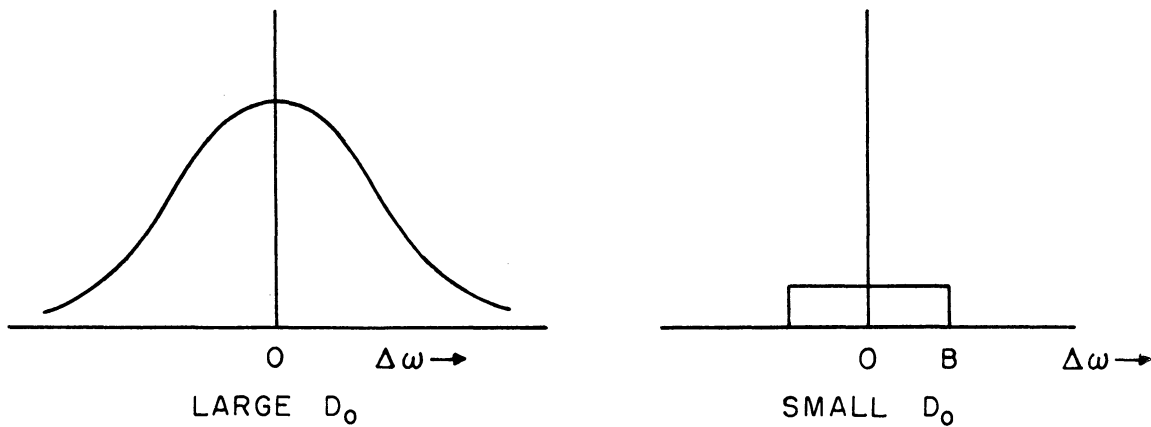
FIG. 3

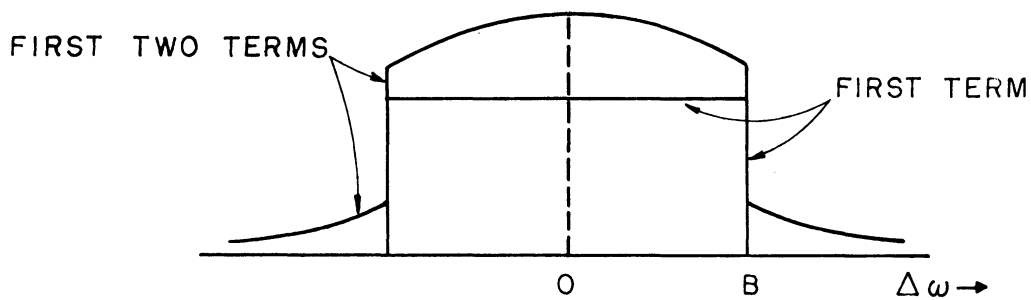SHAPE OF ϕM POWER SPECTRUM.

FIG. 4

CORRECTION FOR SMALL D_0.

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