Scattering by a Paraboloid of Revolution Due to an Interior Axial Point Source

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ABSTRACT

If one considers a paraboloid of revolution of focal length η_0 with an interior point source located anywhere on the axis, the exact solution to the Dirichlet or Neumann problem (Green's function of the first or second kind) may be written in the form of an integral representation. In this report we consider the asymptotic evaluation of these integrals for both low ($k\eta_0 \ll 1$) and high ($k\eta_0 \gg 1$) frequencies. The low frequency results are obtainable from an infinite series over the zeros of a particular Whittaker function, corresponding to a Mie series found in the scattering by closed convex bodies. For high frequencies, we find multiple reflections and caustics arising from saddle point evaluations as well as "whispering gallery" waves, which arise from the nature of the behavior of the above zeros at high frequencies.

The work at high frequencies is only briefly discussed since it covers research that is actually in progress at the present time. The aim of this discussion is to introduce possible approaches to the solution.

This work on the paraboloid is preliminary to a description of scattering by general concave surfaces. Having integrated the formalism in terms of the physical phenomena such as the whispering gallery waves, multiple reflections and caustics, we now are in a position to search for generalizations of these to other concave surfaces. The scheme we propose is to determine the dependence of the physical effects on the local geometry of the paraboloid and then to make the essentially physical argument that this geometric dependence is the same for other concave shapes. This approach is similar to that used in determining creeping waves on general convex shapes and is an application of the physical arguments used in Keller's geometric theory of diffraction.

I
INTRODUCTION

1.1 Preliminary Discussion

For the most part the solutions of scattering problems have been confined to convex surfaces; relatively little has been done in the case of concave surfaces. The latter usually give rise, in the short wavelength limit, to such effects as caustics, multiple reflections and whispering gallery waves (a form of traveling waves). Some of the early considerations of these effects are found in the book by Rayleigh (1945); modern investigations are illustrated by the papers of Kimber (1961a, b) which treat the circular cylinder and sphere respectively. In this paper we will consider the paraboloid of revolution (Dirichlet or Neumann boundary condition) with an interior point source on the axis, but not necessarily at the focal point.

The scattering by a paraboloid of revolution differs from the circular cylinder and sphere mentioned above (both closed bodies). Although it is found that multiple reflections, caustics and whispering gallery waves occur when the point source is not at the focal point, the point source at the focal point gives a concave surface scattering problem which does not exhibit these effects. In the short wavelength limit it shows only a single reflection. The case of a dipole, with moment perpendicular to the axis, at the focal point of a perfectly conducting paraboloid of revolution has been investigated by Fock (1957) and Skalskaya (1955). Pinney (1946a, b) considered the moment both perpendicular and parallel to the axis. Although there is a double reflection, it is natural to consider the plane wave problem in this category. The scattering of a high frequency plane wave by the interior of a parabolic cylinder (Dirichlet boundary condition) was studied by Lamb (1906), who indicated that the method could be extended to the paraboloid of revolution.

Perhaps the best starting point for a study of scattering problems pertaining to the paraboloid of revolution is the book by Buchholz (1953) which includes a complete bibliography. Also worthy of mention is the earlier written paper by Buchholz (1942/3).

1.2 Mathematical Statement of the Problem

Let D be the closed interior domain of the paraboloid of revolution and let $\rho(\underline{r})$ ($\underline{r}$ is the usual position vector) denote the point source distribution. The precise form of $\rho(\underline{r})$ depends on the definition of a point source; it will be specified later. The arguments of Ritt and Kazarinoff (1959, 1960) can be applied to characterize the solution to the problems stated above as the ergodic limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T v(\underline{r}, t)\, dt,$$

where

$$v(\underline{r}, t) = u(\underline{r}, t)\, e^{-i\omega t}$$

and $u(\underline{r}, t)$ is the twice continuously differentiable in D for fixed time t, twice continuously differentiable in time, function in the space time domain $\{D \times 0 < t < \infty\}$ satisfying

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \rho(\underline{r})\, e^{i\omega t}$$

$$u \text{ or } \frac{\partial u}{\partial n} = 0 \text{ on the boundary of } D$$

some prescribed initial conditions.

Ritt and Kazarinoff (Ibid) also show that the above limit, which we call $v(\underline{r})$, is a twice continuously differentiable function in D satisfying

$$\nabla^2 v + k^2 v = \rho(\underline{r})$$

$$v \text{ or } \frac{\partial v}{\partial n} = 0 \text{ on the boundary of } D$$

$$\lim_{r \to \infty} r \left(\frac{\partial v}{\partial r} + ikv \right) = 0 \quad \text{(Sommerfeld radiation condition).}$$

This, of course, is the more familiar formulation of the scattering problem. Ritt and Kazarinoff (Ibid) further show that $v(\underline{r})$ is the limit $\lim_{s \to 0^+} v(\underline{r}, s)$ where $v(\underline{r}, s)$ is the function, differentiable as above, in the domain $\{D \times 0 < s < \infty\}$ satisfying

$$\nabla^2 v + \gamma^2 v = \rho(\underline{r}) \qquad (\gamma = \frac{1}{c}(\omega - is))$$

$$v \text{ or } \frac{\partial v}{\partial n} = 0 \qquad \text{on the boundary of } D$$

$$\int_{D'} |v(\underline{r}, s)|^2 \, dV < \infty \ ;$$

dV is the usual volume element and D' is the closed exterior to D.

1.3 Coordinates of the Paraboloid of Revolution

The natural system of coordinates, i.e. the system for which the wave equation separates and the boundary of D is a level surface, is defined in the following manner: two families of confocal paraboloids of revolution $\xi = \xi_o$, $\eta = \eta_o$ with focal point at the origin, given by the equations

$$\rho^2 = 4\xi(\xi - z), \qquad \rho^2 = 4\eta(\eta + z) \qquad (\rho^2 = x^2 + y^2)$$

together with the usual azimuth angle ϕ. If we make the natural choice for the domains of these variables, $0 \leqslant \xi, \ \eta < \infty, \ 0 \leqslant \phi < 2\pi$, they can be related to the rectangular (x, y, z), cylindrical (ρ, ϕ, z) and spherical (r, θ, ϕ) coordinates by the following equations

$$x = \rho \cos\phi = r \sin\theta \cos\phi = 2\sqrt{\xi\eta} \cos\phi$$

$$y = \rho \sin\phi = r \sin\theta \sin\phi = 2\sqrt{\xi\eta} \sin\phi$$

$$z = z = r\cos\theta = \xi - \eta .$$

This system of coordinates is called the coordinates of the paraboloid of revolution and is illustrated in Fig. 1. Thus if a physical problem is shown in Fig. 2, we see that the domain D under consideration may be represented by $0 \leqslant \eta \leqslant \eta_o$, $0 \leqslant \xi < \infty$, $0 \leqslant \phi < 2\pi$, the boundary of D by $\eta = \eta_o$. The point source is shown lying to the right of the focal point, since this is the case that will occupy most of our attention. The case of the point source lying to the left of the focal point will be introduced, but no calculations performed.

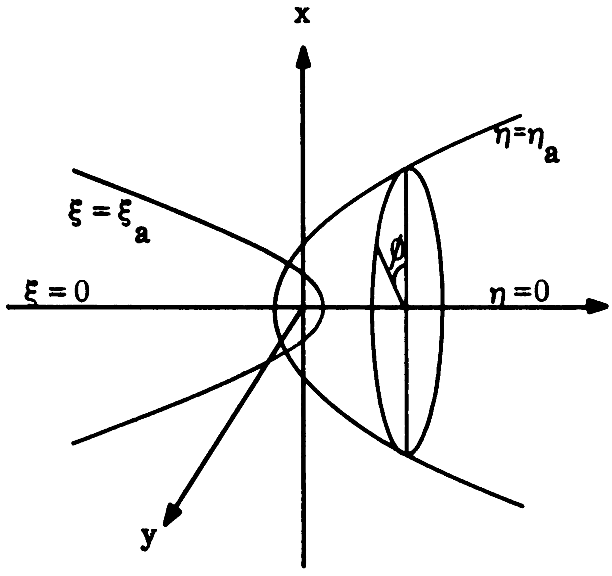

FIG. 1: Coordinates of the Infinite Paraboloid of Revolution

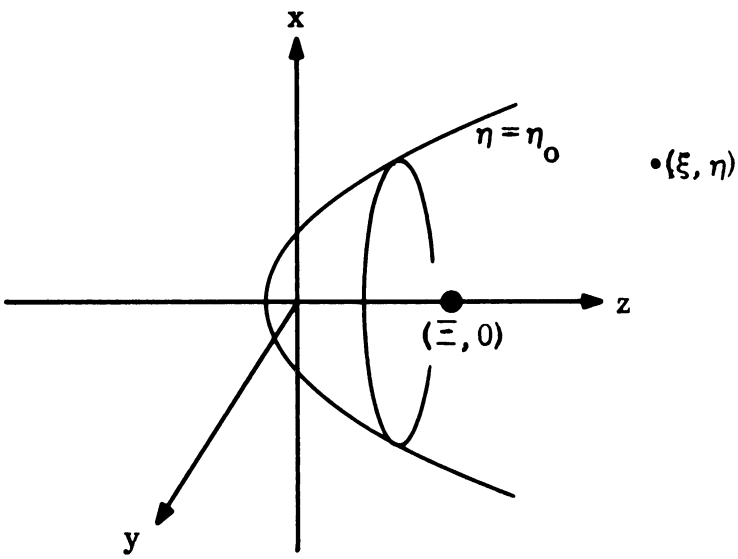

FIG. 2: Point Source at $(\Xi, 0)$, Field Point at (ξ, η)

1.4 Integral Representations of the Solution

Integral representations for the solutions $v(\xi, \eta, \phi)$ can be derived from the second and third formulations of (1.2). We first show how the method of Ritt and Kazarinoff (1959, 1960) can be applied to derive an integral representation from the third formulation. Thus we begin with the inhomogeneous wave equation in which the wave number has an imaginary part.

$$\nabla^2 v + \gamma^2 v = \rho(\underline{r}) \qquad \left(\gamma = \frac{1}{c}(\omega - is)\right) \tag{1.1}$$

In the coordinates of the paraboloid of revolution $\nabla^2 v$ has the representation given by (Buchholz, 1953)

$$\nabla^2 v = \frac{1}{2(\xi+\eta)} \left\{ \frac{\partial}{\partial \xi}\left(2\xi \cdot \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(2\eta \frac{\partial v}{\partial \eta}\right) + \frac{\xi+\eta}{2\xi\eta} \frac{\partial^2 v}{\partial \phi^2} \right\} .$$

But since $\rho(\underline{r})$ represents a point source on the axis of the paraboloid of revolution, the problems have axial symmetry; hence the ϕ dependence can be removed. Therefore equation (1.1) becomes

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial v}{\partial \eta}\right) + \gamma^2(\xi+\eta) = (\xi+\eta)\rho(\xi, \eta) . \tag{1.2}$$

For the point source at $(\Xi, 0)$, $(\xi+\eta)\rho(\xi,\eta) = c\delta(\xi - \Xi)\delta(\eta)$ where c is a constant which depends on the precise form of $\rho(\underline{r})$, i.e. on the definition of a point source. We now make the stipulation (or normalization) that our definition of a point source is such that $c = 1$. (This implies that $\rho(\underline{r}) = 4\pi\delta(\underline{r} - \underline{r}_o)$, $\underline{r}_o$ the position vector of the point source. The normalization is discussed further in Appendix A.1.) Substitution of this choice in (1.2) yields

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial v}{\partial \eta}\right) + \gamma^2(\xi+\eta) = \delta(\xi - \Xi)\delta(\eta)$$

which can be written as

$$-L_\eta v - L_\xi v = \delta(\xi - \Xi)\delta(\eta) \tag{1.3}$$

where

$$L_x y = -\frac{d}{dx}\left(p(x)\,\frac{dy}{kx}\right) + q(x)y \qquad \text{with } p(x) = x,\; q(x) = -\gamma^2 x\,.$$

In order to proceed we must consider the operators L_η, L_ξ defined by the relations

$$L_\eta y = -\frac{d}{d\eta}\left(\eta\,\frac{dy}{d\eta}\right) - \gamma^2 \eta y \qquad 0 \leqslant \eta \leqslant \eta_o$$

$$L_\xi y = -\frac{d}{d\xi}\left(\xi\,\frac{dy}{d\xi}\right) - \gamma^2 \xi y \qquad 0 \leqslant \xi < \infty$$

inasmuch as they do not correspond exactly with the ones studied in Ritt and Kazarinoff (1959, 1960). However, since $p(x) = x$, $q(x) = -\gamma^2 x$ implying $\text{Im}\, q(x) = \frac{2\omega s}{c^2} x$ is $\geqslant$ some q_o, the conditions on p and Im q correspond. For L_η the homogeneous differential equation to be studied is $L_\eta y - \lambda y = 0$. It can be written as

$$\frac{d^2 y}{d\eta^2} + \frac{1}{\eta}\frac{dy}{d\eta} + \left(\gamma^2 + \frac{\lambda}{\eta}\right) y = 0\,. \tag{1.4}$$

The substitution $y = u\eta^{-1/2}$ results in the equation

$$\frac{d^2 u}{d\eta^2} + \left(\gamma^2 + \frac{\lambda}{\eta} + \frac{1}{4\eta^2}\right) u = 0$$

or

$$\frac{d^2 u}{d(\pm 2i\gamma\eta)^2} + \left(-\frac{1}{4} + \frac{\lambda}{\pm 2i\gamma(\pm 2i\gamma\eta)} + \frac{1}{4(\pm 2i\gamma\eta)^2}\right) u = 0$$

which is Whittaker's equation. It has the two Whittaker functions $M_{\pm\lambda/2i\gamma,\,0}(\pm 2i\gamma\eta)$, $W_{\pm\lambda/2i\gamma,\,0}(\pm 2i\gamma\eta)$ as linearly independent solutions. A complete discussion of the equation together with these functions is found in Buchholz (1953). The solutions $M_{\pm\lambda/2i\gamma,\,0}(\pm 2i\gamma\eta)$ are regular at $\eta = 0$ and $\in \mathcal{L}_2(0, \eta_o)$ where $\mathcal{L}_2(a, b)$ is the class of all square integrable functions on (a, b). Except for certain values of λ the solutions $W_{\pm\lambda/2i\gamma,\,0}(\pm 2i\gamma\eta)$ are not regular at zero, but are $\in \mathcal{L}_2(0, \eta_o)$ for all values of λ.

To see which solutions should be considered for the definition of the resolvent Green's function we make use of the condition

$$\int_{D'} |v(\underline{r}, s)|^2 dV < \infty.$$

For large η

$$W_{\pm\lambda/2i\gamma, 0}(\pm 2i\gamma\eta) \sim e^{\mp i\gamma\eta} = e^{\mp i\left[\frac{1}{c}(\omega - is)\right]\eta} = e^{\mp ik\eta} e^{\mp s\eta} ,$$

thus the solution $W_{-\lambda/2i\gamma, 0}(-2i\gamma\eta)$ cannot be used to build a linear component of $v(\underline{r}, s)$ and we must consider the solution $W_{\lambda/2i\gamma, 0}(2i\gamma\eta)$. Since the solutions $M_{\pm\lambda/2i\gamma, 0}(\pm 2i\gamma\eta)$ are linearly dependent, the solution $M_{\lambda/2i\gamma, 0}(2i\gamma\eta)$ can be considered. Therefore $y_1(\eta, \lambda) = \eta^{-1/2} M_{\lambda/2i\gamma, 0}(2i\gamma\eta)$ and $y_2(\eta, \lambda) = \eta^{-1/2} W_{\lambda/2i\gamma, 0}(2i\gamma\eta)$ are two linearly independent solutions of (1.4) such that

(1) $y_1(\eta, \lambda)$ is regular at $\eta = 0$

(2) $y_2(\eta, \lambda)$ is not regular at $\eta = 0$ except for the value $\frac{\lambda}{2i\gamma} = n + \frac{1}{2}$ or $\lambda = i\gamma(2n+1)$.

This leads at once to the properties of L_η, $0 \leqslant \eta \leqslant \eta_o$

(a) $\eta = 0$ is a regular singular point of $L_\eta y - \lambda y = 0$, $p(0) = 0$

(b) $p(\eta_o) \neq 0$

(c) For $\mathrm{Im}\,\lambda < q_o = 0$ (hence λ cannot equal $i\gamma(2n+1)$), the homogeneous equation $L_\eta y - \lambda y = 0$ has exactly one linearly independent solution regular at $\eta = 0$.

Now to find the resolvant Green's function of the operator L_η, $0 \leqslant \eta \leqslant \eta_o$, we need a solution $\phi_1(\eta, \lambda)$ of $L_\eta y - \lambda y = C$ which satisfies the boundary condition at η_o, together with a solution $\phi_2(\eta, \lambda)$ of $L_\eta y - \lambda y = 0$ which is regular at $\eta = 0$. Considering first the Neumann problem, this is accomplished by the choice

$$\phi_1(\eta, \lambda) = y_2(\eta, \lambda)\left(\frac{dy_1(\eta, \lambda)}{d\eta}\right)_{\eta=\eta_o} - y_1(\eta, \lambda)\left(\frac{dy_2(\eta, \lambda)}{d\eta}\right)_{\eta=\eta_o}$$

$$\phi_2(\eta, \lambda) = y_1(\eta, \lambda)$$

(Hereafter we shall denote the derivative $\left(\frac{dF(\eta,\lambda)}{d\eta}\right)_{\eta=\eta_o}$ by $F'(\eta_o,\lambda)$.) With these definitions the Wronskian $W\left[\phi_1(\eta,\lambda),\ \phi_2(\eta,\lambda)\right]$ of $\phi_1(\eta,\lambda)$ and $\phi_2(\eta,\lambda)$ becomes

$$W\left[\phi_1(\eta,\lambda),\phi_2(\eta,\lambda)\right] = (2i\gamma)y_1'(\eta_o,\lambda)\,W\left[y_2(\eta,\lambda),y_1(\eta,\lambda)\right]$$

which reduces to (Buchholz, 1953)

$$W\left[\phi_1(\eta,\lambda),\ \phi_2(\eta,\lambda)\right] = \frac{(2i\gamma)y_1'(\eta_o,\lambda)}{\eta\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} .$$

It should be noted that

$$\phi_1(\eta_o,\lambda) = 2i\gamma\,W\left[y_2(\eta_o,\lambda),y_1(\eta_o,\lambda)\right] = \frac{2i\gamma}{\eta_o\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} .$$

The resolvent Green's function $G(\eta,\eta',\lambda)$ thus has the value

$$G(\eta,\eta',\lambda) = \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)}{(2i\gamma)y_1'(\eta_o,\lambda)}\begin{cases}\phi_1(\eta,\lambda)\phi_2(\eta',\lambda) & \eta>\eta' \\ \phi_1(\eta',\lambda)\phi_2(\eta,\lambda) & \eta<\eta\end{cases}$$

the resolvent operator the representation

$$R_\lambda y = \int_0^{\eta_o} G(\eta,\eta',\lambda)y(\eta')\,d\eta' .$$

For the Dirichlet problem we can choose the two functions

$$\psi_1(\eta,\lambda) = y_2(\eta,\lambda)y_1(\eta_o,\lambda) - y_1(\eta,\lambda)y_2(\eta_o,\lambda)$$

$$\psi_2(\eta,\lambda) = y_1(\eta,\lambda) .$$

Thus

$$W\left[\psi_1(\eta,\lambda),\psi_2(\eta,\lambda)\right] = (2i\gamma)y_1(\eta_o,\lambda)\,W\left[y_2(\eta,\lambda),y_1(\eta,\lambda)\right]$$

which reduces to (Buchholz, 1953)

$$W\left[\psi_1(\eta,\lambda),\,\psi_2(\eta,\lambda)\right] = \frac{(2i\gamma)\,y_1(\eta_o,\lambda)}{\eta\,\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} .$$

The resolvent Green's function then has the value

$$G(\eta,\eta',\lambda) = \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)}{(2i\gamma)\,y_1(\eta_o,\lambda)} \begin{cases} \psi_1(\eta,\lambda)\psi_2(\eta',\lambda) & \eta > \eta' \\ \psi_1(\eta',\lambda)\psi_2(\eta,\lambda) & \eta < \eta' \end{cases} ,$$

the resolvent operator the same representation as above.

The properties of the operator L_ξ can be written down at once.

(a) $\xi = 0$ is a regular singular point for $L_\xi y - \lambda y = 0$, $p(0) = 0$.

(b) For $\mathrm{Im}\,\lambda < q_o = 0$ the homogeneous equation $L_\xi y - \lambda y = 0$ has exactly one linearly independent solution

$$y_1(\xi,\lambda) = \xi^{-1/2} M_{\lambda/2i\gamma,\,0}(2i\gamma\xi)$$

which is regular at $\xi = 0$ and $\epsilon \mathcal{L}_2(0,\xi_o)$ $0 < \xi_o < \infty$, plus exactly one linearly independent solution

$$y_2(\xi,\lambda) = \xi^{-1/2} W_{\lambda/2i\gamma,\,0}(2i\gamma\xi)$$

which is regular at infinity and $\epsilon \mathcal{L}_2(\xi_o,\infty)$.

The resolvent Green's function $\tilde{G}(\xi,\xi',\lambda)$ thus has the value

$$\tilde{G}(\xi,\xi',\lambda) = \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)}{2i\gamma} \begin{cases} y_1(\xi,\lambda)y_2(\xi',\lambda) & \xi < \xi' \\ y_1(\xi',\lambda)y_2(\xi,\lambda) & \xi > \xi' \end{cases} ,$$

the resolvent operator $\tilde{R}_\lambda$ the representation

$$\tilde{R}_\lambda y = \int_0^\infty G(\xi, \xi', \lambda) y(\xi')\, d\xi' .$$

It should be observed that in this case $\tilde{R}_\lambda$ is not only analytic in $\mathrm{Im}\,\lambda < q_o = 0$, but also in the larger domain $\mathrm{Im}\,\lambda < k$.* As in Ritt and Kazarinoff (1959, 1960) the papers by Sims (1957) and Phillips (1952) are the basis for the work on resolvents and resolvent Greeen's functions.

We can now proceed with the method of Ritt and Kazarinoff to find the integral representations for the solutions. Consider the Neumann problem and let Γ be a path in the complex λ-plane defined by the straight line running from

$$-\infty - i\sigma \ \text{ to } \ \infty - i\sigma \qquad 0 < \sigma < k.$$

Then applying the resolvents R_λ, $\tilde{R}_{-\lambda}$ successively to equation (1.3), using the resolvent relation $R_\lambda(L_x y - \lambda y) = y$ and integration along Γ, noting that the singular points of R_λ lie above Γ while those of $\tilde{R}_{-\lambda}$ lie below Γ, we arrive at the integral representation

$$v_N(\xi, \eta, s) = \frac{1}{2\pi i} \int_\Gamma \tilde{G}(\xi, \Xi, -\lambda) G(\eta, 0, \lambda)\, d\lambda$$

where $v_N(\xi, \eta, s)$ will denote the solution to the Neumann problem.

Substituting for $\tilde{G}(\xi, \Xi, -\lambda)$ we obtain

$$v_N(\xi, \eta) = \lim_{s \to 0^+} \frac{1}{2\pi i} \int_\Gamma d\lambda \, \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2i\gamma}\right)}{(2i\gamma)^2 y_1'(\eta_o, \lambda)} \, y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda) y_1(0, \lambda) \phi_1(\eta, \lambda)$$

where $\xi_1 = \min(\xi, \Xi)$, $\xi_2 = \max(\xi, \Xi)$. But

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*See Appendix A.2.

$$y_1(0,\lambda) = (2i\gamma)^{1/2}$$

$$\phi_1(\eta,\lambda) = y_2(\eta,\lambda)y_1'(\eta_o,\lambda) - y_1(\eta,\lambda)y_2'(\eta_o,\lambda)$$

and therefore

$$v_N(\xi,\eta) = \lim_{s\to 0^+} \frac{(2i\gamma)^{-3/2}}{2\pi i} \int_\Gamma d\lambda \, \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_o,\lambda)} \cdot y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda) \cdot$$

$$\cdot \left[y_2(\eta,\lambda)y_1'(\eta_o,\lambda) - y_1(\eta,\lambda)y_2'(\eta_o,\lambda)\right] \qquad (1.5)$$

When $\eta = \eta_o$,

$$\phi_1(\eta_o,\lambda) = \frac{2i\gamma}{\eta_o\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} \quad ,$$

thus the field on the surface is given by the simpler formula

$$v_N(\xi,\eta_o) = \lim_{s\to 0^+} \frac{(2i\gamma)^{-1/2}}{\eta_o(2\pi i)} \int_\Gamma d\lambda \, \frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_o,\lambda)} \, y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda) \qquad (1.6)$$

The above procedure may be repeated exactly to obtain $v_D(\xi,\eta)$, the solution to the Dirichlet problem. Let Γ be a path in the complex λ-plane as before, then

$$v_D(\xi,\eta) = \lim_{s\to 0^+} \frac{(2i\gamma)^{-3/2}}{2\pi i} \int_\Gamma d\lambda \, \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_o,\lambda)} \, y_1(\xi_1,\lambda)y_2(\xi_2,\lambda) \cdot$$

$$\cdot \left[y_2(\eta,\lambda)y_1(\eta_o,\lambda) - y_1(\eta,\lambda)y_2(\eta_o,\lambda)\right] \qquad (1.7)$$

The surface field for the Dirichlet problem is given by the normal derivative $\left(\partial v_D(\xi,\eta)/\partial n\right)_{\eta=\eta_o}$. This derivative is governed by the relation

$$\frac{\partial v_D}{\partial n} = \frac{\eta^{1/2}}{(\xi+\eta)^{1/2}} \frac{\partial v_D}{\partial \eta} ,$$

thus the equation for the surface field is given by

$$\left(\frac{\partial v_D(\xi,\eta)}{\partial n}\right)_{\eta=\eta_o} = \frac{-\lim\limits_{s\to 0^+}}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}} \frac{(2i\gamma)^{-1/2}}{2\pi i} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_o,\lambda)} y_1(\xi_1,-\lambda) y_2(\xi_2,-\lambda) . \tag{1.8}$$

Consider now equations (1.5) through (1.8). We note that there is an essential difference between these equations and the similar equations in Ritt and Kazarinoff (1959, 1960). For these equations Γ is independent of s, thus it may be possible to take the limit as $s \to 0^+$ inside the integral. According to Buchholz (1953) the functions $y_1(x,\lambda)$, $y_2(x,\lambda)$ are entire functions of $\lambda/2i\gamma$; the path Γ is defined so that the functions $\Gamma\left(\frac{\lambda}{2i\gamma}+\frac{1}{2}\right)$, $\Gamma\left(-\frac{\lambda}{2i\gamma}+\frac{1}{2}\right)$ are analytic functions of $\lambda/2i\gamma$ on Γ. Therefore we can take the limit as $s \to 0^+$ inside the integral and equations (1.5) through (1.8) are valid without the $\lim\limits_{s\to 0^+}$ condition when the parameter γ is replaced by the parameter k.

Buchholz (1942/3, 1953) derives an integral representation from the second formulation of section 1.2. For $\rho(\underline{r}) = 4\pi\delta(\underline{r}-\underline{r}_o)$ (where $\underline{r}_o$ denotes the vector to the point source at $(\Xi, 0)$) the free space Green's function has the form $-e^{-ikR}/R$ (time dependence $e^{i\omega t}$, $R = |\underline{r}-\underline{r}_o|$). Thus he first derives (Buchholz, 1953) an integral representation for $-e^{-ikR}/R$ as

$$-\frac{e^{-ikR}}{R} = -\frac{2ik}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\,\Gamma(s+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2})\,m_s^{(0)}(2ik\xi_1)\,w_s^{(0)}(2ik\xi_2)\,w_{-s}^{(0)}(2ik\eta)$$

where $|\sigma'| < 1/2$, $\xi \neq \Xi$, $\eta > 0$, $m_s^{(0)}(2ikx) = (2ikx)^{-1/2} M_{s,0}(2ikx)$ and $w_s^{(0)}(2ikx) = (2ikx)^{-1/2} W_{s,0}(2ikx)$. Then he assumes $v(\xi,\eta)$ has the form

$$v(\xi,\eta) = v'(\xi,\eta) + \left(-\frac{e^{-ikR}}{R}\right)$$

with

$$v'(\xi,\eta) = \frac{1}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\,\Gamma(s+\tfrac{1}{2})\Gamma(-s+\tfrac{1}{2})\, m_s^{(0)}(2ik\xi_1)w_s^{(0)}(2ik\xi_2)A_s m_{-s}^{(0)}(2ik\eta)$$

where again $|\sigma'| < 1/2$ and A_s is an unknown function of s. Thus $v(\xi,\eta)$ formally satisfies the inhomogeneous wave equation while the boundary condition may be satisfied by a suitable choice of A_s. For

$$\frac{\partial v}{\partial n} = 0 \quad \text{on the boundary} \qquad A_s = 2ik\,\frac{w_{-s}^{(0)'}(2ik\eta_o)}{m_{-s}^{(0)'}(2ik\eta_o)}$$

$$v = 0 \quad \text{on the boundary} \qquad A_s = 2ik\,\frac{w_{-s}^{(0)}(2ik\eta_o)}{m_{-s}^{(0)}(2ik\eta_o)}$$

where again

$$F'(2ik\eta_o) = \left(\frac{d}{d\eta}F(2ik\eta)\right)_{\eta=\eta_o} .$$

We continue by considering explicitly only the Neumann problem. The obvious modifications can be made for the Dirichlet problem. Substituting for A_s we have

$$v_N(\xi,\eta) = \frac{2ik}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\,\Gamma(s+\tfrac{1}{2})\Gamma(-s+\tfrac{1}{2})\,\frac{m_s^{(0)}(2ik\xi_1)w_s^{(0)}(2ik\xi_2)}{m_{-s}^{(0)'}(2ik\eta_o)} \cdot$$

$$\cdot\left[w_{-s}^{(0)'}(2ik\eta_o)m_{-s}^{(0)}(2ik\eta) - m_{-s}^{(0)'}(2ik\eta_o)w_{-s}^{(0)}(2ik\eta)\right] \tag{1.9}$$

as a formal solution to the inhomogeneous wave equation which satisfies the boundary condition. From the asymptotic behavior of $w_s^{(0)}(2ik\xi_2)$ at infinity it is seen that $v_N(\xi,\eta)$ satisfies the radiation condition. Thus it remains to show that the integral exists. Let us return to the previous $y_1(x,\lambda)$, $y_2(x,\lambda)$ notation and substitute -s for s in equation (1.9). Then we obtain

$$v_N(\xi,\eta) = -\frac{(2ik)^{-1/2}}{2\pi i}\int_{\sigma'+i\infty}^{\sigma'-i\infty} ds\, \Gamma(s+\frac{1}{2})\,\Gamma(-s+\frac{1}{2})\,\frac{y_1(\xi_1,-2iks)y_2(\xi_2,-2iks)}{y_1'(\eta_o,2iks)}\cdot$$

$$\cdot\left[y_1(\eta,2iks)y_2'(\eta_o,2iks) - y_2(\eta,2iks)y_1'(\eta_o,2iks)\right] \qquad (1.10)$$

It is shown in the Appendix A.3* and Buchholz (1942/3, 1953) that

$$\Gamma(-s+\frac{1}{2})\left[y_1(\eta,2iks)y_2'(\eta_o,2iks) - y_2(\eta,2iks)y_1'(\eta_o,2iks)\right]$$

is analytic in the complex s-plane, $y_1'(\eta_o, 2iks)$ has simple zeros which lie along the imaginary axis and the integrand of equation (1.10) vanishes exponentially on a large semi-circle in the right half s-plane. Thus the integral for $0 < \sigma' < 1/2$ represents the zero solution which can be omitted by the further restriction $-1/2 < \sigma' \leqslant 0$. However, for $\sigma' = 0$ the integral is not defined and so we arrive finally at the restriction $-1/2 < \sigma' < 0$. In addition it is seen in the appendix A.3 that along this path the integral converges. Thus $v_N(\xi,\eta)$ given by equation (1.10) together with the restriction $-1/2 < \sigma' < 0$ is a solution of the Neumann problem given by the second formulation of section 1.2. We write it as

$$v_N(\xi,\eta) = \frac{(2ik)^{-1/2}}{2\pi i}\int_{\substack{\sigma'+i\infty \\ (-1/2<\sigma'<0)}}^{\sigma'-i\infty} ds\, \Gamma(s+\frac{1}{2})\,\Gamma(-s+\frac{1}{2})\,\frac{y_1(\xi_1,-2iks)y_2(\xi_2,-2iks)}{y_1'(\eta_o,2iks)}\cdot$$

$$\cdot\left[y_2(\eta,2iks)y_1'(\eta_o,2iks) - y_1(\eta,2iks)y_2'(\eta_o,2iks)\right] \qquad (1.11)$$

If in (1.11) the substitution $s = \lambda/2ik$ is made, $v_N(\xi,\eta)$ becomes

$$v_N(\xi,\eta) = \frac{(2ik)^{-3/2}}{2\pi i}\int_{-\infty+2ik\sigma'}^{\infty+2ik\sigma'} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_o,\lambda)}\cdot$$

$$\cdot\left[y_2(\eta,\lambda)y_1'(\eta_o,\lambda) - y_1(\eta,\lambda)y_2'(\eta_o,\lambda)\right]$$

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* In the Appendix A.3 we again consider only the Neumann problem. The computations can be easily modified for the Dirichlet problem; the results are essentially the same.

with $-1/2 < \sigma' < 0$ which implies

$$v_N(\xi,\eta) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} \mp \frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_o,\lambda)} \cdot$$

$$(0<\sigma<k)$$

$$\cdot\left[y_2(\eta,\lambda)y_1'(\eta_o,\lambda) - y_1(\eta,\lambda)y_2'(\eta_o,\lambda)\right] \qquad (1.12)$$

As is expected, equation (1.12) agrees with (1.5). Substituting $\eta = \eta_o$ in (1.12) yields the equation for the surface field

$$v_N(\xi,\eta_o) = \frac{(2ik)^{-1/2}}{\eta_o(2\pi i)} \int_{-\infty - i\sigma}^{\infty - io} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_o,\lambda)} \qquad (1.13)$$

$$(0<\sigma<k)$$

which of course agrees with equation (1.6). Repeating the above arguments we obtain for the Dirichlet problem

$$v_D(\xi,\eta) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1(\eta_o,\lambda)} \cdot$$

$$(0<\sigma<k)$$

$$\cdot\left[y_2(\eta,\lambda)y_1(\eta_o,\lambda) - y_1(\eta,\lambda)y_2(\eta_o,\lambda)\right] \qquad (1.14)$$

while for the surface field

$$\left(\frac{\partial v_D(\xi,\eta)}{\partial n}\right)_{\eta=\eta_o} = \frac{-1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}} \frac{(2ik)^{-1/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1(\eta_o,\lambda)}$$

$$(0<\sigma<k) \qquad (1.15)$$

These integral representations (equations (1.12) through (1.15)) will be used throughout the paper in preference to the corresponding representations obtained by the substitution $s = \lambda/2ik$.

As indicated previously the above derivations correspond to the point source of $(\Xi, 0)$. The integral representations corresponding to the same point source at $(0, H)$ can be derived in a like manner without any difficulty. For $\eta_1 = \min(\eta, H)$, $\eta_2 = \max(\eta, H)$ they are

$$v_N(\xi, \eta) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{y_2(\xi, -\lambda) y_1(\eta_1, \lambda)}{y_1'(\eta_0, \lambda)} \cdot \qquad (0 < \sigma < k)$$

$$\cdot \left[y_2(\eta_2, \lambda) y_1'(\eta_0, \lambda) - y_1(\eta_2, \lambda) y_2'(\eta_0, \lambda)\right] \qquad (1.16)$$

for the Neumann problem, and

$$v_D(\xi, \eta) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{y_2(\xi, -\lambda) y_1(\eta_1, \lambda)}{y_1(\eta_0, \lambda)} \qquad (0 < \sigma < k)$$

$$\cdot \left[y_2(\eta_2, \lambda) y_1'(\eta_0, \lambda) - y_1(\eta_2, \lambda) y_2(\eta_0, \lambda)\right] \qquad (1.17)$$

for the Dirichlet Problem

1.5 Discussion of Integral Representations with Respect to Asymptotic Expansions

For the present discussion we consider the integral representation given by equation (1.12). The statements made can easily be changed to apply to the other representations. We have already mentioned that part of the integrand of (1.12),

$$\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \left[y_2(\eta, \lambda) y_1'(\eta_0, \lambda) - y_1(\eta, \lambda) y_2'(\eta_0, \lambda)\right]$$

is analytic in the complex λ-plane, $y_1'(\eta_0, \lambda)$ has simple zeros which lie along the real axis, and the integrand of equation (1.12) vanishes exponentially on a large semi-circle in the upper half λ-plane. Therefore the singularities of the integrand of (1.12) are of two types:

(1) simple poles at the zeros of $y_1'(\eta_o, \lambda)$ which lie on the real axis,

(2) simple poles at the poles of the function $\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)$ which lie at the points $\lambda = -ik(2n+1)$ $n = 0, 1, 2, \ldots$ on the negative imaginary axis.

Since the contour of integration runs between the two sets of poles and the integrand vanishes as described above, the contour can be closed around the poles corresponding to the zeros of $y_1'(\eta_o, \lambda)$. In this manner, by use of the residue theorem, a series expansion for the solution can be obtained (Buchholz, 1953). The same asymptotic expansions used to demonstrate the vanishing of the integrand show that the series converges (Appendix A.3). However, the series does not lend itself to asymptotic analysis when $k\eta_o \gg 1$ and thus corresponds to the situation found in the scattering by closed convex bodies (Ritt and Kazarinoff; 1959, 1960). If the correspondence is complete the series should readily yield the first term of the asymptotic expansion when $k\eta_o \ll 1$. This is found to be true; the calculations are performed in Chapter 3.

There are two cases of interest when $k\eta_o \ll 1$. One is $k\xi_1 \ll 1$, $k\xi_2 \gg 1$ which corresponds to the point source (field point) in the near field and the field point (point source) in the far field. These conditions correspond to the most likely physical situations; a possible application is the use of the surface field to consider scattering from a body whose surface has a concave portion. The other case of interest is $k\xi_1 \ll 1$, $k\xi_2 \ll 1$ and corresponds to both source and field points in the near field. This case is of interest because of the relationship with the potential problem. In Chapter 3 it is shown that the solution to the Dirichlet potential problem agrees with the first term of the asymptotic expansion of the solution to the Dirichlet problem when $k\eta_o \ll 1$, $k\Xi \ll 1$ and $k\xi \ll 1$.

While these low frequency results are complete, further low frequency investigations are in progress. In particular the Neumann potential problem as well as a physical interpretation of the far field results are being considered. The work for $k\eta_o \gg 1$ is being carried out at the present time. Thus, Chapter IV is only a summary of a possible approach to the analysis.

II
UNIFORM ASYMPTOTIC EXPANSIONS

In this section we will consider separately the two asymptotic representations of the Whittaker functions necessary for later use. The cases studied all correspond to those used in the problems investigated in future sections. The results obtained are based on the work of Langer (1935, 1949). More detailed results are obtained in the memoir of Erdélyi and Swanson (1957) which discusses the necessity for the two representations, and the paper by Taylor (1939).

2.1 Airy Function Representation

For this section the Whittaker equation will be written as

$$\frac{d^2u}{dz^2} + \left[-\frac{1}{4} + \frac{\ell}{z} + \frac{1}{4z^2}\right] u = 0 ; \tag{2.1}$$

hence the two Whittaker functions of interest are $M_{\ell,0}(z)$ and $W_{\ell,0}(z)$. We make the substitution $s = z/4\ell$, thus equation (2.1) becomes

$$\frac{d^2u}{ds^2} + \left[4\ell^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2}\right] u = 0$$

and upon defining $\bar{\rho} = -2i\ell$ we obtain

$$\frac{d^2u}{ds^2} + \left[\bar{\rho}^2\left(\frac{s-1}{s}\right) + \frac{1}{4s^2}\right] u = 0 . \tag{2.2}$$

Considering $\bar{\rho}$ to be a complex parameter such that $|\bar{\rho}| \gg 1$ and s belonging to a simply connected, closed domain of the complex plane which includes the point $s = 1$ but excludes the point $s = 0$, we find equation (2.2) can be of the type studied in Langer (1949) and his results may be applicable.

Let $\bar{\phi}^2(s) = \frac{s-1}{s}$ where $\bar{\phi}(s)$ is the root of $\bar{\phi}^2(s)$ such that

$$\lim_{s \to 1} \left\{\frac{1}{(s-1)^{1/2}} \bar{\phi}(s)\right\} = 1.$$

We also define

$$\bar{\Phi}(s) = \int_1^s \bar{\phi}(\tau)d\tau \quad , \qquad \bar{\zeta} = \bar{\rho}\,\bar{\Phi}(s) \tag{2.3}$$

and

$$\bar{\psi}(s) = \left[\bar{\Phi}(s)\right]^{1/6}\left[\bar{\phi}(s)\right]^{-1/2} \text{ with } \bar{\psi}(1) = \lim_{s\to 1} \bar{\psi}(s) \ ; \tag{2.4}$$

these are the functions to be used in the theory. For the applications we must consider the region $\arg s = [-\delta, \delta]$ where δ is a small positive number. Then, provided $|s| \gg \frac{1}{|\bar{\rho}|^{2/3}}$ Langer's results are applicable (Taylor, 1939). For $|s| > 1$ we can write

$$\Phi(s) = \sqrt{s(s-1)} - \log(\sqrt{s-1} + \sqrt{s}) \tag{2.5}$$

while for $|s| \lessdot 1$ we use

$$\Phi(s) = \pm i \int_1^s \left(\frac{1-\tau}{\tau}\right)^{1/2} d\tau \tag{2.6}$$

where the upper (lower) sign corresponds to the argument of negative real s being $\pi(-\pi)$. When $s \sim 1$ $\left[|s-1| = O(1/\bar{\rho}^{-2/3}), \text{(Taylor, 1939; Erdélyi and Swanson, 1957; Buchholz, 1953)}\right]$ we employ the expansions

$$\bar{\phi}(s) = (s-1)^{1/2}\left[1 - \frac{1}{2}(s-1) + O(s-1)^2\right] \qquad \text{as} \quad s \to 1 \tag{2.7}$$

and

$$\bar{\Phi}(s) = \frac{2}{3}(s-1)^{3/2} - \frac{1}{5}(s-1)^{5/2} + O\left((s-1)^{7/2}\right) \quad \text{as} \quad s \to 1 \ . \tag{2.8}$$

In order to derive the desired asymptotic representations we need the behavior of the above functions as $s \to \infty$. Thus expanding $(1 - 1/s)^{1/2}$ about $s = \infty$ leads directly to

$$\bar{\phi}(s) = 1 - \frac{1}{2s} + O(1/s^2) \qquad \text{as} \quad s \to \infty \tag{2.9}$$

and using (2.5) and (2.3) respectively yields

$$\bar{\Phi}(s) = s - \frac{1}{2} - \log 2\sqrt{s} + O(1/s) \qquad \text{as } s \to \infty \qquad (2.10)$$

and

$$\bar{\zeta} = \bar{\rho}\left[s - \frac{1}{2} - \log 2\sqrt{s} + O(1/s)\right] \qquad \text{as } s \to \infty \qquad (2.10a)$$

We wish to find the asymptotic representations of the functions $M_{\ell,0}(z) = M_{\ell,0}(4\ell s)$ asymptotic to

$$\frac{(4\ell s)^{-\ell} e^{2\ell s}}{\Gamma(\frac{1}{2} - \ell)} + \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Gamma(\frac{1}{2} + \ell)} e^{-\pi i(\ell - \frac{1}{2})}$$

for large ℓs, $-\pi/2 < \arg \ell s < 3\pi/2$, and $W_{\ell,0}(4\ell s)$ asymptotic to $\frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Gamma(\frac{1}{2} - \ell)}$ for large ℓs, $|\arg \ell s| < 3\pi/2$. We shall accomplish this by finding the asymptotic representations for $W_{\ell,0}(4\ell s)$ and $W_{-\ell,0}(-4\ell s)$ and using the relation (Buchholz, 1953)

$$M_{\ell,0}(4\ell s) = \frac{e^{-\ell\pi i} W_{-\ell,0}(-4\ell s)}{\Gamma(\frac{1}{2} - \ell)} + \frac{e^{-\ell\pi i} e^{\pi i/2}}{\Gamma(\frac{1}{2} + \ell)} W_{\ell,0}(4\ell s) \ . \qquad (2.11)$$

(In choosing the particular form of (2.11) we make use of the value $\arg \ell s = \pi/2$ which occurs in the applications.) To find these asymptotic representations we must compare $W_{\ell,0}(4\ell s)$ and $W_{-\ell,0}(4\ell s)$ with the functions

$$\bar{V}^{(j)}(s) = (\pi/2)^{1/2} e^{\pm 5\pi i/12} \bar{\psi}(s) \bar{\zeta}^{-1/3} H^{(j)}_{1/3}(\bar{\zeta}) \qquad j = 1, 2$$

which are solutions of the differential equation

$$\frac{d^2 y}{ds^2} + \left[\bar{\rho}^2 \left(\frac{s-1}{s}\right) - \frac{\bar{\psi}''(s)}{\bar{\psi}(s)}\right] y = 0$$

called the related equation of (2.2). These functions have the asymptotic behavior for large $\bar{\zeta}$

$$\bar{V}^{(j)}(s) = \bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{\pm i\bar{\zeta}} \left[1 + O(1/\bar{\zeta})\right] \begin{matrix} j=1\,, & -\pi < \arg\bar{\zeta} < 2\pi \\ j=2\,, & -2\pi < \arg\bar{\zeta} < \pi \end{matrix}$$

Now if $\arg s = \alpha \in [-\delta, \delta]$ then due to the nature of the applications $\arg \ell$ must have the value $\arg \ell = \pi/2 - \alpha$ and thus as $s \to \infty$ $\arg \bar{\zeta} \to 0$. Therefore either of the above expansions is valid as $s \to \infty$; from (2.10a) we see that $\bar{V}^{(2)}(s)$ has the correct exponential dependence in s for $W_{\ell,0}(4\ell s)$, $\bar{V}^{(1)}(s)$ the same for $W_{-\ell,0}(-4\ell s)$ and hence according to Langer (1949)

$$W_{-\ell,0}(-4\ell s) = \bar{C}_1\left[V^{(1)}(s) + O(1/\bar{\rho})\right] \qquad |\bar{\zeta}| \leqslant N \tag{2.12a}$$

$$W_{-\ell,0}(-4\ell s) = \bar{C}_1\left[\bar{V}^{(1)}(s) + \frac{\bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{i\bar{\zeta}} O(1)}{\bar{\rho}}\right] \qquad |\bar{\zeta}| > N \tag{2.12b}$$

$$W_{\ell,0}(4\ell s) = \bar{C}_2\left[\bar{V}^{(2)}(s) + O(1/\bar{\rho})\right] \qquad |\bar{\zeta}| \leqslant N \tag{2.13a}$$

$$W_{\ell,0}(4\ell s) = \bar{C}_2\left[\bar{V}^{(2)}(s) + \frac{\bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{-i\bar{\zeta}} O(1)}{\bar{\rho}}\right] \qquad |\bar{\zeta}| > N \tag{2.13b}$$

where N is a large positive number and the $\bar{C}_j$ $j = 1, 2$ are determined by the relations

$$\bar{C}_1 = \lim_{s \to \infty} \frac{W_{-\ell,0}(-4\ell s)}{\bar{V}^{(1)}(s)} = \frac{(-4\ell s)^{-\ell} e^{2\ell s}}{\bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{i\bar{\zeta}}}$$

$$\bar{C}_2 = \lim_{s \to \infty} \frac{W_{\ell,0}(4\ell s)}{\bar{V}^{(2)}(s)} = \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{-i\bar{\zeta}}} .$$

Using the definition (2.4) and the expansions (2.9) and (2.10) we obtain

$$\bar{C}_1 = (-2i\ell)^{1/6} e^{-\ell \log -\ell/e} , \qquad \bar{C}_2 = (-2i\ell)^{1/6} e^{\ell \log \ell/e} .$$

Thus (2.12) and (2.13) become for $C_\ell = e^{\ell \log \ell/e}$

$$W_{-\ell,0}(-4\ell s) = (-2i\ell)^{1/6} C_{-\ell}\left[\bar{V}^{(1)}(s) + O(1/\bar{\rho})\right] \qquad |\bar{\zeta}| > N \tag{2.14a}$$

$$W_{-\ell,0}(-4\ell s) = (-2i\ell)^{1/6} C_{-\ell}\left[\bar{V}^{(1)}(s) + \frac{\bar{\Psi}(s)\,\bar{\zeta}^{-1/6} e^{i\bar{\zeta}}\, O(1)}{\bar{\rho}}\right] \qquad |\bar{\zeta}| > N \qquad (2.14b)$$

$$W_{\ell,0}(4\ell s) = (-2i\ell)^{1/6} C_{\ell}\left[\bar{V}^{(2)}(s) + O(1/\bar{\rho})\right] \qquad |\bar{\zeta}| \leqslant N \qquad (2.15a)$$

$$W_{\ell,0}(4\ell s) = (-2i\ell)^{1/6} C_{\ell}\left[\bar{V}^{(2)}(s) + \frac{\bar{\Psi}(s)\,\bar{\zeta}^{-1/6} e^{-i\bar{\zeta}}\, O(1)}{\bar{\rho}}\right] \qquad |\bar{\zeta}| > N \qquad (2.15b)$$

Using (2.11) and the relations (Erdélyi et al, 1953)

$$\Gamma(\tfrac{1}{2} \pm \ell) = \sqrt{2\pi}\; e^{\pm \ell \log \pm \ell/e}\left[1 + O(1/\ell)\right]$$

valid for $|\ell| \gg 1$, we obtain

$$M_{\ell,0}(4\ell s) = \frac{(-2i\ell)^{1/6} e^{-\pi i \ell}}{\sqrt{2\pi}}\left[\bar{V}^{(1)}(s) + e^{\pi i/2}\,\bar{V}^{(2)}(s) + O(1/\bar{\rho})\right]$$

$$|\bar{\zeta}| \leqslant N \qquad (2.16a)$$

$$M_{\ell,0}(4\ell s) = \frac{(-2i\ell)^{1/6} e^{-\pi i \ell}}{\sqrt{2\pi}}\left[\bar{V}^{(1)}(s) + e^{\pi i/2}\,\bar{V}^{(2)}(s) + \frac{\bar{\Psi}(s)\,\bar{\zeta}^{-1/6}\left[e^{i\bar{\zeta}} O(1) + e^{-i\bar{\zeta}} O(1)\right]}{\bar{\rho}}\right]$$

$$|\bar{\zeta}| > N \qquad (2.16b)$$

For purposes of calculation it is often more convenient to represent the Hankel functions $H^{(j)}_{1/3}(\bar{\zeta})$ in terms of the Airy function Ai(g) defined by the integral representation

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty} \cos(\tfrac{1}{3}s^3 + zs)\,ds \;.$$

We can use the relations (Abramowitz and Stegun, 1964)

$$H^{(1)}_{1/3}(\bar{\zeta}) = e^{-\pi i/6}\sqrt{3/\bar{\sigma}}\left[Ai(-\bar{\sigma}) - i\,Bi(-\bar{\sigma})\right], \quad \bar{\sigma} = (\tfrac{3}{2}\bar{\zeta})^{2/3}$$

$$\left[Ai(-\bar{\sigma}) - i\,Bi(-\bar{\sigma})\right] = 2e^{-\pi i/3} Ai(-\bar{\sigma} e^{2\pi i/3})$$

$$H^{(2)}_{1/3}(\bar{\zeta}) = e^{\pi i/6}\sqrt{3/\bar{\sigma}}\left[Ai(-\bar{\sigma}) + i\,Bi(-\bar{\sigma})\right], \quad \bar{\sigma} = (\tfrac{3}{2}\bar{\zeta})^{2/3}$$

$$\left[\mathrm{Ai}(-\bar{\sigma}) + i\,\mathrm{Bi}(-\bar{\sigma})\right] = 2e^{\pi i/3}\,\mathrm{Ai}(-\bar{\sigma}e^{-2\pi i/3})$$

(Bi(g) is an Airy function linearly independent of Ai(g) which does not enter in the final result and so it is not defined here) to find

$$H^{(1)}_{1/3}(\bar{\zeta}) = e^{-\pi i/2}\,\frac{(2)^{4/3}(3)^{1/6}}{\bar{\zeta}^{1/3}}\,\mathrm{Ai}(-\bar{\sigma}e^{2\pi i/3})$$

$$H^{(2)}_{1/3}(\bar{\zeta}) = e^{\pi i/2}\,\frac{(2)^{4/3}(3)^{1/6}}{\bar{\zeta}^{1/3}}\,\mathrm{Ai}(-\bar{\sigma}e^{-2\pi i/3})\,.$$

Thus equation (2.14) and (2.15) become

$$W_{-\ell,0}(-4\ell s) = 2(3)^{1/6}\pi^{1/2}(-i\ell)^{1/6}e^{-\pi i/12}C_{-\ell}\left[\bar{\Psi}(s)\mathrm{Ai}(-\bar{\sigma}e^{2\pi i/3}) + O(1/\bar{\rho})\right]$$

$$|\bar{\zeta}| \leq N \qquad (2.17a)$$

$$W_{-\ell,0}(-4\ell s) = 2(3)^{1/6}\pi^{1/2}(-i\ell)^{1/6}e^{-\pi i/12}C_{-\ell}\left[\bar{\Psi}(s)\mathrm{Ai}(-\bar{\sigma}e^{2\pi i/3}) + \frac{\bar{\Psi}(s)\bar{\zeta}^{-1/6}e^{i\bar{\zeta}}O(1)}{\bar{\rho}}\right]$$

$$|\bar{\zeta}| > N \qquad (2.17b)$$

$$W_{\ell,0}(4\ell s) = 2(3)^{1/6}\pi^{1/2}(-i\ell)^{1/6}e^{\pi i/12}C_{\ell}\left[\bar{\Psi}(s)\mathrm{Ai}(-\bar{\sigma}e^{-2\pi i/3}) + O(1/\bar{\rho})\right]$$

$$|\bar{\zeta}| \leq N \qquad (2.18a)$$

$$W_{\ell,0}(4\ell s) = 2(3)^{1/6}\pi^{1/2}(-i\ell)^{1/6}e^{\pi i/12}C_{\ell}\left[\bar{\Psi}(s)\mathrm{Ai}(-\bar{\sigma}e^{-2\pi i/3}) + \frac{\bar{\Psi}(s)\bar{\zeta}^{-1/6}e^{-i\bar{\zeta}}O(1)}{\bar{\rho}}\right]$$

$$|\bar{\zeta}| > N \qquad (2.18b)$$

with

$$\bar{\sigma} = \left[\frac{3}{2}\bar{\zeta}\right]^{2/3}\,.$$

Finally, upon using (Abramowitz and Stegun, 1964)

$$\mathrm{Ai}(-\bar{\sigma}) = e^{-\pi i/3}\mathrm{Ai}(-\bar{\sigma}e^{2\pi i/3}) + e^{\pi i/3}\mathrm{Ai}(-\bar{\sigma}e^{-2\pi i/3})$$

equations (2.16) become

$$M_{\ell,0}(4\ell s) = (2)^{1/3}(6)^{1/6}e^{\pi i/6}(\ell)^{1/6}e^{-\pi i\ell}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma}) + O(1/\bar{\rho})\right]$$

$$|\bar{\zeta}| \leqslant N \qquad (2.19a)$$

$$M_{\ell,0}(4\ell s) = (2)^{1/6}(6)^{1/6}e^{\pi i/6}(\ell)^{1/6}e^{-\pi i\ell}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma}) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\left[e^{i\bar{\zeta}}O(1) + e^{-i\bar{\zeta}}O(1)\right]}{\bar{\rho}}\right]$$

$$|\bar{\zeta}| > N \qquad (2.19b)$$

2.2 Bessel Function Representation

If after the substitution $s = z/4\ell$ in (2.1) we define $\rho = 2\ell$, then equation (2.1) becomes

$$\frac{d^2u}{ds^2} + \left[\rho^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2}\right]u = 0 . \qquad (2.20)$$

Suppose then we consider ρ a complex parameter such that $|\rho| \gg 1$ and s belonging to a simply connected, closed domain R_s of the complex plane which includes the point $s = 0$, but excludes the point $s = 1$, then (2.20) is of the type studied in Langer (1935) and his results may be applied providing R_s has all the properties he requires. We will assume that R_s has these required properties since it is straightforward to show that the domains used in the applications have them.

Let $\phi^2(s) = \frac{1}{s}(1-s)$ where $\phi(s)$ is to be the root of $\phi^2(s)$ determined by the relation

$$\lim_{s\to 0}\left\{s^{1/2}\phi(s)\right\} = 1 .$$

We also define

$$\Phi(s) = \int_0^s \phi(\tau)d\tau , \qquad \zeta = \rho\,\Phi(s) \qquad (2.21)$$

$$\psi(s) = \left[\phi(s)\,\Phi(s)\right]^{-1/2} \quad \text{with} \quad \psi(0) = \lim_{s\to 0}\psi(s) \qquad (2.22)$$

For the applications we must consider three regions of s, defined for δ small, positive as

$$\text{(i)}\ \arg s \in [\pi - \delta, \pi]\ , \qquad \text{(ii)}\ \arg s \in [-\pi, -\pi + \delta]\ , \qquad \text{and (iii)}\ \arg s \in [-\delta, \delta]\ , \ |s| < 1\ .$$

When $\arg s \in [\pi - \delta, \pi]$, equation (2.21) can be written as

$$\Phi(s) = -i \int_0^s \left(\frac{\tau - 1}{\tau}\right)^{1/2} d\tau\ , \tag{2.23a}$$

while for $\arg s \in [-\pi, -\pi + \delta]$ we can write

$$\Phi(s) = i \int_0^s \left(\frac{\tau - 1}{\tau}\right)^{1/2} d\tau\ . \tag{2.23b}$$

In either case (region (i) or (ii))

$$\int_0^s \left(\frac{\tau - 1}{\tau}\right)^{1/2} d\tau = -\sqrt{s(s-1)} + \log(\sqrt{1-s} - \sqrt{-s}) \tag{2.24}$$

where again the square root is such that $\lim_{s \to 0} \sqrt{1-s} = 1$. For region (iii) Langer's results are applicable provided

$$|1 - s| \gg \frac{1}{|\rho|^{2/3}} \qquad \text{(Taylor, 1939)}$$

In order to derive the asymptotic representations, the behavior of these functions as $s \to 0$ and $s \to \infty$ must be known. As $s \to 0$ we expand $(1-s)^{1/2}$ about $s = 0$ resulting in

$$\phi(s) = \frac{1}{s^{1/2}} - \frac{1}{2} s^{1/2} + O(s^{3/2}) \qquad \text{as } s \to 0 \tag{2.25}$$

Therefore equations (2.21) and (2.22) yield

$$\Phi(s) = 2s^{1/2} - \frac{1}{3}s^{3/2} + O(s^{5/2}) \qquad \text{as } s \to 0 \tag{2.26}$$

$$\zeta = \rho\Phi(s) = 2\rho s^{1/2}\left[1 - \frac{1}{6}s + O(s^2)\right] \qquad \text{as } s \to 0 \tag{2.26a}$$

$$\Psi(s) = \frac{1}{2^{1/2}}\left[1 + \frac{1}{3}s + O(s^2)\right] \qquad \text{as } s \to 0 \tag{2.27}$$

and

$$\frac{\Phi(s)}{\phi(s)} = 2s\left[1 + \frac{1}{3}s + O(s^2)\right] \qquad \text{as } s \to 0 \tag{2.28}$$

In examining the behavior at infinity we will only consider regions (i) and (ii) since in region (iii) we are concerned only with $|s| < 1$. Thus expanding $(1 - \frac{1}{s})^{1/2}$ about $s = \infty$ in region (i) yields

$$\phi(s) = -i\left[1 - \frac{1}{2s} + O(1/s^2)\right] \qquad \text{as } s \to \infty \tag{2.29a}$$

while expanding about $s = \infty$ in region (ii) yields

$$\phi(s) = i\left[1 - \frac{1}{2s} + O(1/s^2)\right] \qquad \text{as } s \to \infty\ . \tag{2.29b}$$

Equations (2.21) and (2.24) then given for region (i)

$$\zeta = \rho\Phi(s) = -i\rho\left[s - \frac{1}{2} - \log(2\sqrt{-s}) + O(1/s)\right] \qquad \text{as } s \to \infty \tag{2.30a}$$

and for region (ii)

$$\zeta = \rho\Phi(s) = i\rho\left[s - \frac{1}{2} - \log(2\sqrt{-s}) + O(1/s)\right] \qquad \text{as } s \to \infty\ . \tag{2.30b}$$

We wish to find the asymptotic representations of the functions $M_{\ell,0}(z) = M_{\ell,0}(4\ell s)$ regular at the origin and $W_{\ell,0}(z) = W_{\ell,0}(4\ell s)$ asymptotic to $(4\ell s)^{\ell} e^{-2\ell s}$ for large ℓs, $|\arg \ell s| < 3\pi/2$. Then for $V^o(s) = \Psi(s)\zeta J_o(\zeta)$ the theory in Langer (1935) asserts

$$M_{\ell,0}(4\ell s) = C\left[V^{o}(s) + \frac{\Psi(s)\zeta^{5}O(1)}{\rho^{4}}\right] \qquad |\zeta| \leqslant N \tag{2.31a}$$

$$M_{\ell,0}(4\ell s) = C\left[V^{o}(s) + \frac{\Psi(s)\zeta^{1/2}\left[e^{i\zeta}O(1) + e^{-i\zeta}O(1)\right]}{\rho}\right] \qquad |\zeta| > N \tag{2.31b}$$

where N is a large positive number and C a function of ℓ determined by a comparison of the behavior of $V^{o}(s)$ and $M_{\ell,0}(4\ell s)$ as $s \to 0$. $V^{o}(s)$ is a solution of the differential equation

$$\frac{d^2y}{ds^2} + \left[\rho^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2} - \frac{\Psi''(s)}{\Psi(s)}\right] y = 0$$

which is called the related equation of (2.20). For the value of C we can writ

$$C = \lim_{s \to 0} \frac{M_{\ell,0}(4\ell s)}{V^{o}(s)}$$

and thus

$$C = \frac{(4\ell)^{1/2}}{2^{1/2}\rho} = \frac{1}{\rho^{1/2}} .$$

Therefore, equations (2.31a), (2.31b) become

$$M_{\ell,0}(4\ell s) = \rho^{1/2}\left[\frac{\Phi(s)}{\phi(s)}\right]^{1/2} J_o(\zeta) + \frac{\Psi(s)\zeta^{5}O(1)}{\rho^{9/2}} \qquad |\zeta| \leqslant N \tag{2.32a}$$

$$M_{\ell,0}(4\ell s) = \rho^{1/2}\left[\frac{\Phi(s)}{\phi(s)}\right]^{1/2} J_o(\zeta) + \frac{\Psi(s)\,\zeta^{1/2}\left[e^{i\zeta}O(1) + e^{-i\zeta}O(1)\right]}{\rho^{3/2}} \qquad |\zeta| > N \tag{2.32b}$$

In Chapter 3 we will be primarily concerned with the case $|\rho s| \ll 1$. Hence $s \to 0$ and we can use expansions (2.26) through (2.28) to find for $|\zeta| \leqslant N$

$$M_{\ell,0}(4\ell s) = (2\rho s)^{1/2} J_0(\zeta) \left[1 + O(s)\right] + O\left((\rho s)^{1/2} s^2\right) .$$

Upon retaining only the order term of the lowest order in s this becomes

$$M_{\ell,0}(4\ell s) = (2\rho s)^{1/2} J_0(\zeta) \left[1 + O(s)\right] , \tag{2.33a}$$

except at a zero of $J_0(\zeta)$, where then the additional term is the required estimate. For $|\zeta| > N$ and ζ not a zero of $J_0(\zeta)$ we have

$$M_{\ell,0}(4\ell s) = (2\rho s)^{1/2} J_0(\zeta) \left[1 + O(s) + O(1/\rho)\right] ,$$

while at a zero of $J_0(\zeta)$ the estimate

$$M_{\ell,0}(4\ell s) = O\left(\frac{(\rho s)^{1/4}}{\rho^{5/4}}\right)$$

is valid. Upon comparing order terms the above equation reduces to

$$M_{\ell,0}(4\ell s) = (2\rho s)^{1/2} J_0(\zeta) \left[1 + O(1/\rho)\right] . \tag{2.33b}$$

We note that the above results for $M_{\ell,0}(4\ell s)$ are valid for s in any of the regions (i), (ii) or (iii). However, in deriving the asymptotic representations for $W_{\ell,0}(4\ell s)$ we must be careful to distinguish between regions. In any case we need the following solutions of the related equation

$$V^{(j)}(s) = (\pi/2)^{1/2} e^{\pm i\pi/4} \Psi(s) \zeta H_0^{(j)}(\zeta) \qquad j = 1, 2$$

with asymptotic expansions for large $|\zeta|$ given by

$$V^{(j)}(s) = \Psi(s) \zeta^{1/2} e^{\pm i\zeta} \left[1 + O(1/\rho)\right] \qquad \begin{array}{l} j = 1, \ -\pi < \arg\zeta < 2\pi \\ j = 2, \ -2\pi < \arg\zeta < \pi \ . \end{array}$$

Case 1: $\arg s \in [\pi - \delta, \pi]$

Let $\arg s = \pi - \alpha \in [\pi - \delta, \pi]$. Due to the nature of the applications $\arg \rho = -\pi/2 + \alpha$ and thus as $s \to \infty$, $\arg \zeta \to 0$. Hence, either of the above asymptotic expansions are valid as $s \to \infty$ and from equation (2.30a) we see that $V^{(2)}(s)$ has the correct exponential dependence in s for $W_{\ell,0}(4\ell s)$, $V^{(1)}(s)$ the same for $W_{-\ell,0}(-4\ell s)$. Therefore according to Langer (1935)

$$W_{-\ell,0}(-4\ell s) = C_1 \left[V^{(1)}(s) + \frac{\Psi(s)\,\zeta \log \zeta \, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N \qquad (2.34a)$$

$$W_{-\ell,0}(-4\ell s) = C_1 \left[V^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N \qquad (2.34b)$$

$$W_{\ell,0}(4\ell s) = C_2 \left[V^{(2)}(s) + \frac{\Psi(s)\zeta \log \zeta \, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N \qquad (2.35a)$$

$$W_{\ell,0}(4\ell s) = C_2 \left[V^{(2)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N \qquad (2.35b)$$

where N is a large positive number and the C_j $j = 1, 2$ are determined by the relations

$$C_1 = \lim_{s \to \infty} \frac{W_{-\ell,0}(-4\ell s)}{V^{(1)}(s)} = \frac{(-4\ell s)^{-\ell} e^{2\ell s}}{\Psi(s)\zeta^{1/2} e^{i\zeta}}$$

$$C_2 = \lim_{s \to \infty} \frac{W_{\ell,0}(4\ell s)}{V^{(2)}(s)} = \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Psi(s)\zeta^{1/2} e^{i\zeta}} .$$

Using the definitions (2.21) and the expansions (2.26) and (2.27) we obtain

$$C_1 = \frac{(-i)^{1/2}}{\rho^{1/2}} e^{-\ell \log \ell/e} \qquad \text{and} \qquad C_2 = \frac{(-i)^{1/2}}{\rho^{1/2}} e^{\ell \log -\ell/e} .$$

Thus with

$$D_\ell = e^{\ell \log -\ell/e}$$

equations (2.34) and (2.35) become

$$W_{-\ell,0}(-4\ell s) = \frac{(-i)^{1/2}}{\rho^{1/2}} D_{-\ell}\left[V^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N \qquad (2.36a)$$

$$W_{-\ell,0}(-4\ell s) = \frac{(-i)^{1/2}}{\rho^{1/2}} D_{-\ell}\left[V^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho}\right] \qquad |\zeta| > N \qquad (2.36b)$$

$$W_{\ell,0}(4\ell s) = \frac{(-i)^{1/2}}{\rho^{1/2}} D_{\ell}\left[V^{(2)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N \qquad (2.37a)$$

$$W_{\ell,0}(4\ell s) = \frac{(-i)^{1/2}}{\rho^{1/2}} D_{\ell}\left[V^{(2)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho}\right] \qquad |\zeta| > N \qquad (2.37b)$$

Case 2: $\arg s \in [-\delta, \delta]$ $|s| < 1$, $\arg s = \pi$ for s negative real.

In this region we cannot directly derive an asymptotic representation for $W_{\ell,0}(4\ell s)$. However, we have the representation (2.32) for $M_{\ell,0}(4\ell s)$ and can show (Erdelyi and Swanson, 1957) that the function

$$W(s) = \frac{(-i)^{1/2}}{\rho^{1/2}} D_{\ell}\left[V^{(2)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N$$

$$= \frac{(-i)^{1/2}}{\rho^{1/2}} D_{\ell}\left[V^{(2)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho}\right] \qquad |\zeta| > N$$

is a solution of the differential equation (2.20) in this region. From the definition of $V^{(2)}(s)$ we see that these two solutions are linearly independent and thus

$$W_{\ell,0}(4\ell s) = A\, M_{\ell,0}(4\ell s) + B\, W(s)$$

or

$$W_{\ell,0}(4\ell s) = A \frac{1}{\rho^{1/2}}\left[\Psi(s)\zeta J_0(\zeta) + O(\text{term})\right]$$

$$+ B\frac{(-i)^{1/2}}{\rho^{1/2}} D_{\ell}\left[(\pi/2)^{1/2} e^{-i\pi/4}\Psi(s)\zeta H_0^{(2)}(\zeta) + O(\text{term})\right]$$

where we have used O(term) to denote the two order terms for $|\zeta| \leqslant N$ or $|\zeta| > N$. Now we have the relation

$$J_o(\zeta) = \frac{H_o^{(1)}(\zeta) + H_o^{(2)}(\zeta)}{2} \quad ,$$

thus the above equation may be written as

$$W_{\ell,0}(4\ell s) = A\frac{1}{\rho^{1/2}}\left[\frac{\Psi(s)\zeta H_o^{(1)}(\zeta)}{2} + \frac{\Psi(s)\zeta H_o^{(2)}(\zeta)}{2} + O(\text{term})\right]$$

$$+ B\frac{(-i)^{1/2}}{\rho^{1/2}} D_\ell\left[(\pi/2)^{1/2} e^{-i\pi/4}\Psi(s)\zeta H_o^{(2)}(\zeta) + O(\text{term})\right]. \qquad (2.38)$$

Since the representation (2.32) holds in region (iii), $|1-s| \gg 1/|\rho|^{2/3}$, so does equation (2.38). Now from section 2.1 we know that equations (2.15) hold in region (iii) provided $|s| \gg 1/|\bar{\rho}|^{2/3}$. Then we can substitute this representation in (2.38) provided $|s| \gg 1/|\bar{\rho}|^{2/3}$ and since $|s-1| \gg 1/|\rho|^{2/3}$ the asymptotic expansions of the Hankel functions may be used in equation (2.38) provided $-\pi < \arg\bar{\zeta} < \pi$, $-2\pi < \arg\bar{\zeta} < \pi$. Consider then $\arg s = 0$, from the nature of the applications $\arg\rho = \arg\ell = \pi/2$ implying $\arg\zeta = 0$. But then $\arg\bar{\rho} = 0$ implying $\arg\bar{\zeta} = -\pi/2$. Thus for $\arg s = 0$,

$$(-2i\ell)^{1/6} e^{\ell\log\ell/e}\,\bar{\Psi}(s)\bar{\zeta}^{-1/6} e^{-i\bar{\zeta}}$$

$$\sim A\frac{1}{\rho^{1/2}}\Psi(s)\,\zeta\left[(2/\pi\zeta)^{1/2}\frac{e^{i(\zeta-\frac{\pi}{4})}}{2} + (2/\pi\zeta)^{1/2}\frac{e^{-i(\zeta-\frac{\pi}{4})}}{2}\right]$$

$$+ B\frac{(-i)^{1/2}}{\rho^{1/2}} e^{\ell\log-\ell/e}\,\Psi(s)\zeta^{1/2} e^{-i\zeta} \quad .$$

This then immediately implies the values

$$A = \mp 2(\pi/2)^{1/2}(-i)^{1/2} e^{-i\pi/4} e^{\ell\log-\ell/e}$$

$$B = \pm 1$$

Therefore equation (2.38) becomes

$$W_{\ell,0}(4\ell s) = \mp \frac{(-i)^{1/2} e^{-i\pi/4}}{\rho^{1/2}} (\pi/2)^{1/2} D_\ell \left[\Psi(s)\zeta H_0^{(1)}(\zeta) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N \qquad (2.39a)$$

$$W_{\ell,0}(4\ell s) = \mp \frac{(-i)^{1/2} e^{-i\pi/4}}{\rho^{1/2}} (\pi/2)^{1/2} D_\ell \left[\Psi(s)\zeta H_0^{(1)}(\zeta) + \frac{\Psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N \qquad (2.39b)$$

The question of sign can be resolved by again considering $\arg s = 0$. Then we have $\arg \ell = \pi/2$, thus we can compare equations (2.39) with the result in Erdélyi and Swanson (1957). Therefore we see that the negative sign must be used. Since $-(-i)^{1/2} e^{-i\pi/4} = (i)^{1/2} e^{i\pi/4}$ we obtain

$$W_{\ell,0}(4\ell s) = \frac{(i)^{1/2}}{\rho^{1/2}} D_\ell \left[V^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N \qquad (2.40a)$$

$$W_{\ell,0}(4\ell s) = \frac{(i)^{1/2}}{\rho^{1/2}} D_\ell \left[V^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N \qquad (2.40b)$$

As previously for the case $|\rho s| \ll 1$ we can use expansions (2.26) through (2.28) to find for $|\zeta| \leqslant N$

$$W_{\ell,0}(4\ell s) = i(\pi)^{1/2} D_\ell (2\ell s)^{1/2} H_0^{(1)}(\zeta) \left[1 + O(s) + O(1/\rho) \right]$$

except at a zero of $H_0^{(1)}(\zeta)$ (there are no zeros of $H_0^{(1)}(\zeta)$ on the principal branch, Erdélyi et al, 1953) when the addition term in (2.40a) is the required estimate. Upon comparing order terms this becomes

$$W_{\ell,0}(4\ell s) = i(\pi)^{1/2} D_\ell (2\ell s)^{1/2} H_0^{(1)}(\zeta) \left[1 + O(1/\rho) \right] \qquad (2.41a)$$

Similarly for $|\zeta| > N$ (with the above exception for zeros of $H_0^{(1)}(\zeta)$)

$$W_{\ell,0}(4\ell s) = i(\pi)^{1/2} D_\ell (2\ell s)^{1/2} H_0^{(1)} \left[1 + O(1/\rho)\right] \tag{2.41b}$$

Case 3: $\arg s \in [-\pi, -\pi+\delta]$

For this case, we consider ρ to have been defined by $\rho = -2\ell$. Then equation (2.20) remains the same together with the resulting definitions and expansions. We let $\arg s = -\pi + \alpha \in [-\pi, -\pi+\delta]$. From the nature of the applications $\arg\rho = \pi/2 - \alpha$, and thus as $s \to \infty$ $\arg\zeta \to 0$. The asymptotic expansions for $H^{(j)}(s)$ are valid as $s \to \infty$ and from equation (2.30b) we see that $V^{(2)}(s)$ has the correct exponential dependence in s for $W_{\ell,0}(4\ell s)$, $V^{(1)}(s)$ the same for $W_{-\ell,0}(-4\ell s)$. Therefore according to Langer (1935)

$$W_{-\ell,0}(-4\ell s) = E_1 \left[V^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N \tag{2.42a}$$

$$W_{-\ell,0}(-4\ell s) = E_1 \left[V^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho}\right] \qquad |\zeta| > N \tag{2.42b}$$

$$W_{\ell,0}(4\ell s) = E_2 \left[V^{(2)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N \tag{2.43a}$$

$$W_{\ell,0}(4\ell s) = E_2 \left[V^{(2)}(s) + \frac{\Psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho}\right] \qquad |\zeta| \geqslant N \tag{2.43b}$$

where N is a large positive number and the E_j $j = 1, 2$ are determined by the relations

$$E_1 = \lim_{s\to\infty} \frac{W_{-\ell,0}(-4\ell s)}{V^{(1)}(s)} = \frac{(-4\ell s)^{-\ell} e^{2\ell s}}{\Psi(s)\zeta^{1/2} e^{i\zeta}}$$

$$E_2 = \lim_{s\to\infty} \frac{W_{\ell,0}(4\ell s)}{V^{(2)}(s)} = \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Psi(s)\zeta^{1/2} e^{-i\zeta}}$$

Using the definitions (2.21) and the expansions (2.26) and (2.27) we obtain

$$E_1 = \frac{(i)^{1/2}}{(-2\ell)^{1/2}} e^{-\ell\log\ell/e}, \qquad E_2 = \frac{(i)^{1/2}}{(-2\ell)^{1/2}} e^{\ell\log -\ell/e}$$

But for $\arg s = -\pi + \alpha$, $\arg\ell = -\pi/2 - \alpha$, hence $(-\ell) = e^{i\pi}\ell$. Therefore,

$$E_1 = \frac{1}{(2i\ell)^{1/2}} D_{-\ell}, \qquad E_2 = \frac{1}{(2i\ell)^{1/2}} D_\ell$$

and equations (2.42) and (2.43) become

$$W_{-\ell,0}(-4\ell s) = \frac{1}{(2i\ell)^{1/2}} D_{-\ell}\left[V^{(1)}(s) + \frac{\psi(s)\,\zeta \log \zeta\, O(1)}{\rho}\right] \tag{2.44a}$$

$$W_{-\ell,0}(-4\ell s) = \frac{1}{(2i\ell)^{1/2}} D_{-\ell}\left[V^{(1)}(s) + \frac{\psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho}\right] \tag{2.44b}$$

$$W_{\ell,0}(4\ell s) = \frac{1}{(2i\ell)^{1/2}} D_\ell\left[V^{(2)}(s) + \frac{\psi(s)\zeta \log \zeta\, O(1)}{\rho}\right] \tag{2.45a}$$

$$W_{\ell,0}(4\ell s) = \frac{1}{(2i\ell)^{1/2}} D_\ell\left[V^{(2)}(s) + \frac{\psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho}\right] \tag{2.45b}$$

Having done this case we note that the restriction of $\arg s = \pi$ for s negative real can be lifted from the results of Case 2. To demonstrate this we consider $|s| < 1$, $\arg s \in [-\delta, \delta]$, $\arg s = -\pi$ for s negative real, and show that the result for $W_{\ell,0}(4\ell s)$ is the same as in Case 2. We have the representation (2.32) for $M_{\ell,0}(4\ell s)$ and can show as before that the function

$$W'(s) = \frac{1}{(2i\ell)^{1/2}} D_\ell\left[V^{(2)}(s) + \frac{\psi(s)\,\zeta \log \zeta\, O(1)}{\rho}\right] \qquad |\zeta| \leqslant N$$

$$= \frac{1}{(2i\ell)^{1/2}} D_\ell\left[V^{(2)}(s) + \frac{\psi(s)\zeta^{1/2} e^{-i\zeta} O(1)}{\rho}\right] \qquad |\zeta| > N$$

is a solution of the differential equation (2.20) in this region. But then we must have

$$W_{\ell,0}(4\ell s) = A' M_{\ell,0}(4\ell s) + B' W'(s)$$

and arguing as previously, we find

$$A' = \mp 2(\pi/2)^{1/2} \frac{1}{(i)^{1/2}} e^{-i\frac{\pi}{4}} D_{\ell}$$

$$B' = \pm 1 .$$

This implies

$$W_{\ell,0}(4\ell s) = \mp \frac{e^{-i\frac{\pi}{4}}}{(2i\ell)^{1/2}} (\pi/2)^{1/2} D_{\ell} \left[\psi(s)\zeta H_0^{(1)}(\zeta) + \frac{\psi(s)\zeta \log \zeta \, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N$$

$$W_{\ell,0}(4\ell s) = \mp \frac{e^{-i\frac{\pi}{4}}}{(2i\ell)^{1/2}} (\pi/2)^{1/2} D_{\ell} \left[\psi(s)\zeta H_0^{(1)}(\zeta) + \frac{\psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N$$

whereupon we see that as in Case 2 the negative sign is correct. But since

$$- \frac{e^{-i\frac{\pi}{4}}}{(i)^{1/2}} = (i)^{1/2} e^{i\frac{\pi}{4}}$$

we have

$$W_{\ell,0}(4\ell s) = \frac{(i)^{1/2}}{(2\ell)^{1/2}} D_{\ell} \left[V^{(1)}(s) + \frac{\psi(s)\zeta \log \zeta \, O(1)}{\rho} \right] \qquad |\zeta| \leqslant N$$

$$W_{\ell,0}(4\ell s) = \frac{(i)^{1/2}}{(2\ell)^{1/2}} D_{\ell} \left[V^{(1)}(s) + \frac{\psi(s)\zeta^{1/2} e^{i\zeta} O(1)}{\rho} \right] \qquad |\zeta| > N$$

This agrees with equations (2.40) since there $\rho = 2\ell$ by definition.

III
LOW FREQUENCY (THIN PARABOLOID) SCATTERING

As indicated in section 1.2 the paraboloid of revolution may be characterized by the focal length η_o. Then for a given wave number k, the mathematical condition $k\eta_o \ll 1$ corresponds either to small k (low frequency scattering) or small η_o (thin paraboloid). We wish to investigate the integral representations in this case. For convenience these representations are now given a slightly different form. Let us define

$$v_1(x,\lambda) = m^{(0)}_{\lambda/2ik}(2ikx) = (2ikx)^{-1/2} M_{\lambda/2ik,\,0}(2ikx) = (2ik)^{-1/2} y_1(x,\lambda)$$

$$v_2(x,\lambda) = w^{(0)}_{\lambda/2ik}(2ikx) = (2ikx)^{-1/2} W_{\lambda/2ik,\,0}(2ikx) = (2ik)^{-1/2} y_2(x,\lambda)$$

and

$$v_1'(x_o,\lambda) = \left(\frac{d}{d(2ikx)} v_1(x,\lambda)\right)_{x=x_o} .$$

(This notation is different from that of Chapter 1 and Appendix A.3 where the primes simply refer to differentiation with respect to x.) Then equations (1.12) through (1.15) become

$$v_N(\xi,\eta) = \frac{1}{2\pi i}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)} \cdot$$
$$(0<\sigma<k)$$
$$\cdot\left[v_2(\eta,\lambda)v_1'(\eta_o,\lambda) - v_1(\eta,\lambda)v_2'(\eta_o,\lambda)\right] \qquad (3.1)$$

$$v_N(\xi,\eta_o) = \frac{1}{2\pi i(2ik\eta_o)}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)} \qquad (3.2)$$
$$(0<\sigma<k)$$

$$v_D(\xi,\eta) = \frac{1}{2\pi i}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1(\eta_o,\lambda)},$$
$$(0<\sigma<k)$$
$$\cdot\left[v_2(\eta,\lambda)v_1(\eta_o,\lambda) - v_1(\eta,\lambda)v_2(\eta_o,\lambda)\right] \qquad (3.3)$$

$$\left(\frac{\partial v_D(\xi,\eta)}{\partial n}\right)_{\eta=\eta_o} = \frac{-1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}}\frac{1}{2\pi i}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1(\eta_o,\lambda)}$$
$$(0<\sigma<k) \qquad (3.4)$$

with the conditions of the derivation $\xi \neq \Xi$, $\eta > 0$ mentioned for completeness.

3.1 Low Frequency (Thin Paraboloid) Poles

The low frequency poles are those of the integrands of equation (3.1) through (3.4) which lend themselves to asymptotic analysis only when $k\eta_o \ll 1$. Hence they correspond to the zeros of the functions $v_1'(\eta_o,\lambda)$ and $v_1(\eta_o,\lambda)$ when $k\eta_o \ll 1$. In order to analyze these zeros we consider $M_{\lambda/2ik,0}(2ik\eta)$ when $k\eta \ll 1$. If $\lambda = 0$ then (Buchholz, 1953)

$$M_{\lambda/2ik,0}(2ik\eta) = M_{0,0}(2ik\eta) = (2ik\eta)^{1/2} J_o(k\eta)$$

and thus $v_1(\eta,0) = J_o(k\eta) = 1 + O\left((k\eta)^2\right)$. Hence for $k\eta_o \ll 1$, $\lambda = 0$ is not a zero of $v_1(\eta_o,\lambda)$. But $v_1'(\eta,0) = O(k\eta)$; therefore for $k\eta_o \ll 1, \lambda = 0$ is a zero of $v_1'(\eta_o,\lambda)$. Suppose now $0 < |\lambda/2k| \leqslant O(1)$, then $M_{\lambda/2ik,0}(2ik\eta)$ has the power series expansion (Buchholz, 1953)

$$M_{\lambda/2ik,0}(2ik\eta) = (2ik\eta)^{1/2}\left[1 + O(k\eta)\right]$$

which implies $v_1(\eta,\lambda) = \left[1 + O(k\eta)\right]$ and $v_1'(\eta,\lambda) = O(1)$. Then $v_1(\eta_o,\lambda) = O(1)$ and $v_1'(\eta_o,\lambda) = O(1)$ indicating that there are no zeros for $0 < |\lambda/2k| \leqslant O(1)$.

To investigate the zeros for $|\lambda/2k| \gg 1$ or $|\lambda/k| \gg 1$ we can use the theory developed in Chapter 2. For the Whittaker equation

$$\frac{d^2u}{d(2ik\eta)^2} + \left(-\frac{1}{4} + \frac{\lambda}{2ik(2ik\eta)} + \frac{1}{4(2ik\eta)^2}\right) u = 0$$

$\ell = \lambda/2ik$, $z = 2ik\eta$ and thus $s = \dfrac{2ik\eta}{4\lambda/2ik} = -k^2\eta/\lambda = -k\eta(k/\lambda)$ where $|s| = k\eta|k/\lambda| \ll 1$.

Now for δ and β small positive numbers the results of section 2.2 for $M_{\ell,0}(4\ell s)$ can be applied in the domains defined by

$$|s| \leqslant \beta, \quad \arg s \in [\pi-\delta, \pi] \quad \text{or} \quad \arg s \in [-\delta, \delta]$$

$$|s| \leqslant \beta, \quad \arg s \in [-\pi . -\pi+\delta] \quad \text{or} \quad \arg s \in [-\delta, \delta].$$

But since the zeros of $M_{\lambda/2ik,0}(2ik\eta)$ occur for real λ, $s = -k^2\eta/\lambda$ lies in one of the above domains. In particular $\left|\frac{\lambda}{2k} s\right| = k\eta/2 \ll 1$ and we can use equations (2.33). Therefore, we have

$$M_{\lambda/2ik,0}(2ik\eta) = (2ik\eta)^{1/2} J_0(\zeta)\left[1 + O\left((k\eta)\frac{k}{\lambda}\right)\right] \qquad |\zeta| \leqslant N$$

$$M_{\lambda/2ik,0}(2ik\eta) = (2ik\eta)^{1/2} J_0(\zeta)\left[1 + O(k/\lambda)\right] \qquad |\zeta| > N$$

where ζ is given by equation (2.26a). In order to use this equation we must specify $\arg\lambda = \pi$ for λ real and negative. Then we can write

$$\zeta = 2\lambda^{1/2}\eta^{1/2}\left[1 + \frac{(k\eta)}{6}\frac{k}{\lambda} + O\left((k\eta)^2\frac{k^2}{\lambda^2}\right)\right] \tag{3.5}$$

It immediately follows that

$$v_1(\eta, \lambda) = J_0(\zeta)\left[1 + O\left((k\eta)\frac{k}{\lambda}\right)\right] \qquad |\zeta| \leqslant N \tag{3.6a}$$

$$v_1(\eta, \lambda) = J_0(\zeta)\left[1 + O(k/\lambda)\right] \qquad |\zeta| > N \tag{3.6b}$$

from which

$$v_1'(\eta,\lambda) = J_o'(\zeta)\,\frac{d\zeta}{d(2ik\eta)}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] + J_o(\zeta)\,O(k/\lambda) \qquad |\zeta| \leqslant N$$

$$v_1'(\eta,\lambda) = J_o'(\zeta)\,\frac{d\zeta}{d(2ik\eta)}\left[1+O(k/\lambda)\right] + J_o(\zeta)\,O(k/\lambda) \qquad |\zeta| > N\,.$$

The second term for $|\zeta| > N$ is present since the order term $O\left((k\eta)\frac{k}{\lambda}\right)$ also occurs in equation (3.6b) but is not written explicitly since it is of lower order than $O(k/\lambda)$. Using equation (3.5) we obtain

$$v_1'(\eta,\lambda) = \frac{\lambda^{1/2}}{2ik\eta^{1/2}}\,J_o'(\zeta)\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] + J_o(\zeta)\,O(k/\lambda) \qquad |\zeta| \leqslant N \qquad (3.7a)$$

$$v_1'(\eta,\lambda) = \frac{\lambda^{1/2}}{2ik\eta^{1/2}}\,J_o'(\zeta)\left[1+O(k/\lambda)\right] + J_o(\zeta)\,O(k/\lambda) \qquad |\zeta| > N \qquad (3.7b)$$

Let ζ_r, $r = 1, 2, 3, \ldots$, $r = -1, -2, -3, \ldots$ denote respectively the positive and negative zeros of $J_o'(\zeta)$. Then the zeros of the function $v_1'(\eta,\lambda)$ are given by the equation

$$\zeta = \zeta_r + O\left((k\eta)^{1/2}\,\frac{k^{3/2}}{\lambda^{3/2}}\right) \qquad (3.8)$$

for arbitrary ζ. Since we are explicitly interested in the zeros λ_r in the complex λ-plane, we must solve equation (3.8) for λ_r. Substituting for ζ we obtain the equation

$$2\lambda^{1/2}\eta^{1/2}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] = \zeta_r + O\left((k\eta)^{1/2}\,\frac{k^{3/2}}{\lambda^{3/2}}\right)$$

which has the solution

$$\lambda_r = \frac{\zeta_r^2}{4\eta}\left[1+O\left(\frac{(k\eta)^2}{\zeta_r^2}\right)\right] \qquad (3.9)$$

But since $\zeta_r = -\zeta_{-r}$ we need only consider $r = 1, 2, 3, \ldots$. Then $k\eta \ll 1$ implies

$\lambda_r/k \gg 1$ since

$$\frac{\lambda_r}{k} \sim \frac{\zeta_r^2}{4k\eta} \geqslant \frac{\zeta_1^2}{4k\eta} \gg 1$$

because $\zeta_1 \simeq 3.832$. This demonstrates consistency with the assumption of $|\lambda/k| \gg 1$ and therefore $v_1'(\eta_o, \lambda)$ has positive zeros given by the equation

$$\lambda_r = \frac{\zeta_r^2}{4\eta_o}\left[1 + O\left(\frac{(k\eta_o)^2}{\zeta_r^2}\right)\right] \qquad r = 1, 2, 3, \ldots \tag{3.9a}$$

To find the zeros of $v_1(\eta_o, \lambda)$ we need, in addition to equation (3.6), the estimate

$$O\left((k\eta)^{1/2}(k\eta)^2 \frac{k^2}{\lambda^2}\right)$$

of $v_1(\eta, \lambda)$ at a zero of $J_o(\zeta)$ (Chapter 2). Thus if β_r $r = 1, 2, 3, \ldots,$ $r = -1, -2, -3, \ldots$ denotes respectively the positive and negative zeros of $J_o(\zeta)$, the zeros of $v_1(\eta, \lambda)$ are given by the equation

$$\zeta = \beta_r + O\left((k\eta)^{1/2}(k\eta)^2 \frac{k^2}{\lambda^2}\right) \tag{3.10}$$

for arbitrary ζ. Substituting for ζ we obtain the equation

$$2\lambda^{1/2}\eta^{1/2}\left[1 + O\left((k\eta)\frac{k}{\lambda}\right)\right] = \beta_r + O\left((k\eta)^{1/2}(k\eta)^2 \frac{k^2}{\lambda^2}\right)$$

which has the solution

$$\lambda_r = \frac{\beta_r^2}{4\eta}\left[1 + O\left(\frac{(k\eta)^2}{\beta_r^2}\right)\right] \quad . \tag{3.11}$$

Again since $\beta_r = -\beta_{-r}$ we need only consider $r = 1, 2, 3, \ldots$. Also $k\eta \ll 1$ implies

$\lambda_r/k \gg 1$ since

$$\frac{\lambda_r}{k} \sim \frac{\beta_r^2}{4k\eta} \geqslant \frac{\beta_1^2}{4k\eta} \gg 1$$

because $\beta_1 \simeq 2.405$. Therefore we again have consistency with the assumption of $|\lambda/k| \gg 1$ and $v_1(\eta_o, \lambda)$ has positive zeros given by

$$\lambda_r = \frac{\beta_r^2}{4\eta_o}\left[1 + O\left(\frac{(k\eta_o)^2}{\beta_r^2}\right)\right] \qquad r = 1, 2, 3, \ldots . \tag{3.11a}$$

From $\cos(\zeta - \frac{\pi}{4})$ which governs the zeros of $J_o(\zeta)$ for large ζ we note that the zeros given by equation (3.9a) for large ζ_r go into the zeros given in Appendix A.3. That this limiting relationship exists follows from the fact that $s = -\frac{k^2\eta}{\lambda} = -k\eta(k/\lambda)$ can go to zero for any $k\eta$ provided k/λ is small enough (λ/k is large enough). Then although $\left|\frac{\lambda}{k} s\right| = k\eta$ is not small the asymptotic representations (equations (2.32)) and expansions (equations (2.26) through (2.28)) still apply. Since in this case ζ is large, $\cos(\zeta - \frac{\pi}{4})$ governs the zeros.

3.2 Residue Series for the Near Field

Let $\lambda_o = 0$ and λ_r $r = 1, 2, 3, \ldots$ denote the positive zeros of $v_1'(\eta_o, \lambda)$. Then we have shown in appendix A.3 that equation (3.1) can be written as

$$v_N(\xi, \eta) = \sum_{r=0}^{\infty} \Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1, -\lambda_r) v_2(\xi_2, -\lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\lambda_r}} \cdot$$

$$\cdot \left[v_2(\eta, \lambda_r) v_1'(\eta_o, \lambda_r) - v_1(\eta, \lambda_r) v_2'(\eta_o, \lambda_r)\right]$$

which upon separating the $r = 0$ term reduces to

$$v_N(\xi,\eta) = \frac{\Gamma(1/2)\Gamma(1/2)\, v_1(\xi_1,0)\, v_2(\xi_2,0)}{\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=0}} \left[v_2(\eta,0)\, v_1'(\eta_o,0) - v_1(\eta,0)\, v_2'(\eta_o,0)\right]$$

$$+ \frac{1}{2ik\eta_o} \sum_{r=1}^{\infty} \Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1,-\lambda_r)\, v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r}} \frac{v_1(\eta,\lambda_r)}{v_1(\eta_o,\lambda_r)} \tag{3.12}$$

We make this separation since although $v_1'(\eta_o,0) \sim 0$, $v_2(\eta,0)$ and $v_2'(\eta_o,0) \sim \infty$ and thus the ratio

$$\frac{1}{\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=0}} \left[v_2(\eta,0)\, v_1'(\eta_o,0) - v_1(\eta,0)\, v_2'(\eta_o,0)\right]$$

must be carefully evaluated.

In order to compute the derivative in the $r = 0$ term we need the definition (Buchholz, 1953)

$$z^{-1/2} M_{\chi,0}(z) = e^{-z/2}\, {}_1F_1(\tfrac{1}{2}-\chi; 1; z)$$

and thus

$$m_{\chi,0}(z) = z^{-1/2} M_{\chi,0}(x) = e^{-z/2} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}-\chi)_r\, z^r}{(1)_r\, r!}$$

If we consider small z we can write

$$m_{\chi,0}(z) = \left[1 - \frac{z}{2} + \frac{z^2}{4} + O(z^3)\right]\left[1 + (\tfrac{1}{2}-\chi)z + \frac{(\frac{1}{2}-\chi)(\frac{1}{2}-\chi+1)}{4}\, z^2 + O(z^3)\right]$$

$$m_{\chi,0}(z) = 1 - \frac{z}{2} + (\tfrac{1}{2}-\chi)z + \frac{z^2}{4} - (\tfrac{1}{2}-\chi)\frac{z^2}{2} + \frac{(\frac{1}{2}-\chi)(\frac{1}{2}-\chi+1)}{4}\, z^2 + O(z^3)$$

Hence

$$\frac{d}{dz}\, m_{\chi,0}(z) = -\chi + \frac{z}{2} - (\frac{1}{2} - \chi)z + (\frac{1}{2} - \chi)(\frac{3}{2} - \chi)\frac{z}{2} + O(z^2)$$

and

$$\frac{d}{d\chi}\frac{d}{dz}\, m_{\chi,0}(z) = -1 + \chi z + O(z^2) \;.$$

Using these forms with $\chi = \frac{\lambda}{2ik}$ and $z = 2ik\eta$ we have

$$\frac{d}{d\lambda}\, v_1'(\eta,\lambda) = -\frac{1}{2ik} + \lambda\eta + O\left((k\eta)^2\right)$$

and

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=0} = -\frac{1}{2ik} + O\left((k\eta_o)^2\right) \;.$$

The other members of the $r = 0$ term are given by (Buchholz, 1953)

$$v_1(\xi_1, 0) = J_o(k\xi_1)$$

$$v_2(\xi_2, 0) = -\frac{i\pi^{1/2}}{2}\, H_o^{(2)}(k\xi_2)$$

$$v_1(\eta, 0) = J_o(k\eta)$$

$$v_1'(\eta_o, 0) = -\frac{k\eta_o}{4i}\left[1 + O\left((k\eta_o)^2\right)\right]$$

$$v_2(\eta, 0) = -\frac{i\pi^{1/2}}{2}\, H_o^{(2)}(k\eta) = -\frac{1}{\pi^{1/2}}\ln\frac{k\eta}{2} + O(1) + \frac{(k\eta)^2}{4\pi^{1/2}}\ln\frac{k\eta}{2} + O\left((k\eta)^2\right)$$

$$v_2'(\eta_o, 0) = -\frac{1}{2ik\eta_o\pi^{1/2}} + \frac{k\eta_o}{4i\pi^{1/2}}\ln\frac{k\eta_o}{2} + O(k\eta_o)$$

Therefore equation (3.12) becomes

$$v_N(\xi,\eta) = \frac{i\pi}{2\eta_o} J_o(k\xi_1)H_o^{(2)}(k\xi_2)\left[1+O\left((k\eta_o)^2\right)\right]$$

$$+\frac{1}{2ik\eta_o}\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\;\frac{v_1(\eta,\lambda_r)}{v_1(\eta_o,\lambda_r)}\;. \tag{3.13}$$

To compute the η members of the r^{th} $(r \geqslant 1)$ term we use the asymptotic representations obtained in Section 3.1.

$$v_1(\eta,\lambda_r) = J_o(\zeta_\eta^{(r)})\left[1+O\left(\frac{(k\eta)^2}{\zeta_r^2}\right)\right] \qquad \left|\zeta_\eta^{(r)}\right| \leqslant N \tag{3.6a}$$

$$v_1(\eta,\lambda_r) = J_o(\zeta_\eta^{(r)})\left[1+O\left(\frac{k\eta}{\zeta_r^2}\right)\right] \qquad \left|\zeta_\eta^{(r)}\right| > N \tag{3.6b}$$

$$v_1'(\eta_o,\lambda) = \frac{\lambda^{1/2}}{2ik\eta_o^{1/2}} J_o'(\zeta_{\eta_o})\left[1+O\left((k\eta_o)\frac{k}{\lambda}\right)\right]+J_o(\zeta_{\eta_o})O(k/\lambda) \qquad \left|\zeta_{\eta_o}\right| \leqslant N \tag{3.7a}$$

$$v_1'(\eta_o,\lambda) = \frac{\lambda^{1/2}}{2ik\eta_o^{1/2}} J_o'(\zeta_{\eta_o})\left[1+O(k/\lambda)\right]+J_o(\zeta_{\eta_o})O(k/\lambda) \qquad \left|\zeta_{\eta_o}\right| > N \tag{3.7a}$$

where

$$\zeta_{\eta_o} = 2\lambda^{1/2}\eta_o^{1/2}\left[1+O\left((k\eta_o)\frac{k}{\lambda}\right)\right]$$

$$\zeta_\eta^{(r)} = 2\lambda_r^{1/2}\eta^{1/2}\left[1+O\left((k\eta)\frac{k}{\lambda_r}\right)\right].$$

Thus

$$\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{1}{2ik} J_o''(\zeta_{\eta_o}^{(r)})\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r^2}\right)\right] \qquad \left|\zeta_{\eta_o}^{(r)}\right| \leqslant N$$

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{1}{2ik} J_o''(\zeta_{\eta_o}^{(r)}) \left[1+O\left(\frac{k\eta_o}{\zeta_r^2}\right)\right] \qquad \left|\zeta_{\eta_o}^{(r)}\right| > N$$

and using Bessel's differential equation

$$J_o''(\zeta_{\eta_o}^{(r)}) + \frac{1}{\zeta_{\eta_o}^{(r)}} J_o'(\zeta_{\eta_o}^{(r)}) + J_o(\zeta_{\eta_o}^{(r)}) = 0$$

these derivatives become , since $\zeta_{\eta_o}^{(r)} = \zeta_r$ the positive zeros of $J_o'(\zeta)$,

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = -\frac{1}{2ik} J_o(\zeta_r)\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r^2}\right)\right] \qquad |\zeta_r| \leqslant N \qquad (3.14a)$$

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = -\frac{1}{2ik} J_o(\zeta_r)\left[1+O\left(\frac{(k\eta_o)}{\zeta_r^2}\right)\right] \qquad |\zeta_r| \gtrsim N \qquad (3.14b)$$

In order to further evaluate (3.13) we must use the mathematical conditions of the near field. Since the near field corresponds to the physical problem of both the source and field points near (with respect to wavelength) the origin, these conditions are $k\Xi \ll 1$, $k\xi \ll 1$. Then we have

$$J_o(k\xi_1) = 1 + O\left((k\xi_1)^2\right) \qquad (3.15a)$$

$$H_o^{(2)}(k\xi_2) = -\frac{1}{\pi^{1/2}} \ln\frac{k\xi_2}{2} - \frac{1}{\pi^{1/2}}(\gamma - 1) + O\left((k\xi_2)^2 \ln\frac{k\xi_2}{2}\right) \qquad (3.15b)$$

where γ is the Euler constant. We now find $v_1(\xi_1,-\lambda_r)$ and $v_2(\xi_2,-\lambda_r)$ for $r \gg 1$, $k\xi_i \ll 1$, $i=1,2$. It suffices to obtain $M_{-\lambda_r/2ik,0}(2ik\xi_1)$ and $W_{-\lambda_r/2ik,0}(2ik\xi_2)$. These functions are solutions of the equation

$$\frac{d^2u}{d(2ik\xi_i)^2} + \left(-\frac{1}{4} - \frac{\lambda_r}{2ik(2ik\xi_i)} + \frac{1}{4(2ik\xi_i)^2}\right) u = 0 \qquad i = 1, 2$$

which upon the substitutions

$$s = \frac{2ik\xi_i}{-4\frac{\lambda_r}{2ik}} = \frac{k^2\xi_i}{\lambda_r} , \quad \ell = -\frac{\lambda_r}{2ik}$$

reduces to

$$\frac{d^2u}{ds^2} + \left[4\ell^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2}\right] u = 0 .$$

Since equation (3.9a) implies s positive and

$$s = \frac{k^2\xi_i}{\lambda_r} \simeq \frac{4(k\xi_i)}{\zeta_r^2}(k\eta_o) \ll 1$$

we can apply the results of section 2.2 (equations (2.33) and (2.41)) to obtain

$$M_{-\lambda_r/2ik,\,0}(2ik\xi_1) = (2ik\xi_1)^{1/2} J_o(\zeta^{(r)}_{\xi_1}) \left[1 + O\left(\frac{(k\xi_1)^2}{\zeta_r^2}\right)\right] \qquad \left|\zeta^{(r)}_{\xi_1}\right| \lessdot N$$

$$M_{-\lambda_r/2ik,\,0}(2ik\xi_1) = (2ik\xi_1)^{1/2} J_o(\zeta^{(r)}_{\xi_1}) \left[1 + O\left(\frac{k\xi_1}{\zeta_r^2}\right)\right] \qquad \left|\zeta^{(r)}_{\xi_1}\right| > N$$

$$W_{-\lambda_r/2ik,\,0}(2ik\xi_2) = \frac{i\pi^{1/2}}{2^{1/2}} D_{-\lambda_r/2ik}(2ik\xi_2)^{1/2} H_o^{(1)}(\zeta^{(r)}_{\xi_2}) \left[1 + O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right]$$

As before,

$$D_{-\lambda_r/2ik} = \exp\left\{-\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\}$$

and from equation (2.26a) we have

$$\zeta^{(r)}_{\xi_1} = 2i\lambda_r^{1/2}\xi_1^{1/2}\left[1 + O\left(\frac{(k\xi_1)(k\eta_o)}{\zeta_r^2}\right)\right] .$$

Hence

$$v_1(\xi_1, -\lambda_r) = J_o(\zeta^{(r)}_{\xi_1})\left[1 + O\left(\frac{(k\xi_1)^2}{\zeta_r^2}\right)\right] \tag{3.16a}$$

$$v_1(\xi_1, -\lambda_r) = J_o(\zeta^{(r)}_{\xi_2})\left[1 + O\left(\frac{k\xi_1}{\zeta_r^2}\right)\right] \tag{3.16b}$$

$$v_2(\xi_2, -\lambda_r) = \frac{i\pi^{1/2}}{2^{1/2}} D_{-\lambda_r/2ik} H_o^{(1)}(\zeta^{(r)}_{\xi_2})\left[1 + O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right] \tag{3.17}$$

Now $\left|\lambda_r/2ik\right| \simeq \zeta_r^2/8k\eta_o \gg 1$, thus from Sterling's formula (Erdélyi, et al, 1953)

$$\Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) = \sqrt{2\pi}\ \exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}\left[1 + O(k/\lambda_r)\right] .$$

Substituting this and equations (3.6), (3.14), (3.15), (3.16) and (3.17) into (3.13) we obtain

$$v_N(\xi, \eta) = -\frac{i\pi^{1/2}}{2\eta_o}\ln k\xi_2 - \frac{i\pi^{1/2}}{2\eta_o}(\gamma - 1) - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o(\zeta^{(r)}_{\xi_1})H_o^{(1)}(\zeta^{(r)}_{\xi_2})J_o(\zeta^{(r)}_{\eta})}{\left[J_o(\zeta_r)\right]^2}\left[1 + O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right] \tag{3.18}$$

where only the largest order term has been retained. Expanding $J_o(\zeta^{(r)}_{\xi_1})$ and $H_o^{(1)}(\zeta^{(r)}_{\xi_2})$

about $\zeta^{(r)}_{\xi_i}(0) = 2i\,\lambda_r^{1/2}(0)\xi_i^{1/2}$ $i=1,2$, $\lambda_r(0) = \zeta_r^2/4\eta_o$; expanding $J_o(\zeta^{(r)}_\eta)$ about $\zeta^{(r)}_\eta(0) = 2\lambda_r^{1/2}(0)\,\eta^{1/2}$ and substituting into equation (3.18) we find

$$v_N(\xi,\eta) = -\frac{i\pi^{1/2}}{2\eta_o}\ln k\xi_2 - \frac{i\pi^{1/2}}{2\eta_o}(\gamma-1) - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta^{(r)}_{\xi_1}(0)\right)H_o^{(1)}\left(\zeta^{(r)}_{\xi_2}(0)\right)J_o\left(\zeta^{(r)}_\eta(0)\right)}{\left[J_o(\zeta_r)\right]^2}\cdot$$

$$\cdot\left[1+O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right] \tag{3.19}$$

where again only the largest order term has been retained.

The solution for the Dirichlet problem is found in much the same fashion as above beginning with equation (3.3). If $\lambda_1, \lambda_2, \ldots \lambda_r, \ldots$ denote the positive zeros of $v_1(\eta_o,\lambda)$ (given by equation (3.11a)), then equation (3.3) may be written as

$$v_D(\xi,\eta) = \sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\left[-v_1(\eta,\lambda_r)v_2(\eta_o,\lambda_r)\right]$$

and on using the Wronskian relation this becomes

$$v_D(\xi,\eta) = \frac{1}{(2ik\eta_o)}\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\left[-\frac{v_1(\eta,\lambda_r)}{v_1'(\eta_o,\lambda_r)}\right]. \tag{3.20}$$

In order to write equation (3.20) when $k\xi_i \ll 1$, $i=1,2$ we need only find $\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}$; the remainder of the asymptotic representations have been given previously (equations (3.6), (3.7), (3.16) and (3.17)). From equations (3.6)

$$\frac{d}{d\lambda} v_1(\eta_o,\lambda) = J_o'(\zeta_{\eta_o}) \frac{\eta_o^{1/2}}{\lambda^{1/2}}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] \qquad \left|\zeta_{\eta_o}\right| \leqslant N$$

$$\frac{d}{d\lambda} v_1(\eta_o,\lambda) = J_o'(\zeta_{\eta_o}) \frac{\eta_o^{1/2}}{\lambda^{1/2}}\left[1+O(k/\lambda)\right] \qquad \left|\zeta_{\eta_o}\right| > N .$$

Thus

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{2\eta_o}{\beta_r} J_o'(\beta_r)\left[1+O\left(\frac{(k\eta_o)^2}{\beta_r^2}\right)\right] \qquad |\beta_r| \leqslant N \qquad (3.21a)$$

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{2\eta_o}{\beta_r} J_o'(\beta_r)\left[1+O\left(\frac{k\eta_o}{\beta_r^2}\right)\right] \qquad |\beta_r| > N \qquad (3.21b)$$

Substituting (3.21), the equations mentioned above, and the asymptotic form of $\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)$ into (3.20), we obtain upon expanding as before

$$v_D(\xi,\eta) = -\frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_{\xi_1}^{(r)}(0)\right) H_o^{(1)}\left(\zeta_{\xi_2}^{(r)}(0)\right) J_o\left(\zeta_{\eta}^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2}\left[1+O\left(\frac{k\xi_2}{\beta_r^2}\right)\right] \qquad (3.22)$$

where $\zeta_{\xi_i}^{(r)}(0) = 2i\lambda_r^{1/2}(0)\xi_i^{1/2}$ $i=1,2$, $\lambda_r(0) = \beta_r^2/4\eta_o$, $\zeta_{\eta}^{(r)}(0) = 2\lambda_r^{1/2}(0)\eta^{1/2}$, and only the largest order term has been retained.

3.3 Dirichlet Potential Problem

It can be shown that the term in equation (3.22) which is independent of k agrees with the corresponding Dirichlet potential problem. For the paraboloid of revolution coordinate system defined by

$$x = \lambda\omega\cos\phi , \qquad y = \lambda\omega\sin\phi , \qquad z = \frac{1}{2}(\lambda^2-\omega^2) \qquad \left(r = \frac{1}{2}(\lambda^2+\omega^2)\right)$$

Morse and Feshbach (1953, chapter 10) give an integral representation for 1/R. For the source at the point $(\lambda=\lambda_o,\ \omega=0)$ this reduces to

$$\frac{1}{R} = i\pi \int_0^{\infty} J_o(t\lambda) J_o(t\lambda_o) H_o^{(1)}(it\omega)\, t dt \qquad \omega > 0 .$$

Then if the substitutions $\lambda = \sqrt{2\xi}$, $\omega = \sqrt{2\eta}$, $\lambda_o = \sqrt{2\Xi}$ are made, the above coordinate system reduces to the one which we are considering and the above integral representation for $1/R$ becomes

$$\frac{1}{R} = i\pi \int_0^{\infty} J_o(t\sqrt{2\xi_1}) J_o(t\sqrt{2\xi_2}) H_o^{(1)}(it\sqrt{2\eta})\, t dt \qquad \eta > 0 \qquad (3.23)$$

where we have used $\xi_1 = \min(\xi, \Xi)$, $\xi_2 = \max(\xi, \Xi)$.

The potential problem corresponding to the Dirichlet problem postulated in Chapter 1 is the solution $\phi_D(\xi, \eta)$ of the equation

$$\nabla^2 \phi = 4\pi \delta(\underline{r} - \underline{r}_o) \qquad (\underline{r}_o \text{ is the vector to the point } (\Xi, 0))$$

together with the boundary condition

$$\phi = 0 \qquad \text{on the boundary,}$$

and the condition at infinity

$$\phi = O\left(\frac{1}{|\underline{r} - \underline{r}_o|}\right) \quad \text{as} \quad |\underline{r}| \to \infty .$$

The free space Green's function for the above problem has the form $-1/R$ where $1/R$ is given by (3.23). Then we can assume that $\phi_D(\xi, \eta)$ has the form $\phi_D(\xi, \eta) = \frac{-1}{R} + \phi_s(\xi, \eta)$ where $\phi_s(\xi, \eta)$ is given by

$$\phi_s(\xi, \eta) = i\pi \int_0^{\infty} J_o(t\sqrt{2\xi_1}) J_o(t\sqrt{2\xi_2}) A_t J_o(it\sqrt{2\eta})\, t dt$$

with A_t an unknown function of t. The boundary condition implies

$$A_t = \frac{H_o^{(1)}(it\sqrt{2\eta_o})}{J_o(it\sqrt{2\eta_o})}$$

and hence

$$\phi_D(\xi,\eta) = -i\pi\int_0^\infty t\,dt\,\frac{J_o(t\sqrt{2\xi_1})J_o(t\sqrt{2\xi_2})}{J_o(it\sqrt{2\eta_o})}\left[H_o^{(1)}(it\sqrt{2\eta})J_o(it\sqrt{2\eta_o}) - H_o^{(1)}(it\sqrt{2\eta_o})J_o(it\sqrt{2\eta})\right]$$

is a solution of the inhomogeneous potential equation which satisfies the boundary condition. That $\phi_D(\xi,\eta)$ has the proper behavior at infinity follows from the fact that $1/R$ has the desired behavior and the form of $\phi_s(\xi,\eta)$. Thus we have a solution, in the form of an integral representation, to the potential problem posed above.

If in this integral representation we substitute $t = \sqrt{2v}$, we obtain

$$\phi_D(\xi,\eta) = -2i\pi\int_0^\infty v\,dv\,\frac{J_o(2v\sqrt{\xi_1})J_o(2v\sqrt{\xi_2})}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta})\,J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})\,J_o(2iv\sqrt{\eta})\right] \tag{3.24}$$

In order to analyze equation (3.24) we consider the function

$$\psi(v) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta})\,J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})J_o(2iv\sqrt{\eta})\right]$$

whereupon we note

$$\psi(v\,e^{-\pi i}) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[\left\{2H_o^{(1)}(2iv\sqrt{\eta}) + H_o^{(2)}(2iv\sqrt{\eta})\right\}J_o(2iv\sqrt{\eta_o}) - \left\{2H_o^{(1)}(2iv\sqrt{\eta_o}) + H_o^{(2)}(2iv\sqrt{\eta_o})\right\}J_o(2iv\sqrt{\eta})\right]$$

which reduces to

$$\psi(v\,e^{-\pi i}) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta})\,J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})\,J_o(2iv\sqrt{\eta})\right]$$

or

$$\psi(ve^{-\pi i}) = \psi(v) .$$

Therefore if we write equation (3.24) as

$$\phi_D(\xi,\eta) = -i\pi \int_0^\infty vdv\, J_o(2v\sqrt{\xi_1})\, H_o^{(1)}(2v\sqrt{\xi_2})\psi(v)$$

$$- i\pi \int_0^\infty vdv\, J_o(2v\sqrt{\xi_1})H_o^{(2)}(2v\sqrt{\xi_2})\,\psi(v) ,$$

we can consider the substitution $v = we^{-\pi i}$ in the second integrand to obtain

$$\phi_D(\xi,\eta) = -i\pi \int_0^\infty vdv\, J_o(2v\sqrt{\xi_1})\, H_o^{(1)}(2v\sqrt{\xi_2})\psi(v)$$

$$+ i\pi \int_0^{-\infty} wdw\, J_o(2w\sqrt{\xi_1})\, H_o^{(1)}(2w\sqrt{\xi_2})\,\psi(w) .$$

This reduces to

$$\phi_D(\xi,\eta) = -i\pi \int_{-\infty}^\infty vdv\, J_o(2v\sqrt{\xi_1})H_o^{(1)}(2v\sqrt{\xi_2})\,\psi(v)$$

and thus the equation for the potential becomes

$$\phi_D(\xi,\eta) = -i\pi \int_{-\infty}^\infty vdv\, \frac{J_o(2v\sqrt{\xi_1})\, H_o^{(1)}(2v\sqrt{\xi_2})}{J_o(2iv\sqrt{\eta_o})} \Big[H_o^{(1)}(2iv\sqrt{\eta})\, J_o(2iv\sqrt{\eta_o})$$

$$- H_o^{(1)}(2iv\sqrt{\eta_o})\, J_o(2iv\sqrt{\eta}) \Big] . \tag{3.25}$$

For large $|v|$ in the upper half plane the usual asymptotic representations for $J_o(2v\sqrt{\xi})$, $H_o^{(1)}(2v\sqrt{\xi})$, $H_o^{(1)}(2iv\sqrt{\eta})$ and $J_o(2iv\sqrt{\eta})$ hold, therefore it is straightforward to show that the integrand of equation (3.25) is vanishing exponentially for large $|v|$ in the upper half plane provided $\xi_1 < \xi_2$. Suppose then $\xi_1 < \xi_2$ ($\xi \neq \Xi$) and let β_r $r = 1, 2, \ldots,$ $r = -1, -2, \ldots$ again denote the positive and negative zeros of $J_o(\beta)$. Thus $v_r = i\beta_r/2\sqrt{\eta_o}$ $r = 1, 2, 3, \ldots$ denotes the zeros of $J_o(2iv\sqrt{\eta_o})$ along the positive imaginary axis and hence also the poles of the integrand of equation (3.26) in the upper half plane. Using the residue theorem we obtain

$$\phi_D(\xi, \eta) = 2\pi i(-i\pi) \sum_{r=1}^{\infty} v_r \frac{J_o(2v_r\sqrt{\xi_1})H_o^{(1)}(2v_r\sqrt{\xi_2})}{\left(\frac{d}{dv} J_o(2iv\sqrt{\eta_o})\right)_{v=v_r}} \left[-H_o^{(1)}(2iv_r\sqrt{\eta_o})\, J_o(2iv_r\sqrt{\eta})\right]$$

which upon using the Wronskian relation for the Bessel functions $J_o(2iv\sqrt{\eta_o})$, $H_o^{(1)}(2iv\sqrt{\eta_o})$ reduces to

$$\phi_D(\xi, \eta) = -\frac{i\pi}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o(2v_r\sqrt{\xi_1})\, H_o^{(1)}(2v_r\sqrt{\xi_2})}{\left[J_o'(\beta_r)\right]^2} J_o(2iv_r\sqrt{\eta}) \tag{3.26}$$

Equation (3.26) yields the Dirichlet potential subject to the conditions $\xi \neq \Xi$, $\eta > 0$. These conditions were also assumed in the derivation of equation (3.22). Upon comparing the k independent term in equation (3.22) with the Dirichlet potential of (3.26) we see that they agree.

3.4 Residue Series for the Far Field

Suppose we consider equation (3.13) without any assumptions on $k\xi_i$ $i = 1, 2$. Then using the formulas developed in section 3.2 we can write

$$v_N(\xi, \eta) \sim \frac{i\pi}{2\eta_o} J_o(k\xi_1) H_o^{(2)}(k\xi_2)$$

$$- \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_1(\xi_1, -\lambda_r) v_2(\xi_2, -\lambda_r) \tag{3.27}$$

where the error terms have been omitted for the sake of simplicity. We will continue to omit them throughout most of the section.

The physical situation of interest is the source (field point) in the near field and the field point (source) in the far field. This corresponds to the mathematical conditions $k\xi_1 << 1$, $k\xi_2 >> 1$, thus the behavior of $v_1(\xi_1, -\lambda_r)$ is at once determined by equations (3.16). It also immediately follows that

$$J_o(k\xi_1) \sim 1$$

$$H_o^{(2)}(k\xi_2) \sim \left(\frac{2}{\pi k\xi_2}\right)^{1/2} e^{-ik\xi_2} e^{i\pi/4} .$$

However, we will write equation (3.27) as

$$v_N(\xi, \eta) \sim J_o(k\xi_1) \left[-\frac{1}{\eta_o}\left(\frac{\pi}{2ik\xi_2}\right)^{1/2} e^{-ik\xi_2}\right] - \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} .$$

$$\cdot \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi_2, -\lambda_r) \tag{3.28}$$

so as to retain the dependence on $k\xi_1$ in the first term.

To find $v_2(\xi_2, -\lambda_r)$ we will make use of the methods of Chapter 2 and thus need to determine the order of magnitude of

$$s_{\xi_2} = \frac{k^2\xi_2}{\lambda_r} = \frac{k}{\lambda_r}(k\xi_2) \sim \frac{4k\eta_o}{\zeta_r^2}(k\xi_2) .$$

Since $k\xi_2 >> 1$ there are two different possibilities for $k\xi_2$; they are $k\xi_2 >> 1/k\eta_o$ and $k\xi_2 = O(1/k\eta_o)$. For $k\xi_2 >> 1/k\eta_o$ we can write the series in equation (3.28) as

$$- \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=1}^{M} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi_2, -\lambda_r)$$

$$- \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=M+1}^{N} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi_2, -\lambda_r)$$

$$- \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=N+1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi_2, -\lambda_r) \tag{3.29}$$

where M is such that $r \leqslant M$ implies $\zeta_r^2 = O(1)$ $(s_{\xi_2} \gg 1)$, N is such that $r > N$ implies $\zeta_r^2 \geqslant O(k\xi_2)$ $(s_{\xi_2} \ll 1)$, and $r \epsilon [M+1, N]$ implies $\zeta_r^2 = O\left((k\eta_o)(k\xi_2)\right)$ $\left(s_{\xi_2} = O(1)\right)$. We will refer to the three sums as $\sum_1, \sum_2, \sum_3$. Thus the problem of finding $v_2(\xi_2, -\lambda_r)$ is reduced to finding it for the three sums $\sum_1, \sum_2, \sum_3$.

In order to evaluate $v_2(\xi_2, -\lambda_r)$ for $\sum_1$ we need the expansions (2.9) and (2.10). Then $\bar{\rho} = -2i\ell = -2i(-\lambda_r/2ik) = \lambda_r/k \simeq \zeta_r^2/4k\eta_o$, thus $\bar{\zeta}$ is large with s_{ξ_2} large and so equations (2.15) yield

$$v_2(\xi_2, -\lambda_r) \sim (2ik\xi_2)^{-1/2} e^{-\frac{\lambda_r}{2ik} \log(2ik\xi_2)} e^{-ik\xi_2} \tag{3.30}$$

Hence $\sum_1$ becomes

$$\sum_1 \sim -\frac{1}{\eta_o} \sqrt{\frac{\pi}{ik\xi_2}} \sum_{r=1}^{M} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} \exp\left\{-\frac{\lambda_r}{2ik} \log(2ik\xi_2)\right\} \cdot e^{-ik\xi_2}$$

Evaluating the two exponentials we have

$$\sum_1 \sim -\frac{1}{\eta_o}\sqrt{\frac{\pi}{ik\xi_2}}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{-\frac{\zeta_r^2\pi}{8k\eta_o}\right\}e^{-ik\xi_2}\ .$$

$$\exp\left\{-\frac{i\zeta_r^2}{8k\eta_o}\log\frac{\zeta_r^2}{16(k\xi_2)(k\eta_o)e}\right\}$$

The expansions (2.7) and (2.8) apply to $v_2(\xi_2,-\lambda_r)$ for $\sum_2$. From equations (2.15) we have

$$v_2(\xi_2,-\lambda_r) \sim (2ik\xi_2)^{-1/2}(\lambda_r/k)^{1/6}\exp\left\{-\frac{\lambda_r}{2ik}\log-\frac{\lambda_r}{2ike}\right\}\bar{\Psi}\left(s_{\xi_2}(0)\right)\bar{\zeta}_{\xi_2}^{(r)}(0)\,H_{1/3}^{(2)}\left(\bar{\zeta}_{\xi_2}^{(r)}(0)\right) \tag{3.31}$$

with

$$s_{\xi_2}(0) = \frac{4k\eta_o}{\zeta_r^2}(k\xi_2)$$

and

$$\bar{\zeta}_{\xi_2}^{(r)}(0) = \frac{\lambda_r}{k}\bar{\Phi}\left(s_{\xi_2}(0)\right)\ .$$

Therefore $\sum_2$ becomes

$$\sum_2 \sim -\frac{1}{\eta_o}\sqrt{\frac{\pi}{ik\xi_2}}\sum_{r=M+1}^{N}\left(\frac{\zeta_r^2}{4k\eta_o}\right)^{7/6}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{-\frac{\zeta_r^2\pi}{8k\eta_o}\right\}\cdot$$

$$\cdot\bar{\Psi}\left(s_{\xi_2}(0)\right)\bar{\zeta}_{\xi_2}^{(r)}(0)\,H_{1/3}^{(2)}\left(\bar{\zeta}_{\xi_2}^{(r)}(0)\right) \tag{3.32}$$

In finding $v_2(\xi_2,-\lambda_r)$ for $\sum_3$ we need only use the results of section 3.2, in particular equation (3.17). Then we find for $\sum_3$

$$\sum_3 \sim -\frac{i\pi}{\eta_o} \sum_{r=N+1}^{\infty} \frac{J_o\left(\zeta_{\eta}^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right) H_o^{(1)}\left(\zeta_{\xi_2}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}$$

Since $\zeta_{\xi_1}^{(r)}(0) = i\zeta_r(\xi_1/\eta_o)^{1/2}$ we may write $\sum_3$ as

$$\sum_3 \sim -\frac{i\pi}{\eta_o} \sum_{r=N+1}^{\infty} \frac{J_o\left(\zeta_{\eta}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\zeta_r(\xi_1/\eta_o)^{1/2}\right\} \exp\left\{-\zeta_r(\xi_2/\eta_o)^{1/2}\right\}$$

Thus we can show that $\sum_3$ is vanishing exponentially along with $\sum_1$ and $\sum_2$ by considering the product of the two exponentials. We write it as

$$\frac{e^{\zeta_r} e^{(\xi_1/\eta_o)^{1/2}}}{e^{\zeta_r} e^{(\xi_2/\eta_o)^{1/2}}}$$

which of course reduces to

$$e^{(\xi_1/\eta_o)^{1/2}} \, e^{-(\xi_2/\eta_o)^{1/2}} \quad .$$

But $\xi_1/\eta_o = O(1)$ and so the latter product is

$$O(1)\, e^{-(\xi_2/\eta_o)^{1/2}} \quad .$$

However, $k\xi_2 \gg 1/k\eta_o$ implying $\xi_2/\eta_o \gg 1/k^2\eta_o^2$. Therefore

$$O(1)\, e^{-(\xi_2/\eta_o)^{1/2}} \ll O(1) e^{-1/k\eta_o}$$

and $\sum_3$ is vanishing as asserted.

We have now shown that the contribution from the residue series in equation (3.27) is exponentially vanishing and thus is much smaller than the order terms involved in the approximations for the $r=0$ term. Therefore we should write equation (3.27) as

$$v_N(\xi,\eta) = \frac{i\pi}{2\eta_o} J_o(k\xi_1)H_o^{(2)}(k\xi_2)\left[1 + O\left((k\eta_o)^2\right)\right].$$

Upon using the conditions $k\xi_1 \ll 1$, $k\xi_2 \gg 1$, and $k\xi_2 \gg 1/k\eta_o$ this becomes

$$v_N(\xi,\eta) = -\frac{1}{\eta_o}\left(\frac{\pi}{2ik\xi_2}\right)^{1/2} e^{-ik\xi_2}\left[1 + O(k\xi_1) + O(1/k\xi_2) + O\left((k\eta_o)^2\right)\right] \qquad (3.33)$$

In the case of $k\xi_2 = O(1/k\eta_o)$ we can write the series in equation (3.28) as

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r)$$

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=M+1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r)$$

where M is such that $r \leqslant M$ implies $\zeta_r^2 = O(1)$ $\left(s_{\xi_2} = O(1)\right)$, and for $r > M$ $\zeta_r^2 \gg 1$ $(s_{\xi_2} \ll 1)$. Then the discussion above for $\sum_2$ applies to the first sum here, a similar discussion to the one above for $\sum_3$ applies to the second sum (here we have $O(1)e^{-(\xi_2/\eta_o)^{1/2}} = O(1)e^{-1/k\eta_o}$). Thus both sums are vanishing exponentially, again the contribution from the residue series in equation (3.27) is much smaller than the order terms involved in the approximation from the $r = 0$ term. This time equation (3.27) becomes

$$v_N(\xi,\eta) = -\frac{1}{\eta_o}\left(\frac{\pi}{2ik\xi_2}\right)^{1/2} e^{-ik\xi_2}\left[1 + O(k\xi_1) + O(1/k\xi_2)\right] \qquad (3.34)$$

since $1/k\xi_2 \gg (k\eta_o)^2$.

For the Dirichlet problem we must consider equation (3.20) without any restrictions on $k\xi_i$, $i=1,2$. Using the formulas developed in section 3.2 we have

$$v_D(\xi,\eta) \sim -\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r) \tag{3.35}$$

However, the preceding arguments for the Neumann problem show that this residue series is vanishing exponentially, thus we have the result that

$$v_D(\xi,\eta) \sim 0 .$$

This result is also true for the field on the surface. Writing equation (3.4) as a residue series we find

$$\left(\frac{\partial v_D}{\partial n}(\xi,\eta)\right)_{\eta=\eta_o} = -\frac{1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}}\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}} \tag{3.36}$$

which upon using the results of section 3.2 becomes

$$\left(\frac{\partial v_D}{\partial n}(\xi,\eta)\right)_{\eta=\eta_o} = -\frac{\sqrt{2\pi}}{2\eta_o}\frac{1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}}\sum_{r=1}^{\infty}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)\beta_r}{\left[J_o'(\beta_r)\right]} \tag{3.37}$$

Again the arguments above can be used to show that the residue series is vanishing exponentially and therefore

$$\left(\frac{\partial v_D}{\partial n}(\xi,\eta)\right)_{\eta=\eta_o} \sim 0.$$

IV
HIGH FREQUENCY (FAT PARABOLOID) SCATTERING

For this chapter we will consider only the surface field for the Neumann problem. In addition we will consider the point source to be located anywhere on the axis. Consequently the pertaining integral representations are

$$v_N(\xi,\eta_o) = \frac{1}{2\pi i(2ik\eta_o)} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)} \qquad (4.1)$$
$$(0 < \sigma < k)$$

with $\xi \neq \Xi$, for the point source located at $(\Xi, 0)$ to the right of the focal point, and

$$v_N(\xi,\eta_o) = \frac{1}{2\pi i(2ik\eta_o)} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{v_2(\xi,-\lambda)v_1(H,\lambda)}{v_1'(\eta_o,\lambda)} \qquad (4.2)$$
$$(0 < \sigma < k)$$

with $\eta_o \neq H$, for the point source located at $(0, H)$ to the left of the focal point.

4.1 High Frequency (Fat Paraboloid) Poles

As shown in Chapter III the poles of the above integrands which lead to the residue series readily summable for $k\eta_o \ll 1$ correspond to the zeros of the function $v_1'(\eta_o, \lambda)$. If we wish to consider an analogy with the scattering by closed convex bodies we must now seek an alternative set of poles which, when the geometric term (terms) is (are) removed, lead to a residue series readily summable for $k\eta_o \gg 1$. The other set of poles of the above integrands are the poles of the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ at $\lambda = -ik(2n+1)$, $n = 0, 1, 2, \ldots$ in the lower half plane. However, as seen in Appendix A.3 these poles lend to a residue series only provided $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_o}$ (or $\sqrt{\xi} + \sqrt{H} < \sqrt{\eta_o}$ in the case of equation 4.2). Although this residue series may be analyzed for $k\eta_o \gg 1$, the inequality sharply limits the range of source and field points which may be considered. Therefore we derive an integral representation which can be analyzed for a wider range of source and field points. In performing this analysis we find the poles of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ play only a minor role, while once again the poles of $1/v_1'(\eta_o, \lambda)$ are of paramount importance. Thus we must consider in detail the zeros of $v_1'(\eta_o, \lambda)$ for $k\eta_o \gg 1$.

According to Buchholz (1953) the zeros of $v_1'(\eta_o, \lambda)$ all lie on the real axis to the right of the value $-k^2\eta_o$. In Appendix A.3 we found the positive zeros of $v_1'(\eta_o, \lambda)$ for $\lambda \gg k^2\eta_o$. Thus it remains to consider the three regions:

(i) λ negative, $\lambda \sim O(k^2\eta_o)$

(ii) $|\lambda/k| \leqslant O(1)$

(iii) λ positive, $\lambda \sim O(k^2\eta_o)$

In region (i), $v_1(\eta, \lambda)$ is governed by the Airy function representation while in region (iii) the Bessel function representation determines the behavior of $v_1(\eta, \lambda)$. The explicit representations in these regions are derived from the results of Chapter II. The zeros of $v_1(\eta, \lambda)$ in region (i) give rise to the "whispering gallery" waves and thus can be called the "whispering gallery" poles. In region (ii) we can write the asymptotic representation (Buchholz, 1953)

$$v_1(\eta, \lambda) \sim (2ik\eta)^{-1/2}\left[\frac{(2ik\eta)^{-\frac{\lambda}{2ik}}e^{ik\eta}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)} + \frac{(2ik\eta)^{\frac{\lambda}{2ik}}e^{-ik\eta}e^{-\pi\frac{\lambda}{2k}}e^{\frac{\pi i}{2}}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)}\right]. \tag{4.3}$$

In addition to the zeros of $v_1'(\eta_o, \lambda)$, we have need for the zeros of the function $v_2'(\eta_o e^{\pi i}, -\lambda)$ in the region $-\pi < \arg\lambda < 0$. Again an explicit representation of these zeros can be found using the results of Chapter II. They are found to arise from the Airy function representation of $v_2'(\eta_o e^{\pi i}, -\lambda)$.

4.2 Equivalent Integral Representation

Consider the integral representation (4.1) together with the following representation (Buchholz, 1953) of $v_1'(\eta_o, \lambda)$

$$v_1(\eta_o, \lambda) = \frac{e^{\pi\frac{\lambda}{2k}}\, v_2'(\eta_o e^{\pi i}, -\lambda)}{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right)} + \frac{e^{\pi\frac{\lambda}{2k}}\, e^{-\frac{i\pi}{2}}\, v_2'(\eta_o, \lambda)}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)} .$$

Then we can write

$$v_1'(\eta_o,\lambda) = \frac{e^{\pi\frac{\lambda}{2k}}\, v_2'(\eta_o\, e^{\pi i}, -\lambda)}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} \left[1 + e^{-\frac{i\pi}{2}} \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)} \frac{v_2'(\eta_o,\lambda)}{v_2'(\eta_o e^{\pi i}, -\lambda)}\right]$$

and if we define

$$F(\xi_1, \xi_2, \eta_o, k, \lambda) = \frac{1}{2\pi i(2ik\eta_o)} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) v_1(\xi_1, -\lambda) v_2(\xi_2, -\lambda)$$

$$g(\eta_o, k, \lambda) = \frac{e^{\pi\frac{\lambda}{2k}}\, v_2'(\eta_o e^{\pi i}, -\lambda)}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)}$$

$$X(\eta_o, k, \lambda) = e^{\frac{i\pi}{2}} \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) v_2'(\eta_o,\lambda)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) v_2'(\eta_o e^{\pi i}, -\lambda)}$$

$-\infty - i\sigma$ to $\infty - i\sigma$ $(0 < \sigma < k)$ by c ,
equation (4.1) becomes

$$v_N(\xi, \eta_o) = \int_c d\lambda \, \frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{v_1'(\eta_o,\lambda)} = \int_c d\lambda \, \frac{F}{g(1-X)} \tag{4.4}$$

But then we have the decomposition

$$v_N(\xi, \eta_o) - \int_c d\lambda \frac{F}{g} = \int_c d\lambda \, \frac{F}{g(1-X)} \cdot X$$

$$= \int_c d\lambda \, \frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{v_1'(\eta_o,\lambda)} \cdot X(\eta_o, k, \lambda) \ .$$

If we now assume

$$v_N(\xi,\eta_o) - \sum_{k=0}^{N-1} \int_c d\lambda \frac{F}{g} \cdot X^k = \int_c d\lambda \frac{F}{g(1-X)} \cdot X^N ,$$

then

$$v_N(\xi,\eta_o) - \sum_{k=0}^{N} \int_c d\lambda \frac{F}{g} \cdot X^k = \int_c d\lambda \left\{ \left[\frac{F}{g(1-X)} \cdot X^N - \frac{F}{g} X^N\right] \right\}$$

$$= \int_c d\lambda \frac{F}{g(1-X)} \cdot X^{N+1}$$

and so by mathematical induction we have shown that for any M we can write

$$v_N(\xi,\eta) - \sum_{k=0}^{M} \int_c d\lambda \frac{F}{g} \cdot X^k = \int_c d\lambda \frac{F}{g(1-X)} \cdot X^{M+1}$$

$$= \int_c d\lambda \frac{F(\xi_1,\xi_2,\eta_o,k,\lambda)}{v_1'(\eta_o,\lambda)} \cdot \left[X(\eta_o,k,\lambda)\right]^{M+1} \qquad (4.5)$$

If we examine the integrand of the remainder term in (4.5) we see that it is of the form of the original integrand of equation (4.1) multiplied by the factor $\left[X(\eta_o,k,\lambda)\right]^{M+1}$. Therefore its behavior on a large semi-circle in the lower half plane may be investigated by considering the behavior of $\left[X(\eta_o,k,\lambda)\right]^{M+1}$ together with the already known (Appendix A.3) behavior of $\frac{F(\xi_1,\xi_2,\eta_o,k,\lambda)}{v_1'(\eta_o,\lambda)}$. We find that the contour c may be closed in the lower half plane provided the inequality

$$\sqrt{\xi_1} + \sqrt{\xi_2} - \sqrt{\eta_o} - 2(M+1)\sqrt{\eta_o} < 0 \qquad (4.6)$$

is satisfied. But for any triplet (ξ,Ξ,η_o) there must be some M_o such that the inequality (4.6) is satisfied. For this M_o we can determine the value of the remainder by the sum of its residues in the lower half plane. Now

$$\frac{F(\xi_1,\xi_2,\eta_o,k,\lambda)}{v_1'(\eta_o,\lambda)}\cdot\left[X(\eta_o,k,\lambda)\right]^{M+1} = \frac{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{2\pi i(2ik\eta_o)v_1'(\eta_o,)}\, e^{i(M+1)\frac{\pi}{2}}\cdot\left[\frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right) v_2'(\eta_o,\lambda)}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) v_2'(\eta_o e^{\pi i},-\lambda)}\right]^{M+1}$$

and hence the only poles of the remainder integrand are ones of order $(M+1)$ at the zeros of $v_2'(\eta_o e^{\pi i},-\lambda)$ which lie in the quadrant $-\pi < \arg\lambda < -\pi/2$ of the lower half plane. However, for $k\eta_o \gg 1$ the residues as well as their sum are exponentially small, consequently (4.5) implies

$$v_N(\xi,\eta_o) \sim \sum_{k=0}^{M_o}\int_C d\lambda\,\frac{F}{g}\cdot X^k$$

or to be more precise

$$v_N(\xi,\eta_o) = \sum_{k=0}^{M_o}\int_C d\lambda\,\frac{F}{g}\cdot X^k + O(e^{-k\eta_o}) \qquad (4.7)$$

4.3 Source Located "Far" from Focus, Surface

Since the representation (4.7) was derived from (4.1) for the source to the right of the focal point we already have that the source is "far" from the surface. If in addition we consider the source to be "far" from the focus ($k\Xi \gg 1$) and the field point to be "far" from the tip of the paraboloid ($k\xi \gg 1$), we can evaluate the integrals occurring in (4.7) by saddle point integrations. The first term corresponds to the usual geometric term (twice the incident field) while the higher order terms correspond to the multiple reflections. Since the number of higher order terms is governed by the inequality (4.6), we can for a fixed source identify regions of the surface with the number of multiple reflections received.

4.4 Source Located "Near" Focus

In the representation (4.7) we can consider $k\Xi = O(1)$ (source "near" the focus) or $k\xi = O(1)$ (field "near" the tip of the paraboloid) but not both. The effect obtained is that the saddle points of the various terms of (4.7) disappear. In addition we can estimate the neighborhoods of the focal point or tip where this effect occurs. As the source approaches the focus this neighborhood corresponds to the rays leaving the surface almost parallel so that no multiple reflections occur. In the region of the tip this neighborhood corresponds to the region where no multiple reflected ray is received. In order to explicitly evaluate the terms of (4.7) we note that the portion of the contour contributing most to the integrals corresponds to $|\lambda/k| \leqslant O(1)$ where the functions $v_i(x, \lambda)$ are governed by asymptotic representations of the type (4.3). The functions can then be replaced by these asymptotic representations and the resulting integrals evaluated. In either case only the term corresponsing to twice the incident field remains.

4.5 Source Located "Near" Surface

If we wish to consider the source to the left of the focal point we must derive a representation corresponding to (4.7) starting with equation (4.2). In this case we must pay careful attention to the "whispering gallery" poles, since when H is close to η_o they are no longer of $O(e^{-k\eta_o})$ as in the case of (4.1), but are of order comparable with the other terms. Thus the "whispering gallery" wave which travels along the surface of the paraboloid becomes more evident as H approaches η_o. While this is true for $k\eta_o \gg 1$ it is felt that this "whispering gallery" wave is the key to the behavior for $k\eta_o$ not so large, even when the source is not close to the focal point.

V
APPLICATION OF RESULTS TO CONTINUING INVESTIGATION

This work on the paraboloid is preliminary to a description of scattering by general concave surfaces. Having integrated the formalism in terms of the physical phenomena such as the whispering gallery waves, multiple reflections and caustics we now are in a position to search for generalizations of these to other concave surfaces. The scheme we propose is to determine the dependance of the physical effects on the local geometry of the paraboloid and then to make the essentially physical argument that this geometric dependance is the same for other concave shapes. This approach is similar to that used in determining creeping waves on general convex shapes and is an application of the physical arguments used in Keller's geometric theory of diffraction.

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APPENDIX A.1
NORMALIZATION (POINT SOURCE NORMALIZATION)

In the text we first assumed the point source in question to be represented by $\rho(\underline{r})$. Let $J(\xi, \eta, \phi)$ denote the volume Jacobian in the coordinates of the paraboloid of revolution. Then for the point source at $(\Xi, 0)$

$$\rho(\xi, \eta) = \frac{1}{J(\xi, \eta, \phi)} C' \delta(\xi - \Xi)\delta(\eta) = \frac{C}{(\xi + \eta)} \delta(\xi - \Xi)\delta(\eta) ,$$

and so

$$\iiint \rho(\xi, \eta) dV = \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} 2C\, \delta(\xi - \Xi)\delta(\eta) d\xi\, d\eta\, d\phi = 4\pi C$$

$$= 4\pi \quad \text{for} \quad C = 1 .$$

Therefore for $C = 1$ we must have

$$\iiint \rho(\underline{r}) dV = \iiint C'' \delta(\underline{r} - \underline{r}_0) dV = 4\pi ,$$

implying $C'' = 4\pi$ and $\rho(\underline{r}) = 4\pi\delta(\underline{r} - \underline{r}_0)$. Since the free space Green's function for this $\rho(\underline{r})$ is $\frac{-e^{-ikR}}{R}$ $(R = |\underline{r} - \underline{r}_0|)$, then in order to have consistency (already demonstrated by the agreement of equations (1.5) and (1.12)) we must show the solution to

$$-L_\eta v - L_\xi v = \delta(\xi - \Xi)\delta(\eta)$$

$$\int_{\text{all space}} |v(\xi, \eta, s)|^2 dV < \infty \qquad (*)$$

has $\frac{-e^{-ikR}}{R}$ $(R = |\underline{r} - \underline{r}_0|)$, for the limit as $s \rightarrow 0+$.

The solution to (*) can be represented as

$$v(\xi, \eta, s) = \frac{1}{2\pi i} \int_\Gamma \tilde{G}(\xi, \Xi, -\lambda)\, G(\eta, 0, \lambda) d\lambda$$

where Γ is a straight line contour between the poles of $\tilde{G}(\xi, \Xi, -\lambda)$ and $G(\eta, 0, \lambda)$. But

$$G(\eta, 0, \lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)}{2i\gamma} y_1(0, \lambda) y_2(\eta, \lambda) \qquad 0 < \eta$$

analytic in $\mathrm{Im}\,\lambda < k$,* and

$$\tilde{G}(\xi, \Xi, -\lambda) = \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2i\gamma}\right)}{2i\gamma} \begin{cases} y_1(\xi, -\lambda) y_2(\Xi, -\lambda) & \xi < \Xi \\ y_1(\Xi, -\lambda) y_2(\xi, -\lambda) & \xi > \Xi \end{cases}$$

analytic in $\mathrm{Im}\,\lambda > -k$.* Therefore if Γ is a path defined by

$$-\infty - i\sigma < \lambda < \infty - i\sigma \qquad |\sigma| < k,$$

$$v(\xi, \eta, s) = \frac{1}{2\pi i} (2\pi i)^{-3/2} \int_\Gamma \Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right) \Gamma\left(\frac{1}{2} + \frac{\lambda}{2i\gamma}\right) y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda) y_2(\eta, \lambda) d\lambda.$$

Arguing as in section (1.4) we see that the limit as $s \to 0^+$ may be taken inside the integral by replacing the parameter γ with the parameter k. But then the integral representation for $\frac{e^{-ikR}}{R}$ in Buchholz (1953) shows that

$$\lim_{s \to 0^+} v(\xi, \eta, s) = -\frac{e^{-ikR}}{R}$$

which demonstrates consistency.

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* See Appendix A.2.

APPENDIX A.2
ANALYTICITY OF RESOLVENT GREEN'S FUNCTION $\tilde{R}_\lambda$

The analyticity of $\tilde{R}_\lambda$ follows that of $\tilde{G}(\xi, \xi', \lambda)$ which was represented as

$$\tilde{G}(\xi, \xi', \lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)}{2i\gamma} \begin{cases} y_1(\xi, \lambda) y_2(\xi', \lambda) & \xi < \xi' \\ y_1(\xi', \lambda) y_2(\xi, \lambda) & \xi > \xi' \end{cases}$$

According to Buchholz (1953) the functions y_1 and y_2 are entire functions of λ, thus the singularities of $\tilde{G}(\xi, \xi', \lambda)$ are those of $\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)$ which are simple poles at the points $\frac{\lambda}{2i\gamma} = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$. $\tilde{G}(\xi, \xi', \lambda)$ will be analytic in any domain which excludes these points. Consider then the expression $\lambda/2i\gamma$ with $\lambda = x + iy$ and $\gamma = \frac{1}{c}(\omega - is)$. We have

$$\frac{\lambda}{2i\gamma} = \frac{-i\lambda}{2\gamma} = \frac{-i\lambda\gamma^*}{2|\gamma|^2} = \frac{-i(x+iy)\left(\frac{1}{c}(\omega + is)\right)}{2|\gamma|^2} .$$

Thus

$$\frac{\lambda}{2i\gamma} = \frac{\frac{sx}{c} + \frac{\omega y}{c} + \frac{i\omega x}{c} + \frac{isy}{c}}{2|\gamma|^2}$$

and if this is to be real $\frac{\omega x}{c} = \frac{sy}{c}$ implying $x = \frac{s}{\omega} y$. Then

$$\frac{\lambda}{2i\gamma} = \frac{\frac{s^2 y}{\omega c} + \frac{\omega^2 y}{\omega c}}{2|\gamma|^2} = \frac{y}{2\omega c|\gamma|^2}(s^2 + \omega^2) .$$

Now $|\gamma|^2 = \gamma\gamma^* = \frac{1}{c^2}(\omega^2 + s^2)$ and so for real $\lambda/2i\gamma$ we have $\frac{\lambda}{2i\gamma} = \frac{yc}{2\omega}$.

To exclude the poles of the Γ-function we need for real $\lambda/2i\gamma$

$$\frac{\lambda}{2i\gamma} < \frac{1}{2} \quad \text{implying} \quad \frac{yc}{2\omega} < \frac{1}{2} \quad \text{or} \quad y < \frac{\omega}{c} = k.$$

But since $\lambda = x + iy$, $y = \text{Im}\,\lambda$ and therefore we arrive at $\text{Im}\,\lambda < k$.

APPENDIX A.3

CLOSING THE CONTOUR (CONVERGENCE OF RESIDUE SERIES)

In this section we consider the integral representation given by equation (1.12)

$$v_N(\xi,\eta) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda) y_2(\xi_2,-\lambda)}{y_1'(\eta_o,\lambda)} \cdot$$

$$(0 < \sigma < k) \qquad \cdot \left[y_2(\eta,\lambda) y_1'(\eta_o,\lambda) - y_1(\eta,\lambda) y_2'(\eta_o,\lambda) \right] .$$

We first show that

$$\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \left[y_2(\eta,\lambda) y_1'(\eta_o,\lambda) - y_1(\eta,\lambda) y_2'(\eta_o,\lambda) \right] \tag{A.3.1}$$

is analytic in the complex λ-plane, and for $|\lambda| \to \infty$

(i) the zeros of $y_1'(\eta_o,\lambda)$ lie along the real axis, (in the text we discuss the zeros of $y_1'(\eta_o,\lambda)$ (when $k\eta_o \ll 1$ and $k\eta_o \gg 1$) for other ranges of $|\lambda|$, a discussion for arbitrary $k\eta_o$ appears in Buchholz (1942/3, 1953))

(ii) the integrand vanishes exponentially in the upper half plane ($\mathrm{Im}\,\lambda \geqslant -\sigma$).

Thus the contour may be closed and the residue series obtained. We also show that the residue series converges.

The only possible poles of (A.3.1) occur at the poles of $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$ which lie on the positive imaginary axis at $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$. But at these points

$$y_1(\eta,\lambda) = y_1\left[\eta, ik(2n+1)\right] = \eta^{-1/2} M_{n+1/2,\,0}(2ik\eta) = (2ik)^{1/2} e^{-ik\eta} L_n^{(0)}(2ik\eta)$$

$$y_2(\eta,\lambda) = y_2\left[\eta, ik(2n+1)\right] = \eta^{-1/2} W_{n+1/2,\,0}(2ik\eta) = (-1)^n n!\, (2ik)^{1/2} e^{-ik\eta} L_n^{(0)}(2ik\eta)$$

where $L_n^{(0)}(2ik\eta)$ is the corresponding Laguerre polynomial. Thus

$$y_2\left[\eta, ik(2n+1)\right] y_1'\left[\eta_o, ik(2n+1)\right] - y_1\left[\eta, ik(2n+1)\right] y_2'\left[\eta_o, ik(2n+1)\right]$$

equals

$$(-1)^n n!(2ik)e^{-ik\eta}L_n^{(0)}(2ik\eta)\left[\frac{d}{d\eta}\left(e^{-ik\eta}L_n^{(0)}(2ik\eta)\right)\right]_{\eta=\eta_o} -$$

$$-(-1)^n n!(2ik)e^{-ik\eta}L_n^{(0)}(2ik\eta)\left[\frac{d}{d\eta}\left(e^{-ik\eta}L_n^{(0)}(2ik\eta)\right)\right]_{\eta=\eta_o} = 0$$

and therefore cancels the simple pole at $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$. This implies the analyticity of (A.3.1).

In order to investigate the zeros of $y_1'(\eta_o, \lambda)$ for $|\lambda| \to \infty$ we first note that the λ-plane will be considered to be cut at $\lambda = \pi$ or $\lambda = -\pi$. Thus the upper half plane ($\text{Im}\,\lambda \geqslant -\sigma$) for $|\lambda| \to \infty$ can be characterized by $|\lambda| \to \infty$, $-\delta \leqslant \arg\lambda < \pi$, $\arg\lambda = \pi$ or $-\pi$, $-\pi < \arg\lambda \leqslant -\pi+\delta$, where δ is a small positive angle which decreases as $|\lambda|$ increases. Similarly the lower half plane ($\text{Im}\,\lambda \leqslant -\sigma$) is characterized by $|\lambda| \to \infty$, $-\pi+\delta \leqslant \arg\lambda \leqslant -\delta$, where δ is as above. Therefore we can investigate the zeros of $y_1'(\eta_o, \lambda)$ in these regions.

For $-\pi \leqslant \arg\lambda < 0$, $y_1'(\eta_o, \lambda)$ has the representation (Buchholz, 1953, p. 98, pertaining to equation 17a)

$$y_1'(\eta_o, \lambda) \sim e^{2i\sqrt{\lambda\eta_o}}\, e^{-\pi\frac{i}{4}} \tag{A.3.2}$$

and thus there are no zeros in this region. For $0 \leqslant \arg\lambda \leqslant \pi$, Buchholz (1953, p. 98, eq. 17a) can be used for $y_1(\eta_o, \lambda)$. Thus

$$y_1(\eta, \lambda) \sim \eta^{-1/2}\left(-\frac{4k^2\eta}{\pi^2\lambda}\right)^{1/4}\cos\left[2\sqrt{\eta\lambda} - \frac{\pi}{4}\right] \tag{A.3.3}$$

which implies

$$y_1'(\eta_o, \lambda) \sim -\frac{1}{\eta_o}\left(-\frac{4k^2\eta_o\lambda}{\pi^2}\right)^{1/4}\sin\left[2\sqrt{\eta_o\lambda} - \frac{\pi}{4}\right]. \tag{A.3.4}$$

Therefore for $|\lambda| \to \infty$ the zeros of $y_1'(\eta_o, \lambda)$ are the zeros of $\sin\left[2\sqrt{\eta_o\lambda} - \frac{\pi}{4}\right]$ which obey the equation

$$\lambda_N = \frac{1}{\eta_o^2}\left(\frac{\pi}{8} \pm \frac{N\pi}{2}\right)^2 \qquad N \to \infty$$

and consequently lie on the positive real axis.

To demonstrate the exponential vanishing of the integrand we will examine the factors

$$\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \qquad (A.3.5)$$

and

$$\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\left[y_2(\eta, \lambda) y_1'(\eta_o, \lambda) - y_1(\eta, \lambda) y_2'(\eta_o, \lambda)\right] \qquad (A.3.1)$$

separately. We further divide the upper half plane into the intervals $-\delta \leq \arg\lambda < 0$, $0 \leq \arg\lambda < \pi/2$, $\arg\lambda = \pi/2$, $\pi/2 < \arg\lambda \leq \pi$, $-\pi \leq \arg\lambda \leq -\pi+\delta$. (We make this choice so that either $\arg\lambda = \pi$ or $\arg\lambda = -\pi$ may be considered.) Then in order to estimate (A.3.5) we need to investigate $y_1(\xi_1, -\lambda)$ and $y_2(\xi_2, -\lambda)$ on these intervals. For $0 \leq \arg\lambda \leq \pi$, Buchholz (1953, p. 98, eq.17b) applies to $y_1(\xi_1, -\lambda)$. Therefore as $|\lambda| \to \infty$

$$y_1(\xi_1, -\lambda) \sim \cos\left[2\sqrt{\xi_1\lambda}\, e^{-i\frac{\pi}{2}} - \frac{\pi}{4}\right] \sim e^{2\sqrt{\xi_1\lambda}}\, e^{-i\frac{\pi}{4}} + e^{-2\sqrt{\xi_1\lambda}}\, e^{i\frac{\pi}{4}} . \qquad (A.3.6)$$

When $-\pi \leq \arg\lambda \leq 0$, we can use the same equation with the opposite sign to assert as $|\lambda| \to \infty$

$$y_1(\xi_1, -\lambda) \sim \cos\left[2\sqrt{\xi_1\lambda}\, e^{i\frac{\pi}{2}} - \frac{\pi}{4}\right] \sim e^{-2\sqrt{\xi_1\lambda}}\, e^{-i\frac{\pi}{4}} + e^{2\sqrt{\xi_1\lambda}}\, e^{i\frac{\pi}{4}} \qquad (A.3.7)$$

The function $y_2(\xi_2, -\lambda)$ is not so simple. In order to examine its behavior we must consider the intervals defined above. The first two intervals may be combined, the rest must be considered separately. Upon obtaining the behavior of $y_2(\xi_2, -\lambda)$ in each interval, the factor (A.3.5) can then be examined.

(i) $-\delta \leq \arg\lambda < \pi/2 \Rightarrow \mathrm{Rp}\lambda > 0$

In this region we cannot obtain the behavior of $y_2(\xi_2, -\lambda)$ directly. Instead we must make use of equation (21b), p. 19 of Buchholz (1953) which asserts

$$y_2(\xi_2, -\lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)}{2\pi i} \left\{ e^{-\pi \frac{\lambda}{2k}} y_2(\xi_2 e^{\pi i}, \lambda) - e^{\pi \frac{\lambda}{2k}} y_2(\xi_2 e^{-\pi i}, \lambda) \right\}$$

Now $-\delta \leq \arg\lambda < \pi/2 \Rightarrow -\pi/2 - \delta \leq \arg \frac{\lambda}{2ik} < 0 \Rightarrow \mathrm{Im} \frac{\lambda}{2ik} < 0$. In addition

$$\arg \frac{\lambda}{2ik}\, 2ik\xi_2 e^{\pi i} = \arg(\lambda e^{\pi i}) \in [\pi - \delta, 3\pi/2)$$

$$\arg \frac{\lambda}{2ik}\, 2ik\xi_2 e^{-\pi i} = \arg(\lambda e^{-\pi i}) \in [-\pi - \delta, -\pi/2) \quad ,$$

therefore Buchholz, (1953, p. 99, eq. 19a) applies to both $y_2(\xi_2 e^{\pi i}, \lambda)$ and $y_2(\xi_2 e^{-\pi i}, \lambda)$. But since $\mathrm{Rp}\lambda > 0$ we have

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi \frac{\lambda}{2k}} y_2(\xi_2 e^{-\pi i}, \lambda) \, .$$

Using the above-mentioned equation for $y_2(\xi_2 e^{-\pi i}, \lambda)$

$$y_2(\xi_2 e^{-\pi i}, \lambda) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{\pi \frac{\lambda}{2k}} e^{-2i\sqrt{\lambda \xi_2 e^{-\pi i}}}$$

$$\sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{\pi \frac{\lambda}{2k}} e^{-2\sqrt{\lambda \xi_2}} \quad .$$

we arrive at

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi \frac{\lambda}{2k}} e^{\pi \frac{\lambda}{2k}} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{-2\sqrt{\lambda \xi_2}}$$

The range of $\arg \frac{\lambda}{2ik}$ implies that Stirling's approximation is valid for the Γ-function $\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)$, thus from Erdélyi et al (1953)

$$\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \qquad \text{(A.3.8)}$$

and so $y_2(\xi_2, -\lambda)$ becomes

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) e^{\pi\frac{\lambda}{2k}} \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi\frac{\lambda}{2k}} e^{-2\sqrt{\lambda\xi_2}} .$$

However, from Erdélyi et al (1953) we also have

$$\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) = \frac{\pi}{\cos\pi\frac{\lambda}{2ik}} \tag{A.3.9}$$

and using the exponential representation of the cosine

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi\frac{\lambda}{2k}} e^{-2\sqrt{\lambda\xi_2}} \tag{A.3.10}$$

Therefore using (A.3.2), (A.3.7), (A.3.9) and (A.3.10) we find

$$\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) \frac{y_1(\xi_1, -\lambda)y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

on $-\delta \leqslant \arg\lambda < 0 \Rightarrow -\delta/2 \leqslant \arg\sqrt{\lambda} > 0 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} < 0,$

and using (A.3.3), (A.3.6), (A.3.9) and (A.3.10) we find

$$\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) \frac{y_1(\xi_1, -\lambda)y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

on $0 \leqslant \arg\lambda < \pi/2 \Rightarrow 0 \leqslant \arg\sqrt{\lambda} < \pi/4 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} > 0.$

In either case (A.3.5) is vanishing exponentially since $\xi_2 > \xi_1$.

(ii) $\arg\lambda = \pi/2 \Rightarrow \lambda = i|\lambda|,\ \arg\sqrt{\lambda} = \pi/4 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} > 0.$

For $\arg\lambda = \pi/2$ we note $-\frac{\lambda}{2ik} = \frac{-|\lambda|}{2k}$, $\frac{\lambda}{2ik} = \frac{|\lambda|}{2k}$. Thus Buchholz (1953, p. 100, eq. 20) can be used to give

$$y_2(\xi_2, -\lambda) \sim \exp\left\{-\frac{|\lambda|}{2k}\log\frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{i|\lambda|\xi_2}}$$

$$\sim \exp\left\{-\frac{|\lambda|}{2k}\log\frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}} \qquad \text{(A.3.11)}$$

But (A.3.8) applies for $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$, and using this together with (A.3.3), (A.3.6) and (A.3.11) we have

$$\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)\frac{y_1(\xi_1, -\lambda)y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{-2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}}\; e^{2\sqrt{|\lambda|\xi_1}\cos\frac{\pi}{4}}}{e^{2\sqrt{|\lambda|\eta_o}\cos\pi/4}}$$

which vanishes exponentially since $\xi_2 > \xi_1$.

(iii) $\pi/2 < \arg\lambda \leqslant \pi \implies \pi/4 < \arg\sqrt{\lambda} \leqslant \pi/2 \implies \text{Rp}\sqrt{\lambda} \geqslant 0,\ \text{Im}\sqrt{\lambda} > 0.$

In this region we note that $0 < \arg\frac{\lambda}{2ik} \leqslant \pi/2 \implies \text{Im}\frac{\lambda}{2ik} > 0$. But in addition $-\frac{\lambda}{2ik} = e^{-i\pi}\frac{\lambda}{2ik}$, $\arg\frac{\lambda}{2ik}2ik\xi_2 \mp \arg\lambda \in (-\pi, 3\pi)$. Hence Buchholz (1953, p. 100, eq. 20) applies to $y_2(\xi_2, -\lambda)$ giving

$$y_2(\xi_2, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{-2\sqrt{\lambda\xi_2}}. \qquad \text{(A.3.12)}$$

Now equation (A.3.8) also applies, hence, using it together with (A.3.3), (A.3.6) and (A.3.12) we find

$$\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)\frac{y_1(\xi_1, -\lambda)y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{-2\,\text{Rp}\sqrt{\lambda\xi_2}}\; e^{2\,\text{Rp}\sqrt{\lambda\xi_1}}}{e^{2\,\text{Im}\sqrt{\lambda\eta_o}}}$$

which vanishes exponentially since $\xi_2 > \xi_1$ and $\text{Im}\sqrt{\lambda} > 0$.

(iv) $-\pi \leqslant \arg\lambda \leqslant -\pi+\delta \implies \text{Rp}\lambda < 0$

In this region $-\lambda = e^{i\pi}\lambda$, thus $-\frac{\lambda}{2ik} = e^{i\pi}\frac{\lambda}{2ik} = \frac{i\lambda}{2k}$. In addition

$-\pi \leqslant \arg\lambda \leqslant -\pi+\delta \implies -\pi/2 \leqslant \arg\frac{i\lambda}{2k} \leqslant -\pi/2+\delta \implies \text{Im}\frac{i\lambda}{2k} < 0$. We also have

$\arg 2ik\xi\left(\frac{i\lambda}{2k}\right) = \arg -\lambda \in [0, \delta] \subset [0, 2\pi)$. Then we can use eq. (19a), p. 99 of Buchholz (1953) to assert

$$y_2(\xi_2, -\lambda) \sim \exp\left\{\frac{i\lambda}{2k} \log \frac{i\lambda}{2ke}\right\} e^{-\pi\frac{\lambda}{2k}} e^{-2i\sqrt{-\lambda\xi_2}}$$
$$= \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\} e^{-\pi\frac{\lambda}{2k}} e^{2\sqrt{\lambda\xi_2}} .$$

But for this region Stirling's formula for the Γ-function $\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)$ is applicable (Erdelyi et al, 1953). Thus

$$\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) \sim \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\} \tag{A.3.13}$$

and so

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{-\pi\frac{\lambda}{2k}} e^{2\sqrt{\lambda\xi_2}} \tag{A.3.14}$$

Then using (A.3.2), (A.3.7), (A.3.9) and (A.3.14) we have

$$\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) \frac{y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

since $-\pi/2 \leqslant \arg\sqrt{\lambda} \leqslant \frac{-\pi+\delta}{2} \implies \mathrm{Rp}\sqrt{\lambda} \geqslant 0$, $\mathrm{Im}\sqrt{\lambda} < 0$. But as $|\lambda| \to \infty$ $\mathrm{Rp}\sqrt{\lambda} \leqslant k$ while $\mathrm{Im}\sqrt{\lambda}$ decreases without bound, therefore

$$\frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}} \leqslant \frac{e^{2k\sqrt{\xi_2}}\, e^{2k\sqrt{\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

and (A.3.5) is vanishing exponentially.

We now examine the factor (A.3.1). We consider first the interval $-\pi \leqslant \arg\lambda \leqslant -\pi+\delta$. Thus from Buchholz (1953, p. 98, eq. 17a) we can derive the representations valid in $-\pi \leqslant \arg\lambda \leqslant 0$

$$y_1(\eta,\lambda) \sim e^{2i\sqrt{\lambda\eta}} \tag{A.3.15a}$$

$$y_1'(\eta,\lambda) \sim e^{2i\sqrt{\lambda\eta}} \quad . \tag{A.3.15b}$$

In order to derive $y_2(\eta,\lambda)$, $y_2'(\eta,\lambda)$ in this interval we recall equation (A.3.10) valid in $0 \leqslant \arg\lambda < \pi/2$. If we then consider the substitution $v = \lambda e^{-\pi i}$ and also recall that in $0 \leqslant \arg\lambda < \pi/2$, $-\lambda = e^{-\pi i}\lambda$, (A.3.10) implies for $-\pi \leqslant \arg v < -\pi/2$

$$y_2(\xi_2, v) \sim \Gamma\left(\frac{1}{2}+\frac{v}{2ik}\right) e^{-\pi\frac{v}{2k}} e^{-2\sqrt{e^{\pi i}v\xi_2}} = \Gamma\left(\frac{1}{2}+\frac{v}{2ik}\right) e^{-\pi\frac{v}{2k}} e^{-2i\sqrt{v\xi_2}} \quad .$$

But since v, ξ_2 are dummy variables we can write for $-\pi \leqslant \arg\lambda < -\pi/2$ and in particular for $-\pi \leqslant \arg\lambda \leqslant -\pi+\delta$

$$y_2(\eta,\lambda) \sim \Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) e^{-\pi\frac{\lambda}{2k}} e^{-2i\sqrt{\lambda\eta}} \tag{A.3.16a}$$

$$y_2'(\eta,\lambda) \sim \Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) e^{-\pi\frac{\lambda}{2k}} e^{-2i\sqrt{\lambda\eta}} \quad . \tag{A.3.16b}$$

Therefore

$$(A.3.1) \sim \Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) e^{-\pi\frac{\lambda}{2k}} \left[e^{-2i\sqrt{\lambda\eta}} e^{2i\sqrt{\lambda\eta_o}} + e^{2i\sqrt{\lambda\eta}} e^{-2i\sqrt{\lambda\eta_o}}\right]$$

and using (A.3.9) together with the exponential representation of the cosine, as well as $Rp\lambda < 0$, $Im\sqrt{\lambda} < 0$ and $\eta \leqslant \eta_o$, we find

$$(A.3.1) \sim e^{-2i\sqrt{\lambda\eta}} e^{2i\sqrt{\lambda\eta_o}} \quad .$$

Hence the integrand (product of A.3.1 and A.3.5) behaves as

$$\text{Integrand} \sim e^{2\,Rp\sqrt{\lambda\xi_2}} e^{2\,Rp\sqrt{\lambda\xi_1}} e^{2\,Im\sqrt{\lambda\eta}} \quad .$$

As previously

$$e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta}} \leqslant e^{2k\sqrt{\xi_2}}\, e^{2k\sqrt{\xi_1}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta}}$$

which vanishes exponentially since $\mathrm{Im}\sqrt{\lambda} < 0$.

Consider the interval $-\delta \leqslant \arg\lambda \leqslant 0$. Equations (A.3.15a) and (A.3.15b) for $y_1(\eta,\lambda)$ and $y_1'(\eta,\lambda)$ apply; thus it remains to find $y_2(\eta,\lambda)$ and $y_2'(\eta,\lambda)$. But now we recall equation (A.3.12) valid in $\pi/2 < \arg\lambda \leqslant \pi$. If we again make the substitution $v = \lambda\, e^{-\pi i}$ and recall that in $\pi/2 < \arg\lambda \leqslant \pi$, $-\lambda = e^{-\pi i}\lambda$, equation (A.3.12) implies for $-\pi/2 < \arg v \leqslant 0$

$$y_2(\xi_2, v) = \exp\left\{\frac{v}{2ik}\log\frac{e^{\pi i}v}{2ike}\right\} e^{-2\sqrt{e^{\pi i}v\xi_2}} = \exp\left\{\frac{v}{2ik}\log\frac{v}{2ike}\right\} e^{\pi\frac{v}{2k}}\, e^{-2i\sqrt{v\xi_2}}.$$

Again v and ξ_2 are dummy variables, thus for $-\pi/2 < \arg\lambda \leqslant 0$ and in particular for $-\delta \leqslant \arg\lambda \leqslant 0$

$$y_2(\eta,\lambda) \sim \exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{\pi\frac{\lambda}{2k}}\, e^{-2i\sqrt{\lambda\eta}} \qquad \text{(A.3.17a)}$$

$$y_2'(\eta,\lambda) \sim \exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{\pi\frac{\lambda}{2k}}\, e^{-2i\sqrt{\lambda\eta}} \qquad \text{(A.3.17b)}$$

Now for $-\pi/2 < \arg\lambda \leqslant 0$, $-\pi < \arg\frac{\lambda}{2ik} \leqslant -\pi/2$, hence equation (A.3.8) applies for $\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)$ and equations (A.3.17) become

$$y_2(\eta,\lambda) \sim \Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) e^{\pi\frac{\lambda}{2k}}\, e^{-2i\sqrt{\lambda\eta}} \qquad \text{(A.3.18a)}$$

$$y_2'(\eta,\lambda) \sim \Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right) e^{\pi\frac{\lambda}{2k}}\, e^{-2i\sqrt{\lambda\eta}}\,. \qquad \text{(A.3.18b)}$$

We proceed exactly as before except that in this case $\mathrm{Rp}\lambda > 0$; therefore we find

$$(A.3.1) \sim e^{-2i\sqrt{\lambda\eta}}\, e^{2i\sqrt{\lambda\eta_o}}$$

and the integrand behaves as

$$\text{Integrand} \sim e^{-2Rp\sqrt{\lambda\xi_2}}\, e^{2Rp\sqrt{\lambda\xi_1}}\, e^{2\,Im\sqrt{\lambda\eta}}$$

which vanishes exponentially since $Im\sqrt{\lambda} \leqslant 0$, $Rp\sqrt{\lambda} > 0$ and $\xi_2 > \xi_1$.

We can now examine the behavior of (A.3.1) on the interval $0 \lessgtr \arg\lambda \leqslant \pi$. Equations (A.3.3) and (A.3.4) apply for $y_1(\eta, \lambda)$ and $y_1'(\eta, \lambda)$; we must derive equations for $y_2(\eta, \lambda)$ and $y_2'(\eta, \lambda)$. For $0 \leqslant \arg\lambda \leqslant \pi$, $-\pi/2 \leqslant \arg \frac{\lambda}{2ik} \leqslant \pi/2$, and $\arg \frac{\lambda}{2ik}\,(2ik\eta) = \arg\lambda \in [0, \pi]$, thus Buchholz (1953, p. 99, eq. 19) yields

$$y_2(\eta, \lambda) \sim \frac{(2)^{1/2}}{\eta^{1/2}} \left(-\frac{4k^2\eta}{\lambda}\right)^{1/4} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \cos\left[2\sqrt{\lambda\eta} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right]$$

$$y_2'(\eta, \lambda) \sim -\frac{(2)^{1/2}}{\eta} (-4k^2\eta\lambda)^{1/4} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \sin\left[2\sqrt{\lambda\eta} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right].$$

Now equation (A.3.8) for the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ is valid, hence

$$y_2(\eta, \lambda) \sim \frac{(2)^{1/2}}{\eta^{1/2}} \left(-\frac{4k^2\eta}{\lambda}\right)^{1/4} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \cos\left[2\sqrt{\lambda\eta} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right] \qquad (A.3.19a)$$

$$y_2'(\eta, \lambda) \sim -\frac{2^{1/2}}{\eta} (-4k^2\eta\lambda)^{1/4} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \sin\left[2\sqrt{\lambda\eta} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right]. \qquad (A.3.19b)$$

Therefore

$$(A.3.1) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)\left[\cos\left[2\sqrt{\lambda\eta} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right] \sin\left[2\sqrt{\lambda\eta_o} - \frac{\pi}{4}\right]\right.$$
$$\left. - \cos\left[2\sqrt{\lambda\eta} - \frac{\pi}{4}\right] \sin\left[2\sqrt{\lambda\eta_o} - \pi\frac{\lambda}{2ik} + \frac{\pi}{4}\right]\right],$$

which upon using the exponential representations of the sine and the cosine becomes

$$(A.3.1) \sim \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) e^{\pi\frac{\lambda}{2k}}\left[e^{-2i\sqrt{\lambda\eta}}\, e^{2i\sqrt{\lambda\eta_o}} + e^{-2i\sqrt{\lambda\eta_o}}\, e^{2i\sqrt{\lambda\eta}}\right]$$

$$+\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) e^{-\pi\frac{\lambda}{2k}}\left[e^{2i\sqrt{\lambda\eta}}\, e^{-2i\sqrt{\lambda\eta_o}} + e^{2i\sqrt{\lambda\eta_o}}\, e^{-2i\sqrt{\lambda\eta}}\right] \tag{A.3.20}$$

Let us now divide $0 \leqslant \arg\lambda \leqslant \pi$ into the sub-intervals $0 \leqslant \arg\lambda < \pi/2$, $\arg\lambda = \pi/2$, $\pi/2 < \arg\lambda \leqslant \pi$.

(i) $0 \leqslant \arg\lambda < \pi/2 \implies \mathrm{Rp}\lambda > 0,\ 0 \leqslant \arg\sqrt{\lambda} < \pi/4 \implies \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} \geqslant 0$

Using (A.3.9) and (A.3.20) together with the exponential representation of the cosine and $\eta \leqslant \eta_o$ we have

$$(A.3.1) \sim e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta_o}} .$$

Then the integrand obeys

$$\text{Integrand} \sim e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\, e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}$$

which vanishes exponentially since $\xi_2 > \xi_1$.

(ii) $\arg\lambda = \pi/2 \implies \mathrm{Rp}\lambda = 0,\ \arg\lambda = \pi/4 \implies \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} > 0$

Thus using (A.3.9) and (A.3.20) as above

$$(A.3.1) \sim e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}}\, e^{2\sqrt{|\lambda|\eta_o}\cos\frac{\pi}{4}}$$

while the integrand obeys

$$\text{Integrand} \sim e^{-2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}}\, e^{2\sqrt{|\lambda|\xi_1}\cos\frac{\pi}{4}}\, e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}}$$

which vanishes exponentially since $\xi_2 > \xi_1$.

(iii) $\pi/2 < \arg\lambda \leqslant \pi \implies \mathrm{Rp}\lambda < 0,\ \pi/4 < \arg\sqrt{\lambda} \leqslant \pi/2 \implies \mathrm{Rp}\sqrt{\lambda} \geqslant 0,\ \mathrm{Im}\sqrt{\lambda} > 0$

The arguments here are the same as above. We find

$$\text{Integrand} \sim e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\; e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\; e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}$$

which vanishes exponentially since $\xi_2 > \xi_1$ and $\mathrm{Im}\sqrt{\lambda} > 0$.

If we now consider equation (1.12), the above argument implies that the path of integration may be closed by an infinitely large semi-circle in the upper half plane ($\mathrm{Im}\,\lambda \geqslant -\sigma$). Then by use of the residue theorem $v_N(\xi,\eta)$ can be evaluated as a sum of residues.

Let $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ denote the zeros of $y_1'(\eta_o,\lambda)$ along the real axis. (For finite λ we appeal to Buchholz (1942/3, 1953) for the location of the zeros.) Thus

$$v_N(\xi,\eta) = (2ik)^{-3/2}\sum_{n=1}^{\infty}\Gamma\left(\frac{\lambda_n}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda_n}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda_n)y_2(\xi_2,-\lambda_n)}{\left[\frac{d}{d\lambda}y_1'(\eta_o,\lambda)\right]_{\lambda=\lambda_n}}\cdot\left[-y_1(\eta,\lambda_n)y_2'(\eta_o,\lambda_n)\right]$$

But from the Wronskian relation we have

$$-y_1(\eta_o,\lambda_n)y_2'(\eta_o,\lambda_n) = \frac{2ik}{\eta_o\,\Gamma\left(-\frac{\lambda_n}{2ik}+\frac{1}{2}\right)},$$

therefore $v_N(\xi,\eta)$ becomes

$$v_N(\xi,\eta) = \frac{(2ik)^{-1/2}}{\eta_o}\sum_{n=1}^{\infty}\Gamma\left(\frac{\lambda_n}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda_n)y_2(\xi_2,-\lambda_n)}{\left[\frac{d}{d\lambda}y_1'(\eta_o,\lambda)\right]_{\lambda=\lambda_n}}\;\frac{y_1(\eta,\lambda_n)}{y_1(\eta_o,\lambda_n)}\;.$$

In this series $\arg\lambda_n = 0$ for n sufficiently large. Hence the previously developed asymptotic forms can be used to investigate the convergence. By arguing as above, it is seen without difficulty that the series converges. We should remember, however, the conditions $\xi_1 = \min(\xi,\Xi)$, $\xi_2 = \max(\xi,\Xi)$, $\xi \neq \Xi$, $\eta > 0$ under which the integral representation was developed; they still apply to the series.

For the most obvious attempt to represent the total field as the sum of the incident plus scattered fields, we use the integral representation for $-e^{-ikR}/R$ (Appendix A.1) to write equation (1.12) as

$$v_N(\xi,\eta) = -\frac{e^{-ikR}}{R} + \frac{(2ik)^{-3/2}}{2\pi i}\int_{\substack{-\infty-i\sigma \\ (0<\sigma<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_0,\lambda)} \cdot \left[-y_1(\eta,\lambda)y_2'(\eta_0,\lambda)\right]$$

The integrand of the remaining integral now vanishes exponentially in the upper half plane if and only if $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$. But this integrand posses, in the upper half plane, poles not only at the zeros of $y_1'(\eta_0,\lambda)$, but also at the poles of the Γ-function $\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)$ which lie along the positive imaginary axis at the points $\lambda = ik(2n+1)$, $n = 0,1,2,\ldots$. The contribution of these latter poles to the total residue series (convergent when $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$) simply cancels $-e^{-ikR}/R$. This follows immediately from the existence of a residue series for e^{-ikR}/R when $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$, and the calculation which shows (A.3.1) analytic in the upper half plane. Thus we arrive at the previously obtained residue series. Evaluating the remaining integral by residues in the lower half plane is discussed below.

The other set of poles of the integral representation of (1.12) are the poles of the Γ-function $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$ and lie along the negative imaginary axis at the points $\lambda = -ik(2n+1)$, $n = 0,1,2,\ldots$. We now investigate the behavior of the integrand as $|\lambda| \to \infty$ in the lower half plane ($\mathrm{Im}\,\lambda \leqslant -\sigma$) which we characterized by the relation $|\lambda| \to \infty$, $-\pi+\delta \leqslant \arg\lambda \leqslant -\delta$ where δ is a small positive angle that decreases as $|\lambda|$ increases. It will still be convenient to examine the factors

$$\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_0,\lambda)} \tag{A.3.5}$$

and

$$\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\left[y_2(\eta,\lambda)y_1'(\eta_0,\lambda) - y_1(\eta,\lambda)y_2'(\eta_0,\lambda)\right] \tag{A.3.1}$$

separately.

Let us first consider the factor (A.3.5). Equations (A.3.15) apply for $y_1(\eta, \lambda)$ and $y_1'(\eta, \lambda)$ while equation (A.3.7) is valid for $y_1(\xi_1, -\lambda)$. Thus it remains to find $y_2(\xi_2, -\lambda)$. We can derive an expression for $y_2(\xi_2, -\lambda)$ which is valid for $-\pi < \arg\lambda < 0$. In this region $-\lambda = e^{i\pi}\lambda$ and so $-\frac{\lambda}{2ik} = e^{i\pi}\frac{\lambda}{2ik} = \frac{i\lambda}{2k}$. Then

$-\pi < \arg\lambda < 0 \Rightarrow -\pi/2 < \arg\frac{i\lambda}{2k} < \pi/2$ while $\arg 2ik\xi_2(-\frac{\lambda}{2ik}) = \arg(e^{i\pi}\lambda) = \pi + \arg\lambda \in (0, \pi)$.

Hence Buchholz (1953, p. 99, eq. 19) may be used for $y_2(\xi_2, -\lambda)$ giving

$$y_2(\xi_2, -\lambda) \sim \exp\left\{\frac{i\lambda}{2k}\log\frac{i\lambda}{2ke}\right\}\cos\left[2\sqrt{e^{\pi i}\lambda\xi_2} - \pi i\frac{\lambda}{2k} + \frac{\pi}{4}\right] .$$

We write it as

$$y_2(\xi_2, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\}\cos\left[2i\sqrt{\lambda\xi_2} - \pi i\frac{\lambda}{2k} + \frac{\pi}{4}\right] .$$

Since in this region equation (A.3 13) applies for $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$ we can also write

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\cos\left[2i\sqrt{\lambda\xi_2} - \pi i\frac{\lambda}{2k} + \frac{\pi}{4}\right] \qquad \text{(A.3.21)}$$

In order to calculate the behavior of (A.3.5) as $|\lambda| \to \infty$, the interval $-\pi+\delta \leqslant \arg\lambda \leqslant -\delta$ is divided into the sub-intervals $-\pi/2 < \arg\lambda \leqslant -\delta$, $\arg\lambda = -\pi/2$, $-\pi+\delta \leqslant \arg\lambda < -\pi/2$ and (A.3.5) examined on them separately.

(i) $-\pi/2 < \arg\lambda \leqslant -\delta \Rightarrow Rp\lambda > 0$, $-\pi/4 < \arg\sqrt{\lambda} \leqslant -\delta/2 \Rightarrow Rp\sqrt{\lambda} > 0$, $Im\sqrt{\lambda} < 0$

From (A.3.21) we find

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{\pi\frac{\lambda}{2k}} e^{-2\sqrt{\lambda\xi_2}} \qquad \text{(A.3.22)}$$

and using (A.3.7), (A.3.9) and (A.3.15b)

$$\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)\frac{y_1(\xi_1, -\lambda)y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \sim \frac{e^{2Rp\sqrt{\lambda\xi_1}}\, e^{-2Rp\sqrt{\lambda\xi_2}}}{e^{-2Im\sqrt{\lambda}\eta_o}}$$

which vanishes exponentially since $\xi_2 > \xi_1$.

(ii) $\arg\lambda = -\pi/2 \Rightarrow \lambda = -i|\lambda|,\ \arg\sqrt{\lambda} = -\pi/4 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} < 0$

From (A.3.21) we find

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{|\lambda|}{2k} + \frac{1}{2}\right) e^{2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}} \qquad \text{(A.3.23)}$$

and using (A.3.7), (A.3.9) and (A.3.15b)

$$\text{(A.3.5)} \sim \frac{e^{2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}}\, e^{2\sqrt{|\lambda|\xi_1}\cos\frac{\pi}{4}}}{e^{2\sqrt{|\lambda|\eta_o}\cos\pi/4}}$$

which vanishes exponentially if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_o}$.

(iii) $-\pi+\delta \leqslant \arg\lambda < -\pi/2 \Rightarrow \mathrm{Rp}\,\lambda < 0,\ \frac{-\pi+\delta}{2} \leqslant \arg\sqrt{\lambda} < -\pi/4 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} <$

From (A.3.21) we find

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{-\pi\frac{\lambda}{2k}}\, e^{2\sqrt{\lambda\xi_2}} \qquad \text{(A.3.24)}$$

and using (A.3.7), (A.3.9) and (A.3.15b)

$$\text{(A.3.5)} \sim \frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

But in this region $\mathrm{Rp}\sqrt{\lambda} < -\mathrm{Im}\sqrt{\lambda}$, thus

$$\frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}} < \frac{e^{-2\,\mathrm{Im}\sqrt{\lambda\xi_1}}\, e^{-2\,\mathrm{Im}\sqrt{\lambda\xi_2}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

which vanishes exponentially if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_o}$. But as $\arg\sqrt{\lambda}$ approaches $-\pi/4$, $-\mathrm{Im}\sqrt{\lambda}$ approaches $\mathrm{Rp}\sqrt{\lambda}$; hence (A.3.5) vanishes exponentially only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_o}$.

To obtain the behavior of the integrand the factor (A.3.1) is now investigated. As above, $y_1(\eta, \lambda)$ and $y_1'(\eta, \lambda)$ are governed by equations (A.3.15a) and (A.3.15b). Let us consider the previously defined intervals.

(i) $-\pi/2 < \arg\lambda \leqslant -\delta \Longrightarrow \mathrm{Rp}\lambda > 0, \ -\pi/4 < \arg\sqrt{\lambda} \leqslant -\delta/2 \Longrightarrow \mathrm{Rp}\sqrt{\lambda} > 0, \ \mathrm{Im}\sqrt{\lambda} < 0.$

Here $y_2(\eta, \lambda)$ and $y_2'(\eta, \lambda)$ are governed by equations (A.3.18a) and (A.3.18b). Therefore

$$(A.3.1) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{\pi \frac{\lambda}{2k}} \left[e^{-2i\sqrt{\lambda\eta}} e^{2i\sqrt{\lambda\eta_o}} + e^{2i\sqrt{\lambda\eta}} e^{-2i\sqrt{\lambda\eta_o}}\right]$$

Arguing as previously,

$$(A.3.1) \sim e^{2\,\mathrm{Im}\sqrt{\lambda\eta}}\, e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}} + e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta_o}}$$

Thus for the integrand

$$\text{Integrand} \sim e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta}} +$$

$$+ \frac{e^{-2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}} \; \frac{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

which vanishes exponentially since $\xi_2 > \xi_1$ and $\eta \leqslant \eta_o$.

(ii) $\arg\lambda = -\pi/2 \Longrightarrow \lambda = -i|\lambda|, \ \arg\sqrt{\lambda} = \pi/4 \Longrightarrow \mathrm{Rp}\sqrt{\lambda} > 0, \ \mathrm{Im}\sqrt{\lambda} < 0.$

Here we can use Buchholz (1953, p. 100, eq. 20) to assert

$$y_2(\eta, \lambda) \sim \exp\left\{-\frac{|\lambda|}{2k}\log\frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}} \qquad \text{(A.3.25a)}$$

$$y_2'(\eta, \lambda) \sim \exp\left\{-\frac{|\lambda|}{2k}\log\frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}} \qquad \text{(A.3.25b)}$$

But now equation (A.3.13) holds for the Γ-function $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$, hence

$$(A.3.1) \sim e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}} e^{2\sqrt{|\lambda|\eta_o}\cos\frac{\pi}{4}} .$$

Therefore the integrand obeys

$$\text{Integrand} \sim e^{2\sqrt{|\lambda|\xi_2}\cos\frac{\pi}{4}} e^{2\sqrt{|\lambda|\xi_1}\cos\frac{\pi}{4}} e^{-2\sqrt{|\lambda|\eta}\cos\frac{\pi}{4}}$$

which vanishes exponentially if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$.

(iii) $-\pi+\delta \leqslant \arg\lambda < -\pi/2 \Rightarrow \mathrm{Rp}\lambda < 0,\ \frac{-\pi+\delta}{2} \leqslant \arg\sqrt{\lambda} < -\pi/4 \Rightarrow \mathrm{Rp}\sqrt{\lambda} > 0,$
$\mathrm{Im}\sqrt{\lambda} < 0,\ \mathrm{Rp}\sqrt{\lambda} < -\mathrm{Im}\sqrt{\lambda}$

Here $y_2(\eta,\lambda)$ and $y_2'(\eta,\lambda)$ are governed by equations (A.3.16a) and (A.3.16b). Arguing as previously

$$(A.3.1) \sim e^{2\,\mathrm{Im}\sqrt{\lambda\eta}}\, e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}} + e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}\, e^{2\,\mathrm{Im}\sqrt{\lambda\eta_o}}$$

Then for the integrand

$$\text{Integrand} \sim \frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}} + \frac{e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_2}}\, e^{2\,\mathrm{Rp}\sqrt{\lambda\xi_1}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}\, \frac{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

By arguing as above we see that the first term vanishes exponentially if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$, the second term vanishes exponentially if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{\eta} < 2\sqrt{\eta_o}$. But the first condition implies the second, therefore the integrand vanishes exponentially if and only if it holds.

Suppose then $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$ and we again refer to equation (1.12); the above argument shows that the path of integration may be closed by an infinitely large semi-circle in the lower half plane. Then by the use of the residue theorem, $v_N(\xi,\eta)$ can be evaluated as a sum of residues. The residues of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ at the poles $\lambda = -ik(2n+1)$, $n = 0, 1, 2, \ldots$ are $\frac{2k}{(-i)} \frac{(-1)^n}{n!}$, thus equation (1.12) yields

$$v_N(\xi,\eta) = -(2ik)^{-1/2} \sum_{n=0}^{\infty} (-1)^n \frac{y_1\left[\xi_1, ik(2n+1)\right] y_2\left[\xi_2, ik(2n+1)\right]}{y_1'\left[\eta_o, -ik(2n+1)\right]} \left[y_2\left[\eta, -ik(2n+1)\right] \cdot\right.$$

$$\left. \cdot\, y_1'\left[\eta_o, -ik(2n+1)\right] - y_1\left[\eta, -ik(2n+1)\right] y_2'\left[\eta_o, -ik(2n+1)\right]\right] .$$

From the form of the residue series it is seen that the asymptotic forms and arguments given above imply convergence for $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$.

The restriction $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$ may be slightly eased. Using the integral representation of $-e^{-ikR}/R$ we again write (1.12) as

$$v_N(\xi,\eta) = -\frac{e^{-ikR}}{R} - \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_1(\xi_1,-\lambda) y_2(\xi_2,-\lambda)}{y_1'(\eta_o,\lambda)} \cdot$$

$$(0 < \sigma < k)$$

$$\cdot\, y_1(\eta,\lambda) y_2'(\eta_o,\lambda) .$$

The exact arguments used above show that the path of integration of the integral can be closed in the lower half plane when $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta} - \sqrt{\eta}$. We obtain then

$$v_N(\xi,\eta) = -\frac{e^{-ikR}}{R} + (2ik)^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{y_1\left[\xi_1, ik(2n+1)\right] y_2\left[\xi_2, ik(2n+1)\right]}{y_1'\left[\eta_o, -ik(2n+1)\right]} \cdot$$

$$\cdot\, y_1\left[\eta, -ik(2n+1)\right] y_2'\left[\eta_o, -ik(2n+1)\right] ,$$

and $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta_o} - \sqrt{\eta}$ implies the series converges.

It is instructive to consider the behavior when $\sqrt{\xi_1} + \sqrt{\xi_2} = \sqrt{\eta}$ ($\sqrt{\xi_1} + \sqrt{\xi_2} = 2\sqrt{\eta_o} - \sqrt{\eta}$). Then the exponential amplitude factors are equal to 1 and the behavior of the integrand as $|\lambda| \rightarrow \infty$ is governed by the powers of λ which appear. The arguments above show that we need only consider the behavior for $-\pi+\delta \leqslant \arg\lambda \leqslant -\pi/2$; the pertaining asymptotic forms show that the integrand $\sim 1/|\lambda|^{3/4}$. Thus the path of integration can be closed in the lower half plane. But at the poles $|\lambda| = k(2n+1)$, consequently the residue series, whose terms behave like $|\lambda|^{1/4}$ for large λ, does not converge.

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13. ABSTRACT If one considers a paraboloid of revolution of focal length η_0 with an interior point source located anywhere on the axis, the exact solution to the Dirichlet or Neumann problem (Green's function of the first or second kind) may be written in the form of an integral representation. In this report we consider the asymptotic evaluation of these integrals for both low ($k\eta_0 << 1$) and high ($k\eta_0 >> 1$) frequencies. The low frequency results are obtainable from an infinite series over the zeros of a particular Whittaker function, corresponding to a Mie series found in the scattering by closed convex bodies. For high frequencies, we find multiple reflections and caustics arising from saddle point evaluations as well as 'whispering gallery' waves, which arise from the nature of the behavior of the above zeros at high frequencies.

The work at high frequencies is only briefly discussed since it covers research actually in progress. The aim of this discussion is to introduce possible approaches to the problem.

This work on the paraboloid is preliminary to a description of scattering by general concave surfaces. Having integrated the formalism in terms of the physical phenomena such as the whispering gallery waves, multiple reflections and caustics, we are now in a position to search for generalizations of these to other concave surfaces. The scheme we propose is to determine the dependence of the physical effects on the local geometry of the paraboloid and then to make the essentially physical argument that this geometric dependence is the same for other concave shapes. This approach is similar to that used in determining the creeping waves on general convex shapes and is an application of the physical arguments used in Keller's geometric theory of diffraction.

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