

THE UNIVERSITY OF MICHIGAN  
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

ON LINEAR VISCOELASTIC MATERIALS  
WITH AGING OR ENVIRONMENT-DEPENDENT PROPERTIES

Donald C. Stouffer

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy in the  
University of Michigan  
Department of Engineering Mechanics  
1968

December, 1968

IP-828

## ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Assistant Professor Alan S. Wineman, chairman of the doctoral committee, for his instruction, guidance and personal enthusiasm during the period of this study. Also the author is indebted to all the committee members for their interest and help.

The author is grateful for the financial assistance he received during his graduate studies from the National Science Foundation, the Tecumseh Products Company, the Engineering Mechanics Department of the University of Michigan, and the aid of the Industry Program of the College of Engineering in the final preparation of this manuscript.

Finally, the author wishes to acknowledge his appreciation to his wife and family for their sacrifice during the years of his graduate study.

## TABLE OF CONTENTS

|  | <u>Page</u> |
|--|-------------|
| ACKNOWLEDGMENTS .....  | ii          |
| LIST OF FIGURES .....  | iv          |
| NOMENCLATURE .....   | vi          |
| INTRODUCTION .....   | 1           |
| CHAPTER I. THE CONSTITUTIVE RELATIONSHIPS .....  | 6           |
| 1.1 The General Constitutive Law .....   | 6           |
| 1.2 Mechanical Response of Non-Aging<br>Environmental-Dependent Materials .....                | 12          |
| 1.3 Integral Representation for Non-Aging<br>Linear Material Response .....                    | 18          |
| CHAPTER II. NON-AGING MATERIALS WITH ENVIRONMENTAL<br>DEPENDENT PROPERTIES .....               | 25          |
| 2.1 The Mapping Hypothesis .....   | 25          |
| 2.2 Experimental Verification .....  | 34          |
| 2.3 Time-Dependent Environmental Fields .....  | 51          |
| 2.4 Spatially-Dependent Environmental Fields .....   | 59          |
| CHAPTER III. AGING VISCOELASTIC MATERIALS .....  | 62          |
| 3.1 Mathematical Preliminaries .....   | 62          |
| 3.2 An Integral Constitutive Law for Aging<br>Viscoelastic Materials .....                     | 66          |
| 3.3 Correlation with Material Response Data<br>and Existing Response Functions .....           | 72          |
| CHAPTER IV. APPLICATION TO BOUNDARY VALUE PROBLEMS .....                                       | 78          |
| 4.1 Field Equations .....  | 78          |
| 4.2 Plane Strain and Generalized Plane Stress .....  | 82          |
| CHAPTER V. INCOMPRESSIBLE THICKWALL CYLINDER IN PLANE STRAIN .....                             | 88          |
| 5.1 Formulation of the Problem .....   | 88          |
| 5.2 Evaluation of the Material Response Functional .....                                       | 96          |
| 5.3 Solution of the Unsupported Cylinder for<br>Temporally Constant Environmental Fields ..... | 99          |
| 5.4 Solution of the Supported Cylinder .....   | 109         |
| APPENDIX .....   | 111         |
| LIST OF REFERENCES .....   | 113         |

## LIST OF FIGURES

| <u>Figure</u> |  | <u>Page</u> |
|---------------|--|-------------|
| 1             | Application of the Principle of Material<br>Frame Indifference .....   | 10          |
| 2             | Time Shift of the Strain and Environmental Histories<br>Relative to the Creation Time .....                            | 17          |
| 3             | Hypothetical Dependence of a Relaxation Function on<br>the Variation of a Particular Environmental Parameter ....      | 27          |
| 4             | Reduced Material Response Curves for the Purpose<br>of Calculating the Time Scaling Factor .....                       | 31          |
| 5             | Deflection of Bakelite Beams in Bending under a Con-<br>stant Load as a Function of Time and Temperature .....         | 35          |
| 6             | Experimental and Theoretical Data for the Creep of<br>a Nylon Filament at Several Temperatures .....                   | 37          |
| 7             | Experimental and Theoretical Data for the Creep of<br>a Nylon Filament at Several Relative Humidities .....            | 39          |
| 8             | The $\alpha$ and $\beta$ Scaling Functions for a Humidity<br>Dependent Nylon Filament .....                            | 41          |
| 9             | Effect of Humidity on the Relaxation of Nylon<br>at 25°C and 5 Percent Extension .....                                 | 42          |
| 10            | Experimental and Theoretical Creep Data for Five<br>Types of Pure and Plasticized Cold-Setting Epoxy<br>Polymers ..... | 43          |
| 11            | Reduced Creep Curves for Five Types of Pure and<br>Plasticized Cold-Setting Epoxy Polymers .....                       | 45          |
| 12            | Stress Relaxation Curves for Five Styron<br>Polystyrene Resins at 23°C .....   | 47          |
| 13            | Stress Relaxation Curves of Wet $\gamma$ -Irradiated<br>Nylon Fiber in a Vacuum at 4 Percent Strain .....              | 47          |
| 14            | Mapping of the Time Coordinate for a "Thermo-<br>rheologically Simple" Material .....                                  | 54          |
| 15            | A Typical Relaxation Surface for the Response<br>Functions of Equation (3.2.9) .....                                   | 73          |

LIST OF FIGURES (Continued)

| <u>Figure</u> |  | <u>Page</u> |
|---------------|--|-------------|
| 16            | Early Age Creep Surface of Mortar .....  | 77          |
| 17            | A Thick Wall Cylinder Supported by a Thin Elastic Shell ..   | 94          |
| 18            | Influence of $\epsilon_2$ on the Radial Stress<br>Component when $C_1 = 0$ .....                                 | 104         |
| 19            | Influence of $\epsilon_2$ on the Tangential Stress<br>Component when $C_1 = 0$ .....                             | 105         |
| 20            | Influence of $\epsilon_1$ on the Tangential Stress History<br>at the Inner Boundary $r = a$ when $C_2 = 0$ ..... | 107         |
| 21            | Influence of $\epsilon_1$ on the Tangential Stress History<br>at the Outer Boundary $r = b$ when $C_2 = 0$ ..... | 108         |

## NOMENCLATURE

|   |   |
|---|---|
| $A, B, C_m, D_m,$<br>$E_m, F_1, F_2$  | Material response coefficients                      |
| $B$   | A viscoelastic body                                 |
| $C^N, H^N$  | Function class                                      |
| $E$   | Elastic modulus                                     |
| $F_i$   | Components of the body force                        |
| $\underline{F}, \underline{F}^{(k)}, \underline{K},$<br>$\underline{L}^{(k)}$ | Tensorial constitutive functional                   |
| $F_{ijkl}$  | Components of the tensorial constitutive functional |
| $G, \hat{G}$  | One dimensional relaxation function                 |
| $\underline{G}$   | Tensorial relaxation function                       |
| $G_{ijkl}$  | Components of the tensorial relaxation function     |
| $G_1, G_2$  | Scalar relaxation function                          |
| $I_1, I_2$  | Integrals   |
| $J, \hat{J}$  | One dimensional creep function                      |
| $\underline{J}$   | Tensorial creep function                            |
| $J_{ijkl}$  | Components of the tensorial creep function          |
| $J_1, J_2$  | Scalar creep function                               |
| $J_0$   | Bessel function of the first kind                   |
| $K$   | Material function                                   |
| $P, P_0$  | Density   |
| $P_a, P_b$  | Pressure  |
| $Q, \hat{Q}$  | Scalar function                                     |
| $S$   | Time Variable                                       |

|                      |   |
|----------------------|---|
| $S_1, S_2$           | Boundary surface                                  |
| $S_{ij}$             | Components of the deviatoric stress tensor        |
| $T, T_0, T_a, T_b$   | Temperature                                       |
| $T_i$                | Components of a prescribed surface traction       |
| $\underline{U}$      | Prescribed displacement vector                    |
| $U_i$                | Components of a prescribed displacement vector    |
| $U_0$                | Eigenfunction                                     |
| $Y_0$                | Bessel function of the second kind                |
| $a, a_i, b_i, C_i$   | Constants   |
| $g_0, g_i, p$        |   |
| $e_{ij}$             | Components of the deviatoric strain tensor        |
| $f$                  | Scalar function                                   |
| $h$                  | Shell thickness                                   |
| $k$                  | Coefficient of thermal conductivity               |
| $r$                  | Radial coordinate                                 |
| $s$                  | Transformed time variable                         |
| $t, t'$              | Current time                                      |
| $t'_c, t'_i$         | Creation time                                     |
| $t_L, t'_L$          | Beginning of a non-zero deformation history       |
| $u$                  | Radial component of the displacement vector       |
| $\underline{u}$      | Displacement vector                               |
| $u_i$                | Cartesian components of the displacement vector   |
|                      | $(u_r = u)$                                       |
| $u_r, u_\theta, u_z$ | Cylindrical components of the displacement vector |
|                      | $(u_r = u)$                                       |
| $x$                  | Two dimensional position vector                   |

|  |   |
|--|---|
| $\underline{x}$                                    | Three dimensional position vector                           |
| $x_i$  | Cartesian components of a three dimensional position vector |
| $z$  | Cylindrical height coordinate ( $z = x_3$ )                 |
| $\Phi$   | Reference environmental state                               |
| $\alpha, \beta, \delta$                            | Scaling functions associated with $G$                       |
| $\alpha, \hat{\beta}, \hat{\delta}$                | Scaling functions associated with $J$                       |
| $\alpha_m$   | eigenvalues   |
| $\delta$   | Dirac delta   |
| $\delta_{ij}$                                      | Kronecker delta   |
| $\underline{\epsilon}, \hat{\underline{\epsilon}}$ | Tensorial strain history                                    |
| $\epsilon_{ij}$                                    | Components of the tensorial strain history                  |
| $\epsilon_r, \epsilon_\theta, \epsilon_z$          | Principal strain components in cylindrical coordinates      |
| $\eta_i$   | Components of a unit normal vector                          |
| $\eta_m, \mu_m$                                    | Relaxation times  |
| $\theta$   | Angular coordinate  |
| $\lambda, \rho$                                    | Finite time interval  |
| $\nu$  | Poissons ratio  |
| $\tau$   | Reduced time  |
| $\xi_1, \xi_2$                                     | Influence functions   |
| $\underline{\sigma}, \hat{\underline{\sigma}}$     | Tensorial stress history                                    |
| $\sigma_{ij}$                                      | Components of the tensorial stress history                  |
| $\sigma_r, \sigma_\theta, \sigma_z$                | Principal stress components in cylindrical coordinates      |
| $\tau, \tau_1, \tau_2, \tau'_1, \tau'_2$           | Running time variables                                      |
| $\phi, \Phi$                                       | Environmental history                                       |
| $\phi_p$   | Constant environmental history                              |



$\psi_p$

Deviation of an environmental history from a reference state

$\psi, \omega$

Scalar functions

## INTRODUCTION

It has been known for some time that the mechanical response of a viscoelastic material can vary drastically due to small changes in some environmental parameter; for example, temperature. The experimental work of Leaderman<sup>(1)\*</sup> was one of the first comprehensive studies of the influence of temperature on the creep of filamentous materials. From Leaderman's study, Schwarzl and Staverman<sup>(2)</sup> observed that for some materials the family of time and temperature-dependent creep response curves are similar, in that creep curves obtained from each constant temperature test, when plotted on a logarithmic time axis, all have the same basic shape, and the same initial and residual values. However, the position of each curve along the logarithmic time axis was different. The amount of displacement between curves depended only on the temperature. Materials obeying this logarithmic time-temperature shift rule were called "Thermorheologically Simple."

In Reference 3 Morland and Lee establish a method of predicting the mechanical response of a "Thermorheologically Simple" material when the temperature is a function of both time and spatial coordinates. Their theory is based on the assumption that in each infinitesimal increment of time the material still obeys the time-temperature shift rule. This basic assumption gives rise to a mapped-time coordinate representation, where the mapping function depends on temperature history and the shift properties of the material.

A second type of property that can influence the mechanical response of a viscoelastic material is age. There exists a large class of

---

\* The numbers raised in parenthesis designate references.

materials, for example, concrete, mortar, adhesives, etc., whose mechanical properties depend on an intrinsic material time or age.

Currently in the literature there exist two basic approaches to mathematically model an aging viscoelastic response. The first method is to use a differential equation with variable coefficients.<sup>1</sup> The second, and most popular, approach is with an integral type constitutive equation.<sup>2</sup>

The motivation for the present study comes from the need for contributions to the theory and method of analysis for treating boundary value problems involving viscoelastic materials with aging and environmental-dependent properties. Thus, the objective of this study can be stated in four distinct parts; (1) to establish the basic theoretical framework of constitutive equations for materials with aging and environmental-dependent properties, (2) to specialize this framework for a larger class of environmental-dependent materials and a larger set of environmental properties than currently exist in the literature, (3) to specialize this framework for aging materials and to obtain a constitutive relationship useful for solving boundary value problems, (4) to apply these specialized constitutive equations to boundary value problems.

The first part of this study is based on the constitutive assumption that the stress is related to the strain and environmental histories through a functional that explicitly depends on the current time and the time the material is made. (The environmental history

---

<sup>1</sup> For example, see Reference 4.

<sup>2</sup> Some typical examples in this case are References 5, 6, 7, and 8.

includes all of the constituents of the environment; for example, temperature, humidity, radiation, etc.) This general constitutive law is simplified by application of the Principle of Material Frame Indifference. Next, it is shown that the aging response reduces to a non-aging response when the aging process becomes complete. This transition requires investigating the effect of the environment and aging process. And finally, it is shown that if the non-aging functional is linear in the strain measure, then the functional can be represented by a Riemann-Stieltjes integral. The generating function of the Riemann-Stieltjes integral turns out to be a functional that depends on the environmental history and has the physical interpretation of a creep or relaxation response functional.

The second part of this study establishes an appropriate representation for the environmental-dependent creep and relaxation response functionals. This is approached by first re-examining the time and environmental-dependent creep response data for temporally constant environmental fields. It is found in many cases that the family of creep curves which result from changes in some environmental parameter--say, humidity--have generally the same shape. Thus, the development rests on the hypothesis that the response curve at one constant humidity value can be mapped onto the response curve at another constant humidity value by; (1) displacing the curve along the logarithmic time axis, (2) scaling the initial elastic response, and (3) scaling the long term or residual response.

The mapping hypothesis, as proposed, contains three adjustable coefficients. It is demonstrated that these three coefficients are

sufficient for good correlation between the hypothesis and the experimental data. Further, it is demonstrated that the hypothesis is valid for a larger class of materials and environmental effects than the "Thermorheologically Simple" theory. Also, a set of relationships is established between the adjustable coefficients for associated creep and relaxation data.

The hypothesis is next extended to include time and spatially dependent environmental histories. Like the work of Morland and Lee, this extension is based on the assumption that in each infinitesimal increment of time the material obeys the mapping hypothesis. However, the development and final results are quite different in this case.

The third objective is to specialize the general functional stress-strain law for aging materials. This is also approached by assuming that the functional is linear in the strain measure. Again, it is established that the functional can be represented by a Riemann-Stieltjes integral. The generating function of this integral depends on two time variables and can be physically interpreted as an aging creep or relaxation response function. This integral law is different from any currently existing in the literature. The integral representation which arises from the functional constitutive law gives rise to an operator algebra which is convenient for solving boundary value problems. Further, it is shown that this operator algebra reduces to Riemann-Stieltjes Convolution algebra for non-aging material response functions.

To bridge the gap between the constitutive theory and a boundary value problem, the field equations are presented next. The general conditions that the solution of a three-dimensional boundary value problem

must satisfy are stated in terms of the aforementioned operator algebra. Also, the field equations are given for the plane strain and generalized plane stress type boundary value problems in terms of the operator algebra.

The study is concluded with the analysis of a thick-wall, non-aging, environmental-dependent, viscoelastic cylinder for two different sets of boundary conditions. The first case is for arbitrary internal and external pressures. The second case is for a viscoelastic cylinder supported by a thin elastic shell with an arbitrary internal pressure. The general mapping hypothesis is used to characterize the material response and exact solutions are obtained for temporally constant environmental fields. A parameterization study is included as part of the solution to show the effect of the environment on the material response.

One final comment should be made regarding this study. A phenomenological approach is used throughout. While it is recognized that aging and environmental phenomena may effect the mechanical properties of the material, no attempt is made to correlate these phenomena to changes in the molecular structure of the material.

## CHAPTER I

### THE CONSTITUTIVE RELATIONSHIPS

In Chapter I two equivalent forms of the constitutive equations are established for aging and environmental-dependent viscoelastic materials. Next these functional equations are specialized for non-aging materials. With the assumption of small deformations the functionals are taken to be linear in the strain measure. The constitutive equations then take on a form analogous to the classical theory of linear viscoelasticity.

#### 1.1 The General Constitutive Law

There are several time variables which can appear in any constitutive assumption. In this section the manner in which the constitutive equation depends on these variables is investigated. To begin, denote a sequence of events on a time scale where the following times are measured relative to an arbitrary origin. Let

$t_c$  represent the creation time of the material,

$t_L$  represent the beginning of a non-zero deformation history

$$(t_L \geq t_c),$$

$t$  represent the current time, and

$\mathcal{T}$  represent the running time variable ( $t_c \leq \mathcal{T} \leq t$ ).

Here  $t_c$  denotes an appropriate beginning of any aging phenomenon that may be present. Usually this will be the time when the material is made. However, for a particular problem it may be more suitable to consider  $t_c$  as the time when the material is cast into some shape. In any case,  $t_c$  is the first reference in time and no event is considered prior to  $t_c$ .

Let  $\sigma_{ij}(\underline{x}, t)$  and  $\epsilon_{ij}(\underline{x}, t)$  be the components in some cartesian coordinate system of the stress and infinitesimal strain tensors of a particle occupying position  $\underline{x}$  at time  $t$ , in body  $\mathbb{B}$ ; and, denote them by  $\underline{\sigma}$  and  $\underline{\epsilon}$  respectively. Let the components of  $\underline{\sigma}$  and  $\underline{\epsilon}$  be defined and continuous on  $[t_c, \infty)$ . The notation  $[a, b]$  represents the closed interval with the endpoints  $a$  and  $b$ ;  $(a, b)$  is the open interval without the endpoints  $a$  and  $b$ ;  $[a, b)$  and  $(a, b]$  denotes the mixed intervals open on the right and left ends respectively.

Next, consider the environmental parameters that can influence the mechanical response of a material; for example, temperature, humidity, concentration, radiation intensity, etc. Let  $\phi(\underline{x}, t)$  represent the set of all environmental properties of the particle at position  $\underline{x}$  in body  $\mathbb{B}$ . Assume  $\phi(\underline{x}, t)$  is defined and continuous for all  $t$  in the interval  $[t_c, \infty)$ .

The following constitutive statement can now be made:

ASSUMPTION 1.1. The stress at any given particle  $\underline{x}$  in body  $\mathbb{B}$  at some time  $t$  is completely determined by the strain and environmental histories at the particle on the interval  $[t_c, t]$ , and depends on the present time  $t$  and the creation time  $t_c$  of the material.

The dependence of  $\underline{\sigma}$  on the histories confined to the time interval  $[t_c, t]$  incorporates the property of non-retroactivity which is part of the constitutive definition of Gurtin and Sternberg.<sup>1</sup> Further it will be assumed that the aging and environmental phenomena are not coupled to

---

<sup>1</sup> See Reference 9 page 303.



the deformation process through the constitutive equations. Any coupling that does occur can be accounted for in the appropriate field equations. For example, if a material has a temperature sensitive response, assume this response is influenced only by the temperature field itself and not the source of the temperature field. From the practical aspect of solving boundary value problems, the strain history is assumed to be infinitesimal and usually any coupling between the strain history and environmental field can be neglected. Assumption 1.1 can be characterized by the equation

$$\underline{\sigma}(t) = \underline{\mathcal{F}} \left[ \underline{\underline{\xi}}(\underline{\tau}_1)^t ; \underline{\underline{\phi}}(\underline{\tau}_2)^t ; t, t_c \right] \quad (1.1.1)$$

where  $\underline{\mathcal{F}}$  is a functional relating a stress history  $\underline{\sigma}(t)$  to each strain and environmental history  $\underline{\underline{\xi}}(t)$  and  $\underline{\underline{\phi}}(t)$  for every  $t$  and  $t_c$  in  $(-\infty, \infty)$ . The quantity  $\underline{\underline{\xi}}(t)$  is the total strain history at a particle and represents the sum of the mechanical and environmental strains. Thus if  $\underline{\underline{\xi}} = 0$  for  $t$  in  $(-\infty, \infty)$ , then the function  $\underline{\mathcal{F}}$  must be such that  $\underline{\sigma}(t) = 0$  for all  $t$  in  $(-\infty, \infty)$ .

The constitutive Equation (1.1.1) is subject to the restriction of the principle Material Frame-Indifference.<sup>2</sup> Since the current study is restricted to infinitesimal deformations and is concerned with primarily the temporal aspects of material response only the time-translation portion of the frame-transformation is considered here. This restriction embodies the principle that in a scale transformation of all events, time intervals and the sense of time are preserved. The environmental parameters to be considered are all scalars. The Principle of Material

---

<sup>2</sup> See Truesdall<sup>(10)</sup> page 22.

Frame Indifference imposes restrictions on tensorial quantities also. Although such restrictions can be easily imposed on Equation (1.1.1) by standard methods, this is not the purpose of the study. Hence, the constitutive law given by Equation (1.1.1) must be invariant under a shift of all events with respect to the time origin.

In order to impose the restriction of Material Frame-Indifference consider a second set of events occurring  $a$  time units earlier than the first (See Figure 1). The significant time variables are given by:

$$\begin{aligned} t' &= t - a, \\ \gamma'_m &= \gamma_m - a, \quad (m = 1, 2) \end{aligned} \quad (1.1.2)$$

and

$$t'_c = t_c - a$$

while the strain and environmental histories are defined by

$$\bar{\underline{\underline{\xi}}}(t') = \bar{\underline{\underline{\xi}}}(t - a) = \underline{\underline{\xi}}(t) \quad (1.1.3)$$

and

$$\bar{\phi}(t') = \bar{\phi}(t - a) = \phi(t).$$

The stress at time  $t'$  can be determined from Equation (1.1.1) as

$$\bar{\underline{\underline{\sigma}}}(t') = \underline{\underline{\mathcal{F}}} \left[ \bar{\underline{\underline{\xi}}}(\gamma'_1) ; \bar{\phi}(\gamma'_2) ; t', t'_c \right]. \quad (1.1.4)$$

$\gamma'_1 = t'_c \quad \gamma'_2 = t'_c$

The principle of Material Frame-Indifference requires

$$\underline{\underline{\sigma}}(t) = \bar{\underline{\underline{\sigma}}}(t'). \quad (1.1.5)$$

Equation (1.1.5) together with (1.1.1) and (1.1.4) establishes a restriction on the function  $\underline{\underline{\mathcal{F}}}$  ;

$$\underline{\underline{\mathcal{F}}} \left[ \underline{\underline{\xi}}(\gamma_1) ; \phi(\gamma_2) ; t, t_c \right] = \underline{\underline{\mathcal{F}}} \left[ \bar{\underline{\underline{\xi}}}(\gamma'_1) ; \bar{\phi}(\gamma'_2) ; t', t'_c \right] \quad (1.1.6)$$

$\gamma_1 = t_c \quad \gamma_2 = t_c \quad \gamma'_1 = t'_c \quad \gamma'_2 = t'_c$

Equation (1.1.6) can give rise to two representations of the functional  $\underline{\underline{\mathcal{F}}}$  .

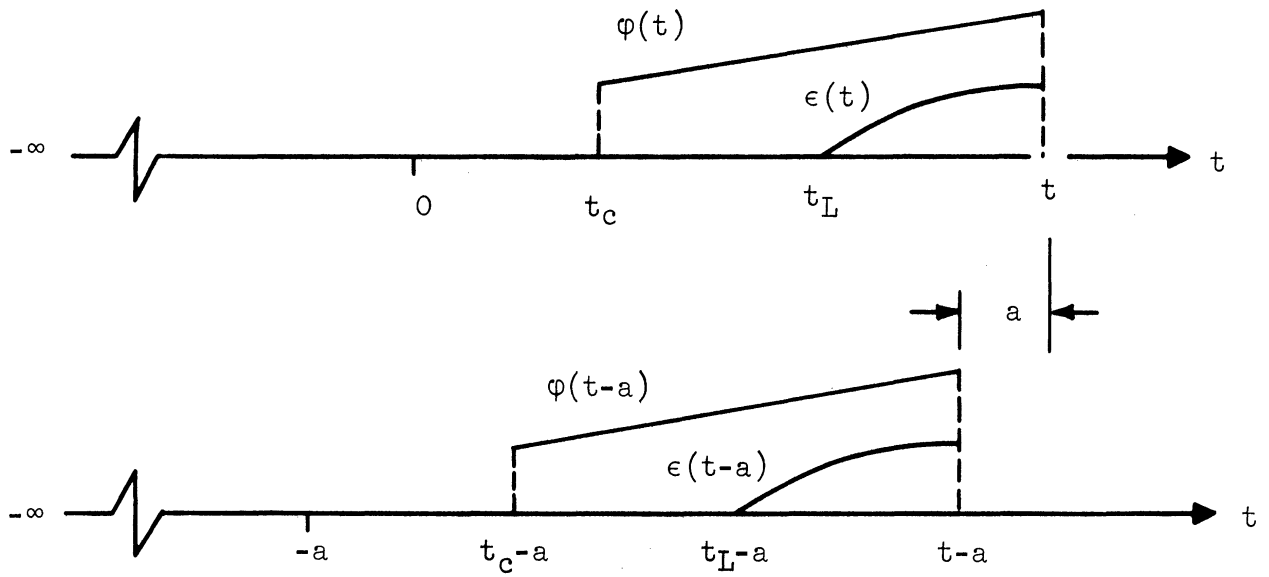


Figure 1. Application of the Principle of Material Frame Indifference.

First let  $a = t$  in Equation (1.1.2). This refers all events to the current time. Then,

$$\begin{aligned} t' &= 0, \\ \gamma'_m &= \gamma_m - t \quad (m=1,2), \end{aligned} \quad (1.1.7)$$

and  $t'_c = t_c - t;$

and Equation (1.1.4) is

$$\underset{\sim}{\mathcal{F}} \left[ \underset{\gamma_1=t_c}{\overset{t}{\mathcal{E}}}(\gamma_1); \underset{\gamma_2=t_c}{\overset{t}{\mathcal{P}}}(\gamma_2); t, t_c \right] = \underset{\sim}{\mathcal{F}} \left[ \underset{\gamma'_1=-(t-t_c)}{\overset{0}{\mathcal{E}}}(t+\gamma'_1); \underset{\gamma'_2=-(t-t_c)}{\overset{0}{\mathcal{P}}}(t+\gamma'_2); 0, t-t_c \right].$$

Defining  $\gamma'_n = -s_n$  ( $n = 1,2$ ) and  $\underset{\sim}{\mathcal{F}}^{(1)}$  to be a new functional, the constitutive equation becomes

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathcal{F}}^{(1)} \left[ \underset{s_1=0}{\overset{t-t_c}{\mathcal{E}}}(t-s_1); \underset{s_2=0}{\overset{t-t_c}{\mathcal{P}}}(t-s_2); t-t_c \right]. \quad (1.1.8)$$

The second form can be derived by letting  $a = t_c$ , i.e., refer all events to the creation time. Equation (1.1.2) becomes

$$\begin{aligned} t' &= t - t_c, \\ \gamma'_m &= \gamma_m - t_c \quad (m=1,2), \end{aligned} \quad (1.1.9)$$

and  $t'_c = 0.$

Equation (1.1.6) can now be written as

$$\underset{\sim}{\mathcal{F}} \left[ \underset{\gamma_1=t_c}{\overset{t}{\mathcal{E}}}(\gamma_1); \underset{\gamma_2=t_c}{\overset{t}{\mathcal{P}}}(\gamma_2); t, t_c \right] = \underset{\sim}{\mathcal{F}} \left[ \underset{\gamma_1=0}{\overset{t-t_c}{\mathcal{E}}}(\gamma_1+t_c); \underset{\gamma_2=0}{\overset{t-t_c}{\mathcal{P}}}(\gamma_2+t_c); t-t_c, 0 \right].$$

Defining  $\underset{\sim}{\mathcal{F}}^{(2)}$  to be a new functional, then

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathcal{F}}^{(2)} \left[ \underset{\gamma_1=0}{\overset{t-t_c}{\mathcal{E}}}(\gamma_1+t_c); \underset{\gamma_2=0}{\overset{t-t_c}{\mathcal{P}}}(\gamma_2+t_c); t-t_c \right]. \quad (1.1.10)$$

In Equations (1.1.8) and (1.1.10) the time transformation demonstrates the fact that for measuring material response there is no

specific time origin. Thus, the Principle of Material Frame Indifference is satisfied if the resulting functionals  $\mathcal{F}^{(1)}$  and  $\mathcal{F}^{(2)}$  depend on the time elapsed since creation  $t - t_c$  (that is, the "age" of material), and the strain and environmental states occurring during this elapsed time. Since  $\mathcal{F}^{(1)}$  and  $\mathcal{F}^{(2)}$  depend only on the quantities measured between  $t_c$  and  $t$ , no generality is lost in setting  $t_c$  equal to zero. Equation (1.1.1) can now be written as

$$\underline{\sigma}(t) = \mathcal{F}^{(1)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ s_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ s_2=0 \end{smallmatrix} \right); t \right] \quad (1.1.11)$$

or

$$\underline{\sigma}(t) = \mathcal{F}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ \gamma_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ \gamma_2=0 \end{smallmatrix} \right); t \right]. \quad (1.1.12)$$

Note that on setting  $t_c = 0$  the simplest form of the constitutive equations for aging materials is obtained. For special considerations it is often more convenient to use some other point on the time scale as the origin.

Equations (1.1.11) and (1.1.12) describe two equivalent forms of the stress-strain law. In Equation (1.1.11) the arguments of the strain and environmental histories are measured relative to the present time. In Equation (1.1.12) the arguments of  $\underline{\varepsilon}$  and  $\phi$  are measured relative to the creation time. For both equations  $t$  denotes the age of the material, since  $t_c$  was chosen as the origin of the time scale.

## 1.2 Mechanical Response of Non-Aging Environmental Dependent Materials

Non-aging response may be thought of as the limit of the aging response as the aging process becomes complete. This transition from an

aging constitutive law to a non-aging constitutive law can be considered to have three phases. (1) The first phase consists of the aging process which occurs in the undeformed state. The environmental history which is present during the phase interacts with the aging phenomenon and affects the mechanical properties on the fully cured material. (2) The second phase is the effect of the environment on the fully cured material prior to any deformation. (3) The third phase is the mechanical response of the fully cured material due to simultaneous deformation and environmental histories.

For the purpose of investigating the first phase of the transition, it is convenient to use the stress-strain law as given by Equation (1.1.8) or (1.1.10), where the origin on the time scale is arbitrary. Now introduce a quantity  $\rho$  that represents the amount of time required for the material to become fully cured. That is, when  $t - t_c \geq \rho$ , the aging process is defined as complete. In general, the amount of time  $\rho$  required for the aging process to become complete will depend on the environmental history occurring simultaneously with the aging process. Also, the resulting mechanical properties of the fully cured material will depend on the environment during the aging process.

Consider now pairs of strain and environmental histories. For some  $t_L > t_c + \rho$  let each strain history satisfy  $\underline{\underline{\epsilon}}(\tau) = 0$  on  $(t_c, t_L)$ , while all environmental histories have the same value  $\phi(\tau)$  on  $(t_c, t_c + \rho)$ . For such histories Equation (1.1.8) and (1.1.10) can be redefined as:

$$\underline{\underline{\sigma}}(t) = \underline{\underline{F}}^{(1)} \left[ \underset{s_1=0}{\underline{\underline{\epsilon}}(t-s_1)} ; \underset{s_2=0}{\phi(t-s_2)} ; \underset{s_2=t-t_c-\rho}{\phi(t-s_2)} ; t-t_c \right] \quad (1.2.1)$$

and

$$\underline{\sigma}(t) = \underline{\mathcal{F}}^{(2)} \left[ \underline{\underline{\epsilon}}(\underline{\gamma}_1); \phi(\underline{\gamma}_2); \phi(\underline{\gamma}_2); t-t_c \right] \quad (1.2.2)$$

$\begin{matrix} t & t & t+\rho \\ \underline{\gamma}_1=t_L & \underline{\gamma}_2=t_c+\rho & \underline{\gamma}_2=t_c \end{matrix}$

The mechanical response of the fully cured material depends on  $\underline{\underline{\epsilon}}(\underline{\gamma})$  for  $\underline{\gamma}$  in  $[t_L, t]$  and on  $\phi(\underline{\gamma})$  in  $[t_c + \rho, t]$ . Changing  $\phi(\underline{\gamma})$  on  $[t_c, t + \rho)$  changes the mechanical response due to the strain history  $\underline{\underline{\epsilon}}(\underline{\gamma})$  on  $[t_L, t]$ . This means the aging process coupled with a particular set of environmental properties produces a cured material with a mechanical response that is completely described by the new functionals. In what follows,  $\phi(\underline{\gamma})$  on  $[t_c, t_c + \rho)$  is to be considered fixed and will be suppressed from Equation (1.2.1) and (1.2.2).

In order to obtain the simplest form for a non-aging constitutive law shift the origin on the time scale to  $t_L$ , and note that  $t_c$  is now measured relative to  $t_L$  and is negative. The age of the material in Equation (1.2.1) and (1.2.2) is given by  $t - t_c$ . The age increment between  $t_c$  and  $t_L = 0$  is fixed for a particular material and strain history, so let  $t$  denote the age of the material relative to the new time origin (see Figure 2). Let  $\lambda = t_c + \rho$  and rewrite Equation (1.2.1) and (1.2.2) as

$$\underline{\sigma}(t) = \underline{\mathcal{F}}^{(1)} \left[ \underline{\underline{\epsilon}}(t-s_1); \phi(t-s_2); t \right] \quad (1.2.3)$$

$\begin{matrix} t & t-\lambda \\ s_1=0 & s_2=0 \end{matrix}$

and

$$\underline{\sigma}(t) = \underline{\mathcal{F}}^{(2)} \left[ \underline{\underline{\epsilon}}(\underline{\gamma}_1); \phi(\underline{\gamma}_2); t \right], \quad (1.2.4)$$

$\begin{matrix} t & t \\ \underline{\gamma}_1=0 & \underline{\gamma}_2=-\lambda \end{matrix}$

where  $\underline{\mathcal{F}}^{(1)}$  and  $\underline{\mathcal{F}}^{(2)}$  are again defined as new functionals.

Now consider the second phase of the transition. Physically, it is reasonable to assume for some materials that the stress at the

present time  $t$  depends only on the strain history and the concurrent environmental history. In other words, the environmental history on the interval  $[-\lambda, 0)$  does not affect the stress at time  $t$  when  $\underline{\underline{\epsilon}}(\gamma) = 0$  on  $[-\lambda, 0)$ . This idea can now be stated formally.

ASSUMPTION 1.2. Let  $\phi(t)$  and  $\bar{\phi}(t)$  be defined and continuous environmental histories on  $[-\lambda, t]$  with the property  $\phi(t) \neq \bar{\phi}(t)$  for  $t$  in  $[-\lambda, 0)$ , and  $\phi(t) = \bar{\phi}(t)$  for  $t$  in  $[0, t]$ . Now assume the stress given by Equation (1.2.3) or (1.2.4) is identical for  $\phi(t)$  and  $\bar{\phi}(t)$ .

Employing Assumption 1.2, Equations (1.2.3) and (1.2.4) become

$$\underline{\underline{\sigma}}(t) = \underline{\underline{L}}^{(1)} \left[ \underline{\underline{\epsilon}}_{s_1=0}^t; \phi_{s_2=0}^t; t \right] \quad (1.2.5)$$

and

$$\underline{\underline{\sigma}}(t) = \underline{\underline{L}}^{(2)} \left[ \underline{\underline{\epsilon}}_{\gamma_1=0}^t(\gamma_1); \phi_{\gamma_2=0}^t(\gamma_2); t \right]. \quad (1.2.6)$$

The final phase of the transition from an aging constitutive law to a non-aging constitutive law rests on the idea that for a fully cured material, the mechanical response is the same under a shift of the strain and environmental histories with respect to the material age. Thus, it can be shown that the constitutive Equations (1.2.5) and (1.2.6) are independent of the explicit time variable  $t$ .

To begin let  $\underline{\underline{\epsilon}}(\gamma)$  denote a strain history which vanishes on  $(-\infty, 0)$  and is non-zero on  $[0, t]$ . Let  $\phi(t)$  be the concurrent environmental history. Then define a second set of strain and environmental histories by



$$\bar{\underline{\underline{\xi}}}(t-a) = \underline{\underline{\xi}}(t) \quad (1.2.7)$$

and

$$\bar{\phi}(t-a) = \phi(t), \quad (1.2.8)$$

where the constant  $\underline{a}$  is arbitrary. The stress state  $\bar{\underline{\underline{\sigma}}}$  which results at time  $t - a$ , due to the strain history  $\bar{\underline{\underline{\xi}}}$  and the concurrent environmental history  $\bar{\phi}$  can be determined from Equation (1.2.5) as:

$$\bar{\underline{\underline{\sigma}}}(t-a) = \underline{\underline{\underline{\mathcal{L}}}}^{(u)} \left[ \bar{\underline{\underline{\xi}}}(t-a-s_1); \bar{\phi}(t-a-s_2); t-a \right]. \quad (1.2.9)$$

Now for  $t - t_c \geq \rho$ , the two stresses are invariant under a time shift with respect to the creation time; i.e.,

$$\underline{\underline{\sigma}}(t) = \bar{\underline{\underline{\sigma}}}(t-a). \quad (1.2.10)$$

Combining Equations (1.2.6), (1.2.9) and (1.2.10) establishes the following restriction on the constitutive functional

$$\begin{aligned} \underline{\underline{\underline{\mathcal{L}}}}^{(u)} \left[ \underline{\underline{\underline{\xi}}}(t-s_1); \phi(t-s_2); t \right] = \\ \underline{\underline{\underline{\mathcal{L}}}}^{(u)} \left[ \bar{\underline{\underline{\xi}}}(t-a-s_1); \bar{\phi}(t-a-s_2); t-a \right]. \end{aligned} \quad (1.2.11)$$

Substituting Equation (1.2.7) and (1.2.8) into Equation (1.2.11) yields

$$\underline{\underline{\underline{\mathcal{L}}}}^{(u)} \left[ \underline{\underline{\underline{\xi}}}(t-s_1); \phi(t-s_2); t \right] = \underline{\underline{\underline{\mathcal{L}}}}^{(u)} \left[ \underline{\underline{\underline{\xi}}}(t-s_1); \phi(t-s_2); t-a \right].$$

Thus, it can be seen that the constitutive functional is independent of the explicit time variable since the constant  $\underline{a}$  is arbitrary.

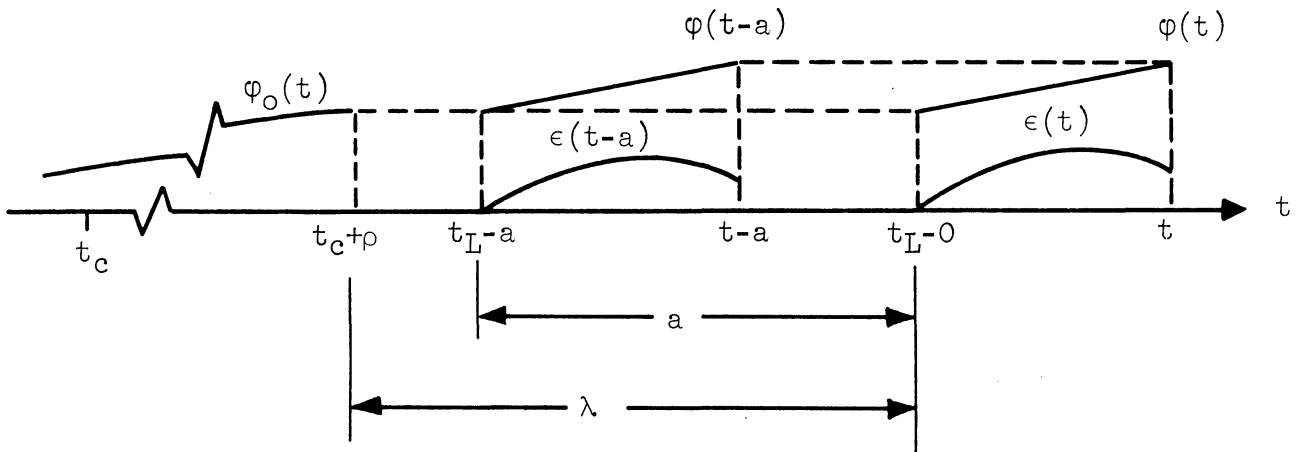


Figure 2. Time Shift of the Strain and Environmental Histories Relative to the Creation Time.

Hence for non-aging materials the constitutive Equation (1.2.1) becomes

$$\underline{\sigma}(t) = \mathcal{K} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ s_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ s_2=0 \end{smallmatrix} \right) \right]. \quad (1.2.12)$$

Now turn to the alternate form of the stress-strain law as given by Equation (1.2.6). Again the stress  $\bar{\underline{\sigma}}(t')$  can be calculated as

$$\begin{aligned} \bar{\underline{\sigma}}(t-a) &= \underline{\mathcal{L}}^{(2)} \left[ \bar{\underline{\varepsilon}} \left( \begin{smallmatrix} t-a \\ \gamma_1=-a \end{smallmatrix} \right); \bar{\phi} \left( \begin{smallmatrix} t-a \\ \gamma_2=-a \end{smallmatrix} \right); t-a \right] \\ &= \underline{\mathcal{L}}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t-a \\ \gamma_1=-a \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t-a \\ \gamma_2=-a \end{smallmatrix} \right); t-a \right]. \end{aligned}$$

Again employing Equation (1.2.10), the above becomes

$$\underline{\mathcal{L}}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ \gamma_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ \gamma_2=0 \end{smallmatrix} \right); t \right] = \underline{\mathcal{L}}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t-a \\ \gamma_1=-a \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t-a \\ \gamma_2=-a \end{smallmatrix} \right); t-a \right]. \quad (1.2.13)$$

Letting  $a = t$  and defining  $\gamma'_n = -s_n$ , Equation (1.2.13) can be written as

$$\underline{\mathcal{L}}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ \gamma_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ \gamma_2=0 \end{smallmatrix} \right); t \right] = \underline{\mathcal{L}}^{(2)} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} 0 \\ s_1=t \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} 0 \\ s_2=t \end{smallmatrix} \right); 0 \right]. \quad (1.2.14)$$

Thus, for non-aging environmental dependent materials the viscoelastic response can again be characterized by the functional equation

$$\underline{\sigma}(t) = \mathcal{K} \left[ \underline{\varepsilon} \left( \begin{smallmatrix} t \\ s_1=0 \end{smallmatrix} \right); \phi \left( \begin{smallmatrix} t \\ s_2=0 \end{smallmatrix} \right) \right]. \quad (1.2.12)$$

Therefore, for the constitutive relationship given in Assumption 1.1 there is only one form of the stress strain law that will satisfy both the Principle of Material Frame Indifference and the condition of translation invariance.

### 1.3 Integral Representation for Non-Aging Linear Material Response

The contents of this section follow closely the work of Gurtin and Sternberg.<sup>(9)</sup> Their developments with the linear theory of viscoelasticity

can be easily modified to include the generalizations presented in this study by allowing all possible strain histories for each fixed environmental history. Thus, for completeness, some of the classical theorems associated with the linear theory of viscoelasticity are extended to include the case at hand and are presented here without proof.

The constitutive Equations (1.2.12) characterize the stress-strain relation of the linear theory of viscoelasticity, if the deformations are small, and if the superposition principle is valid. Assuming this to be true, it is now possible to represent the functional  $\mathcal{K}$  by a Riemann-Stieltjes integral.

THEOREM 1.1.<sup>3</sup> (Linear hereditary stress-strain law) Let the stress be related to the strain by

$$\underline{\sigma}(t) = \mathcal{K} \left[ \underline{\varepsilon}_{s_1=0}(t-s_1); \phi_{s_2=0}(t-s_2) \right];$$

where,

- (a)  $\underline{\varepsilon}$  and  $\phi$  are defined and continuous on  $(-\infty, \infty)$ ;
- (b)  $\underline{\varepsilon}$  vanishes on  $(-\infty, 0)$ .

Further, let  $\mathcal{K}$  have the following properties:

- (c) the stress  $\underline{\sigma}(t)$  depends only on the past values of strain (non-retroactivity);
- (d) for every  $\underline{\varepsilon}(t) = \underline{\varepsilon}(t-a)$  and any fixed "a" and all t in  $(-\infty, \infty)$ , then  $\underline{\sigma}(t) = \underline{\sigma}(t-a)$  for all t in  $(-\infty, \infty)$  (translation-invariance);
- (e) for every  $\underline{\varepsilon}_1(t)$  and  $\underline{\varepsilon}_2(t)$  defined on  $(-\infty, \infty)$ , and every pair of constants  $a_1$  and  $a_2$ , then

---

<sup>3</sup> See Section 2 of Gurtin and Sternberg. (9)

$$\mathcal{K}[a_1 \underline{\varepsilon}_1 + a_2 \underline{\varepsilon}_2; \phi] = a_1 \mathcal{K}[\underline{\varepsilon}_1; \phi] + a_2 \mathcal{K}[\underline{\varepsilon}_2; \phi] \quad (\text{linearity});$$

- (f) for every fixed  $t$  and every  $\alpha > 0$ , there exists a  $\delta_t(\alpha) > 0$  such that  $[\underline{\varepsilon}_{ij}(t) \varepsilon_{ij}(t)]^{1/2} < \delta_t(\alpha)$  for all  $t$  in  $(-\infty, \infty)$  implies  $[\underline{\sigma}_{ij}(t) \sigma_{ij}(t)]^{1/2} < \alpha$  (continuity).

Then

$$\underline{\sigma}(t) = \int_{-\infty}^t \underline{\varepsilon}(t-\gamma) d\underline{\mathcal{G}}[\gamma, \phi(t-s)], \quad (1.3.1)$$

where  $\underline{\mathcal{G}}$  has the following properties:

- (g)  $\underline{\mathcal{G}}$  is a fourth order tensor with  $G_{ijkl} = G_{jikl} = G_{ijlk}$  for all  $t$  in  $(-\infty, \infty)$ ;
- (h)  $\underline{\mathcal{G}}$  is of bounded variation on every sub-interval in  $(-\infty, \infty)$ ;
- (i)  $\underline{\mathcal{G}}$  vanishes on  $(-\infty, 0)$ ; and,
- (j)  $\underline{\mathcal{G}}$  is continuous on the right, that is  $G[\gamma, \phi(t-s)] = \underline{\mathcal{G}}[\gamma^+, \phi(t-s)]$ .

The constitutive equation of Theorem 1.1 does not yet embody the restriction of Assumption 1.2. First, for convenience with the development, let the argument of the environmental history  $t-s$  in Equation (1.3.1) be replaced by  $s$ , then the environmental history can be written as  $\phi(s)$  in the interval  $[0, t]$ . Next, consider a unit step strain history which is applied at time  $t_0 > 0$ , then the stress at time  $t$  as given by Equation (1.3.1) is

$$\underline{\sigma}(t) = \underline{\mathcal{G}}[t-t_0, \phi(s)],$$

Assumption 1.2 implies that the stress at time  $t$  is independent of the environmental history on the time interval  $(-\infty, t_0]$ . Thus, the

equation above can be rewritten as

$$\underline{\sigma}(t) = \underline{G} [t-t_0, \phi(s)]_{s=t_0}^t \quad (1.3.2)$$

The result of Theorem 1.1 must be modified so that Assumption 1.2 is satisfied.

THEOREM 1.2. Let the stress be related to the strain and environmental histories by Equation (1.3.1) and let Equation (1.3.1) satisfy Assumption 1.2. Then the stress at time  $t$  is given by

$$\underline{\sigma}(t) = \int_{-\infty}^t \underline{\underline{\epsilon}}(t-\gamma) d\underline{G} [\gamma, \phi(s)]_{s=t-\gamma}^t \quad (1.3.3)$$

The stress-strain law (1.3.3) can further be specialized by considering the isotropy properties of the material in question. For the full isotropy group of rotations the fourth order tensor  $\underline{G}$  can be replaced by two scalar functions. Rogers and Pipkin<sup>(11)</sup> established the representations of  $\underline{G}$  for many of the sub groups of the full isotropy group of rotations. The extension of these results to include Equation (1.3.3) follows directly since isotropy rotations are valid for each environmental history.

For convenience, assume the material is isotropic and homogeneous. Define the deviatoric stress and strain tensors as

$$S_{ij} = \sigma_{ij}(t) - \frac{1}{3} \sigma_{kk}(t) \delta_{ij} \quad (1.3.4)$$

and

$$C_{ij} = E_{ij}(t) - \frac{1}{3} E_{kk}(t) \delta_{ij}, \quad (1.3.4)$$

where the quantity  $\delta_{ij}$  is the Kronecker delta. Gurtin and Sternberg's theorem for the representation of linear isotropic hereditary stress-strain laws can now be applied.<sup>4</sup>

THEOREM 1.3. If the material is isotropic, then corresponding to every linear hereditary stress-strain law there exists two real-valued functionals  $G_n$  ( $n = 1, 2$ ) such that

$$S_{ij}(t) = \int_{-\infty}^t e_{ij}(t-\gamma) dG_1[\gamma, \phi(S)]_{s=t-\gamma}^t$$

(1.3.5)

and

$$\sigma_{kk}(t) = \int_{-\infty}^t E_{kk}(t-\gamma) dG_2[\gamma, \phi(S)]_{s=t-\gamma}^t$$

The function  $G_1$  is the shear relaxation function and  $G_2$  the dilatational relaxation function.

It is possible to remove the necessity of continuous stress and strain histories. Gurtin and Sternberg treated discontinuous stress and strain histories as the limiting of appropriate sequences of continuous stress and strain histories.<sup>5</sup> This puts the concept of discontinuous stress and strain histories on a sound theoretical basis which leads to a natural physical meaning of the integral law.

---

<sup>4</sup> See Theorem 2.5 of Reference 9.

<sup>5</sup> See Theorem 3.1 of Reference 9.

Finally, the Riemann-Stieltjes integrals in Theorem 1.3 can be reduced to Riemann integrals by making use of this concept of discontinuous strain histories. For convenience, let  $\epsilon_{ij}(t)$  be continuous for all  $t$  in  $(0, \infty)$ . Then from the rule relating Riemann-Stieltjes integrals to Riemann integrals<sup>6</sup> and employing the fact that  $\epsilon_{ij}(t)$  vanishes on  $(-\infty, 0)$ , Equations (1.3.5) become<sup>7</sup>

$$S_{ij} = \epsilon_{ij}(t) G_1[0, \phi(t)] + \int_{0^+}^t \epsilon_{ij}(t-\gamma) G_1[\gamma, \phi(\frac{t}{s})]_{, \gamma} d\gamma \quad (1.3.6)$$

and  $\sigma_{RR} = \epsilon_{RR}(t) G_2[0, \phi(t)] + \int_{0^+}^t \epsilon_{RR}(t-\gamma) G_2[\gamma, \phi(\frac{t}{s})]_{, \gamma} d\gamma.$

In the preceding discussion it was assumed that the stress is determined by the strain and environmental histories. This assumption can be reversed, and one could assume the strain is determined by the stress and environmental histories. This amounts to interchanging the roles of  $\sigma$  and  $\epsilon$  in the preceding development. Thus, it follows immediately that

$$\epsilon_{ij}(t) = S_{ij}(t) J_1[0, \phi(t)] + \int_{0^+}^t S_{ij}(t-\gamma) J_1[\gamma, \phi(\frac{t}{s})]_{, \gamma} d\gamma \quad (1.3.7)$$

and

$$\epsilon_{RR}(t) = \sigma_{RR}(t) J_2[0, \phi(t)] + \int_{0^+}^t \sigma_{RR}(t-\gamma) J_2[\gamma, \phi(\frac{t}{s})]_{, \gamma} d\gamma.$$

Here  $J_1$  and  $J_2$  are the creep functions in shear and isotropic dilatation, respectively.

---

<sup>6</sup> See Theorem 9-8 of Reference 12.

<sup>7</sup> Here  $\frac{d(\quad)}{d\gamma} = (\quad)_{, \gamma}$ .



Finally, it should be stated that many of the theorems of classical linear viscoelasticity will carry over to the constitutive equations presented here. In the remainder of this paper they will be applied when required. At the time of first application, all necessary clarifying remarks will be made.

## CHAPTER II

### NON-AGING MATERIALS WITH ENVIRONMENTAL DEPENDENT PROPERTIES

This chapter discusses in detail the mechanical response of non-aging viscoelastic materials with environmental dependent properties and obtains a specific representation for the creep and relaxations functionals of Equation (1.3.6) and (1.3.7). In Section 2.1 an assumption is made about the influence of time and spatially constant environments on the mechanical response of the material. Section 2.2 presents experimental data to show to what extent the assumption is valid. The last sections extend the results to time and spatially dependent environmental histories.

#### 2.1 The Mapping Hypothesis

The problem now is to deduce how the temporally constant environment affects the mechanical properties of non-aging, linear, viscoelastic materials.

Consider a typical experimental program for the determination of a material response functional. For convenience, let  $G$  represent a one-dimensional relaxation function. The usual procedure is to hold all environmental parameters constant at some value  $\Phi$ , that is,  $\Phi = \phi(t)$  for all  $t$  in  $(-\infty, \infty)$ . Then, if a unit step strain is applied at  $t = 0$ , Equation (1.3.6) gives

$$\sigma(t) = G(t, \Phi). \quad (2.1.1)$$

That is, the measured stress as a function of time is equal to the relaxation function. Also  $G$  must be associated with the particular constant

environmental history  $\bar{\Phi}$  during the test. A typical relaxation curve is shown in Figure 3. For the purpose of this study a relaxation function is assumed to monotonically decrease from a defined initial modulus  $G(0, \bar{\Phi})$  to a defined residual modulus  $G(\infty, \bar{\Phi})$ . Now some environmental property, say temperature, is fixed at  $\phi_1$  for  $t$  in  $(-\infty, \infty)$ . If the new temperature state  $\phi_1$  is not too different from  $\bar{\Phi}$ , it is reasonable to assume the material response will still be linear. Thus, the constitutive equations will still have the same form as Equation (1.3.6); and, a material response function can be determined experimentally at the new environmental parameter  $\phi_1$ .

In this manner a family of relaxation curves, as shown in Figure 3, can be obtained. The notation  $G(t, \phi_p)$  signifies the dependence of the relaxation function on an environmental history  $\Phi(t) = \phi_p$ , which is constant throughout the body for the entire strain history.

The following assumption is now made about the nature of this family of relaxation curves:

ASSUMPTION 2.1. (The Mapping Hypothesis) There exists a set of single-valued functions which map the relaxation function  $G(t, \bar{\Phi})$  onto the relaxation functions  $G(t, \phi_p)$  for each value of  $\phi_p$  in the admissible range. Further, this mapping has the form

$$G(t, \phi_p) = \alpha(\phi_p) \mathbb{1}(t) + \beta(\phi_p) G[\delta(\phi_p)t, \bar{\Phi}], \quad (2.1.2)$$

where the set of functions  $\alpha$ ,  $\beta$  and  $\delta$  depend only on  $\phi_p$ .

The admissible range of  $\phi_p$  corresponds to the range of values the environmental parameters (temperature, humidity, etc.) can take on

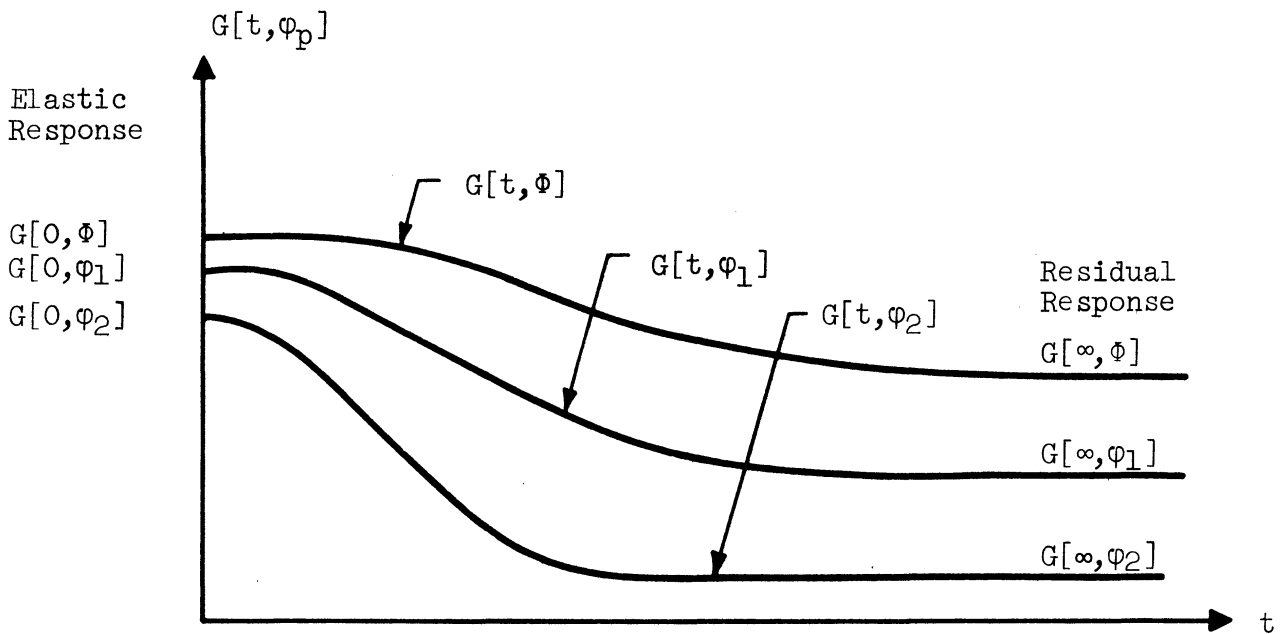


Figure 3. Hypothetical Dependence of a Relaxation Function on the Variation of a Particular Environmental Parameter.

so that Equation (2.1.2) is valid. The quantity  $1(t)$  is the Heaviside unit step function defined by

$$1(t-a) = \begin{cases} 0 & \text{for all } t \text{ in } (-\infty, a) \\ 1 & \text{for all } t \text{ in } [a, \infty) . \end{cases} \quad (2.1.3)$$

The Heaviside step function preserves the property that  $G[t, \phi_p]$  vanishes on the negative time interval. The quantity  $\Phi$  is the reference environmental constant for the mapping hypothesis.

The functions  $\alpha$ ,  $\beta$  and  $\gamma$  are subject to a set of restrictions. The first is that the material response function  $G(t, \phi_p)$  must reduce to the reference state response when  $\phi_p = \Phi$ . This requires that

$$\begin{aligned} \alpha(\Phi) &= 0, \\ \beta(\Phi) &= 1, \\ \text{and } \gamma(\Phi) &= 1. \end{aligned} \quad (2.1.4)$$

Next it is necessary to guarantee that the stress and strain always has the same sense. This requires that  $G(t, \phi_p) \geq 0$  for all  $t$  in  $(-\infty, \infty)$ . Thus the inequality

$$\alpha(\phi_p) + \beta(\phi_p)G[t, \Phi] \geq 0 \quad \text{for all } t \text{ in } [0, \infty), \quad (2.1.5)$$

places a limitation on the range of values that  $\alpha(\phi_p)$  and  $\beta(\phi_p)$  can assume.

Finally, in order that the sense of time be preserved,  $\gamma(\phi_p)$  must satisfy

$$\gamma(\phi_p) > 0 . \quad (2.1.6)$$

Assumption 2.1 states that the instantaneous (or elastic) response of the material for the environment at some value  $\phi_p$  is  $[\alpha(\phi_p) + \beta(\phi_p) G(0, \Phi)]$ , and the long term (or residual) response is given by  $[\alpha(\phi_p) + \beta(\phi_p) G(\infty, \Phi)]$ . The function  $\beta(\phi_p)$  scales the total amount of relaxation, and  $\gamma(\phi_p)$  scales the relaxation time. Thus, analytically, Assumption 2.1 implies that changes in shape of the relaxation curves can be accounted for by scaling.

Having stated the basic mapping hypothesis, next a method for calculating the scaling coefficients from experimental data is presented. Assume that an experimental program has been carried out providing a known set of curves similar to those shown in Figure 3.

First, determine the amplitude scaling factors  $\alpha$  and  $\beta$ . Experimental data demonstrates that the most convenient variable to use is the deviation in  $\phi_p$  from the reference state rather than the variable  $\phi_p$  itself. Therefore, define

$$\psi_p = \phi_p - \Phi; \quad (2.1.7)$$

and, assume that  $\alpha(\phi_p)$  and  $\beta(\phi_p)$  can be approximated by a power series in  $\psi_p$ .<sup>1</sup> Let

$$\alpha(\phi_p) = \sum_{i=1}^N a_i \psi_p^i = \sum_{i=1}^N a_i (\phi_p - \Phi)^i, \quad (2.1.8)$$

and

$$\beta(\phi_p) = 1 + \sum_{i=1}^N b_i \psi_p^i = 1 + \sum_{i=1}^N b_i (\phi_p - \Phi)^i, \quad (2.1.9)$$

where  $a_i$  and  $b_i$  are coefficients that must be determined from the

---

<sup>1</sup> Depending on the experimental data, other series representations for  $\alpha(\phi_p)$  and  $\beta(\phi_p)$  may be more convenient.

experimental data. Equations (2.1.8) and (2.1.9) satisfy the restrictions established by Equation (2.1.4); that is, when  $\phi_p = \Phi$ ,  $\alpha = 0$  and  $\beta = 1$ .

The purpose of the functions  $\alpha$  and  $\beta$  is to map the initial elastic response and the long-term or residual response of  $G(t, \Phi)$  onto the amplitudes of  $G(0, \phi_p)$  and  $G(\infty, \phi_p)$ . Thus, for each value of  $\phi_p = \phi_i$  ( $i = 1, 2, \dots, N$ ), two equations can be written by combining Equation (2.1.8), (2.1.9) and (2.1.3); they are:

$$G(0, \phi_i) = [a_1 \psi_i + a_2 \psi_i^2 + \dots + a_N \psi_i^N] + [1 + b_1 \psi_i + b_2 \psi_i^2 + \dots + b_N \psi_i^N] G(0, \Phi),$$

and

$$G(\infty, \phi_i) = [a_1 \psi_i + a_2 \psi_i^2 + \dots + a_N \psi_i^N] + [1 + b_1 \psi_i + b_2 \psi_i^2 + \dots + b_N \psi_i^N] G(\infty, \Phi).$$

In this manner a system of  $2N$  equations in  $2N$  unknowns is established. The unknowns in the system are the coefficients  $a_i$  and  $b_i$  ( $i = 1, 2, \dots, N$ ); the  $G(0, \phi_p)$ 's and  $G(\infty, \phi_p)$ 's are known from the experimental data, and the  $\psi_p$ 's can be calculated. In general, the functions  $\alpha$  and  $\beta$  can be calculated to the accuracy desired. It requires  $N + 1$  experimental curves to determine Equations (2.1.8) and (2.1.9) to the  $N^{\text{th}}$  power.

Since  $\alpha$  and  $\beta$  have been determined,  $\gamma(\phi_p)$  can now be found. First calculate  $G[\gamma(\phi_p)t, \Phi]$  with the known data from

$$G[\gamma(\phi_p)t, \Phi] = \frac{1}{\beta(\phi_p)} [G(t, \Phi) - \alpha(\phi_p) 1(t)]. \quad (2.1.10)$$

This produces a set of curves reduced to a common amplitude (See Figure 4).

It is convenient to consider the relaxation curves as plotted against

$\log_{10} t$  rather than  $t$ , therefore define a new function  $\tilde{G}$  by

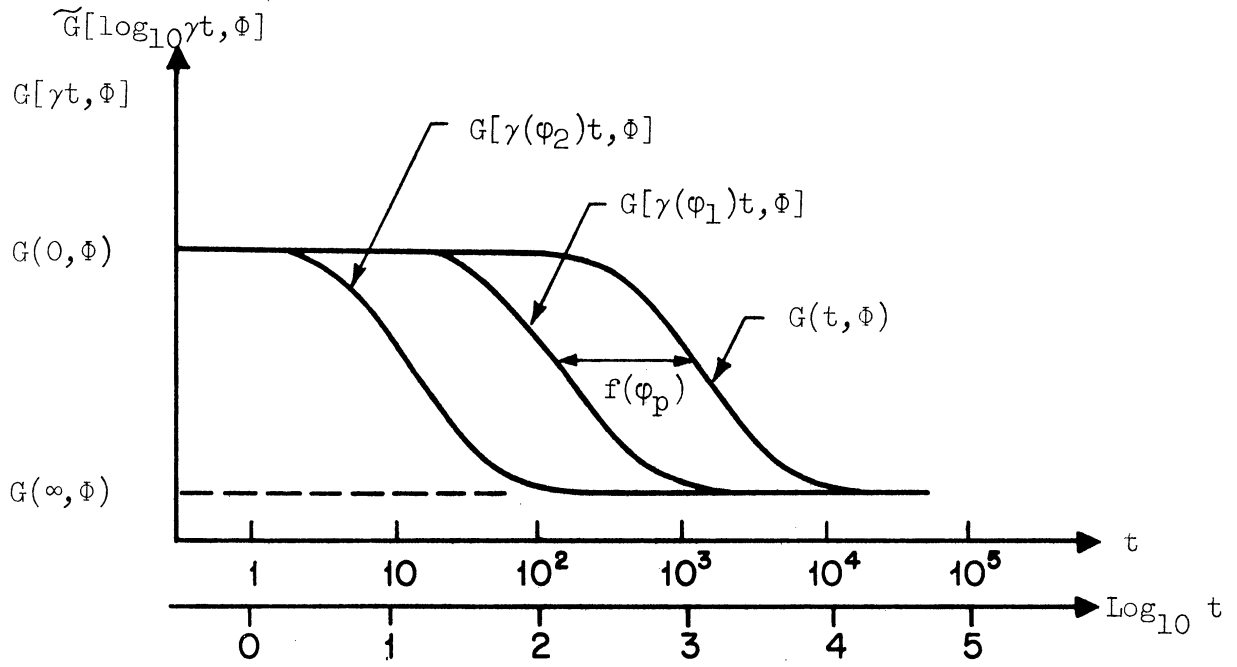


Figure 4. Reduced Material Response Curves for the Purpose of Calculating the Time Scaling Factor.



$$\tilde{G} [\text{Log}_{10} \gamma(\phi_p) t, \Phi] = G [\gamma(\phi_p) t, \Phi]. \quad (2.1.11)$$

Let

$$\text{Log}_{10} \gamma(\phi_p) = f(\phi_p), \quad (2.1.12)$$

then Equation (2.1.11) can be written as:

$$G [\gamma(\phi_p) t, \Phi] = \tilde{G} [\text{Log}_{10} t + f(\phi_p), \Phi]. \quad (2.1.13)$$

Thus, the time scaling factor  $\gamma(\phi_p)$  is equivalent to a rigid shift of the reduced relaxation curves on the logarithmic time scale.

This type of behavior was first studied by Schwarzl and Staverman<sup>(2)</sup> in connection with temperature dependence properties. Such materials were called "Thermorheologically Simple." Therefore, Assumption 2.1 has embodied in it the special case of "Thermorheologically Simple" materials.

What remains now is to determine the function  $f(\phi_p)$  from reduced relaxation curves. Once again assume that  $f(\phi_p)$  can be approximated by a power series in  $\psi_n$ .<sup>2</sup> Let

$$f(\phi_p) = \sum_{i=1}^N c_i \psi_p^i = \sum_{i=1}^N c_i (\phi_p - \Phi)^i, \quad (2.1.14)$$

where the  $c_i$ 's must be determined from the reduced material response functions. This can be done by picking off an appropriate amount of shift  $f(\phi_p)$  for each value of  $\phi_p$ ,<sup>3</sup> and writing a set of  $N$  equations for the  $N$  unknown  $c_i$ 's. These equations are:

<sup>2</sup> Ibid.

<sup>3</sup> See Reference 13.

$$\begin{aligned}
 f(\phi_1) &= c_1 \psi_1 + c_2 \psi_1^2 + \dots + c_N \psi_1^N, \\
 f(\phi_2) &= c_1 \psi_2 + c_2 \psi_2^2 + \dots + c_N \psi_2^N, \\
 &\vdots \\
 f(\phi_N) &= c_1 \psi_N + c_2 \psi_N^2 + \dots + c_N \psi_N^N.
 \end{aligned} \tag{2.1.15}$$

The final form of  $\gamma(\phi_p)$  can now be written as:

$$\text{Log}_{10} \gamma(\phi_p) = \sum_{i=1}^N c_i (\phi_p - \Phi)^i. \tag{2.1.16}$$

Equation (2.1.16) satisfies the restriction that  $\gamma(\phi_p) > 0$  for all  $\phi_p$ . Also, condition (2.1.4) is satisfied since  $\gamma = 1$  when  $\phi_p = \Phi$ .

In Section 2.2 some experimental data are examined regarding Assumption 2.1. Since much of the data is available in both the relaxation and creep forms, it is convenient to introduce a creep function of the same structure as Equation (2.1.2); i.e., assume the creep compliance function for any constant value  $\phi_p$  in the admissible range is given by

$$J(t, \phi_p) = \hat{\alpha}(\phi_p) 1(t) + \hat{\beta}(\phi_p) J[\hat{\gamma}(\phi_p)t, \Phi]. \tag{2.1.17}$$

In Equation (2.1.17),  $J(t, \Phi)$  is the creep function in some reference state  $\Phi$ ;  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\gamma}$  are mapping functions dependent only on  $\phi_p$ . Equation (2.1.17) is also subject to the restrictions of Equation (2.1.4), (2.1.5) and (2.1.6). Further, at this point in the development, the scaling factors  $\alpha$ ,  $\beta$  and  $\gamma$  are completely independent of  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\gamma}$ . At the end of the next section relationships between these quantities are established.

## 2.2 Experimental Verification

The purpose of this section is to point out the various environmental phenomena for which the material response functions (2.1.2) and (2.1.17) may be a valid representation. First, it should be emphasized that they will only be valid for a portion of the possible range of variation of the environmental parameters considered. Also, Equation (2.1.2) and (2.1.17) will not be valid for all possible combinations of materials and environmental effects. However, it is anticipated that this representation can produce a close approximation to a sufficiently large class of material response situations.

The experimental data that follows is viewed phenomenologically, and no attempt is made to correlate the variation in the environmental parameter to the effect on the molecular structure of the material. To be consistent with the linear theory of viscoelasticity, the data is limited to small deformations.

### A. Temperature

Hetenyi<sup>(14)</sup> investigated the deformation of bakelite beams as part of a study in photoelasticity. A constant and uniform bending moment was applied to identical bakelite beams at several temperatures and the deflection was recorded as a function of time. (The Correspondence Principle<sup>4</sup> verifies that the time dependent deflection is proportional to the creep function.) The deformation curves shown in Figure 5 verified that the maximum deflection was the same for all temperatures.<sup>5</sup>

---

<sup>4</sup> For example see Flugge<sup>(17)</sup> page 32.

<sup>5</sup> The study by Hetenyi was originally presented in Reference 14. The curves given here were taken from Leaderman<sup>(1)</sup> who reproduced the data, in collaboration with Hetenyi, in a more extensive form.

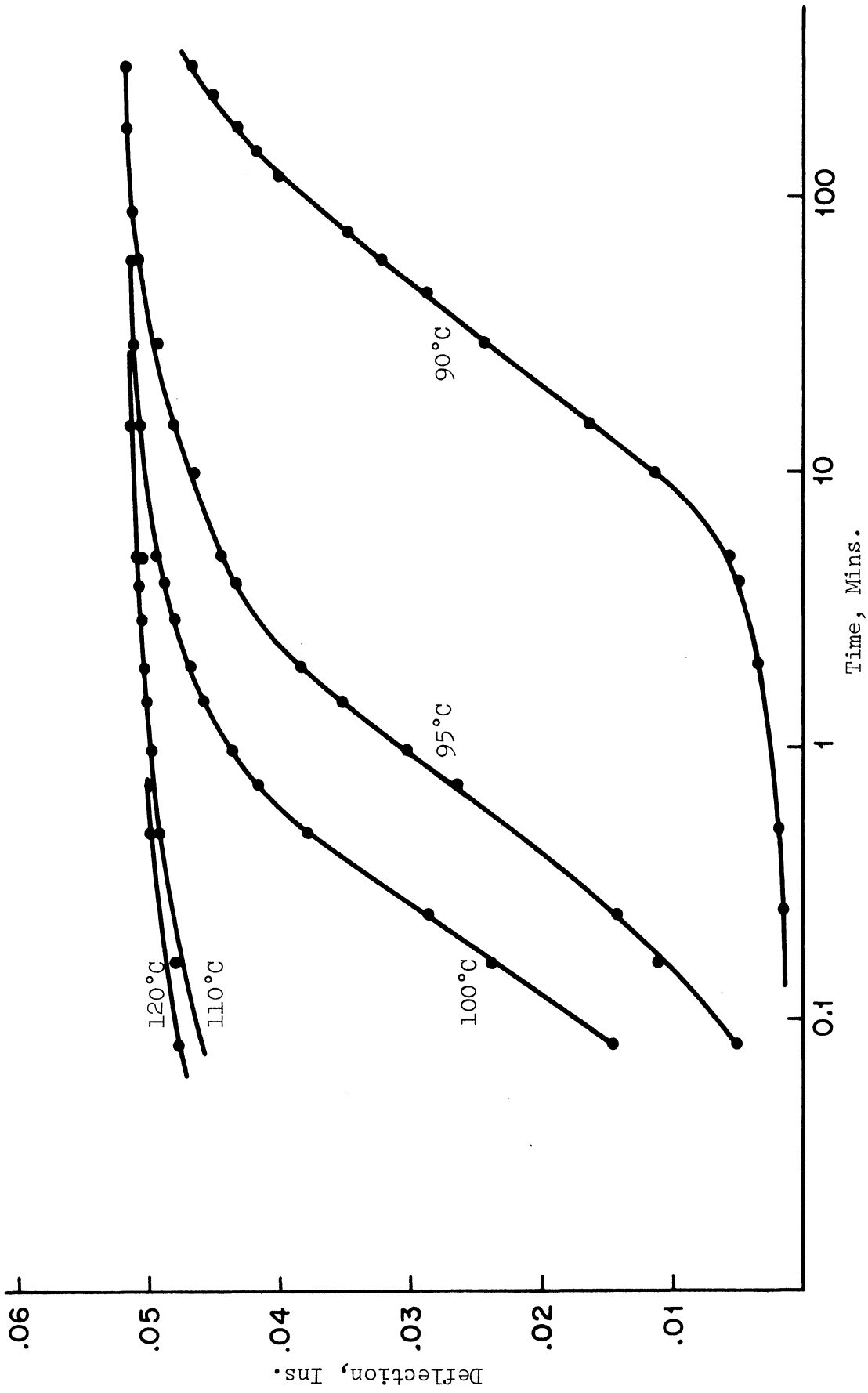


Figure 5. Deflection of Bakelite Beams in Bending Under a Constant Load as a Function of Time and Temperature. (From Leaderman, H, Elastic and Creep Properties of Filamentous Materials and Other High Polymers (1))

Also, the temperature change shifted the curve along the logarithmic time axis to shorter times for higher temperatures.

Schwarzl and Staverman<sup>(2)</sup> called materials obeying this basic time-temperature shift law "Thermorheologically Simple." Later Morland and Lee<sup>(3)</sup> extended the hypothesis to time dependent temperature fields.

Ferry<sup>(15)</sup> arrived at almost the same result from a molecular theory. He considered the viscoelastic response to be dependent on the movement of individual polymer chains. However, his analysis led to a  $\beta$  type vertical scaling as well as a logarithmic time scaling. The vertical shift was given by the ratio  $PT/P_0T_0$ ; where  $P_0$  and  $P$ ,  $T_0$  and  $T$  represented the densities and temperatures in the reference state and an arbitrary state respectively. The time scaling given by the W. L. F. equation (See Reference 15, page 212) was empirically introduced.

Nagamatus and his co-workers<sup>(16)</sup> demonstrated that a more complex vertical logarithmic shifting factor is required when the crystallinity of the polymer is dependent on the temperature.

A specific example of a vertical shift in the creep compliance function due to temperature changes is seen in the viscoelastic behavior of nylon. Figure 6 demonstrates the effect of temperature on the time-dependent response of a nylon filament for total elongations up to seven percent. An increase in temperature displaces the response curve upward. Selecting 21.5 degrees centigrade as the reference temperature, the temperature-dependent creep function (2.1.17) can be written as:

$$J(t, T) = 0.0356(T - 21.5)1(t) + J(t, 21.5). \quad (2.2.1)$$

The data points calculated with Equation (2.2.1) are within 0.5 percent

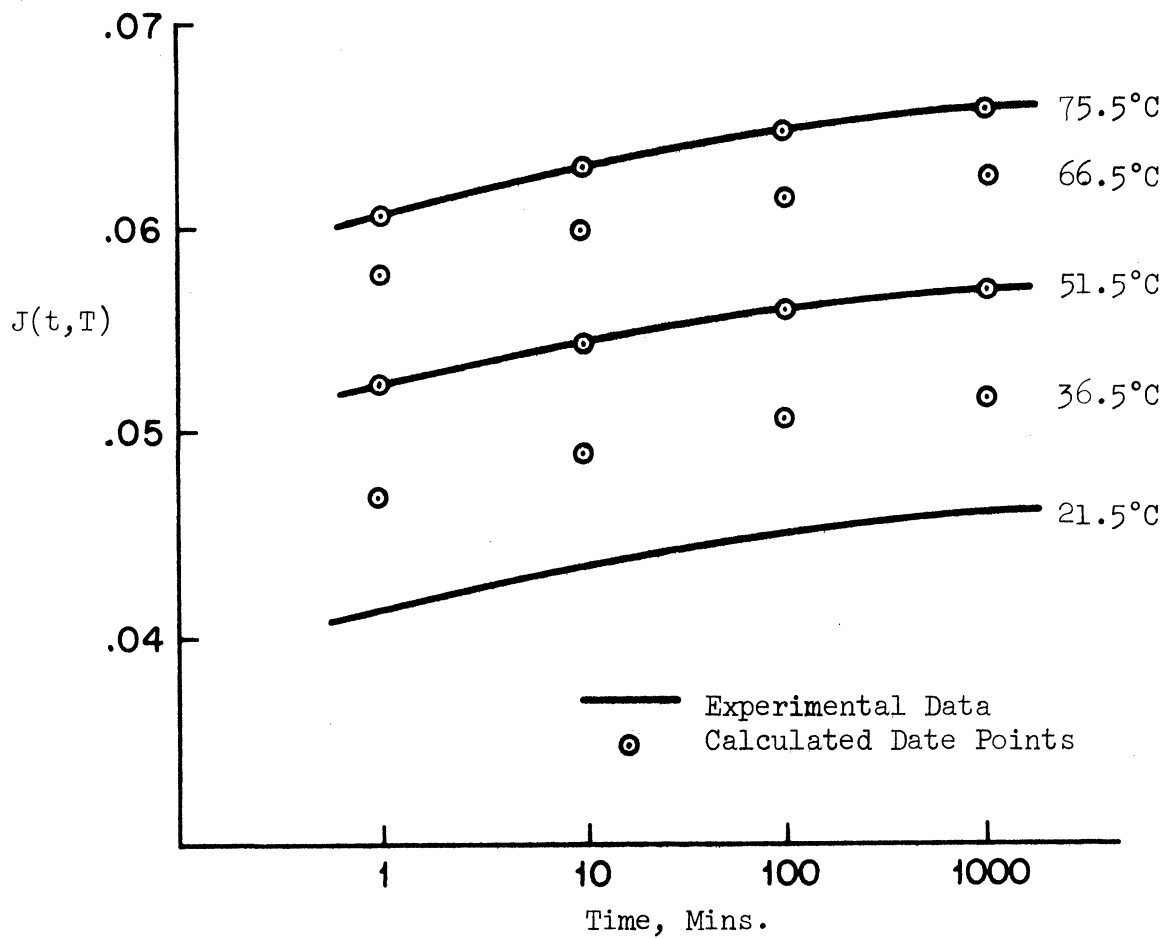


Figure 6. Experimental and Theoretical Data for the Creep of a Nylon Filament at Several Temperatures. (Experimental Data from Leaderman H. Elastic and Creep Properties of Filamentous Materials and Other High Polymers<sup>(1)</sup>)

of the experimental data. This accuracy verifies that a linear vertical shifting factor is reasonable for this application.

#### B. Humidity

Moisture usually affects viscoelastic response of polymers in two ways: first, the rigid shift of the response curve with respect to the log-time axis, and second, the vertical scaling of the response function.

The log-time shift behavior was demonstrated by Steinberger<sup>(18)</sup> in his studies of the viscoelastic response of cellulose acetate rayon. Rayon fibers were loaded with a constant stress of  $6.5 \times 10^8$  dynes per  $\text{cm}^2$  for a range of relative humidities from 0 percent to 100 percent. The measured data indicated that at low relative humidities there was little creep, while at higher relative humidities the creep was more pronounced. Also the curves appeared to be part of a common curve that was displaced along the logarithmic time axis to shorter times for higher relative humidities.

Further, the work of Leaderman<sup>6</sup> demonstrates that rayon acetate responds as a "Thermorheologically Simple" material in the presence of constant temperature fields. Thus, rayon acetate appears to manifest the same general behavior for constant temperature and humidity environments.

Nylon exhibited the second type of behavior in the presence of moisture variations. Leaderman<sup>(1)</sup> investigated the effect of humidity on a nylon filament for small deformations (See Figure 7). The strain curves

---

<sup>6</sup> See Reference 1, Figure 111 (Page 174).

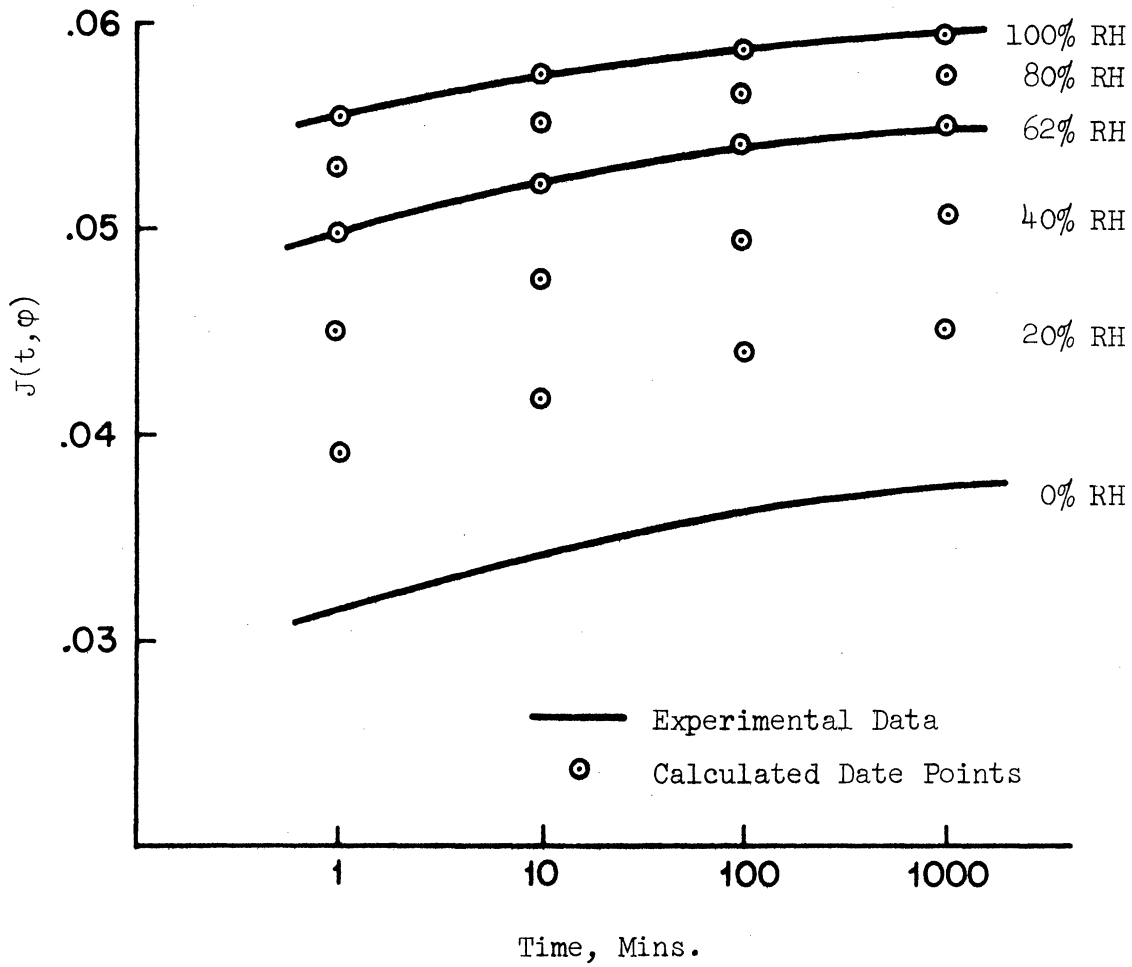


Figure 7. Experimental and Theoretical Data for the Creep of a Nylon Filament at Several Relative Humidities. (Experimental Data from Leaderman, H. Elastic and Creep Properties of Filamentous Materials and Other High Polymers<sup>(1)</sup>)



were displaced upward for increasing relative humidities as shown in Figure 7. The data suggests that nylon is more sensitive to humidity variations at low values of relative humidity.

The strain can be written as a function of time and humidity using the proposed mathematical model. Choosing zero relative humidity as the reference state, Equation (2.1.17) becomes

$$J(t, \phi) = (4.03\phi - .44\phi^2)J(t) + (1 - .045\phi - .342\phi^2)J(t, 0). \quad (2.2.2)$$

Calculated data points are shown superimposed on Figure 7. The maximum difference between experimental and calculated data points is one percent. The dependence of the scaling functions  $\alpha$  and  $\beta$  on the relative humidity is shown in Figure 8. Here again the accuracy of the model for low degree polynomials indicates that the model is quite reasonable.

Meredith and Hsu<sup>(19)</sup> confirm the humidity dependent properties of nylon with their relaxation tests. Figure 9 shows the decay of a tensile load in a nylon filament at five percent elongation for three different environments. The fiber is allowed to reach an equilibrium state with dry, wet and 65 percent relative humidity environments before recording the data. The relaxation curves in Figure 9 embody the same basic features as the creep curves of Figure 7.

### C. Material Composition

In a recent study Theocaris<sup>(20)</sup> investigates the viscoelastic properties of plasticized, cold-setting, epoxy polymers in the transition region. In his study he presents creep and relaxation data of polymers with five different ratios of plasticizer to prepolymer. The creep curves are presented in Figure 10 for a reference temperature of 25 degrees

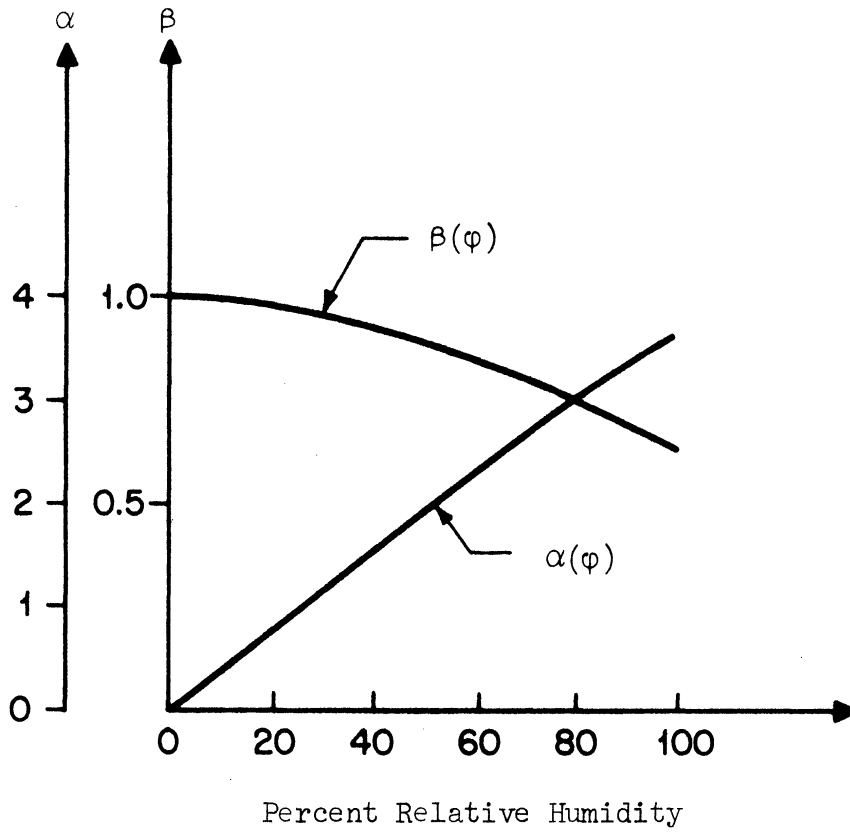


Figure 8. The  $\alpha$  and  $\beta$  Scaling Functions for a Humidity Dependent Nylon Filament.

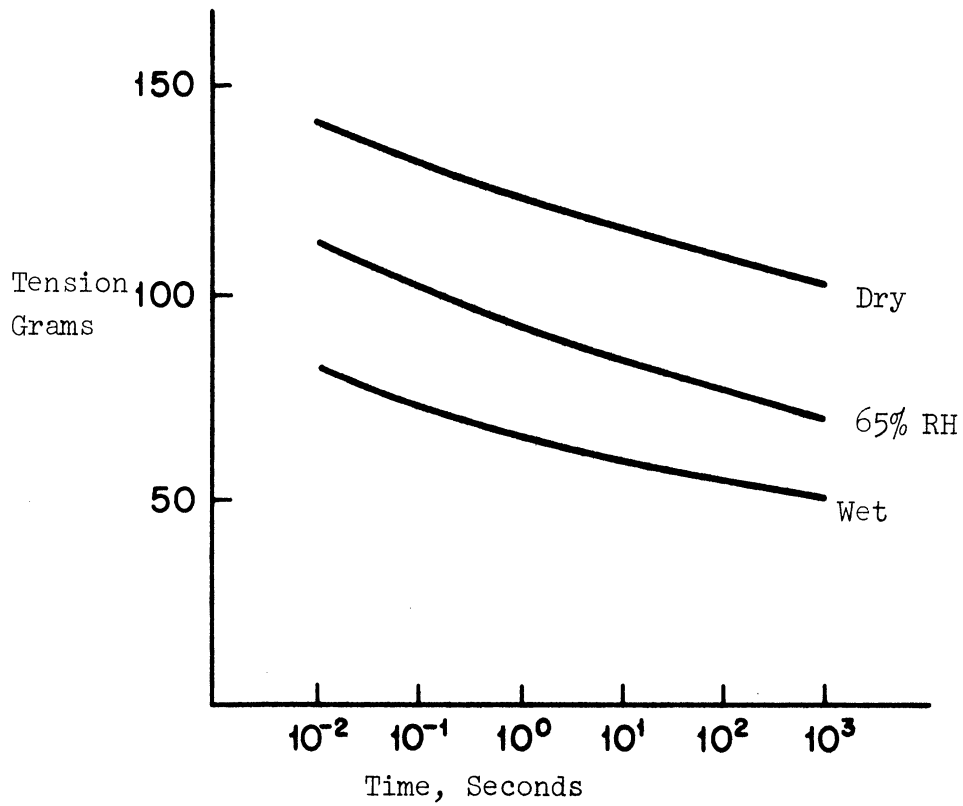


Figure 9. Effect of Humidity on the Relaxation of Nylon at 25°C and 5% Extension. (Data from Meredith, H. and Hsu, B-S, J. of Polymer Science. (18))

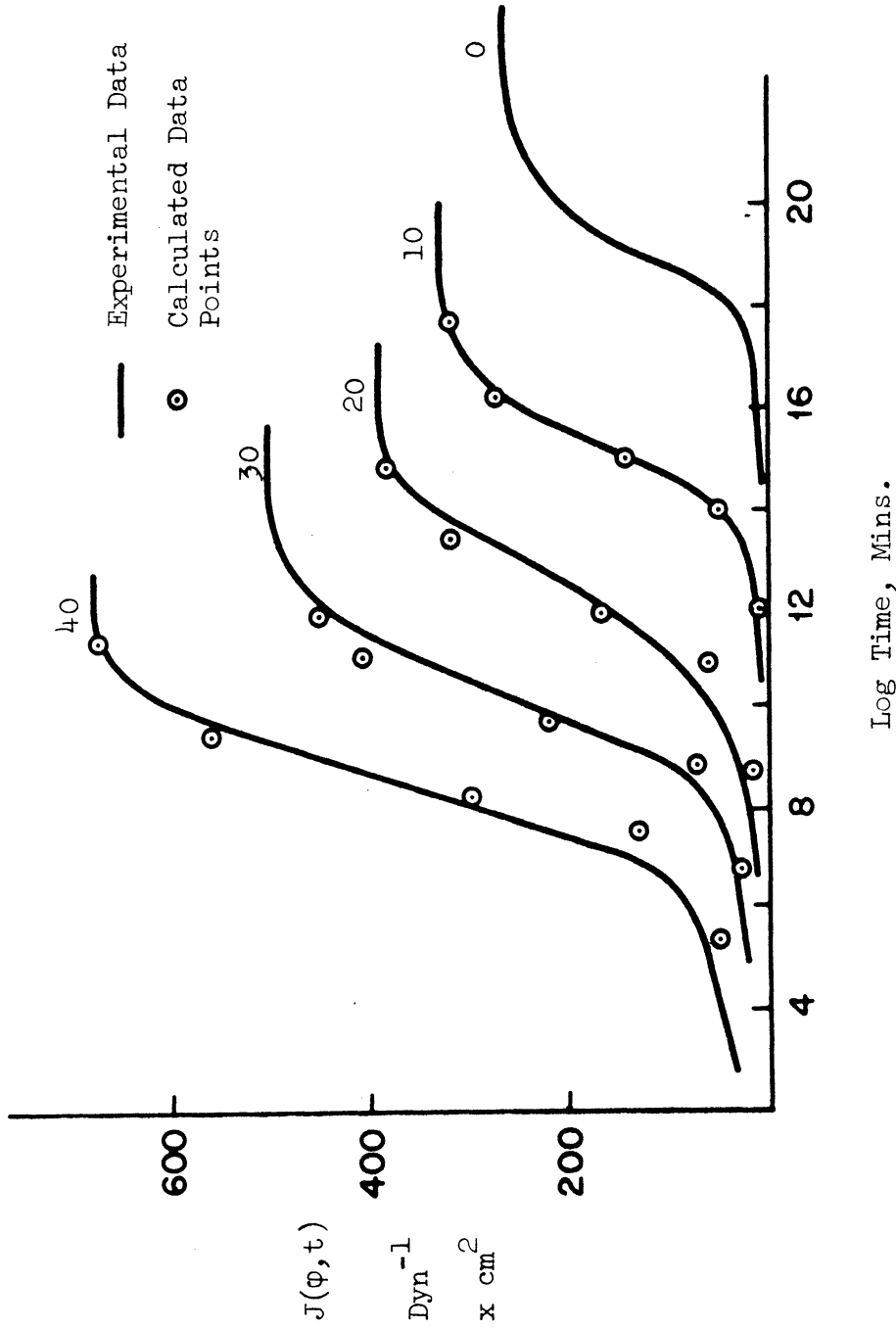


Figure 10. Experimental and Theoretical Creep Data for Five Types of Pure and Plasticized Cold-Setting Epoxy Polymers. (Experimental Data from Theocaris, P.S., "Viscoelastic Behavior of Plasticized Epoxy Polymers in Their Transition Region," Proceedings of the Society of Experimental Stress Analysis (21))

Centigrade. The materials are designated as 0, 10, 20, 30 and 40. They are all cold-setting epoxies with 100 parts by weight of prepolymer and eight parts by weight of T.E.T. hardner. The designation 0, 10, 20, 30 or 40 represents the part by weight of plasticizer.

It is seen that increasing the plasticizer shifts the curves along the log time scale to shorter times. Also Theocaris points out that all of the curves have the same shape and could coincide if the ordinates were multiplied by a constant factor dependent on the amount of plasticizer.

The scaling factors can be determined by the method outlined in Section 2.1. The resulting creep function is

$$J(t, \phi) = (1 + 9 \times 10^{-3} \phi + 7.4 \times 10^{-4} \phi^2) J[\gamma(\phi)t, 0], \quad (2.2.3)$$

where

$$\gamma(\phi) = 0.454\phi - 4.73 \times 10^{-3} \phi^2.$$

Here  $\phi$  is defined as the amount by weight of plasticizer in the epoxy and  $\phi = 0$  is taken as the reference state. The calculated data points are also shown on Figure 10. The fit is not as good here as in the previous cases. The inaccuracies can be seen by examining the reduced creep curves in Figure 11. They do not all have the same shape; hence the hypothesis is not quite correct in the case. However, the model is reasonable for predicting the approximate response of epoxies in this family.

Polystyrene resins exhibit an entirely different behavior than that of epoxy polymers. The relaxation curves for five types of styron

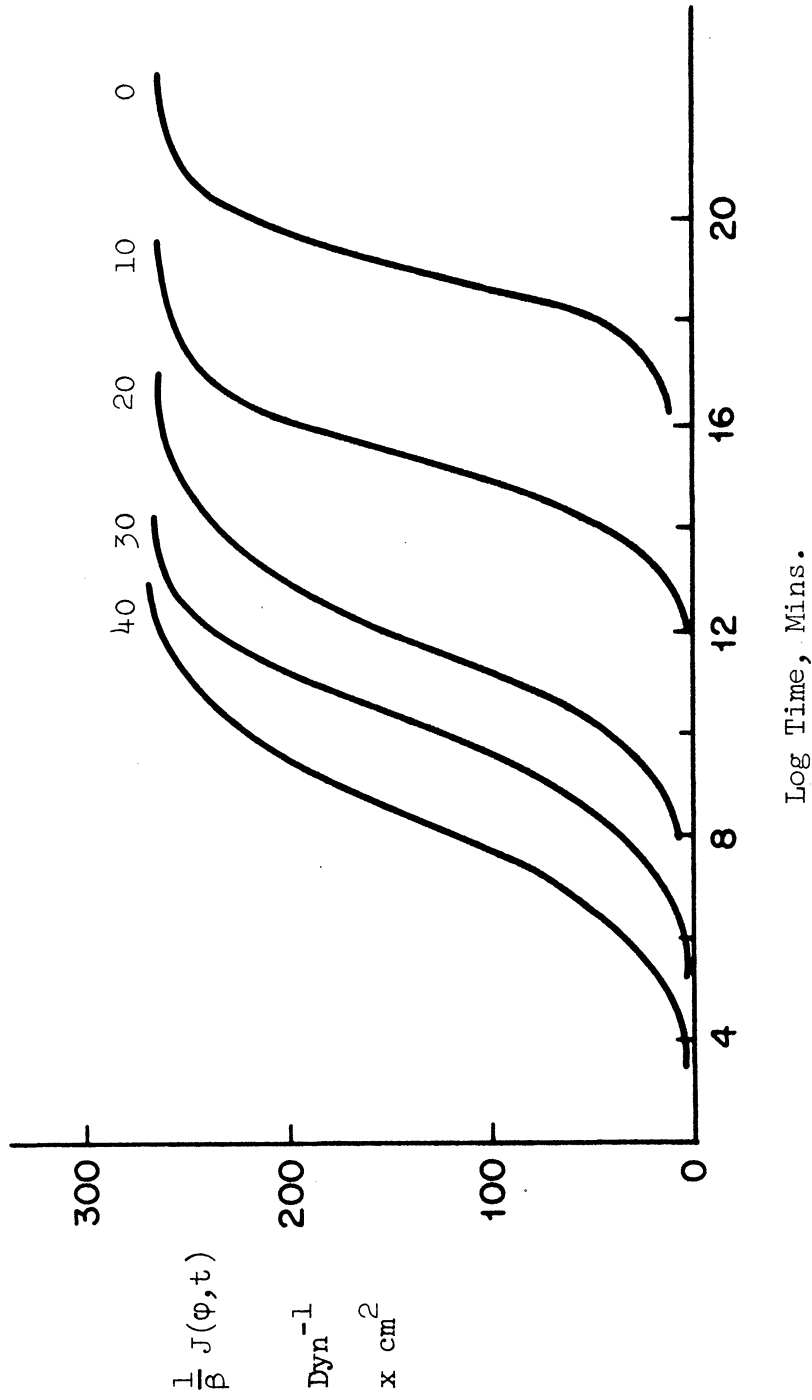


Figure 11. Reduced Creep Curves for Five Types of Pure and Plasticized Cold-Setting Epoxy Polymers. (Experimental Data from Theocaris, P.S. "Viscoelastic Behavior of Plasticized Epoxy Polymers in Their Transition Region."(21))

polystyrene resins manufactured by the Dow Chemical Company<sup>(21)</sup> are shown in Figure 12. The material numbers (345, 475B, etc.) are codes representing the material composition and the curves shown in the figure are all of the same family. Room temperature (23°C) and 0.5 percent strain are constant for all of the tests. The data clearly demonstrates that changes in composition displace the relaxation curve along the ordinate axis. Hence, all curves could be represented by a common stress relaxation function and an additive constant that depends on the composition of the material.

The work on Kashimo<sup>(22)</sup> provides one additional example of how Equation (2.1.2) or (2.1.17) can be used to model material response. He studied the viscoelastic behavior of  $\gamma$ -irradiated and heat-set nylon 6 fibers. A small portion of his results are presented in Figure 13. Shown here are the stress relaxation curves of a wet nylon fiber in a vacuum for various amounts of radiation and four percent strain. The  $\gamma$ -irradiated specimens show a higher stress which is attributed to the formation of cross-links between the chains. All the curves are nearly straight lines and the mapping hypothesis appears to fit rather well. Here an appropriate scale factor variable is the radiation dose in  $\gamma$  rays.

#### D. Data Relationships

The mapping hypothesis, as proposed, contains six adjustable material coefficients; three for creep data and three for relaxation data. In this section the relationships between the creep and relaxation coefficients are established for a given material.

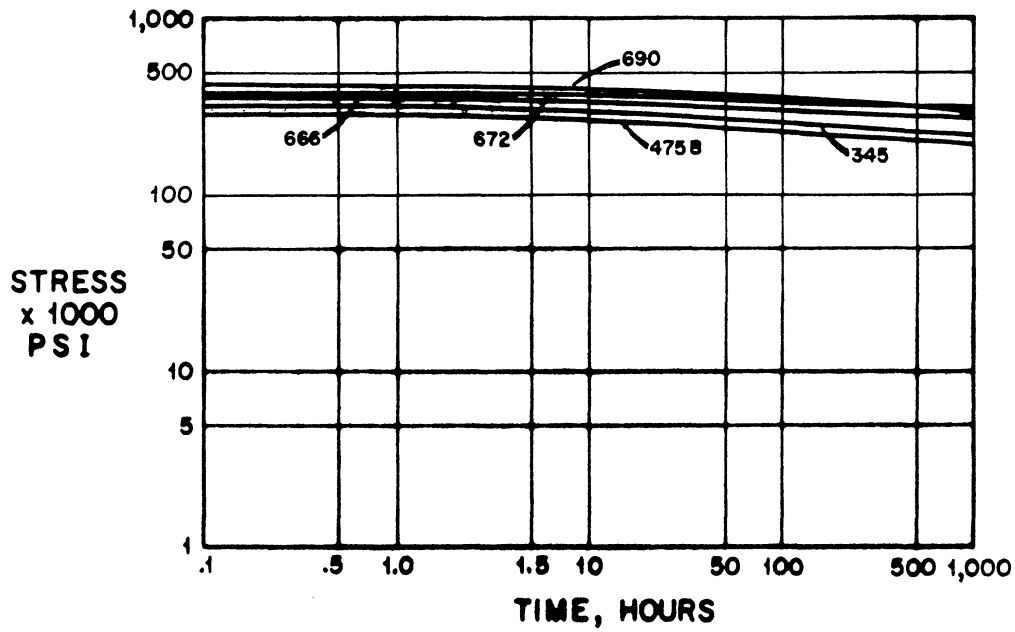


Figure 12. Stress Relaxation Curves for Five Styron Polystyrene Resins at 23°C. (From "Strength and Stiffness," Dow Chemical Company, 1967<sup>(22)</sup>).

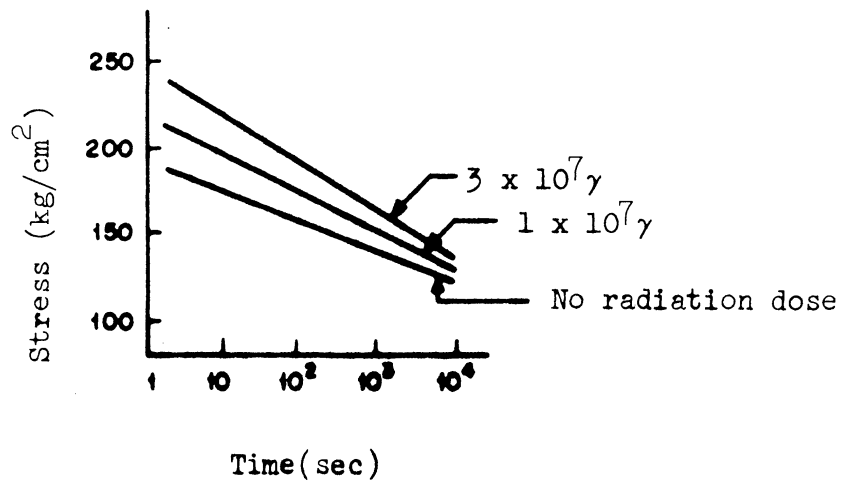


Figure 13. Stress Relaxation Curves of Wet  $\gamma$ -irradiated Nylon Fiber in a Vacuum at 4% Strain (From Koshimo, A. J. of Appl. Poly. Sci. <sup>(20)</sup>).



At this point it is convenient to introduce Riemann-Stieltjes convolution algebra. The properties and definitions associated with a Stieltjes convolution and its inverse are given in the Appendix. Also given in the Appendix are the usual definitions of functions in class  $C^N$  and  $H^N$ .

Now the relation between the associated creep and relaxation functions can be stated.<sup>7</sup>

THEOREM 2.1. Let  $G$  be a relaxation function in  $H^2$  with  $G(0) \neq 0$ . Then  $G$  has a unique Stieltjes inverse  $G^{-1} \stackrel{d}{=} J$ . Thus, every strain history  $\epsilon$  in  $H^0$  giving the stress history  $\sigma = \epsilon * dG$  implies  $\epsilon = \sigma * dJ$ . Further, the associated creep function  $J$  satisfies the equation

$$J * dG = 1(t). \quad (2.2.4)$$

Assuming the required continuity, the relationship between two associated material response functions can now be written for some fixed value of  $\phi_p$  on  $(-\infty, \infty)$  from Theorem 2.1 as:

$$J(t, \phi_p) * dG(t, \phi_p) = 1(t). \quad (2.2.5)$$

Assume that  $G(t, \phi_p)$  has been determined from an experimental program and is given by Equation (2.1.2). Then Equation (2.2.5) is an integral equation to determine the associated material response function  $J(t, \phi_p)$ , which is not necessarily of the form of Equation (2.1.17). The general solution of Equation (2.2.5) can be easily obtained by the method of

---

<sup>7</sup> See Reference 9, Theorem 3.3.

successive approximations<sup>8</sup> for a specific  $G(t, \phi_p)$ . For certain special forms of  $G(t, \phi_p)$  Equation (2.2.5) gives rise to two useful, simple results.

COROLLARY 2.1. For every fixed  $\phi_p$  in the admissible range let  $G(t, \phi_p)$  and  $J(t, \phi_p)$  satisfy Equation (2.2.5), and let  $G(t, \phi_p)$  be given by Equation (2.1.2). Then the associated material response function  $J(t, \phi_p)$  can be written as

$$J(t, \phi_p) = \hat{J}[\gamma(\phi_p)t, \phi_p]. \quad (2.2.6)$$

Proof. First note that  $1(t) = 1[\gamma(\phi_p)t]$ . Then define a relaxation function  $\hat{G}$  for a new time variable  $\bar{\gamma} = \gamma(\phi_p)t$  by rewriting Equation (2.1.2) as

$$\hat{G}(\bar{\gamma}, \phi_p) = G\left(\frac{\bar{\gamma}}{\gamma(\phi_p)}, \phi_p\right) = \alpha(\phi_p)1(\bar{\gamma}) + \beta(\phi_p)G(\bar{\gamma}, \Phi). \quad (2.2.7)$$

Expressing Equation (2.2.5) as an integral, employing Equation (2.2.7) and the change of variable theorem<sup>9</sup> for Riemann-Stieltjes integrals gives

$$\int_{-\infty}^{\bar{\gamma}} J\left(\frac{\bar{\gamma}}{\gamma} - \frac{\bar{\gamma}}{\gamma}, \phi_p\right) d\hat{G}(\bar{\gamma}, \phi_p) = 1(\bar{\gamma}), \quad (2.2.8)$$

where  $\bar{\gamma} = \gamma\tau$ . Next define

$$J\left(\frac{\bar{\gamma}}{\gamma}, \phi_p\right) = \hat{J}(\bar{\gamma}, \phi_p). \quad (2.2.9)$$

The  $\hat{J}$  is the unique inverse of  $\hat{G}$  since  $\hat{J}$  satisfies  $\hat{J} * d\hat{G} = 1(t)$ .

When the relaxation function is given by Equation (2.1.2), the associated creep function defined by Equation (2.2.5) does not have a simple analytic form. In particular, it is different from Equation

<sup>8</sup> For example see Reference 23, Section 9.

<sup>9</sup> See Theorem 9.7, Reference 12.

(2.1.17). For the special case when  $\alpha = 0$  in Equation (2.1.2), the solution of Equation (2.2.5) can easily be obtained.

COROLLARY 2.2. For every fixed  $\phi_p$  in the admissible range, let  $G(t, \phi_p)$  and  $J(t, \phi_p)$  satisfy Equation (2.2.5). Further, let

$$G(t, \phi_p) = \beta(\phi_p) G[\gamma(\phi_p)t, \Phi], \quad (2.2.10)$$

where  $G(t, \Phi)$  is the relaxation function associated with  $J(t, \Phi)$ ; that is  $J(t, \Phi) * dG(t, \Phi) = 1(t)$ . Then

$$J(t, \phi_p) = \frac{1}{\beta(\phi_p)} J[\gamma(\phi_p)t, \Phi]. \quad (2.2.11)$$

Proof. Let  $\tau = \gamma(\phi_p)t$ . Corollary 2.1 and (2.2.10) permits (2.2.5) to be written as

$$\hat{J}(\tau, \phi_p) * d\{\beta(\phi_p) G(\tau, \Phi)\} = 1(\tau). \quad (2.2.12)$$

Convolute (2.2.12) with  $J(\tau, \Phi)$  and note from the change of variable theorem for Stieltjes integrals that  $J(\tau, \Phi) * dG(\tau, \Phi) = 1(\tau)$ . Then, using the linearity and associativity properties of Stieltjes convolutions, Equation (2.2.12) becomes

$$\{\beta(\phi_p) \hat{J}(\tau, \phi_p)\} * d1(\tau) = 1(\tau) * dJ(\tau, \Phi). \quad (2.2.13)$$

Solving for  $\hat{J}(\tau, \phi_p)$  and employing Equation (2.2.6), Equation (2.2.13) can be written as

$$J(t, \phi_p) = \hat{J}(\tau, \phi_p) = \frac{1}{\beta(\phi_p)} J[\gamma(\phi_p)t, \Phi].$$

Thus for the case when  $\alpha = 0$ ,  $J(t, \phi_p)$  is given by Equation (2.1.17) provided

$$\hat{\alpha}(\phi_p) = 0,$$

$$\hat{\beta}(\phi_p) = 1/\beta(\phi_p),$$

and

$$\hat{\gamma}(\phi_p) = \gamma(\phi_p).$$

### 2.3 Time-Dependent Environmental Fields

In this section the results obtained for temporally constant environmental fields are extended to include time dependent environments. Recall that  $G[t, \phi_p]$  is the one-dimensional relaxation response for a material in a temporally constant environment  $\phi_p$ . The relaxation functional which depends on a time-variant environmental history is given by  $G[t-t_0, \overset{t}{\underset{S=t_0}{\phi(S)}}]$  for a unit step strain applied a time  $t_0$ . For the development in this section it is convenient to let  $t_0 = 0$ . The results for the more general case will be stated later. The following assumption permits extension of the mapping hypothesis from temporally constant environmental fields to the case under consideration.

ASSUMPTION 2.2 Let  $\phi'$  be a representative value of the environmental history  $\phi(S)$  in the infinitesimal time interval  $(t, t+\Delta t)$ . Then, assume the response of the material is such that

$$G[t+\Delta t, \overset{t+\Delta t}{\underset{S=0}{\phi(S)}}] = G[t, \overset{t}{\underset{S=0}{\phi(S)}}] + \Delta G, \quad (2.3.1)$$

where the incremental change  $\Delta G$  of the response functional is some portion of the response function  $G(t, \phi')$  which corresponds to the environmental state  $\phi'$ .

Assumption 2.2 implies that the amount of change of the relaxation functional in the infinitesimal interval of time  $[t, t+\Delta t]$  depends only on a representative value of the environment in that interval of time. In effect, the environmental history on  $[0, t]$  has no influence on the stress in the time interval  $[t, t+\Delta t]$  other than to determine the stress at time  $t$ .

In order to examine the underlying aspects of this problem, first consider the effect of time-dependent environmental histories on a "Thermorheologically Simple" type of material. Thus, by definition, the time-invariant mapping is given by Equation (2.1.2) when  $\alpha = 0$  and  $\beta = 1$ ; i.e., the initial and residual responses are unaffected by environmental changes. Further, the relaxation (or creep) functional is assumed to monotonically decrease (or increase) between the initial and residual values for every environmental history.<sup>10</sup> Thus, the environment affects only the rate at which the response occurs.

Let the environmental history  $\phi(t)$  be partitioned into  $N$  sub-intervals. Let  $\Delta t_i$  represent the length of a typical time interval and let  $\phi_i$  be a representative value on the environmental history in that interval. Denote the value of  $\phi(t)$  at  $t = 0$  by  $\phi_0$ . Further assume that this approximation of  $\phi(t)$  is piecewise continuous function.

Now at  $t = 0^+$  we have

$$G[0, \phi(s)]_{s=0} = G[0, \phi_0] = G[0, \Phi]. \quad (2.3.2)$$

Since  $\phi_0$  is the initial value of the environmental history then  $G[0, \Phi]$  follows from Equation (2.1.2).

---

<sup>10</sup> The development presented here is for a relaxation response; however, extension to a creep response follows directly.

During the first interval of time  $(0, \Delta t_1]$ , the relaxation functional will monotonically decrease an amount  $\Delta G_1$  from the initial response. Hence, we can write

$$G[\Delta t_1, \phi(s)]_{s=0}^{\Delta t_1} = G[0, \Phi] + \Delta G_1, \quad (2.3.3)$$

During  $(0, \Delta t_1]$ , the stress is assumed to relax along the  $G[t, \phi_1]$  curve, so

$$\Delta G_1 = G[\Delta t_1, \phi_1] - G[0, \phi_1]. \quad (2.3.4)$$

Using Equation (2.1.2),  $\Delta G_1$  becomes

$$\Delta G_1 = G[\gamma(\phi_1)\Delta t_1, \Phi] - G[0, \Phi]. \quad (2.3.5)$$

This means the amount of relaxation  $\Delta G_1$  that occurs in the time interval  $(0, \Delta t_1]$  for the environmental state  $\phi_1$  will require a time interval  $\gamma(\phi_1)\Delta t_1$  in the reference state  $\Phi$  (See Figure 14). Combining (2.3.3) and (2.3.5) gives:

$$G[\Delta t_1, \phi(s)]_{s=0}^{\Delta t_1} = G[\gamma(\phi_1)\Delta t_1, \Phi]. \quad (2.3.6)$$

In the next time interval  $[\Delta t_1, \Delta t_1 + \Delta t_2]$  and  $\phi$  can be approximated by the value  $\phi_2$ . During this time interval the response functional changes an amount  $\Delta G_2$ , then

$$G[\Delta t_1 + \Delta t_2, \phi(s)]_{s=0}^{\Delta t_1 + \Delta t_2} = G[\Delta t_1, \phi(s)]_{s=0}^{\Delta t_1} + \Delta G_2, \quad (2.3.7)$$

where Assumption 2.2 implies that  $\Delta G_2$  depends on  $G(t, \phi_2)$ . The amount of relaxation  $\Delta G_2$  that occurs in the time interval  $(\Delta t_1, \Delta t_1 + \Delta t_2]$  for the environmental state  $\phi_2$  is equivalent to that which occurs in

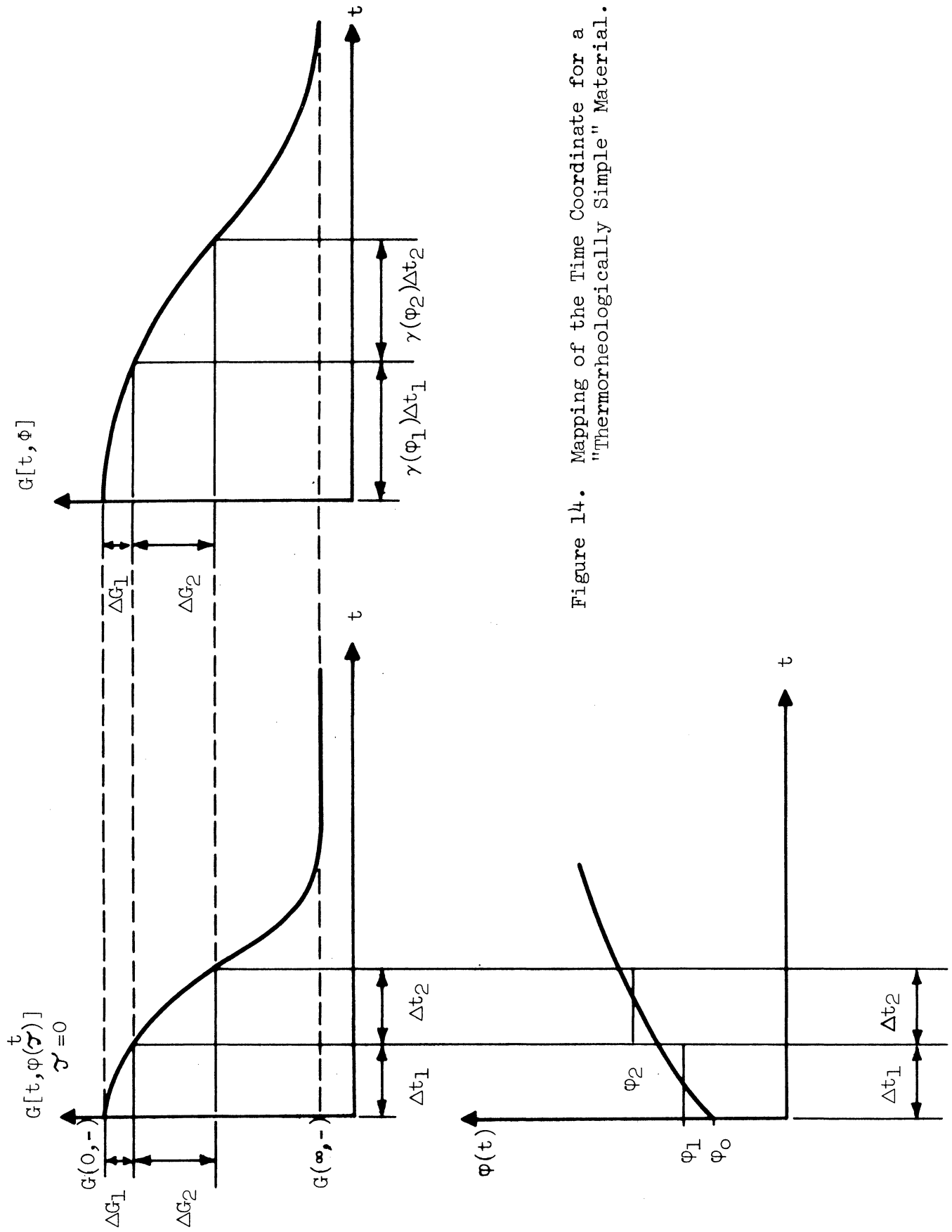


Figure 14. Mapping of the Time Coordinate for a "Thermorheologically Simple" Material.

an elapsed time  $\gamma(\phi_2)\Delta t_2$  in the reference state. Then by accounting for the relaxation in the first time interval  $(0, \Delta t_1]$  by  $\gamma(\phi_1)\Delta t_1$  in the reference state  $\Phi$ , the incremental response  $\Delta G_2$  is

$$\Delta G_2 = G[\gamma(\phi_1)\Delta t_1 + \gamma(\phi_2)\Delta t_2, \Phi] - G[\gamma(\phi_1)\Delta t_1, \Phi]. \quad (2.3.8)$$

Thus, from Equation (2.3.6) and (2.3.8), (2.3.7) becomes

$$G[\Delta t_1 + \Delta t_2, \phi(s)_{s=0}^{\Delta t_1 + \Delta t_2}] = G[\gamma(\phi_1)\Delta t_1 + \gamma(\phi_2)\Delta t_2, \Phi]. \quad (2.3.9)$$

Continuing in this fashion, it is seen that at some time  $t = \sum_{i=1}^N \Delta t_i$ ,

$$G[t, \phi(s)_{s=0}^t] = G[\sum_{i=1}^N \gamma(\phi_i)\Delta t_i, \Phi]. \quad (2.3.10)$$

Letting  $N \rightarrow \infty$  for a fixed value of  $t$ , Equation (2.3.10) goes over to

$$G[t, \phi(s)_{s=0}^t] = G(\xi, \Phi) \quad (2.3.11)$$

where  $\xi$  is defined as the following functional of  $\phi(\gamma)$ :

$$\xi = f(t) = \int_0^t \gamma[\phi(\gamma)] d\gamma. \quad (2.3.12)$$

Thus, the environmental history can be accounted for by a one-to-one<sup>11</sup> mapping of the time coordinate, which depends only on the material function  $\gamma(\phi_p)$ , and this mapping is given by Equation (2.3.12). Although the development above differs, the reduced time  $\xi$  is identical to the results of Morland and Lee.<sup>(3)12</sup>

<sup>11</sup> It was pointed out by Sternberg<sup>(24)</sup> that Equation (2.1.14) and (2.1.16) are sufficient to guarantee that (2.3.12) has a unique inverse.

<sup>12</sup> For additional reference regarding the reduced time concept and its application see References 24, 25, and 26.



Now, it is easy to extend the temporally constant environmental results to time-dependent environmental fields when all three scaling functions are present. In this case the initial elastic response and residual response are influenced by the environmental history.

Let  $\phi_0$  be the initial value of the environmental field. Then at  $t = 0^+$  the elastic response is

$$G[0, \phi(\overset{\circ}{S})_{S=0}] = \alpha(\phi_0) + \beta(\phi_0)G(0, \Phi) \equiv \overset{\circ}{G}. \quad (2.3.13)$$

Again, define  $\phi_1$  to be a representative value of  $\phi$  occurring in the interval  $(0, \Delta t_1]$ . The incremental change in the relaxation functional  $\Delta G_1$ , that occurs in interval of time  $(0, \Delta t_1]$ , is comprised of two parts. The first part is the monotonic decay of the stress that occurs in the time interval  $(0, \Delta t_1]$  at the environmental value  $\phi_1$ . The second part is the vertical shift in the response curve due to  $\phi$  changing from  $\phi_0$  to  $\phi_1$ . That is,

$$\Delta G_1 = [\alpha(\phi_1) - \alpha(\phi_0)] + \beta(\phi_1) \{ G[\gamma(\phi_1)\Delta t_1, \Phi] - G[0, \Phi] \}, \quad (2.3.14)$$

since the second quantity in brackets is the same as in Equation (2.3.5).

Note in this case  $\Delta G_1$  could be positive or negative depending on the value of  $\Delta \alpha_1$ , and hence,  $G[t, \phi(S)]_{S=0}^t$  may not be a monotonically decreasing function. Denoting

$$\Delta \alpha_i = \alpha(\phi_i) - \alpha(\phi_{i-1}),$$

the total response after  $\Delta t_1$  is:

$$\begin{aligned} G[\Delta t_1, \phi(S)_{S=0}^{\Delta t_1}] &= \overset{\circ}{G} + \Delta G_1 \\ &= \alpha(\phi_0) + \Delta \alpha_1 + \beta(\phi_1)G[\gamma(\phi_1)\Delta t_1, \Phi] - G(0, \Phi) \{ \beta(\phi_1) - \beta(\phi_0) \}. \end{aligned} \quad (2.3.15)$$

In the next time interval  $(\Delta t_1, \Delta t_1 + \Delta t_2]$ ,  $\phi$  takes on the representative value  $\phi_2$ . Then

$$G[\Delta t_1 + \Delta t_2, \phi(s)_{s=0}^{\Delta t_1 + \Delta t_2}] = G_0 + \Delta G_1 + \Delta G_2, \quad (2.3.16)$$

where

$$\Delta G_2 = \Delta \alpha_2 + \beta(\phi_2) \{ G[\gamma(\phi_1)\Delta t_1 + \gamma(\phi_2)\Delta t_2, \Phi] - G[\gamma(\phi_1)\Delta t_1, \Phi] \}. \quad (2.3.17)$$

Combining (2.3.14) and (2.3.17) with (2.3.16) gives:

$$\begin{aligned} G[\Delta t_1 + \Delta t_2, \phi(s)_{s=0}^{\Delta t_1 + \Delta t_2}] &= \alpha(\phi_0) + \Delta \alpha_1 + \Delta \alpha_2 + \beta(\phi_2) G[\gamma(\phi_1)\Delta t_1 + \gamma(\phi_2)\Delta t_2, \Phi] \\ &\quad - G(0, \Phi) \{ \beta(\phi_1) - \beta(\phi_0) \} - G[\gamma(\phi_1)\Delta t_1, \Phi] \{ \beta(\phi_2) - \beta(\phi_1) \}. \end{aligned} \quad (2.3.18)$$

Hence, after some time  $t = \sum_{i=1}^N \Delta t_i$ , the relaxation functional can be written as:

$$\begin{aligned} G[t, \phi(s)_{s=0}^t] &= \alpha(\phi_0) + \sum_{i=1}^N \Delta \alpha_i + \beta[\phi(t)] G[\sum_{i=1}^N \gamma(\phi_i)\Delta t_i, \Phi] \\ &\quad - \sum_{i=1}^N G[\sum_{p=0}^{i-1} \gamma(\phi_p)\Delta t_p, \Phi] \{ \beta(\phi_i) - \beta(\phi_{i-1}) \}. \end{aligned} \quad (2.3.19)$$

Letting  $N \rightarrow \infty$  for a fixed time interval  $[0, t]$ , Equation (2.3.19) can be written as:

$$\begin{aligned} G[t, \phi(s)_{s=0}^t] &= \alpha[\phi(t)] + \beta[\phi(t)] G\left\{ \int_{0^+}^t \gamma[\phi(\tau)] d\tau, \Phi \right\} \\ &\quad - \int_{0^+}^t G\left\{ \int_{0^+}^{\tau} \gamma[\phi(\theta)] d\theta, \Phi \right\} d\beta[\phi(\tau)]. \end{aligned} \quad (2.3.20)$$

Integrating Equation (2.3.20) by parts yields:

$$\begin{aligned} G[t, \phi(s)_{s=0}^t] &= \alpha[\phi(t)] + \beta[\phi(0)] G[0, \Phi] \\ &\quad + \int_{0^+}^t \beta[\phi(\tau)] dG\left\{ \int_{0^+}^{\tau} \gamma[\phi(\theta)] d\theta, \Phi \right\}. \end{aligned} \quad (2.3.21)$$

Equation (2.3.21) can be arrived at by interchanging the summation order in Equation (2.3.19). Equation (2.3.21) embodies the reduced time concept, but is more complicated due to the vertical scaling. In fact, a history of the vertical scaling must be included; but, only  $\alpha[\phi(t)]$  enters into Equation (2.3.21).

The two forms of the relaxation function (2.3.20) and (2.3.21) are equivalent; however depending upon the nature of the scaling functions  $\alpha$  and  $\beta$ , Equation (2.3.20) may be more useful in the solution of boundary value problems. In either case it will probably be necessary to use a numerical technique to determine the relaxation functional for a "real" material.

If Equation (2.3.21) does not depend on  $\beta$ , then on setting  $\beta[\phi(t)] = 1(t)$ , Equation (2.3.21) reduces to

$$\begin{aligned} G[t, \phi(\xi)] &= \alpha[\phi(t)] 1(t) + G\left\{ \int_0^t \delta[\phi(\tau)] d\tau, \Phi \right\} \\ &= \alpha[\phi(t)] 1(t) + G(\xi, \Phi), \end{aligned} \quad (2.3.22)$$

with  $\xi$  defined by Equation (2.3.12). For the special case of an elastic material  $G[\xi, \Phi]$  is replaced by  $E_R 1(t)$ , where  $E_R$  is the elastic modulus corresponding to the reference state  $\Phi$ . Then Equation (2.3.22) becomes:

$$G[t, \phi(\xi)] = E[\phi(t)] 1(t). \quad (2.3.23)$$

Here  $E[\phi(t)]$  is an environmental-dependent elastic modulus which depends only on the current value of the environment.

Next the material response functionals given by Equations (2.3.20) and (2.3.21) can be modified to be compatible with the relaxation function given by Equation (1.3.2). Since the relaxation functional in Equation (1.3.2) depends only on the environmental history in the time interval  $[t_0, t]$ . The representation of this functional can be obtained by carrying out the above derivation for the time interval  $[t_0, t]$ . Therefore the relaxation functional of Equation (1.3.2) is given by

$$G[t-t_0, \phi(s)_{s=t_0}^t] = \alpha[\phi(t)] + \beta[\phi(t)] G\left\{ \int_{t_0}^t \gamma[\phi(\theta)] d\theta, \Phi \right\} - \int_{t_0}^t G\left\{ \int_{t_0}^s \gamma[\phi(\theta)] d\theta, \Phi \right\} d\beta[\phi(s)], \quad (2.3.24)$$

when  $t > t_0$ ; also

$$G[t-t_0, \phi(s)_{s=t_0}^t] = \alpha[\phi(t)] + \beta[\phi(t_0)] G[0, \Phi] + \int_{t_0}^t \beta[\phi(s)] dG\left\{ \int_{t_0}^s \gamma[\phi(\theta)] d\theta, \Phi \right\}. \quad (2.3.25)$$

#### 2.4 Spatially-Dependent Environmental Fields

Next consider spatially dependent environmental fields. Recall that Assumption 1.1 stated that the stress of any given particle is completely determined by the strain and environmental histories at that particle. The environmental history of the neighboring particle does not affect this response. This is an application of the Principle of Local Action (See Reference 10, page 34).

Thus, for spatially variant environmental fields  $\phi(t, \underline{x})$  the scaling factors become functions of position. The particle at  $\underline{x}$  is affected only by the history of  $\phi(t, \underline{x})$ . The response function of the particle is given by

$$G[t-t_0, \phi(s)] = \alpha[\phi(t, x)] + \beta[\phi(t_0, x)] G(0, \Phi) + \int_{t_0}^t \beta[\phi(s, x)] dG \left\{ \int_{t_0}^s \gamma[\phi(\theta, x)] d\theta, \Phi \right\}. \quad (2.4.1)$$

The scaling factors  $\alpha$ ,  $\beta$  and  $\gamma$  are now function of time and position.

The scaling factors  $\alpha(\phi)$ ,  $\beta(\phi)$ , and  $\gamma(\phi)$ , and the material response function in the reference state  $G(t, \Phi)$  are known from an experimental program. The environmental history  $\phi(t, x)$  can be found from the appropriate physical law and boundary conditions. Then Equation (2.4.1) can be evaluated to give the material response function.

Finally, the results of this chapter can be easily generalized to include three-dimensional stress and strain histories. For convenience with the notation, write the scaling factors as:

$$\begin{aligned} \alpha(t, x) &\equiv \alpha[\phi(t, x)] 1(t), \\ \beta(t, x) &\equiv \beta[\phi(t, x)], \end{aligned} \quad (2.4.2)$$

and  $\gamma(t, x) \equiv \gamma[\phi(t, x)].$

Now the isotropic relaxation functions in shear and dilatation can be written as:

$$G_i[t-t_0, \phi(s)] = \alpha_i(t, x) + \beta_i(t_0, x) G_i(0, \Phi) + \int_{t_0}^t \beta_i(s, x) dG_i \left\{ \int_{t_0}^s \gamma_i(\theta, x) d\theta, \Phi \right\}. \quad (2.4.3)$$

for  $i = 1, 2.$

If a creep response is considered, a similar development gives the isotropic creep functions in shear and dilatation as:

$$J_i [t-t_0, \phi(s)_{s=t_0}^t] = \hat{\alpha}_i(t, \varepsilon) + \hat{\beta}_i(t_0, \varepsilon) J_i(0, \Phi) + \int_{t_0}^t \hat{\beta}_i(s, \varepsilon) dJ_i \left\{ \int_{t_0}^s \gamma(\theta, \varepsilon) d\theta, \Phi \right\} \quad (2.4.4)$$

for  $i = 1, 2$ .

The development in this chapter was devoted to finding a specific representation for environmental-dependent material response functions. Next it is desirable to correlate the results of this chapter with the constitutive models established in Chapter I. In view of the complicated form of Equations (2.3.24) and (2.3.25), it appears to be convenient to use a modified form of the constitutive law derived in Chapter I. Therefore, integrate Equation (1.3.6) and (1.3.7) by parts to obtain

$$\begin{aligned} S_{ij}(t) &= C_{ij}(0) G_1 [t, \phi(s)_{s=0}^t] + \int_0^t G_1 [t-\gamma, \phi(s)_{s=\gamma}^t] C_{ij}(\gamma), \gamma d\gamma, \\ \sigma_{kk}(t) &= E_{kk}(0) G_2 [t, \phi(s)_{s=0}^t] + \int_0^t G_2 [t-\gamma, \phi(s)_{s=\gamma}^t] \varepsilon_{kk}(\gamma), \gamma d\gamma; \end{aligned} \quad (2.4.5)$$

and

$$\begin{aligned} C_{ij}(t) &= S_{ij}(0) J_1 [t, \phi(s)_{s=0}^t] + \int_0^t J_1 [t-\gamma, \phi(s)_{s=\gamma}^t] S_{ij}(\gamma), \gamma d\gamma, \\ \varepsilon_{kk}(t) &= \sigma_{kk}(0) J_2 [t, \phi(s)_{s=0}^t] + \int_0^t J_2 [t-\gamma, \phi(s)_{s=\gamma}^t] \sigma_{kk}(\gamma), \gamma d\gamma. \end{aligned} \quad (2.4.6)$$

The material response functions  $G_i$  and  $J_i$  in Equations (2.4.5) and (2.4.6) are given by Equations (2.4.3) and (2.4.4) respectively.

## CHAPTER III

### AGING VISCOELASTIC MATERIALS

Chapter III discusses the constitutive relationships for aging and environmental-dependent viscoelastic materials. Section 3.1 introduces an operator algebra and establishes the properties of a Riemann-Stieltjes integral for use throughout the rest of the paper. In Section 3.2 an integral representation is presented for the functional form of the constitutive equations derived in Section 1.1 and the representation is specialized for isotropic materials. Finally, the resulting creep and relaxation functions are related to the other work in the field and some experimental data is presented.

#### 3.1 Mathematical Preliminaries

For use later in this chapter and in subsequent chapters, it is convenient to introduce an algebra for the integral operator arising in the representation of the constitutive equation presented in Section 3.2.

To begin, define the following scalar quantities.

DEFINITION 3.1. Let the scalar function  $f(t)$  be defined and have the following properties on  $(-\infty, \infty)$  :

- (a)  $f(t)$  vanishes on  $(-\infty, 0)$  ;
- (b)  $f(t)$  is continuous on the right in the interval  $[0, \infty)$  ;
- (c)  $f(t), t$  exist and is continuous on  $[0, \infty)$  .

DEFINITION 3.2. Let the scalar function  $\psi(\tau, t)$  have the following properties:

- (a)  $\psi(\tau, t)$  is defined for all  $t$  and  $\tau$  in  $(-\infty, \infty)$ , and is continuous on the right for  $\tau$  in  $(-\infty, \infty)$ ;
- (b)  $\psi(\tau, t)$  vanishes for  $t$  in  $(-\infty, 0)$ , for  $\tau$  in  $(-\infty, 0)$  and for  $\tau > t$ ;
- (c)  $\psi(\tau, t)_{,\tau}$  exists and is continuous with respect to  $\tau$  on the right in  $[0, t]$ .

DEFINITION 3.3. Let the functions  $f$  and  $\psi$  have the properties given by Definitions 3.1 and 3.2. Then the Riemann-Stieltjes integrals

$$I_1(t) = \int_{-\infty}^t f(t-\tau) d\psi_t(\tau) \quad (3.1.1)$$

and

$$I_2(t) = \int_{-\infty}^t \psi(t-\tau, t) df(\tau) \quad (3.1.2)$$

exist for all  $t$  in  $(-\infty, \infty)$ . The notation  $\psi_t(\tau)$  indicates the integration variable is  $\tau$  and  $t$  is fixed at some value in  $(-\infty, \infty)$ . Further, let

$$f : d\psi_t = I_1 \quad (3.1.3)$$

and  $\psi_t : df = I_2 \quad (3.1.4)$

also represent the integrals given in Equation (3.1.1) and (3.1.2) respectively.

THEOREM 3.1<sup>1</sup> (Properties of the Operator Algebra). Let  $f$  have the properties given in Definition 3.1; and, let  $\psi$  have the properties given by Definition 3.2. Then

---

<sup>1</sup> This is the counterpart of Theorem A.1 given in the Appendix.



- (a)  $f:d\psi_t$  and  $\psi_t:df$  both vanish on  $(-\infty, 0)$  and exist on  $[0, t]$  ;
- (b)  $f:d\psi_t = \psi_t:df$  ;
- (c)  $f:d\psi_t$  is linear and homogeneous in  $f$  and  $\psi$  ;
- (d) if  $f = 1(t-t_L)$  then  $\psi_t:df = \psi(t-t_L, t)$  ;
- (e)  $f:d\psi_t = 0$  implies  $f = 0$  ;
- (f)  $f:d\psi_t = \psi(0, t)f(t) + \int_0^t f(t-\gamma)\psi(\gamma, t), \gamma d\gamma$ .

Proof: Parts (a) and (c) follow directly from the properties of Riemann-Stieltjes integrals as given in Chapter 9 of Apostol.<sup>(12)</sup> Part (d) results from the direct application of the theorem on step integrators (Theorem 9-9,<sup>(12)</sup>).

To prove (b) define  $\alpha$  and  $\beta$  to be positive constants. Then

$$f:d\psi_t = \int_{-\beta}^{t+\alpha} f(t-\gamma)d\psi_t(\gamma), \quad (3.1.5)$$

since  $\psi(\gamma, t) = 0$  for  $\gamma$  on  $(-\infty, -\beta)$ , and  $f(t-\gamma) = 0$  when  $\gamma$  is on  $(t, t+\alpha)$ . Now integrate by parts to obtain

$$f:d\psi_t = f(t-\gamma)\psi_t(\gamma) \Big|_{-\beta}^{t+\alpha} - \int_{-\beta}^{t+\alpha} \psi_t(\gamma)df(t-\gamma), \quad (3.1.6)$$

or

$$f:d\psi_t = \int_{t+\alpha}^{-\beta} \psi_t(\gamma)df(t-\gamma).$$

Let  $\xi = t - \gamma$ , and observe that the upper and lower limits of integration become  $t + \beta$  and  $-\alpha$  respectively, then Equation (3.1.6) becomes

$$f:d\psi_t = \int_{-\alpha}^{t+\beta} \psi_t(t-\xi)df(\xi). \quad (3.1.7)$$

This can be rewritten as

$$f: d\psi_t = \int_{-\infty}^t \psi(t-\gamma, t) df(\gamma). \quad (3.1.8)$$

Comparing Equation (3.1.8) with (3.1.2) and (3.1.4) yields part (b).

For part (f) define the function  $\hat{\psi}$  by:

$$\psi(\gamma, t) = \hat{\psi}(\gamma, t) + \psi(0, t) 1(\gamma). \quad (3.1.9)$$

Then, substituting Equation (3.1.9) into (3.1.1), and using part (b) and the theorem on step integrators (Theorem 9-9 of Reference 12), yields

$$f: d\psi_t = \psi(0, t)f(t) + \int_{-\infty}^t f(t-\gamma) d\hat{\psi}_t(\gamma). \quad (3.1.10)$$

Observing  $\psi = 0$  for  $t$  in  $(-\infty, 0)$ , and that  $\psi(\gamma, t)_{, \gamma}$  is continuous on  $[0, t]$ , then

$$\hat{\psi}(\gamma, t)_{, \gamma} = \psi(\gamma, t)_{, \gamma} \quad (3.1.11)$$

for all  $\gamma$  in  $[0, t]$ . Hence, substituting Equation (3.1.11) into (3.1.10) yields

$$f: d\psi_t = \psi(0, t)f(t) + \int_0^t f(t-\gamma) \psi(\gamma, t)_{, \gamma} d\gamma,$$

which is the desired result.

To verify (e) set  $f: d\psi_t$  equal to zero in part (f) to obtain

$$0 = \psi(0, t)f(t) + \int_0^t f(t-\gamma) \psi(\gamma, t)_{, \gamma} d\gamma. \quad (3.1.12)$$

Now make the change of variable

$$\xi = t - \gamma \quad (3.1.13)$$

in Equation (3.1.12) to obtain

$$0 = \psi(0,t)f(t) + \int_0^t f(\tau) \frac{2\psi(t-\tau,t)}{2(t-\tau)} d\tau. \quad (3.1.14)$$

Thus, Equation (3.1.14) is a homogeneous Volterra integral equation of the second kind for  $f(t)$ , when  $\psi(0,t) \neq 0$ . The uniqueness properties (Reference 27 page 35) of Volterra integral equations verify that  $f(t) = 0$  on  $[0,t]$ . For the case when  $\psi(0,t) = 0$  let

$$g(t-\tau,t) = \frac{2\psi(t-\tau,t)}{2(t-\tau)},$$

then differentiate Equation (3.1.14) with respect to  $t$  to get

$$0 = g(0,t)f(t) + \int_0^t f(\tau) \frac{2g(t-\tau,t)}{2t} d\tau.$$

Thus, once again it follows that  $f(t)$  must equal zero.

### 3.2 An Integral Constitutive Law for Aging Viscoelastic Materials

In this section an integral representation of the aging constitutive law is presented. In Section 1.1 two equivalent functional relationships were established for aging and environmental-dependent materials,

$$\underline{\sigma}(t) = \underline{\mathcal{F}}^{(1)} \left[ \underline{\varepsilon} \left( \frac{t}{s_1} \right); \phi \left( \frac{t}{s_2} \right); t \right], \quad (1.1.11)$$

and

$$\underline{\sigma}(t) = \underline{\mathcal{F}}^{(2)} \left[ \underline{\varepsilon} \left( \frac{t}{\gamma_1} \right); \phi \left( \frac{t}{\gamma_2} \right); t \right]. \quad (1.1.12)$$

In both cases  $t = 0$  represents the time when the aging process begins. For the current treatment it is convenient to use the form given by Equation (1.1.11). Also, it is convenient to drop the dependence of the stress

history on environmental history in Equation (1.1.11). Due to the lack of experimental evidence for materials in this area, it is difficult to make a reasonable assumption for a specific form of the functional  $\mathcal{F}^{(1)}$ .

Before proceeding to an integral representation of Equation (1.1.11), it is convenient to show that if  $\mathcal{F}$  is linear in the strain history, then  $\mathcal{F}$  is a fourth order tensor valued functional. Let  $X$  denote a Cartesian frame of reference, then Equation (1.1.11) is written in the  $X$  frame as

$$\sigma_{ij}^X = \mathcal{F}_{ijkl}^X [\epsilon_{kl}^X(t-\gamma), t], \quad (3.2.1)$$

where the repeated indices imply the usual summation convention. Define  $R$  to be the rotation that carries the  $X$  frame into the  $X'$  frame, that is,

$$x'_i = a_{ij} x_j, \quad (3.2.2)$$

and

$$a_{ik} a_{jk} = \delta_{ij} ; \text{DET} [a_{ij}] = 1.$$

Now by direct application of Equation (1.1.11), the stress in the  $X'$  frame can be written as:

$$\sigma_{mm}^{X'} = \mathcal{F}_{mmpq}^{X'} [\epsilon_{pq}^{X'}(t-\gamma), t]. \quad (3.2.3)$$

But  $\sigma_{mn}^{X'}$  and  $\epsilon_{pq}^{X'}$  are second order tensors; therefore,

$$\sigma_{ij}^{X'} = a_{mi} a_{mj} \sigma_{mm}^{X'} \quad (3.2.4)$$

and

$$\epsilon_{pq}^{X'} = a_{pk} a_{ql} \epsilon_{kl}^X. \quad (3.2.5)$$

Substitution of Equation (3.2.4) and (3.2.5) into (3.2.3) yields

$$\sigma_{ij}^X = a_{mi} a_{mj} \mathcal{F}_{mmpq}^{X'} [a_{pk} a_{ql} \epsilon_{kl}^X(t-\gamma), t]. \quad (3.2.6)$$

Employ the linearity property of  $\mathcal{F}$  and compare Equation (3.2.6) to (3.2.1) to get

$$\mathcal{F}_{ijkl}^X [\epsilon_{kl}^X(t-\gamma), t] = a_{mi} a_{mj} a_{pk} a_{ql} \mathcal{F}_{mmpq}^{X'} [\epsilon_{kl}^X(t-\gamma), t].$$

Considering strain histories where all components are zero except  $\epsilon_{kl}^X = g$ , then the equation above becomes

$$\mathcal{F}_{ijkl}^X [g(t-\gamma), t] = a_{mi} a_{mj} a_{pk} a_{ql} \mathcal{F}_{mmpq}^{X'} [g(t-\gamma), t]. \quad (3.2.7)$$

Thus  $\mathcal{F}$  transforms as a fourth order Cartesian tensor. For completeness, from the symmetry properties of  $\sigma_{ij}$  and  $\epsilon_{ij}$  it also follows that

$$\mathcal{F}_{ijkl} = \mathcal{F}_{jikl} = \mathcal{F}_{ijlk}. \quad (3.2.8)$$

Hence, the assertion of linearity allows the usual tensorial properties of a material response function to carry over to a functional type constitutive law.

It is possible to obtain an integral representation for the constitutive equations given in (3.2.1) if the functional  $\mathcal{F}$  is assumed to be linear in the strain history, non-retroactive, and continuous.<sup>2</sup> König and Meixner<sup>(29)</sup> consider a similar functional having all of the above mentioned properties along with that of translation invariance.

---

<sup>2</sup> These properties are defined in Theorem 1.1.

They prove that the linear hereditary law can be expressed as a Riemann-Stieltjes integral whose integrator is uniquely determined from the functional  $\mathcal{F}$  and is independent of the strain history.

Let each strain history belong to a class of functions  $f(\mathcal{H})$  which is continuous on  $(-\infty, \infty)$  and zero on  $(-\infty, 0)$ . For each  $t > 0$  define a linear space of continuous functions on the closed interval  $[0, t]$  with the property that for each function  $f(0) = 0$ . Then for each fixed time interval  $[0, t]$ , every  $\mathcal{F}_{ijkl}$  represents a linear functional defined on the above space of functions. The Hahn-Banach Lemma (Reference 30 page 114) allows extension to a class of functions which is continuous on  $[0, t]$  when  $f(0) \neq 0$ . Then the Riesz Theorem (Reference 30 page 110) can be applied to represent the functional as a Riemann-Stieltjes integral with a generating function that depends on the integration variable  $\mathcal{H}$  in  $[0, t]$  and the size of the integration interval  $[0, t]$ . At this point in the proof a translation-invariant condition for non-aging materials, can be invoked to remove the dependence of the generating function on interval size. Thus, employing an intermediate result of König and Meixner the following constitutive representation can be arrived at for aging materials.

THEOREM 3.2 (Linear hereditary aging stress-strain law) Let the stress history be related to the strain history by (3.2.1) where:

- (a)  $\mathcal{E}$  is defined and continuous on  $(-\infty, \infty)$  ;
- (b)  $\mathcal{E}$  vanishes on  $(-\infty, 0)$  ;

and, the functional  $\mathcal{F}$  has the properties of linearity, non-retroactivity, and continuity.<sup>3</sup> Then Equation (3.2.1) can be written as

---

<sup>3</sup> Ibid.

$$\underline{\sigma}(t) = \int_{-\infty}^t \underline{\xi}(t-\tau) d\underline{G}(\tau, t), \quad (3.2.9)$$

where  $\underline{G}$  has the following properties:

- (c)  $\underline{G}(\mathcal{J}, t)$  is defined for all  $t$  in  $(-\infty, \infty)$  and  $\mathcal{J}$  in  $(-\infty, t]$ , and is continuous on the right<sup>4</sup> for  $\mathcal{J}$  in  $(-\infty, t]$ ;
- (d)  $\underline{G}(\mathcal{J}, t)$  vanishes for  $t$  in  $(-\infty, 0)$ ;
- (e)  $\underline{G}$  is of bounded variation in  $\mathcal{J}$  for every sub-interval in  $(-\infty, \infty)$ ;
- (f)  $\underline{G}$  is a fourth order tensor whose components with respect to a Cartesian coordinate system satisfy

$$G_{ijkl} = G_{jikl} = G_{ijlk}$$

for all  $t$  and  $\mathcal{J}$  in the domain of definition.

The proof of (c), (d), and (e) of Theorem (3.2) follows directly from König and Meixner with the understanding that  $\underline{G}$  is defined as zero before the beginning of the aging process (for negative  $t$ ). The proof of (f) is a consequence of Equation (3.2.7) and (3.2.8) since the König and Meixner representation is valid for each stress and strain component.

The constitutive law presented in Theorem 3.2 is for continuous strain histories. Once again the theorem of Gurtin and Sternberg (Reference 9, page 312) will allow extension of the constitutive law to a class of strain histories which is discontinuous on  $[0, t]$ . Application of the theorem in the case of aging materials is valid since the proof of the theorem does not depend on the size of the time interval  $[0, t]$ .

For convenience, assume the material to be isotropic. (As mentioned earlier, if this is not the case, then the appropriate material restrictions can be placed on the material response functional.) Now from Theorem 2.5 of Reference 9 it is possible to replace the tensorial function  $\underline{G}$  in Equation (3.2.9) by two scalar functions. Introducing the deviatoric and hydrostatic stress and strain components as defined in Equation (1.3.4) and using part (f) of Theorem 3.1 the constitutive equation can be written as

$$S_{ij}(t) = G_1(0,t)E_{ij}(t) + \int_0^t E_{ij}(t-\gamma)G_1(\gamma,t),\gamma d\gamma$$

and

$$\underline{\sigma}_{kk}(t) = G_2(0,t)\underline{E}_{kk}(t) + \int_0^t \underline{E}_{kk}(t-\gamma)G_2(\gamma,t),\gamma d\gamma. \quad (3.2.10)$$

Here  $G_1$  and  $G_2$  are the relaxation functions in shear and dilation respectively.

By interchanging the role of the stress and strain histories in Equation (3.2.1), a similar constitutive relation can be obtained. The counterpart of the relaxation law given in Equation (3.2.9) is the creep law

$$\underline{\epsilon}(t) = \int_{-\infty}^t \underline{\sigma}(t-\gamma) d\underline{J}(\gamma,t). \quad (3.2.11)$$

For an isotropic material the aging hereditary creep law is

$$e_{ij}(t) = J_1(0,t)S_{ij}(t) + \int_0^t S_{ij}(t-\gamma)J_1(\gamma,t),\gamma d\gamma$$

and

$$\underline{E}_{kk}(t) = J_2(0,t)\underline{\sigma}_{kk}(t) + \int_0^t \underline{\sigma}_{kk}(t-\gamma)J_2(\gamma,t),\gamma d\gamma. \quad (3.2.12)$$

where  $J_1$  and  $J_2$  are the creep functions in shear and dilatation respectively.



### 3.3 Correlation with Material Response Data and Existing Response Functions

To investigate the physical meaning of the variables in the relaxation function of Equation (3.2.9), let the strain history be given by

$$\underline{\xi}(t) = \underline{1}(t-t_L). \quad (3.3.1)$$

Then apply parts (b) and (d) of Theorem 3.1 to Equation (3.2.9) to get

$$\underline{\sigma}(t) = \underline{G}(t-t_L, t). \quad (3.3.2)$$

Thus,  $G(t-t_L, t)$  is the stress at the present time  $t$  due to a step strain that was applied  $t_L$ . The quantity  $(t-t_L)$  represents the time that has elapsed since the strain was applied. A typical relaxation surface is shown in Figure 15.

At this point it is easy to introduce the relationships between the creep and relaxation functions for aging materials. The relaxation function  $\underline{G}$  in Equation (3.3.2) may be interpreted as the stress history required to produce the unit-step strain history  $\underline{1}(t-t_L)$ . Thus, substituting (3.3.1) and (3.3.2) into (3.2.11) yields

$$\begin{aligned} \underline{1}(t-t_L) &= \int_{-\infty}^t \underline{G}(t-\gamma-t_L, t-\gamma) d\underline{J}(\gamma, t) \\ &= \int_{-\infty}^t \underline{J}(t-\gamma, t) d\underline{G}(\gamma-t_L, \gamma). \end{aligned} \quad (3.3.3)$$

Two additional relationships may be arrived at when a unit-step stress history is applied to Equation (3.2.11). They are:

$$\begin{aligned} \underline{1}(t-t_L) &= \int_{-\infty}^t \underline{G}(t-\gamma, t) d\underline{J}(\gamma-t_L, \gamma) \\ &= \int_{-\infty}^t \underline{J}(t-\gamma-t_L, t-\gamma) d\underline{G}(\gamma, t). \end{aligned} \quad (3.3.4)$$

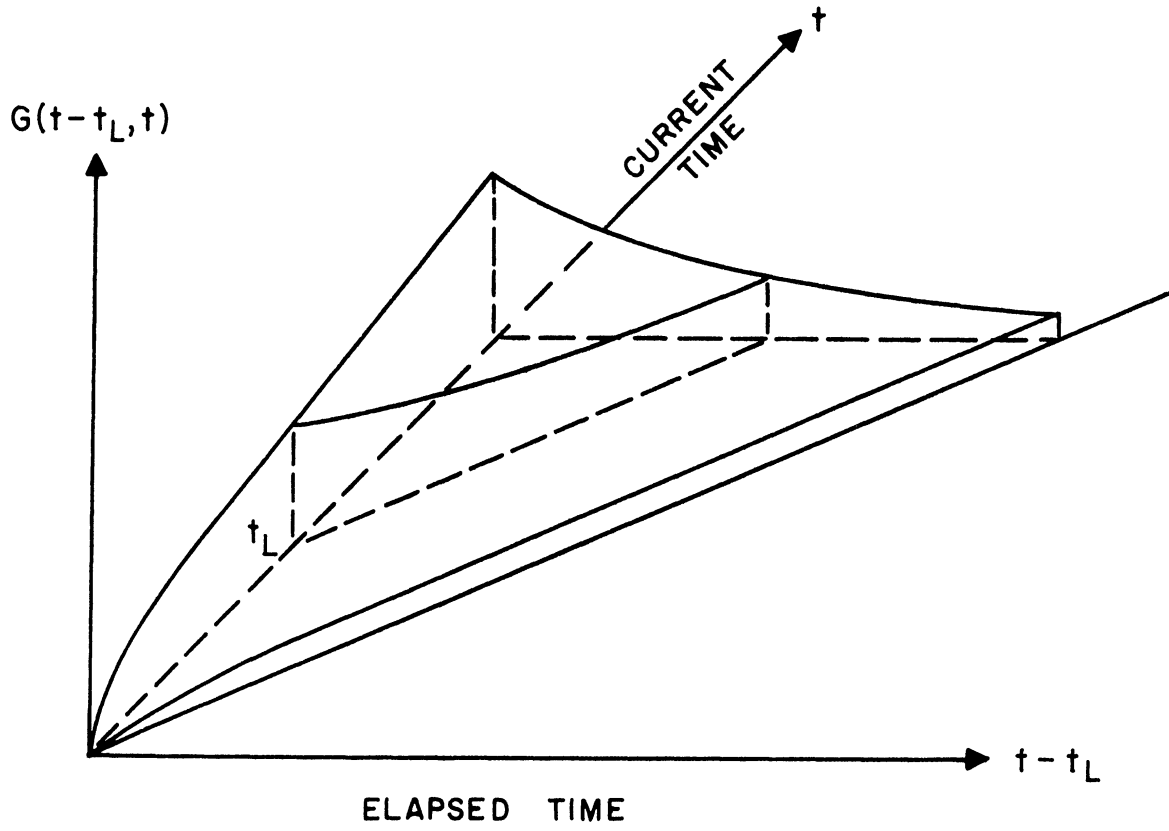


Figure 15. A Typical Relaxation Surface for the Response Functions of Equation (3.2.9).

Equations (3.3.3) and (3.3.4) will yield four equivalent forms of a Volterra integral equation which may be solved for one of the material response functions when the other is determined experimentally.

Alternate constitutive laws for aging materials exist in the literature. These alternate forms can easily be related to the above forms by a change of variable.

One of the first investigations in the field of aging concrete response is given by McHenry in Reference 6. His development is based on an unusual definition of creep. McHenry observed in the laboratory that as concrete ages the elastic modulus increases. That is, if identical loads are applied to similar concrete specimens of different ages, the amount of corresponding instantaneous strain decreases as the age increases. Thus, McHenry defines the true creep to be the difference between the total deformation and the decreasing instantaneous response.

For example, let  $\sigma(t)$  be a unit-step stress applied to the material at time  $t_L$ . Then the strain at some later time  $t$  is given by

$$\epsilon(t) = \frac{1}{E(t)} + J(t_L, t - t_L), \quad (3.3.5)$$

where  $1/E(t)$  is the instantaneous strain resulting from a unit step stress applied at time  $t$ .  $J(t_L, t - t_L)$  is the creep response as a function of the age of the material  $t_L$  at loading, and  $t - t_L$  is the elapsed time after the load was applied.

McHenry constructed a super-position integral from Equation (3.3.5) to determine the response for one-dimensional time dependent stress histories. His results are

$$\epsilon(t) = \frac{\sigma(t)}{E(t)} + \int_{t_L}^t J(\gamma, t - \gamma) \frac{d\sigma(\gamma)}{d\gamma} d\gamma \quad (3.3.6)$$

for the special case when  $\sigma(t_L) = 0$ . Equation (3.3.6) reduces immediately to the linear elastic constitutive law in the absence of a creep response.

An alternative to the McHenry formulation is presented by Sackman.<sup>(5)</sup> His material response functions involve the current time and the time of loading as the two independent variables. The constitutive law from Reference 5 is

$$\epsilon(t) = \int_{t_L^-}^{t^+} H(t, \gamma) \frac{d\sigma(\gamma)}{d\gamma} d\gamma = - \int_{t_L^-}^{t^+} \sigma(\gamma) \frac{\partial H(t, \gamma)}{\partial \gamma} d\gamma, \quad (3.3.7)$$

employing the fact that the derivative of unit step functions give rise to a Dirac delta function. For the stress history  $\sigma(t) = 1(t-t_L)$  the solution (3.3.7) is

$$\epsilon(t) = H(t, t_L). \quad (3.3.8)$$

Thus  $H(t, t_L)$  is the strain that results at time  $t$  due to a unit-step stress applied at time  $t_L$ .

Sackman's result can be related to Equation (3.2.11) by letting

$$J(\gamma, t) = \hat{J}(t-\gamma, t), \quad (3.3.9)$$

where  $t-\gamma$  corresponds to the actual time of loading. This new creep function  $\hat{J}(t-\gamma, t)$  gives rise to an additional form that Equations (2.3.11) and (2.3.12) can assume. For example, the first of (2.3.11) can be re-written as

$$1(t-t_L) = \int_{-\infty}^t \hat{G}(t_L, t-\gamma) d\hat{J}(t-\gamma, t),$$

where  $\hat{G}$  is the relaxation function associated with  $\hat{J}$ .

In another independent study Arutyunyan<sup>(7)</sup> obtained a similar stress-strain relation. He used his result to solve many problems in reinforced concrete structures.

In general, the creep or relaxation response modulus of an aging material must be determined experimentally as a function of the two independent time variables. For example, Sackman's model  $J(t, t_L)$  represents the strain at time  $t$  due to a unit-step stress applied at time  $t_L$ . Thus to determine  $J(t, t_L)$  experimentally it is necessary to perform a series of one-dimensional creep tests initiated at different values of  $t_L$ . This results in a family of curves that may be represented as a creep surface.

Bresler and Selna<sup>(8)</sup> establish a creep surface for mortar using Yashin's data.<sup>(31)</sup> The curves in Figure 16 represent the early age creep of mortar loaded to about 20 percent of its compressive strength. The tests are carried out at a relative humidity of 75 to 80 percent and for temperatures of 60 to 70 degrees Fahrenheit. The surface is presented using the absolute time  $t$  and the age of the material  $t_L$  at which the load is applied. Consequently, the surface can only exist in one half of the first quadrant.

Comparing the three forms of the constitutive relationship presented in this section, it can be seen that they all are valid representations of the stress-strain law. In fact, it is easy to pass from one form to another by appropriately redefining the time variables and response functions. Sackman's model appears to lend itself to experimental investigations rather well, while Equation (3.2.9) may be better for analytical studies since many of the Riemann-Stieltjes convolution algebra properties carry over.

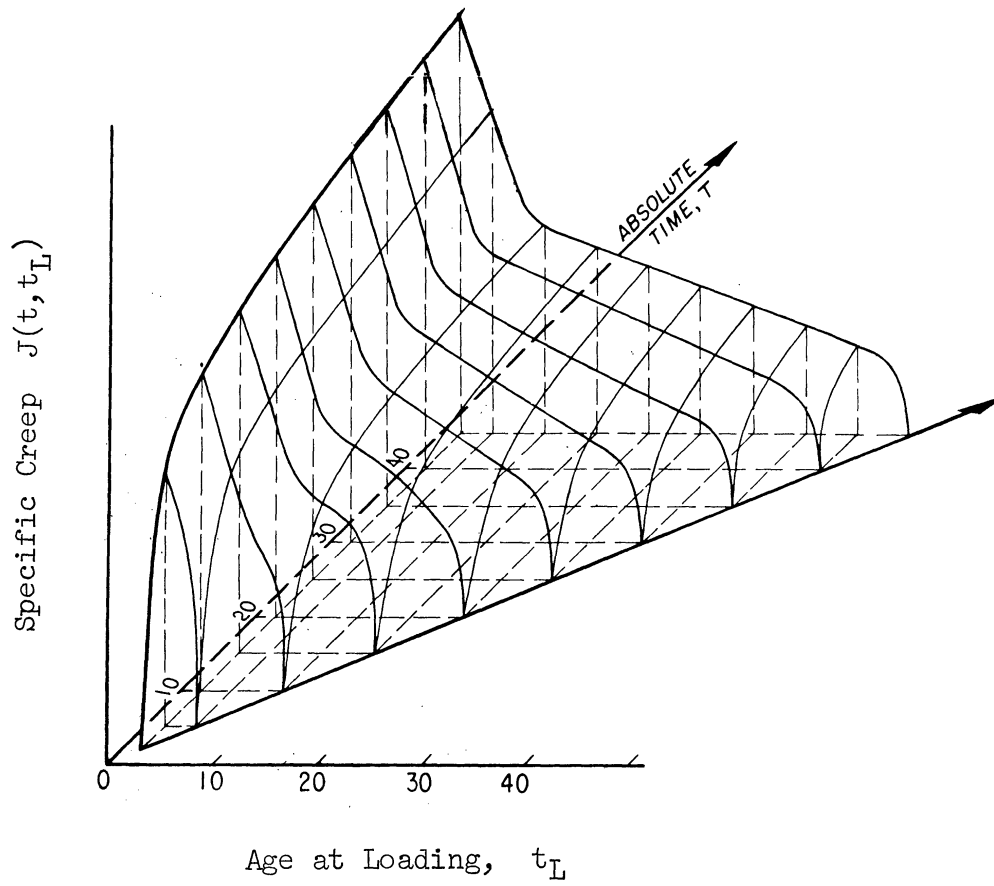


Figure 16. Early Age Creep Surface of Mortar.  
(From Bresler and Selna, "Analysis of Time-Dependent Behavior of Reinforced Concrete Structures," Symposium on Creep of Concrete (8))

## CHAPTER IV

### APPLICATION TO BOUNDARY VALUE PROBLEMS

The first section of Chapter IV gives the quasi-static field equations and boundary conditions which the stress, strain, and displacement field histories must satisfy to qualify as a solution to a boundary value problem. Section 4.2 specializes the field equations for plane strain and generalized plane stress boundary value problems.

#### 4.1 Field Equations

Recall that  $\sigma_{ij}$  and  $\epsilon_{ij}$  are the Cartesian components of the second order, symmetric stress and strain tensors  $\underline{\sigma}$  and  $\underline{\epsilon}$ . Further, let  $u_i$  be the Cartesian components of the displacement vector  $\underline{u}$ . The quantities  $\underline{\sigma}$ ,  $\underline{\epsilon}$  and  $\underline{u}$  are all taken to depend on position and time unless otherwise specified. Let the viscoelastic state defined by  $\underline{\sigma}$ ,  $\underline{\epsilon}$  and  $\underline{u}$  exist in body  $\mathcal{B}$ . The time dependent surface  $S(t)$  of body  $\mathcal{B}$  is comprised two complementary subsets  $S_1(t)$  and  $S_2(t)$ .

For convenience with the notation let  $G_i(\mathcal{J}, t, \underline{x})$  and  $J_i(\mathcal{J}, t, \underline{x})$ , ( $i = 1, 2$ ) represent the material response functions of Equation (2.4.5) and (2.4.6) or (3.2.10) and (3.2.12) respectively. The operational notation introduced in Section 3.1 can be used to represent the constitutive equations mentioned above. For the case when the material is not dependent on the environment or is non-aging then  $G_i$  and  $J_i$  become independent of either  $\underline{x}$  or  $\mathcal{J}$  respectively.

For environmental dependent materials, it is assumed that  $\phi(\underline{x}, t)$  can be determined from the appropriate physical law and boundary conditions. Also the strain history which results from time and spatial gradients in  $\phi(\underline{x}, t)$  is assumed to be known.

Further, assume that the initial conditions are given by

$$\underline{E}_{ij} = \underline{\sigma}_{ij} = u_i = 0 \quad (4.1.1)$$

for all  $\underline{x}$  in body  $\mathbb{B}$  and all  $t$  in  $(-\infty, 0)$ . That is, the body in the undeformed state is stress free.

Now the governing field equations are stated. The equilibrium equations which must be satisfied in  $\mathbb{B}$  for all  $t$  are

$$\underline{\sigma}_{ij,j}(\underline{x}, t) + F_i(\underline{x}, t) = 0, \quad (4.1.2)$$

where  $F_i(\underline{x}, t)$  is the body force history per unit of volume. The strain history is related to the displacement history by

$$\underline{E}_{ij}(\underline{x}, t) = \frac{1}{2} [u_{i,j}(\underline{x}, t) + u_{j,i}(\underline{x}, t)]. \quad (4.1.3)$$

To state the constitutive equations, recall that the deviatoric stress and strain components are

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

and

$$e_{ij} = E_{ij} - \frac{1}{3} E_{kk} \delta_{ij}. \quad (4.1.4)$$

Then the isotropic hereditary relaxation law for a general material is

$$S_{ij} = e_{ij} : dG_1, \quad (4.1.5)$$

$$\sigma_{kk} = E_{kk} : dG_2; \quad (4.1.6)$$

and the integral creep law can be written as



$$E_{ij} = S_{ij} : dJ_1, \quad (4.1.7)$$

$$E_{kk} = \sigma_{kk} : dJ_2. \quad (4.1.8)$$

Equations (4.1.5) through (4.1.8) use the operator notation introduced in Theorem 3.1.

It should be noted that if the volumetric response is assumed to be elastic then Equation (4.1.6) and (4.1.8) can be replaced by

$$\frac{1}{3K} \sigma_{kk}(\underline{x}, t) = E_{kk}(\underline{x}, t), \quad (4.1.9)$$

where  $K$  is the elastic bulk modulus. This assumption is very often a good approximation since many viscoelastic materials demonstrate an elastic dilatational response.

If the material is nearly incompressible, the dilatational effects can be neglected. This amounts to letting  $K \rightarrow \infty$  in Equation (4.1.9). Thus,

$$E_{11} + E_{22} + E_{33} = 0 \quad (4.1.10)$$

and  $\sigma_{kk}$  cannot be determined from Equation (4.1.9). The indeterminacy can be removed by relating  $\sigma_{kk}$  to the boundary conditions. An example of this type is considered in Chapter V.

Next, the boundary conditions are considered. If  $U(\underline{x}, t)$  is the prescribed displacement on the boundary  $S(t)$  or portion of the boundary of body  $\mathbb{B}$  during the time interval  $[0, \infty)$  then,

$$u_i(\underline{x}, t) = U_i(\underline{x}, t) \quad (4.1.11)$$

for all  $\underline{x}$  on  $S$  and all  $t$  in  $[0, \infty)$ . The stress type boundary

condition takes on the form

$$\sigma_{ij}(x,t) \eta_j = T_i(x,t) \quad (4.1.12)$$

where  $\eta_j$  is the unit normal vector at  $\underline{x}$  and  $T_i$  is the prescribed surface traction on remaining boundary of the body in  $[0, \infty)$ . In addition to these boundary conditions, suitable boundary conditions must be prescribed so that it is possible to calculate the value of  $\phi(\underline{x}, t)$  from the appropriate physical law for all  $\underline{x}$  in  $\mathbb{B}$  and  $t$  in  $[0, \infty)$ .

Now the displacement equilibrium equations are established.

From Equation (4.1.4) and (4.1.5) the stress-strain relation can be re-written as:

$$\sigma_{ij} = \epsilon_{ij} : dG_1 + \frac{1}{3} \delta_{ij} \epsilon_{kk} : d(G_2 - G_1), \quad (4.1.13)$$

where  $G_1$  and  $G_2$  represent the general material response functions introduced earlier. Substituting in the strain-displacement Equations (4.1.3), Equation (4.1.13) becomes

$$\sigma_{ij} = \frac{1}{2} (\mu_{i,j} + \mu_{j,i}) : dG_1 + \frac{1}{3} \delta_{ij} \mu_{k,k} : d(G_1 - G_2). \quad (4.1.14)$$

Substitute (4.1.14) into (4.1.2) to obtain

$$\mu_{i,jj} : dG_1 + \mu_{j,jj} : dK + \frac{2}{3} \delta_{ij} \mu_{k,k} : dG_2 : j \quad (4.1.15)$$

$$+ \{ \mu_{i,jj} + \mu_{j,i} - \frac{2}{3} \delta_{ij} \mu_{k,k} \} : dG_1 : j + 2 F_i = 0,$$

where

$$K = \frac{1}{3} (G_1 + 2G_2). \quad (4.1.16)$$

In conclusion of this section, it should be mentioned that many of the existing field theorems in viscoelasticity can be applied here. Gurtin and Sternberg, <sup>(33)</sup> established a reciprocal theorem for anisotropic linear

viscoelastic solids. Then in 1966 Lubliner and Sackman<sup>(34)</sup> extended the theorem to include aging materials. Also Lubliner and Sackman<sup>(35)</sup> developed a general uniqueness theorem for linear viscoelasticity which is valid for both aging and anisotropic materials.

#### 4.2 Plane Strain and Generalized Plane Stress

In this section the general field equations are specialized for the two-dimensional problem. For this purpose introduce the convention that when lower case Greek subscripts appear they take on the values (1,2). Let body  $\mathbb{B}$  occupy a right cylindrical region of space with a lateral boundary  $S$ . Let the generators of  $\mathbb{B}$  be parallel to the  $x_3$  axis. Let  $x_3 = 0$  be the midplane and let  $x_3 = \pm h$  be the boundary surfaces normal to the  $x_3$  axis.

For the general two-dimensional mixed boundary value problem these additional restrictions are necessary on the boundary parallel to the  $x_3$  axis:

$$\begin{aligned} U_\alpha &= U_\alpha(x, t), \quad U_3 = 0; \\ T_\alpha &= T_\alpha(x, t), \quad T_3 = 0; \end{aligned} \tag{4.2.1}$$

where  $x$  denotes  $(x_1, x_2)$ . The applied body force  $\underline{F}$  must have the form

$$F_\alpha = F_\alpha(x, t), \quad F_3 = 0. \tag{4.2.2}$$

Since all mechanical quantities are limited to two spatial variables, let the environmental field history  $\phi$  also be independent of  $x_3$ . Then it follows that the general material functions are of the form

$$G_\alpha = G_\alpha(\gamma, t, \kappa)$$

and

$$J_\alpha = J_\alpha(\gamma, t, \kappa). \quad (4.2.3)$$

PLANE STRAIN. Consider first the case of plane strain. By definition set

$$u_\alpha = u_\alpha(t, \kappa_1, \kappa_2), \quad u_3 = 0. \quad (4.2.4)$$

Then as a result of Equation (4.1.2), (4.1.3), (4.1.5) and (4.1.6) the following are obtained

$$E_{\alpha 3} = 0, \quad \sigma_{\alpha 3} = 0; \quad (4.2.5)$$

$$E_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}); \quad (4.2.6)$$

$$\sigma_{\alpha\beta,\beta} + F_\alpha = 0; \quad (4.2.7)$$

$$\sigma_{\alpha\beta} = E_{\alpha\beta} : dG_1 + \delta_{\alpha\beta} E_{\gamma\gamma} : dG; \quad (4.2.8)$$

where  $G = \frac{1}{3}(G_2 - G_1).$  (4.2.9)

Next determine the relationship between  $\sigma_{33}$  and  $\sigma_{\alpha\alpha}$ . Begin by setting  $i = j = 3$  in Equation (4.1.5). Introduce (4.1.6) into (4.1.7) and employ the linearity properties of the integral operator to get

$$2\sigma_{33} - \sigma_{\alpha\alpha} = -E_{\alpha\alpha} : dG_1, \quad (4.2.10)$$

noting that  $E_{kk} = E_{\alpha\alpha}$  since  $E_{33} = 0$ . Next rewrite (4.1.8) as

$$E_{\alpha\alpha} = (\sigma_{33} + \sigma_{\alpha\alpha}) : dJ_2. \quad (4.2.11)$$

Substituting Equation (4.2.11) into (4.2.10) and rearranging yields

$$2\sigma_{33} + [(\sigma_{33} + \sigma_{\alpha\alpha}) : dJ_2] : dG_1 = \sigma_{\alpha\alpha}. \quad (4.2.12)$$

Equation (4.2.12) establishes the governing Volterratype integral equation to determine  $\sigma_{33}$  once  $\sigma_{\alpha\alpha}$  is known. This equation can be solved when a specific material is considered. The solution will be of the form

$$\sigma_{33}(x, t) = K_1(t, x) + \int_0^t K_2(t, \gamma, x) \sigma_{\alpha\alpha}(\gamma, x) d\gamma, \quad (4.2.13)$$

where  $K_1(t, x)$  and  $K_2(t, \gamma, x)$  are functions of  $J_2$ ,  $G_1$  and  $\sigma_{\alpha\alpha}$ .

For future reference it is useful to have the inverse of the stress-strain relation given by Equation (4.2.8). Substituting Equation (4.1.4) and (4.1.10) into (4.1.7) yields

$$\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} : dJ_1 + \frac{1}{3} \delta_{\alpha\beta} [\sigma_{\gamma\gamma} : dJ + \sigma_{33} : dJ_2], \quad (4.2.14)$$

where  $J = J_2 - J_1$ . In Equation (4.2.14) all of the stress components are required in order to determine the strain components  $\epsilon_{\alpha\beta}$ ; therefore, it is necessary to solve Equation (4.2.12) prior to using (4.2.14).

Thus for the solution of a plane-strain boundary value problem the quantities  $u_\alpha$ ,  $\epsilon_{\alpha\beta}$  and  $\sigma_{\alpha\beta}$  must satisfy Equation (4.2.6), (4.2.7) and (4.2.8) or (4.2.14) and the boundary conditions

$$u_\alpha = \bar{U}_\alpha(x, t) \text{ on } S_1 \text{ for } (0 \leq t < \infty),$$

$$\sigma_{\alpha\beta} n_\beta = \bar{T}_\alpha(x, t) \text{ on } S_2 \text{ for } (0 \leq t < \infty), \quad (4.2.15)$$

together with the initial conditions

$$u_\alpha(x, t) = \sigma_{\alpha\beta}(x, t) = 0 \text{ in } \mathbb{B} \text{ for } (-\infty < t < 0). \quad (4.2.16)$$

In Equation (4.2.15),  $S_1$  and  $S_2$  are complementary subsets of the boundary of body  $B$  and  $\eta_\beta$  is the unit normal.

GENERALIZED PLANE STRESS. The governing condition for a plane stress boundary value problem is

$$\sigma_{\alpha 3}(\mathbf{x}, t) = 0. \quad (4.2.17)$$

Introducing Equation (4.2.17) into (4.1.7) and (4.1.8), and in view of (4.1.4), the constitutive equations become

$$\begin{aligned} \epsilon_{\alpha\beta} &= \sigma_{\alpha\beta} : dJ_1 + \delta_{\alpha\beta} \sigma_{\gamma\gamma} : dK \\ \epsilon_{33} &= \sigma_{\alpha\alpha} : dK \end{aligned} \quad (4.2.18)$$

where the material response function  $K$  is defined as

$$K = \frac{1}{3}(J_2 - J_1). \quad (4.2.19)$$

The equilibrium equations, (4.1.2) reduce to

$$\sigma_{\alpha\beta, \beta} + F_\alpha = 0 \quad (4.2.20)$$

in view of Equations (4.2.2) and (4.2.17).

All of the above equations resemble those for plane strain; but the dependence of all field histories on  $x_3$  is still present. The generalized plane stress boundary value problem eliminates this dependence by the averaging technique

$$\hat{Q}(\mathbf{x}, t) = \hat{Q}(x_1, x_2, t) = \frac{1}{2h} \int_{-h}^h Q(x_1, x_2, x_3, t) dx_3. \quad (4.2.21)$$

In Equation (4.2.21)  $\hat{Q}$  represents any typical field history and  $h$  is small compared to all other dimensions. Further, for convenience, let  $\hat{G}_\alpha$  and  $\hat{J}_\alpha$  represent the relaxation and creep moduli associated with the generalized plane stress problem. Note that  $\hat{J}_\alpha$  and  $\hat{G}_\alpha$  are independent of the  $x_3$  coordinate since  $\phi$  is independent of  $x_3$ .

Applying Equation (4.2.21) to (4.1.3), (4.2.18), (4.2.19) and (4.2.20), the field equations for the generalized plane stress problem becomes

$$\hat{\sigma}_{\alpha\beta,\beta} + F_\alpha = 0 ; \quad (4.2.22)$$

$$\hat{\epsilon}_{\alpha\beta} = \frac{1}{2} (\hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha}) ; \quad (4.2.23)$$

$$\hat{\epsilon}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta} : d\hat{J}_1 + \delta_{\alpha\beta} \hat{\sigma}_{\gamma\gamma} : d\hat{K} ;$$

$$\hat{\epsilon}_{33} = \hat{\sigma}_{\gamma\gamma} : d\hat{K} \quad (4.2.24)$$

where 
$$\hat{K} = \frac{1}{3} (\hat{J}_2 - \hat{J}_1). \quad (4.2.25)$$

Then the field histories  $\hat{\sigma}_{\alpha\beta}$ ,  $\hat{\epsilon}_{\alpha\beta}$  and  $\hat{u}_\alpha$  must satisfy Equations (4.2.22), (4.2.23), and (4.2.24) together with the initial conditions

$$\hat{u}_\alpha(x,t) = \hat{\sigma}_{\alpha\beta}(x,t) = 0 \text{ in } \mathbb{B} \text{ for } (-\infty < t < 0), \quad (4.2.26)$$

and the boundary conditions

$$\hat{u}_\alpha(x,t) = \hat{U}_\alpha(x,t) \text{ on } S_1 \text{ for } (0 \leq t < \infty)$$

and 
$$\hat{\sigma}_{\alpha\beta} n_\beta = T_\alpha(x,t) \text{ on } S_2 \text{ for } (0 \leq t < \infty) \quad (4.2.27)$$

to qualify as a solution of the generalized plane stress boundary value problem.

In view of Equation (4.2.13), the coefficient of  $\delta_{\alpha\beta}$  in Equation (4.2.14) defines a linear integral functional in  $\sigma_{\alpha\beta}$ . Comparing this result to Equation (4.2.24) it is apparent that an analogy between the Plane Strain and Generalized Plane Stress problems exists. This correspondence can be established by replacing  $\hat{J}_1$  and  $\hat{K}$  in Equation (4.2.24) by  $J_1$  and the above mentioned linear integral functional of  $\sigma_{\alpha\beta}$  respectively. In some special cases this correspondence may take on a particularly convenient and useful form. For "Thermorheologically Simple" materials this relationship is given by Gurtin and Sternberg in Reference 25.



## CHAPTER V

### INCOMPRESSIBLE THICKWALL CYLINDER IN PLANE STRAIN

This chapter investigates the influence of temperature on the stress components in a thickwall cylinder composed of a material which obeys the constitutive relationships of Chapter II. Section 5.1 contains the formulation of the cylinder problem for two sets of boundary conditions. First, two arbitrary pressure histories are applied at the internal and external boundary. The second situation considers an internal pressure history with the external boundary supported by a thin elastic shell. In the second section the material response functional is established for a three parameter solid under the influence of a temperature field. Sections 5.3 and 5.4 give the exact solutions for the unsupported and supported cylinders for temporally constant temperatures. Also a parameterization study is included for the unsupported cylinder.

#### 5.1 Formulation of the Problem

To illustrate a method of solution for the viscoelastic models considered earlier, a thickwall cylinder is examined in the state of plane strain. The problem is formulated for a linear incompressible viscoelastic material which has an aging or non-aging environmental-dependent shear response in creep  $J(t, \mathcal{J}, x)$  and relaxation  $G(t, \mathcal{J}, x)$ . Later in the study, the shear response is specialized for non-aging materials with temperature-dependent properties.

Let  $(r, \theta, z)$  denote a generic point in a cylindrical polar coordinate system which is centered in body  $\mathbb{B}$ . Body  $\mathbb{B}$  is a cylinder of inner radius  $r = a$ , outer radius  $r = b$ , and height  $z = \pm Z$ .

The cylinder is analyzed for two sets of boundary conditions. Case I is for arbitrary internal and external pressure histories which are uniform on the boundaries. Case II considers the viscoelastic cylinder to be supported by a thin elastic shell. The internal surface at the elastic shell is assumed to be in a uniform stress-free contact with the external boundary of the viscoelastic cylinder prior to time equal to zero. At time equal to zero an arbitrary pressure history is applied uniformly to the inner boundary of the viscoelastic cylinder. The external boundary of the elastic shell is taken to be stress-free for all time.

Due to the symmetry in the boundary conditions and body, the  $(r, \theta, z)$  coordinate system coincides with the principal directions of the stress and strain tensors. Denote the components of the principal stress and strain tensors by  $(\sigma_r, \sigma_\theta, \sigma_z)$  and  $(\epsilon_r, \epsilon_\theta, \epsilon_z)$  respectively. Also, for this particular coordinate system and for the case of plane strain, all field quantities are independent of the  $\theta$  and  $z$  coordinates. Assume the displacement field is given by

$$u = u_r(r, t) ; \quad u_\theta = u_z = 0 . \quad (5.1.1)$$

Then the strain-displacement equations reduce to

$$\epsilon_r(r, t) = u_{,r} ; \quad \epsilon_\theta(r, t) = \frac{u}{r} ; \quad \epsilon_z = 0 . \quad (5.1.2)$$

In the absence of body forces the only non-trivial equilibrium equation is

$$\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (5.1.3)$$

Since the object of this study is to investigate the effect of temperature

sensitive material response on the stress distribution, assume that the dilatational strain due to temperature is negligible. The stress-strain behavior in shear becomes

$$\sigma_r - \sigma_\theta = (\epsilon_r - \epsilon_\theta) : dG \quad (5.1.4)$$

where  $G$  is the aging, environmental-dependent shear response function and  $:$  denotes the integral operator given the Theorem 3.1.

The incompressibility condition requires that the dilatational response reduce to

$$\epsilon_r + \epsilon_\theta = \mu_{,r} + \frac{\mu}{r} = \frac{1}{r} (r\mu)_{,r} = 0 \quad ;$$

therefore,

$$\mu = \frac{f(t)}{r} \quad , \quad (5.1.5)$$

where  $f(t)$  is an arbitrary function of time. Then from Equation (5.1.2) the strain histories become

$$\epsilon_r = -\epsilon_\theta = -\frac{f(t)}{r^2} \quad . \quad (5.1.6)$$

Further the incompressibility condition in combination with Equation (4.2.10) gives

$$\sigma_z = \frac{1}{2} (\sigma_r + \sigma_\theta) \quad . \quad (5.1.7)$$

Combining Equation (5.1.4) and (5.1.6) yields

$$\sigma_r - \sigma_\theta = -\frac{2f(t)}{r^2} : dG \quad . \quad (5.1.8)$$

Substitute (5.1.8) into the equilibrium Equation (5.1.4) to obtain

$$\sigma_{r,r} = \frac{2f(t)}{\rho^3} : dG. \quad (5.1.9)$$

Integrating with respect to  $r$ , Equation (5.1.9) becomes

$$\sigma_r(r,t) = \int_a^r \left\{ \frac{2f(t)}{\rho^3} : dG(\gamma, t, \rho) \right\} d\rho + h(t). \quad (5.1.10)$$

Thus Equation (5.1.10) is the governing field equation that must be satisfied together with the boundary conditions.

Since the boundary condition on the internal surface is the same in both cases, it is convenient to consider this boundary condition first. Let  $P_a(t)$  denote an arbitrary time dependent pressure function. Then at  $r = a$  the normal stress must satisfy

$$\sigma_r(a,t) = -P_a(t). \quad (5.1.11)$$

Applying Equation (5.1.11) to (5.1.10) gives:

$$h(t) = - \int_{r=a} \left\{ \frac{2f(t)}{\rho^3} : dG(\gamma, t, \rho) \right\} d\rho - P_a(t) \quad (5.1.12)$$

Then eliminating  $h(t)$  from Equation (5.1.12) and (5.1.10), and interchanging time and spatial integrations yields

$$\sigma_r(r,t) = -P_a(t) + f(t) : dK(\gamma, t, r), \quad (5.1.13)$$

where

$$K(\gamma, t, r) = 2 \int_a^r \frac{G(\gamma, t, \rho)}{\rho^3} d\rho. \quad (5.1.14)$$

Combining Equation (5.1.13) and (5.1.8), the tangential stress component is given by

$$\sigma_\theta(r,t) = f(\gamma) : d \left\{ \frac{2G(\gamma, t, r)}{r^2} + K(\gamma, t, r) \right\} - P_a(t). \quad (5.1.15)$$

CASE I. Consider first the unsupported cylinder. Again define  $P_b(t)$  to be an arbitrary time dependent pressure function. The second boundary condition can now be written as

$$\sigma_r(b,t) = -P_b(t). \quad (5.1.16)$$

Introducing Equation (5.1.16) into (5.1.13) gives

$$P_a(t) - P_b(t) = f(r) : dK(r,t,b). \quad (5.1.17)$$

From Definition 3.3 it is seen that Equation (5.1.17) is a Volterra type integral equation which is to be solved for the function  $f(t)$ . A solution of Equation (5.1.17) can easily be obtained for non-aging materials in the presence of at temporally constant environmental fields. In this special case the integral operator in Equation (5.1.17) reduces to a Stieltjes convolution. Thus, employing the notation in the Appendix, the governing field equation for  $f(t)$  becomes

$$P_a(t) - P_b(t) = f(t) * dK(t,b). \quad (5.1.18)$$

The solution of Equation (5.1.18) can formally be written as

$$f(t) = [P_a(t) - P_b(t)] * dK(t,b)^{-1} \quad (5.1.19)$$

where  $K(t,b)^{-1}$  is the Riemann-Stieltjes inverse of  $K(t,b)$ . Introducing Equation (5.1.19) into (5.1.13) and (5.1.15), and using the associativity and commutativity properties of Stieltjes convolutions, yields

$$\sigma_r(r,t) = [K(t,b)^{-1} * dK(t,r)] * d[P_a(t) - P_b(t)] - P_a(t) \quad (5.1.20)$$

and

$$\sigma_{\theta}(r,t) = \left\{ K(t,b)^{-1} * d \left[ K(t,r) + \frac{2G(t,r)}{r^2} \right] \right\} * d [P_a(t) - P_b(t)] - P_a(t). \quad (5.1.21)$$

For the special case when the relaxation function is independent of  $r$ , then Equations(5.1.20) and (5.1.21) reduce to the Lame solution<sup>1</sup> of an elastic thickwall cylinder:

$$\begin{aligned} \sigma_r(t) &= -\frac{a^2 b^2 [P_a(t) - P_b(t)]}{(b^2 - a^2) r^2} + \frac{a^2 P_a(t) - b^2 P_b(t)}{(b^2 - a^2)} \\ \sigma_{\theta}(t) &= \frac{a^2 b^2 [P_a(t) - P_b(t)]}{(b^2 - a^2) r^2} + \frac{a^2 P_a(t) - b^2 P_b(t)}{(b^2 - a^2)} \end{aligned} \quad (5.1.22)$$

CASE II. When the exterior surface of the viscoelastic cylinder is supported by a thin elastic shell, the governing boundary conditions at the interface are:

$$\begin{aligned} u_r(b^-, t) &= u_r(b^+, t) ; \\ \sigma_r(b^-, t) &= \sigma_r(b^+, t) , \end{aligned} \quad (5.1.23)$$

see Figure 17.

Entering the first of Equation (5.1.23) into the strain-displacement equation for the tangential strain yields

$$\epsilon_{\theta}(b^-, t) = \frac{u(b^-, t)}{b} = \frac{u(b^+, t)}{b} = \epsilon_{\theta}(b^+, t) \quad (5.1.24)$$

for all  $t$ .

The radial stress in the elastic shell can now be related to the tangential strain. The general solution of the elastic problem is given

---

<sup>1</sup> See Reference 36, Section 26.

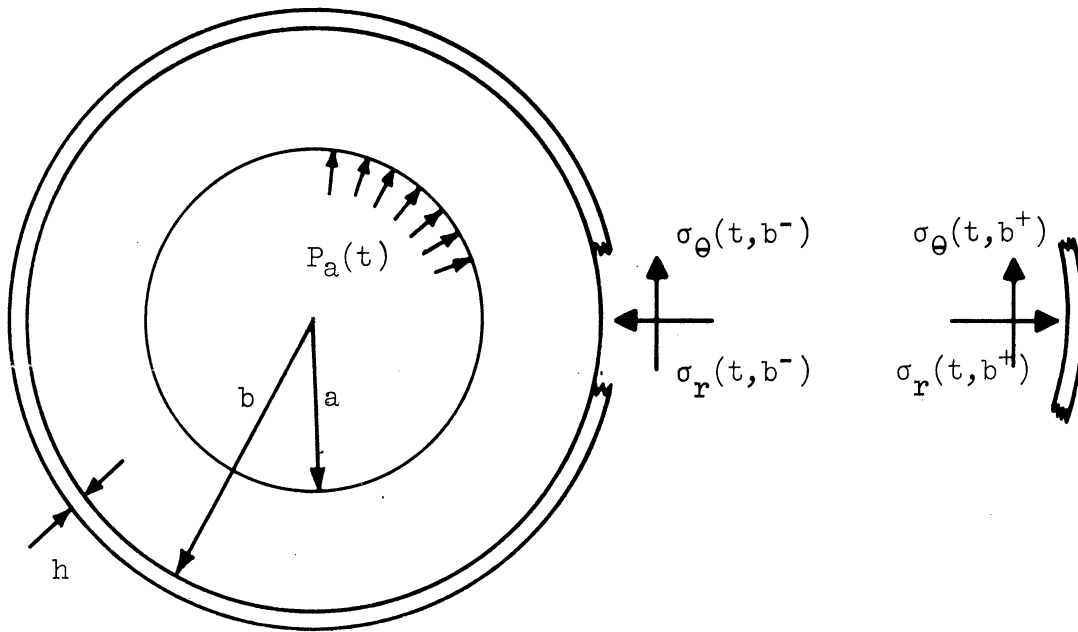


Figure 17. A Thick Wall Cylinder Supported by a Thin Elastic Shell.

by Timoshenko and Goodier,<sup>(36)</sup> thus for  $h \ll b$  it follows that

$$\epsilon_{\theta}(b^+, t) = -\sigma_r(b^+, t) \frac{1+\nu}{E} \left[ 1 + \frac{b}{h}(1-\nu) \right], \quad (5.1.25)$$

where  $E$  and  $\nu$  are the elastic modulus and Poisson's ratio of the shell. Thus employing the second boundary condition of Equation (5.1.23) and (5.1.24), (5.1.25) becomes

$$\sigma_r(b^-, t) = -\lambda \epsilon_{\theta}(b^-, t), \quad (5.1.26)$$

where

$$\frac{1}{\lambda} = \frac{1+\nu}{E} \left[ 1 + \frac{b}{h}(1-\nu) \right]. \quad (5.1.27)$$

But  $\epsilon_{\theta}(b^-, t)$  is given by Equation (5.1.6), so the governing condition on the stress field at  $r = b$  is given by

$$\sigma_r(b^-, t) = -\frac{\lambda f(t)}{b^2}. \quad (5.1.28)$$

Substituting Equation (5.1.28) into (5.1.13) and replacing  $\sigma_r$  by  $\sigma_r^*$  gives

$$-\frac{\lambda f(t)}{b^2} = -P_a(t) + f(t) * dK(t, b).$$

Thus

$$\left[ \frac{\lambda}{b^2} + K(0, b) \right] f(t) = P_a(t) - \int_0^t f(\tau) K(t-\tau, b)_{, t-\tau} d\tau \quad (5.1.29)$$

for non-aging materials in the presence of temporally constant environmental fields. Equation (5.1.29) is a Volterra integral equation of the second kind which is to be solved for  $f(t)$ .



## 5.2 Evaluation of the Material Response Functional

To proceed, it is now necessary to pick a particular material or class of materials. Since one of the objectives of this chapter is to investigate the effect of the vertical scaling and shift factors of mapping hypothesis introduced in Chapter II on the mechanical field histories, let the temporally constant relaxation response be given by

$$G(t, \psi) = C_1 \psi + (1 + C_2 \psi) G(t, \Phi), \quad (5.2.1)$$

where  $\psi$  is defined as the deviation of some environmental parameter from the reference state  $\Phi$ . Thus, Equation (2.1.7) becomes

$$\psi(\underline{x}, t) = \phi(\underline{x}, t) - \Phi. \quad (5.2.2)$$

Further, assume the mechanical response in the reference state is that of a three parameter solid,<sup>2</sup> then

$$G(t, \Phi) = q_0 + q_1 \exp\left(-\frac{t}{\rho}\right). \quad (5.2.3)$$

Now from Equation (2.3.20), the material response functional for a non-aging, environmental-dependent, three parameter, viscoelastic solid is

$$G\left[t, \phi\left(\frac{t}{s}\right)\right] = C_1 \psi + (1 + C_2 \psi) \left[ q_0 + q_1 \exp\left(-\frac{t}{\rho}\right) \right] - C_2 \int_0^t \left[ q_0 + q_1 \exp\left(-\frac{\tau}{\rho}\right) \right] \psi_{,\gamma} d\tau \quad (5.2.4)$$

for  $t > 0$ .

Let the initial temperature be  $T_0$  throughout the cylinder prior to  $t = 0$ . Let  $\phi = T(r, t)$  denote the temperature field which

---

<sup>2</sup> See Reference 17, page 16.

arises when the temperature at the internal boundary  $r = a$  is changed from  $T_b$  to  $T_a$  at  $t = 0$ . The governing equation for this temperature field is

$$\frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right), \quad (5.2.5)$$

where  $k$  is the coefficient of thermal conductivity. Equation (5.2.5) is to be solved together with the boundary conditions

$$T(a, t) = \begin{cases} T_b & \text{for } t \text{ in } (-\infty, 0) \\ T_a & \text{for } t \text{ in } [0, \infty) \end{cases} \quad (5.2.6)$$

and  $T(b, t) = T_b$  for all  $t$ .

The solution of Laplace's equation is a special case of a general solution given by Carslaw and Jaeger,<sup>(37)</sup> thus

$$T(r, t) = \frac{T_a \log\left(\frac{b}{r}\right) + T_b \log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)} + \pi(T_a - T_b) \sum_{m=1}^{\infty} \frac{U_0(r\alpha_m) J_0(\alpha\alpha_m) J_0(b\alpha_m)}{J_0^2(\alpha\alpha_m) - J_0^2(b\alpha_m)} \exp(-k\alpha_m^2 t), \quad (5.2.7)$$

where

$$U_0(\alpha_m r) = J_0(r\alpha_m) Y_0(b\alpha_m) - J_0(b\alpha_m) Y_0(r\alpha_m). \quad (5.2.8)$$

$J_0(r\alpha_m)$  and  $Y_0(r\alpha_m)$  are the Bessel functions of the first and second kind respectively. The  $\alpha_n$ 's are the roots of<sup>3</sup>

$$J_0(\alpha x) Y_0(b\alpha) - J_0(b\alpha) Y_0(\alpha x) = 0. \quad (5.2.9)$$

---

<sup>3</sup> These roots are given in Appendix IV, Table IV of Reference 37.

It is now possible to evaluate  $G[t, (S)]$  . Without loss of generality, let  $T_b$  correspond to the reference state  $\Phi$  , then Equation (5.2.2) becomes

$$\psi(r,t) = \frac{(T_a - T_b)}{\log(b/a)} \log(b/r) + \pi(T_a - T_b) \sum_{m=1}^{\infty} \frac{U_0(r\alpha_m) J_0(\alpha\alpha_m) J_0(b\alpha_m)}{J_0^2(\alpha\alpha_m) - J_0^2(b\alpha_m)} \exp(-k\alpha_m^2 t) . \quad (5.2.10)$$

Then on substituting Equation (5.2.10) into (5.2.4), and integrating term by term, the relaxation response function becomes

$$G[t, \Phi_{s=0}^t] = [q_0 + q_1 \exp(-\frac{t}{\tau})] + [C_1 + C_2 q_0 + C_2 q_1 \exp(-\frac{t}{\tau})] \left\{ \frac{(T_a - T_b) \log(b/r)}{\log(b/a)} + \pi(T_a - T_b) \sum_{m=1}^{\infty} \frac{U_0(r\alpha_m) J_0(\alpha\alpha_m) J_0(b\alpha_m)}{J_0^2(\alpha\alpha_m) - J_0^2(b\alpha_m)} \exp(-k\alpha_m^2 t) \right\} + C_2 \pi(T_a - T_b) \sum_{m=1}^{\infty} \frac{U_0(r\alpha_m) J_0(\alpha\alpha_m) J_0(b\alpha_m)}{J_0^2(\alpha\alpha_m) - J_0^2(b\alpha_m)} \times \left\{ q_0 [\exp(-k\alpha_m^2 t) - 1] + \frac{q_1 p k \alpha_m^2}{(p k \alpha_m^2 + 1)} [\exp(-\frac{p k \alpha_m^2 + 1}{p} t) - 1] \right\} . \quad (5.2.11)$$

The combining Equation (5.2.11) with Equation (5.1.14) gives

$$K(t,r) = A + B \exp(-\frac{t}{\tau}) + \sum_{m=1}^{\infty} \left\{ C_m \exp(-\frac{t}{\gamma_m}) + D_m \exp(-\frac{t}{\eta_m}) \right\} , \quad (5.2.12)$$

where  $A$ ,  $B$ ,  $C_n$ ,  $D_n$ ,  $\gamma_n$ , and  $\eta_n$  are given by

$$A = \frac{q_0(r^2 - a^2)}{a^2 r^2} + \frac{(C_1 + C_2 q_0)(T_a - T_b)}{a^2 r^2} \left[ r^2 - a^2 \frac{\log(b/r)}{\log(b/a)} - \frac{r^2 - a^2}{\log(b^2/a^2)} \right] \\ + 2\pi C_2 (T_a - T_b) \sum_{m=1}^{\infty} \left[ q_1 \frac{\eta_m}{\mu_m} - q_0 \right] E_m ;$$

$$B = \frac{q_1(r^2 - a^2)}{a^2 r^2} + \frac{C_2 q_1 (T_a - T_b)}{a^2 r^2} \left[ r^2 - a^2 \frac{\log(b/r)}{\log(b/a)} - \frac{r^2 - a^2}{\log(b^2/a^2)} \right] ;$$

$$C_m = 2\pi (C_1 + 2C_2 q_0)(T_a - T_b) E_m ;$$

$$D_m = 2\pi C_2 q_1 (T_a - T_b) \left[ 1 + \frac{\eta_m}{\mu_m} \right] E_m ; \quad (5.2.13)$$

$$E_m = \frac{J_0(a\alpha_m) J_2(b\alpha_m)}{J_0^2(a\alpha_m) - J_0^2(b\alpha_m)} \int_a^r \frac{U_0(\rho \alpha_m)}{\rho^3} d\rho ;$$

$$\mu_m = 1/k\alpha_m^2 ;$$

$$\eta_m = 1/(p k \alpha_m^2 + 1) .$$

### 5.3 Solution of the Unsupported Cylinder for Temporally Constant Temperature Fields

The results of the preceding section verify that the details of obtaining an analytic solution are quite involved. Since the objective of this study is to investigate the influence of the vertical scaling coefficients  $C_1$  and  $C_2$  on the stress fields, assume that the temperature field is constant for all time. Hence the solution corresponds to heating the cylinder and letting it come to a steady-state temperature distribution prior to applying the external loads. The thermal boundary conditions for this case are given by

$$\begin{aligned} T(a,t) &= T_a \quad \text{for } t \text{ in } (-\infty, \infty), \\ T(b,t) &= T_b \quad \text{for } t \text{ in } (-\infty, \infty), \end{aligned} \quad (5.3.1)$$

then, the solution of Equation (5.2.5) together with (5.3.3) yields

$$T(r,t) = \frac{T_a \log(b/r) + T_b \log(r/a)}{\log(b/a)}. \quad (5.3.2)$$

The deviation  $\psi$  from the reference state  $T_b$  is then

$$\psi = \frac{(T_a - T_b)}{\log(b/a)} \log(b/r) \quad (5.3.3)$$

and the relaxation function becomes

$$\begin{aligned} G(t,r) &= q_0 + (C_1 + C_2 q_0)(T_a - T_b) \frac{\log(b/r)}{\log(b/a)} \\ &+ q_1 \left[ 1 + C_2 (T_a - T_b) \frac{\log(b/r)}{\log(b/a)} \right] \exp\left(-\frac{t}{p}\right). \end{aligned} \quad (5.3.4)$$

Substituting (5.3.4) into (5.1.12) gives

$$K(t,r) = A + B \exp\left(-\frac{t}{p}\right), \quad (5.3.5)$$

where  $A$  and  $B$  are given by Equation (5.2.13) provided  $E_n = 0$ .

The Stieltjes inverse  $K(t,b)^{-1}$  must satisfy Equation (A1.3).

Applying the Laplace transformation technique to (A1.3), and in view of part (f) of Theorem (A1.3), it follows that the transform of the inverse function  $\overline{K(s,r)}^{-1}$  is given by

$$\begin{aligned} \overline{K(s,r)}^{-1} &= \frac{1}{s^2 \overline{K(s,r)}} = \frac{1}{s^2 \left\{ \frac{A}{s} + \frac{B-p}{1+ps} \right\}} \\ &= \frac{1/A}{s} - \frac{Bp/A^2}{1 + \left(\frac{A+B}{A}\right)ps}. \end{aligned} \quad (5.3.6)$$

Inverting Equation (5.3.6) and setting  $r = b$  yields

$$K(t, b)^{-1} = \frac{1}{A(b)} \left\{ 1 - \frac{B(b)}{A(b)+B(b)} \exp \left[ -\frac{A(b)}{A(b)+B(b)} \frac{t}{p} \right] \right\}. \quad (5.3.7)$$

The resultant stress field can now be determined. Let the pressure functions be given by

$$P_a(t) = P_a 1(t) ; \quad P_b(t) = 0, \quad (5.3.8)$$

where  $P_a$  is an arbitrary constant. Then Equation (5.1.20) and (5.1.21) become

$$\sigma_r(r, t) = P_a \left\{ K(t, b)^{-1} * dK(t, r) - 1(t) \right\}; \quad (5.3.9)$$

$$\sigma_\theta(r, t) = P_a \left\{ K(t, b)^{-1} * d \left[ K(t, r) + \frac{2G(t, r)}{r^2} \right] - 1(t) \right\}. \quad (5.3.10)$$

Substituting Equation (5.3.5) and (5.3.7) into the above and evaluating for  $t$  in  $(0, \infty)$  yields

$$\sigma_r(r, t) = P_a \left\{ \frac{A(r)}{A(b)} + \frac{1}{A(b)} \left[ B(r) - B(b) \frac{A(r) + B(r)}{A(b) + B(b)} \right] \times \exp \left[ -\left( \frac{A(b)}{A(b) + B(b)} \right) \frac{t}{p} \right] - 1 \right\} \quad (5.3.11)$$

and

$$\sigma_\theta(r, t) = P_a \left\{ \frac{A'(r)}{A(b)} + \frac{1}{A(b)} \left[ B'(r) - B(b) \frac{A'(r) + B'(r)}{A(b) + B(b)} \right] \times \exp \left[ -\left( \frac{A(b)}{A(b) + B(b)} \right) \frac{t}{p} \right] - 1 \right\}. \quad (5.3.12)$$

In Equation (5.3.11) and (5.3.12)

$$A'(r) = A(r) + F_1(r)$$

and 
$$B'(r) = B(r) + F_2(r) ; \quad (5.3.13)$$

$A(r)$  and  $B(r)$  are given by Equation (5.2.13) provided  $E_n = 0$  ; and,  $F_1(r)$  and  $F_2(r)$  are given by

$$F_1(r) = \frac{2}{r^2} \left[ q_0 + (c_1 + c_2 q_0)(T_a - T_b) \frac{\log(b/r)}{\log(b/a)} \right] ;$$

$$F_2(r) = \frac{2q_1}{r^2} \left[ 1 + c_2(T_a - T_b) \frac{\log(b/r)}{\log(b/a)} \right] . \quad (5.3.14)$$

It is now possible to evaluate the effect of the Mapping Hypothesis given in Assumption 2.1 on the resulting stress distributions. For the special case when  $T_a = T_b$  it follows that

$$A(r) = \frac{q_0(r^2 - a^2)}{a^2 r^2} ; \quad B(r) = \frac{q_1(r^2 - a^2)}{a^2 r^2} ;$$

$$A'(r) = \frac{q_0(r^2 + a^2)}{a^2 r^2} ; \quad B'(r) = \frac{q_1(r^2 + a^2)}{a^2 r^2} . \quad (5.3.15)$$

Then it is easy to show that the coefficients of the exponential terms in Equations (5.3.11) and (5.3.12) vanish, and that the remaining terms reduce to the temporally constant stress-state given in Equation (5.1.22). Thus the temperature sensitive material properties introduce a time-dependent response.

Next consider the special case when  $C_1 = 0$  in Equation (5.2.1). This corresponds to scaling the initial response and residual response by the same factor. From Equations (5.2.13) and (5.3.13) it can be shown

$$\frac{A(r)}{q_0} = \frac{B(r)}{q_1}$$

and 
$$\frac{A'(r)}{q_0} = \frac{B'(r)}{q_1} . \quad (5.3.16)$$

Introduction of Equation (5.3.16) into (5.3.11) and (5.3.12) also causes the exponential term to vanish and the resulting stress field is again independent of time. The stress components for  $C_1 = 0$  are

$$\sigma_r(r) = -P_a \left\{ 1 - \frac{b^2 \left[ (r^2 - a^2) + C_2 (T_a - T_b) \left( r^2 - a^2 \frac{\log(b/r)}{\log(b/a)} - \frac{r^2 - a^2}{\log(b^2/a^2)} \right) \right]}{r^2 \left[ (b^2 - a^2) + C_2 (T_a - T_b) \left( b^2 - \frac{r^2 - a^2}{\log(b^2/a^2)} \right) \right]} \right\} \quad (5.3.17)$$

Similarly it follows that

$$\sigma_\theta(r) = -P_a \left\{ 1 + \frac{b^2 \left[ (r^2 + a^2) + C_2 (T_a - T_b) \left( r^2 + a^2 \frac{\log(b/r)}{\log(b/a)} - \frac{r^2 - a^2}{\log(b^2/a^2)} \right) \right]}{r^2 \left[ (b^2 - a^2) + C_2 (T_a - T_b) \left( b^2 - \frac{r^2 - a^2}{\log(b^2/a^2)} \right) \right]} \right\} \quad (5.3.18)$$

To study the effect of temperature on the stress components, it is convenient to let

$$a=1, \quad b=2, \quad \text{and} \quad \xi_2 = C_2 (T_a - T_b). \quad (5.3.19)$$

Figures 18 and 19 show the influence of  $\xi_2$  on the radial and tangential stress components. The variable  $\xi_2$  may be viewed in either of two ways. The first is to consider  $C_2$  fixed, then  $\xi_2$  represents the effect of temperature on the response of some particular material. This corresponds to the vertical scaling suggested by Ferry, which is discussed in Section 2.2A. The second approach is to hold  $(T_a - T_b)$  fixed, then  $\xi_2$  shows the response of a general class of materials which exhibit vertical scaling.

As shown in Figures 18 and 19,  $\xi_2$  has very little influence on the radial stress; but, the influence on the tangential stress is quite pronounced. For a given material ( $C_2$  held fixed) the curves indicate that as  $T_a - T_b$  increases, the magnitude of the stress component



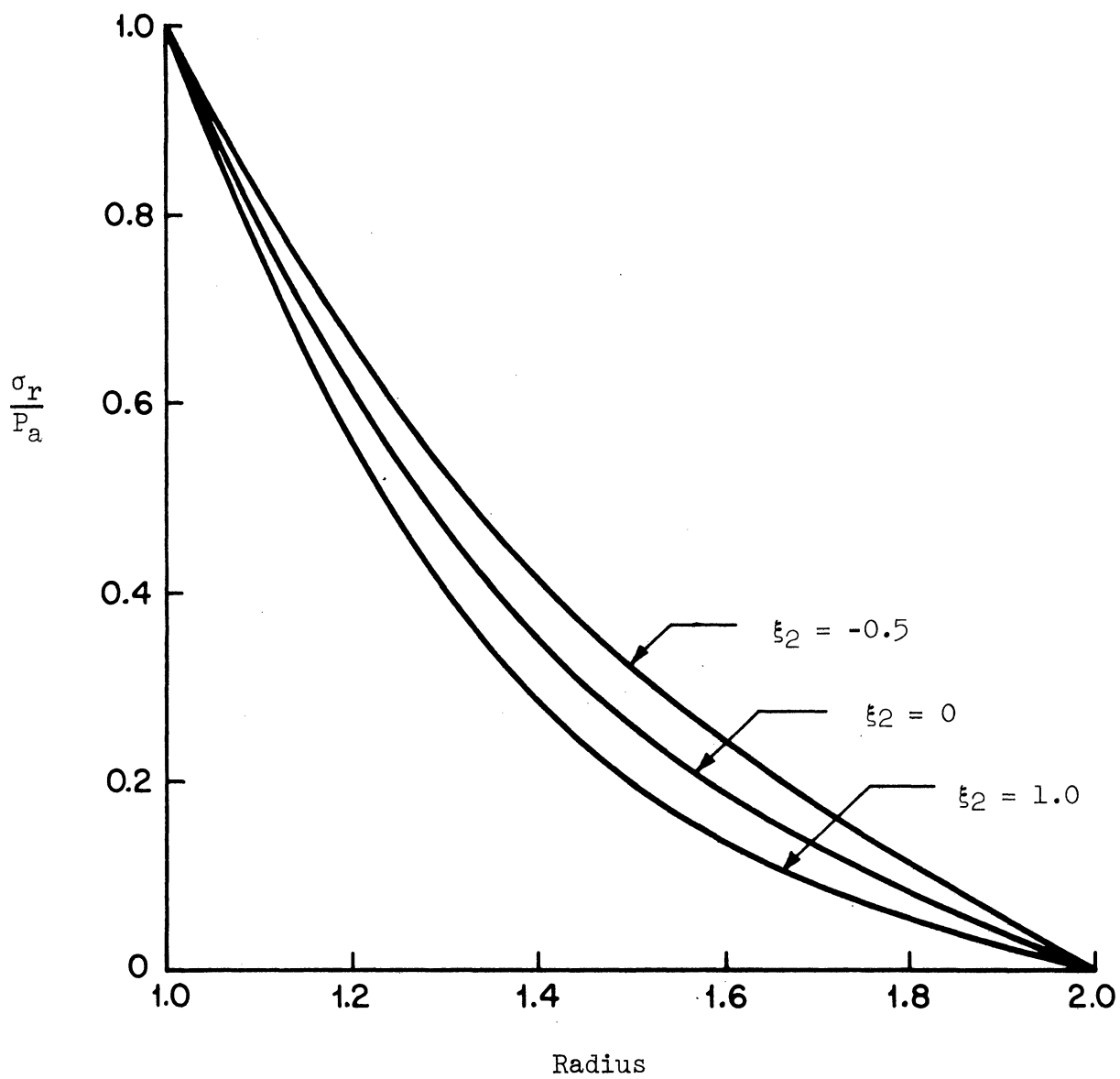


Figure 18. Influence of  $\xi_2$  on the Radial Stress Component  
When  $C_1 = 0$ .

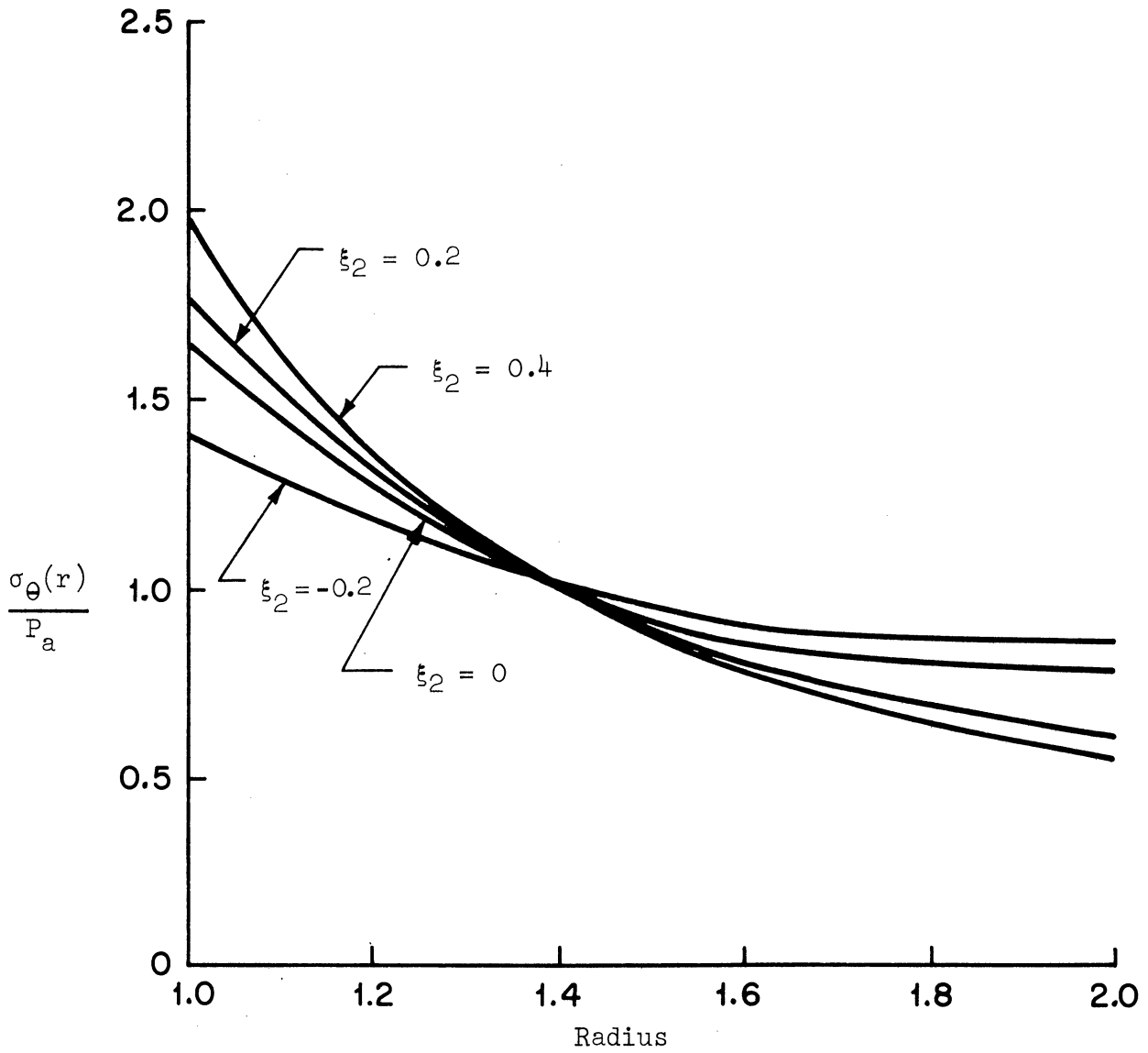


Figure 19. Influence of  $\xi_2$  on the Tangential Stress Component When  $C_1 = 0$ .

increases on the inner boundary and decreases on the outer boundary. If  $T_a - T_b$  is held fixed, then as  $\xi_2$  increases the magnitude of the stress components increase at  $r = a$ . Note that the effect of the vertical scaling causes a significant deviation from the elastic solution which corresponds to  $\xi_2 = 0$ .

Consider next the material response when  $C_1 \neq 0$  and  $C_2 = 0$ . This corresponds to changing the initial and residual response moduli by the same additive constant. Under these circumstances the equations for the stress-state retain the same basic form as given in Equation (5.3.11) and (5.3.12). Thus it is the presence of  $C_1$  that introduces the exponential time term. For both stress components, the time independent behavior, as well as the rate and amount of viscoelastic response, all depend on  $C_1(T_a - T_b)$ .

For the purpose of studying the influence of the temperature term  $C_1(T_a - T_b)$  on the stress histories, again set

$$a=1, \quad b=1, \quad q_0 = q_1 = \frac{1}{2},$$

and, 
$$\xi_1 = C_1(T_a - T_b). \quad (5.3.20)$$

Figures 20 and 21 show the result of introducing Equation (5.3.20) into (5.3.12) and evaluating  $\sigma_\theta(r, t)$  at  $r = a$  and  $r = b$ . As  $\xi_1$  increases the general long time trend is to increase the magnitude of the stress components on the inner boundary and decrease the stress on the outer boundary. The relaxation time does not appear to be altered significantly by changes in  $\xi_1$ .

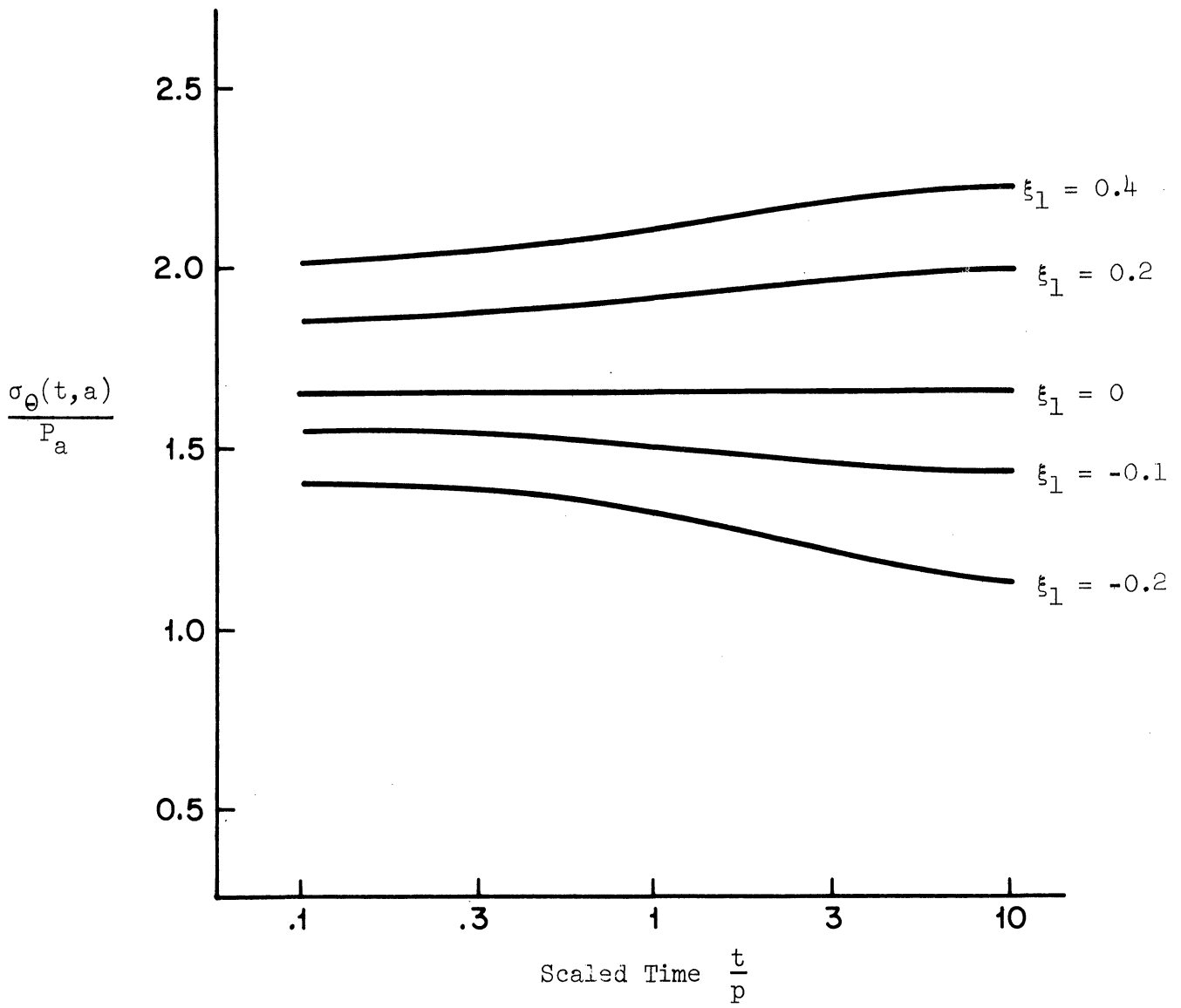


Figure 20. Influence of  $\xi_1$  on the Tangential Stress History at the Inner Boundary  $r = a$  When  $C_2 = 0$ .

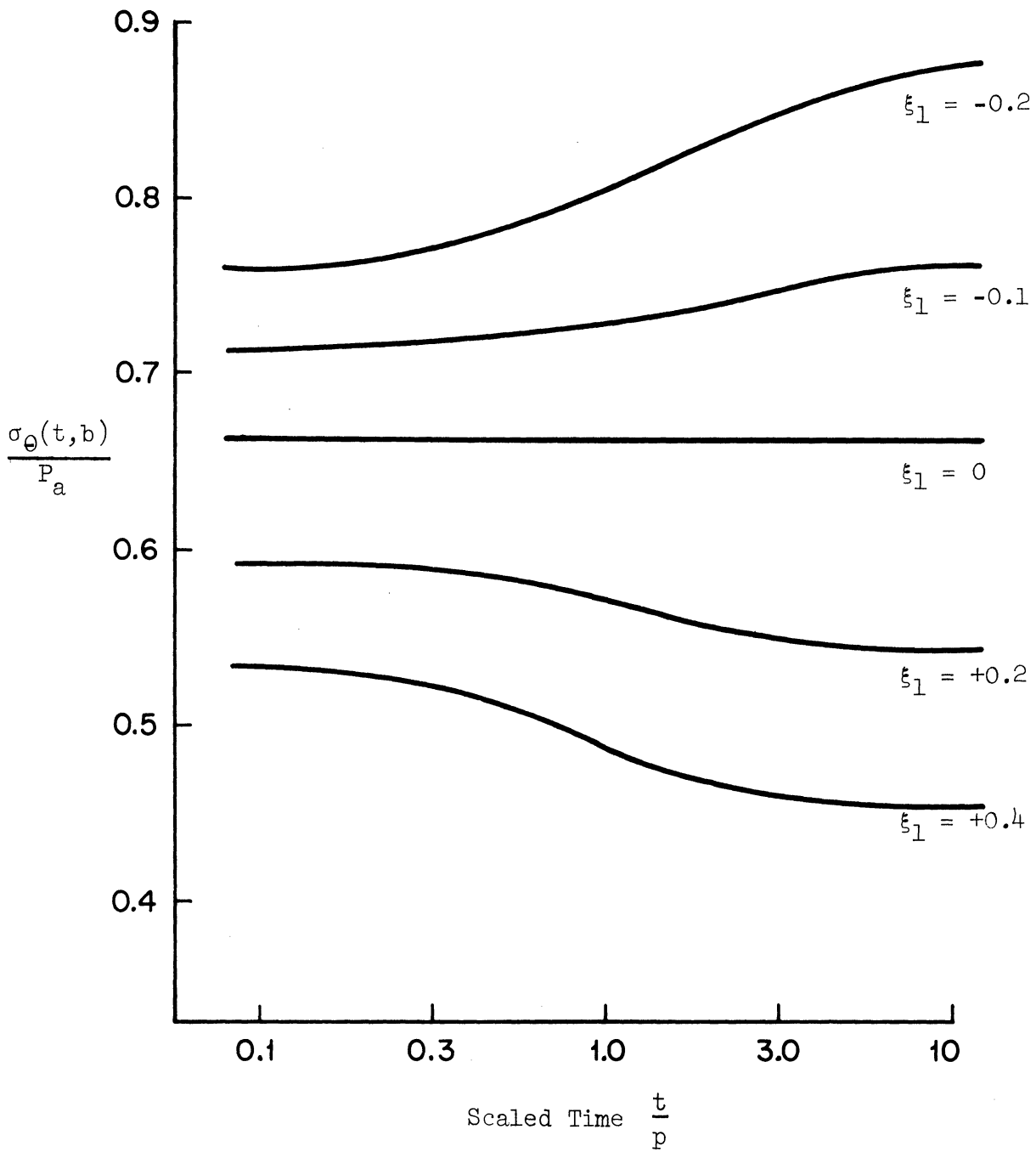


Figure 21. Influence of  $\xi_1$  on the Tangential Stress History at the Outer Boundary  $r = b$  When  $C_2 = 0$ .

#### 5.4 Solution for the Supported Cylinder

Consider now the supported cylindrical shell. Here also an exact solution can be obtained for a temporally constant environmental field. Let  $G(t,r)$  and  $K(t,r)$  be given by Equation (5.3.4) and (5.3.5) respectively. Now the Volterra integral Equation (5.1.29) can be solved for  $f(t)$  since  $K(b,t)$  is known.

Set  $r = b$  in Equation (5.3.8), then the Laplace transform of Equation (5.1.29) is

$$\lambda \overline{f(s)} = \overline{P_a(s)} + \frac{B(b)}{(1+p s)} \overline{f(s)}, \quad (5.4.1)$$

where  $\overline{P_a(s)}$  and  $\overline{f(s)}$  are the transformation of  $P_a(t)$  and  $f(t)$  respectively, and

$$\lambda = \frac{\lambda}{b^2} + A(b) + B(b). \quad (5.4.2)$$

Solving Equation (5.4.1) for  $\overline{f(s)}$  yields

$$\overline{f(s)} = \overline{P_a(s)} \left\{ \frac{1}{\lambda} + \frac{B(b)}{\lambda [\lambda - B(b)] \left[ 1 + \frac{\lambda p}{\lambda - B(b)} s \right]} \right\}. \quad (5.4.3)$$

Inverting Equation (5.4.3) gives

$$f(t) = \frac{P_a(t)}{\lambda} + \frac{B(b)}{\lambda^2 p} \int_0^t P_a(t-\gamma) \exp \left[ - \left( \frac{\lambda - B(b)}{\lambda p} \right) \gamma \right] d\gamma. \quad (5.4.5)$$

For the special case when  $P_a(t) = P_a 1(t)$ , Equation (5.4.5) reduces to

$$f(t) = \frac{P_a}{[\lambda - B(b)]} \left\{ 1 - \frac{B(b)}{\lambda} \exp \left[ - \left( \frac{\lambda - B(b)}{\lambda p} \right) t \right] \right\} \quad (5.4.6)$$

for  $t > 0$ .

The radial and tangential normal stresses can now be determined. Substitute Equation (5.3.4), (5.3.5) and (5.4.6) into (5.1.13) and (5.1.15),

and evaluate to get

$$\sigma_r(r,t) = P_a \left\{ \frac{b^2 A(r)}{\lambda + b^2 A(b)} + \frac{b^2}{\lambda + b^2 A(b)} \left[ B(r) - \frac{b^2 B(b) [A(r) + B(r)]}{\lambda + b^2 [A(b) + B(b)]} \right] \times \right. \\ \left. \exp \left[ - \frac{\lambda + b^2 A(b)}{\lambda + b^2 [A(b) + B(b)]} \frac{t}{p} \right] - 1 \right\}; \quad (5.4.7)$$

$$\sigma_\theta(r,t) = P_a \left\{ \frac{b^2 A'(r)}{\lambda + b^2 A(b)} + \frac{b^2}{\lambda + b^2 A(b)} \left[ B'(r) - \frac{b^2 B(b) [A'(r) + B'(r)]}{\lambda + b^2 [A(b) + B(b)]} \right] \times \right. \\ \left. \exp \left[ - \frac{\lambda + b^2 A(b)}{\lambda + b^2 [A(b) + B(b)]} \frac{t}{p} \right] - 1 \right\}, \quad (5.4.8)$$

where  $A'(r)$  and  $B'(r)$  are given by Equation (5.3.14). From Equation (5.1.28) it can be seen that  $\lambda = 0$  corresponds to stress free boundary condition at  $r = b$ . Therefore, on setting  $\lambda = 0$  in Equation (5.4.7) and (5.4.8);  $\sigma_r$  and  $\sigma_\theta$  reduce to the unsupported cylinder solution.

## APPENDIX

### RIEMANN-STIELTJES CONVOLUTION ALGEBRA

For the convenience of the reader, the definitions and properties of Riemann-Stieltjes convolution algebra are stated here. The results given below are verified by Gurtin and Sternberg in Reference 9.

DEFINITION A1. Let the function  $f$  be in class  $C^N$  if it is defined, continuous and has  $N$  continuous derivatives on  $(a,b)$ .

DEFINITION A2. Define the function  $f$  to be in class  $H^N$  if it is defined on  $(-\infty, \infty)$ , vanishes on  $(-\infty, 0)$ , and is in class  $C^N$  on  $[0, \infty)$ .

DEFINITION A3. Let  $\psi$  be in class  $H^1$  and  $\phi$  in  $H^0$ . This is sufficient to guarantee the existence of the Riemann-Stieltjes integral

$$\Theta(t) = \int_{-\infty}^t \phi(t-\tau) d\psi(\tau) \quad (A1.1)$$

for all  $t$  in  $(-\infty, \infty)$ . The function  $\Theta(t)$  defined by Equation (A1.1) is the Stieltjes convolution of  $\phi$  and  $\psi$ . Let the function  $\Theta(t)$  be denoted by

$$\Theta = \phi * d\psi. \quad (A1.2)$$

THEOREM A1. Let  $\phi$  be in class  $H^0$  and  $\psi$  and  $\omega$  in  $H^1$ . Then

- (a)  $\phi * d\psi$  is in  $H^0$  and  $\omega * d\psi$  is in  $H^1$ ;
- (b)  $\phi * d\psi = \psi * d\phi$ ;



- (c)  $\phi * d(\psi * d\omega) = (\phi * d\psi) * d\omega = \phi * d\psi * d\omega;$
- (d)  $\phi * d(\psi + \omega) = \phi * d\psi + \phi * d\omega;$
- (e)  $\phi * d\psi = 0$  implies  $\phi = 0$  or  $\psi = 0;$
- (f)  $\phi * d\psi = \phi(t)\psi(0) + \int_0^t \phi(t-\gamma)\psi(\gamma), \gamma d\gamma;$
- (g)  $\phi * d1(t) = \phi(t).$

THEOREM A2. Let  $\phi$  be in  $H^1$ . Then there is only one function  $\psi$  in  $H^1$  such that

$$\phi * d\psi = 1(t) \quad (A1.3)$$

for all  $t$  in  $(-\infty, \infty)$ . If  $\psi$  exist it is called the Stieltjes inverse of  $\phi$  and it is denoted by  $\psi = \phi^{-1}$ . Let the functions  $\alpha$  and  $\beta$  be in  $H^0$ , then

$$\alpha * d\phi = \beta \quad \text{implies} \quad \alpha = \beta * d\phi^{-1}. \quad (A1.4)$$

THEOREM A3. If  $\phi$  is in  $H^2$ , then a necessary and sufficient condition that  $\phi^{-1}$  exist is that  $\phi(0) \neq 0$ .

## LIST OF REFERENCES

1. Leaderman, H. Elastic and Creep Properties of Filamentous Materials and Other High Polymers. Washington, D.C.: Textile Foundation, 1943.
2. Schwarzl, F., and Staverman, A.J. "Time-Temperature Dependence of Linear Viscoelastic Behavior," J. Appl. Phys., 23.8 (1952), page 838.
3. Morland, L. W., and Lee, E. H. "Stress Analysis of Linear Viscoelastic Materials with Temperature Variation," Tran. Soc. of Rheol., 4(1960), page 233.
4. Lubliner, J., and Sackman, J. L. "On Aging Viscoelastic Materials," J. Mech. Phys. Solids, 14 (1966), page 25.
5. Sackman, J. L. "Creep of Concrete and Concrete Structures." Proceedings of the Princeton University Conference on Solid Mechanics, Princeton, N. J. (1963).
6. McHenry, D. "A New Aspect of Creep in Concrete and its Application to Design." Proceeding of the Forty-Sixth Annual Meeting of the American Society for Testing Materials, Pittsburgh, Penn., 43 (1943), page 1069.
7. Arutyunyan, N. KH. Some Problems in the Theory of Creep. Translated by H. E. Nowotny. New York, Pergamon Press, 1966.
8. Bresler, B. and Selna, L. "Analysis of Time-Dependent Behavior of Reinforced Concrete Structures," Symposium on Creep of Concrete, American Concrete Institute Special Publication No. 9 (1964).
9. Gurtin, M. E., and Sternberg, Eli. "On the Linear Theory of Viscoelasticity," Arch. Rational Mech. Anal., 11.4 (1962), page 291.
10. Truesdell, C. The Elements of Continuum Mechanics. New York: Springer-Verlag, 1966.
11. Rogers, T. G. and Pipkin, A. C. "Asymmetric Relaxation and Compliance Matrices in Linear Viscoelasticity," ZAMP, 14 (1963), page 334.
12. Apostol, T. M. Mathematical Analysis. Reading Mass.: Addison-Wesley, 1964.
13. Bischoff, J., Catsiff, E., and Tobolsky, A. V. "Elastoviscous Properties of Amorphous Polymers in the Transition Region I." J. Amer. Chem. Soc. 74 (1952) page 3378.
14. Hetenyi, M. "The Fundamentals of Three-Dimensional Photoelasticity," Trans. Amer. Soc. Mech. Engrs., 60 (1938), page A-149.

15. Ferry, J. D. Viscoelastic Properties of Polymers. New York: J. Wiley and Sons, 1961.
16. Nagamatsu, K., Takemure, T., Yoshitomi, T., and Takemoto, T. "Effect of Crystallinity on the Viscoelastic Properties of Polyethylene," Jour. of Poly. Sci., 33 (1958), page 513.
17. Flügge, W. Viscoelasticity. Waltham, Mass.: Blaisdell, 1967.
18. Steinberger, R. L. Creep in Cellulose Acetate Filaments. Textile Research, 6 (1936), page 191.
19. Meredith, R. and B.-S. Hsu. "Stress Relaxation in Nylon and Terry-lene; Influence of Strain, Temperature and Humidity," Jour. of Poly. Sci., 61 (1962), page 262.
20. Theocaris, P. S. "Viscoelastic Behavior of Plasticized Epoxy Polymers in Their Transition Region." Proceedings of the Society of Experimental Stress Analysis, (22.1), Westport, Conn., 1965.
21. Strength and Stiffness, Mechanical Properties. Dow Chemical Company, Form Number 153-5288A-3M-267; Midland, Michigan, 1967.
22. Koshimo, A. "Steam and Heat Setting of Nylon 6 Fiber. VIII. Viscoelastic and Thermal Behavior of Heat-Set or  $\gamma$ -Irradiated Nylon 6 Fiber," Jour. of Applied Polymer Sci., 9 (1965), page 69.
23. Lovitt, W. V. Linear Integral Equations. New York: Dover, 1950.
24. Sternberg, E. "On the Analysis of Thermal Stresses in Viscoelastic Solids," Proceeding of the Third Symposium on Naval Structural Mechanics, New York (1963).
25. Sternberg, E. and Gurtin, M. E. "Further Study of Thermal Stresses in Viscoelastic Materials with Temperature-Dependent Properties." Proceeding I.V.T.A.M., Symposium on Second Order Effects of Elasticity, Plasticity, and Fluid Dynamics; Haifa (1962).
26. Muki, R. and Sternberg, E. "On Transient Thermal Stresses in Viscoelastic Materials with Temperature-Dependent Properties," Journal of Applied Mechanics, 28 (1961), page 193.
27. Smithies, F. Integral Equations. Cambridge University Press; London, 1958.
28. Mikusinski, J. Operational Calculus. Pergamon Press; New York, 1959.
29. König, H., and Meixner, J. "Lineare Systeme und Lineare Transformationen," Mathematische Nachrichten 19 (1958), page 256.

30. Riesz, F., and B. Sz-Nagy. Functional Analysis. Translated by L. Boron. New York, Frederich Ungar Pub., 1955.
31. Yashin, A. V. "Polzuchest Betona V Rannem Vozraste" (Creep of Concrete at an Early Age), Trudy Instituta vyp. 4, Nizhb, Moscow, 1959.
32. Churchill, R. V. Operational Mathematics. McGraw-Hill; New York, 1958.
33. Gurtin, M. E. and Sternberg, E. "A Reciprocal Theorem in the Linear Theory of Anisotropic Viscoelastic Solids," Jour. of the Soc. of Industrial and Applied Mathematics, 11.3 (1963), page 607.
34. Sackman, J. L., and Lubliner, J. "A Reciprocal Theorem of an Aging Viscoelastic Body," Journal of the Franklin Institute, 282.3 (1966) page 161.
35. Lubliner, J., and Sackman, J. L. "On Uniqueness in General Linear Viscoelasticity," Quarterly of Applied Mathematics, 25.2 (1967) page 129.
36. Timoshenko, S. and Goodier, J. N. Theory of Elasticity. McGraw-Hill; New York, 1951.
37. Carslaw, H. S., and Jaeger, J. C. Conductron of Heat in Solids. Oxford University Press; London, 1948.