# FJRW Rings and Landau-Ginzburg Mirror Symmetry

by

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# CHAPTER I

# Introduction

During the last twenty years, mirror symmetry has been a driving force for much progress in geometry and physics. This thesis contributes a version of mirror symmetry purely in the Landau-Ginzburg (LG) setting. Roughly speaking, given a singularity $W\colon \mathbb{C}^N \to \mathbb{C}$ and a symmetry group $G$ we produce a 'mirror pair' $(W^T, G^T)$ such that the LG $A$-model of the pair $(W, G)$ is isomorphic to the LG $B$-model of $(W^T, G^T)$.

Landau-Ginzburg mirror symmetry is not a new idea; it was an important physical tool used to verify Calabi-Yau mirror symmetry in the early investigations of this phenomenon. Throughout the literature, a striking construction [BH] was Berglund–Hübsch's transposed potential $W^T$, which applies in the case of a so-called *invertible* potential $W$ (Definition 1).

Berglund–Hübsch proposed almost twenty years ago that $W$ and $W^T$ form a mirror pair. It was known that *orbifold* LG models $(W, G)$ and $(W^T, G^T)$ must be considered for this proposition to be valid, and the construction of the dual group $G^T$ was known in many cases (e.g. for the Fermat Quintic). We present a general construction in Section 2.7.

Equipped with the correct notion of duality for orbifold LG theories, we prove a mirror theorem relating the FJRW theory of Fan–Jarvis–Ruan–Witten (which we denote 'FJRW theory') [FJR1] and the orbifold $B$-model of Intriligator–Vafa [IV]:

**Theorem 1.1.** *Let $W$ be a non-degenerate invertible potential and $G$ a group of diagonal symmetries of $W$. There is an isomorphism of bi-graded vector spaces*

$$\mathscr{H}_{W,G} \cong \mathscr{Q}_{W^T,G^T},$$

*where $\mathscr{H}_{W,G}$ is the FJRW A-model of $(W,G)$ and $\mathscr{Q}_{W^T,G^T}$ is the orbifold B-model of $(W^T,G^T)$.*

Furthermore, we establish a mirror isomorphism at the level of Frobenius algebras when $G$ is the maximal diagonal symmetry group and $G^T$ is the trivial group.

**Theorem 1.2.** *Let $W$ be a non-degenerate invertible potential and $G^{max}$ its maximal group of diagonal symmetries. There is an isomorphism of Frobenius algebras*

$$\mathscr{H}_{W,G^{max}} \cong \mathscr{Q}_{W^T},$$

*where $\mathscr{Q}_{W^T}$ is the unorbifolded B-model of $W^T$.*

Invertible potentials include, for example, Arnol'd's list of simple, unimodal and bimodal singularities [AGV]. Theorem 1.2 has already been proven for the simple and parabolic singularities [FJR1] and the unimodal and bimodal singularities [KP+]. The 14-families of exceptional (unimodal) singularities exhibit the famous *Arnol'd strange duality*. It seems natural to consider this duality from the LG mirror symmetry perspective. For example, we apply Theorem 1.2 to show that strange duality indeed agrees with LG mirror symmetry.

**Corollary 1.3.** *Let $W$ be one of Arnol'd's 14 exceptional singularities with strange dual $W^{SD}$, and $J$ its exponential grading operator. Then*

$$\mathscr{H}_{W,\langle J\rangle} \cong \mathscr{Q}_{W^{SD}}.$$

*i.e. the LG A-model for $W$ orbifolded by $J$ is isomorphic (as a Frobenius algebra) to the* *unorbifolded LG $B$-model of $W^{SD}$.*

We should emphasize that the subject of LG mirror symmetry was never fully developed in physics because (i) a construction of the $A$-model was absent, and (ii) although the orbifold $B$-model state space was given by Intriligator–Vafa [IV], the ring structure was still lacking. The first problem was solved recently by Fan–Jarvis–Ruan–Witten [FJR1]-[FJR3] with the construction of FJRW theory. As for the second problem, Kaufmann wrote down the multiplication in many cases and proposed a general recipe [Ka1]-[Ka3].

Guided by his recipe, we produce a multiplication for non-degenerate invertible potentials $W$ and $G \subset SL_N\mathbb{C}$. Our definition of multiplication has an important restriction not present in Kaufmann's recipe, namely that the $B$-model orbifold group should be a subgroup of $SL_N\mathbb{C}$. This is dual to the fact that in Fan–Jarvis–Ruan–Witten's construction, every admissible $A$-model orbifold group must contain the exponential grading operator $J$.

It is worth noting that Theorem 1.1 specializes to the main result of Kreuzer [K] in the case where $G$ is the maximal group of diagonal symmetries of $W$. That work considers only a single grading, and appeals to physically motivated 'twist selection rules' to argue that the mirror map is degree-preserving. We clarify the physical picture, and establish our theorems in a more general context (bi-grading, dual group, Frobenius algebra structure) which may facilitate future applications of LG mirror symmetry.

Compared to the other forms of mirror symmetry such as Calabi-Yau to Calabi-Yau and toric to LG, our version is more general and has the benefit of not having any poorly behaved exceptional cases. For example, the LG orbifold theories under consideration do not have to correspond to Calabi-Yau manifolds. Even if they do correspond to Calabi-Yau manifolds (orbifolds), they may be embedded in non-Gorenstein orbifolds, so Batyrev's proof [B] of mirror symmetry may not apply.

This generality, combined with a proof of LG / CY correspondence, has been exploited by

Chiodo-Ruan [CR] to generalize Batyrev's theorem on Calabi-Yau hypersurfaces of Gorenstein weighted projective spaces.

Another important application is the integrable hierarchies problem. Recall that for the unorbifolded $B$-model of $W^T$, there is Saito's semi-simple Frobenius manifold theory (in genus zero) [S] and the high-genus theory due to Givental [Gi]. Theorem 1.2 naturally suggests the following conjecture

**Conjecture.** *Let $W$ be a non-degenerate invertible potential and $G^{max}$ be its maximal group of diagonal symmetries. Then the full FJRW-theory of $(W, G^{max})$ is isomorphic to the Saito–Givental theory of $W^T$.*

In many cases, the Saito–Givental theory of $W^T$ is expected to satisfy certain integrable hierarchies. The study of these examples leads to a generalization of Witten's famous ADE integrable hierarchies conjecture solved by Fan–Jarvis–Ruan [FJR1]. We refer the interested reader to [R2] for the details.

This thesis is organized as follows.

## 1.1 Organization of thesis

We present some basic notions regarding invertible potentials in Chapter I, including Kreuzer–Skarke's classification of invertible potentials.

In Chapter II we review the construction of the FJRW $A$-model Frobenius algebra, as well as the orbifold $B$-model state space of Intriligator–Vafa. We introduce a multiplication on the orbifold $B$-model and show that this multiplication respects a suitably shifted version of the bi-grading of Intriligator–Vafa.

In Section 2.8 we prove Landau-Ginzburg mirror symmetry for state-spaces, after introducing a suitable notion of duality between the symmetry groups of Berglund–Hübsch dual potentials.

In Chapter III, we show that there is an isomorphism of Frobenius algebras between the maximally orbifolded $A$-model of a potential and the unorbifolded $B$-model of the Berglund–Hübsch dual. We present evidence that the Frobenius algebra isomorphism extends beyond this maximally orbifolded case, and demonstrate a relation between Arnol'd's strange duality and Landau-Ginzburg mirror symmetry.

## 1.2 Preliminaries on Invertible Potentials

**Definition 1.** Given $c_1, \dots, c_s \in \mathbb{C}$, we call

$$W\colon \mathbb{C}^N \to \mathbb{C} \tag{1.1}$$

$$(X_1, \dots, X_N) \longmapsto \sum_{i=1}^{s} c_i \prod_{j=1}^{N} X_j^{a_{ij}}, \tag{1.2}$$

an *invertible non-degenerate quasihomogeneous potential* if

- $W$ is quasi-homogenous of degree 1 with respect to a unique set of weights $(q_1, \dots, q_N) \in \mathbb{Q}^N$.
- $W$ has an isolated singularity at the origin in $\mathbb{C}^N$.
- The number of monomials equals the number of variables (i.e. $s = N$).

Non-degeneracy encompasses the isolation of the singularity and the uniqueness of the weights. Invertibility is the condition on the number of monomials.

The exponent matrix $A = (a_{ij})$ encodes the potential, modulo the coefficients $c_i$ of the monomials.

The weights (or *charges*) $q_i$ are determined by the condition that

$$A\begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{1.3}$$

Since the matrix $A_W$ is square, the uniqueness of the weights is equivalent to invertibility of $A_W$. Note that the $c_i$ may therefore be absorbed by rescaling the variables, so in what follows we will take $c_i = 1$ without loss of generality.

The transposed matrix $A_W^T$ will also correspond to a quasi-homogeneous polynomial, which we denote by $W^T$. In fact, $W^T$ will also be an invertible potential. The only condition that is not obvious is that $W^T$ has an isolated singularity at the origin in $\mathbb{C}^N$, but this will follow easily from the Kreuzer-Skarke classification of invertible potentials in Section 1.2.

**Definition 2.** We define the *maximal group of diagonal symmetries of* $W$, denoted by $G^{\max}$, to be the kernel of the homomorphism

$$(\mathbb{C}^*)^N \to (\mathbb{C}^*)^N \tag{1.4}$$

$$(\lambda_1, \ldots, \lambda_N) \mapsto (\prod_{j=1}^{N} \lambda_j^{a_{1j}}, \ldots, \prod_{j=1}^{N} \lambda_j^{a_{Nj}}). \tag{1.5}$$

(As above, the matrix $A_W = (a_{ij})$ is the $N \times N$ matrix whose $(i, j)$ entry is the exponent of $X_j$ in the $i^{\text{th}}$ monomial of $W$.)

An element of $G^{\max}$ will be called a *diagonal symmetry of* $W$. Concretely, $(\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N$ is a diagonal symmetry of $W$ if

$$W(\lambda_1 X_1, \ldots, \lambda_N X_N) = W(X_1, \ldots, X_N)$$

for all $(X_1, \ldots, X_N) \in \mathbb{C}^N$.

*Remark.* It is immediate from invertibility of $A_W$ that a diagonal symmetry $g \in G^{\max}$ is of the form $(e^{2\pi i \Theta_1^g}, \dots, e^{2\pi i \Theta_N^g})$. We we call $\Theta_i^g \in [0,1)$ the *phase* of the action of $g$ on the variable $X_i$.

**Definition 3.** We write

$$(1.6) \qquad A^{-1} = \left( \rho_1 \middle| \rho_2 \middle| \cdots \middle| \rho_N \right), \qquad \text{with column vectors} \qquad \rho_k = \begin{bmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_N^{(k)} \end{bmatrix}$$

Then each $\rho_k$ defines a symmetry of $W$ via

$$\rho_k X_j = \exp(2\pi i \varphi_j^{(k)}) X_j.$$

We will abuse notation and use the same symbol to denote the symmetry and the column vector.

*Remark.* Suppose $g \in (\mathbb{C}^*)^N$ is a diagonal symmetry of $W$, with $gX_k = \exp(2\pi i g_k) X_k$. Since $g$ preserves $W$,

$$A \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} \in \mathbb{Z}^N,$$

so the phase vector $(g_1, \dots, g_N)^T$ is a linear combination of the columns of $A^{-1}$. This implies that the $\rho_k$ generate the group $G^{\max}$ of diagonal symmetries of $W$, and for any $g \in G^{\max}$, we can write $g = \prod_{i=1}^N \rho_i^{\alpha_i}$.

*Remark.* The group $G^{\max}$ is non-trivial, as it contains the *exponential grading operator* $J$, which acts on $X_k$ with phase $q_k$.

Multiplying Equation (1.3) by $A^{-1}$, we see that $J$ is given by

$$J = \prod_{i=1}^{N} \rho_i. \tag{1.7}$$

In [KS], Kreuzer and Skarke prove that an invertible potential is non-degenerate if and only if it can be written as a sum of (decoupled) invertible potentials of one of the following three types, which we will refer to as *atomic types*:

$$W_{\text{Fermat}} = X^a.$$

$$W_{\text{loop}} = X_1^{a_1}X_2 + X_2^{a_2}X_3 + \cdots + X_{N-1}^{a_{N-1}}X_N + X_N^{a_N}X_1.$$

$$W_{\text{chain}} = X_1^{a_1}X_2 + X_2^{a_2}X_3 + \cdots + X_{N-1}^{a_{N-1}}X_N + X_N^{a_N}.$$

Although this classification allows for terms $X_kX_{k+1}$ (i.e. $a_k = 1$), we will only consider the case $a_i \geq 2$ so that the weights satisfy $q_i \leq \frac{1}{2}$, as this condition is necessary for the construction of the FJRW $A$-model.

*Remark.* The proof of Theorem 3.1 (Chain potentials) is valid only if $a_N > 2$, so that all weights are strictly less than $\frac{1}{2}$. The omitted only non-trivial case thus omitted is the chain potential with $a_N = 2$.

It is clear that the transpose construction $W^T$ preserves the above types. Our arguments will rely heavily on an understanding of these 'atomic' potentials and their symmetry groups, and we will recall some elementary facts from [K] without proof. Because the Fermat potential is particularly straightforward, our discussion focuses on Loops and Chains.

**Notation.** We use the following notation for quantities which take value either 0 or 1:

$$\delta_{\alpha,\beta} := \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else.} \end{cases}$$

Also

$$\delta_i^{\text{even}} := \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{else,} \end{cases}$$

with $\delta_i^{\text{odd}}$ defined similarly.

**Definition 4.** The fixed locus of $g \in (\mathbb{C}^*)^N$ acting on $\mathbb{C}^N$ is either $\{0\}$ or a co-ordinate subspace of $\mathbb{C}^N$. We define $F_g \subset 1, \ldots, N$ to be the set of indices corresponding to coordinates fixed by $g$. That is,

$$\text{Fix}(g) = \{(x_1, \ldots, x_n) \,|\, x_i = 0 \text{ for } i \notin F_g\}.$$

We now recall without proof the facts from [K] which will be useful in what follows.

The following lemma facilitates the computation of the phase of a given symmetry on a variable $X_j$.

**Lemma 1.1.** *Let $W \in \mathbb{C}[X_1, \ldots, X_N]$ be a non-degenerate invertible potential of atomic type, with exponent matrix $A_W$ and generators of $G^{max}$ given by $\rho_1, \ldots, \rho_N$ corresponding to the columns of $A_W^{-1}$. Let $g = \prod_{i=1}^N \rho_i^{\alpha_i}$, with $0 \leq \alpha_i < a_i$. For $j \in \{1, \ldots, N\}$ with $X_j$ not fixed by $gJ$ (i.e. $j \notin F_{gJ}$),*

$$\Theta_j^{gJ} = \sum_{i=1}^{N} (\alpha_i + 1)\varphi_j^{(i)}.$$

i.e. *The phase of $gJ$ on $X_j$ is given by the exponent-weighted algebraic sum of the phases of the $\rho_i$ on $X_j$, without the need to reduce this sum modulo* 1.

*If $X_j$ is fixed by $gJ$, $\Theta_j^{gJ} = 0$ although the algebraic sum of phases may equal either 0 or 1.*

**Definition 5.** For $W\colon \mathbb{C}^N \to \mathbb{C}$, the *Milnor Ring* (or local algebra) $\mathscr{Q}_W$ of $W$ is the quotient of $\mathbb{C}[X_1, \ldots, X_N]$ by the Jacobian ideal of $W$. That is

$$\mathscr{Q}_W = \mathbb{C}[X_1, \ldots, X_N] / \left\langle \partial_{X_1} W, \ldots, \partial_{X_N} W \right\rangle.$$

It is a fact that if $W$ has an isolated singularity at the origin, $\mathscr{Q}_W$ is finite-dimensional over $\mathbb{C}$. More details will be given in Chapter 2.1.

The following lemma gives explicit vector space generators over $\mathbb{C}$ for the Milnor ring of a loop or chain potential.

**Lemma 1.2.**

- *The Milnor ring $\mathscr{Q}_{W_{loop}}$ for a loop potential is generated over $\mathbb{C}$ (as a vector space) by $\{\prod_{i=1}^N X_i^{\alpha_i} \,|\, 0 \le \alpha_i < a_i\}$, and has dimension $\prod_{i=1}^N a_i$.*
- *The Milnor ring $\mathscr{Q}_{W_{chain}}$ for a chain potential is generated over $\mathbb{C}$ (as a vector space) by $\{\prod_{i=1}^N X_i^{\alpha_i} \,|\, 0 \le \alpha_i < a_i\}$ subject to the condition that the largest set $\{1, \dots, s\}$ of consecutive indices for which $\alpha_i = \delta_i^{odd}(a_i - 1)$ has an even number of elements (possibly zero). Its dimension is $\sum_{i=1}^{N+1}(-1)^{i-1}\prod_{j=i}^N a_j$, where we interpret the empty product as equal to 1.*

The next lemma explicitly identifies the diagonal symmetry groups of atomic invertible potentials.

**Lemma 1.3.**

- *Let $W_{loop}$ be a loop potential. Then $G_W^{max}$ has order $\prod_{i=1}^N a_i - (-1)^N$.*

  *If $N$ is even, any diagonal symmetry $g$ of $W_{loop}$ may be written*

  $$g = \prod_{i=1}^N \rho_i^{\alpha_i} \qquad \text{with} \qquad 0 \le \alpha_i < a_i.$$

  *This presentation is unique, except in the case of $J^{-1}$, where*

  $$J^{-1} = \prod_{i\ even} \rho_i^{a_i - 1} \qquad \text{and} \qquad J^{-1} = \prod_{i\ odd} \rho_i^{a_i - 1}.$$

*If $N$ is odd, any diagonal symmetry $g \neq J^{-1}$ of $W_{loop}$ may be written uniquely as*

$$g = \prod_{i=1}^{N} \rho_i^{\alpha_i} \qquad \textit{with} \qquad 0 \leq \alpha_i < a_i.$$

- *Let $W_{chain}$ be a chain potential. Then $G_W^{max}$ has order $\prod_{i=1}^{N} a_i$, and any $g \in G_W^{max}$ may be written uniquely as*

$$g = \prod_{i=1}^{N} \rho_i^{\alpha_i} \qquad \textit{with} \qquad 0 \leq \alpha_i < a_i.$$

*Remark.* Lemmas 1.2, and 1.3 combine to show that for loop and chain potentials, the image of the $\mathbb{C}$-linear map

$$\Omega^N(\mathbb{C}^N)/(dW^T \wedge \Omega^{N-1}) \longrightarrow \mathbb{C}[G_W^{\max}]$$

generated by

$$\bigwedge_{i=1}^{N} Y_i^{\alpha_i} dY_i \longmapsto \left( \prod_{i=1}^{N} \rho_i^{\alpha_i} \right) J$$

is the subring of the group ring generated by group elements with *even dimensional* fixed loci. The map is injective for chains, and for loops with $N$ odd. For loops with $N$ even, $J^{-1}$ has two preimages, while every other group element has a single pre-image.

As complex vector spaces, $\Omega^N(\mathbb{C}^N)/(dW^T \wedge \Omega^{N-1})$ and $\mathscr{Q}_{W^T}$ are clearly isomorphic. However, the presentation in terms of forms is more natural because it reflects the identification [Wa1] between the space of Lefschetz thimbles and the space of versal deformations (*i.e.* the local algebra) which plays a central role in the construction of the FJRW $A$-model [FJR1].

Two observations are worth bearing in mind. First, a key distinction between an element of the local algebra and the corresponding $N$-form is that the natural $G^{\max}$ action differs by a determinant twist coming from the volume form; it is this twisted action which is appropriate in the LG mirror symmetry setting. Second, as is evident from the above map, there is a compensating shift by $J$ in the group-grading of the $A$-model, which serves to produce an

$A$-model multiplicative identity in the $J$-graded summand. For the evident duality between monomials and group elements to preserve the bi-grading (Section 2.5), the volume form is necessary.

For $g \in G^{\max}$, the next lemma identifies the $G^{\max}$-invariants in $\mathscr{Q}_{\mathrm{Fix}(gJ)}$.

**Lemma 1.4.**

- *For a loop potential $W_{loop}$, the only symmetry $gJ$ with non-trivial fixed locus is $gJ = \mathrm{id}$, which has fixed locus $\mathbb{C}^N$. Generators of the $G^{max}$ invariants as a $\mathbb{C}$-vector space are given by*

$$\mathscr{Q}_{\mathrm{Fix}(gJ)}^{G^{max}} = \begin{cases} \emptyset & \text{if } gJ = \mathrm{id}, \text{ and } N \text{ is odd.} \\ \left\{\bigwedge_{i=1}^N X_i^{\delta_i^{even}(a_i-1)} dX_i,\ \bigwedge_{i=1}^N X_i^{\delta_i^{odd}(a_i-1)} dX_i\right\} & \text{if } gJ = \mathrm{id}, \text{ and } N \text{ is even.} \\ \{1\} & \text{otherwise.} \end{cases}$$

- *For a chain potential, $W_{chain}$, if a symmetry $gJ$ fixes $X_t$, it must fix $\{X_t, \ldots, X_N\}$. Now, $Fix(gJ) = \{X_t, \ldots, X_N\}$ implies $g = \prod_{i=1}^N \rho_i^{\alpha_i}$ has $\alpha_i = \delta_{N-i}^{even}(a_i - 1)$ for $i \geq t$, and this relation does not hold for $i = t-1$.*

  *The $G^{max}$-invariants in $\mathscr{Q}_{\mathrm{Fix}(gJ)}$ are generated by*

$$\mathscr{Q}_{\mathrm{Fix}(gJ)}^{G^{max}} = \begin{cases} \emptyset & \text{if } \mathrm{Fix}(gJ) = \{X_t, \ldots, X_N\} \text{ is odd-dimensional.} \\ \{\prod_{i=t}^N X_i^{\delta_{i-t}^{even}(a_i-1)} dX_i\}, & \text{if } \mathrm{Fix}(gJ) = \{X_t, \ldots, X_N\} \text{ is even-dimensional,} \\ \{1\} & \text{if } \mathrm{Fix}(gJ) = \emptyset. \end{cases}$$

# CHAPTER II

# The $A$ and $B$ models

## 2.1 FJRW $A$-model

Let $W$ be a non-degenerate quasi-homogeneous potential (Definition 1 in the variables $x_1, x_2, \ldots, x_N$ with weights $q_1, q_2, \ldots, q_N$ respectively. Recall that non-degeneracy requires that these weights are uniquely determined by the condition that each monomial in $W$ has total weight 1, and that $W$ has an isolated singularity at the origin.

**Definition 6.** The central charge of $W$ is

$$\hat{c} := \sum_{j=1}^{N} (1 - 2q_j).$$

**Definition 7.** The Jacobian ideal $\mathcal{J}(W)$ is given by

$$\mathcal{J}(W) = \left\langle \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \ldots, \frac{\partial W}{\partial x_N} \right\rangle .$$

**Definition 8.** The Milnor ring $\mathcal{Q}_W$ is given by

$$\mathcal{Q}_W := \mathbb{C}[x_1, x_2, \ldots, x_N]/\mathcal{J}(W).$$

The Milnor ring is a finite dimensional vector space over $\mathbb{C}$, of dimension

$$\mu = \prod_{j=1}^{N} \left( \frac{1}{q_j} - 1 \right).$$

It is graded by weighted degree, where the weighted degree of a monomial $\prod_i X_i^{\alpha_i}$ is $\sum_i \alpha_i q_i$. The elements of top degree form a one-dimensional subspace generated by $\operatorname{hess}(W) = \det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$. One can check directly that the top degree is equal to $\hat{c}$.

**Definition 9.** For $f, g \in \mathscr{Q}_W$, the residue pairing $\langle f, g \rangle$ is given by

$$fg = \frac{\langle f, g \rangle}{\mu} \operatorname{hess}(W) + \text{ terms of weighted degree} < \hat{c}. \tag{2.1}$$

This pairing is non-degenerate, and endows the Milnor ring with the structure of a Frobenius algebra (i.e. $\langle fg, h \rangle = \langle f, gh \rangle$). For more details, see [AGV].

To define the FJRW ring, we require in addition to $W$ a choice of a group of diagonal symmetries of $W$. The choice of group heavily affects the resulting structure of the FJRW ring.

Recall the maximal group of diagonal symmetries

$$G_W^{\max} = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_N) \subseteq (\mathbb{C}^*)^N \,|\, W(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_N x_N) = W(x_1, x_2, \ldots, x_N) \right\},$$

which always contains the exponential grading element $J = (e^{2\pi i q_1}, e^{2\pi i q_2}, \ldots, e^{2\pi i q_N})$. In general, the theory requires that the symmetry group be *admissible* (see [FJR1] section 2.3). In Theorem 2.5 we prove that admissible groups of diagonal symmetries are precisely those containing $J$.

The Landau–Ginzburg Mirror Symmetry Conjecture states the following:

**Conjecture** (Landau–Ginzburg Mirror Symmetry Conjecture). *For a non-degenerate, quasi-homogeneous, potential $W$ and diagonal symmetry group $G$, there is a dual potential $W^T$ with dual symmetry group $G^T$ so that the FJRW-ring of $(W, G)$ is isomorphic to an orbifolded Milnor ring of $(W^T, G^T)$.*

*Remark.* We use the notation $W^T$ suggestively for the dual potential, as one of our main theorems is that the Berglund–Hübsch transposed potential is the appropriate dual in the context of LG-to-LG mirror symmetry for non-degenerate invertible potentials.

## 2.2 The A-model state space

We now outline the definition of $\mathscr{H}_{W,G}$ as a $\mathbb{C}$-vector space, after which we will define the pairing, grading, and multiplication that make $\mathscr{H}_{W,G}$ a Frobenius algebra.

In [FJR1], the state space $\mathscr{H}_{W,G}$ is defined in terms of Lefschetz thimbles:

$$\mathscr{H}_{W,G} = \bigoplus_{\gamma \in G} H^{\text{mid}}(\operatorname{Fix}\gamma, W_\gamma^{-1}(\infty), \mathbb{Q})^G.$$

Here, $W_\gamma^{-1}(\infty)$ is a generic smooth fiber of the restriction of $W$ to $\operatorname{Fix}\gamma$; for further details, see [FJR1]. For our purposes, it will be most convenient to give a presentation in terms of Milnor rings, but we should point out that the isomorphism between the two presentations is not canonical ([Wa1], [Wa2]).

**Definition 10.** Let $G$ be an admissible group. ($i.e.$ $G \subseteq G_W^{\max}$ and $J \in G$). For $h \in G$, let $\operatorname{Fix} h \subset \mathbb{C}^N$ be the fixed locus of $h$, and let $N_h$ be its dimension. Define

$$H_h := \Omega^{N_h}(\mathbb{C}^{N_h}) / \left( dW|_{\operatorname{Fix} h} \wedge \Omega^{N_h - 1}(\mathbb{C}^{N_h}) \right) \cong \mathscr{Q}_{W|_{\operatorname{Fix} h}} \cdot \omega$$

where $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{N_h}}$ is a volume form.

The group $G$ acts on $H_h$ via its action on the coordinates*.

The FJRW state space is then given by

$$\mathscr{H}_{W,G} = \left( \bigoplus_{\gamma \in G} H_\gamma \right)^G .$$

The state space $\mathscr{H}_{W,G}$ is $\mathbb{Q}$-graded by the so-called $W$-degree, which depends only on the $G$-grading. To define this grading, first recall that each element $h \in G$ can be uniquely expressed as

$$h = (e^{2\pi i \Theta_1^h}, e^{2\pi i \Theta_2^h}, \dots, e^{2\pi i \Theta_N^h})$$

with $0 \leq \Theta_i^h < 1$.

**Definition 11.** For $\alpha_h \in \mathscr{H}_h$, the $W$-degree of $\alpha_h$ is defined by

$$\deg_W(\alpha_h) := \dim \operatorname{Fix} h + 2 \sum_{j=1}^{N} (\Theta_j^h - q_j). \tag{2.2}$$

*Remark.* We introduce the $A$-model bi-grading in definition 2.10, with respect to which the $W$-degree is simply the sum of the gradings. For the moment, the reader may note that the $W$-degree ensures that the summand corresponding to the distinguished group element $J$ has degree zero; it is this summand which will contain the identity for the $A$-model product.

Since $\operatorname{Fix} h = \operatorname{Fix} h^{-1}$, we have $\mathscr{H}_h \cong \mathscr{H}_{h^{-1}}$, and the residue pairing on $\mathscr{Q}_{W|_{\operatorname{Fix} h}}$ induces a pairing

$$\mathscr{H}_h \otimes \mathscr{H}_{h^{-1}} \to \mathbb{C}.$$

The pairing on $\mathscr{H}_{W,G}$ is the direct sum of these pairings. Fixing a basis for $\mathscr{H}_{W,G}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle \alpha, \beta \rangle$, with inverse $\eta^{\alpha,\beta}$.

*Note the volume form encodes a determinant-twist on the natural $G$-action on $\mathscr{Q}_{W|_{\operatorname{Fix} h}}$.

## 2.3 The A-model Frobenius Algebra

For each pair of non-negative integers $g$ and $n$, with $2g-2+n>0$, the FJRW cohomological field theory produces for each $N$-tuple $(\alpha_1,\alpha_2,\dots,\alpha_N)\in(\mathscr{H}_{W,G})^N$ classes $\Lambda^W_{g,n}(\alpha_1,\alpha_2,\dots,\alpha_N)\in H^*(\overline{\mathscr{M}}_{g,n})$ of complex degree $3g-3+n-D$, where the 'homology degree' $D$ is given by

$$D:=\hat{c}_W(g-1)+\frac{1}{2}\sum_{i=1}^{N}\deg_W(\alpha_i).$$

The $n$-point correlators are defined to be

$$\langle\alpha_1,\dots,\alpha_N\rangle_{g,n}:=\int\limits_{\overline{\mathscr{M}}_{g,n}}\Lambda^W_{g,n}(\alpha_1,\dots,\alpha_N),$$

so $\langle\alpha_1,\dots,\alpha_N\rangle_{g,n}$ obviously vanishes unless the homology degree of $\Lambda^W_{g,n}(\alpha_1,\dots,\alpha_N)$ is zero. The ring structure on $\mathscr{H}_{W,G}$ is determined by the genus-zero three-point correlators. In other words, if $r,s\in\mathscr{H}_{W,G}$, then

$$r\star s:=\sum_{\alpha,\beta}\langle r,s,\alpha\rangle_{0,3}\,\eta^{\alpha,\beta}\beta \tag{2.3}$$

where the sum is taken over all choices of $\alpha$ and $\beta$ in a fixed basis of $\mathscr{H}_{W,G}$. This product endows $\mathscr{H}_{W,G}$ with the structure of a Frobenius algebra. That is, the product interacts well with the pairing in the sense that for $\alpha,\beta,\gamma\in\mathscr{H}_{W,G}$,

$$\langle\alpha\star\beta,\gamma\rangle=\langle\alpha,\beta\star\gamma\rangle$$

The classes $\Lambda^W_{g,n}(\alpha_1,\dots,\alpha_N)$ satisfy the following axioms which facilitate the computation of the genus zero three-point correlators $\langle\alpha_1,\alpha_2,\alpha_3\rangle$. In particular, we show in Chapter III that these axioms completely determine the product structure on the $A$-model state space. The reader may wish to skip to Section 2.4 on a first reading.

**Axiom 1.** *Dimension: If the homology degree* $D \notin \frac{1}{2}\mathbb{Z}$*, then* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$*. In particular, if* $g = 0$ *and* $n = 3$*, then* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0$ *unless* $D = 0$*, which occurs if and only if* $\sum_{i=1}^{3} \deg_W \alpha_i = 2\hat{c}$*.*

**Axiom 2.** *Symmetry: Let* $\sigma \in S_N$*. Then*

$$\left\langle \alpha_1, \ldots, \alpha_n \right\rangle_{g,n} = \left\langle \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)} \right\rangle_{g,n}.$$

The next few axioms relate to the degrees of line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ endowing a $k$-pointed orbicurve $\mathscr{C}$ with a so-called $W$*-structure*. This means that for each monomial $\prod_{j=1}^{N} z_j^{a_{ij}}$ of $W$, $\bigotimes_{j=1}^{N} \mathscr{L}_j^{\otimes a_{ij}} \cong \omega_{\mathscr{C},\log}$. Here, $\omega_{\mathscr{C},\log}$ is obtained by pulling back from the underlying curve $C = |\mathscr{C}|$ the bundle $\omega_C \otimes \mathscr{O}(p_1) \otimes \cdots \otimes \mathscr{O}(p_k)$. The identification of monomials in the $\mathscr{L}_j$ with $\omega_{\mathscr{C},\log}$ arises naturally in the attempt to solve the Witten equation on the orbicurve $\mathscr{C}$. The details may be found in [FJR1] and provide geometric background to the present construction.

Consider the class $\Lambda^W_{g,k}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, with $\alpha_j \in \mathscr{H}_{h_j}$ for each $j \in \{1, \ldots, k\}$. For each variable $X_j$, the rational number $l_j := \deg |\mathscr{L}_j|$ is given by

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^{k} \Theta_j^{h_i}. \tag{2.4}$$

($|\mathscr{L}_j|$ denotes the pushforward of the bundle $\mathscr{L}_j$ on the orbicurve $\mathscr{C}$ to the underlying coarse curve).

**Axiom 3.** *Integer degrees: If* $l_j \notin \mathbb{Z}$ *for some* $j \in \{1, \ldots, N\}$*, then* $\Lambda^W_{g,k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = 0$.

*Remark.* This axiom has the following important consequence, which follows immediately from examining Equation (2.4).

**Corollary 2.1.** *Suppose* $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) \neq 0$*, with* $\alpha_i \in \mathscr{H}_{h_i}$*. Then*

$$\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \tilde{\alpha}_k) = 0 \text{ for any } \tilde{\alpha}_k \notin \mathscr{H}_{h_k}.$$

*Proof.* Since $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) \neq 0$, we know that for all $j$

$$l_j = q_j(2g-2+k) - \sum_{i=1}^{k} \Theta_j^{h_i} \in \mathbb{Z}.$$

Suppose $\alpha_k \in \mathscr{H}_{\tilde{h}_k}$, where $\tilde{h}_k = (\tilde{h}_k h_k^{-1})h_k$. In order to have

$$\tilde{l}_j = q_j(2g-2+k) - \sum_{i=1}^{k-1} \Theta_j^{h_i} - \Theta_j^{\tilde{h}_k} \in \mathbb{Z},$$

we need $\Theta_j^{\tilde{h}_k h_k^{-1}} \in \mathbb{Z}$.

Now, by Axiom 3, $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \tilde{\alpha}_k) = 0$ unless this holds for all $j$, which is equivalent to $\tilde{h}_k = h_k$. □

**Axiom 4.** *Concavity: If* $l_j < 0$ *for all* $j \in \{1, 2, \dots, N\}$*, then* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1$*.*

The next axiom is related to the Witten map. When $H^0(\bigoplus_{j=1}^N \mathscr{L}_j)$ and $H^1(\bigoplus_{j=1}^N \mathscr{L}_j)$ have the same rank, the Witten map is given by:

$$\begin{aligned} \mathcal{W} : H^0(\bigoplus_{j=1}^{N} \mathscr{L}_j) &\to H^1(\bigoplus_{j=1}^{N} \mathscr{L}_j) \\ (s_1, \dots, s_N) &\mapsto \left( \overline{\frac{\partial W}{\partial x_1}}, \overline{\frac{\partial W}{\partial x_2}}, \dots, \overline{\frac{\partial W}{\partial x_N}} \right) |_{x_i = s_i}. \end{aligned}$$

Concretely, for each $k \in \{1 \dots, N\}$, $\overline{\frac{\partial W}{\partial x_k}}|_{x_i=s_i}$ is a poloynomial in the $\overline{s}_i$. The degree of $\mathcal{W}$ is minus one times the highest exponent occuring among the $\overline{\frac{\partial W}{\partial x_k}}|_{x_i=s_i}$. (The sign comes from complex conjugation).

*Example.* If $W = x_1^3 x_2 + x_2^5$, then $\mathcal{W}(s_1, s_2) = (3\overline{x_1}^2\overline{x_2}, \overline{x_1}^3 + 4\overline{x_2}^5)$, and the degree of the

Witten map is $\deg \mathcal{W} = -5$.

The fact that the Witten map is well-defined is a consequence of the geometric conditions on the $\mathscr{L}_j$ considered in [FJR1]. For further details, we refer readers to the original paper.

Put $h^i_j = \operatorname{rank} H^i(\mathscr{L}_j)$.

**Axiom 5.** *Index-Zero: Consider the class $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $\alpha_i \in \mathscr{H}_{\gamma_i}$. If $\operatorname{Fix} \gamma_i = \{0\}$ for each $i \in \{1, 2, \ldots, n\}$ and $\Lambda_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is of homology degree*

$$\sum_{j=1}^{N} (h^0_j - h^1_j) = 0,$$

*then $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle_{g,n}$ is equal to the degree of the Witten map.*

**Axiom 6.** *Composition: If the four-point class, $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is of homology degree zero, then the correlator $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ decomposes in terms of three-point correlators in the following way:*

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \sum_{\beta,\delta} \langle \alpha_1, \alpha_2, \beta \rangle \, \eta^{\beta,\delta} \, \langle \delta, \alpha_3, \alpha_4 \rangle \, ,$$

*where the sum is taken over a basis for $\mathscr{H}_{W,G}$.*

As indicated earlier, the exponential grading operator $J$ plays a special role in the $A$-model product. Note that $\operatorname{Fix} J = \{0\}$ so $\mathscr{H}_J \cong \mathbb{C}$ and $\deg \mathscr{H}_J = 0$. The identity element in the FJRW-ring is an element of $\mathscr{H}_J$, and we denote this element by $\mathbf{1}$.

**Axiom 7.** *Pairing: For $\alpha_1, \alpha_2 \in \mathscr{H}_{W,G}$, $\langle \alpha_1, \alpha_2, \mathbf{1} \rangle = \eta(\alpha_1, \alpha_2)$, where $\eta$ is the pairing in $\mathscr{H}_{W,G}$.*

**Axiom 8.** *Sums of potentials: If $W_1 \in \mathbb{C}[x_1, \ldots, x_r]$ and $W_2 \in \mathbb{C}[y_1, \ldots, y_s]$ are two non-degenerate, quasi-homogeneous potentials with maximal symmetry groups $G_1$ and $G_2$, then the maximal symmetry group of $W = W_1 + W_2$ is $G = G_1 \times G_2$, and there is an isomorphism* *of Frobenius algebras*

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{W_1,G_{W_1}} \otimes \mathscr{H}_{W_2,G_{W_2}}$$

*Remark.* We note an important consequence of Axiom 8. Under the same hypotheses as in the statement of the axiom, we have a Frobenius Algebra isomorphism

$$\mathscr{Q}_W \cong \mathscr{Q}_{W_1} \otimes \mathscr{Q}_{W_2},$$

and similarly

$$\mathscr{Q}_{W^T} \cong \mathscr{Q}_{W_1^T} \otimes \mathscr{Q}_{W_2^T}.$$

Consequently, in order to prove the Mirror Symmetry Conjecture for $W = W_1 + W_2$ a sum of decoupled potentials (with maximal $A$-model orbifold group, dual to the trivial $B$-model orbifold group), it suffices to prove it for $W_1$ and $W_2$ individually.

**Axiom 9.** *Deformation Invariance: $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is independent of the representative $W$ of a fixed deformation-equivalence class of potentials.*

## 2.4 Orbifold $B$-model

Let $W \in \mathbb{C}[y_1, \ldots, y_N]$ be a non-degenerate quasi-homogeneous potential, where $y_i$ has weight $q_i \in \mathbb{Q}$.

We will take $W$ to be an invertible potential, so $W = \sum W_j$ where each $W_j \in \mathbb{C}[y_1^{(j)}, \ldots, y_{n_j}^{(j)}]$ is of loop, chain, or Fermat type.

**Definition 12.** Let $G \subset (\mathbb{C}^*)^N$ be a group of diagonal symmetries of $W$.

For $g \in G$, $\mathrm{Fix}(g)$ is a $N_g$-dimensional co-ordinate subspace of $\mathbb{C}^N$, where $N_g = \dim \mathrm{Fix}(g)$. Put $\mathscr{Q}_g := \mathscr{Q}_{W|_{\mathrm{Fix}\, g}} \omega_{\mathrm{Fix}\, g}$, where as before the presence of the volume form $\omega_{\mathrm{Fix}\, g}$ encodes a determinant twist of the natural $G$-action on $\mathscr{Q}_{W|_{\mathrm{Fix}\, g}}$.

The unprojected state space of the Landau–Ginzburg orbifold $B$-model of $(W, G)$ is defined to be

$$\mathscr{Q} = \bigoplus_{g \in G} \mathscr{Q}_g.$$

This defines $\mathscr{Q}$ as a $G$-graded $\mathbb{C}$-vector space. $\mathscr{Q}$ also possesses a $\mathbb{Q}$ bi-grading, which we discuss in the next section. We will show that the multiplication defined in this section respects the bi-grading.

Pairings $\mathscr{Q}_g \otimes \mathscr{Q}_{g^{-1}} \to \mathbb{C}$ are induced by the residue pairing under the identification $\mathscr{Q}_g \cong \mathscr{Q}_{g^{-1}}$. The sum of these pairings endows $\mathscr{Q}$ with a non-degenerate pairing $\langle\, ,\, \rangle$.

We aim to equip $\mathscr{Q}$ with an algebra structure which preserves both the $G$-grading and the $\mathbb{Q}$ bi-grading. We observe that for $g \in G$, we have a restriction homomorphism $\mathscr{Q}_{\mathrm{id}} \to \mathscr{Q}_g$ given by setting variables not fixed by $g$ equal to zero. This induces on $\mathscr{Q}_g$ the structure of a $\mathscr{Q}_{\mathrm{id}}$ module, with $1 \cdot \omega_{\mathrm{Fix}\, g} \in \mathscr{Q}_g$ as the generator of the $g$-graded summand.

So to define an algebra structure on $\mathscr{Q}$, it suffices to define a compatible multiplication

$$1_g \star 1_h = \gamma_{g,h} 1_{gh}.$$

Since $1_e$ will be the identity for the multiplication, we require

$$1_e \star 1_g = 1_g \quad \text{so} \quad \gamma_{e,g} = 1_g = \gamma_{g,e}. \tag{2.5}$$

For the multiplication to be associative, we must have

$$(1_g \star 1_h) \star 1_k = 1_g \star (1_h \star 1_k) \quad \text{so} \quad \gamma_{g,h}\gamma_{gh,k} = \gamma_{g,hk}\gamma_{h,k}. \tag{2.6}$$

We propose the following definition of $\gamma$ and check that it satisfies (2.5) and (2.6).

**Definition 13.** For $g \in G$, recall $F_g = \{i : gX_i = X_i\}$. Define $\gamma$ through the equation

$$\gamma_{g,h} \frac{\operatorname{hess} W|_{\operatorname{Fix} g \cap \operatorname{Fix} h}}{\mu_{W|_{\operatorname{Fix} g \cap \operatorname{Fix} h}}} = \begin{cases} \frac{\operatorname{hess} W|_{\operatorname{Fix} gh}}{\mu_{W|_{\operatorname{Fix} gh}}} & \text{if } F_g \cup F_h \cup F_{gh} = \{1, \cdots, n\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

*Remark.* By definition, $\gamma_{g,h}$ has non-zero pairing with the determinant of the Hessian of $W$ on the common fixed locus of $g$ and $h$, provided each variable is fixed by at least one of $g$, $h$ and $gh$.

The denominators are dimensions of the local algebra of $W$ restricted to appropriate fixed loci. The choice of denominator has no bearing on the associativity of the product, as one observes that a triple-product vanishes unless one of the factors comes from the identity summand. Nevertheless, the choice *is* of some consequence in the context of mirror symmetry. While the above prescription is intrinsic and consistent, it is certainly possible that a different *scaling* of the $\gamma_{g,h}$ will be useful for studying compatibility of the $B$-model product with that on the $A$-model. To date, direct computation of the $A$-model product beyond the maximally orbifolded case studied in Section 3.1 has been elusive.

**Proposition 2.1.** *The above multiplication $\star$ is associative.*

*Proof.* This definition obviously satisfies (2.5), and it remains to check the associativity (2.6) of the candidate cocycle $\gamma$.

We see here the benefit of restricting our attention to invertible potentials (sums of loops, chains, and Fermat types).

We first check associativity of multiplication when $W$ is of one of these atomic types. The key point here is that if a symmetry of $W$ fixes $y_1$, then it acts trivially on all of $\mathbb{C}^N$. So $1_g \star 1_h = \gamma_{g,h} 1_{gh}$ can be non-zero only if one of $g$, $h$, or $gh$ is the identity.

If $g = \mathrm{id}$, $h = \mathrm{id}$, or $k = \mathrm{id}$ then associativity is obvious.

Suppose $g \neq \mathrm{id}$, $h \neq \mathrm{id}$ and $k \neq \mathrm{id}$. We show that both sides of (2.6) vanish. Consider the left hand side. If $gh \neq \mathrm{id}$ then by the above remark, $1_g \star 1_h = 0$. If $gh = \mathrm{id}$, the left hand side is $\gamma_{g,g^{-1}} 1_k$. Now, $\gamma_{g,g^{-1}}$ pairs with $\operatorname{hess} W|_{\mathrm{Fix}(g)}$, so depends on the variables not fixed by $g$ (in particular $y_1$). Since $k \neq \mathrm{id}$, $y_1$ is not fixed by $k$, and $\gamma_{g,g^{-1}} 1_k = 0 \in \mathscr{Q}_k$. A similar argument applies to the right hand side.

Thus we have an associative multiplication on $\mathscr{Q}$ for $W$ a loop, chain, or Fermat potential. In fact, we have shown furthermore that a triple-product vanishes unless one of the factors is in the identity sector, and the other two factors are in sectors corresponding to mutually inverse group elements.

This multiplication (Definition 13) extends to any invertible potential, as the product may be decomposed into contributions from each atomic summand, and associativity on the summands implies associativity for the whole invertible potential. $\square$

In the next section, we show that the multiplication on the unprojected state space descends to a multiplication on invariants, without making any assumptions about the potential being of atomic type.

#### 2.4.1 Projecting to invariants

Now we turn our attention to the $G$-invariants in $\mathscr{Q}$ for the determinant-twisted $G$ action. We make the important restriction that $G \subseteq SL_N\mathbb{C}$, so that the $G$-invariants in $\mathscr{Q}_{\mathrm{id}}$ are the same whether or not we twist by the determinant on $\operatorname{Fix} \mathrm{id} = \mathbb{C}^N$. This means that the $\mathscr{Q}_{\mathrm{id}}$-module structure on $\mathscr{Q} = \bigoplus_{g \in G} \mathscr{Q}_g$ descends to a $(\mathscr{Q}_{\mathrm{id}})^G$-module structure on the determinant-twisted $G$ invariants $\left(\bigoplus_{g \in G} \mathscr{Q}_g\right)^G$. This '$SL$' hypothesis will be justified later when we see that admissible $A$-model orbifold groups correspond to subgroups of $SL_N\mathbb{C}$ on the $B$-side.

To see that the product descends to invariants, we prove the following lemma.

**Lemma 2.2.** *Suppose $H, K \in Q_e$ are monomials such that $H1_h \in \mathscr{Q}_h$ and $K1_k \in \mathscr{Q}_k$ are (determinant-twisted) $G$-invariants. Then $HK1_h \star 1_k$ is a (determinant-twisted) $G-$ invariant.*

*Proof.* The lemma is trivially true if $HK1_h \star 1_k = 0$. We may therefore suppose that for each $i \in \{1, \dots, n\}$, at least one of $h_i$, $k_i$ or $h_i k_i$ equals 1.

$G$-invariance of the $H1_h$ and $K1_k$ yields

$$g(H) \prod_{i \in F_h} g_i = 1 \tag{2.8}$$

$$g(K) \prod_{i \in F_k} g_i = 1, \tag{2.9}$$

where $g(H)$ denotes the phase of the action of $g$ on the monomial $H$, and similarly for $g(K)$. We need to compute the action of $g$ on $HK1_h \star 1_k$.

Since we assume $1_h \star 1_k \neq 0$, Equation (2.7) applies. The phase of $g$ on either side of this relation must coincide, so

$$g(\gamma_{h,k}) = \frac{\prod_{i \in F_{hk}} g_i^{-2}}{\prod_{i \in F_h \cap F_k} g_i^{-2}}.$$

Then, using (2.8) and (2.9), the phase of $g$ on

$$(H1_h) \star (K1_k) = HK\gamma_{h,k} 1_{hk}$$

is

$$\begin{aligned} g(H)g(K)g(\gamma_{h,k}) \prod_{i \in F_{hk}} g_i &= \prod_{i \in F_h} g_i^{-1} \prod_{i \in F_k} g_i^{-1} \frac{\prod_{i \in F_h \cap F_k} g_i^2}{\prod_{i \in F_{hk}} g_i^2} \prod_{i \in F_{hk}} g_i \\ &= \prod_{i \in \{1, \dots, n\}} g_i^{-1} \\ &= 1, \end{aligned}$$

by the assumption $G \subseteq SL_N\mathbb{C}$. (For the penultimate equality, recall that at least one of $h$, $k$, and $hk$ equals id.)

So the $\star$-product of $G$-invariants is again $G$-invariant. $\square$

### 2.4.2 Pairing and Frobenius Algebra

The pairing $\langle\, , \rangle$ on $\mathscr{Q}_{W,G}$ is the sum of the pairings $\mathscr{Q}_g \otimes \mathscr{Q}_{g^{-1}} \to \mathbb{C}$, which are induced by the residue pairing under the identification $\mathscr{Q}_g \cong \mathscr{Q}_{g^{-1}}$.

The orbifold Milnor ring (after projecting to $G$ invariants) is a Frobenius Algebra. This follows from the definition of the pairing and the associativity of multiplication.

By construction, the above multiplication preserves the $G$-grading, and we will show in the next section that it preserves the $\mathbb{Q}$ bi-grading also.

## 2.5 Bi-grading

To introduce the bi-gradings for Landau–Ginzburg theories, we introduce the following standard notations.

**Notation.** A form $\alpha = \bigwedge X_i^{\alpha_i} dX_i \in \mathscr{Q}_{W|_{\mathrm{Fix}\, g}} \cdot \omega_{\mathrm{Fix}\, g}$ has *weighted degree*

$$\deg \alpha := \sum (\alpha_i + 1) q_i.$$

Note that the volume form contributes to the degree on an equal footing with the monomial.

A symmetry $g = (e^{2\pi i \Theta_1^g}, \ldots, e^{2\pi i \Theta_N^g}) \in (\mathbb{C}^*)^N$ with $\Theta_i^g \in [0, 1)$ has *age*

$$\mathrm{age}(g) := \sum_{i=1}^{N} \Theta_i^g.$$

**Definition 14.** For an invariant $\alpha$ in the fixed locus of a symmetry $h = (e^{2\pi i \Theta_1^h}, \ldots, e^{2\pi i \Theta_N^h})$, we define the bi-gradings as follows.

$$(Q_+^A, Q_-^A) := (\deg \alpha, N_h - \deg \alpha) + (\mathrm{age}\, h, \mathrm{age}\, h) - (\sum_{i=1}^{N} q_i, \sum_{i=1}^{N} q_i) \tag{2.10}$$

$$(Q_+^B, Q_-^B) := (\deg \alpha, \deg \alpha) + (\mathrm{age}\, h, \mathrm{age}\, h^{-1}) - (\sum_{i=1}^{N} q_i, \sum_{i=1}^{N} q_i) \tag{2.11}$$

*Remark.* Note this grading recovers the $A$-model grading of Equation 2.2 as the sum of the $A$-model bi-gradings.

**Lemma 2.1.** *The $B$-model multiplication (Def. 13) preserves the bigrading $(Q_+^B, Q_-^B)$ of Eq. (2.10).*

*Proof.* Consider a product

$$H1_h \star K1_k = \gamma_{h,k} HK1_{hk}$$

If the product vanishes, it trivially preserves bi-degree. If it is non-vanishing, we consider the two cases $hk \neq \mathrm{id}$ and $hk = \mathrm{id}$.

If $hk \neq \mathrm{id}$, then we may suppose $h = \mathrm{id}$ so $\gamma_{h,k} = 1$.

$$\begin{aligned}
\left(\deg(H1_{\mathrm{id}}) + \operatorname{age}\mathrm{id} - \sum_{i=1}^{N} q_i\right) + \left(\deg(K1_k) + \operatorname{age} k - \sum_{i=1}^{N} q_i\right) \\
= \deg(HK1_k) + \deg 1_{\mathrm{id}} + \operatorname{age} k - 2\sum_{i=1}^{N} q_i \\
= \deg(HK1_k) + \operatorname{age} k - \sum_{i=1}^{N} q_i,
\end{aligned}$$

and similarly with $\operatorname{age} k$ replaced by $\operatorname{age} k^{-1}$, so bi-degree is preserved.

If $hk = \mathrm{id}$ then

$$\operatorname{age} h + \operatorname{age} k = \operatorname{age} h + \operatorname{age} h^{-1} = N - N_h.$$

This case is symmetric in $h$ and $h^{-1}$, so to show the bi-grading is preserved by multiplication we need only show $\deg + \operatorname{age} - \sum_{i=1}^{N} q_i$ is preserved. This follows from the following computation.

$$\begin{aligned}
\left(\deg(H1_h) + \operatorname{age}(h) - \sum_{i=1}^{N} q_i\right) + \left(\deg(K1_{h^{-1}}) + \operatorname{age}(h^{-1}) - \sum_{i=1}^{N} q_i\right) \\
= \deg(HK) + \deg 1_h + \deg 1_{h^{-1}} + (N - N_h) - 2\sum_{i=1}^{N} q_i \\
= \deg(HK1_{\mathrm{id}}) + (N - 2\deg_{1_{\mathrm{id}}}) - (N_h - 2\deg_{1_h}) - \sum_{i=1}^{N} q_i \\
= \deg(\gamma_{h,k}HK1_{\mathrm{id}}) + \operatorname{age}\mathrm{id} - \sum_{i=1}^{N} q_i,
\end{aligned}$$

because $\operatorname{age}(\mathrm{id}) = 0$, and by definition of $\gamma_{h,k}$, we have

$$\begin{aligned}
\deg(\gamma_{h,k}) &= \deg \operatorname{hess} W - \deg \operatorname{hess} W|_{\operatorname{Fix} h} \\
&= (N - 2\deg_{1_{\mathrm{id}}}) - (N_h - 2\deg_{1_h}).
\end{aligned}$$

The last equality holds because $\operatorname{hess} W$ is the determinant of the matrix $(\partial_i \partial_j W)$, so

$$\deg \operatorname{hess} W = \deg \prod_{i=1}^{N} \partial_i^2 W = \sum_{i=1}^{N} \deg \partial_i^2 W = \sum_{i=1}^{N} (1 - 2q_i) = N - 2 \deg 1_{\mathrm{id}},$$

and similarly for $W|_{\mathrm{Fix}\, h}$. $\square$

## 2.6 Relation between $A$ and $B$ model for a fixed potential

Note that the state spaces of the $A$ and $B$ models for a fixed potential are isomorphic as vector spaces. The relationship between the bi-gradings is:

$$deg_+^A = deg_+^B, \qquad deg_-^A = -deg_-^B + \hat{c}.$$

This simple relation is particularly relevant in the Calabi-Yau case ($\sum q_i = 1$) where the same relation holds for the Calabi-Yau hypersurface defined by $W = 0$, giving further evidence of Landau–Ginzburg mirror symmetry.

## 2.7 Duality of Groups

As before, let

$$W = \sum_{i=1}^{N} \prod_{j=1}^{N} X_j^{a_{ij}}$$

be a non-degenerate quasi-homogeneous potential, with exponent matrix $A = (a_{ij})$.

Following Berglund–Hübsch [BH], we consider the transposed potential

$$W^T = \sum_{i=1}^{N} \prod_{j=1}^{N} Y_j^{a_{ji}},$$

which has exponent matrix $A^T$.

If the $i^{\text{th}}$ row of $A^{-1}$ is given by $\overline{\rho}_i = (\overline{\varphi}_1^{(i)}, \cdots, \overline{\varphi}_N^{(i)})$, we get diagonal symmetries of $W^T$ as before:

$$\overline{\rho}_k = (e^{2\pi i \overline{\varphi}_1^{(k)}}, \cdots, e^{2\pi i \overline{\varphi}_N^{(k)}}).$$

As above,the $\overline{\rho}_k$'s are symmetries of $W^T$ and generate $G_{W^T}^{\max}$. The exponential grading operator is $\overline{J} = \prod_{i=1}^{N} \overline{\rho}_i$.

The following lemma is straightforward, but essential to what follows.

**Lemma 2.1.**

$$\prod_{i=1}^{N} \rho_i^{\alpha_i} \textit{ preserves the monomial } \prod_{j=1}^{N} X_j^{r_j}$$

*if and only if*

$$\prod_{j=1}^{N} \overline{\rho}_j^{r_j} \textit{ preserves the monomial } \prod_{i=1}^{N} Y_i^{\alpha_i}.$$

*Proof.* Both statements are equivalent to

$$(r_1, \ldots, r_N) A_W^{-1} (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{Z}$$

□

In particular, since each $\overline{\rho}_k$ preserves every monomial $\prod_i Y_i^{a_{ij}}$ appearing in $W^T$, we have that $\prod_i \rho_i^{a_{ij}}$ preserves every $X_k$. That is:

**Corollary 2.2.** *Let* $W = \sum_{i=1}^{N} \prod_{j=1}^{N} X_j^{a_{ij}}$ *be a non-degenerate, invertible potential with exponent matrix* $A = (a_{ij})$. *Let the symmetry* $\rho_k$ *be given by the* $k$*th column of* $A^{-1}$ *as above.*

*Then*

$$\prod_{i=1}^{N} \rho_i^{a_{ij}} = 1$$

*for every* $j \in \{1, \ldots, n\}$.

*Remark.* In [K], this observation is attributed to Skarke in the special case of Loop potentials.

**Definition 15.** For a group $G$ of symmetries of $W$, we define the dual group $G^T$ as

$$(2.12)\qquad G^T := \left\{ \prod_{i=1}^{N} \overline{\rho}_i^{r_i} \in G_{W^T} \;\middle|\; (r_1, \ldots, r_N) A_W^{-1} (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{Z} \text{ for all } \prod_{i=1}^{N} \rho_i^{\alpha_i} \in G \right\}.$$

If $g = \prod_{i=1}^{N} \rho_i^{\alpha_i'}$ is a different presentation, $\prod_{i=1}^{N} \rho_i^{\alpha_i - \alpha_i'} = 1$. Hence

$$A_W^{-1}[\alpha_1 - \alpha_1', \cdots, \alpha_N - \alpha_N']^T \in \mathbb{Z}^N$$

and the above definition is independent of presentation of elements of $G$.

The following lemma will be used to show that $(G^T)^T = G$.

**Lemma 2.3.** *Let $G, H \subset (\mathbb{C}^*)^N$ be groups acting diagonally on $\mathbb{C}[X_1, \ldots, X_N]$ with*

$$\mathbb{C}[X_1, \ldots, X_N]^G = \mathbb{C}[X_1, \ldots, X_N]^H.$$

*Then the fixed fields for the induced actions of $G$ and $H$ on $\mathbb{C}(X_1, \ldots, X_N)$ also coincide, i.e.*

$$\mathbb{C}(X_1, \ldots, X_N)^G = \mathbb{C}(X_1, \ldots, X_N)^H.$$

*Proof.* Given an element $k = a/b$ of $\mathbb{C}(X_1, \ldots, X_N)$ with $a$ a sum of distinct monomials $a_i$, $G$-invariance of $k$ is equivalent to $G$-invariance of each summand $a_i/b$, so we may suppose the numerator is a monomial. Since invariance of $k$ is equivalent to invariance of $1/k$, we may suppose that both numerator and denominator are monomials.

Now given $k$ in $\mathbb{C}(X_1, \ldots, X_N)$ a ratio of monomials, we may (since the $G$-action is diagonal) augment numerator and denominator by the same monomial to express $k$ as $f/g$, where $f$ is a monomial and $g$ is a *$G$-invariant* monomial, so that $G$-invariance of $k$ is equivalent to $G$

invariance of the monomial $f$. This is then equivalent to $H$-invariance of $f$ by hypothesis, and since $g$ is also $H$-invariant, this is equivalent to the $H$-invariance of the rational function $f/g = k$. $\square$

**Lemma 2.4.** *Let $G$ be a group of diagonal symmetries of the non-degenerate invertible potential $W$, and $G^T$ the dual group of symmetries of $W^T$. Then*

$$(G^T)^T = G.$$

*Proof.* It is clear from the definition that $G \subseteq (G^T)^T$ and $\mathbb{C}[X_1, \ldots, X_N]^G \subseteq \mathbb{C}[X_1, \ldots, X_N]^{(G^T)^T}$. This implies that $G$ and $(G^T)^T$ have equal invariant rings, and the actions on $\mathbb{C}[X_1, \ldots, X_N]$ extend to actions on the fraction field with the same fixed field. Because the groups acting are finite, it follows (*e.g.* [Ar], Corollaries to Theorem 14) that $G = (G^T)^T$. $\square$

It is also obvious $\{1\}^T = G^{\max}$. Now we compute $\langle J\rangle^T$. Since $J = \prod_{i=1}^N \rho_i$, $h = \prod_{i=1}^N \bar{\rho}_i^{r_i} \in \langle J\rangle^T$ if and only if $\sum_i r_i q_i \in \mathbb{Z}$. Since $\sum_i r_i q_i$ is precisely the phase of $\det(h)$, we have $\langle J\rangle^T = SL_N\mathbb{C} \cap G^{\max}_{W^T}$.

This explains the $SL$ restriction made in the proof of Lemma 2.2 that the orbifold $B$-model multiplication descends to the invariants under the action of the orbifold group.

We can use the argument from the proof of Lemma 2.4 to settle a question suggested in [FJR1], namely whether any diagonal symmetry group containing $J$ satisfies the following definition of admissible groups.

**Definition 16** ([FJR1] Defn 2.3.2)**.** We say that a subgroup $G \leq G^{\max}_W$ is *admissible* or is an *admissible group of Abelian symmetries of* $W$ if there exists a Laurent polynomial $Z$, quasi-homogeneous with the same weights $q_i$ as $W$, but with no monomials in common with $W$, and such that $G = G_{W+Z}$.

*Remark.* In our discussion of the moduli space $\overline{\mathscr{M}}^W_{g,k}$ of orbifold curves endowed with line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ forming a $W$ structure, we glossed over the orbifold structure at the marked points $p_1, \ldots, p_k$. In general, the marked points should have isotropy in $G_W^{\max}$. Restricting the isotropy to some lie in some subgroup $G$ of $G_W^{\max}$, it is not clear that the moduli space one obtains is a proper stack. The above notion of admissibility is a sufficient condition for properness of this stack, which allows the construction of the Landau-Ginzburg $A$-model for proper subgroups of $G_W^{\max}$.

The following proposition therefore indicates that the construction of the Landau-Ginzburg $A$-model in [FJR1] is valid for any subgroup $G \subseteq G_W^{\max}$ containing $J$.

**Proposition 2.5.** *For $W \in \mathbb{C}[X_1, \ldots, X_N]$ a non-degenerate (not-necessarily invertible) potential, any group of diagonal symmetries of $W$ containing $J$ is admissible.*

*Proof.* For a group $G$ of diagonal symmetries of $W$ containing $J$ to be admissible, we require the existence of a Laurent polynomial $Z$ in $X_1, \ldots, X_N$, quasi-homogeneous with the same weights as $W$, such that $G$ is the *maximal* diagonal symmetry group of $W + Z$.

Now, the ring of $G$-invariants is finitely generated by monomials; let us fix a generating set. Suppose $W = \sum_i W_i$ is a sum of distinct monomials $W_i$. If we let $Z$ be the sum of those generators not divisible (in $\mathbb{C}[X_1, \ldots, X_N]$) by any $W_i$, then $G$ is the maximal diagonal symmetry group of $W + Z$. (Otherwise there is a diagonal symmetry group $H$, with $G \subseteq H$ and $\mathbb{C}[X_1, \ldots, X_N]^G \subseteq \mathbb{C}[X_1, \ldots, X_N]^H$, implying $G = H$ as before). Since $J$ preserves each of the constituent monomials of $Z$, each of these monomials has integral quasi-homogeneous degree. We may correct each of these monomials by a (negative) power of any monomial in $W$ to ensure that each of the monomials has quasi-homogeneous degree equal to 1, and since we are correcting by $G$-invariants not dividing the monomials of $Z$, we do not change the maximal symmetry group of $W + Z$. $\square$

## 2.8 Mirror Symmetry for State Spaces

We propose in this section a 'Total Unprojected Mirror map'. The adjective 'total' indicates that the map involves a sum over maximal symmetry groups, while 'unprojected' indicates that the map is defined without taking invariants for any diagonal action. Mirror Symmetry for LG state-spaces will be obtained by restricting the orbifold groups and taking invariants.

First, we need a lemma, which will allow us to exploit the Remark following Lemma 1.2, in which we noted that the natural map from $B$-model Milnor ring elements $\prod_i Y_i^{\alpha_i} dY_i$ to $A$-model symmetries $\prod_i \rho_i^{\alpha_i} J$ maps onto the collection of symmetries with even-dimensional fixed locus.

**Lemma 2.1.** *Let $W$ be an invertible potential, and let $h \in G_{W^T}^{max}$ and $\bigwedge_{j \in F_h} Y_j^{\alpha_j} dY_j \in \mathscr{Q}_{W^T|_{\mathrm{Fix}\, h}}$. Define $g = \prod_{j \in F_h} \rho_j^{\alpha_j+1} \in G_W^{max}$. Then $F_g \cup F_h = \{1, \ldots, N\}$ and $F_g \cap F_h$ has an even number of elements.*

*Proof.* It suffices to prove this for atomic potentials. For Fermat type, we have $N = 1$ and either $h = \mathrm{id}$ or $g = \mathrm{id}$, so the lemma is clear.

For loop potentials, the only symmetry with non-trivial fixed locus is the identity. So if $h \neq \mathrm{id}$, then $g = \mathrm{id}$ and the lemma holds. If $h = \mathrm{id}$ and $g \neq \mathrm{id}$, the lemma also holds, so the only case to check is $h = \mathrm{id}$ and $g = \mathrm{id}$. But by Lemma 1.2, this can only happen if $N$ is even, so the result follows.

The only non-trivial case is the chain potential. Here, $F_h = \{1, \ldots, t\}$ and $F_g = \{s, \ldots, N\}$ for some $s$ and $t$. If $F_g = \{s, \ldots, N\}$, we must have $s \leq t + 1$ and $\alpha_j = \delta_{t-j}^{\mathrm{even}}(a_j - 1)$ for $s \leq j \leq t$. Further, since $Y_{s-1} \prod_{j=s}^{t} Y_j^{\delta_{t-j}^{\mathrm{even}}(a_j-1)}$ vanishes in $\mathscr{Q}_{W^T|_{\{Y_1, \cdots, Y_t\}}}$ when $t - s$ is even, we see that $t - s$ must be odd. *i.e.* There is an even number of elements (possibly zero) in $\{s, s+1, \ldots, t\}$, and the result follows. $\square$

**Lemma 2.2** (Definition of mirror elements)**.** *Let $W$ be an atomic invertible potential, $h \in$*

*$G^{max}_{W^T}$ and $H = \bigwedge_{j\in F_h} Y_j^{\alpha_j} dY_j \in \mathscr{Q}_{W^T|_{\mathrm{Fix}\,h}}$. Put $g_H := \prod_{j\in F_h} \rho_j^{\alpha_j+1} \in G^{max}_W$. If $W$ is of chain or Fermat type, there is a unique element $G_h = \bigwedge_{j\in F_{g_H}} X_j^{r_j} dX_j \in \mathscr{Q}_{W|_{\mathrm{Fix}\,g_H}}$ such that $g_H = \prod_{j\in F_{g_H}} \overline{\rho}_j^{r_j+1}$.*

*If $W$ is a loop potential we have a unique choice of $G_h$ as above unless $h = \mathrm{id}$ and $g_H = \mathrm{id}$. In this case there are two choices of $\alpha_1, \dots, \alpha_N$ for which $g_H = \mathrm{id}$, and correspondingly two choices of $G_h$.*

*Proof.* (Note it follows from Corollary 2.2 that different monomial representatives $H$ of a given element in the Milnor ring of $W^T$ correspond to the same element $g_H \in G^{\max}_W$.)

If $W$ is a loop potential with $N$ even, and $h = \mathrm{id}$, we may have $g_H = \mathrm{id}$, in which case $\alpha_j = \delta_j^{\mathrm{even}}(a_j - 1)$ or $\alpha_j = \delta_j^{\mathrm{odd}}(a_j - 1)$. In each instance, we prescribe $r_j = \alpha_j$ for all $j$.

Otherwise, Lemma 2.1 implies that $F_h \cup F_{g_H} = \{1, \dots, N\}$ and $\# F_h \cap F_{g_H}$ is even, and by the remark following Lemma 1.4, there is a *unique* generator $G_h = \bigwedge_{j\in F_{g_H}} X_j^{r_j} dX_j \in \mathscr{Q}_{W|_{\mathrm{Fix}\,g_H}}$ such that $h = \prod_{j\in F_{g_H}} \overline{\rho}_j^{r_j+1}$. □

**Notation.** We use 'ket' notation to indicate group grading. For example, we denote by $G_h \,|\, g_H\rangle$ the element $G_h \in \mathscr{Q}_{W|_{\mathrm{Fix}\,g_H}} \subset \oplus_{g\in G^{\max}_W} \mathscr{Q}_{W_g}$ whose existence was asserted in the preceding lemma.

**Definition 17** (Total Unprojected Mirror Map)**.** Let $W$ be a non-degenerate, invertible potential of atomic type. We define the Total Unprojected Mirror Map as a linear map given on monomial generators by

$$\bigoplus_{h\in G^{\max}_{W^T}} \mathscr{Q}_{W^T|_{\mathrm{Fix}\,h}} \longrightarrow \bigoplus_{g\in G^{\max}_W} \mathscr{Q}_{W|_{\mathrm{Fix}\,g}} \tag{2.13}$$
$$H \,|\, h\rangle \longmapsto G_h \,|\, g_H\rangle\,.$$

If $W = \sum_j W_j$ is a sum of atomic potentials, note that $G^{\max}_W = \bigoplus_j G^{\max}_{W_j}$ and $\mathscr{Q}_{W|_{\mathrm{Fix}(\oplus_j g_j)}} = \bigotimes_j \mathscr{Q}_{W_j|_{\mathrm{Fix}\,g_j}}$.

So

$$\bigoplus_{g \in G_W^{\max}} \mathscr{Q}_{W|_{\mathrm{Fix}\, g}} = \bigotimes_j \left( \bigoplus_{g \in G_{W_j}^{\max}} \mathscr{Q}_{W_j|_{\mathrm{Fix}\, g}} \right),$$

and similarly for $W^T$. We may therefore define the Total Unprojected Mirror Map for an arbitrary invertible potential $W = \sum W_j$ as the tensor product of the map 2.13 on the atomic summands $W_j$.

**Theorem 2.3.** *The total unprojected mirror map is an isomorphism.*

*Proof.* Applying the above definition with $G^T$ in place of $G$, and recalling that $(G^T)^T = G$, we obtain the inverse map. □

**Theorem 2.4** (Projected Mirror Map)**.** *Let $W$ be a non-degenerate, invertible potential and $G$ an admissible A-model diagonal symmetry group of $W$. Restricting the total unprojected mirror map to $\mathscr{Q}_{W^T,G^T}$ yields an isomorphism*

$$\mathscr{Q}_{W^T,G^T} \cong \mathscr{H}_{W,G}.$$

*Proof.* Certainly, the restriction of the total unprojected mirror map to $\mathscr{Q}_{W^T,G^T}$ yields and isomorphism onto its image. By definition of $G^T$, this image is contained in $\mathscr{H}_{W,G}$. On the other hand, restricting the inverse mirror map to $\mathscr{H}_{W,G}$ yields an isomorphism onto its image, which is contained in $\mathscr{Q}_{W^T,G^T}$. □

*Remark.* For $G = G_W^{\max}$, this recovers the main result of [K].

*Example.* We present here the example of the two-variable loop potential $W = x^3y + xy^5$, orbifolded by $J = (e^{(2\pi i)2/7}, e^{(2\pi i)1/7})$ on the $A$-side and by the dual group $J^T = \langle(-1,-1)\rangle$ on the $B$-side. The table below presents the vector space generators for the $A$-model $(W, \langle J \rangle)$ and the $B$-model $(W^T, \langle J^T \rangle)$, along with the bi-grading. We denote the standard volume form on $\mathrm{Fix}\, \rho_x^a \rho_y^b$ by $e_{\rho_x^a \rho_y^b}$. The $A$ and $B$ model invariants in each column correspond to each other under the Mirror Map (Equation (2.13)), and evidently the bi-grading is preserved.

| $(W, \langle J \rangle)$ | $e_{\rho_x^1 \rho_y^1}$ | $e_{\rho_x^2 \rho_y^2}$ | $e_{\rho_x^0 \rho_y^2}$ | $e_{\rho_x^1 \rho_y^3}$ | $e_{\rho_x^2 \rho_y^4}$ | $e_{\rho_x^0 \rho_y^4}$ | $x^2 e_{\rho_x^0 \rho_y^0}$ | $xy^2 e_{\rho_x^0 \rho_y^0}$ | $y^4 e_{\rho_x^0 \rho_y^0}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\deg_+^A$ | 0 | $\frac{6}{7}$ | $\frac{12}{7}$ | $\frac{4}{7}$ | $\frac{10}{7}$ | $\frac{16}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ |
| $\deg_-^A$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(W^T, SL)$ | $e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xye_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2y^2e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $y^2e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xy^3e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2y^4e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2e_{\bar{\rho}_x^3 \bar{\rho}_y^1}$ | $e_{\bar{\rho}_x^2 \bar{\rho}_y^3}$ | $y^4e_{\bar{\rho}_x^1 \bar{\rho}_y^5}$ |
| $\deg_+^B$ | 0 | $\frac{6}{7}$ | $\frac{12}{7}$ | $\frac{4}{7}$ | $\frac{10}{7}$ | $\frac{16}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ |
| $\deg_-^B$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

*Example.* Now we present the example of the two-variable chain potential $W = x^3y + y^4$, orbifolded by $J = (e^{2\pi i/4}, e^{2\pi i/4})$ on the $A$-side and by the dual group $J^T = \left\langle (e^{2\pi i/3}, e^{-2\pi i/3}) \right\rangle$ on the $B$-side. The table below presents the vector space generators for the $A$-model $(W, \langle J \rangle)$ and the $B$-model $(W^T, \left\langle J^T \right\rangle)$, along with the bi-grading. The $A$ and $B$ model invariants in each column correspond to each other under the Mirror Map (Equation (2.13)), and we see again that the bi-grading is preserved.

| $(W, \langle J \rangle)$ | $e_{\rho_x^1 \rho_y^1}$ | $e_{\rho_x^2 \rho_y^2}$ | $e_{\rho_x^0 \rho_y^3}$ | $x^2 e_{\rho_x^0 \rho_y^0}$ | $xye_{\rho_x^0 \rho_y^0}$ | $y^2 e_{\rho_x^0 \rho_y^0}$ |
|---|---|---|---|---|---|---|
| $\deg_+^A$ | 0 | 1 | 2 | 1 | 1 | 1 |
| $\deg_-^A$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(W^T, SL)$ | $e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xye_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2y^2e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $y^3e_{\bar{\rho}_x^3 \bar{\rho}_y^1}$ | $e_{\bar{\rho}_x^2 \bar{\rho}_y^2}$ | $e_{\bar{\rho}_x^1 \bar{\rho}_y^3}$ |
| $\deg_+^B$ | 0 | 1 | 2 | 1 | 1 | 1 |
| $\deg_-^B$ | 0 | 0 | 0 | 0 | 0 | 0 |

We now prove that the (total, unprojected) Mirror Map (Equation (2.13)) preserves bi-degree. Of course, the LG state-space isomorphisms corresponding to different pairs $(H, H^T)$ of orbifold groups inherit this property.

**Theorem 2.5.** *Let $W$ be a non-degenerate invertible potential. The Unprojected Mirror Map defined on generators by Equation (2.13) is a bi-degree preserving isomorphism of vector spaces.*

*Proof.* The following lemma will be useful to reduce the amount of direct computation required.

**Lemma 2.6.** *If the mirror map sends $H\,|\,h\rangle \mapsto G\,|\,g\rangle$ with $\deg H = \operatorname{age} g$, $\operatorname{age} h = \deg G$ and $N_h + N_g = N$ then $Q_B^{\pm}(H\,|\,h\rangle) = Q_A^{\pm}(G\,|\,g\rangle)$.*

*Proof.* It is clear that under the above hypothesis, the '+' grading is preserved. To show the '$-$' grading is also preserved, it suffices to observe that

$$N_h - \deg H = N - N_g - \operatorname{age} g = \operatorname{age} g^{-1}.$$

$\square$

Inspection of Equations (2.10) indicates that the bi-degrees are simply sums of contributions from each atomic summand, so if the total unprojected mirror map preserves bidegree for atomic potentials, it does so for all invertible potentials.

We may therefore restrict our attention to the invertible potentials of Fermat, Loop and Chain type. For each of these cases, we will prove that Equation (2.13) is a bi-degree preserving vector space isomorphism.

**Fermat:** $W = X^N$

The total unprojected mirror map is defined on generators by:

$$Y^k dY \mid \mathrm{id}\,\rangle \longmapsto 1 \,\big|\, \rho^{k+1} \big\rangle\,, \qquad 0 \le k < N-1,$$

and

$$1 \,\big|\, \rho^{k+1} \big\rangle \longmapsto X^k dX \mid \mathrm{id}\,\rangle\,, \qquad 0 \le k < N-1.$$

Lemma 2.6 in this case, after noting that the mirror map exchanges degree and age, and the fixed loci on either side of the mirror map have complementary dimension.

**Loop:** $W = \sum_{i=1}^{N} X_i^{a_i} X_{i+1}$

(The subscripts are taken modulo $N$).

The structure of the loop potential means that the only group element with non-trivial fixed locus is the identity. Therefore we study the total unprojected mirror map out of the $B$-model identity sector and twisted sectors separately.

*Identity $B$-model sector:*

$$\bigwedge_{j=1}^{N} Y_j^{\alpha_j} dY_j \mid \mathrm{id}\,\rangle \longmapsto \bigwedge X_j^{r_j} dX_j \left| \prod_{j=1}^{N} \rho_j^{\alpha_j+1} \right\rangle ,$$

where we are purposefully vague about the range of the product for the $A$-model monomial, since it may either be empty (in which case the monomial should be interpreted as 1) or it may run from 1 to $N$ (when the $B$-model monomial corresponds to the $A$-model identity group element).

In the first case, we have

$$\bigwedge_{j=1}^{N} Y_j^{\alpha_j} dY_j \mid \mathrm{id}\,\rangle \longmapsto 1 \left| \prod_{j=1}^{N} \rho_j^{\alpha_j+1} \right\rangle ,$$

so

$$\begin{aligned}(Q_+^B, Q_-^B) &= (\sum_{i=1}^N (\alpha_i+1)\overline{q}_i, \sum_{i=1}^N (\alpha_i+1)\overline{q}_i) + (\operatorname{age}\mathrm{id}, \operatorname{age}\mathrm{id}) - (\sum_{i=1}^N \overline{q}_i, \sum_{i=1}^N \overline{q}_i)\\ &= (\operatorname{age} g, \operatorname{age} g) + (\deg 1, \deg 1) - (\sum_{i=1}^N q_i, \sum_{i=1}^N q_i)\\ &= (Q_+^A, Q_-^A).\end{aligned}$$

On the other hand, if $\prod_{j=1}^N \rho_j^{\alpha_j+1} = \mathrm{id}$, $N$ is even and the mirror map looks like

$$\bigwedge_{j=1}^N Y_j^{\delta_j^{\mathrm{even}}(a_j-1)} dY_j \mid \mathrm{id}\rangle \longmapsto \bigwedge_{j=1}^N X_j^{\delta_j^{\mathrm{even}}(a_j-1)} dX_j \mid \mathrm{id}\rangle\,,$$

or a similar expression with 'even' replaced by 'odd'. We note that

$$\deg \bigwedge_{j=1}^N X_j^{\delta_j^{\mathrm{even}}(a_j-1)} dX_j = \sum_{i \text{ even}} (a_j q_j + q_{j+1}) = N/2,$$

since $a_j q_j + q_{j+1} = 1$ by definition of the weights. (Here we take indices modulo $N$.) A similar statement holds with 'even' replaced by 'odd', and the bi-degrees are

$$\begin{aligned}(Q_+^B, Q_-^B) &= (N/2, N/2) + (\operatorname{age}\mathrm{id}, \operatorname{age}\mathrm{id}) - (\sum_{i=1}^N \overline{q}_i)\\ &= (N/2, N - N/2) + (\operatorname{age}\mathrm{id}, \operatorname{age}\mathrm{id}) - (\sum_{i=1}^N q_i)\\ &= (Q_+^A, Q_-^A)\end{aligned}$$

*Twisted $B$-model sectors:*

Since the $B$-model twisted sectors have trivial fixed loci, the mirror map sends them all to the $A$-model untwisted sector.

$$1\left|\prod_{j=1}^{N}\overline{\rho}_j^{r_j+1}\right\rangle \longmapsto \bigwedge_{j=1}^{N} X_j^{r_j}dX_j \,|\, \mathrm{id}\,\rangle\,,$$

In this case, the fixed loci are of complementary dimension, $\deg 1 = \operatorname{age}\mathrm{id}$ and $\deg \bigwedge_{j=1}^{N} X_j^{r_j}dX_j = \operatorname{age}\prod_{j=1}^{N}\overline{\rho}_j^{r_j+1}$ by Lemma 1.1, so Lemma 2.6 ensures that bi-degree is preserved.

**Chain:** $W = X_1^{a_1}X_2 + X_2^{a_2}X_3 + \cdots + X_{N-1}^{a_{N-1}}X_N + X_N^{a_N}$

This case is more involved than the others, because a symmetry of the chain potential may fix $\{X_s, X_{s+1}, \ldots, X_N\}$ for any $s = 1, \ldots, N$ or it may have trivial fixed locus.

The total mirror map acts on generators via

$$\bigwedge_{j=1}^{t} Y_j^{\alpha_j}dY_j\left|\prod_{j=s}^{N}\overline{\rho}_j^{r_j+1}\right\rangle \longmapsto \bigwedge_{j=s}^{N} X_j^{r_j}dX_j\left|\prod_{j=1}^{t}\rho_j^{\alpha_j+1}\right\rangle,$$

where $\{Y_1, \cdots, Y_t\}$ are the $B$-model fixed variables and $\{X_s, \cdots, X_N\}$ are the $A$-model fixed variables. We will consider $t = 0$ and $s = N+1$ to denote trivial fixed loci, and empty products and sums will be assumed to equal 1 and 0 respectively.

We now proceed to compare the bi-gradings on either side of the total mirror map. This is facilitated by the following lemma.

**Lemma 2.7.** *Consider the mirror map acting on a B-model generator via*

$$H\,|\,h\rangle \mapsto G_h\,|\,g_H\rangle\,.$$

*Then*

$$\deg H = \operatorname{age} g_H + \frac{1}{2}(N_g + N_h - N),$$

*and*

$$\deg G_h = \operatorname{age} h + \frac{1}{2}(N_g + N_h - N).$$

*Proof.* This lemma is proved by a direct computation similar to that in the proof of Lemma 1.1. □

We then have

$$\begin{aligned}
(Q_+^B, Q_-^B) &= (\deg H, \deg H) + (\operatorname{age} h, N - N_h - \operatorname{age} h) - (\sum_{i=1}^N q_i, \sum_{i=1}^N q_i) \\
&= (\operatorname{age} g_H, \operatorname{age} g_H) + \frac{1}{2}(N_g + N_h - N, N_g + N_h - N) + \\
&\qquad (\deg G_h, -\deg G_h) + \frac{1}{2}(N - N_g - N_h, N_g - N_h + N), -(\sum_{i=1}^N \overline{q}_i, \sum_{i=1}^N \overline{q}_i) \\
&= (\operatorname{age} g_H, \operatorname{age} g_H) + (\deg G_h, N_g - \deg G_h) - (\sum_{i=1}^N q_i, \sum_{i=1}^N q_i) \\
&= (Q_+^A, Q_-^A),
\end{aligned}$$

as desired. □

# CHAPTER III

# Mirror Symmetry for Frobenius Algebras

In the previous chapter, we proved the mirror isomorphism for Landau-Ginzburg state spaces. This isomorphism holds at the level of bi-graded vector spaces, but it is interesting to ask about the relationship between Frobenius Algebra structures for the A-model of $(W, G)$ and the B-model of $(W^T, G^T)$.

We prove the following theorems in this direction.

**Theorem 3.1.** *Let $W: \mathbb{C}^N \to \mathbb{C}$ be a non-degenerate, invertible potential with maximal diagonal symmetry group $G^{max}$ and all charges $q_j < \frac{1}{2}$. Let $W^T$ be the Berglund–Hübsch dual potential of $W$, with Milnor ring $\mathscr{Q}_{W^T}$. Then*

$$\mathscr{Q}_{W^T} \cong \mathscr{H}_{W,G^{max}}$$

*as Frobenius algebras. i.e., The maximally orbifolded A-model of $W$ is isomorphic to the unorbifolded B-model of $W^T$.*

Note that by the classification of

**Theorem 3.2.** *Let $W(X_1, \ldots, X_N)$ be a loop potential with $N$ odd, satisfying the Calabi-Yau condition: $\sum_i q_i = 1$. Let $G$ be an admissible A-model orbifold group such that $G \subset SL_N\mathbb{C}$. Then the mirror map (Equation (2.13)) is a Frobenius algebra isomorphism.*

*Remark.* It is not clear whether the Frobenius algebra structures on the $A$-model and the $B$-model are compatible with the mirror map in general. The difficulty lies in the computation of the $A$-model product structure when the algebra generators are in summands graded by group elements with non-trivial fixed locus, and this case remains essentially open.

**Notation.** Following the physical literature on Landau-Ginzburg models, we introduce the following terminology. We refer to the $G$-graded summands in $\mathscr{H}_{W,G}$ as *sectors*. We call summands with trivial fixed locus *Neveu-Schwarz* sectors, and summands with non-trivial fixed locus are called *Ramond* sectors.

## 3.1 Maximal Symmetry Group

**Theorem 3.1.** *Let $W\colon \mathbb{C}^N \to \mathbb{C}$ be a non-degenerate, invertible potential with maximal diagonal symmetry group $G^{max}$ and all charges $q_j < \frac{1}{2}$. Let $W^T$ be the Berglund–Hübsch dual potential of $W$, with Milnor ring $\mathscr{Q}_{W^T}$. Then*

$$\mathscr{Q}_{W^T} \cong \mathscr{H}_{W,G^{max}}$$

*as bi-graded Frobenius algebras.*

The restriction to $q_j < \frac{1}{2}$ ensures that the ring generators of $\mathscr{H}_{W,G^{\max}}$ are in Neveu-Schwarz sectors, for which the FJRW multiplication can be computed using algebro-geometric methods. As remarked earlier, the only non-trivial case for which this hypothesis fails is the chain potential with $a_N = 2$. It is unclear whether the conclusion of the theorem holds in this case.

Note this corresponds to the duality of state spaces, since $G^{\max}$ is dual to the trivial group. However, the linear isomorphism in Theorem 3.1 may in general differ from that of Theorem 2.5. In the earlier theorem, there was a choice of parity involved in the presentation of the

Mirror Map for loop potentials. We have been unable to determine whether this choice is compatible with the FJRW product structure on the $A$-model.

*Remark.* We would like to note that in the case $N = 2$, Theorem 3.1 has been proven independently by Fan-Shen [FS] in the case of chain potentials, and Acosta [A] in the case of loop potentials.

**Notation.** To make the notation less cumbersome in this case, we will omit the notation $dY_1 \wedge \cdots \wedge dY_N \,|\, \mathrm{id}\rangle$ for the $B$-model sector.

To prove the theorem, we recall that by combining the remark following Axiom 8 and the classification of invertible potentials ([KS], recalled in Section 1.2), it suffices to prove Theorem (3.1) for potentials of Fermat, Loop and Chain type, which we address individually below.

**Fermat Potentials:** $W = X^a$

The Mirror Theorem in this case was proved as the $A_r$ case of the 'self-duality' theorem in [FJR1]. The essential point here is that the exponent matrix is equal to its transpose in the self-dual cases proved in [FJR1]. Our results show that self-duality is in a sense coincidental, and that in general it is the transposed potential $W^T$ which is the $B$-model mirror to $W$.

**Loop Potentials:** $W = \sum_{i=1}^{N} X_i^{a_i} X_{i+1}$ **(indices taken** $\mod N$**)**

Since degree is additive under multiplication in $\mathscr{Q}_{W^T}$ and in $\mathscr{H}_{W,G^{\max}}$, the isomorphism (2.13) of graded vector spaces suggests that the desired ring isomorphism

$$\mathscr{Q}_{W^T} \xrightarrow{\cong} \mathscr{H}_{W,G^{\max}} \tag{3.1}$$

should be induced by the map

$$\begin{aligned} \mathbb{C}[Y_1, \dots, Y_N] &\longrightarrow \mathscr{H}_{W,G^{\max}} \\ Y_i &\longmapsto 1_{\rho_i J}, \end{aligned} \tag{3.2}$$

where for $g \in G^{\max}$, $1_g$ denotes the identity in $\mathscr{H}_g^{G^{\max}} \cong \mathscr{Q}_{W|_{\mathrm{Fix}(g)}}^{G^{\max}}$, and the map is extended to $\mathbb{C}[Y_1, \dots, Y_N]$ by multiplicativity.

The following two lemmas show that $\mathscr{H}_{W,G^{\max}}$ is generated by the elements $1_{\rho_i J}$, subject to the relations

$$(1_{\rho_k J})^{\star a_k} + a_{k-1} 1_{\rho_{k-2} J} \star (1_{\rho_{k-1} J})^{\star (a_{k-1}-1)} = 0.$$

This means that the kernel of the above map is precisely the Jacobian ideal $dW^T$, yielding the desired isomorphism.

We proceed to prove the necessary lemmas.

**Notation.** Define

$$\boldsymbol{\rho}^{\boldsymbol{\alpha}} := \prod_{i=1}^{N} \rho_i^{\alpha_i}.$$

**Lemma 3.2.** *If $\alpha_i + \beta_i \leq a_i - 1$ for $i \in \{1, \dots, N\}$, and $\mathrm{id} \notin \{\boldsymbol{\rho}^{\boldsymbol{\alpha}} J, \boldsymbol{\rho}^{\boldsymbol{\beta}} J, \boldsymbol{\rho}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} J\}$, then*

$$1_{\boldsymbol{\rho}^{\boldsymbol{\alpha}} J} \star 1_{\boldsymbol{\rho}^{\boldsymbol{\beta}} J} = 1_{\boldsymbol{\rho}^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} J}.$$

*Proof.* The lemma is obviously true when $\boldsymbol{\rho}^{\boldsymbol{\alpha}} = \mathrm{id}$ or $\boldsymbol{\rho}^{\boldsymbol{\beta}} = \mathrm{id}$, since $1_J$ is the multiplicative identity in $\mathscr{H}_{W,G^{\max}}$.

By definition (Equation (2.3)),

$$1_{\boldsymbol{\rho}^{\boldsymbol{\alpha}} J} \star 1_{\boldsymbol{\rho}^{\boldsymbol{\beta}} J} = \sum_{\mu,\nu} \left\langle 1_{\boldsymbol{\rho}^{\boldsymbol{\alpha}} J}, 1_{\boldsymbol{\rho}^{\boldsymbol{\beta}} J}, \mu \right\rangle \eta^{\mu\nu} \nu.$$

For the three point correlator $\left\langle 1_{\boldsymbol{\rho}^{\alpha}J}, 1_{\boldsymbol{\rho}^{\beta}J}, \mu\right\rangle$ to be non-zero, we must have

$$\deg 1_{\boldsymbol{\rho}^{\alpha}J} + \deg 1_{\boldsymbol{\rho}^{\beta}J} + \deg \mu = 2\hat{c}.$$

By Corollary 2.1, $\mu \in \mathscr{H}_{gJ}$ for the unique $g = \boldsymbol{\rho}^{\gamma} \in G^{\max}$ satisfying the condition

$$\sum_{i=1}^{N}(\alpha_i + \beta_i + \gamma_i)\overline{q}_i = 2\sum_{i=1}^{N}(a_i - 1)\overline{q}_i$$

and having line bundles $|\mathscr{L}_j|$ of integral degree.

Note that since $\sum_i q_i = \sum_i \overline{q}_i$ and $a_i q_i + q_{i+1} = 1$ we have

$$2\hat{c} = 2\sum_{i=1}^{N}(1 - 2q_i) = 2\sum_{i=1}^{N}(a_i q_i - q_i) = 2\sum_{i=1}^{N}(a_i - 1)\overline{q}_i = \deg 1_{\boldsymbol{\rho}^{\max}J}.$$

Since $0 \leq \alpha_i + \beta_i \leq a_i - 1$ by hypothesis, $\gamma_i = a_i - 1 - \alpha_i - \beta_i$ potentially prescribes the group element $g$, and we demonstrate below that the corresponding line bundles indeed have integral degree.

We compute the degrees $l_j$ of the line bundles $|\mathscr{L}_j|$, using the formula

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^{k}\Theta_j^{h_i},$$

Where $g$ is the genus of the correlator (zero in this case), $k$ is the number of insertions (i.e. three), $h_i \in G^{\max}$ is the group grading of the $i^{\text{th}}$ insertion, and $\Theta_j^{h_i}$ is the phase of the action of $h_i$ on $X_j$. Recalling that $\varphi_j^{(i)}$ is the phase of $\rho_i$ on the variable $X_j$, we have

$$l_j = q_j - (\Theta_j^{\boldsymbol{\rho}^{\alpha}J} + \Theta_j^{\boldsymbol{\rho}^{\beta}J} + \Theta_j^{\boldsymbol{\rho}^{\gamma}J}) = q_j - \sum_{i=1}^{N}(\alpha_i + 1)\varphi_j^{(i)} - \sum_{i=1}^{N}(\beta_i + 1)\varphi_j^{(i)} - \sum_{i=1}^{N}(\gamma_i + 1)\varphi_j^{(i)}$$
$$= -2q_j - \sum_{i=1}^{N}(\alpha_i + \beta_i + \gamma_i)\varphi_j^{(i)} = -2q_j - \sum_{i=1}^{N}(a_i - 1)\varphi_j^{(i)} = -2q_j - \sum_{i=1}^{N}(\delta_{i,j} - \varphi_j^{(i-1)} - \varphi_j^{(i)}) = -1$$

By the concavity axiom (Axiom 4), $\left\langle 1_{\boldsymbol{\rho}^{\alpha}J}, 1_{\boldsymbol{\rho}^{\beta}J}, \mu \right\rangle = 1$. Since $\mu$ and $\nu$ correspond to sectors with trivial fixed loci, $\eta^{\mu\nu} = 1$.

We conclude on substituting into Equation (3.1) that

$$1_{\boldsymbol{\rho}^{\alpha}J} \star 1_{\boldsymbol{\rho}^{\beta}J} = 1_{\boldsymbol{\rho}^{\alpha+\beta}J},$$

as claimed. □

**Lemma 3.3.** *For $N > 2$,*

$$1^{a_k}_{\rho_k J} = -a_{k-1} 1_{\rho_{k-2}\rho_{k-1}^{a_{k-1}-1} J}.$$

*Proof.* By definition,

$$1_{\rho_k^{(a_k-1)} J} \star 1_{\rho_k J} = \sum_{\mu,\nu} \left\langle 1_{\rho_k^{(a_k-1)} J}, 1_{\rho_k J}, \mu \right\rangle \eta^{\mu\nu} \nu. \tag{3.3}$$

The only non-zero term in the sum occurs when

$$\deg 1_{\rho_k^{(a_k-1)} J} + \deg 1_{\rho_k J} + \deg \mu = 2\hat{c}$$

and the corresponding line bundles have integral degree.

This first condition is equivalent to

$$(a_k - 1)\overline{q}_k + \overline{q}_k + \deg \mu = \sum_{i=1}^{N} (a_i - 1)\overline{q}_i.$$

Recalling that for all $i$, $a_i \overline{q}_i + \overline{q}_{i-1} = 1$, so

$$\overline{q}_{k-2} + (a_{k-1} - 1)\overline{q}_{k-1} = 1 - \overline{q}_{k-1} = a_k \overline{q}_k,$$

and denoting $g = \boldsymbol{\rho}^{\gamma}$, we see that

$$\sum_{i=1}^{N} \gamma_i \overline{q}_i = \sum_{i=1}^{N} (a_i - 1)\overline{q}_i - (a_k - 1)\overline{q}_k - \overline{q}_k = \sum_{i=1}^{N} (a_i - 1)\overline{q}_i - \overline{q}_{k-2} - (a_{k-1} - 1)\overline{q}_{k-1}.$$

Thus we can solve for $\gamma_i$ in the range $0 \leq \gamma_i < a_i$, namely

$$\gamma_i = (a_i - 1) - \delta_{i,k-1}(a_{k-1} - 1) - \delta_{i,k-2}.$$

We now confirm that the line-bundles which determine the correlator in question have integral degree, via

$$\begin{aligned}
l_j &= q_j - (\Theta_j^{\boldsymbol{\rho}^{\alpha} J} + \Theta_j^{\boldsymbol{\rho}^{\beta} J} + \Theta_j^{\boldsymbol{\rho}^{\gamma} J}) \\
&= q_j - \sum_{i=1}^{N} (\alpha_i + \beta_i + \gamma_i + 3)\varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - \sum_{i=1}^{N} \gamma_i \varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - \sum_{i=1}^{N} \left((a_i - 1) - \delta_{i,k-1}(a_{k-1} - 1) - \delta_{i,k-2}\right) \varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - (1 - 2q_j) + (a_{k-1} - 1)\varphi_j^{(k-1)} + \varphi_j^{(k-2)} \\
&= -1 + \delta_{j,k-1} - \delta_{j,k} \\
&= \begin{cases} 0 & \text{if } j = k - 1 \\ -2 & \text{if } j = k \\ -1 & \text{else.} \end{cases}
\end{aligned}$$

By the index-zero axiom (Axiom 5), the non-vanishing three-point correlator is given by $-1$ times the $X_{k-1}$-degree of

$$\frac{\partial W}{\partial X_k} = X_{k-1}^{a_{k-1}} + a_k X_k^{a_k - 1} X_{k+1},$$

so

$$\left\langle 1_{\rho_k^{(a_{k-1}-1)}J}, 1_{\rho_k J}, \mu \right\rangle = -a_{k-1}.$$

As in the preceding lemma, we have $\mu$ and $\nu$ necessarily in sectors with trivial fixed loci (i.e. not in the untwisted sector), so $\eta^{\mu\nu} = 1$.

Substituting into Equation (3.3), we conclude that

$$1_{\rho_k^{a_k-1}J} \star 1_{\rho_k J} = -a_{k-1} 1_{\rho_{k-2}\rho_{k-1}^{a_{k-1}-1}J},$$

as claimed. $\square$

For completeness, we address the case of two-variable loop potentials in Lemma 3.4 below. As already indicated, this result has been obtained independently by Acosta [A].

**Lemma 3.4.** *For $N = 2$, $k \in \{1, 2\}$,*

$$1_{\rho_k^{a_k}J} = -a_{k-1} 1_{\rho_{k-2}\rho_{k-1}^{a_{k-1}-1}J}.$$

*Proof.* The method of proof for the preceding lemma is not directly applicable here, because for a two-variable loop, $\rho_k^{a_{k-1}}J = \mathrm{id}$, so the multiplicands used in the proof do not all lie in Neveu-Schwarz sectors and the index-zero axiom is not directly applicable.

However, if $a_k > 3$, so $\mathscr{H}_{\rho_k^{a_k-2}J}$ and $\mathscr{H}_{\rho_k^2 J}$ are Neveu-Schwarz sectors, the proof of Lemma 3.3 is easily amended to yield the same conclusion by considering the product $1_{\rho_k^{a_k}J} = 1_{\rho_k^{a_k-2}J} \star 1_{\rho_k^2 J}$.

So it remains only to consider the cases of two-variable loop potentials with one of the exponents (which we may take to be $a_2$) equal to 2 or 3.

For convenience of notation, we will use variables $x := x_1$ and $y := x_2$, and change the subscripts in the obvious way, so for example $\rho_x := \rho_1$ and $\rho_y = \rho_2$.

- $W = x^{a_x}y + xy^3$, with $a_x \geq 3$. To prove the lemma, we need to show

$$1^{\star 3}_{\rho_y J} = -3\left(1_{\rho_x^2 \rho_y J}\right).$$

(If $a_x = 3$, then by symmetry of the $W$, the corresponding relation will hold with $x$ and $y$ exchanged.) Using Corollary 2.1, we see that

$$1^{\star 2}_{\rho_y J} = \left\langle 1_{\rho_y J}, 1_{\rho_y J}, \mu_{\mathrm{id}} \right\rangle \eta^{\mu,\nu} \nu_{\mathrm{id}},$$

and

$$1^{\star 3}_{\rho_y J} = \left\langle 1_{\rho_y J}, 1_{\rho_y J}, \mu_{\mathrm{id}} \right\rangle \eta^{\mu,\nu} \left\langle \nu_{\mathrm{id}} 1_{\rho_y J}, 1_{\rho_y J} \right\rangle 1_{\rho_x^2 \rho_y J}.$$

The coefficient of $1_{\rho_x^2 \rho_y J}$ is, by the composition axiom (Axiom 6) equal to $\left\langle 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J} \right\rangle$. The line bundle degrees for this correlator are

$$l_x = 2q_x - 4\Theta_x^{\rho_y J} = 2(\tfrac{2}{8}) - 4(\tfrac{1}{8}) = 0,$$
$$l_y = 2q_y - 4\Theta_y^{\rho_y J} = 2(\tfrac{2}{8}) - 4(\tfrac{5}{8}) = -2.$$

So the correlator is given by $-1$ times the $x$-degree of $\partial W/\partial y$. *i.e* $\left\langle 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J} \right\rangle = -3$, as required.

- $W = x^2y + xy^3$. The composition axiom argument used to compute $1^{\star 3}_{\rho_y J}$ above applies here, yielding $1^{\star 3}_{\rho_y J} = -2(1_{\rho_x \rho_y J})$.

  For degree reasons, we see that $\mathscr{H}_{W,G^{\max}}$ has a ring generator $\mu = \alpha x^2 dx \wedge dy + \beta y^2 dx \wedge dy \in \mathscr{H}_{\mathrm{id}}$ that is not in the vector subspace generated by $1^{\star 2}_{\rho_x J} = \gamma x^2 dx \wedge dy + \delta y dx \wedge dy$.

Here $\gamma$ and $\delta$ are determined by the $\star$-product, and we seek $\alpha$ and $\beta$ so that

$$\mu^2 = -3(1_{\rho_x J} \star \mu).$$

It turns out that the matrix of the pairing $\mathscr{H}_{\mathrm{id}} \otimes \mathscr{H}_{\mathrm{id}} \to \mathbb{C}$ is given by the symmetric matrix $-\frac{1}{6}A_W^{-1}$. Then, by the pairing axiom (Axiom 7), the desired relation is equivalent to

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -3 \begin{pmatrix} \gamma & \delta \end{pmatrix} A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Consider a non-zero vector $v$ orthogonal to $(\gamma, \delta)$ with respect to the inner product with matrix $A^{-1}$ on $\mathbb{C}^2$. Putting $(\alpha, \beta) = (\gamma, \delta) + \lambda v$ and substituting into the above relation, we obtain the quadratic equation

$$\lambda^2 v^T A^{-1} v + 4 \begin{pmatrix} \gamma & \delta \end{pmatrix} A^{-1} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0.$$

The coefficients in this equation are non-zero, as the vanishing of either of them would contradict non-degeneracy of the form $A^{-1}$. Either solution specifies $\mu$, which is not a multiple of $1_{\rho_x J}^{(\star 2)}$ because $\lambda \neq 0$.

With $\mu$ determined, the lemma follows.

- $W = x^2y + xy^2$. In this case, $G^{\max} = \langle J \rangle$, with $J = (e^{2\pi i/3}, e^{2\pi i/3})$. The sectors $\mathscr{H}_J$ and $\mathscr{H}_{J^{-1}}$ are Neveu-Schwarz, respectively of minimal and maximal degree ($\deg_+^A$). The identity sector is $\mathscr{H}_{\mathrm{id}} = \mathbb{C}[xdx \wedge dy, ydx \wedge dy]$, and the multiplication of the generators into $\mathscr{H}_{J^{-1}}$ is determined by the pairing axiom (Axiom 7), from which it is easy to see that

$$(xdx \wedge dy)^2 = -2(xdx \wedge dy)(ydx \wedge dy) = (ydx \wedge dy)^2.$$

These are precisely the defining relations for the generators of $\mathscr{Q}_{W^T}$, so the desired isomorphism holds.

$\square$

Using associativity of $A$-model multiplication to avoid the identity (Ramond) sector, it is easy to see that the mirror map is surjective. The dimension count of Lemma 1.4 then guarantees that the relations are generated by those in Lemma 3.3 if $N > 2$ or Lemma 3.4 if $N = 2$, from which the desired isomorphism follows.

**Chain Potentials:** $W = \sum_{i=1}^{N-1} X_i^{a_i} X_{i+1} + X_N^{a_N}$

Since degree is additive under multiplication in $\mathscr{Q}_{W^T}$ and in $\mathscr{H}_{W,G^{\max}}$, the isomorphism (2.13) of graded vector spaces suggests that the desired ring isomorphism

$$\mathscr{Q}_{W^T} \xrightarrow{\cong} \mathscr{H}_{W,G^{\max}} \tag{3.4}$$

should be induced by the map

$$\begin{aligned} \mathbb{C}[Y_1, \dots, Y_N] &\longrightarrow \mathscr{H}_{W,G^{\max}} \\ Y_i &\longmapsto 1_{\rho_i J}, \end{aligned} \tag{3.5}$$

which is extended to $\mathbb{C}[Y_1, \dots, Y_N]$ by multiplicativity. The following two lemmas show that $\mathscr{H}_{W,G^{\max}}$ is generated by the elements $1_{\rho_i J}$, subject to the relations

$$(1_{\rho_k J})^{\star a_k} + a_{k-1} 1_{\rho_{k-2} J} \star (1_{\rho_{k-1} J})^{\star (a_{k-1} - 1)} = 0.$$

This means that the kernel of the mirror map is precisely the Jacobian ideal $dW^T$, yielding the desired isomorphism.

*Remark.* Note the assumption that $q_N < \frac{1}{2}$ is essential to our arguments, as we will use the fact that $\text{Fix}(\rho_N J)$ is trivial.

We proceed to prove the necessary lemmas.

**Notation.** Define

$$\boldsymbol{\rho}^{\boldsymbol{\alpha}} := \prod_{i=1}^{N} \rho_i^{\alpha_i}.$$

**Lemma 3.5.** *If $\alpha_i + \beta_i \leq a_i - 1$ for $i \in \{1, \ldots, N\}$, and $\boldsymbol{\rho}^{\boldsymbol{\alpha}} J, \boldsymbol{\rho}^{\boldsymbol{\beta}} J$ and $\boldsymbol{\rho}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} J$ have trivial fixed loci, then*

$$1_{\boldsymbol{\rho}^{\boldsymbol{\alpha}} J} \star 1_{\boldsymbol{\rho}^{\boldsymbol{\beta}} J} = 1_{\boldsymbol{\rho}^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} J}.$$

*Proof.* The argument here is practically identical to the one used to prove Lemma 3.2. $\square$

**Lemma 3.6.**

$$1_{\rho_N^{a_N - 1} J} \star 1_{\rho_{N-1} J} = 0$$

*Proof.* Note this relation corresponds to the Jacobian relation $\frac{\partial W_{\text{chain}}^T}{\partial Y_N} = 0$.

By definition (Equation (2.3)),

$$1_{\rho_N^{a_N - 1} J} \star 1_{\rho_{N-1} J} = \sum_{\mu,\nu} \left\langle 1_{\rho_N^{a_N - 1} J}, 1_{\rho_{N-1} J}, \mu \right\rangle \eta^{\mu\nu} \nu.$$

For the three point correlator $\left\langle 1_{\rho_N^{a_N - 1} J}, 1_{\rho_{N-1} J}, \mu \right\rangle$ to be non-zero, the line bundles $|\mathscr{L}_j|$ must have integral degree.

We know from Corollary 2.1 that there is at most one group element $gJ$ for which $\mu \in \mathscr{H}_{gJ}$ yields a non-zero three point correlator. For the sector $\mathscr{H}_{gJ}$, let us consider the implication of integrality of the line bundles $|\mathscr{L}_j|$, for $j \in \{N, N-1\}$:

$$\begin{aligned} l_N &= q_N - \Theta_N^{\rho_N^{a_N-1}J} - \Theta_N^{\rho_{N-1}J} - \Theta_N^{gJ} \\ &= q_N - (a_N+1)\varphi_N^{(N)} - \Theta_N^{gJ} \\ &= -1 - \Theta_N^{gJ}. \end{aligned}$$

For this to be integral, we require $\Theta_N^{gJ} \in \mathbb{Z}$, *i.e.* $gJ$ fixes $X_N$. Furthermore,

$$\begin{aligned} l_{N-1} &= q_{N-1} - \Theta_{N-1}^{\rho_N^{a_N-1}J} - \Theta_{N-1}^{\rho_{N-1}J} - \Theta_{N-1}^{gJ} \\ &= q_{N-1} - (a_N+1)\varphi_{N-1}^{(N)} - 3\varphi_{N-1}^{(N-1)} - \Theta_{N-1}^{gJ} \\ &= -\varphi_{N-1}^{(N-1)} - \Theta_{N-1}^{gJ}. \end{aligned}$$

For this to be integral, we require $\Theta_{N-1}^{gJ} = 1 - \varphi_{N-1}^{(N-1)} \notin \mathbb{Z}$, *i.e.* $gJ$ *does not fix* $X_{N-1}$.

Since a chain potential fixes consecutive variables, we conclude that $gJ$ has one-dimensional fixed locus, and consequently $\mathscr{H}_{gJ}$ is empty, and the product vanishes as claimed. $\square$

**Lemma 3.7.** *For* $k \in \{2, \ldots, N\}$,

$$1_{\rho_k J}^{\star(a_k)} = -a_{k-1} 1_{\rho_{k-2}\rho_{k-1}^{a_{k-1}-1}J}.$$

*Proof.* Note these relations correspond to the Jacobian relations $\frac{\partial W^T{}_{\text{chain}}}{\partial Y_{k-1}} = 0$.

For $2 \leq k \leq N-1$, the proof proceeds exactly as in Lemma 3.3.

For $k = N$, we face the obstacle that $\rho_N^{a_N-1}J$ is a Ramond Sector, so we cannot use the index-zero axiom as before. We could realize $1_{\rho_N J}^{\star(a_N)}$ as the product of $1_{\rho_N J}^{\star 2}$ and $1_{\rho_N J}^{\star(a_N-2)}$, but this fails to avoid the Ramond sector when $a_N = 3$. Instead, we mimic the computation in [FJR1], where the composition axiom (Axiom 6) is used to determine the ring structure of $\mathscr{H}_{E_7, G^{\max}}$.

The reader may check using Corollary 2.1 that

$$1_{\rho_N^{(a_N-2)}J} \star 1_{\rho_N J} = \left\langle 1_{\rho_N^{(a_N-2)}J},\, 1_{\rho_N J},\, \mu 1_{\prod \rho_i^{\gamma_i}J} \right\rangle \eta^{\mu,\nu} \nu 1_{\rho_N^{a_N-1}J},$$

with

$$\gamma_i = \begin{cases} 0 & \text{if } i = N-1 \\ a_i - 2 & \text{if } i = N-2 \cdot \\ a_i - 1 & \text{else.} \end{cases}$$

Multiplying by $1_{\rho_N J}$, we see

$$1_{\rho_N J}^{\star a_N} = \left\langle 1_{\rho_N^{(a_N-2)}J},\, 1_{\rho_N J},\, \mu 1_{\prod \rho_i^{\gamma_i}J} \right\rangle \eta^{\mu,\nu} \left\langle \nu 1_{\rho_N^{a_N-1}J},\, 1_{\rho_N J},\, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}}J} \right\rangle 1_{\rho_{N-2}\rho_{N-1}^{a_{N-1}}}.$$

Now, by the composition axiom,

$$\left\langle 1_{\rho_N^{(a_N-2)}J},\, 1_{\rho_N J},\, \mu 1_{\prod \rho_i^{\gamma_i}J} \right\rangle \eta^{\mu,\nu} \left\langle \nu 1_{\rho_N^{a_N-1}J},\, 1_{\rho_N J},\, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}}J} \right\rangle = \left\langle 1_{\rho_N^{(a_N-2)}J},\, 1_{\rho_N J},\, 1_{\rho_N J},\, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}}J} \right\rangle.$$

Since all the sectors in this four-point correlator are Neveu-Schwarz, we may use the index-zero axiom to determine its value. A calculation similar to the other index-zero calculations yields for the degrees of the line bundles $|\mathscr{L}_j|$:

$$l_j = \begin{cases} -2 & \text{if } j = N \\ 0 & \text{if } j = N-1 \\ -1 & \text{else.} \end{cases}$$

So, the four-point correlator is $-1$ times the $X_{N-1}$ degree of $\partial W / \partial X_N = a_N X_N^{a_N-1} + X_{N-1}^{a_{N-1}}$, namely $-a_{N-1}$. This completes the proof of Lemma 3.7. $\square$

Surjectivity of the mirror map is again clear from associativity of $A$-model multiplication, where we avoid Ramond sectors (so we can apply the preceding lemmas) by noting that $\boldsymbol{\rho}^{\gamma} J$ has trivial fixed locus as long as $\gamma_N < a_{N-1}$. A dimension count using Lemma 1.4 then indicates that the relations in $\mathscr{H}_{W,G^{\max}}$ are generated by those in the lemmas, and the desired isomorphism follows.

## 3.2 $SL$ symmetries for Calabi-Yau Loop Potentials

As evidence that the $B$-model multiplication defined in Section 2.4 is the appropriate product to consider in the context of LG-to-LG mirror symmetry, we prove the following theorem.

**Theorem 3.1.** *Let $W(X_1, \ldots, X_N)$ be a loop potential with $N$ odd, satisfying the Calabi-Yau condition: $\sum_i q_i = 1$. Let $G$ be an admissible $A$-model orbifold group such that $G \subset SL_N\mathbb{C}$. Then the mirror map (Equation* (2.13)*) is a Frobenius algebra isomorphism.*

*Remark.* Note that the group generated by the exponential grading operator $J$ is automatically a subgroup of $SL_N\mathbb{C}$ in the Calabi-Yau case.

The theorem is applicable more generally than the statement initially suggests, as the FJRW $A$-model depends only on the charges and the orbifold group, not the presentation of the potential [R1]. So, for example, the $J$-orbifolded $A$-models coincide for $W_{\text{loop}} = X_1^4X_2 + X_2^4X_3 + X_3^4X_4 + X_4^4X_5 + X_5^4X_1$ and $W_{\text{Fermat}} = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5$, and the $J$-orbifolded $A$-model of the latter maybe computed as the $SL$-orbifolded $B$-model of the former.

*Proof.* Recall that because of the loop structure of the potential, the fixed locus for $g \in G$ is trivial unless $g = \text{id}$.

By Theorem 2.5, we know the mirror map is a bijection. To see that it is an isomorphism of Frobenius algebras, we consider $B$-model multiplication between untwisted sectors, between twisted sectors, and between an untwisted sector and a twisted sector.

*Untwisted $B$-model sector:*

$$\prod_{j=1}^{N} Y_j^{\alpha_j} dY_j \mid \mathrm{id} \rangle \longmapsto 1 \left| \prod_{j=1}^{N} \rho_j^{\alpha_j+1} \right\rangle,$$

where we note that since $N$ is odd, the $A$-model sector corresponding to the monomial $\prod_{j=1}^{N} Y_j^{\alpha_j}$ is *not* the identity sector, so has trivial fixed locus.

Note that on the $A$-model side, the identity sector has degree $\hat{c} = N - 2\sum q_i$, which is an odd integer, while the twisted sector corresponding to a group element $g \in SL_N\mathbb{C}$ has degree $2\sum_i(\Theta_i^g - q_i)$, an even integer. Since degree is additive under multiplication, the product of two Neveu-Schwarz invariants has no contribution from the identity sector.

Consequently, in the $A$-model product (Equation (2.3)), all invariants appearing with non-zero coefficient on the right-hand side are Neveu-Schwarz invariants for the action of the maximal $A$-model symmetry group, and the correlators required to determine the multiplication are as computed in the section on Loop potentials in Section III. *i.e.* The multiplicative relations on the $A$-model twisted sectors correspond precisely to the Jacobian relations in the $B$-model untwisted sector.

We must now consider the *Twisted $B$-model sectors:*

Since the $B$-model twisted sectors have trivial fixed loci, the mirror map sends them all to the $A$-model untwisted sector.

$$1 \left| \prod_{j=1}^{N} \overline{\rho}_j^{r_j+1} \right\rangle \longmapsto \prod_{j=1}^{N} X_j^{r_j} dX_j \mid \mathrm{id} \rangle,$$

On the $A$-model side, by the pairing axiom (Axiom 7),

$$\left( \prod_{j=1}^{N} X_j^{r_j} dX_j \right) 1_{\mathrm{id}} \star \left( \prod_{j=1}^{N} X_j^{s_j} dX_j \right) 1_{\mathrm{id}} = \left\langle \prod_{j=1}^{N} X_j^{r_j}, \prod_{j=1}^{N} X_j^{s_j} \right\rangle 1_{J^{-1}},$$

On the $B$-model side,

$$\left|\prod_{j=1}^{N}\overline{\rho}_j^{r_j+1}\right\rangle \star \left|\prod_{j=1}^{N}\overline{\rho}_j^{s_j+1}\right\rangle$$

vanishes unless every variable is fixed in $\left|\prod_{j=1}^{N}\overline{\rho}_j^{r_j+s_j+2}\right\rangle$, which means precisely that $\prod_{j=1}^{N}X_j^{r_j+s_j} = \lambda \operatorname{hess} W$ for some $\lambda \in \mathbb{C}$. *i.e.* we have the same condition for the non-vanishing of the $A$ and $B$ model products above. When this condition is satisfied, the $B$-model product is given by

$$\left|\prod_{j=1}^{N}\overline{\rho}_j^{r_j+1}\right\rangle \star \left|\prod_{j=1}^{N}\overline{\rho}_j^{s_j+1}\right\rangle = (\operatorname{hess} W)1_{\mathrm{id}},$$

which clearly corresponds up to scalars with the above $A$-model product under the mirror map.

It remains only to check that the multiplication between the twisted and untwisted $B$-model sectors satisfies the same relations as the corresponding $A$-model products. The $B$-model $\mathscr{Q}_{\mathrm{id}}$-module structure means the only way such a product can be non-trivial is if the multiplicand from the untwisted sector is $1_{\mathrm{id}}$ – the multiplicative identity. Since the mirror map preserves the identity, we need only show that on the $A$-model side,

$$\left|\prod_{j=1}^{N}\rho_j^{\alpha_j+1}\right\rangle \star \prod_{j=1}^{N}X_j^{r_j}\,|\,\mathrm{id}\rangle = 0.$$

This holds for degree reasons: the untwisted sector is the only sector with odd degree, and the twisted sectors all have even degree; by additivity of degree, the product has odd degree, so since it does not lie in the untwisted sector it must vanish. □

*Remark.* The hypotheses for this theorem ensure that there are no non-zero contributions from the Ramond sector to products of Neveu-Schwarz invariants. The above argument may be adapted whenever such a situation is established, so it should be possible to extend this result beyond the case of Calabi-Yau potentials orbifolded by subgroups of $SL_N\mathbb{C}$.

## 3.3 Strange Duality

Arnol'd's list of 14 exceptional singularities provides a source of interesting examples of Landau–Ginzburg Mirror Symmetry. In particular, we have the following:

**Proposition 3.1.** *Let $W$ be one of the 14 exceptional unimodal singularities, and $W^{SD}$ its Strange Dual. Then, there is a Frobenius algebra isomorphism*

$$\mathscr{H}_{W,\langle J\rangle} \cong \mathscr{Q}_{W^{SD}}.$$

*Proof.* Of course, when $J$ generates $G_W^{\max}$ and $W^{\mathrm{SD}} = W^T$, this is just a restatement of Theorem 3.1. However, examining Table 3.1, we see this is only the case for $S_{12}$, $Z_{12}$ and $E_{12}$ (which are self-dual), and $Z_{11}$ and $E_{13}$ (which are strange dual to each other).

To realize the observation for the remaining singularities in Arnol'd's list, we choose an equivalent singularity $W'$ for each singularity $W$ in such a way that

- $\mathscr{Q}_{W'} \cong \mathscr{Q}_W$.
- The charges of $W'$ coincide with the charges of $W$, so $J_{W'} = J_W$.
- The maximal symmetry group of $W'$ is generated by $J_{W'}$.
- Transposition yields the Strange Dual class in the updated list of exceptional singularities.

The Landau Ginzburg $A$-model $\mathscr{H}_{W,G}$ constructed in [FJR1] depends only on $G \subset (\mathbb{C}^*)^N$ and the charges $q_1, \ldots, q_N$, and not on the specific choice of representative of the equivalence class of the singularity $W$ [R1]. This means we are free to compute the FJRW ring of $W$ orbifolded by $\langle J\rangle$ as $\mathscr{H}_{W',\langle J\rangle}$.

Table 3.1: Arnold's list of the 14 exceptional unimodal singularities $W$, with representatives $W'$ chosen so Strange Duality is compatible with transposition.

| Class | $W$ | $\langle J\rangle = G_W^{\max}$ | $W'$ |
|---|---|---|---|
| $Q_{10}$ | $x^2z+y^3+z^4$ | Yes | $x^2z+y^3+z^4$ |
| $E_{14}$ | $x^2+y^3+z^8$ | No | $x^2+y^3+xz^4$ |
| $Q_{11}$ | $x^2z+y^3+yz^3$ | Yes | $x^2z+y^3+yz^3$ |
| $Z_{13}$ | $x^2+y^3z+z^6$ | No | $x^2+y^3z+z^3x$ |
| $Q_{12}$ | $x^2z+y^3+z^5$ | No | $x^2z+y^3+xz^3$ |
| $S_{11}$ | $x^2y+y^2z+z^4$ | Yes | $x^2y+y^2z+z^4$ |
| $W_{13}$ | $x^2+y^4+yz^4$ | No | $x^2+xy^2+yz^4$ |
| $S_{12}$ | $x^2y+y^3z+xz^2$ | Yes | $x^2y+y^3z+xz^2$ |
| $U_{12}$ | $x^3+y^3+z^4$ | No | $x^2y+xy^2+z^4$ |
| $Z_{11}$ | $x^2+y^3z+z^5$ | Yes | $x^2+y^3+yz^5$ |
| $E_{13}$ | $x^2+y^3+yz^5$ | Yes | $x^2+y^3z+z^5$ |
| $Z_{12}$ | $x^2+y^3z+yz^4$ | Yes | $x^2+y^3z+yz^4$ |
| $W_{12}$ | $x^2+y^4+z^5$ | No | $x^2+xy^2+z^5$ |
| $E_{12}$ | $x^2+y^3+z^7$ | Yes | $x^2+y^3+z^7$ |

Since $W'$ is chosen so that $J$ generates $G_{W'}^{\max}$, we can then apply Theorem 3.1 to $W'$ to yield the isomorphisms

$$\mathscr{H}_{W,\langle J\rangle} \cong \mathscr{H}_{W',\langle J\rangle} \cong \mathscr{Q}_{(W')^T} \cong \mathscr{Q}_{W^{\mathrm{SD}}}.$$

Appropriate choices of $W'$ are indicated in Table 3.1. $\square$

One can attempt to use the original representative $W$ where $\langle J\rangle$ is not necessarily $G^{\max}$. Indeed, it fits into the general Landau–Ginzburg Orbifold Mirror Conjecture using the orbifold $B$-model of Section 2.4.

*Example.* We present here the case of $U_{12}$, which exhibits the general features of the other examples. We use Proposition 2.5 to show that we have a *Frobenius algebra isomorphism*

$$\mathscr{H}_{U_{12},\langle J\rangle} \cong \mathscr{Q}_{U_{12}^T,\mathbb{Z}/3\mathbb{Z}},$$

rather than just the bi-graded vector space isomorphism guaranteed by Proposition 2.5.

We know from the preceding discussion that

$$\mathscr{H}_{U_{12},\langle J\rangle} \cong \mathscr{Q}_{U_{12}^{SD}} \cong \mathbb{C}[x,y,z]/\left\langle x^2,\, y^2,\, z^3\right\rangle.$$

*Remark.* Note the isomorphism claimed between the Milnor rings of $U_{12}' = X^2Y + XY^2 + Z^4$ and $U_{12} = x^3 + y^3 + z^4$ is induced by the map

$$\mathbb{C}[X,Y,Z] \to \mathscr{Q}_{U_{12}}$$

which sends $X \mapsto \omega x + \omega^2 y$, $X \mapsto \omega^2 x + \omega y$, and $Z \mapsto z$. (Here $\omega = e^{2\pi i/3}$).

For the $B$-model of $U_{12}^T = x^3 + y^3 + z^4$, we note that the group dual to $\langle J\rangle$ is the $SL$ subgroup of $G^{\max}_{U_{12}^T}$, namely $\mathbb{Z}/3\mathbb{Z}$ generated by $\rho := (\omega, \omega^2, 1)$.

Now,

$$\rho^k = \begin{cases} \mathbb{C}^3_{xyz} & \text{if } k = 0 \\ \mathbb{C}_z & \text{if } k = 1,\, 2 \end{cases}$$

and

$$\mathscr{Q}_{\rho^k}^{U_{12}^T} = \begin{cases} \langle e_0,\, ze_0,\, z^2e_0,\, xye_0,\, xyze_0,\, xyz^2e_0\rangle & \text{if } k = 0 \\ e_k & \text{if } k = 1,\, 2, \end{cases}$$

where $e_0 = dx \wedge dy \wedge dz$ and $e_1 = dz = e_2$.

We put $X = e_1$, $Y = e_2$ and $Z = ze_0$.

Note

$$\deg X = \deg Y = \tfrac{1}{2}(1 - 2q_x) + \tfrac{1}{2}(1 - 2q_y) = \tfrac{1}{3},$$

while

$$\deg Z = q_z = \tfrac{1}{4}.$$

We observe immediately that $Z^3 = 0$ (since multiplication in the untwisted sector is just multiplication in the unorbifolded Milnor ring).

Further, $X^2 = 0 = Y^2$, since the variables $x$ and $y$ are fixed in neither the $(\omega, \omega^2, 1)$-sector, nor the $(\omega^2, \omega, 1)$-sector.

Meanwhile, $XY = \alpha xye_0$ for $\alpha \neq 0$, since $xy$ has non-zero pairing with $\mathrm{hess}(U_{12}|_{\mathbb{C}_z}) = 12z^2$.

Thus we see the degree preserving map

$$\mathbb{C}[X, Y, Z] \mapsto \mathscr{Q}_{U_{12}^T, \mathbb{Z}/3\mathbb{Z}}$$

generated by $X \mapsto e_1$, $Y \mapsto e_2$ and $Z \mapsto ze_0$ is surjective, with kernel $\langle Z^3,\, X^2,\, Y^2\rangle$, and induces an isomorphism

$$\mathscr{H}_{U_{12}, \langle J\rangle} \cong \mathscr{Q}_{U_{12}} \mapsto \mathscr{Q}_{U_{12}^T, \mathbb{Z}/3\mathbb{Z}}.$$

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