THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

PULSATILE PRESSURE AND FLOW THROUGH DISTENSIBLE VESSELS

Victor L. Streeter, Sc.D.
W. Ford Keitzer, M.D.
David F. Bohr, M.D.

December, 1962

IP-595

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES CONT'D

I. INTRODUCTION

The problems of characterizing pulsatile patterns of pressure and flow in the arterial system are intriguing and complex. Ejection of blood from the left ventricle initiates non-linear transients in pressure and flow at the root of the aorta. These transients initiate complex pulse patterns that are propagated throughout the arterial tree. Some of the factors that must be reckoned with in an analysis of these patterns are itemized in the list that follows: 1) The force initiating the transients is itself complex; the velocity of ventricular ejection increases rapidly with the opening of the aortic valve, then declines slowly to reach a negative nadir with the closure of the aortic valve. 2) The distensibility of the walls of the arteries receiving this positive increment of pressure and flow has an important influence on pulse patterns. This physical property of the arterial wall is also responsible for changes in configuration and velocity of the transients as they pass over the arterial tree. The pulsatile patterns are further distorted by 3) fractional losses of both positive and negative flow of the blood, and by the 4) branching and tapering architecture of the arterial tree.

5) The resistances to forward motion of blood through the distal arteriolar beds also have their effects on the contours of the arterial pulses observed upstream.

Both experimental and theoretical methods have been used to analyze and to quantify the influence of these factors on the arterial pulse pattern. The experimental approach has profited substantially from new instrumentation which permits a recording of the transients of both pressure and flow with a high degree of fidelity at various levels of the arterial tree[1,2,3,4]. The theoretical approach employs established mathematical relations between known physical parameters which permit the prediction of pressure and flow at specific points in the arterial tree and specific times in the cardiac cycle. If an adequate mathematical expression were available to describe these relations, the fit of these predictions to the actual measured values would then become a valuable tool both for assessing the accuracy of the selected values for the physical parameters and for checking the significance to the various factors that influence the transients in the arterial tree.

The theoretical analysis of these transients then has two requirements: 1) realistic values for these factors that influence the transients, and 2) a mathematical statement of the inter-relationship of these factors which will define pressure and flow with respect to time and position in the arterial tree. Approximations of the required quantitative values for the physical factors involved can be found in the literature. However, one of the more unyielding problems has been the development of a mathematical expression for the inter-relation of these factors in pulsatile flow in a distensible vessel. Owing to the difficulties encountered, greatly simplifying assumptions have been made, such as lumped parameters, laminar flow, linear frictional resistance, and steady oscillatory flow[1,2]. The resulting equations, despite the restrictive assumptions, are very complicated and do not lend themselves to ease of computation when practical boundary conditions are introduced.

The current study presents a new mathematical approach which permits a more realistic solution of specific equations describing these inter-relations. Specific values for pressure and flow with respect to time and position in the arterial tree can be computed. This approach has not previously been applied to a study of the transients in blood vessels. It starts with two established equations dealing with these transients and the physical factors that influence them. These simultaneous equations are: 1) the continuity equation which equates the net influx of blood entering a small segment of the arterial tree with the increase of volume of that segment, and 2) the momentum equation (Newton's second law) which equates the forward force acting on this segment of blood with the backward force plus the force exerted by the arterial wall and the force required to overcome friction in the artery (Figure 3). The inclusion in this equation of the statement for friction makes it a non-linear, partial differential equation which could not be solved by previous methods. The new aspect of the current approach is the application of the method of characteristics which permits the solution of these two simultaneous equations. Specific values can be computed for the unknown functions of pressure and velocity along characteristic lines relating the independent variables of time and distance (Figures 4 and 5). With the aid of a high speed computer these unknowns have been determined for frequent periods of time (1/400'th of a physiological pulse cycle) and for short segments along the arterial tree (1 cm).

The problem of reflections and their interpretation in an analysis of unsteady flow become less significant with this approach. A pressure pulse is transmitted through the vessel at a speed that depends upon the tube properties and the pressure within the tube. As the pressure varies both with distance along the tube and with time, the speed of the pulse wave changes continuously with the independent variables, distance along the tube and time. For any change in speed of the pulse wave, reflections are set up which move in the reverse direction, and the transmitted wave is affected was well. As reflections also occur at boundaries, i.e., entrances, exits, branches and obstructions, previous methods of keeping track of all these reflections became hopelessly complicated.

The new method outlined here automatically takes all these reflections into account, not by keeping track of them separately, but by satisfying all of the basic equations at closely-spaced sections along the vessel at frequent time intervals, and by satisfying the boundary equations. Owing to the relative ease of handling the computations for interior sections and for boundaries, solutions may be programmed simulating branching arteries with satisfactory accuracy.

The theory of characteristics, which applies to the solution of hyperbolic partial differential equations, first gained prominence in solution of supersonic flow problems by Courant and Friedricks.(5) These methods were extended to applications of free-surface flow cases later, by Stoker.(6) More recently they have been applied to water hammer situations by Lai(7), Streeter and Lai(8), and Streeter(9), in which non-linear terms for wall expansion and for wall friction have been retained in the equations. This paper presents an extension of the theory to flexible vessels, tapering vessels, and vessels with distributed outflow along their length, together with in vivo experimental confirmation of the method.

II. THEORETICAL METHODS AND RESULTS

In this section the mathematical and physical relationships for flow through flexible tubes are derived, including equations for tapering vessels with distributed outflow.

A. Derivation of Basic Equations

The assumptions required for analytical handling of pulsatile flow are first discussed, followed by development of elasticity relations and the continuity and momentum equations for vessels of constant initial diameter. From these relationships the characteristic equations are obtained and finite difference methods applied to develop the equations for the method of specified time intervals. After discussion of boundary conditions, an example is presented.

1. Basic Assumptions

The basic assumptions required in developing the working equations are:

a. One-dimensional flow; the velocity at a cross section is given by the average velocity at the section at a given instant.

b. The vessel walls are elastic, with a Poisson ratio of 0.5, and they are tethered; hence the volume of elastic vessel wall per unit length remains constant.

c. The fluid density is constant. Compressibility of blood is small compared with expansion of the vessel walls under increased pressure, and

d. Pressure losses due to wall friction may be expressed as proportional to some power of the velocity at a cross section.

2. Elasticity Relationships

Although arteries have viscoelastic properties, their effect seems to be minor and previous investigators[10] have concluded that the stress-strain curve may be linearized without introducing appreciable error.

If D is the inside diameter of the vessel, and t' the effective wall thickness, then as a consequence of the constant volume of elastic wall material resulting from the assumption of Poisson's ratio of 0.50,

$$Dt' = D_o t_o' \qquad (1)$$

D_o is the unstressed diameter and t_o' is the corresponding wall thickness. If the tube is subjected to an increment of pressure dP internally, the tensile force dT resisting this pressure increment per unit length of vessel, Figure 1, is

$$dT = \frac{d(PD)}{2}$$

Since the diameter change, on a percentage basis, is much less than the pressure change, the term dD/2 is neglected in expanding the right hand side, leaving

$$dT = \frac{D}{2}\, dP \qquad (2)$$

By dividing through by the wall thickness the change in tensile stress dS (force per unit area) is obtained

$$dS = \frac{dT}{t'} = \frac{DdP}{2t'}$$

Now, by dividing through by the elastic modulus of the vessel wall, Y, the unit strain is obtained, which is the change in length per unit length caused by dP. Since circumference changes are proportional to diameter changes, the unit strain is dD/D,

$$\frac{dD}{D} = \frac{D}{2t'} \frac{dP}{Y} \tag{3}$$

After use of Equation (1) to eliminate t', and after separating variables

$$dP = 2Yt'_o D_o \frac{dD}{D^3} \tag{4}$$

Integrating

$$P = Yt'_o D_o \left(\frac{1}{D_o^2} - \frac{1}{D^2}\right) \tag{5}$$

in which the condition has been used that $D = D_o$ when $P = 0$. By multiplying and dividing the right-hand side by $\pi/4$ to introduce the vessel cross sectional area A, and correspondingly A_o,

$$\frac{A}{A_o} = \frac{1}{1 - \frac{P D_o}{t'_o Y}} \tag{6}$$

The pressure P has been expressed as force per unit area (dynes/cm^2). It is customary to express it in terms of the height of a liquid column (the fluid flowing). These are related by $P = \rho g H$ in which ρ is the mass density (gm/cc), g is gravity (980 cm/sec^2), H is the pressure in height of fluid flowing (cm). Two additional substitutions are made, let

$$a_o^2 = \frac{t'_o Y}{\rho D_o} \qquad a^2 = \frac{t'Y}{\rho D} \tag{7}$$

Then

$$\frac{A}{A_o} = \frac{D^2}{D_o^2} = \frac{1}{1 - gH/a_o^2} \tag{8}$$

By use of Equations (1) and (7)

$$\frac{a_o^2}{a^2} = \frac{t_o'}{t'} \frac{D}{D_o} = \frac{D^2}{D_o^2} = \frac{A}{A_o} \tag{9}$$

After equating expressions (8) and (9)

$$a^2 = a_o^2 - gH \tag{10}$$

and from Equation (9)

$$D = \frac{D_o\, a_o}{a} \tag{11}$$

a is the speed of the pressure pulse wave through the vessel. It changes both with time t and with distance x along the vessel, and whenever a changes reflections are produced.

The partial derivatives of A with respect to the two independent variables, time t and distance x are needed later and result from Equations (8) to (11) (a variable subscript x or t represents partial differentiation with respect to that variable, i.e., $A_x = \partial A/\partial x$.

$$\frac{A}{A_o} = \frac{a_o^2}{a^2}$$

$$\frac{A_x}{A_o} = -\frac{2\,a_o^2}{a^3}\, a_x\,, \qquad \frac{A_t}{A_o} = -\frac{2\,a_o^2}{a^3}\, a_t$$

In these equations A_o is considered to be constant.

Then, from Equation (10)

$$2a \cdot a_x = -gH_x \qquad 2a \cdot a_t = -gH_t$$

Now, by eliminating a_x, a_t, and A_o from the last three sets of equations

$$\frac{A_x}{A} = \frac{gH_x}{a^2} \qquad \frac{A_t}{A} = \frac{gH_t}{a^2} \tag{12}$$

Also, from the relation $P = \rho gH$

$$P_x = \rho gH_x \tag{13}$$

These elastic relationships are used in the continuity and momentum equations that are now developed.

3. Continuity Equation

The continuity equation is a material balance for a small segment of vessel, which states that the net mass inflow per unit time is just equal to the time rate of increase of mass within the segment, Figure 2.

If V is the average velocity at the entrance to the element, the rate of mass inflow is ρAV, and the rate of mass outflow is $\rho AV + (\rho AV)_x dx$. The net mass inflow per unit time is then $-(\rho AV)_x dx$ and must just equal the time rate of increase of mass within the segment $(\rho Adx)_t$. Equating these expressions

$$-(\rho AV)_x dx = (\rho Adx)_t \tag{14}$$

ρ is constant for all practical purposes when considering flow in a distensible vessel. A and V are dependent variables. After expanding Equation (14), remembering that x is independent of t, and then dividing through by the mass of the fluid segment, ρ Adx,

$$\frac{VA_x}{A} + V_x + \frac{A_t}{A} = 0 \tag{15}$$

which is the continuity equation and must hold throughout the vessel.

4. Momentum Equation

The momentum equation when applied in the x-direction to the fluid in a segmental volume, Figure 3, is a statement that the resultant x-component of force on the segment of fluid is just equal to the net efflux of x-momentum plus the time rate of increase of x-momentum within the segment. The resultant force component, Figure 3, is

$$F = PA + (P + P_x \frac{dx}{2})\ A_x dx - [PA + (PA)_x dx] - \tau_o \pi D dx$$

The first term is due to pressure within the fluid acting over the cross section at x, the second term is the force component in the x-direction due to the tube wall pushing against the fluid (zero for constant A, as $A_x dx$ is the increase in cross section in the length dx). The term in brackets is the force pushing against the element on the distal side. The action of fluid friction at the tube wall is given by the product of shear stress τ_o at the wall and area of wall surface πDdx. This force is assumed to act wholly in the x-direction. By expanding the expression for F it becomes

$$F = -P_x A dx - \tau_o \pi D dx + P_x A_x \frac{(dx)^2}{2}$$

Since the term with square of dx becomes of a higher order of smallness as dx approaches zero, it may be dropped from the equation.

The momentum influx at x is ρAV^2 and the momentum efflux at x + dx is $\rho AV^2 + (\rho AV^2)_x dx$, with a net efflux of $(\rho AV^2)_x dx$. The time rate of increase of momentum within the segment is given by $(\rho AdxV)_t$. After combining the force and momentum terms,

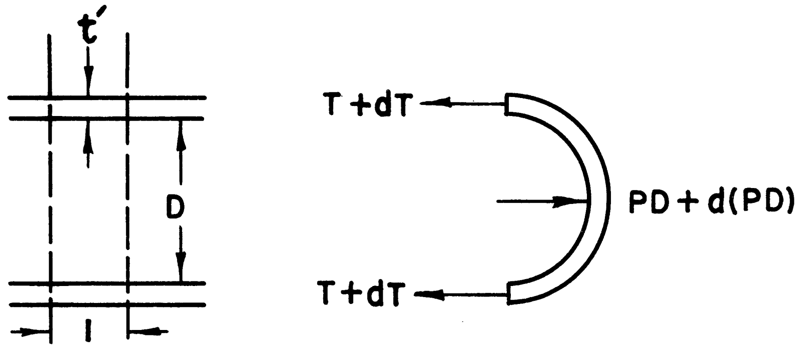

Figure 1. Relation Between Tensile Force Change in Wall dT and Pressure Change dP.

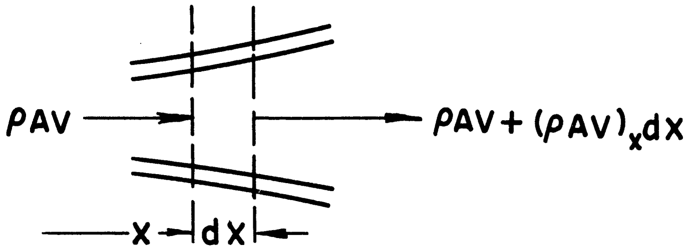

Figure 2. Material Balance Showing Mass Per Unit Time Entering and Leaving an Elemental Volume.

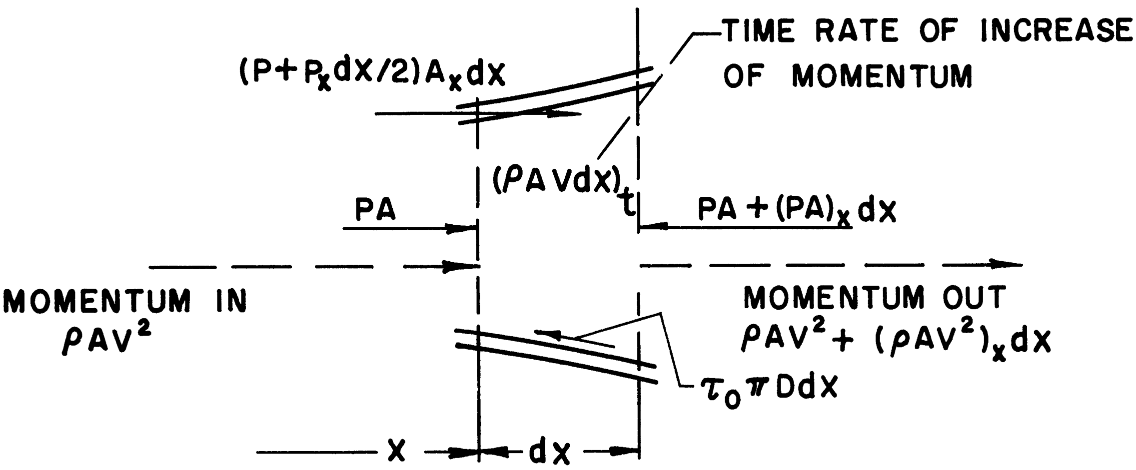

Figure 3. Force Acting on Segment of Fluid (Solid Lines), and Momentum Relationships (Dotted Lines).

$$- P_x A dx - \tau_o \pi D dx = (\rho A V^2)_x dx + (\rho A dx V)_t$$

By expanding the partial derivatives, then dividing through by the mass of the segment $\rho A dx$, and after replacing P_x by $\rho g H_x$,

$$gH_x + \frac{\tau_o \pi D}{\rho A} + \frac{A_x}{A} V^2 + 2V\, V_x + V_t + V \frac{A_t}{A} = 0 \qquad (16)$$

The wall shear stress τ_o may be written in the form

$$\tau_o = k \frac{\rho V^2}{2} \qquad (17)$$

For established, steady laminar flow $k = 16/R$, with R the Reynolds number $VD\rho/\mu$, in which μ is the fluid viscosity (Poise). For turbulent flow $k = f/4$, with f the commonly used Darcy-Weisbach friction factor.[11] By inserting Equation (17) into Equation (16), using f,

$$gH_x + \frac{fV^2}{2D} + \frac{A_x}{A} V^2 + 2V\, V_x + V_t + V \frac{A_t}{A} = 0 \qquad (18)$$

This is the momentum equation for flow through a distensible vessel. By multiplying Equation (15) by V and subtracting it from Equation (18), substantial simplification results*

$$gH_x + V\, V_x + V_t + \frac{fV^2}{2D} = 0 \qquad (19)$$

Equations (15) and (19) contain the continuity and momentum principles. After substituting the elastic relationships given by Equations (12) into Equation (15) it becomes, upon simplification,

$$VH_x + H_t + \frac{a^2}{g} V_x = 0 \qquad (20)$$

*By writing the V^2 term in the friction expression as $V|V|$, it will reverse sign if the flow reverses, and hence act in the proper direction at all times.

Equations (19) and (20) are used to develope the final equations.

5. Development of Characteristic Equations

By calling Equation (19) L_1 and Equation (20) L_2, they may be combined linearly using an unknown multiplier λ, as follows:

$$L = L_1 + \lambda L_2 = \lambda[H_x(\frac{g}{\lambda} + V) + H_t] + [V_x(V + \frac{\lambda a^2}{g}) + V_t] + \frac{fV^2}{2D} = 0 \quad (21)$$

If two distinct values of λ are taken, two equations result which contain the momentum and continuity principles. The theory of characteristics determines two special values of λ which result in great simplification of the equations. To review some fundamental relations in calculus, if $H = H(x,t)$ and $V = V(x,t)$, then the total derivatives of H and V with respect to t are

$$\frac{dH}{dt} = H_x \frac{dx}{dt} + H_t \qquad \frac{dV}{dt} = V_x \frac{dx}{dt} + V_t$$

where in these relationships H and V are pressure and velocity of a particle as it moves (x becomes a function of t). By examining Equation (21) it is seen that the first bracket contains dH/dt if

$$\frac{g}{\lambda} + V = \frac{dx}{dt} \quad (22)$$

and the second bracket contains dV/dt if

$$V + \frac{\lambda a^2}{g} = \frac{dx}{dt} \quad (23)$$

These expressions must be the same. By equating them and solving for λ

$$\frac{g}{\lambda} + V = V + \frac{\lambda a^2}{g}$$

and

$$\lambda = \pm \frac{g}{a} \tag{25}$$

Now, by restricting the applicability of Equation (21) to those characteristic lines for which Equations (22) and (23) are satisfied, it may be written in the simple form

$$L = \lambda \frac{dH}{dt} + \frac{dV}{dt} + \frac{fV^2}{2D} = 0 \tag{25}$$

By applying $\lambda = +g/a$ to Equations (22) and (25)

$$\left.\begin{aligned} &\frac{g}{a}\frac{dH}{dt} + \frac{dV}{dt} + \frac{fV^2}{2D} = 0 && (26)\\ &\frac{dx}{dt} = V + a && (27)\end{aligned}\right\} C_+$$

and by applying $\lambda = -g/a$ to the same equations

$$\left.\begin{aligned} &\frac{-g}{a}\frac{dH}{dt} + \frac{dV}{dt} + \frac{fV^2}{2D} = 0 && (28)\\ &\frac{dx}{dt} = V - a && (29)\end{aligned}\right\} C_-$$

Equation (26), a total differential equation, is valid only along a line defined by Equation (27), if a plot is made on an x - t plane, as shown in Figure 4. If C_+ in the figure represents the characteristic line defined by $dx/dt = V + a$ and passing through a point R where V, H, x and t are known then Equation (26) may be used as one relation between pressure H and velocity V along this line. Similarly Equation (28) is valid only along a line defined by Equation (29), Figure 4. Now with V and H known at the known points R and S, four equations are available at the intersection P of the two characteristic curves, for computing V, H, x and t.

6. The Method of Specified Time Intervals

For computation purposes, Equations (26) to (29) are written as finite difference equations. For use with a high-speed digital computer, the theory of characteristics method may be extended to a system in which the time and distance intervals are preselected.(12) This is called the method of specified time intervals, and entails interpolation of values of V and H at unknown points R and S, Figure 5, such that P occurs at equally spaced distances Δx along the vessel for equal time increments. In Figure 5 consider that V and H have been computed for the first two rows at the equally spaced sections. Values of V and H are to be computed next for point P at time t_P, which is $t_C + \Delta t$, and values of V and H are known at A, B, and C.

Equations (26) to (29) are written as finite difference equations, for points R and P on C_+ and for points S and P on C_-, with $\Delta t = t_P - t_R = t_P - t_S$. The mesh $(\Delta x, \Delta t)$ is assumed to be so fine that the velocity in the friction term may be evaluated at known conditions at C. Also a_R and a_S are replaced by a_C, as well as D_R and D_S by D_C.

$$\frac{g}{a_C}(H_P - H_R) + V_P - V_R + \frac{f_C V_C \, |V_C| \, \Delta t}{2D_C} = 0 \tag{30}$$

$$x_P - x_R = (V + a)_C \, \Delta t \tag{31}$$

$$\frac{-g}{a_C}(H_P - H_S) + V_P - V_S + \frac{f_C V_C \, |V_C| \, \Delta t}{2D_C} = 0 \tag{32}$$

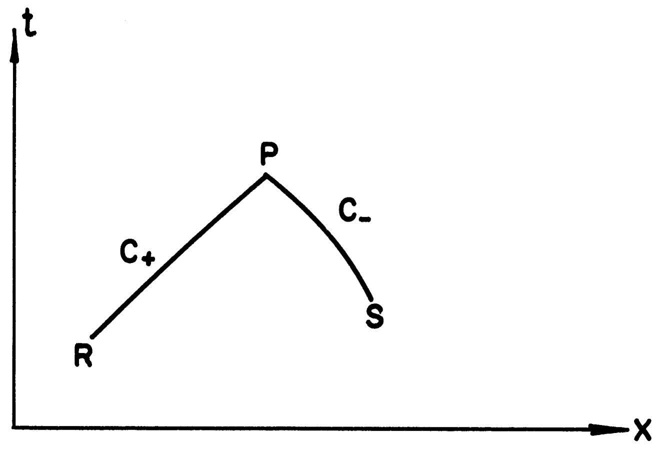

Figure 4. Characteristic Lines C_+ and C_- Drawn Through Points R and S Respectively, Where V and H are Known. Their Intersection P is a Point Where x, t, V, and H May Be Determined From the Equations.

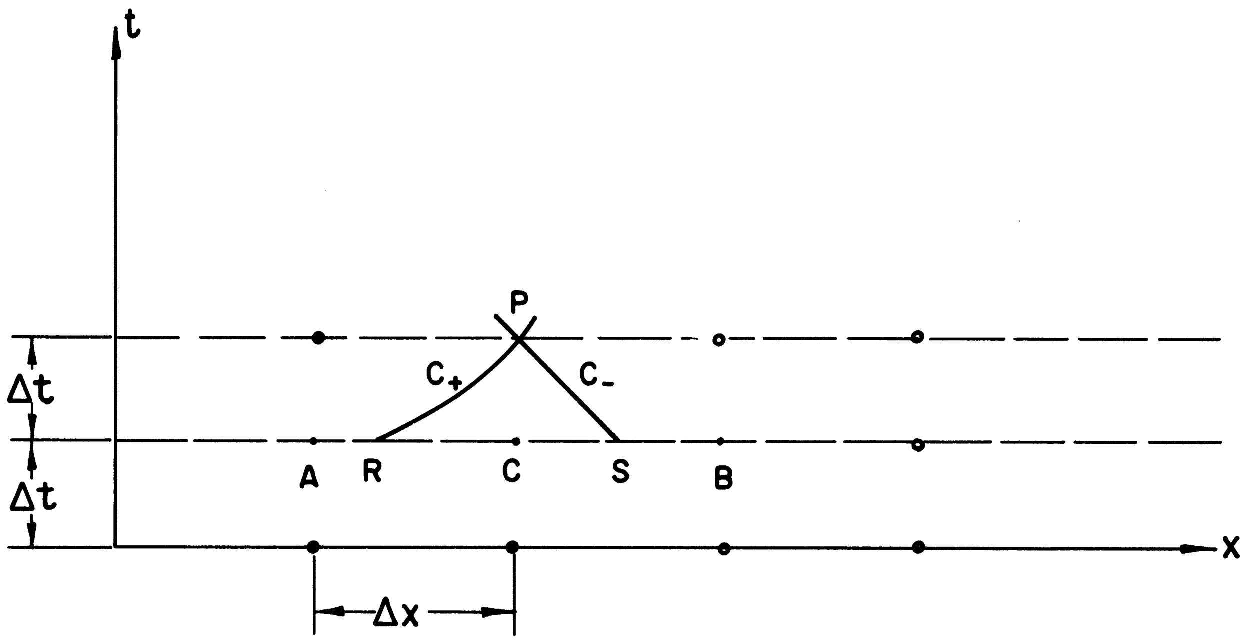

Figure 5. By Specifying Δt and Δx, Which Locates Point P, Values of V and H at R and S May Be Found by Interpolation From Values at A, C, and B.

$$x_P - x_S = (V - a)_C \Delta t \tag{33}$$

The V^2 friction term has been replaced by $V|V|$.

V_R, H_R, V_S, and H_S are now to be computed from the equations, with reference to Figure 5, by linear interpolation. The mesh (values of Δx, Δt) is assumed to be fine enough so that the slopes of the characteristics lines are given adequately by evaluating $V + a$ and $V - a$ at C. From Figure 5

$$\frac{V_R - V_A}{V_C - V_A} = \frac{x_R - x_A}{x_C - x_A} = \frac{x_R - x_A}{\Delta x} \tag{34}$$

By remembering that $x_P = x_C$, from Equation (31)

$$x_C - x_R = (V + a)_C \Delta t$$

Then

$$x_R - x_A = \Delta x - (x_C - x_R) = \Delta x - (V + a)_C \Delta t \tag{35}$$

For convenience the grid mesh ratio $\Delta t/\Delta x$ is called Θ. Substituting into Equation (35), the first of the following four equations is obtained,

$$V_R = V_A + (V_C - V_A)(1 - \Theta(V + a)_C) \tag{36}$$

$$H_R = H_A + (H_C - H_A)(1 - \Theta(V + a)_C) \tag{37}$$

$$V_S = V_B + (V_C - V_B)(1 + \Theta(V - a)_C) \tag{38}$$

$$H_S = H_B + (H_C - H_B)(1 + \Theta(V - a)_C) \tag{39}$$

The last three equations are found in a manner similar to that used in obtaining Equation (36). In

$$\theta = \frac{\Delta t}{\Delta x} \tag{40}$$

Δt must be selected so that R and S lie within the reach defined by points A and B.

With the interpolated values of V_R, H_R, V_S, H_S known, Equations (30) and (32) may now be solved for H_P and V_P,

$$H_P = \frac{H_R + H_S}{2} + \frac{a_C}{2a_C}(V_R - V_S) \tag{41}$$

$$V_P = \frac{V_R + V_S}{2} + \frac{g}{2a_C}(H_R - H_S) - \frac{f_C V_C |V_C| \Delta t}{2D_C} \tag{42}$$

Equations (36) through (41) permit all interior points of the grid to be computed, i.e., all points where corresponding points, A, B, and C are known. At the end points an additional condition is needed to solve for V_P and H_P, since only one of the Equations (30) and (32) is available at a given boundary. This additional condition is called the boundary condition.

7. Boundary Conditions

Boundary conditions may take on many forms. Some examples are given here to illustrate procedures in developing them. At the proximal end of the vessel Equation (32) applies, Figure 6.

$$V_P = V_S + \frac{g}{a_C}(H_P - H_S) - \frac{f_C V_C |V_C| \Delta t}{2D_C} \tag{43}$$

in which V_P and H_P are the unknowns, as V_S and H_S are given by the interpolation Equations (38) and (39). If a known pulse inflow Q_P into the vessel is known at this instant, then a second equation becomes available, as follows:

$$Q_P = V_P \frac{\pi}{4} D_P^2 \tag{44}$$

But from Equations (10) and (11)

$$D_P^2 = \frac{D_o^2 a_o^2}{a^2} = \frac{D_o^2 a_o^2}{a_o^2 - gH_P} \tag{45}$$

By substituting Equation (45) into Equation (44) to eliminate D_P, and after solving for V_P,

$$V_P = \frac{4Q_P}{\pi D_o^2 a_o^2} (a_o^2 - gH_P) \tag{46}$$

which is the second relationship required. With the pulse inflow given for each time increment, V_P and H_P at the proximal end may be computed progressively as the rest of the solution proceeds. Another example is to have the pressure pulse specified as a function of time at $x = 0$. In this case the known H_P is inserted into Equation (43) and V_P may be computed.

At the distal end, Figure 7, the outflow may be expressed in terms of the pressure difference between the vessel H_P and the terminal bed H_B as

$$Q_P = k_1 (H_P - H_B)^m = V_P \frac{\pi}{4} D_P^2 \tag{47}$$

Equation (30), solved for V_P is

$$V_P = V_R - \frac{g}{a_C} (H_P - H_R) - \frac{f_C V_C |V_C| \Delta t}{2D_C} \tag{48}$$

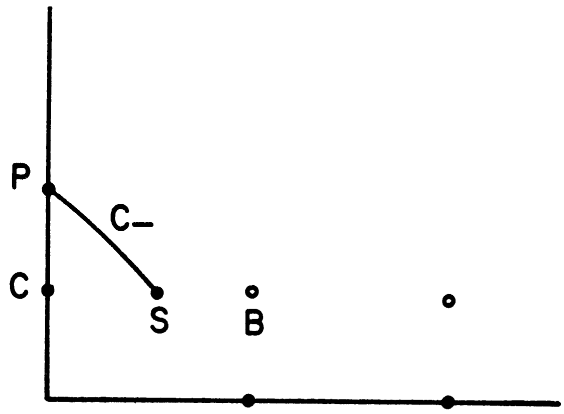

Figure 6. Proximal Boundary Relations. One Relation Between V and H at P is Obtained from Values of V and H at S.

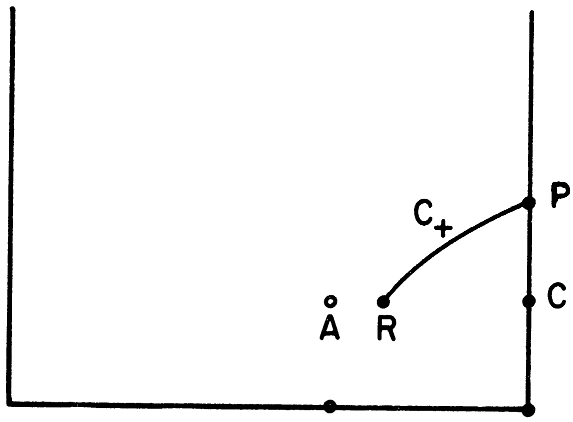

Figure 7. Distal Boundary Relations.

By use of Equations (10) and (11), Equation (47) may be solved for V_P in terms of H_P

$$V_P = \frac{4k_1}{\pi D_o^2 a_o^2} (H_P - H_B)^m (a_o^2 - gH_P) \qquad (49)$$

A trial solution may be required, depending upon the value of exponent m in the terminal bed resistance relationship. Care must be taken in applying a terminal bed Q - H relation. Within the vessel proximal to the bed, there is no simple Q - H relation, primarily due to the complex reflections. If a segment of an arterial system is to be analyzed, the boundary conditions should generally be expressed as H versus t, Q versus t, or V_P versus t.

For known pressure as a function of time, V_P is found directly from Equation (48).

8. Example

To illustrate the application of the characteristics theory to a flow situation, a known pulse flow is injected into a distensible vessel, with a linear terminal bed at the distal end. The computer program, Figure 8, is in the MAD (Michigan Algorithmic Decoder) language and an IBM 709 is used in performing the calculations. The pulse flow from the heart, measured at the ascending aorta of a dog using the Square Wave electrio magnetic flowmeter** has been expressed by a series of empirical formulas. The average flow is 35.65 cc/sec over the 0.4 sec. pulse cycle, with a maximum inflow of 160 cc/sec. A friction factor

**Carolina Medical Electronics, Inc., Winston-Salem, North Carolina.

```
                   PULSATILE FLOW THROUGH A DISTENSIBLE TUBE
         DIMENSION V(20),VP(20),H(20),HP(20),D(20),Q(20),AA(20)             *001
         INTEGER I,U,N,P,KK                                                 *002
         PRINT COMMENT$1                   GIVEN DATA FOR PROBLEM$          *003
 A1      READ DATA                                                          *004
         PRINT RESULTS Y,THICK,DO,L,HB,HO,QAVE,RHO,G,SG,F,N,DELT,PTIME      *005
        2,P,KK                                                              *005

         PRELIMINARY CONSIDERATIONS

         TM=.12*PTIME                                                       *006
         PMT=.25*PTIME                                                      *007
         PPT=.32*PTIME                                                      *008
         PTT=.34*PTIME                                                      *009
         QM=160.                                                            *010
         AO=SQRT.(Y*THICK/(RHO*DO))                                         *011
         VO=.7854*DO*DO*AO*AO                                               *012
         TH=DELT*N/L                                                        *013
         A=SQRT.(AO*AO-G*SG*HO)                                             *014
         D=DO*AO/A                                                          *015
         VP=QAVE/(.7854*D*D)                                                *016
         DHF=F*L*VP*VP/(N*D*2.*G*SG)                                        *017
         THROUGH A2,FOR I=0,1,I.G.N                                         *018
         V(I)=VP                                                            *019
         H(I)=HO-DHF*I                                                      *020
         D(I)=D                                                             *021
         Q(I)=QAVE                                                          *022
 A2      AA(I)=SQRT.(AO*AO-G*SG*H(I))                                       *023
         K=QAVE/(HO-HB)                                                     *024
         T=-5.*DELT                                                         *025
         U=-5                                                               *026
         C1=K/(.7854*DO*DO*AO*AO)                                           *027
         C2=C1*(AO*AO+G*SG*HB)                                              *028
         C3=C1*G*SG                                                         *029
         C4=C1*HB*AO*AO                                                     *030
         ACN=SQRT.(AO*AO-G*SG*H(N))                                         *031
         PRINT COMMENT$0                   HEADS(CM HG),    VELOCIIIES(CM/  *032
        2SEC),    DIAMETERS(CM),    AND FLOW(CC/SEC)$                       *032
         PRINT COMMENT$0     TIME     FLOW    X/L=     .0       .1       .2 *033
        2    .3       .4       .5       .6       .7       .8       .9    1. *033
        3$                                                                  *033
 A3      PRINT FORMAT B1,T,Q,H(0),H(2),H(4),H(6),H(8),H(10),H(12),H(14      *034
        2),H(16),H(18),H(20)                                                *034
         PRINT FORMAT B2,V(0),V(2),V(4),V(6),V(8),V(10),V(12),V(14),V(      *035
        216),V(18),V(20)                                                    *035
         PRINT FORMAT B3,D(0),D(2),D(4),D(6),D(8),D(10),D(12),D(14),D(      *036
        216),D(18),D(20)                                                    *036
         PRINT FORMAT B4,Q(0),Q(2),Q(4),Q(6),Q(8),Q(10),Q(12),Q(14),Q(      *037
        216),Q(18),Q(20)                                                    *037
         VECTOR VALUES B1=$1HO,F8.3,F9.2,S2,3H H=,11F8.3*$                  *038
         VECTOR VALUES B2=$1H ,S20,2HV=,11F8.3*$                            *039
         VECTOR VALUES B3=$1H ,S20,2HD=,11F8.3*$                            *040
         VECTOR VALUES B4=$1H ,S20,2HQ=,11F8.3*$                            *041
 A4      T=T+DELT                                                           *042
         U=U+1                                                              *043
         WHENEVER U.G.P,TRANSFER TO A1                                      *044

         CALCULATION OF INTERIOR POINTS

         THROUGH A5,FOR I=1,1,I.E.N                                         *045
         HR=H(I-1)+(H(I)-H(I-1))*(1.-TH*(V(I)+AA(I)))                       *046
         VR=V(I-1)+(V(I)-V(I-1))*(1.-TH*(V(I)+AA(I)))                       *047
         VS=V(I+1)+(V(I)-V(I+1))*(1.+TH*(V(I)-AA(I)))                       *048
         HS=H(I+1)+(H(I)-H(I+1))*(1.+TH*(V(I)-AA(I)))                       *049
         HP(I)=(HR+HS)/2.+AA(I)*(VR-VS)/(2.*G*SG)                           *050
         VP(I)=(VR+VS)/2.+G*SG*(HR-HS)/(2.*AA(I))-F*V(I)*.ABS.V(I)*DEL      *051
        2T/(2.*D(I))                                                        *051
         AA(I)=SQRT.(AO*AO-G*SG*HP(I))                                      *052
         D(I)=DO*AO/AA(I)                                                   *053
 A5      Q(I)=.7854*D(I)*D(I)*VP(I)                                         *054

         CALCULATION OF PROXIMAL BOUNDARY CONDITION
         WHENEVER T.L.O.,Q=QAVE                                             *055
         WHENEVER T.GE.O..AND.T.LE.PMT,Q=QM*T/(TM*EXP.(T/TM-1.))            *056
         WHENEVER T.G.PMT.AND.T.LE.PPT,Q=QM*T/(TM*EXP.(T/TM-1.))-.56*Q      *057
        2M*(T-PMT)/(PPT-PMT)                                                *057
         WHENEVER T.G.PPT.AND.T.LE.PTT,Q=QM*(T-PTT)/PPT                     *058
         WHENEVER T.G.PTT.AND.T.LE.PTIME,Q=0.                               *059
         WHENEVER .ABS.(T-PTIME).L..001,T=0.                                *060
         AC=SQRT.(AO*AO-G*SG*H)                                             *061
         VS=V(1)+(V-V(1))*(1.+TH*(V-AC))                                    *062
         HS=H(1)+(H-H(1))*(1.+TH*(V-AC))                                    *063
         HP=(Q/(.7854*DO*DO)-VS+G*SG*HS/AC+F*V*.ABS.V*DELT/(2.*D))/(G*      *064
        2SG*(1./AC+Q/VO))                                                   *064
         A=AO*AO-G*SG*HP                                                    *065
         VP=Q*A/VO                                                          *066
         D=DO*AO/SQRT.(A)                                                   *067

         DISTAL BOUNDARY CONDITION

         VR=V(N-1)+(V(N)-V(N-1))*(1.-TH*(V(N)+ACN))                         *068
         HR=H(N-1)+(H(N)-H(N-1))*(1.-TH*(V(N)+ACN))                         *069
         VEE=VR+G*SG*HR/ACN-F*V(N)*.ABS.V(N)*DELT/(2.*D(N))                 *070
         C5=G*SG/ACN                                                        *071
         C6=(C5+C2)/C3                                                      *072
         C7=(VEE+C4)/C3                                                     *073
         HP(N)=C6/2.-SQRT.(C6*C6/4.-C7)                                     *074
         VP(N)=VEE-C5*HP(N)                                                 *075
         ACN=SQRT.(AO*AO-G*SG*HP(N))                                        *076
         D(N)=DO*AO/ACN                                                     *077
         Q(N)=VP(N)*.7854*D(N)*D(N)                                         *078
         THROUGH A6,FOR I=0,1,I.G.N                                         *079
         V(I)=VP(I)                                                         *080
 A6      H(I)=HP(I)                                                         *081
         WHENEVER U/KK*KK.E.U,TRANSFER TO A3                                *082
         TRANSFER TO A4                                                     *083
         END OF PROGRAM                                                     *084
```

Figure 8 (Cont'd)

GIVEN DATA FOR PROBLEM

Y =	5.000000E 06,	THICK =	.100000,	DO =	1.300000,	L =	50.00000C
HB =	10.000000,	HO =	14.000000,	QAVE =	35.650000,	RHO =	1.00000C
G =	980.000000,	SG =	13.600000,	F =	.300000,	N =	2C
DELT =	4.000000E-03,	PTIME =	.400000,	P =	300,	KK =	2

HEADS(CM HG), VELOCITIES(CM/SEC), DIAMETERS(CM), AND FLOW(CC/SEC)

TIME	FLOW	X/L=	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
.000	.00	H=	12.945	12.959	13.009	13.112	13.289	13.543	13.846	14.138	14.364	14.496	14.536
		V=	.000	2.891	5.573	7.863	9.600	10.716	11.390	12.038	12.946	14.033	15.116
		D=	1.751	1.751	1.754	1.760	1.770	1.785	1.802	1.820	1.834	1.843	1.845
		Q=	.000	6.964	13.469	19.128	23.619	26.803	29.062	31.325	34.214	37.428	40.427
.008	61.36	H=	13.713	12.899	12.962	13.083	13.275	13.533	13.823	14.096	14.309	14.442	14.485
		V=	24.261	2.135	3.803	4.672	4.748	4.542	4.910	6.453	9.029	12.022	15.000
		D=	1.794	1.748	1.752	1.758	1.769	1.784	1.801	1.818	1.831	1.839	1.842
		Q=	61.359	5.125	9.164	11.344	11.670	11.352	12.510	16.743	23.772	31.942	39.975
.016	103.88	H=	14.203	13.458	12.948	13.093	13.297	13.540	13.793	14.027	14.218	14.345	14.390
		V=	39.743	18.362	1.591	1.083	-.154	-1.161	-.783	1.510	5.350	9.965	14.779
		D=	1.824	1.780	1.751	1.759	1.770	1.784	1.799	1.813	1.825	1.833	1.836
		Q=	103.879	45.673	3.831	2.630	-.380	-2.903	-1.991	3.900	13.999	26.300	39.130
.024	131.90	H=	14.531	14.047	13.366	13.134	13.339	13.553	13.756	13.940	14.098	14.210	14.253
		V=	49.334	34.381	10.433	-2.881	-4.875	-6.044	-5.389	-2.563	2.121	8.015	14.453
		D=	1.845	1.815	1.774	1.761	1.773	1.785	1.797	1.808	1.818	1.825	1.827
		Q=	131.898	88.915	25.796	-7.017	-12.032	-15.125	-13.669	-6.580	5.505	20.958	37.906
.032	148.87	H=	14.760	14.468	14.004	13.446	13.384	13.558	13.710	13.843	13.958	14.044	14.080
		V=	54.790	44.065	24.657	.401	-9.090	-9.899	-8.809	-5.621	-.420	6.379	14.028
		D=	1.860	1.841	1.812	1.779	1.775	1.785	1.794	1.802	1.809	1.814	1.817
		Q=	148.865	117.297	63.581	.998	-22.500	-24.782	-22.274	-14.340	-1.080	16.493	36.361
.040	157.51	H=	14.931	14.772	14.517	14.034	13.568	13.550	13.654	13.738	13.806	13.857	13.882
		V=	57.272	48.927	34.603	12.226	-8.132	-12.697	-11.097	-7.590	-2.077	5.225	13.527
		D=	1.871	1.861	1.844	1.814	1.786	1.785	1.791	1.796	1.800	1.803	1.805
		Q=	157.515	133.045	92.424	31.591	-20.373	-31.771	-27.957	-19.228	-5.285	13.341	34.598
.048	160.00	H=	15.067	15.001	14.877	14.570	14.009	13.610	13.587	13.624	13.646	13.661	13.674
		V=	57.605	50.660	39.903	23.704	1.939	-12.017	-12.362	-8.488	-2.770	4.625	12.980
		D=	1.881	1.876	1.868	1.848	1.812	1.788	1.787	1.789	1.791	1.791	1.792
		Q=	160.000	140.041	109.327	63.548	5.002	-30.188	-31.006	-21.343	-6.975	11.657	32.746
.056	158.01	H=	15.182	15.179	15.127	14.941	14.502	13.888	13.554	13.503	13.484	13.468	13.471
		V=	56.414	50.487	42.366	31.446	15.210	-3.545	-11.313	-8.421	-2.595	4.545	12.426
		D=	1.888	1.888	1.885	1.872	1.843	1.805	1.785	1.782	1.781	1.780	1.780
		Q=	158.010	141.377	118.190	86.553	40.583	-9.070	-28.315	-21.007	-6.465	11.313	30.932
.064	152.86	H=	15.280	15.316	15.299	15.176	14.871	14.302	13.688	13.400	13.325	13.289	13.263
		V=	54.184	49.269	43.332	36.230	26.047	10.333	-4.154	-6.794	-1.773	4.855	11.898
		D=	1.895	1.898	1.897	1.888	1.867	1.830	1.793	1.776	1.772	1.770	1.770
		Q=	152.860	139.357	122.410	101.431	71.330	27.191	-10.489	-16.835	-4.371	11.945	29.262
.072	145.57	H=	15.358	15.415	15.411	15.318	15.095	14.662	14.000	13.427	13.190	13.129	13.119
		V=	51.303	47.573	43.567	39.271	33.548	23.801	9.307	-.746	-.129	5.372	11.422
		D=	1.901	1.905	1.904	1.898	1.882	1.854	1.812	1.778	1.764	1.761	1.760
		Q=	145.567	135.552	124.100	111.094	93.368	64.222	23.994	-1.852	-.316	13.081	27.799
.080	136.91	H=	15.412	15.478	15.476	15.395	15.215	14.889	14.343	13.650	13.162	12.999	12.980
		V=	48.061	45.716	43.475	41.353	38.732	33.984	24.347	11.306	4.416	6.193	11.009
		D=	1.904	1.909	1.909	1.903	1.891	1.868	1.833	1.791	1.763	1.754	1.753
		Q=	136.911	130.871	124.443	117.655	108.741	93.185	64.250	28.476	10.776	14.957	26.558
.088	127.48	H=	15.436	15.505	15.502	15.425	15.264	15.001	14.588	13.980	13.332	12.958	12.873
		V=	44.671	43.833	43.242	42.915	42.536	41.148	36.540	26.253	14.109	8.925	10.686
		D=	1.906	1.911	1.911	1.905	1.894	1.876	1.849	1.811	1.772	1.751	1.747
		Q=	127.482	125.737	124.021	122.370	119.858	113.744	98.084	67.589	34.807	21.501	25.607
.096	117.72	H=	15.428	15.498	15.494	15.416	15.266	15.040	14.722	14.266	13.656	13.096	12.873
		V=	41.273	41.964	42.935	44.175	45.489	46.240	44.981	39.174	27.549	15.984	10.686
		D=	1.906	1.911	1.910	1.905	1.894	1.879	1.857	1.828	1.791	1.759	1.747
		Q=	117.721	120.312	123.059	125.883	128.195	128.181	121.886	102.836	69.414	38.844	25.607

Figure 8 (Cont'd)

.104	95.15	H=	15.257	15.458	15.453	15.376	15.235	15.035	14.781	14.449	13.986	13.431	13.123
		V=	33.786	40.119	42.558	45.195	47.817	49.912	50.529	48.052	40.219	26.690	11.434
		D=	1.894	1.908	1.907	1.902	1.892	1.878	1.861	1.840	1.811	1.778	1.761
		Q=	95.153	114.678	121.604	128.410	134.445	138.312	137.489	127.740	103.589	66.269	27.834
.112	60.01	H=	14.900	15.314	15.382	15.309	15.179	15.005	14.797	14.549	14.236	13.881	13.667
		V=	21.867	36.025	42.089	45.947	49.574	52.501	54.100	53.496	48.837	35.952	12.961
		D=	1.869	1.898	1.902	1.897	1.888	1.876	1.862	1.846	1.826	1.805	1.792
		Q=	60.010	101.884	119.630	129.900	138.821	145.168	147.373	143.201	127.934	91.951	32.681
.120	25.25	H=	14.485	14.971	15.243	15.220	15.107	14.959	14.790	14.604	14.434	14.356	14.335
		V=	9.476	26.831	40.230	46.356	50.728	54.168	56.279	56.477	52.540	39.295	14.648
		D=	1.842	1.874	1.893	1.891	1.883	1.873	1.862	1.850	1.839	1.834	1.833
		Q=	25.252	74.006	113.185	130.193	141.301	149.286	153.232	151.768	139.530	103.791	38.632
.128	-9.01	H=	14.008	14.549	14.930	15.089	15.021	14.903	14.771	14.656	14.648	14.801	14.906
		V=	-3.494	16.400	33.612	45.628	51.233	54.999	57.331	57.222	51.515	36.964	15.927
		D=	1.812	1.846	1.871	1.882	1.877	1.869	1.861	1.853	1.853	1.863	1.870
		Q=	-9.013	43.901	92.440	126.930	141.825	150.970	155.894	154.335	138.860	100.722	43.728
.136	-.00	H=	14.045	14.067	14.521	14.821	14.912	14.841	14.757	14.745	14.903	15.173	15.301
		V=	-.000	5.165	24.715	40.711	50.651	55.024	57.123	55.345	46.689	32.113	16.721
		D=	1.814	1.816	1.844	1.864	1.870	1.865	1.860	1.859	1.869	1.888	1.897
		Q=	-.000	13.375	66.032	111.092	139.118	150.361	155.169	150.211	128.151	89.887	47.244
.144	.00	H=	13.940	13.900	14.053	14.450	14.700	14.769	14.769	14.891	15.173	15.450	15.548
		V=	.000	4.109	14.555	32.732	46.535	53.872	55.139	50.627	39.861	27.417	17.182
		D=	1.808	1.806	1.815	1.840	1.856	1.861	1.861	1.869	1.888	1.907	1.914
		Q=	.000	10.522	37.658	87.019	125.900	146.466	149.910	138.848	111.567	78.325	49.446
.152	.00	H=	13.815	13.799	13.774	14.020	14.388	14.641	14.825	15.083	15.409	15.634	15.698
		V=	.000	5.199	10.237	23.014	38.955	49.252	50.696	43.687	32.988	23.875	17.448
		D=	1.801	1.800	1.798	1.813	1.836	1.852	1.864	1.882	1.904	1.920	1.925
		Q=	.000	13.223	25.995	59.410	103.121	132.698	138.379	121.484	93.953	69.158	50.787
.160	.00	H=	13.679	13.674	13.654	13.698	14.026	14.456	14.881	15.277	15.582	15.747	15.788
		V=	.000	5.567	10.400	16.326	28.922	40.102	42.365	35.793	27.232	21.403	17.602
		D=	1.792	1.792	1.791	1.794	1.813	1.840	1.868	1.895	1.917	1.929	1.932
		Q=	.000	14.044	26.201	41.250	74.691	106.661	116.103	100.950	78.570	62.530	51.587
.360	.00	H=	13.470	13.507	13.601	13.709	13.794	13.840	13.857	13.874	13.913	13.960	13.971
		V=	.000	7.545	15.155	22.433	28.587	32.857	34.743	33.861	29.862	22.823	13.754
		D=	1.780	1.782	1.788	1.794	1.799	1.802	1.803	1.804	1.806	1.809	1.810
		Q=	.000	18.826	38.047	56.723	72.690	83.803	88.716	86.562	76.536	58.680	35.389
.368	.00	H=	13.308	13.338	13.421	13.535	13.652	13.753	13.841	13.934	14.042	14.138	14.169
		V=	.000	6.033	12.558	19.440	25.834	30.596	32.738	31.724	27.678	21.464	14.248
		D=	1.771	1.773	1.777	1.784	1.791	1.797	1.802	1.808	1.814	1.820	1.822
		Q=	.000	14.890	31.161	48.596	65.077	77.587	83.503	81.422	71.557	55.855	37.153
.376	.00	H=	13.177	13.199	13.264	13.370	13.506	13.659	13.821	13.991	14.154	14.273	14.310
		V=	.000	4.891	10.333	16.378	22.351	27.027	29.159	28.147	24.527	19.658	14.589
		D=	1.764	1.765	1.768	1.775	1.782	1.791	1.801	1.811	1.821	1.829	1.831
		Q=	.000	11.965	25.382	40.506	55.766	68.111	74.276	72.518	63.891	51.628	38.414
.384	.00	H=	13.069	13.084	13.133	13.227	13.374	13.570	13.799	14.035	14.235	14.362	14.398
		V=	.000	4.094	8.552	13.465	18.385	22.302	24.108	23.387	20.858	17.770	14.797
		D=	1.758	1.758	1.761	1.766	1.775	1.786	1.800	1.814	1.826	1.834	1.837
		Q=	.000	9.942	20.830	32.997	45.481	55.878	61.323	60.435	54.637	46.953	39.197
.392	.00	H=	12.975	12.987	13.029	13.118	13.275	13.502	13.778	14.054	14.273	14.402	14.439
		V=	.000	3.503	7.075	10.702	14.114	16.729	17.988	17.872	16.968	15.901	14.892
		D=	1.752	1.753	1.755	1.760	1.769	1.782	1.798	1.815	1.829	1.837	1.839
		Q=	.000	8.453	17.120	26.046	34.692	41.729	45.691	46.244	44.564	42.133	39.561
.000	.00	H=	12.894	12.907	12.952	13.049	13.217	13.463	13.757	14.043	14.265	14.396	14.436
		V=	.000	2.939	5.644	7.919	9.622	10.726	11.432	12.121	13.012	13.987	14.866
		D=	1.748	1.749	1.751	1.756	1.766	1.780	1.797	1.814	1.828	1.836	1.839
		Q=	.000	7.058	13.592	19.187	23.565	26.686	28.997	31.338	34.155	37.046	39.536

Figure 8. Computer Program and Sample of Calculations for Pulse Flow Into a Flexible Tube. Pulse Time 0.4 sec, Unstressed Tube i.d. = 1.3 cm. t' = 0.1 cm, Y = 5. x 10^6 dynes/cm^2. Problem was started as a steady state flow with flow of 35.65 cc/sec. and head of 14. cm Hg at proximal end. Friction factor F = 0.3 and losses varying as square of the velocity. The calculations are printed out for the third pulse after steady flow, with values given for each 0.008 sec. time interval.

f = 0.3 was used and a time increment of Δt = 0.004 sec. taken with twenty equal reaches of Δx = 2.5 cm. The pressure H, velocity V, diameter D and flow rate Q were calculated, and values printed out for every .008 sec. time interval and at 5 cm distances along the tube. The terminal bed pressure was taken as 100 mm Hg, and the problem was iniated by first setting up a steady flow equal to the average flow (QAVE).

The sequence of main calculations in the program are as follows (see Figure 8):

1. Calculation of steady state problem, storing of H, V, D, and Q for the 21 sections 2.5 cm apart for time t = 0.
2. Calculation of interior points, statements 45 through 54, for the next time increment.
3. Calculation of the proximal boundary condition, statements 55 through 67.
4. Calculation of the distal boundary condition equations, statements 68 through 78.
5. Print out of every second set of results, incrementation of T and U, and check on determination of end of solution.

The program was run through 3 pulse cycles. The end of the second and third pulses showed rather close agreement, indicating that steady oscillatory flow has almost been established. A gross check on continuity was made as a measure of the accuracy of the finite difference method. The total inflow in the 3rd pulse is 35.65 x 0.4 = 14.26 cc. The total outflow is 14.34 cc, and the volume of fluid within the tube has decreased by 0.29 cc. This indicates an error in continuity of about 1.5%, which could be further reduced by taking shorter reaches and smaller time increments, or by going to a method of calculation[8] having second order accuracy.

B. Equations for Tapered Distensible Vessel with Distributed Outflow

Since the vascular system is so complex, and several computer statements are generally required for any boundary condition, a special program has been developed for flow through a vessel having its unstressed diameter varying with length along the tube, Figure 9, and with flow leaving the vessel in a distributed manner. Such a vessel could represent the aorta, with branches of various sizes along its length. The outflow along the vessel is set up as a flow per unit length of vessel, with rate proportional to the head difference inside and outside the tube.

If q represents the outflow per unit length, then

$$q = k_i \, (H - HB) \tag{50}$$

in which k_i has an assigned value for each of the N sections of the vessel. The continuity equation, Equation (14), now has an extra term

$$-(\rho AV)_x dx - \rho q dx = (\rho A dx)_t \tag{51}$$

which simplifies to

$$V \frac{A_x}{A} + V + \frac{A_t}{A} + \frac{q}{A} = 0 \tag{52}$$

By assuming that the fluid leaving through the walls has its axial momentum reduced by contact with the branch, the momentum equation becomes

$$- P_x A dx - \tau_o \pi D dx = (\rho A V^2)_x \, dx + (\rho A V dx)_t + \rho q V dx$$

After reducing this equation in a manner similar to Equations (18) and (19), one obtains Equation (19) as before.

In order to take the tapering effect into account A_x is evaluated as follows:

$$A = A_o \frac{a_o^2}{a^2}$$

with A_o a function of x. Then

$$A_x = A_{ox} \frac{a_o^2}{a^2} - 2A_o \frac{a_o^2}{a^3} \, a_x$$

But

$$a^2 = a_o^2 - gH$$

and

$$2a \cdot a_x = - gH_x$$

Hence

$$\frac{A_x}{A} = \frac{A_{ox}}{A_o} + \frac{gH_x}{a^2} = \frac{\alpha}{A_o} + \frac{gH_x}{a^2} \qquad (53)$$

in which α is the rate of change of unstressed vessel area per unit length. A_t/A is obtained as before.

Following the previous procedures in developing the finite difference equations, the two controlling equations become (the interpolation equations V_R, H_R, V_S, H_S are unchanged):

$$V_P = V_R - \frac{g}{a_C}(H_P - H_R) - \frac{fV_C\ |V_C|\ \Delta t}{2D} - \alpha \frac{a_C}{A_o}\ V_C\Delta t - \frac{a_C{}^3 q\Delta t}{A_o a_o{}^2} \qquad (54)$$

$$V_P = V_S + \frac{g}{a_C}(H_P - H_S) - \frac{fV_C\ |V_C|\ \Delta t}{2D} + \alpha \frac{a_C}{A_o}\ V_C\Delta t + \frac{a_C{}^3 q\Delta t}{A_o a_o{}^2} \qquad (55)$$

By addition, and then by subtraction, the two equations yield values of V_P and H_P at interior points:

$$V_P = \frac{V_R + V_S}{2} + \frac{g}{2a_C}(H_R - H_S) - \frac{fV_C\ |V_C|\ \Delta t}{2D} \qquad (56)$$

$$H_P = \frac{H_R + H_S}{2} + \frac{a_C}{2g}\left[V_R - V_S - \frac{2a_C\alpha V\Delta t}{A_o} - \frac{2a_C{}^3 q\Delta t}{A_o a_o{}^2}\right] \qquad (57)$$

The boundary conditions are handled exactly as before, except that either Equation (59) or Equation (55) is used, depending upon whether it is a right-end or a left-end boundary, respectively. For actual computation Equation (50) is used to eliminate q from the working equations. As very small time increments are generally used (.001 sec.), it is a satisfactory approximation to allow H to equal H_C in Equation (50), although this is not absolutely necessary. With this type of distributed outflow (Equation 50), each branch is treated as if it were a terminal bed, i.e., a definite relation between pressure H and flow q is established for each branch.

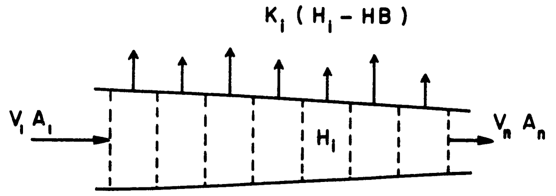

Figure 9. **Tapered Vessel with Distributed Outflow Along the Walls. Flow Per Unit Length Through i-th Reach Shown. HB is Terminal Bed Pressure.**

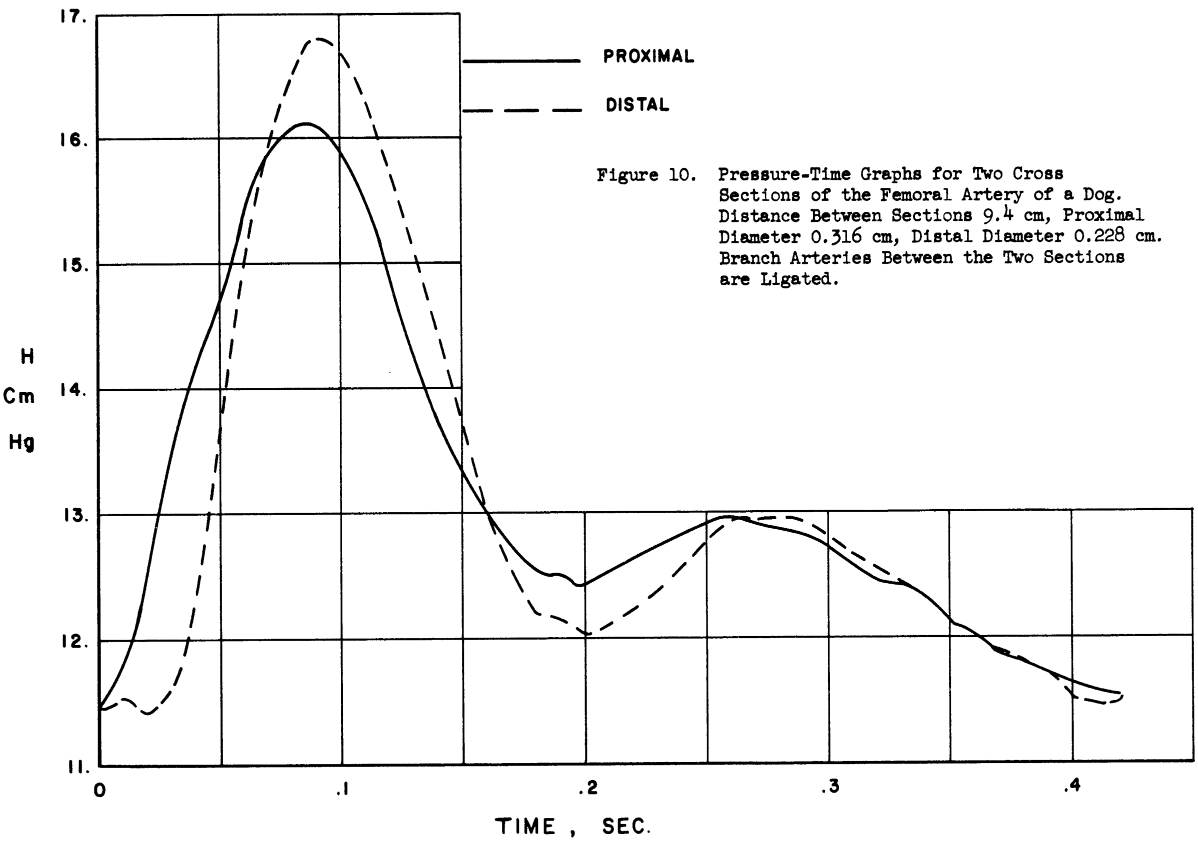

Figure 10. Pressure-Time Graphs for Two Cross Sections of the Femoral Artery of a Dog. Distance Between Sections 9.4 cm, Proximal Diameter 0.316 cm, Distal Diameter 0.228 cm. Branch Arteries Between the Two Sections are Ligated.

III. COMPARISON OF MEASURED FLOW WITH FLOW COMPUTED FROM PRESSURE-TIME DATA IN A TAPERING VESSEL

To compare a calculated flow with experimentally measured flow an in vivo experiment was performed. Pressure and flow data were obtained from the femoral artery of a dog. Pressure was determined by inserting short needles at two points in the artery 9.4 cm apart. Small branches between were ligated at the vessel wall. Suitable catheters and fittings were connected between the needles and two Sanborn differential transducers. Flow was measured by means of a non-cannulating electromagnetic flow probe placed immediately upstream from the proximal pressure needle. Data was recorded on a four-channel Sanborn (350) system. Figure 10 shows the proximal and distal pressure obtained for one cycle. The solid line in Figure 11 is the flow obtained for the same cycle. The dashed line in Figure 11 is the flow calculated by using the experimental pressure data as boundary conditions.

In computing the flow from the pressure-time data values of pressure were read off the strip chart for each 0.01 sec. By parabolic interpolation these values were replaced by 0.001 sec. interval values over the pulse cycle. These computed values then provided the boundary conditions needed to compute flow during the cycle. The diameter of artery was assumed to decrease linearly along the length under consideration, and the length was split into seven equal reaches. Initially the flow was assumed to be steady at about the average flow, with the head (pressure) constant along the artery. After two cycles have been computed the flow has almost achieved its steady oscillatory character.

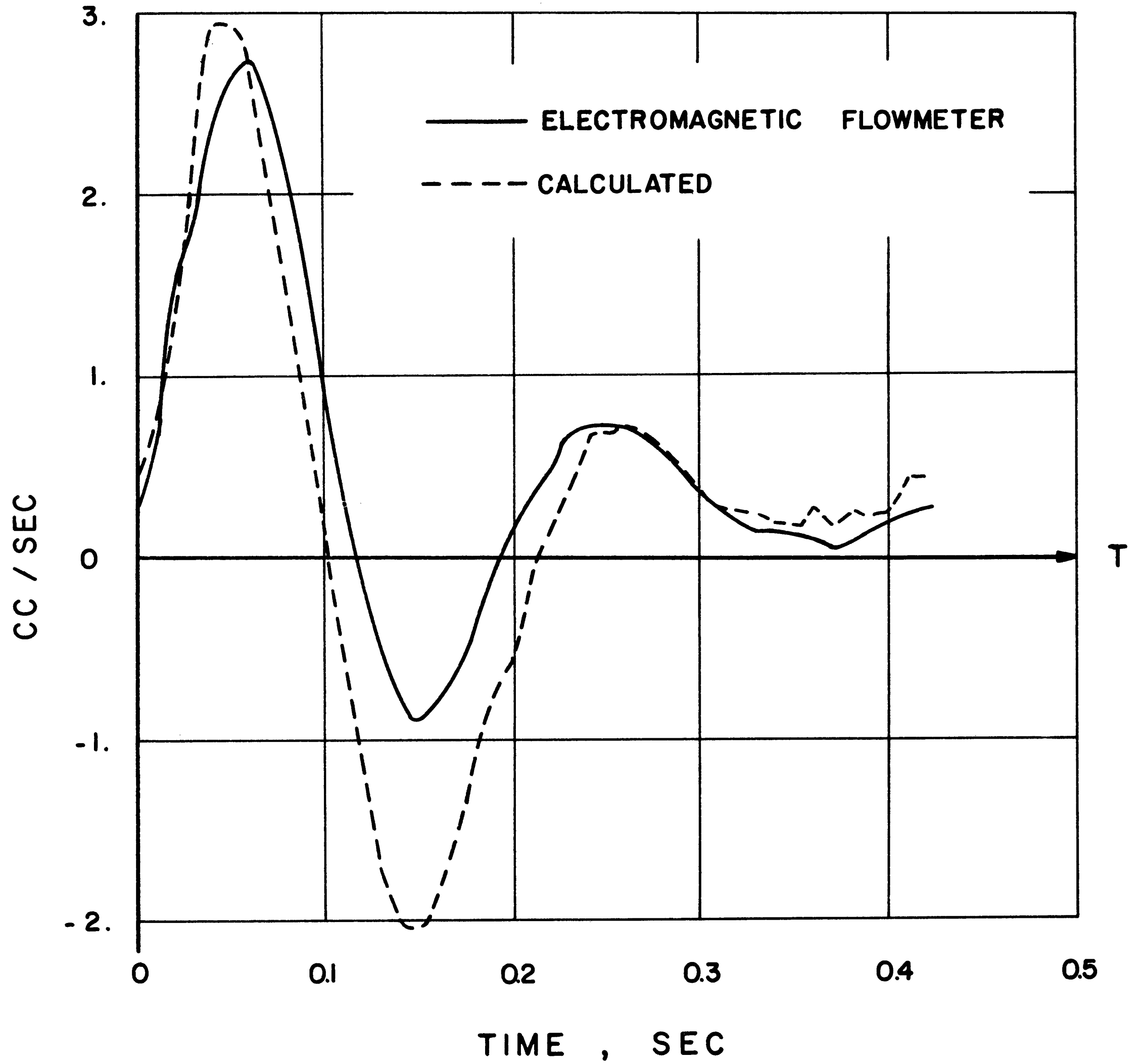

Figure 11. Electromagnetic Flow Meter Data For the Upstream Section of the Femoral Artery (Figure 10) and computed Flow Using the Pressure-Time Data of Figure 10 for Boundary Conditions. In the Computer Solution the Speed of Pulse Wave at H = 11.5 cm Hg was a = 1200 cm/sec. The Friction Factor was f = 0.4 and the Energy Dissipation was Taken to Vary as the Square of the Velocity.

The calculated flow does not coincide exactly with the experimental: several factors contribute to this discrepancy. Pressure measurements are extremely critical; errors of 1 mm in locating the zero pressure datum line can result in a computation that reverses the direction of average flow. Physical dimensions of the transducer fittings and lines may well introduce their own values and transients that alter the recorded pressure pulse. Dampening of the recorded flow from amplifier filtering factors in the flowmeter must also be considered.

Several of these factors have been made an object of special study using programming and the theoretical model. Results of these studies, though not a subject of this paper, have shown that more refined raw data will produce better correspondence between calculated and experimental results.

In making the calculations the frictional effects are quite unknown, but very important. By a gradient method, values of f and n in the friction term

$$\frac{f}{2D_C} V_C |V_C|^{n-1}$$

were obtained that gave the best fit (least square method) with the flowmeter data. These results yielded values of f of about 0.4 and n about 2.0.

SUMMARY

The basic differential equations for elastic wall material and for continuity and momentum are derived, including fluid frictional resistance of the wall of the tubes, based on one-dimensional flow. These partial differential equations are transformed into four ordinary differential equations using the theory of characteristics. Then difference equations are developed and by an interpolation method (method of specified time intervals) equations are obtained for computation of velocity and pressure at equally-spaced sections along the vessel at specified equal time intervals. The equations are first applied to a flexible tube of initial constant diameter, with a pulse flow taken from in vivo experiments.

Equations are then developed for tapering tubes with distributed outflow along their lengths (to simulate branches). Pressure-time data from femoral artery measurements are then used to compute flow through the artery, and the results are compared with electromagnetic flow-meter data.

ACKNOWLEDGEMENTS

The authors are indebted to the University of Michigan Computing Center for use of its digital computer, and to the NIH for its support through project No. HE07443-01 on "Pulsalite Flow Through Distensible Tubes". The experimental work was completed while Dr. Keitzer was supported in part by projects U.S.P.H.S.B-2290C1 and U.S.P.H.S.H-4260C3.

NOTATION

a	=	speed of pressure pulse through vessel
a_o	=	speed of pressure pulse through vessel at zero pressure
A	=	area of vessel cross section, point on x-t plot
A_o	=	area of vessel cross section at zero pressure
B	=	point on x-t plot
C	=	designation of characteristic curve, point on x-t plot
D	=	diameter of vessel
D_o	=	diameter of vessel at pressure zero
f	=	Darcy-Weisbach friction factor
F	=	force on fluid element
g	=	acceleration due to gravity
H	=	pressure expressed in length of column of fluid flowing
K	=	constant
L	=	label for a differential equation
m	=	exponent
P	=	pressure, or point to be computed
q	=	outflow per unit length
Q	=	flow through vessel
R	=	known point on characteristic curve
$\underset{\sim}{R}$	=	Reynolds number
S	=	tensile stress in vessel wall; known point on characteristic curve
t	=	time

t'	=	thickness of vessel wall
t'_o	=	thickness of vessel wall at pressure zero
T	=	tensile force per unit length in tube wall
V	=	velocity in vessel
X	=	distance along vessel
Y	=	elastic modulus of vessel wall
α	=	rate of change of unstressed vessel area per unit length
θ	=	ratio $\Delta t/\Delta x$
λ	=	undetermined multiplier
μ	=	dynamic viscosity
ρ	=	density of fluid
τ_o	=	frictional wall shear stress

NOTATION FOR MAD PROGRAM

A = speed of pressure pulse at pressure HO

AC = speed of pressure pulse at pressure H

AO = speed of pressure pulse at zero pressure

ACM = speed of pressure pulse at pressure H(N)

C1, C2, C3, C4, C5, C6, C7 = constants

D = diameter

DELT = time increment

DHF = steady state pressure drop in reach Δx

F = friction factor

G = gravity

H = pressure, from previous calculation

HO = bed pressure

HB = initial pressure

HP = pressure to be calculated

HR = interpolated head

HS = interpolated head

I = integer

K = constant in terminal bed relation

KK = constant

N = number of reaches

P = constant

PMT, PPT, PTT = constants in computing pulse

PTIME = pulse period

Q = flow

QAVE = steady-state flow

QM = maximum pulse flow

RHO = density

SG = specific gravity of mercury in terms of fluid flowing

T = time

TM = constant in computing pulse

U = integer

V = velocity, from previous calculation

VP = velocity to be calculated

VR = interpolated velocity

VS = interpolated velocity

REFERENCES

1. Womersley, J. R. "An Elastic Tube Theory of Pulse Transmission and Oscillatory Flow in Mammalian Arteries." WADC Technical Report Tr 56-614, January 1957.

2. McDonald, D. A. "Blood Flow Through Arteries," The Williams and Wilkins Co., Baltimore, 1960.

3. Warner, Homer R. "Use of Analogue Computers in the Study of Control Mechanisms in the Circulation" Federation Proceedings Vol. 21, No. 1, January - February, 1962 Symposium on Computer Techniques in Physiology.

4. Remington, J. W. "Contour Changes of the Aortic Pulse During Propagation" Am. J. Phys. Vol. 199, No. 2, August 1960, pp. 331-334.

5. Courant, R., and Friedrichs, K. O. "Supersonic Flow and Shock Waves," Interscience Publishers, New York, 1948.

6. Stoker, J. J. "The Formation of Breakers and Bores," Communications on Applied Mathematics, Vol. 1. No. 1, 1948.

7. Lai, Chintu, "A Study of Waterhammer Including Effect of Hydraulic Losses." Doctoral Dissertation, The University of Michigan, 1961.

8. Streeter, V. L., and Lai, Chintu. "Water-Hammer Analysis Including Fluid Friction." Journal of the Hydraulics Division, Proc. ASCE, Paper 3135, HY3, pp. 79-112, May 1962.

9. Streeter, V. L. "Waterhammer Analysis with Nonlinear Frictional Resistance." Conference on Hydraulics and Fluid Mechanics, University of Western Australia (In press, presented December 1962).

10. Peterson, L. H., Jensen, R. E., and Parnell, John. "Mechanical Properties of Arteries in Vivo" Circulation Research, Vol. VIII, No. 3, pp. 622-639, 1960.

11. Streeter, V. L. "Fluid Mechanics," 3rd ed. McGraw-Hill Book Co. Inc., New York, 1962, pp. 210-222.

12. Lister, M. "The Numerical Solution of Hyperbolic Partial Differential Equations by the Methods of Characteristics." in Mathematical Methods for Digital Computers, John Wiley and Sons, New York, 1960.