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INTEGRALS OF THE CALCULUS OF VARIATIONS

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PREFACE

This thesis is dedicated to

Professor Lamberto Cesari
Professor D. B. Sawyer
My parents
My wife

I must express special gratitude to Professor Cesari for his patient, yet inspiring supervision of my work over the past two and a half years. Part I of this thesis was carried out under partial support of NSF research grant GP-57 at The University of Michigan. I also wish to thank my committee, and particularly Professor G. W. Hedstrom, the second reader.

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ABSTRACT

The general purpose of this thesis is to study—in an abstract and unified formulation—properties of the integrals of the calculus of variations which are usually discussed separately in a number of particular situations (parametric and non-parametric curves, surfaces, varieties, with differential elements of orders one, two, etc.). An abstract form of the integrals of the calculus of variations has been given by Cesari in two recent papers, where Burkill-type or BC-integrals of vector-valued set functions relative to a mesh function are treated in a very general setting. Cesari introduced a condition of "quasi additivity" on the set function, that is sufficient for the existence of the corresponding BC-integral. In particular, the formulation includes Weierstrass-type integrals $\int f(T,\phi)$ over a Euclidean variety T with a quasi additive set function ϕ of bounded variation, and therefore the Weierstrass integrals of the calculus of variations for curves and surfaces studied by Tonelli, Bouligand, Menger, and Pauc. Under suitable conditions, the integral $\int f(T,\phi)$ can be expressed both as a BC-integral and as a Lebesgue-Stieltjes integral with respect to a measure induced by ϕ.

In this thesis, a modification of the Weierstrass-type integral is made to allow a more convenient expression of the integrals of the calculus of variations. Specifically, Cesari's results for the integral $\int f(T,\phi)$ are extended to an integral of the form $\int f(\zeta,\phi)$, where now ζ is a set function with appropriate properties.

The particular properties considered in this thesis are invariance, behavior under rotation, and semicontinuity. Here invariance means that the integrals corresponding to two systems (ζ,ϕ), (ζ',ϕ') of set functions have the same value. We introduce relations between the systems (ζ,ϕ), (ζ',ϕ') that ensure invariance. Fréchet invariance for parametric curve and surface integrals is framed under these invariance theorems. Concerning behavior under rotation, the integrals are proved invariant under rotations in Euclidean space provided such rotation leads to pairs (ζ,ϕ), (ζ',ϕ') of set functions related in an appropriate sense.

Some very general theorems of lower semicontinuity of the integrals $\int f(\zeta,\phi)$ in the Lebesgue-Stieltjes form are proved under suitable convexity conditions on f. The semicontinuity is relative to a topology appropriate to the formulation. These general theorems are then shown to contain as corollaries the particular lower semicontinuity theorems of Tonelli and Turner for parametric curves, of Cesari and Turner for parametric surfaces, of Tonelli for non-parametric curves, and of Cinquini for parametric curves in E_3 depending on differential elements of orders two, or three.

In addition, further functionals are considered in Cesari's formulation. The Lebesgue-Stieltjes integral is shown to be a BC-integral of a quasi additive set function relative to the standard mesh function. The bend or total curvature of a curve is expressed as a BC-integral of various set functions relative to appropriate mesh functions.

The last part of the thesis concerns the shape of level surfaces of harmonic functions in three dimensions. In terms of the corresponding regions of higher potential, or "regions of potential," the results can be summarized as follows. If two regions of potential are convex, then every intermediate region of potential is convex. If two regions of potential are star-shaped relative to some point, then every intermediate region of potential is similarly star-shaped. On the other hand, we prove by an example that if two regions of potential are merely simply connected, the intermediate regions of potential need not be simply connected.

I. INTRODUCTION

The techniques used in the direct method of the calculus of variations show an underlying similarity and unity which has long been noted by many authors such as Bouligand,[2] Tonelli,[26] and Menger.[17] These similarities can be seen in the so far parallel but quite separated discussions of the integrals of the calculus of variations for parametric and non-parametric curves in E_m, for parametric and non-parametric surfaces in E_m, for the same integrals depending on differential elements of higher orders, and so on.

Our objective in this thesis has been to give a unified discussion of the main properties of the integrals of the calculus of variations in the frame of an axiomatic treatment of the same integrals. We shall discuss essentially the properties of semicontinuity, invariance with respect to representation, and invariance with respect to orthogonal linear transformations in E_m.

A major step in this direction was made by Cesari[6,7] who introduced the concept of quasi additive set function ϕ with respect to a given mesh function δ, and developed an axiomatic treatment of the corresponding integral $B = \int\phi$ for set functions ϕ which are quasi additive and of bounded variation. We shall call $\int\phi$ a Burkill-Cesari integral, or BC-integral. This integral includes both the usual Burkill-type integrals, and the apparently unrelated parametric Weierstrass-type integrals

$\int f(T,\phi)$ relative to a mapping $T: A \to E_m$ and a set function ϕ which is quasi additive and of bounded variation with respect to a mesh function δ. Indeed, as Cesari proved in Ref. 6 under general assumptions, the set function $\Phi = f(T,\phi)$ is again quasi additive and of bounded variation with respect to the same mesh function δ, and hence $\int f(T,\phi)$ can be defined as a BC-integral $\int f(T,\phi) = \int \Phi$. Under a convenient system of axioms, the BC-integral can be represented[7] as a Lebesgue-Stieltjes integral, in particular $\int f(T,\phi) = (A) \int f(T,\Theta)d\mu$ in a convenient measure space $(A,\mathcal{B},\mu)$. The line integrals and surface integrals of the calculus of variations in parametric form are included in the axiomatic treatment of Refs. 6 and 7, together with a number of other familiar concepts such as total variation of a function of one real variable, Jordan length of a curve, Lebesgue area of a surface, and Lebesgue-Stieltjes integral of a μ-measurable function $f: A \to E_1$ in a measure space $(A,\mathcal{B},\mu)$. Nishiura has shown in his thesis[19] that this process also covers integrals over k-varieties in E_m. Cesari's results in Refs. 6 and 7 are partially surveyed in Chapter 2 of this thesis.

Chapter 3 complements the remark made by Cesari[6] that the Lebesgue-Stieltjes integral of a μ-integrable function $f: A \to E_1$ in any measure space $(A,\mathcal{B},\mu)$ can be interpreted as a BC-integral, $B = \int \phi$ of a convenient interval function ϕ which is quasi additive with respect to a conveniently chosen mesh function δ. Here we show that the same result can be accomplished by means of an interval function which is quasi additive with respect to the mesh function of the usual theory of Lebesgue-

Stieltjes integrals.

In Chapter 4 we consider the bend Ω, or total curvature, of a curve X in E_n, as defined by Iseki.[13] Under various sets of general assumptions on the curve, we show that Iseki's bend can be expressed as the BC-integral of an appropriate quasi additive function ϕ. By this approach, it is shown that the bend Ω of the curve X is completely similar to the Jordan length L, and partakes with L the same formal properties and axiomatic treatment. An analogous result can be expected for possible extensions to bends Ω_k of any order k, $1 \leq k \leq n$ ($k = 1 \leq n$, length; $k = 2 \leq n$, bend or total curvature; $k = 3 \leq n$, total torsion; etc.).

In Chapter 5 we take into consideration a modified form of Weierstrass-type integrals $\int f(\cdot\zeta,\phi)$, where now both ζ and ϕ are set functions and ϕ is quasi additive and of bounded variation with respect to a given mesh function δ. Under general hypotheses on f, ζ, and ϕ, we show that the set function $\Phi = f(\cdot\zeta,\phi)$ is again quasi additive and of bounded variation with respect to the same mesh function δ, and hence $\int f(\zeta,\phi) = \int \Phi$ is again a BC-integral. By this process, the line and surface integrals depending on differential elements of first and second, or higher orders, of the calculus of variations can be included in the same axiomatic treatment mentioned above.

In Chapter 6 we discuss, in the frame of the same axiomatic treatment, invariant properties of the present integrals with respect to change of the generating set functions. In Chapter 7 we discuss the invariant character of the same integrals with respect to linear orthogonal transformations in E_m. Both the results of Chapters 6 and 7 extend results proved by Cesari and Turner in surface area theory, and show, therefore, that also these results hold in the present general axiomatic treatment.

In Chapter 8 we discuss the difficult question of the semicontinuity of "regular" integrals. In the same present axiomatic treatment we prove that convenient properties of convexity on f (regularity) assure properties of semicontinuity of $\int f(\zeta, \phi)$ with respect to appropriate topologies. These general results do not require differentiability conditions on f, in harmony with recent work on line and surface integrals of the calculus of variations (L. Tonelli,[25] and L. Turner[29]). The complexity of the present axiomatic treatment is to be expected in view of the generality of the results. In Chapter 9, we apply the results of Chapter 8 to the particular line and surface integrals of the calculus of variations. We deduce in each case corresponding sufficient conditions for lower semicontinuity proved by Tonelli, Cinquini, Turner, and others by a number of separate arguments.

Chapter 10 (Part II of the thesis) deals with properties of harmonic functions in E_3. It concerns the situation of a function $\phi(P)$, $P \epsilon E_3$, continuous in E_3 and harmonic in an open connected set $D \subset E_3$, such that the complement $D' = E_3 - D$ is the union of two closed disjoint sets C_o and C_1, C_1 compact, and $\phi = 1$ on C_1, $\phi = 0$ on C_o. If both C_1 and $C_o' = E_3 - C_o$ are star-shaped with respect to the origin, that is, the intersections of every half-line from the origin with C_1 and C_o' are segments, then the regions $\{P:\phi(P) > k\}$, $0 < k < 1$, are also star-shaped with respect to the origin. In addition, if C_1 and C_0' are convex, then each region $\{P:\phi(P) > k\}$, $0 < k < 1$, is also convex. This work was initiated and practically completed at the University of Otago. The research of this Chapter 10 was suggested to the writer by Professor D. B. Sawyer of the University of Otago, and continues previous work of R. M. Gabriel.[31] The work on this chapter was completed—particularly the rigorous treatment of the counter example—at The University of Michigan.

P A R T I

2. THE BC-INTEGRAL

2.1 INTRODUCTION

In this chapter, we partially review results of Cesari.[6,7] The proofs are not given. Some of the results in Sections 2.3 and 2.5 will be extended in Chapter 5 and there proved in a modified setting.

Consider a set A, a collection {I} of subsets I ("intervals") of A, and a non-empty family $\mathcal{D}$ of finite systems D = [I] of sets $I \in \{I\}$. We shall make the following general assumptions:

(b_1) either (b_1') A is any set, and the sets I of each $D \in \mathcal{D}$, D = [I], are disjoint, that is, $I \in D$, $J \in D$ implies $I \cap J = \emptyset$; or (b_1'') A is a topological space, $\mathcal{U}$ is the collection of its open sets, the sets I of {I} possess interior points, and the sets of each D are non-overlapping, that is, $I \in D$, $J \in D$ implies $\bar{I} \cap J^{o} = I^{o} \cap \bar{J} = \emptyset$ where $^{-}$ and ^{o} denote closure and interior respectively in the topology $(A, \mathcal{U})$.

Let δ be a real function ("mesh") on $\mathcal{D}$ (that is, defined for every system $D \in \mathcal{D}$), such that

(d_1) $0 < \delta(D) < \infty$;

(d_2) for each $\varepsilon > 0$, there is a system $D \in \mathcal{D}$ with $\delta(D) < \varepsilon$.

Let ϕ be a vector function on {I} with values $\phi(I) = [\phi_r(I)] = [\phi_1(I), \ldots, \phi_k(I)]$ in E_k. We call such ϕ an "interval" function, and denote its Euclidean norm function by $\|\phi\|$.

Define

$$\underline{B}_r(\phi,A,\delta) = \liminf_{\delta(D)\to 0} \sum_{I\in D} \phi_r(I) ,$$

$$\overline{B}_r(\phi,A,\delta) = \limsup_{\delta(D)\to 0} \sum_{I\in D} \phi_r(I) ,$$

$$\underline{B}(\phi,A,\delta) = [\underline{B}_r(\phi,A,\delta)] ,$$

$$\overline{B}(\phi,A,\delta) = [\overline{B}_r(\phi,A,\delta)] .$$

If $\underline{B}(\phi,A,\delta) = \overline{B}(\phi,A,\delta)$, we call their common value the Burkill-Cesari or BC-integral $B(\phi,A,\delta) = \int\phi$. We shall include ϕ, A, and δ in the notation for B only where necessary. We denote $B(\|\phi\|)$ by V, and $B(|\phi_r|)$ by V_r. The number V is called the total variation of ϕ. If $V < \infty$, we say that ϕ is of bounded variation.

These integrals have the following obvious properties relative to a given system $A, \mathcal{U}, \{I\}, \mathcal{D}, \delta$:

(i) If ϕ and ϕ' have finite BC-integrals, and α,α' are constants, then $\alpha\phi+\alpha'\phi'$ has finite BC-integral $\alpha B(\phi)+\alpha' B(\phi')$.

(ii) If ψ and ψ' have BC-integrals and are positive, and β,β' are positive constants, then $\beta\psi+\beta'\psi'$ has BC-integral $\beta B(\psi) +\beta' B(\psi')$. Hence, for ψ real, and defining $\psi^+ = (|\psi|+\psi)/2$, $\psi^- = (|\psi|-\psi)/2$,

$$B(|\psi|) = B(\psi^+) + B(\psi^-), \quad B(\psi^+) = B(\psi)+B(\psi^-) ,$$

where, in each case, the integrals on the right hand side are assumed to exist.

If ψ, ψ' are real with $\psi(I) \leq \psi'(I)$ for every I, then $B(\psi) \leq B(\psi')$ when these integrals exist. Also

$$V_r \leq V \leq \Sigma V_r$$
$$\|B\| \leq (\Sigma V_r^2)^{1/2} \leq V .$$

2.2 QUASI ADDITIVE FUNCTIONS

In the setting of Section 2.1, an interval function ϕ on {I} is called "quasi additive" with respect to a mesh function δ on $\mathcal{D}$ if, (ϕ) for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, for every $D_0 \in \mathcal{D}$ with $\delta(D_0) < \eta(\varepsilon)$, there exists $\lambda(\varepsilon, D_0) > 0$ such that, for every $D \in \mathcal{D}$ with $\delta(D) < \lambda(\varepsilon, D_0)$,

$$\sum_{I \in D_0} \|\phi(I) - \Sigma^{(I)}\phi(J)\| < \varepsilon$$

and

$$\Sigma' \|\phi(J)\| < \varepsilon$$

where

$\Sigma^{(I)}$ is the sum over all $J \in D$ with $J \subset I$,

and

Σ' is the sum over all $J \in D$ contained in no I.

A real interval function ψ on {I} is called "quasi subadditive" with respect to a mesh function δ on $\mathcal{D}$ if, under similar conditions,

$$(\psi) \qquad \sum_{I \in D_0} [\psi(I) - \Sigma^{(I)}\psi(J)]^+ < \varepsilon .$$

For these properties relative to a given system $A, \mathcal{U}$, $\{I\}$, $\mathcal{D}$, δ, the following results have been proved:[6]

(iii) If ϕ, ϕ' are quasi additive and α, α' are constants, then $\alpha\phi + \alpha'\phi'$ is quasi additive.

(iv) If ψ, ψ' are quasi subadditive and β, β' are positive constants, then $\beta\psi + \beta'\psi'$ is quasi subadditive.

(v) If ϕ is quasi additive, then each ϕ_r is quasi additive, and conversely.

(vi) If ψ^+ and ψ^- are quasi additive, then ψ and $|\psi|$ are quasi additive, and conversely.

(vii) If ϕ is quasi additive, then $\|\phi\|$, $|\phi_r|$, ϕ_r^+, ϕ_r^- are quasi subadditive.

(viii) If each ϕ_r is positive and quasi subadditive, then $\|\phi\|$ is quasi subadditive.

(ix) If ϕ is quasi additive, then ϕ has a finite BC-integral.

If ψ is positive and quasi subadditive, then ψ has a BC-integral. Hence if ϕ is quasi additive, then $\|\phi\|$, $|\phi_r|$, ϕ_r^+, and ϕ_r^- have BC-integrals.

Note that if ψ is positive, then the following strengthening of the quasi subadditive condition (ψ*) for each $\mathcal{E} < 0$ and each $D_0 \in \mathcal{D}$, there exists $\lambda(\mathcal{E}, D_0) > 0$ such that, for every $D \in \mathcal{D}$ with $\delta(D) < \lambda(\mathcal{E}, D_0)$,

$$\sum_{I \in D_0} [\psi(I) - \Sigma^{(I)}\psi(J)]^+ < \mathcal{E},$$

gives

$$B(\psi) = \sup_{D\in\mathcal{D}} \sum_{I\in D} \psi(I) .$$

Under condition (ψ) only, $B(\psi)$ may not be the supremum of the corresponding sums, but only the limit as $\delta \to 0$, as was proved by examples in Ref. 6.

(x) If ψ is positive and quasi subadditive, and $B(\psi) < \infty$, then ψ is quasi additive.

As a consequence, we have the following results.

(xi) If a vector interval function ϕ is quasi additive and $B(\|\phi\|) < \infty$, then $\|\phi\|$, $|\phi_r|$, ϕ_r^+, and ϕ_r^- are quasi additive.

(xii) If each $|\phi_r|$ is quasi additive, then $\|\phi\|$ is quasi additive. Hence, if each ϕ_r^+, ϕ_r^- is quasi additive, then $\|\phi\|$ is quasi additive.

In Ref. 6, Section 4, Cesari shows how the following functionals can be expressed as BC-integrals of quasi additive interval functions with respect to appropriate mesh functions:

The Jordan length of continuous and discontinuous curves in E_n.

The Cauchy integral in an interval in E_m.

The Lebesgue-Stieltjes integral of a μ-integrable function $f(x)\colon A \to E_1$, in a measure space $(A,\mathcal{B},\mu)$.

The parametric line integrals over curves C in E_n assumed to be only continuous and rectifiable (or Weierstrass integrals on C). These integrals can be thought of as depending on generalized differential elements of order one of C.

The parametric surface integrals over surfaces S in E_3 assumed to be only continous and of finite Lebesgue area (or Cesari-Weierstrass integrals on S). These integrals can be thought of as depending on generalized differential elements of order one of S.

2.3 THE WEIERSTRASS-TYPE INTEGRAL $\int f(T,\phi)$ AS A BC-INTEGRAL

In the setting of Section 2.1, let ϕ be a vector interval function from $\{I\}$ to E_k.

Let $T = T(w)$, $w \in A$, be a mapping from A to E_m. For each $I \in \{I\}$, define

$$\omega(I) = \sup_{u,v \in I} \|T(u)-T(v)\| ,$$

and for each $D \in \mathcal{D}$, define

$$\omega(D) = \max_{I \in D} \omega(I) .$$

We shall assume that the following condition holds:

$$(\omega) \qquad \omega(D) \leq \delta(D) .$$

Let $f(p,q)$ be a real function on $T(A) \times E_k$. We shall denote by U the unit sphere $\{q: \|q\| = 1\}$ in E_k. Assume that

(f) $f(p,q)$ is positively homogeneous of degree one in q, that is, $f(p,tq) = tf(p,q)$ for all $t > 0$, $p \in T(A)$, $q \in E_k$; and $f(p,q)$ is bounded and uniformly continuous on $T(A) \times U$.

Define

$$\Phi(I) = f(T(\tau), \phi(I)) ,$$

where τ is any fixed point of I.

Cesari[6] has proved the following fundamental theorem:

(xiii) If ϕ is quasi additive and of bounded variation with respect to δ, and conditions (ω), (f) hold, then

$$\Phi(I) = f(T(\tau), \phi(I)), \qquad \tau \epsilon I$$

is quasi additive and of bounded variation with respect to δ, and the elements λ, η of the definition (ϕ) can be defined independently of the choice of the points τ in I. Thus the BC-integral of Φ exists and is finite, that is

$$\int \Phi = \int f(T, \phi) = \lim_{\delta(D) \to 0} \Sigma f(T(\tau), \phi(I)) .$$

Also, $\int f(T, \phi)$ is independent of the choice of τ in I.

By means of this theorem, the Weierstrass-type integrals $\int f(T, \phi)$ relative to a mapping T and a quasi additive set function ϕ are reduced to the standard BC-integrals of Section 2.1.

Let us mention here that line integrals for rectifiable continuous curves have been treated as Weierstrass integrals by Tonelli[23] in view of applications to calculus of variations, and again more recently by N. Aronszajn,[1] G. Bouligand,[2] K. Menger,[17] C. Pauc.[21] Surface integrals for continuous surfaces of finite Lebesgue area have been treated as Weierstrass-type integrals by Cesari.[3,5,6]

2.4 INDUCED MEASURES

We shall assume here that A is a topological space. As in Ref. 7, we localize the properties of the system A, $\mathcal{U}$, $\{I\}$, $\mathcal{D}$, δ in Section 2.1 to a class $\mathcal{G}$ of subsets of A, including A, as follows. For each G in $\mathcal{G}$, let $D_G = \{I: I \in D, I \subset G\}$, $\mathcal{D}_G = \{D_G: D \in \mathcal{D}\}$. We require that

(b_2) $\mathcal{D}_G$ is non-empty for each non-empty $G \in \mathcal{G}$. For G non-empty, let δ_G be a mesh function on $\mathcal{D}_G$, such that

(d_3) for each $\tau > 0$, there exists $\nu(\tau, G) > 0$ such that, for every $D \in \mathcal{D}$ with $\delta(D) < \nu(\tau, G)$, $\delta_G(D_G) < \tau$ and D_G is non-empty.

For a vector function ϕ on $\{I\}$, we can consider, as in Section 2.1, the existence of the following limits:

$$B_r(G) = \lim_{\delta_G(D_G) \to 0} \sum_{I \in D_G} \phi_r(I),$$

$$V(G) = \lim_{\delta_G(D_G) \to 0} \sum_{I \in D_G} \|\phi(I)\|,$$

$$V_r(G) = \lim_{\delta_G(D_G) \to 0} \sum_{I \in D_G} |\phi_r(I)|,$$

$$V_r^+(G) = \lim_{\delta_G(D_G) \to 0} \sum_{I \in D_G} \phi_r^+(I),$$

$$V_r^-(G) = \lim_{\delta_G(D_G) \to 0} \sum_{I \in D_G} \phi_r^-(I) .$$

We shall call V(G) the total variation of ϕ in G. If these limits exist, then, by the relation (d_3) between δ and δ_G, they also exist for $\delta(D) \to 0$.

The properties of BC-integrals given in Section 2.1 obviously apply also for each $G \in \mathcal{G}$. Also, if $G_1, G_2 \in \mathcal{G}$ with $G_1 \subset G_2$, then

$$V(G_1) \leq V(G_2)$$

whenever these exist, and similarly for V_r, V_r^+, V_r^-.

From now on, we shall assume also that $(\mathcal{G}\phi)\phi(I)$ is quasi additive with respect to each δ_G. In other words, we assume that, given $\mathcal{E} > 0$ and $G \in \mathcal{G}$, there is a number $\eta(\mathcal{E}, G) > 0$ such that, if $D_{OG} = [I]$ is any system in $\mathcal{D}_G$ with $\delta_G(D_{OG}) < \eta(\mathcal{E}, G)$, then there is also a number $\lambda(\mathcal{E}, D_{OG}, G) > 0$ such that, for every system $D_G = [J]$ in $\mathcal{D}_G$ with $\delta_G(D_G) < \lambda(\mathcal{E}, D_{OG}, G)$, we have

$$\sum_{I \in D_{OG}} \|\phi(I) - \sum_{J \subset I} \phi(J)\| < \mathcal{E}$$

and

$$\Sigma' \|\phi(J)\| < \mathcal{E}$$

where Σ' ranges over all $J \in D_G$ not completely contained in any $I \in D_{OG}$.

If ϕ satisfies condition $(\mathcal{G}\phi)$, then $B(G)$, $V(G)$, $V_r(G)$, $V_r^+(G)$, and $V_r^-(G)$ exist for each $G \in \mathcal{G}$. If also $V(A) < \infty$, then $\|\phi\|$, $|\phi_r|$, ϕ_r^+, ϕ_r^- satisfy condition $(\mathcal{G}\phi)$, and all the integrals above are finite.

From now on, we shall require (a) $\mathcal{G} \subset \mathcal{U}$; (c) each $I \in \{I\}$ is $\mathcal{U}$-connected.

Under hypotheses (a), (b), (c), (d), $(\mathcal{G}\phi)$, and $V(A) < \infty$, every disjoint sequence $\{G_i\}$ with G_i and $\cup G_i \in \mathcal{G}$ has

$$V(\bigcup G_i) = \sum V(G_i)$$

and similarly for B, V_r, V_r^+, V_r^-.

Consider the following conditions for sequences G_i in $\mathcal{G}$.

(H_1) If $G_i \to \emptyset$, then $V(G_i) \to 0$ and similarly for B, V_r, V_r^+, V_r^-.

(H_2) If $G_i \subset G_{i+1}$ and $G_i \to G \epsilon \mathcal{G}$, then $V(G_i) \to V(G)$ and similarly for B, V_r, V_r^+, V_r^-.

(H_3) If $\bigcup_{i=1}^{n} G_i \epsilon \mathcal{G}$ for each n and $\bigcup G_i \epsilon \mathcal{G}$, then $V(\bigcup G_i) \leq \sum V(G_i)$ and similarly for V_r, V_r^+, V_r^-.

The condition

(e) "For every pair of distinct sets $G_1, G_2 \epsilon \mathcal{G}$ with $G = G_1 \cup G_2 \epsilon \mathcal{G}$, $G_1 \cap G_2 \neq \emptyset$, and any $I \epsilon \{I\}$ with $I \subset G$, $I \cap G_1 \neq \emptyset$, $I \cap G_2 \neq \emptyset$, there exists $X(I, G_1, G_2) > 0$ such that any $D_G \epsilon \mathcal{D}_G$ with $\delta_G(D_G) < X(I, G_1, G_2)$ and $J \epsilon D_G$ with $J \subset I$ have $J \subset G_1$ or $J \subset G_2$ or both," with (H_2) and $V(A) < \infty$ implies (H_3).

Also, the condition

(g) "The sets $I \epsilon \{I\}$ are $\mathcal{U}$-compact" with (e) and $V(A) < \infty$ implies (H_2).

From now on, we require that

(a') $\mathcal{G}$ is a subtopology of $\mathcal{U}$.

For each $X \subset A$, define

$$\mu(X) = \inf_{G \supset X} V(G) ,$$

and similarly for μ_r, μ_r^+, μ_r^- from V_r, V_r^+, V_r^-. If $V(A) < \infty$, all these are finite, and we can define

$$\nu_r(X) = \mu_r^+(X) - \mu_r^-(X), \qquad \nu(X) = [\nu_r(X)] .$$

It can be shown that, for every set $X \subset A$, there is a sequence $\{G_i\}$, $G_i \in \mathcal{G}$, $X \subset G_i$, with $V(G_i) \to \mu(X)$, $V_r(G_i) \to \mu_r(X)$, $V_r^+(G_i) \to \mu_r^+(X)$, $V_r^-(G_i) \to \mu_r^-(X)$, and, if $V(A) < \infty$, $B(G_i) \to \nu(X)$. Also

$$\mu_r(X) = \mu_r^+(X) + \mu_r^-(X) ;$$

and

$$\mu_r(X) \leq \mu(X) \leq \sum_{r=1}^{k} \mu_r(X) ,$$

$$\|\nu(X)\| \leq [\Sigma\mu_r^2(X)]^{1/2} \leq \mu(X) .$$

In addition, for every $G \in \mathcal{G}$, $\mu(G) = V(G)$, and similarly for μ_r, μ_r^+, μ_r^-, and, if $V(A) < \infty$, ν.

(xiv) If condition (H_1) holds, then $\mu(\emptyset) = 0$, and similarly for μ_r, μ_r^+, μ_r^-.

(xv) If conditions (H_1), (H_2), (H_3) hold, and $V(A) < \infty$, then μ, μ_r, μ_r^+, and μ_r^- are other measures.

We shall then define measurable sets in the standard way.

Consider the condition

(H_4) For every $G \in \mathcal{G}$, there exists a sequence $\{G_i\}$, $G_i \in \mathcal{G}$, such that $G_i \subset G$, $\overline{G}_i \subset G_{i+1}$ (where $\overline{G}_i$ is the $\mathcal{G}$-closure of G_i), and $V(G_i) \to V(G)$, and similarly for B, V_r, V_r^+, V_r^-.

The condition

(P) "For every $G \in \mathcal{G}$, there exists a sequence $\{G_i\}$, $G_i \in \mathcal{G}$, such that

$G_i \subset G$, $G_i \subset G_{i+1}$, and $G_i \to G'$ with (H_2) implies (H_4).

(xvi) If conditions (H_1), (H_2), (H_3), (H_4) hold and $V(A) < \infty$, then all $G \in \mathcal{G}$ (and so all sets of the minimal σ-algebra $\mathcal{B}$ containing $\mathcal{G}$) are μ, μ_r, μ_r^+, and μ_r^- measurable, so that the restrictions to $\mathcal{B}$ are measures; these measures on $\mathcal{B}$ are $\mathcal{G}$-regular; and the ν_r on $\mathcal{B}$ are signed measures. Also, for each r, there is a Hahn decomposition of A into two disjoint measurable sets A_r^+, A_r^-, such that, for every $H \in \mathcal{B}$, $\nu_r(A_r^+ \cap H) \geq 0$, $\nu_r(A_r^- \cap H) \leq 0$. Writing

$$\nu_r^+(H) = \nu_r(A_r^+ \cap H), \qquad \nu_r^-(H) = -\nu_r(A_r^- \cap H) ,$$

$$\nu_r^* = \nu_r^+ + \nu_r^-,$$

we have

$$0 \leq \nu_r^+ \leq \mu_r^+ ,$$

$$0 \leq \nu_r^- \leq \mu_r^- ,$$

$$0 \leq \nu_r^* \leq \mu_r .$$

From now on, we suppose that ϕ satisfies a stronger quasi additivity condition (ϕ') in which sums are taken over $J \subset$ (or $\not\subset$) I^o, the $\mathcal{G}$-interior of I, rather than just I.

(xvii) If (H_1), (H_2), (H_3), (H_4) hold and $V(A) < \infty$, $\mu_r^+ = \nu_r^+$, $\mu_r^- = \nu_r^-$, $\mu_r = \nu_r^*$ on $\mathcal{B}$; and for any $H \in \mathcal{B}$,

$$\mu(H) = \sup_{[H]} \sum_{H_i \in [H]} (\Sigma \mu_r^2(H_i))^{1/2} = \sup_{[H]} \sum_{H_i \in [H]} \|\nu(H_i)\|$$

where [H] is any finite decomposition of H into disjoint sets $H_i \in \mathcal{B}$.

Earlier inequalities imply absolute continuity appropriate to the existence of the Radon-Nikodym derivatives

$$\theta_r = d\nu_r/d\mu , \qquad \beta_r = d\mu_r/d\mu ,$$
$$\beta_r^+ = d\mu_r^+/d\mu , \qquad \beta_r^- = d\mu_r^-/d\mu$$

μ-almost everywhere in A, and

$$\gamma_r = d\nu_r/d\mu_r ,$$

$$\gamma_r^+ = d\mu_r^+/d\mu_r , \qquad \gamma_r^- = d\mu_r^-/d\mu$$

μ_r-almost everywhere, together with their respective measurability. We have $-1 \leq \theta_r, \gamma_r \leq 1$, $0 \leq \beta_r, \beta_r^+, \beta_r^-, \gamma_r^+, \gamma_r^- \leq 1$. Let $\theta = [\theta_r]$.

(xviii) If (H_1), (H_2), (H_3), (H_4) hold and $V(A) < \infty$, then

$$\beta_r = \beta_r^+ + \beta_r^- , \qquad \theta_r = \beta_r^+ - \beta_r^- ,$$
$$\beta_r^+ = \gamma_r^+\beta_r , \qquad \beta_r^- = \gamma_r^-\beta_r , \qquad \beta_r^+\beta_r^- = 0 ,$$
$$|\theta_r| = \beta_r , \qquad \|\theta\|^2 = \sum\beta_r^2 = 1$$

μ-almost everywhere, and

$$\gamma_r^+ + \gamma_r^- = 1 , \qquad \gamma_r = \gamma_r^+ - \gamma_r^- ,$$
$$\gamma_r^+\gamma_r^- = 0, \qquad |\gamma_r| = 1$$

and either $\gamma_r^+ = 1$, $\gamma_r^- = 0$ or $\gamma_r^+ = 0$, $\gamma_r^- = 1$

μ_r-almost everywhere.

2.5 REPRESENTATION OF BC-INTEGRALS

We mention here the following main theorems proved in Ref. 7. By $T = T(w)$, $w \in A$, is meant a mapping from A into E_m.

(xix) Under hypotheses (a'), (b), (c), (d), (ϕ'), (H_1), (H_2), (H_3), (H_4), $V(A) < \infty$, (ω), (f), the integral $\int f(T,\phi)$ can be expressed as

$$\lim_{\delta(D)\to 0} \sum_{I \in D} f(T(\tau),\nu(I^0)) \ .$$

(xx) Under the same hypotheses as in (xix), the function $f(T(w),\Theta(w))$, $w \in A$, is defined μ-almost everywhere in A, is μ-measurable and μ-integrable in A, and the integral $\int f(T,\phi)$ has the following representation as a Lebesgue-Stieltjes integral in the measure space $(A, \mathcal{B}, \mu)$:

$$\int f(T,\phi) \quad = \quad (A) \int f[T(w),\Theta(w)] d\mu \ .$$

In particular, if we take T constant and $f(T,\phi) = \phi_r$, then $\omega(D) = 0$ for every D, so the relation (ω) is certainly satisfied, and we have

$$\int \phi_r \quad = \quad (A) \int d\nu_r \ , \qquad r = 1,2,\ldots,k \ ,$$

or, in vector form

$$\int \phi \quad = \quad (A) \int d\nu \ .$$

3. THE LEBESGUE-STIELTJES INTEGRAL AS A BC-INTEGRAL

In Ref. 6, Section 4, Cesari has shown how the finite Lebesgue-Stieltjes integral on any measure space can be expressed as the BC-integral of a certain quasi additive interval function ψ with respect to a certain mesh function δ. The objective of this chapter is to show that this can be accomplished also by another interval function $\bar{\psi}$ with the usual mesh function $\bar{\delta}$ of Lebesgue-Stieltjes theory.

3.1 THE INTERVAL FUNCTION $\bar{\Psi}$

Let $(X, \mathcal{M}, \mu)$ be a measure space, and f a μ-measurable real function on X, and let $(X)\int f\, d\mu$ be the corresponding Lebesgue-Stieltjes integral. Since f can be decomposed into its positive and negative parts, we shall assume $f \geq 0$. Let μ_p denote the μ-measure of the set $\{x: f(x) > p\}$.

In the setting of Section 2.1, let us take for the set A the extended set of non-negative real numbers, $A = [0 \leq y \leq \infty]$. Let us take for $\{I\}$ the collection of all half-open half-closed intervals $I(p,q) = (p,q]$. Let $\mathcal{D}$ be the collection of all finite systems

$$D = \{(p_{i-1}, p_i]: i = 1,2,\ldots,n-1\},$$

$$0 = p_0 < p_1 < \ldots < p_{n-1} < \infty, \qquad n \geq 2,$$

of non-overlapping intervals I covering some finite interval $(0, p_{n-1}]$.

Let $\bar{\Psi}(I) = \bar{\Psi}(p,q]$ be the interval function

$$\overline{\Psi}(p,q] = (q-p)\mu\{x: f(x) > \nu_{pq}\} ,$$

where ν_{pq} denotes any number with $p \leq \nu_{pq} \leq q$.

Let $\overline{\delta}(D)$ be the usual mesh function of Lebesque-Stieltjes theory,

$$\overline{\delta}(D) = \max_{i=1,..n-1} (p_i - p_{i-1}) + 1/p_{n-1}$$

This function is obviously a mesh function.

Theorem 3.1. If $\mu_0 < \infty$, then the function $\overline{\Psi}$ is quasi subadditive with respect to the mesh function $\overline{\delta}$, and the corresponding BC-integral coincides with the Lebesque-Stieltjes integral of f:

$$\int\overline{\Psi} = (X) \int f(x)\, d\mu .$$

If $\mu_0 = \infty$, then the same is true with the particular choice $\nu_{pq} = q$.

Proof: The proof is divided into parts (a), (b), (c), and (d).

(a) For $\mu_0 < \infty$, $\overline{\Psi}$ is quasi subadditive with respect to $\overline{\delta}$. In fact, we shall prove a more general result which will be used in part (b).

For any $\varepsilon > 0$, take any

$$D_0 = \{(p_{i-1},p_i]: i = 1,2,\ldots,n-1\}$$

with $\overline{\delta}(D_0) < \eta(\varepsilon) = \varepsilon/\mu_0$; and any

$$D = \{(q_{j-1},q_j]: j = 1,2,\ldots,m-1\}$$

with $\overline{\delta}(D) < \lambda(\varepsilon,D_0)$, where

$$\lambda(\mathcal{E}, D_o) = \min \left\{ \mathcal{E} / \sum_{i=1}^{n-1} \mu_{p_i},\ 1/p_{n-1},\ p_i - p_{i-1} \text{ for } i=1,2,\ldots n-1 \right\}$$

Let

$$q_{J(i)} = \max \{q_j : q_j \leq p_i\} ,$$

$$q_{j(i)} = \min \{q_j : p_{i-1} \leq q_j\} ;$$

these exist in $[p_{i-1}, p_i]$ since $q_{m-1} > p_{n-1}$ and $q_j - q_{j-1} < p_i - p_{i-1}$.

Let $\overline{\Psi}_1$, $\overline{\Psi}_2$ denote the interval functions corresponding to two choices of ν. Denote a sum over $(q_{j-1}, q_j] \subset (p_{i-1}, p_i]$ by $\Sigma^{(i)}$. Then

$$\overline{\Psi}_1(p_{i-1}, p_i] - \Sigma^{(i)} \overline{\Psi}_2(q_{j-1}, q_j]$$

$$\leq (p_i - p_{i-1})\mu_{p_{i-1}} - (q_{J(i)} - q_{j(i)})\mu_{p_i}$$

$$= (p_i - p_{i-1})(\mu_{p_{i-1}} - \mu_{p_i}) + (p_i - q_{J(i)} + q_{j(i)} - p_{i-1})\mu_{p_i} .$$

Hence

$$\Sigma\{\overline{\Psi}_1(p_{i-1}, p_i] - \Sigma^{(i)} \overline{\Psi}_2(q_{j-1}, q_j]\}^+$$

$$\leq \max(p_i - p_{i-1})(\mu_o - \mu_{p_{n-1}}) + 2\max(q_j - q_{j-1}) \sum_{1}^{n-1} \mu_{p_i}$$

$$< 3\mathcal{E}.$$

The particular case $\overline{\Psi}_1 = \overline{\Psi}_2$ gives the required quasi subadditivity.

(b) $\int\overline{\Psi}$ is independent of the choice of ν_{pq} in $[p, q]$.

Let the notation be as in part (a), but denote $(p_{i-1}, p_i]$ by I_i, and $(q_{j-1}, q_j]$ by J_j.

Since $\overline{\Psi}_2$ is non-negative,

$$\Sigma\overline{\Psi}_1(I_i) \leq \sum_i [\overline{\Psi}_1(I_i) - \Sigma^{(i)}\overline{\Psi}_2(J_j)]^+ + \Sigma\overline{\Psi}_2(J_j) .$$

Hence, for any $\mathcal{E} > 0$, if $\delta(D_o) < \eta(\mathcal{E})$ and $\delta(D) < \lambda(\mathcal{E}, D_o)$, then $\Sigma\overline{\Psi}_1(I_i) < \mathcal{E} + \Sigma\overline{\Psi}_2(J_j)$. Hence, if $\delta(D_o) < \eta(\mathcal{E})$, then

$$\Sigma\overline{\Psi}_1(I_i) \leq \mathcal{E} + \int\overline{\Psi}_2 ,$$

so

$$\int\overline{\Psi}_1 \leq \mathcal{E} + \int\overline{\Psi}_2 .$$

Thus

$$\int\overline{\Psi}_1 \leq \int\overline{\Psi}_2 ,$$

so, by symmetry,

$$\int\overline{\Psi}_1 = \int\overline{\Psi}_2 .$$

(c) For $\mu_o < \infty$, $\int\overline{\Psi} = (X)\int f(x)d\mu$.

By part (b), we need prove this only for the choice $\nu_{pq} = q$. For this choice, we shall not assume $\mu_o < \infty$.

Consider the set

$$S(D) = \bigcup_{i=1}^{n-1} \{(x,y): p_{i-1} < y \leq p_i, f(x) > p_i\} .$$

Let m be the product measure $\ell \times \mu$, where ℓ is real Lebesgue measure. Then

$$m\{S(D)\} = \sum_{I \in D} \overline{\Psi}(I) ,$$

where $\nu_{pq} = q$, so

$$\lim_{\overline{\delta}(D)\to 0} m\{S(D)\} = \int \overline{\Psi} .$$

Now

$$\{(x,y)\colon 0 < y < f(x)\} \subseteq \liminf_{\overline{\delta}(D)\to 0} S(D) .$$

Hence

$$(X) \int f(x) d\mu \leq m\{\liminf_{\overline{\delta}(D)\to 0} S(D)\}$$

$$\leq \lim_{\overline{\delta}(D)\to 0} m\{S(D)\} .$$

Also

$$S(D) \subset \{(x,y)\colon 0 \leq y \leq f(x)\} .$$

Hence

$$\lim_{\overline{\delta}(D)\to 0} m\{S(D)\} \leq (X) \int f \, d\mu .$$

Thus

$$\int \overline{\Psi} = \lim_{\overline{\delta}(D)\to 0} m\{S(D)\} = (X) \int f \, d\mu .$$

Remark: The condition $\mu_0 < \infty$ cannot be omitted here when ν_{pq} is chosen arbitrarily in $[p,q]$. For example, let $f(x) = x^{-2}$ on $\{x \geq 1\}$ in E_1. Then $\int f \, d\mu = 1$. However, if $p_1 < 1$, $p_2 = p_1^{1/4}$, and $\nu_{p_1 p_2} = p_1$, then

$$(p_2-p_1)\mu\{x: f(x) > \nu_{p_1p_2}\} = (p_1^{1/4}-p_1)(p_1^{-1/2}-1)$$

$$\to \infty \quad \text{as} \quad p_1 \to 0 .$$

(d) For μ_o possibly infinite, if $\nu_{pq} = q$, then

$$\overline{\Psi}(p,q] = (q-p)\mu\{x: f(x) > q\}$$

is quasi subadditive with respect to $\overline{\delta}$.

Let the notation be as in part (a). For any $\mathcal{E} > 0$, take any D_o; and take any D with $\overline{\delta}(D) < \lambda(\mathcal{E},D_o)$. Then

$$\overline{\Psi}(p_{i-1},p_i] - \Sigma^{(i)}\overline{\Psi}(q_{j-1},q_j]$$

$$\leq (p_i-q_{J(i)}+q_{j(i)}-p_{i-1})\mu_{p_i} .$$

Hence

$$\Sigma\{\overline{\Psi}(p_{i-1},p_i] - \Sigma^{(i)}\overline{\Psi}(q_{j-1},q_j]\}^+$$

$$\leq 2 \max (q_j-q_{j-1}) \ \Sigma \mu_{p_i}$$

$$< 2\mathcal{E}.$$

3.2 COMPARISON WITH PREVIOUS RESULTS

For $(X,\mathcal{M},\mu)$ as in Section 3.1, and f a non-negative μ-integrable function on X, Cesari considers the following situation (Ref. 6, p. 101-105).

The set A = X.

Intervals $I(p,q) = \{x: x\epsilon X,\ p < f(x) \leq q\}$, where $0 \leq p < q \leq \infty$.

Systems $D = \{I(p_{i-1},p_i): i = 1,2,\ldots,n\}$ with $0 = p_o < p_1 < \ldots < p_n=\infty$.

The interval function

$$\psi(p,q) = p\mu\{x: p < f(x) \leq q\} .$$

The mesh function

$$\delta(D) = \max_{i=1,\ldots n-1} (p_i-p_{i-1}) + 1/p_{n-1} + \sum_{i=1}^{n-1} p_i\mu\,\{x: f(x) = p_i\} .$$

Then ψ is quasi additive with respect to δ (Ref. 6).

Classical theory, for example (Ref. 22, p. 117-121), (Ref. 11, p. 179-183), shows, though in a somewhat different and restricted form, that $(X)\int f(x)\,d\mu$ is the BC-integral of ψ with respect to $\overline{\delta}$. Since $\overline{\delta}(D) \leq \delta(D)$, the BC-integral of ψ with respect to δ is also $(X)\int f(x)\,d\mu$.

In fact, provided $\mu_o < \infty$, $(X)\int f\,d\mu$ (even possibly infinite) is also the BC-integral of the more general interval function

$$\psi'(p,q] = \nu_{pq}\,\mu\{x: p < f(x) \leq q\}$$

with respect to $\overline{\delta}(D)$, for any ν_{pq} satisfying

$$p \leq \nu_{pq} \leq q , \qquad \eta_{p\infty} = p .$$

The following example shows that ψ, and more generally ψ', need not be even quasi subadditive with respect to $\overline{\delta}$.

Let $(X,\mathcal{M},\mu)$ be a measure space with $\mu(X) = 1$, and consider

$f(x) = 1$ for every x in X. Then $\psi'(p,q) = \nu_{pq}$ for $p < 1 \leq q$, 0 otherwise. If ψ' were quasi subadditive with respect to $\overline{\delta}$, then

$$\sum_{I \in D_o} [\psi'(I) - \Sigma^{(I)}\psi'(J)]^+ < \mathcal{E}$$

for $\overline{\delta}(D_o)$ less than some $\eta(\mathcal{E})$ and $\overline{\delta}(D)$ less than some $\lambda(\mathcal{E}, D_o)$. Take $\mathcal{E} = 1/2$; D_o with $\overline{\delta}(D_o) < \eta(1/2)$ and some member $I(\alpha,1)$ with $\alpha > 1/2$ (so $\psi'(I_o) = \nu_{\alpha 1} > 1/2$); and D with $\overline{\delta}(D) < \lambda(1/2, D_o)$ and some member $J(\beta,\gamma)$ with $\beta < 1 < \gamma$ (so $\psi'(J) = 0$ for $J \subset I \in D_o$). Then

$$\sum_{I \in D_o} [\psi'(I) - \Sigma^{(I)}\psi'(J)]^+ = \nu_{\alpha 1} > 1/2 .$$

4. THE BEND OF A CURVE AS A BC-INTEGRAL

Iseki[13] has introduced the concept of "bend" of a curve in E_n, as a generalization of total curvature. He develops the theory of this bend in Ref. 13-15, and other papers. Results relevant to this chapter are given in Section 4.1 below.

The objective of this chapter is to obtain the bend of a curve as a BC-integral from a generating interval function as simple as possible. This leads to a systematic treatment of the bend. As is to be expected, the simplicity of the generating function that can be achieved depends on the strength of conditions imposed on the curve. Sections 4.2, 4.3, and 4.4 below deal with the problem under different sets of conditions.

In the present chapter, the set A of Section 2.1 will be a fixed interval [a,b] with the Euclidean topology, and the systems D will be finite subdivisions of [a,b], so that the subsets I are closed subintervals of [a,b].

4.1 THE BEND OF A CURVE

Consider a curve $X(t)$, $a \leq t \leq b$, in E_n. Its bend $\Omega = \Omega(a,b)$ can be taken as the supremum of angle sums

$$\sum_{i=1}^{N-1} \left\langle X(a_i)-X(a_{i-1}),\ X(a_{i+1})-X(a_i)\right\rangle$$

over all subdivisions $a_0, a_1, \ldots, a_N$ of [a,b] with $a = a_0 < a_1 < \ldots < a_N = b$, for which $X(a_{i-1}) \neq X(a_i)$, $i=1,\ldots N$. Here $\langle A,B \rangle$ denotes the geometric angle between the non-zero vectors A, B, $0 \leq \langle A,B \rangle \leq \pi$; angles involving zero vectors are not defined. This is not quite the same as Iseki's definitions in Ref. 13, p. 141 and Ref. 14, p. 115, but is easily seen to be equivalent.

We shall need a continuity property of the bend for continuous curves with finite bend. This is given essentially in Ref. 13, Section 32. We shall cast it as follows: For every positive $\mathcal{E}$ and t in [a,b], there is a positive $\Delta(\mathcal{E},t)$ such that $\Omega(t-\delta,t) < \mathcal{E}$ and $\Omega(t,t+\delta) < \mathcal{E}$ for $0 < \delta < \Delta(\mathcal{E},t)$. (When t = a or b, one of these must be omitted.)

4.2 CONTINUOUS LIGHT CURVES WITH FINITE BEND

Let X(t), $a \leq t \leq b$, be a light curve, that is, X(t) is constant on no subinterval of [a,b].

The simple interval function

$$\sup \left\langle X(t)-X(u),\ X(v)-X(t) \right\rangle$$

where the supremum is taken over all t in (u,v) with $X(t) \neq X(u)$ or X(v), cannot in general generate the bend. This can be seen by considering a circle, where $\Omega = 2\pi$, while every D sum of the interval function has value π.

Consider the interval function

$$\psi(u,v) = \limsup_{(\delta,\delta')\to(0+,0+)} \left\{ \left\langle X(u+\delta)-X(u),\ X(v-\delta')-X(u+\delta) \right\rangle + \left\langle X(v-\delta')-X(u+\delta),\ X(v)-X(v-\delta') \right\rangle \right\} .$$

Here $\limsup_{(\delta,\delta')\to(0+,0+)} f(\delta,\delta')$ means $\inf_{r>0} \sup_{\substack{\|(\delta,\delta')\|<r \\ \delta>0,\ \delta'>0}} f(\delta,\delta')$.

ψ is defined, because, if not,

(i) $X(u+\delta) = X(u)$ for all δ in some positive neighborhood of 0,

or

(ii) $X(v-\delta') = X(v)$ for all δ' in some positive neighborhood of 0,

or

(iii) $X(v-\delta') = X(u+\delta)$ for all δ,δ' in some positive neighborhood

$\|(\delta,\delta')\| < r$, $\delta > 0$, $\delta' > 0$, of $(0,0)$.

In all three cases, X could not be light.

Note that the simpler symmetric lim sup, that is, with $\delta' = \delta$, need not be defined; for example, for $u = 0$, $v = 1$ with $X(t) = (t-t^2,0)$ on $0 \leq t \leq 1$.

Using subadditivity of angle and lim sup, one can easily prove that

$$\psi(u,w) \leq \psi(u,v) + \psi(v,w) + \gamma(v) ,$$

where $u < v < w$, and

$$\gamma(v) = \limsup_{(\delta,\delta')\to(0+,0+)} \langle X(v)-X(v-\delta'),\ X(v+\delta)-X(v)\rangle .$$

Now $\Sigma\gamma(v) \leq \Omega$ for any sum over a finite number of v. Hence, if $\Omega < \infty$, $\gamma(v) = 0$ except at a countable number of points. Thus

$$\delta(D) = \max(a_{i+1}-a_i) + \sum_{i=1}^{N-1} \gamma(a_i)$$

is a mesh function on the family $\mathcal{D}$ of finite subdivisions $D = [I_i]$,

$I_i = [a_i, a_{i+1}]$, $a = a_0 < a_1 < \ldots < a_N = b$, of $[a,b]$.

Theorem 4.1. If $X(t)$, $a \le t \le b$, is a light continuous curve with finite bend Ω, then ψ is quasi additive with respect to δ, and $\int\psi = \Omega$.

Proof: Consider any positive ε. Take any

$$D_0 = [I_i],\ I_i = [a_i, a_{i+1}],\ a = a_0 < a_1 < \ldots < a_N = b,$$

and any

$$D = [J_j],\ J_j = [b_j, b_{j+1}],\ a = b_0 < b_1 < \ldots < b_M = b,$$

with

$$\delta(D) < \min\{a_{i+1} - a_i \text{ for } 0 \le i < N;\ \Delta(\varepsilon/N, a_i) \text{ for } 0 < i < N;\ \varepsilon\}\ .$$

Here Δ is the function involved in the continuity of bend in Section 4.1.

Let

$$b_{j(i)} = \min(b_j\colon b_j \ge a_i)\ ,$$

$$b_{J(i)} = \max(b_j\colon b_j \le a_{i+1})\ .$$

These exist in $[a_i, a_{i+1}]$ since $b_{j+1} - b_j < a_{i+1} - a_i$. Then

$$\psi(I_i) - \Sigma^{(i)}\psi(J_j)$$
$$\le \psi(a_i, b_{j(i)}) + \psi(b_{J(i)}, a_{i+1}) + \sum_{j(i)}^{J(i)} \gamma(b_j)$$

with special simplification at a for $i = 0$ and at b for $i = N-1$. Now

$$\psi(u,v) \leq \Omega(u,v) ,$$

so

$$\Sigma[\psi(I_i) - \Sigma^{(i)}\psi(J_j)]^+ < 3\varepsilon .$$

Thus ψ is quasi subadditive with respect to δ, and so has a BC-integral $\int\psi$. Now $\int\psi \leq \Omega$ immediately, so $\int\psi$ is finite. Hence, since ψ is non-negative, ψ is quasi additive with respect to δ.

To prove that $\int\psi = \Omega$, consider any angle sum Σ_1 appearing in the definition of Ω. For any positive ε, using continuity of angle and of X, shift the points of subdivision to where $\gamma = 0$ (such points are dense), while keeping $\Sigma_1 < \Sigma_2 + \varepsilon$, where Σ_2 is the angle sum for the adjusted subdivision. Since $\int\psi < \infty$, $|\Sigma\psi(I) - \int\psi| < \varepsilon$ for $\delta(D)$ less than some $\zeta(\varepsilon)$. Subdivide the second subdivision further at points where $\gamma = 0$ to obtain a subdivision D_3 with $\delta(D_3) < \zeta(\varepsilon)$. Now put in pairs of points about the points of subdivision of D_3, sufficiently close to make the part of the new angle sum Σ_4 corresponding to the points of D_3 less than ε (this is possible since $\gamma = 0$ at these points) and the rest of Σ_4 less than

$$\sum_{D_3} \psi(I) + \varepsilon .$$

By subadditivity of angle, $\Sigma_2 \leq \Sigma_3 \leq \Sigma_4$. Hence

$$\Sigma_1 < \Sigma_4 + \varepsilon < \sum_{D_3} \psi(I) + 3\varepsilon < \int\psi + 4\varepsilon .$$

Thus $\Sigma_1 \leq \int\psi$, so $\Omega \leq \int\psi$.

This completes the proof of Theorem 4.1.

Continuous light curves with finite bend have right and left tangents R, L at every point (with obvious restriction at a, b) (Ref. 13, p. 162). Hence, by continuity of angle,

$$\psi(u,v) = \left\langle R(u),\ X(v)-X(u)\right\rangle + \left\langle X(v)-X(u), L(v)\right\rangle$$

and

$$\gamma(v) = \left\langle L(v), R(v)\right\rangle .$$

4.3 GENERAL LIGHT CURVES

Let us consider the discontinuous plane curve X(t), $0 \leq t \leq 2$, defined by

$$X(t) = (t,0) \text{ on } 0 \leq t < 1 \text{ and } 1 < t \leq 2,$$
$$(1,1) \text{ at } t = 1.$$

We have here $\sup \Sigma\psi = \pi$. If we consider the modified interval function ψ' defined by

$$\psi'(u,v) = \sup\left\{\left\langle X(s)-X(u),\ X(t)-X(s)\right\rangle + \left\langle X(t)-X(s),\ X(v)-X(t)\right\rangle\right\}$$

where the supremum is taken over all s,t for which $u < s < t < v$, $X(u) \neq X(s)$, $X(s) \neq X(t)$, $X(t) \neq X(v)$, we have $\sup \Sigma\psi' = 3\pi/2$. Finally $\Omega = 2\pi$. The discrepancy between these values shows not only that we cannot generate Ω from ψ by adjusting the mesh function, but that any adjustment of ψ keeping to "two angles" will fail to generate Ω.

Thus, in order to deal with discontinuous light curves $X(t)$, $a \leq t \leq b$, we shall consider the "three angle" generator

$$\overline{\psi}(u,v) = \sup_{u<t<v} \phi(u,t,v)$$

where

$$\phi(u,t,v) = \limsup_{\delta\to 0+} \Big\{ \langle X(u+\delta)-X(u),\ X(t)-X(u+\delta)\rangle + \langle X(t)-X(u+\delta),\ X(v-\delta)-X(t)\rangle + \langle X(v-\delta)-X(t),\ X(v)-X(v-\delta)\rangle \Big\} .$$

$\overline{\psi}$ is defined, because, if not, ϕ would not be defined for any t. Then, for all δ in some positive neighborhood of 0,

(i) $X(u+\delta) = X(u)$,

or

(ii) $X(t) = X(u+\delta)$

or

(iii) $X(v-\delta) = X(t)$

or

(iv) $X(v) = X(v-\delta)$.

In all four cases, X would not be light.

Some manipulation shows that $\overline{\psi}$ has the same property

$$\overline{\psi}(u,w) \leq \overline{\psi}(u,v) + \overline{\psi}(v,w) + \overline{\gamma}(v) \qquad (u < v < w)$$

as ψ, but with

$$\overline{\gamma}(v) = \limsup_{\delta \to 0+} \left\langle X(v)-X(v-\delta),\ X(v+\delta)-X(v) \right\rangle .$$

For $a < u < b$, define

$$\lambda^{+}(u) = \limsup_{\delta \to 0+} \overline{\psi}(u,u+\delta),$$

$$\lambda^{-}(u) = \limsup_{\delta \to 0+} \overline{\psi}(u-\delta,u),$$

$$\overline{\lambda}(u) = \lambda^{+}(u) + \lambda^{-}(u) .$$

Then, for any positive ε,

$$\overline{\psi}(u,u+\delta) < \lambda^{+}(u) + \varepsilon$$

and

$$\overline{\psi}(u-\delta,u) < \lambda^{-}(u) + \varepsilon$$

for all positive δ less than some $\Delta(\varepsilon,u)$.

Define

$$\Gamma = \sup \Sigma\overline{\gamma}(t)$$

$$\Lambda = \sup \Sigma\overline{\lambda}(t)$$

where the sums are over a finite number of distinct t. In order that

$$\overline{\delta}(D) = \max(a_{i+1}-a_i) + \sum_{1}^{N-1} \overline{\gamma}(a_i) + \sum_{1}^{N-1} \overline{\lambda}(a_i)$$

be a mesh function on the finite subdivisions

$$D = [I_i],\ I_i = [a_i, a_{i+1}],\ a = a_0 < a_1 < \ldots < a_N = b\ ,$$

it is sufficient that $\{t: \bar{\gamma}(t) = \bar{\lambda}(t) = 0\}$ be dense in $[a,b]$. This will be true if Γ and Λ are finite. Since Γ and $\Lambda \leq \Omega$, $\bar{\delta}$ is certainly a mesh function for a curve with finite bend.

Theorem 4.2. If $X(t)$, $a \leq t \leq b$, is any light curve (not necessarily continuous) for which $\bar{\delta}$ is a mesh function (in particular, for $\Gamma < \infty$, $\Lambda < \infty$), then $\bar{\psi}$ is quasi subadditive with respect to $\bar{\delta}$, and $\int\bar{\psi} = \Omega$.

Proof: For any positive $\mathcal{E}$, take any

$$D_0 = [I_i],\ I_i = [a_i, a_{i+1}],\ a = a_0 < a_1 < \ldots < a_N = b$$

with $\bar{\delta}(D_0) < \mathcal{E}$, and any

$$D = [J_j],\ J_j = [b_j, b_{j+1}],\ a = b_0 < b_1 < \ldots < b_M = b$$

with $\bar{\delta}(D) < \min\{a_{i+1}-a_i \text{ for } 0 \leq i < N;\ \Delta(\mathcal{E}/N, a_i) \text{ for } 0 < i < N;\ \mathcal{E}\}$.

Then

$$\begin{aligned}\bar{\psi}(I_i) - \Sigma^{(i)}\bar{\psi}(J_j) &\leq \bar{\psi}(a_i, b_{j(i)}) + \bar{\psi}(b_{J(i)}, a_{i+1}) + \sum_{j(i)}^{J(i)} \bar{\gamma}(b_j)\\ &< \lambda^+(a_i) + \mathcal{E}/N + \lambda^-(a_{i+1}) + \mathcal{E}/N + \sum_{j(i)}^{J(i)} \bar{\gamma}(b_j)\end{aligned}$$

with special simplification at a for $i = 0$ and at b for $i = N-1$. Hence

$$\Sigma[\overline{\Psi}(I_i) - \Sigma^{(i)}\overline{\Psi}(J_j)]^+$$
$$< \sum_1^{N-1} \overline{\lambda}(a_i) + \sum_1^{N-1} \overline{\gamma}(b_j) + 2\varepsilon$$
$$< 4\varepsilon .$$

Thus $\overline{\Psi}$ is quasi subadditive with respect to $\overline{\delta}$, and so has a BC-integral $\int\overline{\Psi}$.

Now $\int\overline{\Psi} \leq \Omega$ immediately, so $\int\overline{\Psi} = \infty$ gives $\int\overline{\Psi} = \Omega$. For $\int\overline{\Psi} < \infty$, we shall prove that $\Omega \leq \int\overline{\Psi}$.

Consider any subdivision D_1 of $[a,b]$ with angle sum Σ_1. For any positive ε,

$$|\Sigma\overline{\Psi}(I) - \int\overline{\Psi}| < \varepsilon$$

for $\overline{\delta}(D)$ less than some $\eta(\varepsilon)$. Subdivide D_1 further to get D_2 with $\max\,(a_{i+1}-a_i) < \frac{1}{2}\,\eta(\varepsilon)$. Form a subdivision D_2' with $\overline{\delta}(D_2') < \eta(\varepsilon)$ by taking a point with $\overline{\gamma} = \overline{\lambda} = 0$ in each subinterval of D_2. From D_2 and D_2' combined, form D_3 by adding pairs of points symmetrically about the points of D_2', close enough to make that part of the angle sum Σ_3 corresponding to the points of D_2' less than ε , and the rest less than

$$\sum_{D_2'} \overline{\Psi}(I) + \varepsilon .$$

Then

$$\Sigma_1 \leq \Sigma_2 \leq \Sigma_3 < \varepsilon + \sum_{D_2'} \overline{\Psi}(I) + \varepsilon < \int\overline{\Psi} + 3\varepsilon .$$

Hence $\Sigma_1 \leq \int\overline{\Psi}$, so $\Omega \leq \int\overline{\Psi}$.

4.4 ANGLE SWEPT OUT BY DIRECTION

Consider a field of directions on [a,b], that is, a function T(u) defined almost everywhere on [a,b], with values unit vectors in E_n. We wish to express the "angle swept out by T" as a BC-integral. The present discussion of the problem is similar to Cesari's treatment of the length of a discontinuous curve in Ref. 6, Section 4.

Let U be the set in [a,b] on which T is defined. Define on U

$$\lambda^+(u) = \limsup \langle T(u),T(u')\rangle \text{ as } u' \to u+ \text{ on } U,$$
$$\lambda^-(u) = \limsup \langle T(u'),T(u)\rangle \text{ as } u' \to u- \text{ on } U,$$
$$\lambda^+(b) = 0 , \quad \lambda^-(a) = 0 \text{ if these are relevant,}$$
$$\lambda(u) = \lambda^+(u) + \lambda^-(u) .$$

Then, for any $\varepsilon > 0$ and any u in U, there exists $\Delta(\varepsilon,u) > 0$ such that

$$\langle T(u),T(u')\rangle < \lambda^+(u) + \varepsilon \text{ for } 0 < u'-u < \Delta(\varepsilon,u),\ u'\in U$$

and

$$\langle T(u'),T(u)\rangle < \lambda^-(u) + \varepsilon \text{ for } 0 < u-u' < \Delta(\varepsilon,u),\ u'\in U .$$

Define $\Lambda = \sup \Sigma\lambda(u)$, where Σ is taken over a finite number of u. If $\Lambda < \infty$, then $\lambda(u) = 0$ except at a countable number of points. Hence, if $\Lambda < \infty$,

$$\delta(D) = \max(a_{i+1}-a_i)+a_1-a+b-a_N+\Lambda-\Sigma\lambda(a_i)$$

is a mesh function on the partial subdivisions

$$D = [I_i], \quad I_i = [a_i, a_{i+1}],$$

$$a \leq a_1 < a_2 < \ldots < a_N \leq b, \quad a_i \epsilon U .$$

<u>Theorem</u> <u>4.3</u>. The interval function

$$\theta(u,v) = \langle T(u), T(v) \rangle$$

is quasi subadditive with respect to δ.

<u>Proof</u>: For any $\mathcal{E} > 0$, take any

$$D_0 = [I_i], \; I_i = [a_i, a_{i+1}], \; a \leq a_1 < a_2 < \ldots < a_N \leq b,$$

and any

$$D = [J_j], \; J_j = [b_j, b_{j+1}], \; a \leq b_1 < b_2 < \ldots < b_M \leq b$$

with

$$\delta(D) < \min\{a_{i+1} - a_i \text{ for } 1 \leq i < N; \; \Delta(\mathcal{E}/N, a_i) \text{ for } 1 \leq i \leq N; \mathcal{E}\} .$$

Then, by subadditivity of angle,

$$\theta(I_i) - \Sigma^{(i)}\theta(J_j)$$

$$\leq \langle T(a_i), T(b_{j(i)}) \rangle + \langle T(b_{J(i)}), T(a_{i+1}) \rangle$$

$$< \lambda^+(a_i) + \mathcal{E}/N + \lambda^-(a_{i+1}) + \mathcal{E}/N$$

with special simplification when $b_{j(i)} = a_i$ or $b_{J(i)} = a_{i+1}$.

Thus

$$\sum_i [\Theta(I_i) - \Sigma^{(i)}\Theta(J_j)]^+$$

$$< 2\mathcal{E} + \bar{\Sigma}\lambda(a_i) ,$$

where $\bar{\Sigma}$ is taken over $a_i \neq$ any b_j, so that

$$\bar{\Sigma}\lambda(a_i) \leq \Lambda - \Sigma\lambda(b_j) < \mathcal{E} .$$

Hence the BC-integral of Θ with respect to δ exists, and equals

$$C(T) \equiv \sup \Sigma\Theta(I_i).$$

Note that $\Lambda \leq C$, so $\Lambda < \infty$ if $C < \infty$.

Now consider the variation

$$V(T) = \sup \Sigma \|T(a_i)-T(a_{i+1})\|$$

of T, that is, the length of the curve traced out by T on the unit sphere. Since

$$2\Theta/\pi \leq \Delta \equiv \|T(u)-T(v)\| \leq \Theta ,$$

$$2C/\pi \leq V \leq C.$$

Simple examples in which T is discontinuous show that V can be less than C.

However, for T defined everywhere and continuous on [a,b], we shall identify C with V. Obviously $V = \infty$ if $C = \infty$. Consider $C < \infty$. For any

$\mathcal{E} > 0$, $\theta \leq \pi\Delta/2 < \mathcal{E}$ for $|u-v|$ less than some $\zeta(\mathcal{E})$. Since $\sum\theta \to C$ as $\delta(D) \to 0$, $|\sum\theta - C| < \mathcal{E}$ for $\delta(D)$ less than some $\eta(\mathcal{E})$. Take D with $\delta(D) < \min[\zeta(\mathcal{E}), \eta(\mathcal{E})]$. We have

$$\theta - \Delta = \theta - 2\sin(\theta/2) \leq \theta^3/24 ,$$

so

$$\sum\theta \leq \sum\Delta + \sum\theta^3/24 .$$

Hence

$$C - \mathcal{E} < V + \mathcal{E}^2 C/24 ,$$

which gives $C \leq V$.

We now apply these results to any continuous curve X on [a,b] that has a right derived direction T (Ref. 13, Section 73) at all but a countable number of points of [a,b]. Then $C(T) = \Omega(X)$ (Ref. 13, Section 95), so that we have another formulation for Ω as a BC-integral.

Also, if X is continuous with tangent directions T (Ref. 13, Section 42) everywhere in [a,b], and $V(T) < \infty$, then T is continuous (Ref. 13, Section 67), so that we can identify C(T) with V(T).

5. GENERALIZED WEIERSTRASS-TYPE INTEGRALS $\int f(\zeta,\phi)$ AS BC-INTEGRALS

In the definition of the Weierstrass-type integral $\int f(T,\phi)$ in Section 2.3, the expression $T(\tau)$ is essentially an interval function. Our objective in this chapter is to replace $T(\tau)$ by a new interval function $\zeta(I)$, and so consider the BC-integral of the interval function

$$\Phi(I) \quad = \quad f(\zeta(I),\phi(I)) \ .$$

The original reason for doing this was in order to express more conveniently integrals involving higher derivatives in the Weierstrass form, and thus as BC-integrals. Let $f(p,q,r)$ be homogeneous in q, and let $D(w)$ be a point function which is a derivative. In the previous formulation, the integral

$$(A) \ \int f(T(w),\Theta(w),D(w))d\mu$$

would be expressed in the Weierstrass form as

$$\lim_{\delta(D)\to 0} \sum_{I\in D} f(T(\tau),\phi(I),D(\tau)) \ , \qquad \tau\in I \ ,$$

with explicit use of the derivative $D(w)$. In the new formulation, we can consider limits of the form

$$\lim_{\delta(D)\to 0} \sum_{I\in D} f(T(\tau),\phi(I),\Delta(I))$$

where $\Delta(I)$ is a quotient of interval functions giving the derivative $D(w)$ in the limit. Here $T(\tau)$ and $\Delta(I)$ can be thought of as a single (vector valued) interval function $\zeta(I)$.

Thus we shall consider interval functions of the form

$$\Phi(I) = f(\zeta(I), \phi(I))$$

where $f(u,v)$ is a real function of two vectors u,v, positively homogeneous of degree one in v, where $\zeta(I)$, $\phi(I)$ are vector-valued interval functions, and $\phi(I)$ is quasi additive with respect to some mesh function δ.

5.1 A LEMMA

We shall need here and in later chapters a lemma that appears in a concrete form in Ref. 6, p. 109 (the first member of relation (5.1)). We shall state it here in an abstract form, and prove it directly.

Lemma 5.1. Let $\{\phi_i: i = 1,2,\ldots,n\}$ and $\{\phi'_j: j = 1,2,\ldots,m\}$ be two sets of vectors in E_k. Define

$$\alpha_i = \begin{cases} \phi_i/\|\phi_i\| & \text{for } \phi_i \neq 0, \\ \text{any unit vector otherwise,} & \end{cases}$$

and similarly α'_j. Let J be a mapping from $\{1,2,\ldots,n\}$ into the subsets of $\{1,2,\ldots,m\}$. Denote by $\Sigma^{(i)}$ a sum of terms over j for which $j \epsilon J(i)$, and by $\Sigma^{(i)}_{\gamma+}$ a sum of terms over j for which $j \epsilon J(i)$ and $\|\alpha_i - \alpha'_j\| \geq \gamma$.

Then

$$\frac{1}{2}\ \gamma^2 \sum_i \Sigma_{\gamma+}^{(i)} \|\phi_j'\|$$

$$\leq \sum_i \|\phi_i - \Sigma^{(i)}\phi_j'\| + \sum_i \left|\ \|\phi_i\| - \Sigma^{(i)}\|\phi_j'\|\ \right| .$$

Proof: We shall denote by $a \cdot b$ the inner product of two k-vectors a,b. For $\gamma \leq \|\alpha_i - \alpha_j'\|$, we have

$$\gamma^2 \leq \|\alpha_i\|^2 - 2\alpha_i \cdot \alpha_j' + \|\alpha_j'\|^2 = 2 - 2\alpha_i \cdot \alpha_j' ,$$

and hence

$$\frac{1}{2}\ \gamma^2 \|\phi_j'\| \leq \|\phi_j'\| - \alpha_i \cdot \phi_j' .$$

Quite generally,

$$0 \leq \|\phi_j'\| - \alpha_i \cdot \phi_j' .$$

Hence

$$\frac{1}{2}\ \gamma^2 \Sigma_{\gamma+}^{(i)} \|\phi_j'\| \leq \Sigma^{(i)}\|\phi_j'\| - \alpha_i \cdot \phi_i + \alpha_i \cdot (\phi_i - \Sigma^{(i)}\phi_j')$$

$$\leq \left|\Sigma^{(i)}\|\phi_j'\| - \|\phi_i\|\ \right| + \|\phi_i - \Sigma^{(i)}\phi_j'\| .$$

The required result follows with summation over i.

5.2 EXISTENCE OF THE INTEGRAL $\int f(\zeta,\phi)$

In the setting of Section 2.1, let ζ,ϕ be vector functions on $\{I\}$, ζ into E_m, ϕ into E_k. Consider a real function f on $K \times E_k$, $\zeta\{I\} \subset K \subset E_m$, satisfying condition (f) of Section 2.3, namely f is positively homogeneous of degree one on E_k, and bounded and uniformly continuous on $K \times U$ where $U = \{q: \|q\| = 1, q \in E_k\}$ is the unit sphere of E_k.

We shall now formulate for the set function $\zeta(I)$ a condition which extends condition (ω) of Section 2.3. For every $D_0 \in \mathcal{D}$, let $\bar{\omega}(D_0)$ be the number

$$\bar{\omega}(D_0) = \max_{I \in D_0} \limsup_{\delta(D)\to 0} \max_{\substack{J \subset I \\ J \in D}} \|\zeta(I)-\zeta(J)\| .$$

We shall assume that

(ζ)

$$\lim_{\delta(D_0)\to 0} \bar{\omega}(D_0) = 0 .$$

We consider now the set function

$$\Phi(I) = f(\zeta(I),\phi(I)), \qquad I \in \{I\} .$$

If the limit

$$\lim_{\delta(D)\to 0} \sum_{I \in D} \Phi(I) = \lim_{\delta(D)\to 0} \sum_{I \in D} f(\zeta(I),\phi(I))$$

exists, we denote this limit by $\int f(\zeta,\phi)$.

We extend now the result (xiii) of Section 2.3.

<u>Theorem 5.1</u>. If ϕ is quasi additive and of bounded variation with respect to a mesh function δ (Section 2.2), and conditions (ζ) and (f) hold, then

$$\Phi(I) = f(\zeta(I),\phi(I))$$

is quasi additive and of bounded variation with respect to δ. Thus the

BC-integral of Φ exists and is finite, that is,

$$\int\Phi = \int f(\zeta,\phi) = \lim_{\delta(D)\to 0} \sum_{I\in D} f(\zeta(I),\phi(I)) .$$

Proof: The conditions on ϕ give $\|\phi\|$ quasi additive also, so, for any $\varepsilon > 0$,

$$\sum_{I\in D_0} \|\phi(I) - \Sigma^{(I)}\phi(J)\| < \varepsilon ,$$

$$\sum_{I\in D_0} \left|\, \|\phi(I)\| - \Sigma^{(I)}\|\phi(J)\| \,\right| < \varepsilon ,$$

and

$$\Sigma' \|\phi(J)\| < \varepsilon$$

for $\delta(D_0)$ less than some $\lambda(\varepsilon)$ and $\delta(D)$ less than some $\eta(\varepsilon,D_0)$; and

$$\left| \sum_{J\in D} \|\phi(J)\| - V \right| < \varepsilon$$

for $\delta(D)$ less than some $\sigma(\varepsilon)$.

The condition (ζ) gives, for any $\varepsilon > 0$, that there exists $\Delta(\varepsilon) > 0$ such that, for every D_0 with $\delta(D_0) < \Delta(\varepsilon)$, there exists $\chi(\varepsilon,D_0) > 0$ such that, if $J\subset I$, $I\in D_0$, $J\in D$ with $\delta(D) < \chi(\varepsilon,D_0)$, then

$$\|\zeta(I) - \zeta(J)\| < \varepsilon .$$

The conditions on f give that there exist M and, for any $\varepsilon > 0$,

$\xi(\mathcal{E}) > 0$ such that

$$|f(p,q)| \leq M \quad \text{on} \quad K \times U$$

and $|f(p,q)-f(p',q')| < \mathcal{E}$ for $\|p-p'\|$ and $\|q-q'\| < \xi(\mathcal{E})$, $p,p' \in K$, $q,q' \in U$.

Take any D_o with $\delta(D_o) < \min[\Delta(\xi(\mathcal{E})), \lambda(\mathcal{E}), \lambda(\mathcal{E}\,\xi^2(\mathcal{E}))]$, and any D with $\delta(D) < \min[\sigma(\mathcal{E}), \eta(\mathcal{E}, D_o), \eta(\mathcal{E}\,\xi^2(\mathcal{E}), D_o), \chi(\xi(\mathcal{E}), D_o)]$. Denote by $\Sigma^{(I)}_{\xi+}$ the sum over $J \subset I$ for which $\|\alpha(I)-\alpha(J)\| \geq \xi$, and by $\Sigma^{(I)}_{\xi-}$ the corresponding sum for which $\|\alpha(I)-\alpha(J)\| < \xi$. Then

$$\sum_{I \in D_o} |f(\zeta(I),\phi(I)) - \Sigma^{(I)} f(\zeta(J),\phi(J))|$$

$$\leq \sum_{I \in D_o} |f(\zeta(I),\alpha(I))| \; | \, \|\phi(I)\| - \Sigma^{(I)} \|\phi(J)\| \, |$$

$$+ \sum_{I \in D_o} (\Sigma^{(I)}_{\xi(\mathcal{E})-} + \Sigma^{(I)}_{\xi(\mathcal{E})+}) |f(\zeta(I),\alpha(I)) - f(\zeta(J),\alpha(J))| \; \|\phi(J)\|$$

$$< M \sum_{I \in D_o} | \, \|\phi(I)\| - \Sigma^{(I)} \|\phi(J)\| \, | + \mathcal{E} \sum_{J \in D} \|\phi(J)\|$$

$$+ 4M\xi^{-2}(\mathcal{E}) \left[\sum_{I \in D_o} \|\phi(I) - \Sigma^{(I)} \phi(J)\| + \sum_{I \in D_o} | \, \|\phi(I)\| - \Sigma^{(I)} \|\phi(J)\| \, | \right]$$

$$< M\mathcal{E} + \mathcal{E}(V+\mathcal{E}) + 8M\mathcal{E}$$

$$= (9M+V+\mathcal{E})\mathcal{E}$$

$$\Sigma' |f(\zeta(J),\phi(J))|$$

$$= \Sigma' |f(\zeta(J),\alpha(J))| \; \|\phi(J)\|$$

$$\leq M\Sigma' \|\phi(J)\|$$

$$< M\mathcal{E}.$$

Hence both sums are less than any positive $\mathcal{E}'$ by taking

$$\mathcal{E} = \min(1, \mathcal{E}'/(9M+V+1), \mathcal{E}'/M) .$$

Since the conditions (f) on f carry over to $|f|$, $|\Phi|$ is also quasi additive. Hence Φ is quasi additive and of bounded variation.

Remark: By this theorem, the Weierstrass-type integral $\int f(\zeta,\phi)$ depending on two set functions ζ,ϕ, of which ϕ is quasi additive and of bounded variation with respect to a mesh function δ, is defined as a BC-integral. Only the axioms (ϕ), (ζ), and (f) are used, as for the integral $\int f(T,\phi)$ only the axioms (ϕ), (ω), and (f) were used in Section 2.3.

We shall show that Theorem 5.1 extends the result (xiii) of Section 2.3. There $T(w)$, $w \in A$, is a map from A into E_m. Take

$$\zeta(I) = T(\tau)$$

where τ is some point of I. Then

$$\max_{\substack{J \subset I \\ J \in D}} \|\zeta(I)-\zeta(J)\| \leq \sup_{u,v \in I} \|T(u)-T(v)\| ,$$

so that

$$\overline{\omega}(D_o) \leq \omega(D_o)$$

where $\omega(D_o)$ was defined in Section 2.3. Thus our condition (ζ) is satisfied if $\omega(D_o) \to 0$ as $\delta(D_o) \to 0$, which is certainly the case if $\omega(D_o) \leq \delta(D_o)$.

5.3 TRANSFORMATION OF THE INTEGRAL $\int f(\zeta,\phi)$

We now wish to transform $\int f(\zeta,\phi)$ to the form

$$\lim_{\delta(D)\to 0} \sum_{I\epsilon D} f(\zeta(I),\ B(I^o))$$

as in Section 2.5. For this we need a lemma that is essentially relation (6.2) of Ref. 7, p. 141.

Lemma 5.2. Under hypotheses (a'), (b), (d), and (ϕ') of Section 2.4, we have

$$\sum_{I\epsilon D_o} \|\phi(I)-B(I^o)\| \to 0 \quad \text{as} \quad \delta(D_o) \to 0 \ .$$

Proof: Our conditions give, for any $\mathcal{E} > 0$,

$$\sum_{I\epsilon D_o} \|\phi(I)-\Sigma^{(I^o)}\phi(J)\| \ < \ \mathcal{E}$$

for $\delta(D_o) < \lambda(\mathcal{E})$ and $\delta(D) < \eta(\mathcal{E},D_o)$;

$$\|\Sigma^{(I^o)}\phi(J)-B(I^o)\| \ < \ \mathcal{E}$$

for

$$\delta_{I^o}(D_{I^o}) \ < \ \Delta(\mathcal{E},I^o) \ ,$$

and

$$\delta_{I^o}(D_{I^o}) \ < \ \mathcal{E} \quad \text{for} \quad \delta(D) \ < \ \nu(\mathcal{E},I^o) \ .$$

Now take any D_o with $\delta(D_o) < \lambda(\mathcal{E}/2)$, and having n members, say; and take D with $\delta(D) < \min[\eta(\mathcal{E}/2,D_o),\nu(\Delta(\mathcal{E}/2n,I^o),I^o)$ for $I\epsilon D_o]$. Then

$$\delta_{I^o}(D_{I^o}) < \Delta(\mathcal{E}/2n,I^o) ,$$

so

$$\|\Sigma^{(I^o)}\phi(J)-B(I^o)\| < \mathcal{E}/2n .$$

Hence

$$\sum_I \|\phi(I) - B(I^o)\|$$

$$\leq \sum_I \|\phi(I)-\Sigma^{(I^o)}\phi(J)\| + \sum_I \|\Sigma^{(I^o)}\phi(J)-B(I^o)\|$$

$$< \mathcal{E}/2 + n\mathcal{E}/2n = \mathcal{E} .$$

Corollaries:

(a) $\sum_{I\epsilon D_o} B(I^o) \to B(A)$ as $\delta(D_o) \to 0$.

(b) Since $|\ \|\phi(I)\| - \|B(I^o)\|\ | \leq \|\phi(I)-B(I^o)\|$,

$$\sum_{I\epsilon D_o} |\ \|\phi(I)\| - \|B(I^o)\|\ | \to 0 \quad \text{as} \quad \delta(D_o) \to 0,$$

so $\sum_{I\epsilon D_o} \|B(I^o)\| \to V(A)$ as $\delta(D_o) \to 0$.

(c) If $\|\phi\|$ satisfies axiom (ϕ') (which is so if ϕ satisfies axiom (ϕ') and $V(A) < \infty$), then

$$\sum_{I\epsilon D_o} |\ \|\phi(I)\| - V(I^o)| \to 0 \quad \text{as} \quad \delta(D_o) \to 0,$$

so $\sum_{I \in D_o} V(I^o) \to V(A)$ as $\delta(D_o) \to 0$.

We are now in a position to extend the result (xix) of Section 2.5.

Theorem 5.2. Under hypotheses (a'), (b), (d), (ϕ'), $V(A) < \infty$, (ζ), (f), we have

$$\sum_{I \in D_o} f(\zeta(I), B(I^o)) \to \int f(\zeta, \phi) \quad \text{as} \quad \delta(D_o) \to 0 .$$

Proof: We have, from Lemma 5.2,

$$\sum_{I \in D_o} \|\phi(I) - B(I^o)\| < \mathcal{E}'$$

for $\delta(D_o)$ less than some $\rho(\mathcal{E}')$. Let $\beta(I) = B(I^o)/\|B(I^o)\|$. Otherwise we use the notation of Theorem 5.1.

For any $\mathcal{E} > 0$, let $\mathcal{E}' = \min(\mathcal{E}, \mathcal{E}\xi^2(\mathcal{E}))$, and take any D_o with $\delta(D_o) < \min(\rho(\mathcal{E}'), \sigma(\mathcal{E}))$. Then

$$\begin{aligned} \Sigma\, |f(\zeta(I),\phi(I)) - f(\zeta(I),B(I^o))| & \\ \leq \Sigma\, |f(\zeta(I),\beta(I))|\ \big|\, \|\phi(I)\| - \|B(I^o)\|\, \big| & \\ + \Sigma\, |f(\zeta(I),\alpha(I)) - f(\zeta(I),\beta(I))|\ \|\phi(I)\| & \\ < M\Sigma\, \big|\, \|\phi(I)\| - \|B(I^o)\|\, \big| + \mathcal{E}\Sigma\|\phi(I)\| & \\ + 4M\xi^{-2}(\mathcal{E})\Sigma(\|\phi(I)-B(I^o)\| + \big|\, \|\phi(I)\| - \|B(I^o)\|\, \big|) & \\ < M\mathcal{E} + \mathcal{E}(V+\mathcal{E}) + 8M\mathcal{E} & \\ = \mathcal{E}(V+9M+\mathcal{E}) . & \end{aligned}$$

Hence $\Sigma f(\zeta(I),B(I^o))$ converges to the same limit as $\Sigma f(\zeta(I),\phi(I))$.

5.4 THE INTEGRAL $\int f(\zeta,\phi)$ AS A LEBESGUE-STIELTJES INTEGRAL

We now express, as in Section 2.5, $\int f(\zeta,\phi)$ as $(A)\int f(Z(w),\Theta(w))d\mu$, where Z is an appropriate limiting point function of ζ. For this we need a lemma, which is the result (5.iii) of Ref. 7.

For a given D, define

$$\eta_D(w) = \begin{cases} \nu(I^o)/\mu(I^o) \text{ for } w\in I^o,\ I\in D,\ \mu(I^o) \neq 0, \\ 0 \text{ otherwise} . \end{cases}$$

Lemma 5.3. Under hypotheses (a'), (b), (c), (d), (H_1), (H_2), (H_3), (H_4), (ϕ') and $V(A) < \infty$,

$$(A) \int \|\Theta(w)-\eta_D(w)\|^2 \, d\mu \to 0 \quad \text{as} \quad \delta(D) \to 0.$$

Proof:

$$\begin{aligned} &(A) \int \|\Theta - \eta_D\|^2 \, d\mu \\ &\quad = (A) \int d\mu - 2(A) \int \Theta\cdot\eta_D \, d\mu + (A) \int \|\eta_D\|^2 \, d\mu \\ &\quad = \mu(A) - \Sigma \, \|\nu(I^o)\|^2/\mu(I^o) \\ &\quad = \mu(A) - \Sigma \, \|\nu(I^o\| + \Sigma \, \|\nu(I^o)\|(\mu(I^o)-\|\nu(I^o)\|)/\mu(I^o) \\ &\quad \leq 2[\mu(A) - \Sigma \, \|\nu(I^o)\|] \\ &\quad = 2[V(A) - \Sigma \, \|B(I^o)\|] \\ &\quad \to 0 \text{ as } \delta(D) \to 0 \text{ by Corollary (b) of Lemma 5.2.} \end{aligned}$$

We are now in a position to extend the main result (xx) of Section 2.5, that is, to prove that the integral $\int f(\zeta,\phi)$—which was defined in Section 5.2 as a BC-integral—admits a representation as a Lebesgue-Stieltjes integral.

We suppose that the interval function ζ converges to a mapping Z from A to E_m in the following sense. (Z) For μ-almost every w in A and any $\mathcal{E} > 0$, there exists $\gamma(\mathcal{E}, w) > 0$ such that if $w \epsilon I^o$, $I \epsilon D$ with $\delta(D) < \gamma(\mathcal{E}, w)$, then

$$\|\zeta(I) - Z(w)\| < \mathcal{E} .$$

We also assume that K is closed.

Theorem 5.3. Under hypotheses (a'), (b), (c), (d), (H_1), (H_2), (H_3), (H_4), (ϕ'), $V(A) < \infty$, (ζ), (Z), (f), the function $f(Z(w), \Theta(w))$ is μ-integrable on A, and

$$\int f(\zeta, \phi) = (A) \int f(Z(w), \Theta(w)) d\mu .$$

Proof: We have $(A)\int \|\Theta - \eta_D\|^2 d\mu \to 0$ as $\delta(D) \to 0$. Take a sequence $\{D_n\}$ with $\delta(D_n) < 1/n$. Then there exists a subsequence $\{D_{n_m}\}$ with corresponding $\eta_m \to \Theta$ μ-almost everywhere in A (see, for example, Ref. 18, pp. 226, 229). Thus there exists a set $A^- \subset A$ with $\mu(A^-) = \mu(A)$, $\|\Theta\| = 1$, and $\eta_m \to \Theta$ on A^-.

For $w \epsilon A^-$, $1/2 < \|\eta_m(w)\|$ for $m >$ some $N(w)$, so $w \epsilon I^o$ for some $I \epsilon D_{n_m}$. Define

$$Z_m(w) = \begin{cases} \zeta(I) \text{ for } w \epsilon I^o,\ I \epsilon D_{n_m} , \\ \text{an arbitrary point in K otherwise} . \end{cases}$$

Then, for $m > N(w)$, $\|Z_m(w) - Z(w)\| = \|\zeta(I) - Z(w)\| < \mathcal{E}$ for $1/n_m < \gamma(\mathcal{E}, w)$, so $Z_m(w) \to Z(w)$. Since K is closed, $Z(w) \epsilon K$.

Although f is assumed uniformly continuous and bounded on K x U only, its positive homogeneity gives f uniformly continuous and bounded (M') on K x $\{q: a \leq \|q\| \leq b\}$ for any a,b satisfying $0 < a < b < \infty$. In particular, f is continuous at each point of K x U, so, on A^-,

$$f(Z_m(w),\eta_m(w)) \to f(Z(w),\Theta(w)) .$$

Now Z_m, η_m are obviously μ-measurable (coordinate-wise) on A^-, so $f(Z_m,\eta_m)$ is μ-measurable, so $f(Z,\Theta)$ is μ-measurable. Indeed, f is bounded and $\mu(A) < \infty$, so $f(Z,\Theta)$ is μ-integrable on A.

Also $(A)\int f(Z_m(w),\eta_m(w))d\mu \to (A)\int f(Z(w),\Theta(w))d\mu$. But

$$(A)\int f(Z_m(w),\eta_m(w))d\mu$$
$$= (A-\bigcup I^o)\int f(Z_m(w),\eta_m(w))d\mu + \sum f(\zeta(I),\nu(I^o)) .$$

Now

$$|(A-\bigcup I^o)\int f(Z_m(w),\eta_m(w))d\mu|$$
$$\leq M'(\mu(A) - \sum\mu(I^o))$$
$$= M'(V(A) - \sum V(I^o))$$
$$\to 0 \text{ as } m \to \infty$$ by Corollary (c) of Lemma 5.2.

Also $\nu(I^o) = B(I^o)$, and, by Theorem 5.2,

$$\sum f(\zeta(I), B(I^o)) \to \int f(\zeta,\phi) .$$

Hence $(A)\int f(Z(w),\Theta(w))d\mu = \int f(\zeta,\phi)$.

Remark: The relation

$$\int f(\zeta,\phi) \quad = \quad (A)\int f(Z(w),\Theta(w))d\mu$$

applies to "non-parametric" integrals of the form

$$(A)\ \int g(Z(w))d\mu \ .$$

For this, we put

$$f(p,q) \quad = \quad g(p)\|q\| \ ,$$

so that $f(Z(w),\Theta(w)) = g(Z(w))$. Then

$$(A)\ \int g(Z(w))d\mu \quad = \quad \lim_{\delta(D)\to 0} \sum_{I\in D} g(\zeta(I))\|\phi(I)\| \quad = \quad \int g(\zeta)\|\phi\|$$

provided $g(p)$ is uniformly continuous and bounded on K. Of course, the conditions on ζ and ϕ must still be satisfied.

5.5 THE CONDITIONS (ζ) AND (Z)

We wish to examine the relation between the condition (ζ) used in Theorem 5.1 and the condition (Z) used in Theorem 5.3. We shall show that a slight strengthening of each implies the other. In order to do this, we need the following lemma.

Lemma 5.4. Let μ be a measure induced as in Section 2.4 by an interval function ϕ under hypotheses (a'), (b), (c), (d), (H_1), (H_2), (H_3), (H_4), (ϕ') and $V(A) < \infty$. Then for μ-almost every w in A and any $\varepsilon > 0$, there exists D with $\delta(D) < \varepsilon$ and $w\in I^o$ for some $I\in D$.

Proof: By Corollary (c) of Lemma 5.2, $\sum_{I \in D} \mu(I^o) \to \mu(A)$ as $\delta(D) \to 0$, so for any $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$ such that

$$\mu(A) - \mu(\bigcup I^o) < \varepsilon$$

for $\delta(D) < \rho(\varepsilon)$. Take $\rho(\varepsilon) \leq \varepsilon$. For each positive integer n, take D_n with $\delta(D_n) < \rho(1/n)$. Let $A_n = \bigcup I^o$: $I \in D_n$. Then $\mu(A) - \mu(A_n) < 1/n$. Take $B = \limsup A_n = \bigcap_m \bigcup_{n>m} A_n$. We have $\mu(A) < \infty$, so $\mu(B) \geq \limsup \mu(A_n) = \mu(A)$, so $\mu(B) = \mu(A)$. For $w \in B$ and any $\varepsilon > 0$, $w \in$ some A_n with $n > 1/\varepsilon$, so $w \in I^o$ for some $I \in D$ with $\delta(D) < 1/n < \varepsilon$.

Now consider the following strengthening of condition (ζ):

(ζ') For any $\varepsilon > 0$, there exists $\Delta(\varepsilon) > 0$ such that, for every D_o with $\delta(D_o) < \Delta(\varepsilon)$, there exists $\chi(\varepsilon, D_o) > 0$ such that, if $J^o \cap I^o \neq \emptyset$, $I \in D_o$, $J \in D$ with $\delta(D) < \chi(\varepsilon, D_o)$, then

$$\|\zeta(I) - \zeta(J)\| < \varepsilon .$$

Theorem 5.4. If ζ satisfies condition (ζ'), then there exists a mapping Z from A to E_m satisfying condition (Z).

Proof: Consider any w in A to which the result of Lemma 5.4 applies. Then, for any $\varepsilon > 0$, there exists I, $w \in I^o$, $I \in D_o$ with $\delta(D_o) < \Delta(\varepsilon)$. By condition ($\zeta'$), if $w \in J_1^o$, $J_1 \in D_1$ with $\delta(D_1) < \chi(\varepsilon, D_o)$, and $w \in J_2^o$, $J_2 \in D_2$ with $\delta(D_2) < \chi(\varepsilon, D_o)$, then $\|\zeta(J_1) - \zeta(J_2)\| < 2\varepsilon$. Hence

$$Z(w) \equiv \lim_{\delta(D) \to 0} [\zeta(J): w \in J^o \in D]$$

exists. Again from condition (ζ'), if $w \in I^o$, $I \in D_o$ with $\delta(D_o) < \Delta(\varepsilon)$,

then

$$\|\zeta(I) - Z(w)\| \leq \mathcal{E} .$$

Now consider the following strengthening of condition (Z):

(Z') For any $\mathcal{E} > 0$, there exists $\gamma(\mathcal{E}) > 0$ such that if $w \in I^o$, $I \in D$ with $\delta(D) < \gamma(\mathcal{E})$, then

$$\|\zeta(I) - Z(w)\| < \mathcal{E} .$$

Theorem 5.5. If there exists $Z(w)$, $w \in A$, satisfying condition (Z'), then $\zeta(I)$ satisfies condition (ζ').

Proof: Consider any D_o and D with $\delta(D_o)$ and $\delta(D) < \gamma(\mathcal{E}/2)$. For $I \in D_o$, $J \in D$ with $J^o \cap I^o \neq \emptyset$, take $w \in J^o \cap I^o$. Then

$$\begin{aligned} \|\zeta(I)-\zeta(J)\| &\leq \|\zeta(I)-Z(w)\| + \|Z(w)-\zeta(J)\| \\ &< \mathcal{E} . \end{aligned}$$

6. INVARIANCE PROPERTIES OF INTEGRALS $\int f(\zeta,\phi)$

In this chapter, we study relations between interval functions ζ,ϕ and ζ',ϕ' which ensure that the corresponding integrals $\int f(\zeta,\phi)$, $\int f(\zeta',\phi')$ have the same value. The present discussion was suggested by Cesari's treatment of the invariance of surface integrals under Fréchet equivalence in Ref. 3. Integrals over rectifiable curves present an analagous invariance under Fréchet equivalence.

We consider a system $A, \mathcal{U}, \{I\}, \mathcal{D}, \delta$ as in Section 2.1, with ζ a vector function from $\{I\}$ to $K \subset E_m$ and ϕ a vector function from $\{I\}$ to E_k; and then a second system $A', \mathcal{U}', \{I'\}, \mathcal{D}', \delta'$, with ζ' from $\{I'\}$ to K and ϕ' from $\{I'\}$ to E_k.

Let $f(p,q)$ be a real function on $K \times E_k$, satisfying the conditions (f) of Section 2.3. Sufficient conditions on ζ,ϕ and ζ',ϕ' for the existence of the integrals $\int f(\zeta,\phi)$, $\int f(\zeta',\phi')$ were given in Section 5.2.

6.1 RELATIONS R BETWEEN INTERVAL FUNCTIONS

We consider three relations (R_1), (R_2), (R_3) of ζ',ϕ' to ζ,ϕ with increasing strength.

(R_1) For any $\mathcal{E} > 0$, there exists a homeomorphism h from A to A' and systems $D \in \mathcal{D}$, $D' \in \mathcal{D}'$ with $\delta(D) < \mathcal{E}$, $\delta'(D') < \mathcal{E}$, satisfying conditions (α) and (β) below.

(R_2) For any $\mathcal{E} > 0$, there exists a homeomorphism h from A to A' and a number $\lambda(\mathcal{E}) > 0$ such that, for every $D' \in \mathcal{D}'$ with $\delta'(D') < \lambda(\mathcal{E})$, there exists $D \in \mathcal{D}$ with $\delta(D) < \mathcal{E}$, satisfying conditions (α) and (β) below.

(R_3) For any $\mathcal{E} > 0$, there exists a homeomorphism h from A to A' and a number $\lambda(\mathcal{E}) > 0$ such that, for every $D' \in \mathcal{D}'$ with $\delta'(D') < \lambda(\mathcal{E})$, there exists a number $\eta(\mathcal{E}, D') > 0$ such that every $D \in \mathcal{D}$ with $\delta(D) < \eta(\mathcal{E}, D')$ satisfies conditions (α) and (β) below.

Condition (α): $\|\zeta'(I') - \zeta(I)\| < \mathcal{E}$ for $hI \subset I'$, $I \in D$, $I' \in D'$.

Condition (β):

$$\text{(i)} \quad \sum_{I' \in D'} \|\phi'(I') - \Sigma^{[I']}\phi(I)\| < \mathcal{E},$$

$$\text{(ii)} \quad \sum_{I' \in D'} \left| \|\phi'(I')\| - \Sigma^{[I']}\|\phi(I)\| \right| < \mathcal{E},$$

and

$$\text{(iii)} \quad \bar{\Sigma} \|\phi(I)\| < \mathcal{E},$$

where $\Sigma^{[I']}$ denotes a sum over all $I \in D$ such that $hI \subset I'$, and $\bar{\Sigma}$ denotes a sum over all $I \in D$ such that hI is contained in no $I' \in D'$.

We shall say that ζ', ϕ' are related to ζ, ϕ in the sense (R_i), or ζ', ϕ' are (R_i)-related to ζ, ϕ, or $(\zeta', \phi')R_i(\zeta, \phi)$, i = 1, or 2, or 3.

For point functions $T(w)$, $T'(w')$ respectively from A,A' to K, inducing interval functions

$$\zeta(I) = T(\tau) \quad \text{for some} \quad \tau \in I,$$

$$\zeta'(I') = T'(\tau') \quad \text{for some} \quad \tau' \in I',$$

condition (α) becomes

$$\|T'(\tau')-T(\tau)\| < \mathcal{E} \quad \text{for} \quad hI \subset I',\ I \epsilon D,\ I' \epsilon D' .$$

This condition is closely related to the standard condition in Fréchet equivalence:

$$(\alpha') \qquad \sup_{w \epsilon A} \|T'(hw)-T(w)\| < \mathcal{E} .$$

<u>Lemma 6.1</u>. Assume that $\omega'(D') \to 0$ as $\delta'(D') \to 0$ (see Sections 2.3, 5.2); that is, for any $\mathcal{E}' > 0$, $\omega'(D') < \mathcal{E}'$ for $\delta'(D')$ less than some $\rho(\mathcal{E}')$. Then, for D' with $\delta'(D') < \rho(\mathcal{E}/2)$, condition (α') relative to $\mathcal{E}/2$ implies condition (α).

<u>Proof</u>: For $hI \subset I'$,

$$\begin{aligned} \|\zeta'(I')-\zeta(I)\| &= \|T'(\tau')-T(\tau)\| \\ &\leq \|T'(\tau')-T'(h\tau)\| + \|T'(h\tau)-T(\tau)\| \\ &< \omega'(I') + \mathcal{E}/2 \\ &< \mathcal{E}. \end{aligned}$$

6.2 INVARIANCE OF INTEGRALS $\int f(\zeta,\phi)$ UNDER RELATIONS R

<u>Theorem 6.1</u>. Let f(p,q) be a real function on K x E_k satisfying condition (f). Let ζ,ϕ, ζ',ϕ' be vector interval functions such that the integrals $\int f(\zeta,\phi)$, $\int f(\zeta',\phi')$ exist, and

$$\overline{V} = \limsup_{\delta(D)\to 0} \sum_{I \epsilon D} \|\phi(I)\| < \infty .$$

Then, if ζ',ϕ' are related to ζ,ϕ in the sense (R_1),

$$\int f(\zeta',\phi') = \int f(\zeta,\phi) .$$

Proof: The conditions on f give that there exist M and, for any $\mathcal{E}' > 0$, $\xi(\mathcal{E}') > 0$ such that

$$|f(p,q)| \leq M \text{ on } K \times U$$

and

$$|f(p,q)-f(p',q')| < \mathcal{E}'$$

for $\|p-p'\|$ and $\|q-q'\| < \xi(\mathcal{E}')$, $p,p' \in K$, $q,q' \in U$.

Further,

$$|\Sigma f(\zeta(I),\phi(I)) - \int f(\zeta,\phi)| < \mathcal{E}'$$

for $\delta(D)$ less than some $\rho(\mathcal{E}')$;

$$|\Sigma f(\zeta'(I'),\phi'(I')) - \int f(\zeta',\phi')| < \mathcal{E}'$$

for $\delta'(D')$ less than some $\rho'(\mathcal{E}')$; and

$$\Sigma \|\phi(I)\| < \bar{V} + \mathcal{E}'$$

for $\delta(D)$ less than some $\sigma(\mathcal{E}')$.

In the relation (R_1) in Section 6.1, take

$$\mathcal{E} = \min(\mathcal{E}',\rho(\mathcal{E}'),\rho'(\mathcal{E}'),\sigma(\mathcal{E}'),\xi(\mathcal{E}'),\mathcal{E}'\xi^2(\mathcal{E}'))$$

to get h, D, D' with $\delta(D) < \rho(\mathcal{E}')$ and $\sigma(\mathcal{E}')$; $\delta'(D') < \rho'(\mathcal{E}')$; and conditions ($\alpha$), ($\beta$) satisfied; in particular, $\|\zeta'(I')-\zeta(I)\| < \mathcal{E} \leq \xi(\mathcal{E}')$ for $hI \subset I'$.

Then

$$\left|\sum_{I'\in D'} f(\zeta'(I'),\phi'(I')) - \sum_{I\in D} f(\zeta(I),\phi(I))\right|$$

$$\leq \sum_{I'\in D'} |f(\zeta'(I'),\phi'(I'))-\Sigma^{[I']}f(\zeta(I),\phi(I))|$$

$$+ \bar{\Sigma}\ |f(\zeta(I),\phi(I))|$$

$$= \sum_{I'\in D'} \Big| f(\zeta'(I'),\alpha'(I'))\ \big[\|\phi'(I')\|-\Sigma^{[I']}\|\phi(I)\|\big]$$

$$+ \Sigma^{[I']}\ \big[f(\zeta'(I'),\alpha'(I'))-f(\zeta(I),\alpha(I))\big]\ \|\phi(I)\|\ \Big|$$

$$+ \bar{\Sigma}\ |f(\zeta(I),\alpha(I))|\ \|\phi(I)\|$$

$$< M \sum_{I'\in D'} \big|\ \|\phi'(I')\|-\Sigma^{[I']}\|\phi(I)\|\ \big| + \mathcal{E}'\Sigma\|\phi(I)\|$$

$$+ 4M\xi^{-2}(\mathcal{E}') \sum_{I'\in D'} (\|\phi'(I')-\Sigma^{[I']}\phi(I)\|$$

$$+ \big|\ \|\phi'(I')\|-\Sigma^{[I']}\|\phi(I)\|\ \big|) + M\bar{\Sigma}\ \|\phi(I)\|$$

$$< M\mathcal{E} + \mathcal{E}'(\bar{V}+\mathcal{E}') + 4M\xi^{-2}(\mathcal{E}')\cdot 2\mathcal{E} + M\mathcal{E}$$

$$\leq (10M+\bar{V}+\mathcal{E}')\,\mathcal{E}' .$$

Hence

$$|\int f(\zeta',\phi')-\int f(\zeta,\phi)| \quad < \quad (10M+\bar{V}+2+\mathcal{E}')\,\mathcal{E}'$$

for any $\mathcal{E}' > 0$, so $\int f(\zeta',\phi') = \int f(\zeta,\phi)$.

Note that we have assumed that $\int f(\zeta,\phi)$ and $\int f(\zeta',\phi')$ are finite. This is certainly so for the first, by the conditions of f and $\overline{V}$. The possibilities $\int f(\zeta',\phi') = \pm \infty$ are easily eliminated by examination of our argument above.

In Theorem 6.1, we have assumed that both integrals $\int f(\zeta,\phi)$ and $\int f(\zeta',\phi')$ exist. If we use the stronger relation (R_2), then the existence of $\int f(\zeta',\phi')$ follows from the existence of $\int f(\zeta,\phi)$.

Therem 6.2. If $\int f(\zeta,\phi)$ exists, $\overline{V} < \infty$, and ζ',ϕ' are related to ζ,ϕ in the sense (R_2), then $\int f(\zeta',\phi')$ exists and equals $\int f(\zeta,\phi)$.

Proof: For any $\mathcal{E}' > 0$, take

$$\mathcal{E} = \min(\mathcal{E}',\rho(\mathcal{E}'),\ \sigma(\mathcal{E}'),\ \xi(\mathcal{E}'),\ \mathcal{E}'\xi^2(\mathcal{E}'))$$

in relation (R_2) in Section 6.1. This gives h and $\lambda(\mathcal{E})$, such that, for every $D'\epsilon\mathcal{D}'$ with $\delta'(D') > \lambda'(\mathcal{E}') \equiv \lambda(\mathcal{E})$, there exists D with $\delta(D) < \mathcal{E} \leq \rho(\mathcal{E}'),\sigma(\mathcal{E}')$, satisfying conditions (α) and (β); in particular $\|\zeta'(I') - \zeta(I)\| < \mathcal{E} \leq \xi(\mathcal{E}')$ for $hI \subset I'$.

Then, as before,

$$\left| \sum_{I'\epsilon D'} f(\zeta'(I'),\phi'(I')) - \sum_{I\epsilon D} f(\zeta(I),\phi(I)) \right| < (10M+\overline{V}+\mathcal{E}')\mathcal{E}'$$

so

$$\left| \sum_{I'\epsilon D'} f(\zeta'(I'),\ \phi'(I)) - \int f(\zeta,\phi) \right| < (10M+\overline{V}+1+\mathcal{E}')\mathcal{E}'$$

For any $\bar{\mathcal{E}} > 0$, take $\mathcal{E}' = \min(1, \bar{\mathcal{E}}/(10M+\bar{V}+2))$. Then, for $\delta'(D') < \bar{\lambda}(\bar{\mathcal{E}}) \equiv \lambda'(\mathcal{E}')$,

$$\left| \sum_{I'\epsilon D'} f(\zeta'(I'), \phi'(I')) - \int f(\zeta,\phi) \right| < \bar{\mathcal{E}}$$

Hence

$$\sum_{I'\epsilon D'} f(\zeta'(I'),\phi'(I')) \to \int f(\zeta\ \phi) \text{ as } \delta'(D') \to 0.$$

As particular cases, we consider in turn

$$f(p,q) \quad = \quad q_r, \ |q_r|, \ \|q\|$$

The conditions on f are satisfied. Then, under the other conditions, B, V_r, and V are invariant. However, some of the conditions assumed in the general theorems are superfluous here. We shall prove directly the invariance of B and V under these relaxed conditions.

Theorem 6.3. If B,B' exist and ϕ' is related to ϕ in the sense (R_1) restricted to conditions (β)(i) and (β)(iii) only, then B' = B.

If V,V' exist and ϕ' is related to ϕ in the sense (R_1) restricted to conditions (β)(ii) and (iii) only, then V' = V .

If B exists and ϕ' is related to ϕ in the sense (R_2) restricted to conditions (β)(i) and (iii) only, then B' exists and equals B.

If V exists and ϕ' is related to ϕ in the sense (R_2) restricted to conditions (β)(ii) and (iii) only, then V' exists and equals V.

<u>Proof</u>: We shall prove only the first result. The method of proof for the three other results will then be fairly obvious.

For any $\mathcal{E}' > 0$, $\|\Sigma\phi(I) - B\| < \mathcal{E}'$ for $\delta(D)$ less than some $\rho(\mathcal{E}')$, and $\|\Sigma\phi'(I') - B'\| < \mathcal{E}'$ for $\delta'(D')$ less than some $\rho'(\mathcal{E}')$. Take $\mathcal{E} = \min(\mathcal{E}', \rho(\mathcal{E}'), \rho'(\mathcal{E}'))$ in the restricted relation (R_1), to get h, D, D' with $\delta(D) < \rho(\mathcal{E}')$, $\delta'(D') < \rho'(\mathcal{E}')$, and conditions (ρ)(i) and (iii) satisfied.

Then

$$\begin{aligned} &\|B' - B\| \\ &\leq \|B' - \Sigma\phi'(I')\| + \sum_{I' \in D'} \|\phi'(I') - \Sigma^{[I']}\phi(I)\| \\ &\quad + \bar{\Sigma}\,\|\phi(I)\| + \|\Sigma\phi(I) - B\| \\ &< \mathcal{E}' + \mathcal{E} + \mathcal{E} + \mathcal{E}' \\ &\leq 4\mathcal{E}'. \end{aligned}$$

Hence

$$B' = B.$$

6.3. SUBSTITUTION OF THE INVARIANCE OF V IN THE RELATIONS

The following result is of some importance, in that the invariance of V can often be proved independently, for example in Ref. 5, p. 457 by semicontinuity.

<u>Theorem 6.4</u>. Relations (R_1), (R_2), (R_3) are equivalent respectively to the same relations with (β)(ii) and (iii) replaced by $V' = V$.

<u>Proof</u>: The forward implication has already been considered.

For the reverse, we shall make use of the relations $|m| = 2m^{+}-m$, $[\|A\|-\Sigma\|B\|]^{+} \leq \|A-\Sigma B\|$. We have, for any $\mathcal{E}' > 0$, $|\Sigma\|\phi(I)\|-V| < \mathcal{E}'$ for $\delta(D)$ less than some $\sigma(\mathcal{E}')$, and $|\Sigma\|\phi'(I')\|-V'| < \mathcal{E}'$ for $\delta'(D')$ less than some $\sigma'(\mathcal{E}')$.

In the case of (R_1), for any $\mathcal{E}' > 0$, take

$$\mathcal{E} = \min(\mathcal{E}'/4, \sigma(\mathcal{E}'/4), \sigma'(\mathcal{E}'/4))$$

in the adjusted conditions to get h, D, D' with $\delta(D) < \mathcal{E} \leq \mathcal{E}'$ and $\sigma(\mathcal{E}'/4)$; $\delta'(D') < \mathcal{E} \leq \mathcal{E}'$ and $\sigma'(\mathcal{E}'/4)$;

$$\sum_{I'\in D'} \|\phi'(I') - \Sigma^{[I']}\phi(I)\| < \mathcal{E} < \mathcal{E}';$$

and $\|\zeta'(I') - \zeta(I)\| < \mathcal{E} < \mathcal{E}'$ for $hI \subset I'$.

Now

$$\begin{aligned}\sum_{I'\in D'} \big|\|\phi'(I')\| - \Sigma^{[I']}\|\phi(I)\|\big| + \bar{\Sigma}\|\phi(I)\| \\ &= 2\Sigma[\|\phi'(I')\| - \Sigma^{[I']}\|\phi(I)\|]^{+} - \Sigma(\|\phi'(I')\| - \Sigma^{[I']}\|\phi(I)\|) \\ &\quad + \bar{\Sigma}\|\phi(I)\| \\ &\leq 2\Sigma\|\phi'(I')-\Sigma^{[I']}\phi(I)\| + V' - \Sigma\|\phi'(I')\| + \Sigma\|\phi(I)\| - V \\ &< 2\mathcal{E}'/2 + \mathcal{E}'/4 + \mathcal{E}'/4 \\ &= \mathcal{E}'.\end{aligned}$$

Hence (β)(ii) and (iii).

The (R_2) and (R_3) cases can be treated by similar techniques.

6.4. PROPERTIES OF THE RELATIONS R

We shall discuss in this section some properties of the relations R.

First, we consider the reflexive property; that is, we consider whether the relations (R_i) have the property $(\zeta,\phi)R_i(\zeta,\phi)$ for i = 1, or 2, or 3. For $(\zeta',\phi') = (\zeta,\phi)$ and h the identity homeomorphism, the relation (R_3) reduces to condition (ζ) for ζ and quasi additivity of ϕ and $\|\phi\|$ with respect to δ. Hence, if ϕ and $\|\phi\|$ are quasi additive with respect to δ (or, equivalently, ϕ is quasi additive and of bounded variation) and ζ satisfies condition (ζ), then $(\zeta,\phi)R_3(\zeta,\phi)$, and so also $(\zeta,\phi)R_2(\zeta,\phi)$ and $(\zeta,\phi)R_1(\zeta,\phi)$.

The relations (R_i) could be made symmetrical by adding the respective transposed relations.

The relations (R_i) themselves in general do not appear to be transitive. However, in the case in which

$$\zeta(I) = T(\tau), \qquad \tau\epsilon I$$

for a point function T in the form discussed in Section 6.1, we can show that (R_2) and (R_3) are transitive.

<u>Theorem 6.5</u>. Suppose that T',ϕ' are (R_2)-related to T,ϕ, and that T'',ϕ'' are (R_2)-related to T',ϕ'. Then T'',ϕ'' are (R_2)-related to T,ϕ.

<u>Proof</u>: For any $\mathcal{E} > 0$, we get h and $\lambda(\mathcal{E})$; and for any $\mathcal{E}' > 0$, we get h' and $\lambda'(\mathcal{E}')$. Now for any $\bar{\mathcal{E}} > 0$, take $\mathcal{E} = \bar{\mathcal{E}}/5$, $\mathcal{E}' = \min(\bar{\mathcal{E}}/5, \lambda(\mathcal{E}))$. Then

$$\sup_{w\in A} \|T''(h'hw) - T(w)\| < \mathcal{E}' + \mathcal{E} < \overline{\mathcal{E}}.$$

For any D" with $\delta''(D'') < \overline{\lambda}(\overline{\mathcal{E}}) \equiv \lambda'(\mathcal{E}')$, there exists D' with $\delta'(D') < \mathcal{E}' \leq \lambda(\mathcal{E})$ and satisfying the condition (β) on ϕ', ϕ''. Hence there exists D with $\delta(D) < \mathcal{E} < \overline{\mathcal{E}}$ and satisfying the condition (β) on ϕ, ϕ'. Then, denoting by $\Sigma^{[[I'']]}$ a sum over all I with $h'hI \subset I''$, we have

$$\sum_{I''} \|\phi''(I'') - \Sigma^{[[I'']]}\phi(I)\|$$

$$= \sum_{I''} \|\phi''(I'') - \Sigma^{[I'']}\phi'(I') + \Sigma^{[I'']}(\phi'(I') - \Sigma^{[I'']}\phi(I))$$

$$- \left\{\Sigma^{[[I'']]} - \Sigma^{[I'']}\,\Sigma^{[I']}\right\}\phi(I)\|$$

$$\leq \sum_{I''}\|\phi''(I'') - \Sigma^{[I'']}\phi'(I')\| + \sum_{I'} \|\phi'(I') - \Sigma^{[I']}\phi(I)\|$$

$$+ \overline{\Sigma}\, \|\phi(I)\| + \sum_{I''} \{\Sigma\|\phi(I)\| \;:\; h'hI \subset I'', hI \subset I', h'I' \not\subset I''\}$$

$$\leq \sum_{I''} \|\phi''(I'') - \Sigma^{[I'']}\phi'(I')\| + \sum_{I'} \|\phi'(I') - \Sigma^{[I']}\phi(I)\|$$

$$+ \overline{\Sigma}\, \|\phi(I)\| + \sum_{I'} \left|\Sigma^{[I']}\|\phi(I)\| - \|\phi'(I')\|\right| + \overline{\Sigma}\, \|\phi'(I')\|$$

$$< \mathcal{E}' + \mathcal{E} + \mathcal{E} + \mathcal{E} + \mathcal{E}'$$

$$\leq \overline{\mathcal{E}}.$$

Also $V' = V$, $V'' = V'$, so $V'' = V$. We can thus use Theorem 6.4 (although it can be done directly) to conclude that T'', ϕ'' are (R_2)-

related to T,ϕ.

This proof can easily be adapted to prove transitivity in the (R_3) case.

6.5. PARAMETRIC CURVE INTEGRALS

As an application of the theory in this chapter, we shall prove the invariance of integrals over Fréchet equivalent parametric curves.

Let T, T' be continuous mappings of bounded variation into E_m from closed intervals $A = [a,b]$, $A' = [a',b']$ respectively, such that

(F) for any $\mathcal{E}' > 0$, there exists a sense-preserving homeomorphism h from A to A' such that

$$\sup_{w \in A} \|T'(hw) - T(w)\| < \mathcal{E}'.$$

Here (Ref. 6, p. 105), $\{I\}$ is the class of all closed subintervals of A, $\mathcal{D}$ is the class of all finite subdivisions $D = [I_i : i = 1,\ldots N]$, $I_i = [a_{i-1},a_i]$, $a = a_0 < a_1 <\ldots< a_N = b$, and $\delta(D) = \max(a_i - a_{i-1})$. We take

$$\phi(I) = T(v) - T(u)$$

where $I = [u,v]$. A similar system is taken for T'.

We shall show that T',ϕ' are related to T,ϕ in the sense (R_3) restricted to conditions (α') and $(\beta)(i)$. Then, if we assume the invariance of the curve length $L = V(\phi)$, the invariance of the curve integral $\int f(T,\phi)$ follows from Theorems 6.4 and 6.1.

Theorem 6.6. If the mappings T, T' are related in the sense (F), then T',ϕ' are related to T,ϕ in the sense (R_3) restricted to conditions (α') and (β)(i).

Proof: For any $\mathcal{E} > 0$, consider any subdivision $D' = [I'_j : j = 1,\ldots M]$ of A', $I'_j = [a'_{j-1}, a'_j]$, $a'_0 = a'_1 < \ldots < a'_M = b'$.

The condition (F) gives a sense-preserving homeomorphism h such that

$$\sup_{w \in A} \|T'(hw) - T(w)\| < \mathcal{E}/4M$$

Condition (R_3)(α') is obviously satisfied.

The mapping $h : A \to A'$ is increasing with $h(a) = a'$, $h(b) = b'$. It is continuous and so uniformly continuous on A, that is, for any $\mathcal{E}' > 0$, $|\Delta h| < \mathcal{E}'$ for $|\Delta w|$ less than some $\lambda(\mathcal{E}')$. Uniform continuity of T on A also gives $\|\Delta T\| < \mathcal{E}'$ for $|\Delta w|$ less than some $\gamma(\mathcal{E}')$.

Take any subdivision $D = [I_i : i = 1,\ldots N]$ of A, $I_i = [a_{i-1}, a_i]$, $a = a_0 < a_1 < \ldots < a_N = b$, with

$$\delta(D) < \min[\gamma(\mathcal{E}/4M), \lambda(a'_j - a'_{j-1}) \text{ for } j = 1,\ldots,M]$$

Define

$$a_{I(j)} = \max a_i : ha_i \leq a'_j ,$$

$$a_{i(j)} = \min a_i : ha_i \geq a'_{j-1} ;$$

these exist with $a'_{j-1} \leq ha_{i(j)} \leq ha_{I(j)} \leq a'_j$ by the condition on $\delta(D)$.

Then

$$\phi'(I'_j) - \Sigma^{[I'_j]}\phi(I_i)$$

$$= T'(a'_j) - T'(a'_{j-1}) - T(a_{I(j)}) + T(a_{i(j)})$$

$$= T'(a'_j) - T(h^{-1}a'_j) - T'(a'_{j-1}) + T(h^{-1}a'_{j-1})$$

$$+ T(h^{-1}a'_j) - T(a_{I(j)}) + T(a_{i(j)}) - T(h^{-1}a'_{j-1})$$

Hence

$$\|\phi'(I'_j) - \Sigma^{[I'_j]}\phi(I_i)\| < \mathcal{E}/M,$$

so

$$\Sigma \|\phi'(I'_j) - \Sigma^{[I'_j]}\phi(I_i)\| < \mathcal{E}.$$

Thus condition (β)(i) of relation (R_3) is satisfied.

6.6. PARAMETRIC SURFACE INTEGRALS

As a second application of the theory of this chapter, we shall prove the invariance of integrals over Fréchet equivalent parametric surfaces.

Let T,T' be continuous mappings of bounded variation into E_3 from respective admissible sets A,A' in E_2 (Ref. 5, p. 27), such that

(F) for any $\mathcal{E}' > 0$, there exists an orientation-preserving homeomorphism h from A to A' such that

$$\sup_{w \in A} \|T'(hw) - T(w)\| < \mathcal{E}'.$$

Here (Ref. 6, p. 106), {I} is the class of all closed simple polygonal regions $I \subset A$, and $\mathcal{D}$ is the class of all finite systems D = [I] of non-overlapping regions I ∈ {I}. Let τ_r, r = 1,2,3, be the projections from E_3 onto the coordinate planes Γ_r of E_3. Put $C_r = \tau_r\, T\, I^*$, where I^* is the oriented boundary curve of I. Let $O(p,C_r)$ be the topological index of the point $p \in \Gamma_r$ with respect to C_r. Then $O(p,C_r)$ is integrable on Γ_r with respect to Lebesgue 2-measure m. Put

$$\phi_r(I) = (\Gamma_r)\int O(p,C_r)\,dm, \qquad r = 1,2,3,$$

$$\phi(I) = [\phi_r(I)],$$

$$U_r = \sup_{D \in \mathcal{D}} \sum_{I \in D} |\phi_r(I)|,$$

$$U = \sup_{D \in \mathcal{D}} \sum_{I \in D} \|\phi(I)\|.$$

Define

$$d(D) = \max_{I \in D} \sup_{u,v \in I} \|T(u) - T(v)\|,$$

$$m(D) = \max_{r=1,2,3} m\Big(\bigcup_{I \in D} [C_r]\Big)$$

$$\mu(D) = \max \left\{ U - \sum_{I \in D} \|\phi(I)\|, U_r - \sum_{I \in D} |\phi_r(I)| \text{ for } r = 1,2,3 \right\}.$$

Then (Ref. 5, p. 358), $\delta(D) = d(D)+m(D)+\mu(D)$ is a mesh function.

We shall use the result (Ref. 5, pp. 186,296) that $N_r(p) = \sup_{D\in\mathscr{D}} \sum_{I\in D} |O(p,C_r)|$ is m-integrable on Γ_r.

Similar definitions and remarks apply to T'.

We shall translate the first part of Ref. 3, Section 3, to show that T,ϕ are related to T',ϕ' in the sense (R_2). Then the invariance of the surface integral $\int f(T,\phi)$ follows by Theorem 6.2 (the second part of Ref. 3, Section 3, is essentially the proof of Theorem 6.2).

<u>Theorem</u> <u>6.7</u>. If T,T' are related in the sense (F), then T,ϕ are related to T',ϕ' in the sense (R_2).

<u>Proof</u>: By absolute continuity of the integrals, for any $\mathcal{E} > 0$ and r = 1,2,3,

$$(E)\int[N_r(p) + N_r'(p)]dm < \mathcal{E}$$

for m(E) less than some $\tau(\mathcal{E})$. Take $\tau(\mathcal{E}) < \mathcal{E}$.

Consider any $D = [I] \in \mathscr{D}$ with $\delta(D) < \tau(\mathcal{E})/2$. Let $\lambda(\mathcal{E}) < \mathcal{E}$ be such that, for r = 1,2,3, the closed $\lambda(\mathcal{E})$-neighbourhood Λ_r of $\bigcup[C_r]$ has

$$m(\Lambda_r) < m(\bigcup[C_r]) + \tau(\mathcal{E})/2 < \tau(\mathcal{E})$$

Take an orientation-preserving homeomorphism $h : A \to A'$ such that

$$\sup_{w\in A}\|T'(hw) - T(w)\| < \lambda(\mathcal{E})/2.$$

Put $\overline{C}_r = \tau_r T' h I^*$; then the Fréchet distance $\|C_r, \overline{C}_r\| < \lambda(\mathcal{E})/2$. The sets hI are compact; hence T' is uniformly continuous on $\bigcup hI$, that is, for any $\mathcal{E}' > 0$, $\|T'(v')-T'(u')\| < \mathcal{E}'$ for $\|v'-u'\|$ less than some $\eta(\mathcal{E}')$, $v',u' \in \bigcup hI$. For each $I \in D$, we can take $I' \subset hI$, $I' \in \{I'\}$, with $\|I'^*, hI^*\| < \eta(\lambda(\mathcal{E})/2)$. Since the hI are non-overlapping, $D' = [I'] \in \mathcal{D}'$. Put $C_r' = \tau_r T' I'^*$. Then $\|\overline{C}_r, C_r'\| < \lambda(\mathcal{E})/2$. Hence $\|C_r, C_r'\| < \lambda(\mathcal{E})$.

Thus $O(p, C_r') = O(p, C_r)$ for $p \in \Gamma_r - \Lambda_r$ (Ref. 5, p. 85).

Hence

$$\phi_r(I) - \phi_r'(I') = (\Lambda_r)\int [O(p,C_r) - O(p,C_r')]dm,$$

so

$$\Sigma|\phi_r(I) - \phi_r'(I')| \leq (\Lambda_r)\int [N_r(p)+N_r'(p)]dm < \mathcal{E}.$$

Thus $\Sigma|\phi_r(I)| < \Sigma|\phi_r'(I)|+\mathcal{E}$, so $U_r - \mathcal{E}/2 < U_r' + \mathcal{E}$. Hence $U_r \leq U_r'$, so $U_r' = U_r$ by symmetry. Also $\Sigma\|\phi(I)-\phi'(I')\| < 3\mathcal{E}$, so $\Sigma\|\phi(I)\| < \Sigma\|\phi'(I')\|+3\mathcal{E}$, so $U - \mathcal{E}/2 < U'+3\mathcal{E}$. Hence $U \leq U'$, so $U' = U$ by symmetry.

From these relations, we have

$$\begin{aligned} U_r' - \Sigma|\phi_r'(I')| &\leq U_r - \Sigma|\phi_r(I)| + \Sigma|\phi_r(I) - \phi_r'(I')| \\ &< \mu(D) + \mathcal{E}, \\ U' - \Sigma\|\phi'(I')\| &\leq U - \Sigma\|\phi(I)\| + \Sigma\|\phi(I) - \phi'(I')\| \\ &< \mu(D) + 3\mathcal{E}. \end{aligned}$$

Also

$$d'(D') \leq \max_{I \in D} \sup_{u,v \in I} \|T'(hu) - T'(hv)\|$$

$$< d(D) + \lambda(\mathcal{E}),$$

$$m'(D') < m(D) + \tau(\mathcal{E})/2.$$

Thus $\delta'(D') < \delta(D) + \lambda(\mathcal{E}) + \tau(\mathcal{E})/2 + 3\mathcal{E} < 5\mathcal{E}$. Hence for any $\mathcal{E}' > 0$, $\delta'(D') < \mathcal{E}'$ by taking $\mathcal{E} = \mathcal{E}'/5$.

We have taken I' so that $h^{-1}I' \subset I$, so a sum $\Sigma^{[I]}$ over I' with $h^{-1}I' \subset I$ is over only one member. Thus

(β) (i) $\Sigma\|\phi(I) - \Sigma^{[I]}\phi'(I')\| = \Sigma\|\phi(I) - \phi'(I')\| < 3\mathcal{E} < \mathcal{E}'$,

(ii) $\Sigma|\,\|\phi(I)\| - \Sigma^{[I]}\|\phi'(I')\|\,| = \Sigma|\,\|\phi(I)\| - \|\phi'(I')\|\,| \leq \Sigma\|\phi(I) - \phi'(I')\| < \mathcal{E}'$,

(iii) $\bar{\Sigma}\|\phi'(I')\| = 0$.

In addition, if $w \in I$, $w' \in I'$, $h^{-1}I' \subset I$, then $w' = hw_o$ for some $w_o \in I$. Hence

(α) $\|T(w) - T'(w')\|$

$$\leq \|T(w) - T(w_o)\| + \|T(w_o) - T'(hw_o)\|$$

$$< d(D) + \lambda(\mathcal{E})/2$$

$$< \mathcal{E}$$

$$< \mathcal{E}'.$$

We have thus shown that $(T,\phi)R_2(T',\phi')$.

7. ROTATIONAL PROPERTIES OF INTEGRALS $\int f(\zeta,\phi)$

We now apply the results of the last chapter to the problem of the behaviour of $\int f(\zeta,\phi)$ under rotations in E_m and E_k. This problem arises classically when the interval functions ζ,ϕ are generated from a variety in E_n, that is, a mapping T from A to E_n. Any orthogonal transformation R in E_n will give a second variety $T' = RT$ in E_n, which will generate corresponding ζ',ϕ', and often δ'. One expects simple rotational relations between B' and B, V' and V, $\int f(\zeta',\phi')$ and $\int f(\zeta,\phi)$, but not beween the interval functions ϕ' and ϕ, ζ' and ζ, because of their approximative nature (see, for example, the remark in Ref. 28, p. 923). However, one would expect approximative rotational relations between ϕ' and ϕ, ζ' and ζ; such approximations are just what were considered generally in the last chapter. From these approximative rotational relations, we deduce the rotational relations between the BC-integrals.

Relative to a system A, $\mathcal{U}$, $\{I\}$, $\mathcal{D}$, we shall consider interval functions ζ, ζ' from $\{I\}$ to E_m and ϕ, ϕ' from $\{I\}$ to E_k, and mesh functions δ, δ' on $\mathcal{D}$.

7.1. APPROXIMATIVE ROTATIONAL RELATIONS

For P, Q orthogonal transformations on E_m and E_k respectively, we consider three relations (O_1), (O_2), (O_3) between ζ', ϕ' and ζ, ϕ with increasing strength.

(O_1) For any $\mathcal{E} > 0$, there exist systems D, $D' \in \mathcal{D}$ with $\delta(D)$ and

$\delta'(D) < \mathcal{E}$ and satisfying conditions (p), (q) below.

(O_2) For any $\mathcal{E} > 0$, there exists $\lambda(\mathcal{E}) > 0$ such that, for any D' with $\delta'(D') < \lambda(\mathcal{E})$, there exists D with $\delta(D) < \mathcal{E}$ and satisfying conditions (p), (q) below.

(O_3) For any $\mathcal{E} > 0$, there exists $\lambda(\mathcal{E}) > 0$ such that, for any D' with $\delta'(D') < \lambda(\mathcal{E})$, there exists $\eta(\mathcal{E}, D') > 0$ such that, for any D with $\delta(D) < \eta(\mathcal{E}, D')$, conditions (p), (q) below are satisfied.

Condition (p): $\|\zeta'(I') - P\zeta(I)\| <$ for $I \subset I'$, $I \in D$, $I' \in D'$.

Condition (q): (i) $\sum_{I' \in D'} \|\phi'(I') - Q\, \Sigma^{(I')} \phi(I)\| < \mathcal{E}$,

$$\text{(ii)} \quad \sum_{I' \in D'} \left| \|\phi'(I')\| - \Sigma^{(I')} \|\phi(I)\| \right| < \mathcal{E},$$

and

$$\text{(iii)} \quad \Sigma' \|\phi(I)\| < \mathcal{E}.$$

7.2. RELATION BETWEEN INTEGRALS

Let $K \subset E_m$ be such that $\zeta(I)$ and $P^{-1}\zeta'(I) \in K$ for each $I \in \{I\}$. Suppose that $f(p,q)$ is a real function on $K \times E_k$, satisfying the conditions (f) of Section 2.3. Define

$$g(p,q) \quad = \quad f(P^{-1}p, Q^{-1}q)$$

<u>Theorem 7.1</u>. If $\int f(\zeta,\phi)$, $\int g(\zeta',\phi')$ exist, $\overline{V} < \infty$, and ζ',ϕ' are related to ζ,ϕ in the sense (O_1), then $\int g(\zeta',\phi') = \int f(\zeta,\phi)$.

Proof: In the setting of Chapter 6, relation (O_1) states that $P^{-1}\zeta'$, $Q^{-1}\phi'$ are related to ζ, ϕ in the sense (R_1), with h the indentity homeomorphism. Hence, by Theorem 6.1,

$$\int f(\zeta,\phi) = \int f(P^{-1}\zeta',Q^{-1}\phi') = \int g(\zeta',\phi')$$

Theorem 7.2. If $\int f(\zeta,\phi)$ exists, $\overline{V} < \infty$, and ζ',ϕ' are related to ζ,ϕ in the sense (O_2), then $\int g(\zeta',\phi')$ exists and equals $\int f(\zeta,\phi)$.

Proof: In the setting of Chapter 6, relation (O_2) states that $P^{-1}\zeta'$, $Q^{-1}\phi'$ are related to ζ,ϕ in the sense (R_2), with h the identity homeomorphism. Hence $\int f(P^{-1}\zeta',Q^{-1}\phi') = \int g(\zeta',\phi')$ exists and equals $\int f(\zeta,\phi)$.

As particular cases, we consider in turn

$$f(p,q) = (Qq)_r,\ |(Qq)_r|,\ \|q\|.$$

The conditions on f are satisfied, so that, under the other conditions,

$$B' = QB,$$
$$V'_r = B(|(Q\phi)_r|),$$
$$V' = V.$$

For the second, $g(\zeta',\phi') = f(P^{-1}\zeta',Q^{-1}\phi') = \phi'_r$; the rest are obvious.

However, corresponding to Theorem 6.3, the relations $B' = QB$, $V' = V$ can be proved from fewer assumptions. These we now set out.

If B,B' exist and ϕ,ϕ' satisfy relation (O_1) restricted to (q)(i) and (iii), then $B' = QB$. If V,V' exist and ϕ,ϕ' satisfy relation (O_1) restricted to (q)(ii) and (iii), then $V' = V$.

If B exists and ϕ,ϕ' satisfy relation (O_2) restricted to (q)(i) and (iii), then B' exists and equals QB. If V exists and ϕ,ϕ' satisfy relation (O_2) restricted to (q)(ii) and (iii), then V' exists and equals V.

7.3. SUBSTITUTION OF SPECIAL RELATIONS

Corresponding to Theorem 6.4, conditions (q)(ii) and (iii) in relations (O_1), (O_2), (O_3) can be replaced by $V' = V$. This is important because invariance of V and often be proved independently, for example in Ref. 5, p. 355.

Furthermore, in the case in which $\zeta(I) = T(\tau)$, $\tau\epsilon I$, for a point function T, with $T' = PT$, we can deduce the relation between the Weierstrass integrals from the B relation, as in Ref. 28, p. 925 and Ref. 20.

Theorem 7.3. Assume conditions on f, T, and ϕ as in Section 2.5 in order that

$$\Sigma f(T(\tau),B(I^o)) \rightarrow \int f(T,\phi) \quad \text{as } \delta(D) \rightarrow 0,$$

$$\Sigma g(T'(\tau),B'(I^o)) \rightarrow \int g(T',\phi') \text{ as } \delta'(D) \rightarrow 0.$$

Also assume that for any $\mathcal{E} > 0$, there exists D with $\delta(D)$ and $\delta'(D) < \mathcal{E}$. Then, if $B' = QB$,

$$\int g(T',\phi') = \int f(T,\phi).$$

Proof: $\Sigma g(T'(\tau),B'(I^o)) = \Sigma f(P^{-1}PT(\tau),Q^{-1}QB(I^o))$.

The first converges to $\int g(T',\phi')$ as $\delta'(D) \to 0$

The second converges to $\int f(T\ \phi)$ as $\delta(D) \to 0$.

Hence the result.

8. SEMICONTINUITY OF INTEGRALS

We now prove semicontinuity theorems for our integrals relative to suitable topologies. For these, the form (A) $\int f(Z(w), \Theta(w))d\mu$ is most convenient. In this form, we shall relax many of the conditions that we have imposed previously. The measure μ will be arbitrary. The continuity and boundedness condition on f will be relaxed to a measurability condition. Homogeneity of f will not be assumed at first.

To distinguish this situation from the previous situation, we shall take the function f of the form $f(r,s)$, $r\epsilon E_n$, $s\epsilon E_\ell$. New requirements of convexity relative to s will be imposed on f in order to obtain the semicontinuity theorems.

We shall show in particular cases in Chapter 9 that our general semicontinuity theorems give the standard ones.

8.1. THE TOPOLOGY τ

Consider a set A and a class $\mathcal{S}$ of triplets $T = (\rho,\sigma,\mu)$, where, in each T, μ is a measure on A, σ is a μ-integrable mapping from A to E_ℓ, and ρ is a μ-measurable mapping from A to E_n. By "measurable" here, we mean that each of the component functions is measurable. Denote the class of μ-measurable sets in A by $\mathcal{M}$.

We shall denote the distance of a point r from a set R in E_n by $d(r,R)$. Let U be the unit sphere $\{s:\|s\| = 1\}$ in E_ℓ.

Define an ecart on $\mathcal{S}$ by

$$t(T_1,T_o) = \sup_{w\in A} \|\rho_1(w)-\rho_o(w)\| +$$

$$\sup_{\substack{M_o\in\mathcal{M}_o\\ u\in U}} \inf_{\substack{M_1\in\mathcal{M}_1\\ M_1\subset M_o}} \left\{ \begin{array}{l} |\mu_1(M_1)-\mu_o(M_o)| + \\ |[(M_1)\int\sigma_1 d\mu_1-(M_o)\int\sigma_o d\mu_o].u| \end{array} \right\}$$

This has the properties

$$t(T_o,T_o) = 0,$$

$$t(T_2,T_o) \leq t(T_2,T_1) + t(T_1,T_o);$$

but $$t(T_1,T_o) = t(T_o,T_1)$$

$$T_1 = T_o \text{ whenever } t(T_1,T_o) = 0$$

need not be valid. However, the two valid properties ensure that the neighbourhoods

$$\eta_{\mathcal{E}}(T_o) = \{T: t(T,T_o) < \mathcal{E}\}$$

form a basis for a topology τ on $\mathcal{S}$ (Ref. 16, p. 47).

Remark: If μ_o is regular with respect to a topology $\mathcal{G}_o \subset \mathcal{M}_o$ on A, as will be the case in our major theorem, then $\mathcal{M}_o$ can be replaced by $\mathcal{F}_o$, the class of $\mathcal{G}_o$-closed sets, in the expression defining $t(T_1,T_o)$. To prove this, denote the corresponding second parts of $t(T_1,T_o)$ by η,η'. Obviously $\eta' \leq \eta$. For any $\mathcal{E} > 0$, we have

$$\eta - \mathcal{E} < |\mu_1(M_1) - \mu_o(M_o)| + |[(M_1)\int\sigma_1 d\mu_1 - (M_o)\int\sigma_o d\mu_o].u|$$

for some $u \in U$, some M_0, and all $M_1 \subset M_0$. Now $\|(E)\int \sigma_0 d\mu_0\| < \mathcal{E}$ for $\mu_0(E)$ less than some $\kappa(\mathcal{E})$ by absolute continuity of the component integrals. Take $F_0 \subset M_0$, $F_0 \in \mathcal{F}_0$, with $\mu_0(M_0 - F_0) < \min(\mathcal{E}, \kappa(\mathcal{E}))$. Then

$$\eta - \mathcal{E} < |\mu_1(M_1) - \mu_0(F_0)| + \mathcal{E} + |[(M_1)\int \sigma_1 d\mu_1 - (F_0)\int \sigma_0 d\mu_0].u| + \mathcal{E}$$

for all $M_1 \subset F_0$. But for all $u \in U$, all F_0, and some $M_1 \subset F_0$,

$$\eta' + \mathcal{E} > |\mu_1(M_1) - \mu_0(F_0)| + |[(M)\int \sigma_1 d\mu_1 - (F_0)\int \sigma_0 d\mu_0].u| .$$

Hence

$$\eta - \mathcal{E} < \eta' + 3\mathcal{E}, \qquad \text{so } \eta \leq \eta'.$$

8.2. THE FIRST SEMICONTINUITY THEOREM

Let $f(r,s)$ be a real function on $E_n \times E_\ell$ such that, for each $T = (\rho,\sigma,\mu) \in \mathcal{J}$,

(f') $f(\rho(w),\sigma(w))$ is μ-measurable on A.

It will be obvious that it is sufficient for f to be defined on $\bigcup_{\mathcal{J}} (\rho \times \sigma)(A)$ only.

In particular, condition (f') is satisfied if f is continuous.

<u>Theorem</u> <u>8.1</u>. Consider a particular triplet $T_0 = (\rho_0,\sigma_0,\mu_0)$ satisfying the following conditions.

(1) There exists $\delta > 0$ such that $f(r,s) \geq 0$ for $d(r,\rho_0(A)) < \delta$, $s \in E_\ell$.

(2) For μ_0-almost every $w_0 \in A$ and any $\mathcal{E} > 0$, there exist $\delta(\mathcal{E},w_0) > 0$, $\beta(\mathcal{E},w_0) \in E_1$, $b(\mathcal{E},w_0) \in E_\ell$, such that, for $\|r-\rho_0(w_0)\| < \delta(\mathcal{E},w_0)$ (and $(r,s) \in \bigcup_{\mathcal{S}} (\rho x \sigma)(A)$—this will be implicit throughout),

(a) $f(r,s) \geq \beta(\mathcal{E},w_0) + b(\mathcal{E},w_0).s$ for all s,

(b) $f(r,s) \leq \beta(\mathcal{E},w_0) + b(\mathcal{E},w_0).s + \mathcal{E}$ for $\|s-\sigma_0(w_0)\| < \delta(\mathcal{E},w_0)$.

(3) $\mu_0(A) < \infty$.

(4) The measure μ_0 is regular in some topology $\mathcal{G}_0$ on A, $\mathcal{G}_0 \subset \mathcal{M}_0$. (We use "regular" in the sense that, for any $\mathcal{E} > 0$ and any $M \in \mathcal{M}_0$, there exists a set F $\mathcal{G}_0$-closed, $F \subset M$, with $\mu_0(M-F) < \mathcal{E}$).

(5) For any $\mathcal{E} > 0$, there exists a set $K \subset A$, $\mathcal{G}_0$-compact, $K \in \mathcal{M}_0$, with $\mu_0(A-K) < \mathcal{E}$.

Then $I(T) = (A) \int f(\rho(w),\sigma(w))d\mu$ is lower semicontinuous at T_0 in the topology τ.

<u>Proof</u>: <u>The case $I(T_0) < \infty$</u>

Take any $\mathcal{E} > 0$. By absolute continuity of the finite integral, there exists $\kappa > 0$ such that

$$(E)\int f(\rho_0(w),\sigma_0(w))d\mu_0 < \mathcal{E}$$

for $E \subset A$ with $\mu_0(E) < \kappa$.

Since ρ_0 and σ_0 are μ_0-measurable, $\mu_0(A) < \infty$, and μ_0 is $\mathcal{G}_0$-regular, we can apply Lusin's theorem(Refs. 12 and 18) to the set $\{w_0: w_0 \in A$, condition (2) holds$\}$ to obtain a $\mathcal{G}_0$-closed set K with $\mu_0(A-K) < \kappa$, ρ_0 and

σ_o $\mathcal{G}_o$-continuous on K, and (2) holding for every $w_o \epsilon K$. By (5), we can take K $\mathcal{G}_o$-compact.

By $\mathcal{G}_o$-continuity of ρ_o and σ_o on K, for each w_o in K, there is a set $H(w_o) \epsilon \mathcal{G}_o$, containing w_o, such that, for w in $H(w_o) \cap K$,

$$\|\rho_o(w) - \rho_o(w_o)\| < \delta(\mathcal{E}, w_o)/2$$

and

$$\|\sigma_o(w) - \sigma_o(w_o)\| < \delta(\mathcal{E}, w_o)$$

Consider any r such that

$$d(r, \rho_o(H(w_o) \cap K)) < \delta(\mathcal{E}, w_o)/2,$$

Then there is a w' in $H(w_o) \cap K$ such that

$$\|r - \rho_o(w')\| < \delta(\mathcal{E}, w_o)/2, \text{ so that } \|r - \rho_o(w_o)\| < \delta(\mathcal{E}, w_o).$$

Hence

(2') (a) $f(r,s) \geq \beta(\mathcal{E}, w_o) + b(\mathcal{E}, w_o) \cdot s$ for all s;

(b) $f(r, \sigma_o(w)) \leq \beta(\mathcal{E}, w_o) + b(\mathcal{E}, w_o) \cdot \sigma_o(w) + \mathcal{E}$ for $w \epsilon H(w_o) \cap K$.

The collection $\{H(w_o)\colon w_o \text{ in } K\}$ covers K, which is $\mathcal{G}_o$-compact. Hence we can take a finite sub-cover $\{H(w_i)\colon i = 1, 2, \ldots, \nu\}$. We shall write $H(w_i)$ as H_i, $\delta(\mathcal{E}, w_i)$ as δ_i, $\beta(\mathcal{E}, w_i)$ as β_i, and $b(\mathcal{E}, w_i)$ as b_i.

Put $E_i = H_i - H_1 - H_2 - \ldots - H_{i-1}$. Then the E_i are disjoint, $E_i \subset H_i$, and $K \subset \bigcup E_i = \bigcup H_i$. Next, put $B_i = E_i \cap K$, so that $\bigcup B_i = K$. Also $B_i \subset H_i \cap K$, so that for $w \in B_i$, $d(\rho_o(w), \rho_o(H_i \cap K)) = 0$ and $w \in H_i \cap K$.

Hence, from (2'b),

$$f(\rho_o(w), \sigma_o(w)) \leq \beta_i + b_i \cdot \sigma_o(w) + \mathcal{E}$$

Consequently $I(T_o)$

$$\begin{aligned} &< (\bigcup B_i) \int f(\rho_o(w), \sigma_o(w)) d\mu_o + \mathcal{E} \\ &\leq \Sigma\, (B_i) \int [\beta_i + b_i \cdot \sigma_o(w) + \mathcal{E}\,] d\mu_o + \mathcal{E} \\ &\leq \Sigma\, \beta_i \mu_o(B_i) + \Sigma\, b_i \cdot (B_i) \int \sigma_o(w) d\mu_o + \mathcal{E}\, \mu_o(A) + \mathcal{E}. \end{aligned}$$

Consider any T in $\mathcal{S}$ such that

$$t(T, T_o) < \min(\delta;\ \delta_i/2 \text{ for } i = 1,2,\ldots,\nu;\ \mathcal{E}/\Sigma|\beta_i|;\ \mathcal{E}/\Sigma\|b_i\|).$$

Then $\|\rho(w) - \rho_o(w)\| < \delta$ for all w in A, so $f(\rho(w), \sigma(w)) \geq 0$. Also $\|\rho(w) - \rho_o(w)\| < \delta_i/2$ for all w in A, so that for w in B_i,

$$d(\rho(w), \rho_o(H_i \cap K)) < \delta_i/2.$$

Hence, by (2'a),

$$f(\rho(w), s) \geq \beta_i + b_i \cdot s$$

for w in B_i and all s. We can take M_i in $\mathcal{M}$, $M_i \subset B_i$, with

$$|\mu(M_i) - \mu_o(B_i)| < \mathcal{E}/\Sigma|\beta_i|,$$

and

$$\left|[(M_i)\int\sigma(w)d\mu - (B_i)\int\sigma_o(w)d\mu_o]\cdot b_i/\|b_i\|\right| < \mathcal{E}/\Sigma\|b_i\|.$$

Then

$$\begin{aligned} I(T) & \\ & \geq \Sigma\ (M_i)\int f(\rho(w),\sigma(w))d\mu \\ & \geq \Sigma\ (M_i)\int(\beta_i + b_i\cdot\sigma(w))d\mu \\ & = \Sigma\ \beta_i\mu(M_i) + \Sigma\ b_i\cdot(M_i)\int\sigma(w)d\mu \\ & \geq \Sigma\ \beta_i\mu_o(B_i) - \Sigma\ |\beta_i|\mathcal{E}/\Sigma|\beta_i| \\ & + \Sigma\ b_i\cdot(B_i)\int\sigma_o(w)d\mu_o - \Sigma\|b_i\|\mathcal{E}/\Sigma\|b_i\| \\ & > I(T_o) - \mathcal{E}\,\mu_o(A) - 3\mathcal{E}. \end{aligned}$$

<u>The case $I(T_o) = \infty$.</u>

The essential difference from the treatment of the first case lies in getting a substitute for absolute continuity of I.

<u>Lemma</u> <u>8.1</u>. Suppose a non-negative function $g(w)$, $w\epsilon A$, is μ-measurable on A, but $(A)\int g(w)d\mu = \infty$. Then, for any ζ, there exists $\kappa > 0$ such that $(E)\int g(w)d\mu > \zeta$ for all E such that $\mu(A-E) < \kappa$.

<u>Proof</u>: $(A)\int g_m(w)d\mu > 2\zeta$ for m greater than some $m(\zeta)$, where $g_m(w) = \min(m,g(w))$. Then

$$\begin{aligned} (E)\int g(w)d\mu & \geq (E)\int g_m(w)d\mu \\ & = (A)\int g_m(w) - (A-E)\int g_m(w)d\mu \\ & > 2\zeta - \mu\ (A-E)m \end{aligned}$$

for $m > m(\zeta)$. Hence, for $\mu\ (A-E) < \zeta/(m(\zeta)+1)$ we have the required inequality.

Returning to the proof of Theorem 8.1 in the second case, take any ζ. By Lemma 8.1, there exists $\kappa > 0$ such that, for $\mu_o(A-E) < \kappa$,

$$(E)\int f(\rho_o(w),\sigma_o(w))d\mu_o > \zeta.$$

Now apply Lusin's theorem to get K as in the first case. Get $H(w)$, H_i, b_i, β_i, δ_i as in the first case, but with $\mathcal{E} = \zeta/2(2+\mu_o(A))$. Construct E_i, B_i as in the first case. Then

$$\begin{aligned} \zeta &< (\bigcup B_i)\int f(\rho_o(w),\sigma_o(w))d\mu_o \\ &\leq \sum\beta_i\mu_o(B_i) + \sum b_i\cdot(B_i)\int\sigma_o(w)d\mu_o + \mathcal{E}\,\mu_o(A). \end{aligned}$$

Consider T as in the first case, but with $\mathcal{E} = \zeta/2(2+\mu_o(A))$, to get

$$I(T) > \zeta - \mathcal{E}\mu_o(A) - 2\mathcal{E} = \zeta/2.$$

Remark: In Chapter 9, condition (1) will be described by saying that f is "non-negative near ρ_o" and condition (2) by saying that f is "T_o-convex".

8.3. THE SECOND SEMICONTINUITY THEOREM

In applying our general results to particular systems, we shall wish to obtain the standard semicontinuity theorems of the calculus of variations. In its present form, our topology on $\mathcal{S}$ is too fine for this. With additional conditions, we can use a coarser topology on $\mathcal{S}$ to obtain a theorem (8.2) which actually contains the standard semicontinuity theorems as we shall see in Chapter 9.

We shall consider a neighbourhood system $\mathcal{V}$ of T_0 satisfying the condition

$$(\mathcal{V}) \qquad \sup_{w\in A} \|\rho(w) - \rho_0(w)\| \to 0$$

and

$$\inf_{\substack{M\in\mathcal{M} \\ M\subset G_0}} \left\{ \begin{array}{l} |\mu(M) - \mu_0(G_0)| + \\ |[(M)\int\sigma d\mu - (G_0)\int\sigma_0 d\mu_0]\cdot u| \end{array} \right\} \to 0$$

for each $G_0 \in \mathcal{G}_0$ and $u \in U$,

as $T \to T_0$ in $\mathcal{V}$.

<u>Remark</u>: The "local" écart

$$t'(T,T_0) = \sup_{w\in A} \|\rho(w) - \rho_0(w)\| + \sup_{\substack{G_0\in\mathcal{G}_0 \\ u\in U}} \inf_{\substack{M\in\mathcal{M} \\ M\subset G_0}} \left\{ \begin{array}{l} |\mu(M) - \mu_0(G_0)| + \\ |[(M)\int\sigma d\mu - (G_0)\int\sigma_0 d\mu_0].u| \end{array} \right\}$$

gives a neighbourhood system $\mathcal{V}'$ coarser than that given by $t(T,T_0)$, since $t'(T,T_0) \leq t(T,T_0)$. The neighbourhood system $\mathcal{V}'$ satisfies condition $(\mathcal{V})$, but it is still too fine for our purpose in Chapter 9 because it involves uniform convergence with respect to G_0 and u.

<u>Theorem</u> <u>8.2</u>. Assume that the conditions of Theorem 8.1 are satisfied, together with the following conditions.

(6) The mapping ρ_0 is $\mathcal{G}_0$-continuous on A.

(7) For any $G \in \mathscr{S}_o$ and any $\mathcal{E} > 0$, there exists $B \in \mathscr{S}_o$ with its $\mathscr{S}_o$-closure $\bar{B} \subset G$ and $\mu_o(G-B) < \mathcal{E}$.

Then $I(T)$ is lower semicontinuous at T_o in any neighbourhood system $\mathcal{V}$ of T_o satisfying condition $(\mathcal{V})$.

Proof: We proceed at first as in Theorem 8.1, except that in the construction of K, we use $\kappa/2$ instead of κ. Also, since ρ_o is continuous on A, $\|\rho_o(w)-\rho_o(w_o)\| < \delta(\mathcal{E},w_o)/2$ for w in $H(w_o)$, instead of $H(w_o) \cap K$. This adjustment is to be made throughout.

Having reached the construction of H_i, δ_i, β_i, b_i, we put $m = \max[|\beta_i|,\|b_i\|: i = 1,2,\ldots\nu]$. For any $\mathcal{E}' > 0$, we have $(E)\int\|\sigma_o\|d\mu < \mathcal{E}'$ for $\mu_o(E) <$ some $\lambda(\mathcal{E}')$. Take $G \in \mathscr{S}_o$, $K \subset G$, with

$$\mu_o(G-K) < \min[\lambda(\mathcal{E}/m), \mathcal{E}/m, \kappa];$$

this is possible by the regularity condition (4) in complementary form. Let $G_i = G \cap H_i$. Use condition (7) to take $B_1 \in \mathscr{S}_o$ with $\bar{B}_1 \subset G_1$ and $\mu_o(G_1-B_1) < \kappa/2\nu$. Then $G_1-\bar{B}_1 \in \mathscr{S}_o$. Inductively, take $B_i \in \mathscr{S}_o$ with

$$\bar{B}_i \subset G_i - \bigcup_1^{i-1} \bar{B}_j,\ \mu_o(G_i-\bigcup_1^{i-1} \bar{B}_j - B_i) < \kappa/2\nu$$

for $i = 2,3,\ldots\nu$. The sets B_i are disjoint and contained in G, while $B_i \subset H_i$. Note that they are not the same as in Theorem 8.1.

Then $\mu_o(A-\bigcup B_i)$

$$< \mu_o(K) + \kappa/2 - \Sigma\mu_o(B_i)$$

$$\leq \mu_o(\bigcup G_i) - \Sigma\mu_o(B_i) + \kappa/2$$

$$= \Sigma\mu_o(G_i - \bigcup_1^{i-1} G_j) - \Sigma\mu_o(B_i) + \kappa/2$$

$$\leq \Sigma\mu_o(G_i - \bigcup_1^{i-1} \bar{B}_j - B_i) + \kappa/2$$

$$< \kappa.$$

Also $|\Sigma(B_i-K)\int(\beta_i+b_i\cdot\sigma_o)d\mu_o|$

$$\leq \Sigma(B_i-K)\int m(1+\|\sigma_o\|)d\mu_o$$

$$\leq m(G-K)\int(1+\|\sigma_o\|)d\mu_o$$

$$< 2\mathcal{E}.$$

Hence $I(T_o)$

$$< (\bigcup B_i)\int f(\rho_o,\sigma_o)d\mu_o + \mathcal{E}$$

$$< (\bigcup B_i\cap K)\int f(\rho_o,\sigma_o)d\mu_o + 2\mathcal{E}$$

$$\leq \Sigma(B_i\cap K)\int[\beta_i+b_i\cdot\sigma_o+\mathcal{E}]d\mu_o + 2\mathcal{E}$$

$$\leq \Sigma(B_i)\int[\beta_i+b_i\cdot\sigma_o]d\mu_o - \Sigma(B_i-K)\int[\beta_i+b_i\cdot\sigma_o]d\mu_o + \mathcal{E}\mu_o(A) + 2\mathcal{E}$$

$$< \Sigma\beta_i\mu_o(B_i) + \Sigma b_i\cdot(B_i)\int\sigma_o d\mu_o + 2\mathcal{E} + \mathcal{E}\mu_o(A) + 2\mathcal{E}.$$

Now consider a $\mathcal{V}$ neighbourhood of T_o such that, for any $T = (\rho,\sigma,\mu)$ in that neighbourhood,

$$\sup_{w\in A} \|\rho(w) - \rho_o(w)\| < \min(\delta;\ \delta_i/2 \text{ for } i = 1,2,\ldots\nu),$$

and

$$\inf_{\substack{M\in\mathcal{M}\\ M\subset B_i}} \left\{ \begin{matrix} |\mu(M) - \mu_o(B_i)| + \\ |[(M)\int\sigma d\mu - (B_i)\int\sigma_o d\mu_o]\cdot b_i/\|b_i\|| \end{matrix} \right\}$$

$$< \quad \min\,(\mathcal{E}/\Sigma|\beta_i|,\ \mathcal{E}/\Sigma\|b_i\|)$$

for $i = 1,2,\ldots\nu$.

We now proceed as in Theorem 8.1, noting that in using inequality (2'a), it does not matter that B_i is not contained in K, because ρ_o is continuous on A.

<u>Remark</u>: Condition (7) is related to axiom (H_4) on V.

8.4. CONVEXITY CONDITIONS

The following condition is closely related to convexity of f in s.

($\bar{2}$) For any (r_o,s_o) in $R \times E_\ell (R\subset E_n)$ and any $\mathcal{E} > 0$, there exist $\delta > 0$, $\beta \in E_1$, $b \in E_\ell$, such that, for $\|r-r_o\| < \delta$ and $r \in R$,

(a) $f(r,s) \geq \beta + b.s$ for all s,

(b) $f(r,s) \leq \beta + b.s + \mathcal{E}$ for $\|s-s_o\| < \delta$.

Condition ($\bar{2}$) implies convexity. For, consider any r_o, s_1, s_2. Take $s_o = \alpha s_1+(1-\alpha)s_2$ with $0 \leq \alpha \leq 1$, and suppose

$$f(r_o,s_o) > \alpha f(r_o,s_1) + (1-\alpha)\, f(r_o,s_2).$$

Put $\mathcal{E} = (1/2)[f(r_o,s_o) - \alpha f(r_o,s_1) - (1-\alpha)\, f(r_o,s_2)]$ in condition

$(\bar{2})$ to get β, b, δ. Then

$$f(r_o,s_1) \geq \beta + b.s_1, \; f(r_o,s_2) \geq \beta + b.s_2.$$

But then $f(r_o,s_o) = \alpha f(r_o,s_1) + (1-\alpha) f(r_o,s_2) + 2\mathcal{E}$

$$> \beta + b.s_o + \mathcal{E}.$$

However, the condition $(\bar{2})$ is stronger than convexity, as the example

$$f(r,s) \;=\; rs \qquad \text{on } E_1 \times E_1$$

shows for $r_o = 0$. If $f(r,s) \geq 0$ is required, take

$$f(r,s) \;=\; [rs+1]^+$$

For f continuous, condition $(\bar{2})$ is weaker than the following strengthened convexity condition:

f is convex in s; and for each r_o, the graph of $f(r_o,s)$ contains no whole straight lines.

This strengthened convexity condition can be put in the analytic form:

For every $r \in R$, s_1, $s_2 \in E_\ell$, $0 \leq \alpha \leq 1$,

$$f(r,\alpha s_1+(1-\alpha)s_2) \leq \alpha f(r,s_1) + (1-\alpha)f(r,s_2);$$

and for no $r \in R$, $s \in E_\ell$, $s' \neq 0$ in E_ℓ, is

$$f(r,s) \;=\; (1/2)f(r,s+\lambda s') + (1/2)f(r,s-\lambda s')$$

for all λ.

First, for f continuous, the strengthened convexity condition is equivalent to:

For any $(r_o,s_o) \in R \times E_\ell$ and any $\mathcal{E} > 0$, there exist $\delta > 0$, $\nu > 0$, $\beta \in E_1$, $b \in E_\ell$, such that for $\|r-r_o\| < \delta$ and $r \in R$,

(a) $f(r,s) \geq \beta + b.s + \nu \|s-s_o\|$ for all s,

(b) $f(r,s) \leq \beta + b.s + \mathcal{E}$ for $\|s-s_o\| < \delta$

(see Ref. 27, p. 9). This condition obviously implies condition $(\bar{2})$.

Second, $f(r,s) = \beta + b.s$ for β,b constant satisfies condition $(\bar{2})$, but its graph contains straight lines.

Thus, when f is continuous, the strengthened convexity condition can replace condition (2) in Theorem 8.1. However, on certain subclasses of $\mathcal{S}$, the semicontinuity theorem is valid when condition (2) is replaced by only convexity of f in s.

<u>Theorem</u> <u>8.3</u>. Suppose the hypotheses of Theorem 8.1 hold, except that condition (2) is replaced by f continuous and f convex in s. Then I(T) is lower semicontinuous at T_o on any subclass of $\mathcal{S}$ with $(A)\int\|\sigma\|d\mu$ bounded.

<u>Proof</u>: Take any $\mathcal{E} > 0$, and put

$$F(r,s) \quad = \quad f(r,s) + \eta\|s\|$$

where $\eta = \mathcal{E}/2m$, m is the upper bound of $(A)\int\|\sigma\|d\mu$. Then F is continuous and satisfies the strengthened convexity condition. Hence $(A)\int F(\rho(w),\sigma(w))d\mu$ is lower semicontinuous at T_o on the class mentioned.

Thus $\qquad (A)\int F(\rho(w),\sigma(w))d\mu > (A)\int F(\rho_o(w),\sigma_o(w))d\mu_o - \mathcal{E}/2$

for $t(T,T_0)$ less than some δ. That is,

$$I(T) + \eta\int\|\sigma\|d\mu > I(T_0) + \eta\int\|\sigma_0\|d\mu_0 - \mathcal{E}/2,$$

which gives

$$I(T) > I(T_0) - \mathcal{E}.$$

A similar adjustment applies to Theorem 8.2.

8.5. THE HOMOGENEOUS CASE

If $f(r,s)$ is positively homogeneous of degree one in s, then condition $(\bar{2})$ reduces to: for any $(r_0,s_0) \in R \times E_\ell$ and any $\mathcal{E} > 0$, there exist $\delta > 0$, $b \in E_\ell$ such that, for $\|r-r_0\| < \delta$ and $r \in R$,

(a) $f(r,s) \geq b.s$ for all δ,

(b) $f(r,s) \leq b.s + \mathcal{E}$ for $\|s-s_0\| < \delta$.

To prove this, condition (2a) with homogeneity gives

$$\alpha f(r,s) \geq \beta + \alpha\ b.s$$

for all $\alpha > 0$, so $0 \geq \beta$. Hence

$$f(r,s) \leq b.q + \mathcal{E}$$

for $\|s-s_0\| < \delta$, $\|r-r_0\| < \delta$, $r \in R$.

Also,

$$f(r,s) \geq \beta/\alpha + b.s$$

for all $\alpha > 0$, so

$$f(r,s) \geq b.s$$

for all $s, \|r-r_o\| < \delta$, $r \in R$.

Now the term $|\mu_1(M_1)-\mu_o(M_o)|$ in $t(T_1,T_o)$ is brought into the proof of Theorem 8.1 by the β terms. Hence, if $f(r,s)$ is positively homogeneous in s, Theorem 8.1 is valid with the improved écart obtained by omitting the above term. A similar adjustment can be made to condition $(\mathcal{V})$ for Theorem 8.2.

9. SEMICONTINUITY IN PARTICULAR CASES

In this chapter, we shall obtain known semicontinuity theorems for curve and surface integrals from our general theorems. In each case, we shall have the particular topology on $\mathcal{T}$ which is used in the corresponding section of the calculus of variations. In each case, we shall verify the conditions (3), (4), (5), (6), (7), and ($\mathcal{V}$) of Theorem 8.2. Hence, if the conditions (f'), (1) and (2) on f are satisfied, we have semicontinuity theorems of the corresponding section of the calculus of variations.

9.1 PARAMETRIC CURVE INTEGRALS $\int f(X,X')d\ell$

Let the set A be a finite closed interval $\{w\colon a \leq w \leq b\}$ in E_1, with Euclidean topology $\mathcal{U}$.

Let the mappings $\rho\colon A \to E_n$ be continuous in $\mathcal{U}$ and of bounded variation.

Take $\mathcal{G} = \mathcal{U}$ for all T. The measures μ and corresponding signed measures ν are constructed for the interval function

$$\phi[u,v] = \rho(v) - \rho(u)$$

on the intervals $I = [u,v]$ in A, by the process described in Chapter 2. The conditions for this are easily verified in this case.

Let $\sigma = \Theta = d\nu/d\mu$; thus σ is μ-integrable.

The triplet $T = (\rho,\sigma,\mu)$ is now determined by ρ.

Since the mappings ρ are of bounded variation, $\mu(A) < \infty$. The measures μ are $\mathcal{G}$-regular by the general theory of Ref. 7. Condition (5) is trivial here. Condition (7) follows from (H_4) of Ref. 7.

The function $f(r,s)$ is assumed to be positively homogeneous of degree one in s, so we shall consider the adjusted condition ($\mathcal{V}$) mentioned in Section 8.5. We shall prove that the neighbourhood system induced by the uniform topology on ρ satisfies that condition. Note that $\int \sigma d\mu = \nu$.

<u>Theorem 9.1</u>. For each $G \in \mathcal{U}$

$$\inf_{\substack{M \in \mathcal{M} \\ M \subset G}} \|\nu(M) - \nu_o(G)\| \to 0$$

as

$$\sup_{w \in A} \|\rho(w) - \rho_o(w)\| \to 0 \quad .$$

<u>Proof</u>: We have

$$\sum_{I \in D_G} \phi_o(I) \to \nu_o(G)$$

as

$$\delta_G(D_G) \to 0 \quad .$$

Hence, for any $\mathcal{E} > 0$,

$$\|\nu_o(G) - \Sigma \phi_o(I)\| < \mathcal{E}$$

for some finite number (m, say) of non-overlapping intervals $I \subset G$.

Here, $\phi[u,v] = \rho(v)-\rho(u) = \nu(u,v)$.

Consider any ρ with $\sup_{w \in A} \|\rho(w)-\rho_o(w)\| < \mathcal{E}/m$. Then

$$\|\nu(I^o) - \nu_o(I^o)\| < 2\,\mathcal{E}/m \quad .$$

Hence

$$\|\nu(\cup I^o) - \nu_o(G)\| < 3\mathcal{E} \quad .$$

For the purposes of Theorem 8.3,

$$(A)\int \|\sigma\| d\mu = \mu(A) = L \quad ,$$

the length of the corresponding curve.

We can treat each mapping ρ as a continuous rectifiable curve C in E_n. The measure μ corresponds to the arc length ℓ on C, and $\mu(A)$ is the Jordan length L of C. Thus C also has the representation

$$X(\ell):\ 0 \leq \ell \leq L \quad .$$

Since $\nu[\ell,\ell'] = X(\ell')-X(\ell)$, we can take σ also as $X' = dX/d\ell$. Thus our integral has the form

$$I(C) = (A)\int f(\rho(w),\Theta(w))d\mu = \int_0^L f(X(\ell),X'(\ell))d\ell \quad .$$

As proved in Ref. 7, if f is also bounded and uniformly continuous on KxU, then our integral also has the form of a BC-integral

$$I(C) = \int f(\rho,\phi) \quad .$$

We can now deduce from Theorem 8.2 the Tonelli-Turner theorem:

Theorem 9.2. Let $f(r,s)$: $K \times E_n \to E_1$, $K \subset E_n$, be positively homogeneous of degree one in s. Let $\mathcal{C}$ be the class of all continuous BV mappings $\rho(w)$: $A \to K$, $w \in A = [a,b] \subset E_1$ (in other words, continuous rectifiable curves C in K) for which $f(\rho(w),\Theta(w))$ is measurable in the corresponding measure μ on A. If $C_0 \in \mathcal{C}$ is such that f is non-negative near C_0 and is C_0-convex, then the integral I(C) is lower semicontinuous at C_0 in $\mathcal{C}$ with the uniform topology.

9.2 PARAMETRIC SURFACE INTEGRALS $\int f(X,J)dudv$

We shall show that Theorem 8.2 covers the semicontinuity results of Cesari in Ref. 4 and Turner in Ref. 29. In the latter paper, our system has the following form.

The set A is any admissible set in E_2 (see Ref. 5, p. 27).

The dimensions n and ℓ are both 3.

The mappings ρ: $A \to E_3$ are continuous in the Euclidean topology $\mathcal{U}$ and of bounded variation.

Each ρ determines a topology $\mathcal{G}$ on A, namely the class of $\mathcal{U}$-open ρ-whole sets in A (see Ref. 5, Section 10.2).

The measures μ and signed measures ν are constructed by the process described in Chapter 2, from an interval function ϕ defined from ρ (see Ref. 6, p. 107).

Let $\sigma = d\nu/d\mu = \Theta$; thus σ is μ-integrable.

The triplet $T = (\rho,\sigma,\mu)$ is now determined by ρ.

Since the mappings ρ are of bounded variation, $\mu(A) < \infty$. The measures μ are $\mathcal{G}$-regular by the general theory of Ref. 7. Condition (5) is satisfied (see the remark at the bottom of p. 196 in Ref. 29). Each ρ is $\mathcal{G}$-continuous. Condition (7) follows from (H_4) of Ref. 7.

The function f(r,s) is assumed to be positively homogeneous in s, so we shall consider the adjusted condition ($\mathcal{V}$) mentioned in Section 8.5. We shall prove that the neighbourhood system induced by the uniform topology on ρ satisfies that condition.

<u>Theorem 9.3</u>. For each $G_0 \epsilon \mathcal{G}_0$ and each unit vector $u \epsilon E_\ell$,

$$\inf_{\substack{M \epsilon \mathcal{M} \\ M \subset G_0}} |[\nu(M) - \nu_0(G_0)].u| \to 0$$

as

$$\sup_{w \epsilon A} \| \rho(w) - \rho_0(w) \| \to 0 \qquad .$$

<u>Proof</u>: Let P be a rotation taking u to the z-axis. Then

$$\begin{aligned} \nu_0(G_0).u &= P\nu_0(G_0).Pu \\ &= \nu^3(G_0, P\rho_0) \\ &= \nu(G_0, \rho_0') \end{aligned}$$

where we have expressed ν explicitly as a function of ρ, ν^3 is the z-component, and ρ_0' is the projection of $P\rho_0$ on the (x,y) plane. The second equality follows from a rotational property of ν, Ref. 28, Theorem 3.

According to Ref. 29, Lemma 3, simplified for our purpose, for any $\mathcal{E} > 0$ and any plane mapping ρ_0' of bounded variation, there exists $\delta > 0$ such that, for any other plane mapping ρ' with

$$\sup_{w \in G_0} \|\rho'(w) - \rho_0'(w)\| < \delta,$$

there exists $M \subset G_0$ with

$$|\nu(M,\rho') - \nu(G_0,\rho_0')| < \mathcal{E} \quad .$$

Now, if we take any ρ with

$$\sup_{w \in A} \|\rho(w) - \rho_0(w)\| < \delta \quad ,$$

and ρ' is projection of $P\rho$ on the (x,y) plane, then

$$\begin{aligned} \|\rho'(w) - \rho_0'(w)\| &\leq \|P\rho(w) - P\rho_0(w)\| \\ &= \|\rho(w) - \rho_0(w)\| \\ &< \delta \end{aligned}$$

on A and so certainly on G_0. Hence

$$\begin{aligned} &|[\nu(M)-\nu_0(G_0)].u| \\ &\quad = \|\nu(M,\rho') - \nu(G_0,\rho_0')| \\ &\quad < \mathcal{E} \quad . \end{aligned}$$

Note that here, for the purposes of Theorem 8.3, $(A)\int\|\sigma\|du = \mu(A)$

$= V = L$, the Lebesgue area of the corresponding surface.

We can treat each mapping ρ as a continuous surface S with finite Lebesgue area (see Ref. 5). The measure μ is the same considered in Ref. 5, Section 25.5, and $\mu(A)$ is the Lebesgue area of S.

As proved in Ref. 7, if f is also bounded and uniformly continuous on KxU, then our integral

$$I(S) = (A)\int f(\rho(w),\Theta(w))d\mu$$

also has the form of a BC-integral

$$I(S) = \int f(\rho,\phi) \quad .$$

As proved in Ref. 5, Section 37 and Appendix B,5(ii), under the same conditions on f, S always admits a representation

$$X(w): \quad w = (u,v) \epsilon A \subset E_2$$

such that

$$I(S) = (A)\int f(X(w),J(w))dudv \quad ,$$

where $J = (J_1,J_2,J_3)$ are the usual Jacobians.

We can now deduce from Theorem 8.2 the Turner theorem (see Ref. 29, Theorem 1):

<u>Theorem</u> <u>9.4</u>. Let $f(r,s) = KxE_3 \to E_1$, $K \subset E_3$, be positively homogeneous of degree one in s. Let $\mathcal{S}$ be the class of all continuous BV mappings

$\rho(w)$: $A \to K$, $w \in A$(in other words, continuous surfaces S in K with finite Lebesgue area) for which $f(\rho(w),\Theta(w))$ is measurable in the corresponding measure μ on A. If $S_o \in \mathscr{A}$ is such that f is non-negative near S_o and is S_o-convex, then the integral I(S) is lower semicontinuous at S_o in $\mathscr{A}$ with the uniform topology.

9.3 NON-PARAMETRIC INTEGRALS $\int f(w,X,\text{grad } X)d\mu$

Let A be any open set in E_k with finite Lebesgue k-measure μ. Let $\mathcal{U}$ be the Euclidean topology on A.

Consider mappings $X(w)$: $A \to E_m$, $w = [w_i] \in A$, absolutely continuous in the sense of Tonelli. Thus $\partial X/\partial w_i$ exists μ-almost everywhere in A for each coordinate w_i, is μ-integrable, and

$$\int_{\alpha}^{\beta} \frac{\partial X}{\partial w_i}\, dw_i = X_{(w_i=\beta)} - X_{(w_i=\alpha)}$$

for each segment $\{\alpha \leq w_i \leq \beta\}$ in A on μ^*-almost every line parallel to the w_i-axis. Here μ^* is Lebesgue (k-1)-measure; if k = 1, we take μ^* as enumeration.

Let the mappings ρ be of the form $(w,X(w))$. Thus the dimension $n = k+m$. In the case $k = 1$, our mappings ρ are essentially non-parametric curves in E_{m+1} on A. In the case $m = 1$, our mappings ρ are essentially non-parametric hypersurfaces in E_{k+1} on A.

Let $\sigma = \text{grad } X = [\partial X/\partial w_i]$. This is a kxm matrix, but here we have to treat it as a km-vector; thus the dimension $\ell = km$. The vector-valued function σ is μ-integrable by the condition on X.

The measure μ is $\mathcal{U}$-regular. In fact, the closed set F in the regularity condition can be taken compact, since its compact intersections F_n with the spheres $\{w: \|w\| \leq n\}$ have $\mu(F_n) \to \mu(F)$. Thus condition (5) of Section 8.2 is also satisfied.

To show that condition (7) is satisfied, we use again the compact regularity of μ. For any $G \in \mathcal{U}$ and any $\mathcal{E} > 0$, we take $F \subset G$, F compact, with $\mu(G-F) < \mathcal{E}$. G is the union of a countable number of closed intervals, and also the union of the corresponding open intervals. Since F is compact, we can take a finite number of the open intervals I_i covering F. Put $B = \bigcup I_i$. Then $\bar{B} = \bigcup \bar{I}_i \subset G$. Of course, B is open, and $\mu(G-B) \leq \mu(G-F) < \mathcal{E}$.

We shall now show that the neighbourhood system induced by the uniform topology on X satisfies the condition $(\mathcal{V})$.

<u>Theorem 9.5</u>. For each $G \in \mathcal{U}$ and each X_o,

$$\inf_{\substack{M \in \mathcal{M} \\ M \subset G}} \left(|\mu(M)-\mu(G)| + \|(M)\int \text{grad } X d\mu - (G)\int \text{grad } X_o d\mu\| \right) \to 0$$

as

$$\sup_{w \in A} \|X(w) - X_o(w)\| \to 0 \quad .$$

<u>Proof</u>: By absolute continuity of the integral, for any $\mathcal{E} > 0$, $\|(E)\int \text{grad } X_o d\mu\| < \mathcal{E}$ for $\mu(E)$ less than some $\lambda(\mathcal{E})$.

Let F be a compact set in G with $\mu(G-F) < \min(\mathcal{E}, \lambda(\mathcal{E}))$. G is the union of a countable number of open intervals. Because F is compact, we can take a finite number of these intervals covering F. We can contract them to closed intervals still covering F, and decompose these into closed non-overlapping intervals J_i. Then

$$\mu(G-\cup J_i) < \mathcal{E} \quad \text{and} \quad \lambda(\mathcal{E}) \quad .$$

Hence

$$\|(G-\cup J_i)\int \text{grad } X_0 d\mu\| < \mathcal{E} \quad .$$

Consider any X with

$$\sup_{w\in A} \|X(w) - X_0(w)\| < \mathcal{E}/\Sigma\mu*(J_i^*)$$

where J_i^* is the boundary of J_i. Then

$$\begin{aligned} &\|(\cup J_i)\int(\text{grad } X-\text{grad } X_0)d\mu\| \\ &= \|\Sigma(J_i^*)\int(X-X_0)d\mu*\| \\ &< \mathcal{E} \quad . \end{aligned}$$

Hence

$$|\mu(\cup J_i) - \mu(G)| + \|(\cup J_i)\int \text{grad } Xd\mu - (G)\int \text{grad } X_0 d\mu\| < 3\mathcal{E} \quad .$$

The triplet $T = (\rho,\sigma,\mu)$ is determined by X. Thus we can describe conditions in terms of X. Specifically, "f is X_0-convex" will mean that f is T_0-convex in the sense of Section 8.2.

We can now deduce from Theorem 8.2 the following theorem.

Theorem 9.6. Consider $f(p,q,s)$: $AxE_m xE_{km} \to E_1$, A open in E_k. Let χ be the class of all ACT mappings $X(w)$: $A \to E_m$, $w \in A$, for which $f(w,X(w), \text{grad } X)$ is measurable with respect to Lebesgue k-measure μ on A. If $X_o \in \chi$ is such that f is non-negative near $(A, X_o(A))$ and is X_o-convex, then the integral

$$I(X) = (A)\int f(w,X(w),\text{grad } X)d\mu$$

is lower semicontinuous at X_o in χ with the iniform topology.

9.4 CURVE INTEGRALS INVOLVING HIGHER DERIVATIVES

Cinquini, in Refs. 8 and 9, deals with variational problems for curve integrals of functions involving derivatives up to the third order. We shall show how our theorems cover Cinquini's results for semicontinuity in these cases.

Second Order Problems: Corresponding to the second order problems of Ref. 9, we have the following system.

The set A is a closed interval in E_1, with Euclidean topology $\mathcal{U}$.

Let X be any absolutely continuous mapping from A to E_3 such that, when parametrized by its arc length s, $X' = dX/ds$ is also absolutely continuous. We put $\rho = (X,X')$, and $\sigma = X' \wedge X''$, where $\wedge$ denotes the vector product in E_3.

The measures μ correspond to the arc lengths s. $\mu(A) = L < \infty$ by absolute continuity of X.

Cinquini calls curves X satisfying the above conditions "ordinary," and uses a topology on them defined by the following neighbourhoods.

If $L_0 > 0$, a δ-neighbourhood of X_0 is the class of ordinary curves X for which, considering s as a function of s_0,

$$|\dot{s}-1| \leq \delta,$$
$$\|X(s) - X_0(s_0)\| \leq \delta,$$
$$\|X'(s) - \dot{X}_0(s_0)\| \leq \delta \quad .$$

To avoid confusion, we use the dot to denote differentiation with respect to s_0.

If $L_0 = 0$, so that X_0 is constant, a δ-neighbourhood of X_0 is the class of ordinary curves X for which either

$$L > 0,$$
$$\|X(s) - X_0\| \leq \delta,$$
$$\|X'(\alpha) - X'(\beta)\| \leq \delta \qquad \text{for all } \alpha, \beta \text{ in } [0,L];$$

or

$$L = 0,$$
$$\|X - X_0\| \leq \delta \quad .$$

If we restrict our considerations to a class of curves for which $(A)\int \|X''\| d\mu$ is bounded (by C, say), we can show that Cinquini's neighbourhoods satisfy condition ($\mathscr{V}$). This is essentially a result of Ref. 9,

p. 33, but we prove it in our form.

Theorem 9.7. For any ordinary curve X_0, any $\mathcal{E} > 0$, and any $G \in \mathcal{U}$, there exists $\delta > 0$ such that

$$\eta \equiv \inf_{\substack{M \in \mathcal{M} \\ M \subset G}} \left(|\mu(M) - \mu_0(G)| + \|(M)\int X' \wedge X'' d\mu - (G)\int \dot{X}_0 \wedge \ddot{X}_0 d\mu_0\| \right) < \mathcal{E}$$

for all X in the δ-neighbourhood of X_0.

Proof: First, consider the case in which $L_0 > 0$.

By absolute continuity of the integral, for any $\mathcal{E}' > 0$, $\|(E)\int \dot{X}_0 \wedge \ddot{X}_0 d\mu_0\| < \mathcal{E}'$ for $\mu_0(E)$ less than some $\lambda(\mathcal{E}')$. As in Theorem 9.1, we can construct a finite number (m, say) of closed non-overlapping intervals J in G with $\mu_0(G - UJ) < \lambda(\mathcal{E}/3)$ and $\mathcal{E}/3$.

For the moment, take any $\delta < 1$, and consider any X in the δ-neighbourhood of X_0. From the condition $|\dot{s} - 1| < \delta$, we have

$$|s - s_0| \leq \delta s_0 \leq \delta L_0 \quad .$$

Now

$$(J)\int X' \wedge X'' d\mu - (J)\int \dot{X}_0 \wedge \ddot{X}_0 d\mu_0$$

$$= (J)\int (X' - \dot{X}_0) \wedge X'' d\mu + (J)\int (X' - \dot{X}_0) \wedge \ddot{X}_0 d\mu_0$$

$$+ (J)\int (\dot{X}_0 \wedge X'' \dot{s} + \ddot{X}_0 \wedge X') d\mu_0 \quad .$$

In this,

$$\begin{aligned} &\|(J)\int(X'-\dot{X}_o)\wedge X''d\mu\| \\ \leq\ &\delta(A)\int\|X''\|d\mu \\ \leq\ &\delta C \quad . \end{aligned}$$

Similarly

$$\|(J)\int(X'-\dot{X}_o)\wedge\ddot{X}_o d\mu_o\| \leq \delta C \quad .$$

And

$$\begin{aligned} &\|(J)\int(\dot{X}_o\wedge X''\dot{s}+\ddot{X}_o\wedge X')d\mu_o\| \\ =\ &\|(\dot{X}_o\wedge X')(J)\| \ = \ \|(\dot{X}_o\wedge[X'-\dot{X}_o])(J)\| \\ \leq\ &2\delta \quad . \end{aligned}$$

Also

$$\mu(J) - \mu_o(J) = (J)\int(\dot{s}-1)d\mu_o \quad ,$$

so

$$|\mu(J) - \mu_o(J)| \leq \delta L_o \quad .$$

Then

$$\begin{aligned} &\|(UJ)\int X'\wedge X''d\mu - (G)\int\dot{X}_o\wedge\ddot{X}_o d\mu_o\| \\ \leq\ &\Sigma\|(J)\int X'\wedge X''d\mu - (J)\int\dot{X}_o\wedge\ddot{X}_o d\mu_o\| + \mathcal{E}/3 \\ \leq\ &m(2C+2)\delta + \mathcal{E}/3 \quad ; \end{aligned}$$

and

$$\begin{aligned} &|\mu(UJ) - \mu_o(G)| \\ \leq\ &\Sigma|\mu(J) - \mu_o(J)| + \mathcal{E}/3 \\ \leq\ &mL_o\delta + \mathcal{E}/3 \quad . \end{aligned}$$

Hence $\eta < \mathcal{E}$ if

$$\delta < \mathcal{E}/3m(L_o+2C+2) \quad .$$

Next, consider the case in which $L_o = 0$. Then $\mu_o(G) = (G)\int \dot{X}_o \wedge \ddot{X}_o d\mu_o = 0$.

For the moment, take any $\delta < 1/2\sqrt{3}$, and consider any X in the δ-neighbourhood of X_o.

If $L = 0$, then the required result is trivial.

If $L > 0$, then, as in Ref. 9, p. 32, $L < 4\sqrt{3}\,\delta$. Take any $J \subset G$. Then $\mu(J) < 4\sqrt{3}\,\delta$, and

$$\|(J)\int X' \wedge X'' d\mu\| < 4\sqrt{3}\,\delta C \quad .$$

Hence $\eta < \mathcal{E}$ if $\delta < \mathcal{E}/4\sqrt{3}(C+1)$.

We can now deduce from Theorem 8.2 the following theorem.

Theorem 9.8. Consider $f(p,q,t): E_3 x E_3 x E_3 \to E_1$. Let A be a closed interval in E_1. Let $\mathcal{C}_2$ be a class of AC mappings $X(w)$: $A \to E_3$, $w \in A$, such that

(i) when parametrized by the arc length s, $X' = dX/ds$ is also AC;

(ii) $(A)\int\|X''\|ds$ is bounded;

(iii) $f(X,X',X'\wedge X'')$ is measurable with respect to s on A.

If $X_0\in \mathcal{C}_2$ is such that f is non-negative near $(X_0 x X_0')(A)$ and is X_0-convex with respect to t in the sense of Section 8.2, then the integral

$$I(X) \quad = \quad (A)\int f(X,X',X'\wedge X'')ds$$

is lower semicontinuous at X_0 in $\mathcal{C}_2$ with the Cinquini topology.

Third Order Problems: In this case, let X by any absolutely continuous mapping from A to E_3 such that, when parametrized by its arc length s, X' and X'' are also absolutely continuous. We put $\rho = (X,X',X'\wedge X'')$, and $\sigma = X'\wedge X'''$.

Cinquini defines a topology on these curves by the following neighbourhoods.

If $L_0 > 0$, a δ-neighbourhood of X_0 is as in the second order case, but with the extra condition

$$\|X''(s) - \ddot{X}_0(s_0)\| \leq \delta \quad .$$

If $L_0 = 0$, a δ-neighbourhood of X_0 is as in the second order case, but with the extra condition

$$\|X''(\alpha) - X''(\beta)\| \leq \delta \qquad \text{for all } \alpha,\beta \text{ in } [0,L]$$

when $L > 0$.

We shall now show that Cinquini's neighbourhoods in the third order case satisfy condition ($\mathcal{V}$). This is essentially a result in Ref. 9, p. 54.

Theorem 9.9. For any X_o, any $\mathcal{E} > 0$ and any $G \in \mathcal{U}$, there exists $\delta > 0$ such that

$$\eta \equiv \inf_{\substack{M \in \mathcal{M} \\ M \subset G}} \left(|\mu(M)-\mu_o(G)| + \|(M)\int X' \wedge X''' d\mu - (G)\int \dot{X}_o \wedge \dddot{X}_o d\mu_o\| \right) < \mathcal{E}$$

for all X in the third order δ-neighbourhood of X_o.

Proof: Consider first the case $L_o > 0$.

By absolute continuity, for any $\mathcal{E}' > 0$,

$$\|(E)\int \dot{X}_o \wedge \dddot{X}_o d\mu_o\| < \mathcal{E}'$$

for $\mu_o(E)$ less than some $\lambda(\mathcal{E}')$. Construct a finite number (m, say) of closed non-overlapping intervals J in G with $\mu_o(G-\bigcup J) < \lambda(\mathcal{E}/3)$ and $\mathcal{E}/3$. Denote by K the maximum of $\|\ddot{X}_o\|$ at the end points of the Js.

For any $\delta < 1$, consider any X in the δ-neighbourhood of X_o. We have

$$\begin{aligned} &\|(J)\int X' \wedge X''' d\mu - (J)\int \dot{X} \wedge \dddot{X} d\mu_o\| \\ &= \|(X' \wedge X'' - \dot{X}_o \wedge \ddot{X}_o)(J)\| \\ &= \|[X' \wedge (X'' - \ddot{X}_o) + (X' - \dot{X}_o) \wedge \ddot{X}_o](J)\| \\ &\leq 2\delta + 2\delta K. \end{aligned}$$

Also $|\mu(J)-\mu_o(J)| \leq \delta L_o$ as in Theorem 9.7.

Then

$$\|(\bigcup J)\int X'\wedge X'''d\mu - (G)\int \dot{X}_o \wedge \dddot{X}_o d\mu_o\|$$

$$\leq \Sigma\|(J)\int X'\wedge X'''d\mu - (J)\int \dot{X}_o \wedge \dddot{X}_o d\mu_o\| + \mathcal{E}/3$$

$$\leq m(2+2K)\delta + \mathcal{E}/3 \quad ;$$

and

$$|\mu(\bigcup J) - \mu_o(G)| \leq mL_o\delta + \mathcal{E}/3 \quad .$$

Hence $\eta < \mathcal{E}$ if

$$\delta < \mathcal{E}/3m(L_o+2K+2) \quad .$$

Next, consider the case $L_o = 0$. For the moment, take any $\delta < 1/2\sqrt{3}$, and consider any X in the δ-neighbourhood of X_o.

If L = 0, the required result is trivial.

If L > 0, take any closed interval J in G.

Then $\mu(J) < 4\sqrt{3}\,\delta$, and

$$\begin{aligned}(J)\int X'\wedge X'''d\mu &= (X'\wedge X'')(J)\\ &= X'(\beta)\wedge X''(\beta) - X'(\alpha)\wedge X''(\alpha)\\ &= X'(\beta)\wedge(X''(\beta)-X''(\alpha)) + (X'(\beta)-X'(\alpha))\wedge X''(\alpha)\end{aligned}$$

where α, β are the s-coordinates of the ends of J.

Considering each component, we have

$$X_i'(L) - X_i'(0) = LX_i''(\Theta)$$

for some Θ in $(0,L)$, so

$$|X_i''(s)| \leq \delta + \delta/L$$

and

$$\|X''(s)\| \leq \sqrt{3}\,\delta(1+1/L) \quad .$$

Also

$$(X'(\beta)-X'(\alpha))\wedge X''(\alpha) = (\beta-\alpha)\Big([X_1''(\Theta_1),X_2''(\Theta_1),X_3''(\Theta_3)] - X''(\alpha)\Big)\wedge X''(\alpha)$$

for some Θ_1, Θ_2, Θ_3 in (α,β), so

$$\begin{aligned} &\|(X'(\beta)-X'(\alpha))\wedge X''(\alpha)\| \\ \leq\ & L\sqrt{3}\,\delta\sqrt{3}\,\delta(1+1/L) \\ <\ & 3\delta^2(4\sqrt{3}\,\delta+1) \\ <\ & 9\delta^2 \quad . \end{aligned}$$

Thus

$$\|(J)\int X'\wedge X''' d\mu\| \leq \delta + 9\delta^2 < 4\delta \quad .$$

Hence $\eta < \mathcal{E}$ if

$$\delta < \mathcal{E}/4(1+\sqrt{3}) \quad .$$

We can now deduce from Theorem 8.2 the following theorem.

Theorem 9.10. Consider $f(p,q,r,t)$: $E_3 x E_3 x E_3 x E_3 \to E_1$. Let A be a closed interval in E_1. Let $\mathcal{C}_3$ be the class of all AC mappings $X(w)\colon A \to E_3$, $w \in A$, such that

(i) when parametrized by the arc length s, X' and X" are also AC;

(ii) $f(X,X',X' \wedge X'',X' \wedge X''')$ is measurable with respect to s on A.

If $X_0 \in \mathcal{C}_3$ is such that f is non-negative near $(X_0 \times X_0' \times X_0' \wedge X_0'')(A)$ and is X_0-convex with respect to t in the sense of Section 8.2, then the integral

$$I(X) = (A)\int f(X,X',X' \wedge X'',X' \wedge X''')ds$$

is lower semicontinuous at X_0 in $\mathcal{C}_3$ with the Cinquini topology.

PART II

10. THE SHAPE OF LEVEL SURFACES OF HARMONIC FUNCTIONS IN THREE DIMENSIONS

10.1 INTRODUCTION

Let ϕ be a harmonic function in E_3. We shall describe the shape of the level surfaces $\{P: \phi(P) = k\}$ of ϕ in terms of the corresponding regions $\{P: \phi(P) > k\}$ of higher potential, or "regions of potential." The results of this chapter can be summarized as follows. If two regions of potential are star-shaped relative to some point, then every intermediate region of potential is similarly star-shaped. If two regions of potential are convex, then every intermediate region of potential is convex. On the other hand, we prove by an example that if two regions of potential are merely simply connected, the intermediate regions of potential need not be simple connected.

These results extend previous work of Gergen[33] and Gabriel[31] for Green's functions in three dimensions.

HYPOTHESIS H: Let C_1 and C_0 be two closed subsets of E_3 (C_1 not empty), and let $\phi(P)$ be a real-valued function of E_3, subject to the following conditions.

(i) $\phi(P)$ is continuous on E_3,

(ii) $\phi(P) = 1$ on C_1,

(iii) $\phi(P) = 0$ on C_0,

(iv) $\phi(P) \to 0$ as $P \to \infty$,

(v) $\phi(P)$ is harmonic on $D = (C_0 \cup C_1)' = E_3 - (C_0 \cup C_1)$.

Since the set C_0 may be empty, the situation just described includes the case where $\phi(P) = 1$ on a closed non-empty set C_1, $\phi(P) \to 0$ as $P \to \infty$, and $\phi(P)$ is harmonic on $C_1' = E_3 - C_1$ (see Ref. 31, pp. 397, 401). We assume the existence of a function satisfying the stated conditions; some conditions on C_1 and C_0 sufficient for the existence are given, for example, in Ref. 30, pp. 290-312).

Note that C_1 and C_0 are disjoint because of conditions (ii) and (iii), and that C_1 is bounded because of conditions (ii) and (iv). In addition, by an application of the principle of the maximum in the strong form, we can deduce from our conditions that $0 \leq \phi(P) \leq 1$ on E_3.

We shall denote the Euclidean distance of a point P from the origin by $\|P\|$, the Euclidean distance of a point P from a set C by $d(P,C)$.

10.2 STAR-SHAPED REGIONS

By definition, a set C is star-shaped relative to the origin O if λP is in C whenever P is in C and $0 \leq \lambda \leq 1$.

<u>Theorem</u> <u>10.1</u>. Let C_1, C_0, and ϕ satisfy Hypothesis H, and let C_1 and $C_0' = E_3 - C_0$ be star-shaped relative to O. Then the regions $D_k = \{P: \phi(P) > k\}$ are star-shaped relative to O.

<u>Lemma</u> <u>10.1</u>. Under the hypotheses of Theorem 10.1, D is connected.

<u>Proof</u>: Let δ be the distance between C_0 and C_1 (for C_0 empty, let δ be any positive number). Since C_0 is closed and C_1 is compact, δ is

positive. Take a point R in C_1 at maximum distance from 0, and any real number Δ greater than $\|R\|$. On each plane through OR, start from OR to divide the disc $\{P: \|P\| \leq \Delta\}$ into closed acute sectors A_i determined by circular arcs of length less than δ. Let R_i be a point on the compact set $A_i \cap C_1$ at maximum distance from 0. Since no point of C_0 is at distance less than δ from R_i, there exists an arc L_i across A_i not meeting $C_0 \cup C_1$. Since C_1 and C_0' are star-shaped, the arcs L_i can be joined by radial segments to form a curve K not meeting $C_0 \cup C_1$. For the same reason, every point of D can be joined by a radial segment to some K, and the curves K can be joined by a segment on the extended segment OR. Hence D is arc-wise connected.

Lemma 10.2. Under the hypotheses of Theorem 10.1, $0 < \phi(P) < 1$ on D.

Proof: Since D is connected, the strong form of the principle of the maximum gives both inequalities.

Lemma 10.3. Under the hypotheses of Theorem 10.1, ϕ is non-increasing on each radius.

Proof: Suppose Lemma 10.3 is false. Then there are points P_0, $\lambda_0 P_0$ in D with $\phi(\lambda_0 P_0) < \phi(P_0)$, $0 < \lambda_0 < 1$. Hence the function $\psi(P) = \phi(P) - \phi(\lambda_0 P)$ has a positive least upper bound m on E_3. By condition (iv) in Hypothesis H, $|\phi(P)| < m/2$ for $\|P\|$ greater than some positive δ. Hence $\psi(P) < m/2$ also for $\|P\| > \delta$. Hence m is the least upper bound of ψ on the compact set $\{P: \|P\| \leq \delta\}$, and so is attained there. But m is not attained when P is in C_0, since $\psi \leq 0$ in C_0. Nor is m attained when P is in C_1, since C_1 is star-shaped so that $\psi = 0$ in C_1. Also, when $\lambda_0 P$ is in C_0, $\psi(P) = 0$ since C_0' is star-shaped; thus m is not attained in that case. Lastly, m cannot be attained at P when $\lambda_0 P$ is in C_1, since $\psi(P) \leq 0$ then. Hence m is attained at some point P_1, where P_1 and $\lambda_0 P_1$ are in D.

Let d be the lesser of $d(P_1, C_0)$, $d(\lambda_0 P_1, C_1)/\lambda_0$; the second is certainly finite. Then the set $N = \{P: \|P-P_1\| < d\}$ is contained in C_0'. Also $\lambda_0 N = \{\lambda_0 P: P \text{ in } N\}$ is contained in C_0', since C_0' is star-shaped. But $\lambda_0 N = \{Q: \|Q-\lambda_0 P_1\|/\lambda_0 < d\}$, and hence is contained in C_1'. Therefore N is contained in C_1', since C_1 is star-shaped. Thus $\psi(P)$ is harmonic in N. By the principle of the maximum, $\psi(P) = m$ on N. Now either (a) $\|P_1-R\| = d$ for some R in C_0, or (b) $\|P_1-R\| = d$ for some $\lambda_0 R$ in C_1. In case (a), $\psi(R) = 0-\phi(\lambda_0 R) \leq 0$, while in case (b), $\psi(R) = \phi(R)-1 \leq 0$. However, $\psi(P) = m$ for some points in any neighbourhood of R. This contradicts continuity.

Theorem 10.1 follows immediately from Lemma 10.3.

Corollary: Under the hypotheses of Theorem 10.1, the radial derivative $\partial\phi/\partial r$ is strictly negative in D. Thus grad $\phi \neq 0$ throughout D.

Proof: The function $r\partial\phi/\partial r$ is harmonic and non-positive in D. Thus if $r\partial\phi/\partial r$ were zero at some point of D, $r\partial\phi/\partial r$ would be zero throughout D, so that ϕ would be radially constant in D. Since each radius meets the set C_1, it would then follow that $\phi(P) = 1$ throughout D, contrary to Lemme 10.2.

10.3 CONVEX REGIONS

Theorem 10.2. Let C_1, C_o, and ϕ satisfy Hypothesis H, and let C_1 and C_o' be convex. Then the sets $D_k = \{P: \phi(P) > k\}$ are convex.

Lemma 10.4. If the hypotheses of Theorem 10.2 are satisfied, and if P and Q are two points in D such that $\phi(P) = \phi(Q)$, then $\phi(R) > \phi(P)$ for every point R on the open segment PQ.

Proof: For all point pairs P,Q with $\phi(P) = \phi(Q)$ and for all points R on the corresponding closed segment PQ, define

$$\Theta(P,Q,R) = \phi(P) + \phi(Q) - 2\phi(R) \quad .$$

The function $\Theta(P,Q,R)$ is continuous and bounded on its domain of definition, and its least upper bound m is non-negative.

If $m = 0$, then $\phi(R) \geq \phi(P) = \phi(Q)$ for all P,Q,R in the domain of Θ. If we assume that Lemma 10.4 is false, then there would be some P_o, Q_o in D, and R_o in the open segment P_oQ_o, with $\phi(R_o) \leq \phi(P_o) = \phi(Q_o)$. Thus if $m = 0$ and Lemma 10.4 is false, we have $\phi(R_o) = \phi(P_o) = \phi(Q_o)$. Hence $\Theta(P_o,Q_o,R_o) = 0$, and $m = 0$ is attained. Since P_o,Q_o are in D, then $0 < \phi(P_o) = \phi(Q_o) < 1$ by Lemma 10.2, hence $0 < \phi(R_o) < 1$. Thus R_o is in D.

If $m > 0$, condition (iv) in Hypothesis H implies the existence of $\delta > 0$ such that $\Theta(P,Q,R) < m/2$ whenever $\|P\| > \delta$, or $\|Q\| > \delta$, and therefore m is the maximum value of Θ on the compact set

$$\{(P,Q,R): \|P\| \leq \delta, \|Q\| \leq \delta, \phi(P) = \phi(Q), R \epsilon PQ\} \quad .$$

Now $\Theta(P,Q,R) = 0$ whenever two of the points P, Q, and R coincide. Also, $\Theta(P,Q,R) \leq 0$ whenever P or Q lies in C_0; and $\Theta(P,Q,R) = 0$ when P or Q lies in C_1, since C_1 is convex. If R lies in C_0, then (by convexity of C_0') either P or Q lies in C_0, hence $\phi(P) = \phi(Q) = \phi(R) = 0$, and again $\Theta(P,Q,R) = 0$. If R lies in C_1, then $\Theta(P,Q,R) \leq 0$ because $\phi(R) = 1$. Thus, for $m > 0$, Θ takes its maximum at some P, Q, R distinct and in D.

It follows that in both cases, either $m = 0$ and Lemma 10.4 assumed false, or $m > 0$, we could conclude that Θ takes its maximum at some P,Q,R with P,Q,R distinct and in D, R in PQ, and $\phi(P) = \phi(Q)$. By the corollary in Section 10.2 we have, on the other hand, grad $\phi \neq 0$ everywhere in D. Then, by a theorem of R. M. Gabriel (see Ref. 31, p. 389), ϕ is radially constant in D with respect to some centre.

For any point S in D, consider a half straight line J from S on which ϕ is constant on each segment lying in D. If J is completely contained in D, then ϕ is constant on J, and, by condition (iv), then $\phi = 0$ on J and $\phi(S) = 0$. If J is not contained in D, then the minimum of $\|S-P\|$ for P in $J \cap (C_0 \cup C_1)$ is attained either in C_0, in which case $\phi(S) = 0$, or in C_1, in which case $\phi(S) = 1$. Hence, in all cases, the results contradict Lemma 10.2. This proves that $m = 0$ and Lemma 10.4 is true.

Proof of Theorem 10.2. If $\phi(P) \geq \phi(Q) > k$ and $\phi(R) \leq k$ for some R in PQ, then there exists a point P' in PR with $\phi(P') = \phi(Q)$. This situation is impossible by Lemma 10.4.

10.4 A COUNTER EXAMPLE

In relation to the results in Section 10.2, it is appropriate to consider an example suggested by W. J. Wong, which shows that if C_1 and C_0' are merely assumed to be simply connected, then the regions D_K need not be simply connected, and grad ϕ can be zero in D. We shall require bounds for the change in ϕ with change in C_1. The technique used is an adaption of a method used by Gergen in Ref. 32.

Suppose C_1^- is C_1 with a piece removed, with corresponding ϕ^-. Then $\phi(P)-\phi^-(P)$ is harmonic in D, continuous in E_3, 0 on C_0, and non-negative on C_1. Hence $\phi(P)-\phi^-(P)$ is non-negative on D. Let A be the piece of the boundary D* of D removed in forming C_1^-, and g(Q;P,D) the Green's function of D with pole P. If D* is sufficiently smooth (see Ref. 34, p. 237), then, for P in D,

$$\phi(P) - \phi^-(P) = (D^*)\int [\phi(Q)-\phi^-(Q)] \frac{\partial g(Q;P,D)}{-4\pi\partial n} d\sigma$$

$$\leq (A)\int \frac{\partial g(Q;P,D)}{-4\pi\partial n} d\sigma \quad .$$

Let K by any compact set in D. Again provided D* is sufficiently smooth (see Ref. 35, p. 259), $\frac{\partial g(Q;P,D)}{-4\pi\partial n}$ has finite upper bound M_K for P in K and Q in C_1^*. Hence

$$\phi(P) - \phi^-(P) \leq M_K a(A) \quad ,$$

where a(A) is the area of A.

Now apply this result to the following system. Let the set C_0' be an open sphere with centre X, and the set C_1 a solid torus inside C_0', with the same centre of symmetry X. We form C_1^- from C_1 by removing a section bounded by two half-planes having the major axis of C_1 as common edge. Then C_1^- is a simply connected continuum. It has only one axis of symmetry, which cuts the inner surface of C_1 at Y and Z, say, the latter being removed in forming C_1^-. Since $\phi(X) < 1$, we can take k between $\phi(X)$ and 1.

First, take $K = \{P,P'\}$, where P is in XY and P' is in XZ with $k < \phi(P) = \phi(P') < 1$. By forming C_1^- appropriately, make $M_K a(A) < \phi(P)-k$. This gives $\phi^-(P) > k$ and $\phi^-(P') > k$, while $\phi^-(X) < k$. Hence the component of grad ϕ^- along YZ is zero somewhere in PP'. With symmetry, this shows that grad $\phi^- = 0$ there.

Second, take $K = \{P: \phi(P) = k\}$. For suitably formed C_1^-, $M_K a(A) < k-\phi(X)$. Hence, $\phi^-(P) > \phi(X)$ on K. On the major axis of C_1, $\phi^-(P) \leq \phi(P) \leq \phi(X)$. This shows that $\{P: \phi^-(P) > \phi(X)\}$ is not simply connected.

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