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NECESSARY CONDITIONS FOR OPTIMIZATION PROBLEMS WITH HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In the present paper we consider a system of nonlinear hyperbolic partial differential equations (state equations) of the form

$$\partial z^i/\partial x\partial y = f_i(x, y, z, z_x, z_y, v), \qquad (x, y) \in G,$$

$$i = 1,\dots,n, \qquad z(x, y) = (z^1,\dots,z^n), \qquad v(x, y) = (v^1,\dots,v^m),$$

$$G = [a \leq x \leq a + h, \;\; b \leq y \leq b + k], \tag{1.1}$$

with Darboux-type boundary conditions

$$z(x, b) = \phi(x), \quad a \leq x \leq a+h, \quad z(a,y) = \psi(y), \quad b \leq y \leq b + k \tag{1.2}$$

with constraints

$$v(x, y) \in U, \tag{1.3}$$

and we are concerned with the minimum of a functional of the form

$$I[z, v] = \sum_{i=1}^n A_i z^i(a + h, b + k). \tag{1.4}$$

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Here $\varphi(x) = (\varphi^1,\ldots,\varphi^n)$, $a \leq x \leq a + h$, and $\psi(y) = (\psi^1,\ldots,\psi^n)$, $b \leq y \leq b + k$, are given absolutely continuous functions (AC) in the respective intervals, with $\phi(a) = \psi(b)$. The control space U above is a given fixed set of the u-space E_m. The constants A_i, $i = 1,\ldots n$, are given.

The minimum of the functional $I[z, v]$ is sought in suitable classes Ω of pairs $z(x, y) = (z^1,\ldots,z^n)$, $v(x, y) = (v^1,\ldots,v^m)$, $(x, y) \in G$, satisfying (1.1), (1.2), (1.3), the functions z^i belonging to a Sobolev space $W_p^1(G)$ on G, $1 \leq p \leq +\infty$, and continuous on G, and the functions v^j being measurable on G. In the present paper we give Pontryagin-type necessary condition for the minimum.

First, in no. 3 we obtain an existence and uniqueness statement for the solution $z(x, y) = (z^1,\ldots,z^n)$, $(x, y) \in G$, $z \in (W_p^1(G))^n$, of the Darboux problem (1.1-2) (the original problem) for a given p, $1 \leq p \leq +\infty$, for given φ, ψ, and for a given measurable function $v(x, y) = (v^1,\ldots,v^m)$, $(x, y) \in G$. We derive this existence and uniqueness statement from our previous paper [6b] on multidimensional integral equations of the Volterra type.

The optimization problem (1.1-4) can be written in the form proposed by Cesari [2] with state equations of the Dieudonné-Rashevsky type, the Hamiltonian function H then containing 2n multipliers λ_i, μ_i, $i = 1,\ldots,n$. As shown in [2], these 2n multipliers are expected to satisfy a suitable system of linear partial differential equations and corresponding boundary conditions (the conjugate problem).

In no. 4 we formulate the conjugate problem pertinent to the optimization problem (1.1-4), and for the first time we prove, in the present situation, an existence theorem for the solutions λ_i, μ_i of the conjugate problem. In other words, we prove, under hypotheses, that there are multipliers λ_i, μ_i, $i = 1,\dots,n$, in $L_\infty(G)$, satisfying in a suitable sense the partial differential equations and boundary conditions pertaining to the conjugate problem of problem (1.1-4). As in no.3, again we derive the existence statement from our previous paper [6b] on multidimensional integral equations of the Volterra type.

In no.5 we give a new proof of the increment formula of [2], under a set of hypotheses different from those in [2]. In no.6 we derive as in [2] the Pontryagin-type necessary condition for the optimization problem (1.1-4) with the existence of suitable multipliers actually proved.

In no.7 we make a number of remarks on the obtained results, particularly in relation to the previous papers by Cesari [2] and A. I. Egorov [3a]. In particular, we show that the present necessary condition yields—under strong smoothness hypotheses—the necessary condition previously proved by A. I. Egorov [3a]. On the other hand we show (no. 7, example 3) that these smoothness hypotheses under which Egorov's condition has been proved are not known a priori, while our necessary condition holds.

2. NOTATIONS

If $(X, \| \|)$ denotes any normed linear space, then X^n, $n \geq 1$ denotes the cartesian product of X with itself, n times; for $x = (x^1,\dots,x^n) \in X^n$, we define $\|x\| = \sum_{i=1}^n \|x^i\|$. If $x \in E^n$, the n-dimensional Euclidean space, then we take $\|x\| \equiv |x| = \sum_{i=1}^n |x^i|$. We shall denote by Γ, a rather arbitrary family of measurable control functions. Precisely, let Γ be any set of measurable functions; $v:G \to U$, $v = (v^1,\dots,v^m)$, with the following property:

(*) For every function $v \in \Gamma$, any point $u \in U$, and any closed subset $S \subset G$, the control function v_ϵ, defined by $v_\epsilon = v$ in $G - S$, and $v_\epsilon = u$ in S, belongs to Γ.

Thus, every constant function $v:G \to \{u\}$, $u \in U$, belongs to Γ.

For functions $\varphi \in L_p(G)$, $1 \leq p \leq +\infty$, we denote by $\|\varphi\|$ or $\|\varphi\|_p$ the usual L_p- norm; in particular, $\|\varphi\|_\infty = \text{Ess Sup } |\varphi|$. For functions in a Sobolov space $[W_p^1(G)]^n$ in G, say, $z(x, y) = (z^1,\dots,z^n)$, we shall denote by $z_x = (z_x^1,\dots,z_x^n)$, and $z_y = (z_y^1,\dots,z_y^n)$, the usual generalized first order partial derivatives of z, and we take $\|z\|_W = \|z\|_{W_p^1(G)} = \|z\|_p + \|z_x\|_p + \|z_y\|_p$.

3. THE ORIGINAL PROBLEM

We shall need the following hypotheses:

(H_1): The functions $\varphi(x) = (\varphi^1,\dots,\varphi^n)$ and $\psi(y) = (\psi,\dots,\psi^n)$ are defined and absolutely continuous on $[a, a + h]$ and $[b, b + k]$, respectively. The derivatives φ_x and ψ_y which exist almost everywhere belong to $L_p([a, a + h])$ and $L_p([b, b + k])$, respectively, for some p, $1 \leq p \leq +\infty$. Furthermore $\varphi(a) = \psi(b)$.

(H_2): The function $f = f(x, y, z_1, z_2, z_3, u) = (f_1,\dots,f_n)$ is defined on $G \times E^{3n} \times U$, and each f_i is continuous in u and measurable in (x, y) for fixed $(z_1, z_2, z_3) \in E^{3n}$.

(H_3): For each $v \in \Gamma$ the function $s_o(x, y) = f(x, y, o, o, o, v(x, y))$ belongs to $L_p(G)$ with p as in (H_1).

(H_4): The functions f_i are differentiable as functions of (z_1, z_2, z_3) and the derivatives $\partial f_i/\partial z_1^j$, $\partial f_i/\partial z_2^j$, df_i/dz^j_3, $i,j = 1,\dots,n$, are continuous in (z_1, z_2, z_3) for fixed (x,y,v). Clearly they are measurable in (x, y).

(H_5): There are functions $K_{jr}(x, y, u)$, $j = 1,2,3$, $r = 1,\dots,n$, such that for all $(x, y, u) \in G \times U$ and $(z_1, z_2, z_3) \in E^{3n}$ we have

$$|\partial f_i/\partial z_j^r(x, y, z_1, z_2, z_3, u)| \leq K_{jr}(x, y, u) \qquad (3.1)$$

and such that $K_{jr}(x, y, v(x, y)) \in L_\infty(G)$ for $v \in \Gamma$, $j = 1,2,3$, $i,r = 1,\dots,n$.

We state below an existence theorem (3.i) for the solution z of the Darboux problem (1.1-2) and a theorem (3.ii) concerning their behavior. We refer to [6 b] (or [6 a]) for proofs of these and other statements. Theorem (3.i) provides norm estimates on the solution z as an element of $W_p^1(G)$, along with pointwise estimates on z, z_x, and z_y. Theorem (3.ii) shows the dependence of the solution on the data.

(3.i) <u>Theorem</u>: Let $v \in \Gamma$ be given. If H_1-H_5 hold, then there exists a unique $z \in (W_p^1(G))^n$ (with p as in H_1), continuous on G, satisfying (1.2), for which the generalized partial derivatives z_x, z_y, z_{xy} exist and satisfy (1.1) a.e. in G. Furthermore, there are constants B_1 and B_2 depending only on h, k, p and on $K = \{\|K_{ij}(x, y, v(x, y)\|_\infty;\ i = 1,2,3;\ j = 1,\dots,n\}$ such that

$$\|z\|_W \leq B_1[k^{1/p}(\|\varphi_x\|_p + 2^{-1}\|\varphi\|_p) + h^{1/p}(\|\psi_y\|_p + 2^{-1}\|\psi\|_p)$$

$$+ (h + k)\|s_o(x, y)\|_p, \qquad (3.2)$$

$$|z(x, y)| \leq 2^{-1}[\|\varphi\|_c + \|\psi\|_c + BB_2(h + k)$$

$$+ e^{K(h + k)}(B + \iint_G s_o(\alpha,\beta)d\alpha d\beta)] \qquad (3.3)$$

$$|z_x(x, y)| \leq \Theta_1(x) + BB_2; \quad |z_y(x, y)| \leq \Theta_2(y) + BB_2 \qquad (3.4)$$

where $B = \|\varphi_x\|_p h^{1/q} + \|\psi_y\|_p k^{1/q} + Khk(\|\varphi\|_c + \|\psi\|_c) + \iint_G s_o(\alpha,\beta)$

and $\Theta_1(x) = e^{Kk}[|\varphi_x(x)| + Kk|\varphi(x)| + \int_b^{b+k} s_o(x,\beta)d\beta]$,

$$\Theta_2(y) = e^{Kh}[|\psi_y(y)| + Kh|\psi(y)| + \int_a^{a+h} s_o(\alpha,y)d\alpha].$$

The existence of the solution and the norm estimate (3.2) follow from [6b, Appendix, Theorem 5, A·2] while pointwise estimates are a consequence of the absolute continuity (in the sense of Tonelli) of the solution z, and a repeated application of Gronwall's lemma. [See 6b, Appendix, (A·10)] or [6a].

(3.ii) <u>Theorem</u>: For i = 1,2, let z_i denote the solution of (1.1-2) corresponding to the data (φ_i, ψ_i) satisfying (H_1) and control function v_i in Γ. Let $z = z_1 - z_2$, $\varphi = \varphi_1 - \varphi_2$, $\psi = \psi_1 - \psi_2$ and $s(x, y) = |f(x, y, z_1, z_{1x}, z_{1y}, v_1) - f(x, y, z_1, z_{1x}, z_{1y}, v_2)|$. With this notation, the above inequalities (3.2), (3.3), (3.4) again hold with s_o replaced by s and no further changes. [See 6b, Appendix, A.10; or [6a].]

Furthermore, if $\varphi_1 = \varphi_2$; $\psi_1 = \psi_2$ and $v_1 = v_2$ outside a square $S = [\bar{x} - \delta, \bar{x} + \delta] \times [\bar{y} - \delta, \bar{y} + \delta] \subset G$, then the pointwise estimates become

$$|z(x, y)| \equiv |z_1(x, y) - z_2(x, y)| \leq B_1 \iint_S s(\alpha,\beta)d\alpha d\beta$$

$$|z_x(x, y)| \leq e^{Kk} \int_{\bar{y} - \delta}^{\bar{y} + \delta} s(x,\beta)d\beta + B_2 \iint_S s(\alpha,\beta)d\alpha d\beta$$

$$|z_y(x, y)| \leq e^{Kh} \int_{\bar{x} - \delta}^{\bar{x} + \delta} s(\alpha,y)d\alpha + B_2 \iint_S s(\alpha,\beta)d\alpha d\beta \tag{3.5}$$

where B_1 and B_2 depend only on h, k, and on $K = \max\ \{\|K_{ij}(x, y, v_r(x, y)\|_\infty,\ i = 1,2,3;\ \ j = 1,\ldots,n;\ \ r = 1,2\}$. We shall need in the sequel these particular pointwise estimates.

Remark 1. In view of the uniqueness of the solution z of the Darboux problem (1.1-2) for any given element $v \in \Gamma$, we shall denote the functional $I[z,u]$ of (1.4) simply by $I[v]$, or $I:\Gamma \to E_1$.

Remark 2. By introducing the notation $z_1 = z$; $z_2 = z_x$; $z_3 = z_y$, the Darboux problem (1.1, 1.2) can be written in the equivalent Dieudonné-Rashevsky form:

$$z_{1x} = z_2;\quad z_{3x} = f(x, y, z_1, z_2, z_3, v);\quad z_{1y} = z_3;$$

$$z_{2y} = f(x, y, z_1, z_2, z_3, v) \tag{3.6}$$

with boundary conditions,

$$z_1(x, b) = \varphi(x);\quad z_2(x, b) = \varphi_x(x);\quad z_1(a, y) = \psi(y);$$

$$z_3(a, y) = \psi_y(y) \tag{3.7}$$

It is to be noted that even though the system (3.6) seems overdetermined (four equations in three unknowns), it is not actually so, since the second and the fourth equations are equivalent.

4. THE CONJUGATE PROBLEM

Cesari [2] has proved Pontryagin-type necessary conditions for problems of optimization with state equations in the Dieudonné-Rashevsky form $z_{ix} = f_i(x, y, z, v)$; $z_{iy} = g_i(x, y, z, v)$, $z = (z_1,\dots,z_n)$, $i = 1,2\dots n$; and taking as Hamiltonian the expression $H = \lambda_1 f_1 + \dots + \lambda_n f_n + \mu_1 g_1 + \dots + \mu_n g_n$. By assuming the z_i, λ_i, μ_i to be in suitable Sobolev spaces, Cesari [2] showed that the multipliers λ_i, μ_i should satisfy the "conjugate problem," i.e., partial differential equations of the form $\lambda_{ix} + \mu_{iy} = -\partial H/\partial z_i$, $i = 1,2\dots n$, along with boundary conditions, which are complementary to those for $z_1,\dots z_n$ and in relation to the cost functional under consideration. In view of remark 2 of no. 3, we use here the same Hamiltonian with the remark that since z_{2x} and z_{3y} do not appear in (3.6) we take $\lambda_2 = \mu_3 = 0$, and the Hamiltonian reduces to

$$H = \lambda_1 z_2 + \lambda_3 f + \mu_1 z_3 + \mu_2 f, \tag{4.1}$$

where $\lambda_i = (\lambda_i^1,\dots,\lambda_i^n)$, $\mu_i = (\mu_i^1,\dots,\mu_i^n)$ and the products are inner products in E_n. By taking the cost functional I in (1.4) in the equivalent form (Cesari, [2])

$$I = 2^{-1} \int_a^{a+h} A\, z_2(x, b + k)dx + 2^{-1}\int_b^{b+k} A\, z_3(a + h, y)dy \tag{4.2}$$

where $A = (A_1,\dots A_n)$, the conjugate problem becomes

$$\lambda^i_{1x} + \mu^i_{1y} = - \Sigma^n_{j=1} (\lambda^j_3 + \mu^j_2) \, \partial f_j/\partial z^i_1,$$

$$\mu^i_{2y} = -\lambda^i_1 - \Sigma^n_{j=1} (\lambda^j_3 + \mu^j_2) \, \partial f_j/\partial z^i_2,$$

$$\lambda^i_{3x} = -\mu^i_1 - \Sigma^n_{j=1} (\lambda^j_3 + \mu^j_2) \, \partial f_j/\partial z^i_3, \qquad i = 1,2...n \tag{4.3}$$

$$\lambda_1(a + h, y) = \mu_1(x, b + k) = 0;$$

$$\mu_2(x, b + k) = \lambda_3(a + h, y) = A/2 \tag{4.4}$$

In the present paper, we show first that the conjugate problem (4.3), (4.4) is equivalent to a system of two-dimensional Volterra-type linear integral equations of the type we have studied in a previous paper [6b]. The results obtained there will enable us to prove in this paper, the existence of multipliers λ_i, μ_i as solutions of the conjugate problem in a suitable class of functions, in $L_\infty(G)$ (not a Sobolev space).

In order to obtain the equivalent system of integral equations, we treat λ_{1x} as arbitrary and formally integrate both sides of (4.3) as follows (in conjunction with boundary conditions in (4.4)):

$$\lambda^i_1(x, y) = \int^x_{a+h} \lambda^i_{1x} (\alpha, y)d\alpha,$$

$$\mu^i_1(x, y) = -\int^y_{b+k} \lambda^i_{1x}(x, \beta) - \int^y_{b+k} w. \, (\partial f/\partial z^i_1)(x, \beta)d\beta,$$

$$\lambda^i_3(x, y) = 2^{-1}A_i + \int^x_{a+h} \int^y_{b+k} \lambda^i_{1x} (\alpha, \beta)d\alpha d\beta +$$

$$\int^x_{a+h} \int^y_{b+k}(w\cdot \, (\partial f/\partial z^i_1))(\alpha, \beta)d\alpha d\beta - \int^x_{a+h}(w\cdot(\partial f/\partial z^i_3))(\alpha, y)d\alpha,$$

$$\mu_2^i(x, y) = 2^{-1}A_i - \int_{b+k}^{y} \int_{a+h}^{x} \lambda_{1x}^i(\alpha, \beta)d\alpha d\beta$$

$$- \int_{b+k}^{y} (w \cdot (\partial f/\partial z_2^i))(x, \beta)d\beta, \qquad (4.5)$$

where w stands for $\lambda_3 + \mu_2$. It is clear that w satisfies the integral equation $w = Tw$ where

$$(Tw)^i(x, y) = A_i + \int_x^{a+h} \int_y^{b+k} w \cdot \partial f/\partial z_1^i + \int_y^{b+k} w \cdot \partial f/\partial z_2^i$$

$$+ \int_x^{a+h} w \cdot \partial f/\partial z_3^i \qquad (4.6)$$

(4.i) Theorem: If H_2, H_4, H_5 hold, if $v \in \Gamma$ and z is the corresponding solution of the Darboux problem (1.1), then there exist infinitely many sets of solutions λ_1, λ_3, μ_1, μ_2 in $(\lambda_\infty(G))^n$, with $\lambda_1(x, y)$, $\lambda_3(x, y)$, $(x, y) \in G$, AC with respect to x for almost all y, $\mu_1(x, y)$, $\mu_2(x, y)$, $(x, y) \in G$, AC with respect to y for almost all x, λ_1, λ_3, μ_1, μ_2 satisfying the boundary conditions (4.4), and having generalized partial derivatives, λ_{1x}, λ_{3x}, μ_{1y}, μ_{2y}.

Proof: As a consequence of (Theorem 3, no. 5; [6b]), it is seen that there is a unique $w \in [\lambda_\infty(G)]^n$ with $w = Tw$, where T is defined by (4.6). The conclusion of the theorem follows now by defining the functions λ_1, μ_1, λ_3, μ_2 as in (4.5), in terms of the unique solution w of (4.6), and an arbitrarily chosen $[\lambda_\infty(G)]^n$ - function λ_{1x}.

Remarks: (a) Since $w = \lambda_3 + \mu_2$ is uniquely determined as the fixed point of T, for different choices of λ_{1x}, we still get the same $\lambda_3 + \mu_2$.

(b) The solutions of (4.5) need not belong to a Sobolev class since, for example, $\partial f/\partial z_3^i$ and hence λ_3^i as given in (4.5) need not possess derivatives with respect to y. (See example 3, no. 7.)

(c) If f does not depend on (z_1, z_2, z_3), then by choosing $\lambda_{1x} = 0$, it is seen that a possible set of multipliers is given by the constant functions $\lambda_1 = 0 = \mu_1$; $\lambda_3 = \mu_2 = A/2$. If $\partial f/\partial z_r^i$, r = 1,2,3, are continuous in (x,y), as is the case when they depend on z only, then the multipliers can be chosen to be continuous. Finally, if f is linear in (z_1, z_2, z_3) with coefficients analytic in (x,y), then the multipliers can be chosen to be analytic in (x,y) (see [6b]).

5. THE INCREMENT FORMULA AND AN ERROR ESTIMATE

Let v_o and v_ϵ be any two elements of Γ, the set of control functions Let z and z_ϵ be the solutions of (1.1-2) corresponding to v_o and v_ϵ, respectively. Let $(\lambda, \mu) = (\lambda_1, \lambda_3, \mu_1, \mu_2)$ and $(\lambda_\epsilon, \mu_\epsilon) = (\lambda_{1\epsilon}, \lambda_{3\epsilon}, \mu_{1\epsilon}, \mu_{2\epsilon})$ be solutions of (4.3-4) corresponding to (v_o, z) and (v_ϵ, z_ϵ) respectively, $(\lambda_1, \lambda_3, \mu_1, \mu_2) \in [L_\infty(G)]^{4n}$ as in (4.1).

In the sequal, when there is no confusion, the symbol H(u) stands for the expression

$$H(u) = H(x, y, z(x, y), z(s, y), z_y(x, y), u, \lambda(x, y), \mu(x, y))$$

$$= H(x, y, z_1(x, y), z_2(x, y), z_3(x, y), u, \lambda(x, y), \mu(x, y).$$

where z, λ, μ are related to v_o and u denotes a point of U Also, for the sake of simplicity, we shall denote by $\dot{z}$ the expression

$$\dot{z} = (z(x, y), z_x(x, y), z_y(x, y)) = (z_1(x, y), z_2(x, y), z_3(x, y)).$$

In any case, we have $z_1 = z$, $z_2 = z_x$, $z_3 = z_y$.

In order to obtain a necessary condition of the Pontryagin type we express the increment $I[v_\epsilon] - I[v_o]$ in terms of the integral of $H(v_\epsilon(x, y))$ over G.

To this end, let us observe that, by simple calculations involving integration by parts of the expression

$$\int_a^{a+h} \int_b^{b+k} [\lambda_1(z_{1\epsilon x} - z_{1x}) + \mu_1(z_{1\epsilon y} - z_{1y}) + \lambda_3(z_{3\epsilon x} - z_{3x})$$

$$+ \mu_2(z_{2\epsilon y} - z_{2y})]dxdy$$

and the boundary conditions (1.2) and (4.5) we obtain

$$I[v_\epsilon] - I[v_o] = \eta + \iint_G [H(v_\epsilon(x, y)) - H(v_o(x, y))]dxdy,$$

where

$$\eta = \Sigma_{j=1}^{3} \iint_G [\partial H/\partial z_j(x, y, z_\Theta, v_\epsilon, \lambda, \mu) - \partial H/\partial z_j(x, y, z_1, v_o,$$

$$\lambda, \mu)](z_{j\epsilon} - z_j) \text{ dxdy}$$

and $z_{j\Theta}(x, y) = z_j(x, y) + \Theta(x, y)[z_{j\epsilon}(x, y) - z_j(x, y)]$, $0 \leq \Theta(x, y) \leq 1$. For details we refer to Cesari [2], or the author [6a].

Error Estimate: It is clear that if f_i (in the state equations (1.1)) are linear, i.e., of the form $Az + Bz_x + Cz_y + D(x, y, u)$ where A, B, C are matrix-valued functions on G, then η reduces to zero. For the nonlinear case, we shall now obtain an estimate on η, and for this purpose, we need the following hypotheses:

(H_6): There exists a function $M(x, y, u)$, $(x, y, u) \in G \times U$, with $M(x, y, v(x, y)) \in L_4(G)$ for any $v \in \Gamma$, such that, for $(x, y, z_1, z_2, z_3, u) \in G \times E^{3n} \times U$ and $1 \leq p < +\infty$, we have $|f(x, y, z_1, z_2, z_3, u)| \leq M(x, y, u) + B_3 [|z_1| + |z_2| + |z_3|]^{p/4}$ for some constant $B_3 \geq 0$. For $p = +\infty$, we require $|f| \leq M(x, y, u) + \phi(|z_1| + |z_2| + |z_3|)$ for some function $\phi(\xi) \geq 0$, $0 \leq \xi < +\infty$ with $\phi(\xi) \leq K\xi$ for some K.

(H_7): There exist functions $K'_{ij}(x, y, u)$, $i = 1,2,3$; $j = 1,\dots,n$, such that $K'_{ij}(x, y, v(x,y)) \in L_\infty(G)$ for any $v \in \Gamma$, and such that for $(x, y, u) \in G \times U$ and $z, \bar{z} \in E^{3n}$, $i = 1,2,3$, $j = 1,\dots,n$, we have,

$$|\partial f/\partial z_j^i (x, y, z_1, z_2, z_3, u) - \partial f/\partial z_j^i(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_3, u)| \leq K'_{ij}(x, y, u) \textstyle\sum_{s=1}^3 |z_s - \bar{z}_s| \qquad (5.2)$$

<u>Remark</u> 1: Let, as before, $s(x, y)$ denote $|f(x, y, \dot{z}(x, y), v_1(x, y)) - f(x, y, \dot{z}(x, y), v_2(x, y))|$ where $\dot{z} = (z, z_x, z_y)$ and z is the solution of (1.1, 1.2) corresponding to v_1 and $v_1, v_2 \in \Gamma$. Then it is seen from H_3, H_4 and H_5 that $s \in L_p(G)$; (p as in H_1). Indeed, $|f(x, y, \dot{z}(x, y), v_1(x, y))| \leq K|\dot{z}(x, y)| + |f(x, y, o, v_1(x, y))|$ where $K = \max \{\|K_{jr}(x, y, v_1(x,y))\|_\infty : j = 1, 2,3;\ r = 1,\dots n\}$; (see H_5). Since $|\dot{z}| \in L_p$ and $|f(x, y, o, v_1(x, y))| \in L_p$ (by H_3), it follows that $s \in L_p(G)$.

The assumption (H_6) is made only to guarantee that in addition, $s \in L_4(G)$. The same conclusion can be made under the following hypothesis:

(H'_6): There is a function $M(x, y)$ defined on G such that (i) $M(x, y)\, v\,(x, y) \in L_4(G)$ for every $v \in \Gamma$, and (ii) for $(x, y, z_1, z_2, z_3) \in G \times E^{3n}$

and $u_1, u_2 \in U$, we have $|f(x, y, z_1, z_2, z_3, u_1) - f(x, y, z_1, z_2, z_3, u_2)| \leq M(x, y) \mid u_1 - u_2 \mid$.

Indeed, $s(x, y) = |f(x, y, \dot{z}(x, y), v_1(x, y)) - f(x, y, \dot{z}(x, y), v_2(x, y))| \leq M(x, y)|v_1(x, y) - v_2(x, y| \leq M|v_1| + M|v_2|$ and $s \in L_4(G)$.

Further, if (H_6') and (H_7) hold with $M(x, y) \in L_\infty(G)$ and $\Gamma \subset [L_\infty(G)]^m$, then statement (3.5) in theorem (3.ii) can be replaced by

$$|z(x,y)| + |z_x(x,y)| + |z_y(x,y) \quad B_4\delta\|v_1 - v_2\|_\infty, \ (x,y) \in G, \tag{5.3}$$

where the constant B_4 depends only on $\|M\|_\infty$, h, k and on all K_{ij}, K'_{ij}.

We shall denote below by u an arbitrary fixed point $u \in U$. Let v_o be an element of Γ, let $z(x, y)$, $(x, y) \in G$, be the corresponding solution of the Darboux problem (1.1-2), and let $\dot{z}$ denote the 3n-vector function $\dot{z}(x, y) = (z, z_x, z_y) = (z_1, z_2, z_3)$ as above. Let $(\bar{x}, \bar{y})$ be an interior point of G. Let $\delta_o > 0$ be the minimum distance of $(\bar{x}, \bar{y})$ from the boundary of G. Let K/n denote the maximum of the 12 n numbers $\|K_{ij}(x, y, u)\|_\infty$, $\|K'_{ij}(x, y, u)\|_\infty$, $\|K_{ij}(x, y, v_o(x,y))\|_\infty$, $\|K'_{ij}(x, y, v_o(x,y))\|_\infty$, $i = 1,2,3$, $j = 1,\dots,n$; where the K_{ij} are as in (H_5) and the K'_{ij} as in (H_7). By (H_6) the function $s(x, y; u) = |f(x, y, \dot{z}(x, y), u) - f(x, y, \dot{z}(x, y), v_o(x, y))|$ belongs to $L_4(G)$. Thus, given $\xi > 0$ there is a $\delta > 0$ such that $\iint_C |s(x, y; u)|^2 dxdy \leq \xi^2$ and $\iint_C |s(x, y; u)|^4 dxdy \leq \xi^2$ for every measurable set $C \subset G$ of measure $\leq 4\delta^2$. We may well assume $0 < \delta \leq \delta_o$. Let S_δ denote the square $[\bar{x} - \delta \leq x \leq \bar{x} + \delta, \bar{y} - \delta \leq y \leq \bar{y} + \delta]$ and let C be any closed subset of S_δ. Let v_ϵ be the function defined by $v_\epsilon = v_o$ in G-C, and $v_\epsilon = u$ in C. Then the function $s(x, y) = |f(x, y, \dot{z}(x, y), v_\epsilon(x, y)) - f(x, y, \dot{z}(x, y), v_o(x, y))|$ is zero outside C and equals $s(x, y; u)$ in C. Note that $v_\epsilon \in \Gamma$, we denote by z_ϵ the solution of problem (1.1-2) relative to v_ϵ, and as usual we write $\dot{z}_\epsilon = (z_\epsilon, z_{\epsilon x}, z_{\epsilon y}) = (z_{1\epsilon}, z_{2\epsilon}, z_{3\epsilon})$. Inequalities (3.5) yield in this case

$$|z_{j\epsilon}(x, y) - z_j(x, y)| \leq B[\iint_{S_\delta} s(\alpha,\beta)d\alpha d\beta + \int_{\bar{y}-\delta}^{\bar{y}+\delta} s(x, \beta)d\beta + \int_{\bar{x}-\delta}^{\bar{x}+\delta} s(\alpha,y)d\alpha] \tag{5.4}$$

for all $(x, y) \in G$, $j = 1,2,3$, independently of the particular closed set $C \subset S_\delta$, and where B depends only on h, k and K. This inequality and the fact that $s \in L_4(G)$ under H_6 (or (H_6')) can now be used to estimate η.

The integrand in the expression for η can be written as

$$[\Sigma_{i=1}^n |\lambda_3^i + \mu_2^i||\partial f_i/\partial z_j(x, y, \dot{z}_\Theta, v_\epsilon) - \partial f_i/\partial z_j(x, y, \dot{z}, v_\epsilon)|$$

$$+ \Sigma_{i=1}^n |\lambda_3^i + \mu_2^i||\partial f_i/\partial z_j(x, y, \dot{z}, v_\epsilon) - \partial f_i/\partial z_j(x, y, \dot{z}, v_o)|] \cdot (z_{j\epsilon} - z_j)$$

which yields, using (H_7)

$$|\eta| \leq \|\lambda_3 + \mu_2\|_\infty \cdot K \cdot \iint_G \Sigma_{i=1}^3 \Sigma_{j=1}^3 |z_{i\Theta} - z_i|\ |z_{j\epsilon} - z_j|(x, y)dxdy$$
$$+ \|\lambda_3 + \mu_2\|_\infty \iint_G \Sigma_{j=1}^3 |z_{j\epsilon} - z_j||\partial f/\partial z_j(x, y, z, \dot{v}_\epsilon)$$
$$- \partial f/\partial z_j(x, y, z, v_o)|dxdy \tag{5.5}$$

But, since the integrand in the last term above is zero outside S_δ, we have $|\eta| \leq \|\lambda_3 + \mu_2\|_\infty \cdot K \cdot (\eta_1 + 2\eta_2)$ where

$$\eta_1 = \iint_G \Sigma_{i,j=1}^3 |z_{i\Theta} - z_i| \, |z_{j\epsilon} - z_j|(x, y)dxdy \text{ and}$$

$$\eta_2 = \iint_{S_\delta} \Sigma_{j=1}^3 |z_{j\epsilon} - z_j|(x, y)dxdy$$

To obtain an estimate on η, we observe that $|z_{i\Theta} - z_i| \leq |z_{i\epsilon} - z_i|$ and then, by (5.4), we get

$$|\eta_1| \leq \iint_G (\Sigma_{i=1}^3 |z_{i\epsilon} - z_i|(x, y))^2 \, dxdy$$

$$\leq 36 \, B^2 \iint_G [\iint_{S_\delta} s(\alpha,\beta)d\alpha d\beta + \int_{\underline{y}-\delta}^{\bar{y}+\delta} s(x,\beta)d\beta$$

$$+ \int_{\underline{x}-\delta}^{\bar{x}+\delta} s(\alpha,y)d\alpha]^2 dxdy$$

Using Hölders' inequality and the fact that $s \in L_4(G)$ under (H_6), it is seen that

$$|\eta| \leq M_2 \, \delta^2 \, [\iint_{S_\delta} s^2(\alpha,\beta)d\alpha d\beta + (\iint_{S_\delta} s^4(\alpha,\beta)d\alpha d\beta)^{1/2}] \tag{5.6}$$

for some positive constant M_2. Similarly, using (5.4) and Hölders' inequality, we get

$$|\eta_2| \leq \iint_{S_\delta} 3B[\iint_{S_\delta} s(\alpha,\beta)d\alpha d\beta + \int_{\underline{y}-\delta}^{\bar{y}+\delta} s(x,\beta)d\beta + \int_{\underline{x}-\delta}^{\bar{x}+\delta} s(\alpha,y)d\alpha]dxdy$$

$$\leq 3B(4\delta^2 + 2\delta + 2\delta)\iint_{S_\delta} s(\alpha,\beta)d\alpha d\beta$$

$$\leq 6B(4\delta + 4)\delta^2 \, (\iint_{S_\delta} s^2(\alpha,\beta)d\alpha d\beta)^{1/2} \tag{5.7}$$

Using (5.6) and (5.7), the inequality (5.5) can now be written as

$$|\eta| \leq M \, \delta^2[\iint_{S_\delta} s^2(\alpha,\beta)d\alpha d\beta + (\iint_{S_\delta} s^2(\alpha,\beta)d\alpha d\beta)^{1/2}$$

$$+ \left(\iint_{S_\delta} s^4(\alpha,\beta)\,d\alpha d\beta\right)^{1/2}\Big] \tag{5.8}$$

where M is a constant depending only on K, B, and $\|\lambda_3 + \mu_2\|_\infty$.

Given $\epsilon > 0$, let us now choose a positive number ζ with $0 < \zeta^2 \leq \zeta < \epsilon/6M$. Let $\delta > 0$ be now chosen as before with $\iint_{S_\delta} s^2 \leq \zeta^2$ and $\iint_{S_\delta} s^4 \leq \zeta^2$. Then $|\eta| \leq M\,\delta^2(\zeta^2 + \zeta + \zeta) \leq M\,\delta^2 \cdot 3 \cdot (\epsilon/6M) = \epsilon\,\delta^2/2$. In conclusion, if $v_0 \in \Gamma$, $u \in U$, and $\epsilon > 0$ are given, then there exists a $\delta > 0$ such that for a function $v_\epsilon = v_0$ outside S_δ and $v_\epsilon = u$ in a closed subset of S_δ

$$I(v_\epsilon) - I(v_0) = \eta + \iint_G [H(v_\epsilon(x,y)) - H(v_0(x,y))]dxdy \tag{5.9}$$

with $|\eta| \leq \delta^2\,\epsilon/2$.

6. A NECESSARY CONDITION FOR OPTIMALITY

In this section, we shall state and prove a necessary condition for optimality, analogous to the one-dimensional Pontryagin's necessary condition. We need the concept of "minimum condition" for the class of problems under consideration, and this is made precise in the following definition:

Definition 6.1: Let $v_0 \in \Gamma$. Then v_0 is said to satisfy the "minimum condition" if there is a set $B \subset G$ with meas B = meas G such that for $(x, y) \in B$, we have $H(v_0(x, y)) \leq H(u)$ for all $u \in U$. We recall that $H(u)$ stands for $H(x, y, z(x, y), z_x(x, y), z_y(x, y), u, \lambda(x, y), \mu(x, y))$ where z and $(\lambda, \mu) = (\lambda_1, \lambda_3, \mu_1, \mu_2)$ satisfy (1.1, 1.2) and (4.3, 4.4) respectively. The following hypothesis is needed in the proof of the necessary condition:

(H_8): The functions $f_i = f_i(x, y, z_1, z_2, z_3, u)$, $i = 1,\ldots,n$, are continuous on $G \times E^{3n} \times U$.

Remark: In the proof of the necessary condition, the inequality (3.1) of (H_5) is needed only for $(z_1, z_2, z_3) = (z(x, y), z_x(x, y), z_y(x, y))$ where z is an optimal trajectory. Further, the hypothesis (H_6) can be replaced by (H_6').

(6.i) Theorem: (Pontryagin-type necessary condition): Let $v_o \in \Gamma$ be optimal for I; i.e., $I(v_o) \leq I(v)$ for all $v \in \Gamma$. Let conditions H_1 - H_8 hold. Then, there exists a unique function $z \in [W_p^1(G)]^n$ satisfying the Darboux problem (1.1-2) and ∞ - many sets of multipliers $(\lambda_1, \lambda_3, \mu_1, \mu_2) \in (L_\infty(G))^{4n}$ satisfying (4.4-5) with v replaced by v_o. With this z and any of these sets of multipliers, the optimal control v_o necessarily satisfies the minimum condition.

Proof: The existence of z and of λ_1, λ_3, μ_1, μ_2 under the hypotheses H_1 - H_5 has been shown in no. 3 and no. 4. Before proving the necessary condition, let us note that throughout this proof, $z_1 = z$, $z_2 = z_x$, $z_3 = z_y$, $(\lambda,\mu) = (\lambda_1, \lambda_3, \mu_1, \mu_2)$ have the same meaning and they correspond to v_o. Further, as in no. 5, $H(u) = H(x, y, u) = H(x, y, z(x, y), z_x(x, y), z_y(x, y), u, \lambda(x, y), \mu(x, y))$.

For each natural number n, let C_n be a closed subset of G such that (i) meas $(C_n) > (1 - n^{-1})$ meas G, and (ii) on C_n the functions v_o, $\dot{z} = (z_1, z_2, z_3)$, $(\lambda, \mu) = (\lambda_1, \lambda_3, \mu_1, \mu_2)$ are all continuous. Let C_n' be the set of all points of density of C_n so that meas $C_n' =$ meas C_n and the functions v_o, z, λ, μ are continuous on C_n' with respect to itself. Now, for any $u \in U$, let $R(x, y; u) = H(x, y, v_o(x, y)) - H(x, y, u)$. Then, this function is continuous on C_n' for each n. Let $B = (\text{interior of } G) \cap (\bigcup_{n=1}^{\infty} C_n')$. Then meas $B \geq$ meas $C_n' > (1-n^{-1})$ meas G for all n and hence meas $B \geq$ meas G. Further, since $B \subset G$, it follows that meas B = meas G. We shall prove that B is the required set, i.e., for $(x,y) \in B$, we have $H(x, y, v_o(x,y)) \leq H(x, y, u)$ for all $u \in U$. Let (x_o, y_o) be an arbitrary point of B. Then there exists an N such that $(x_o, y_o) \in C_N'$. Now, let us choose δ_1, δ_2, δ_3, δ_4 as follows:

(i) Since $(x_o, y_o) \in C_N'$, it is a point of density for C_N' and hence there is a $\delta_1 > 0$ such that $0 < \delta < \delta_1$ implies meas $(C_N' \cap S_\delta(x_o, y_o)) > \frac{1}{2}$ meas $(S_\delta(x_o, y_o))$ where, as before, $S_\delta(x_o, y_o)$ is a square of side length 2δ with center at (x_o, y_o).

(ii) Let us suppose that the minimum condition does not hold at (x_o, y_o). Then, there is a $u \in U$ with $\epsilon = R(x_o, y_o; u) > 0$. Using the continuity of $R(x, y; u)$ we obtain a $\delta_2 > 0$ such that $|R(x, y; u) - R(x_o, y_o; u)| < \epsilon/2$ whenever $(x, y) \in C_N$ and $|(x, y) - (x_o, y_o)| < 2\,\delta_2$. Thus, $R(x, y; u) > \epsilon/2$ for all $(x, y) \in C_N \cap S_{\delta_2}(x_o, y_o)$.

(iii) The function $s(x, y; u) = |f(x, y, \dot{z}(x, y), u) - f(x, y, \dot{z}(x, y), v_o(x, y))|$, with u as in (ii), belongs to $L_4(G)$ (by (H_6)); and hence there exists a $\delta_3 > 0$ such that for $0 < \delta < \delta_3$, we have $\iint_{S_\delta} s^2(u, \alpha, \beta)d\alpha d\beta < \zeta^2$ and $\iint_{S_\delta} s^4(u, \alpha, \beta)d\alpha d\beta < \zeta^2$ show ζ is some number with $0 < \zeta^2 \leq \zeta \leq \epsilon/6M$

(ϵ as in (ii) and M as in the equality (5.6)).

(iv) Since (x_o, y_o) is in the interior of G there is a $\delta_4 > 0$ such that $S_\delta = S_\delta(x_o, y_o) \subset G$ for $0 < \delta < \delta_4$.

Let $\sigma > 0$ be such that $\sigma < \min(\delta_1, \delta_2, \delta_3, \delta_o)$ and let v_ϵ be a function defined by $v_\epsilon(x, y) = u$ if $(x, y) \in C_N \cap S_\sigma$ and $= v_o(x, y)$ otherwise. Clearly, v_ϵ is an element of Γ. Also, $R(x, y; v_\epsilon(x, y))$ is zero outside $C_N \cap S_\sigma$ and is $> \epsilon/2$ for all $(x, y) \in C_N \cap S_\sigma$. Thus

$$
\begin{aligned}
I[v_\epsilon] - I[v_o] &= \eta + \iint_G [H(v_\epsilon(x, y)) - H(v_o(x, y))]dxdy \\
&= \eta - \iint_{C_N \cap S_\sigma} R(x, y; v_\epsilon(x, y))dxdy \\
&< \eta - 2^{-1}\epsilon \text{ meas } (C_N \cap S_\sigma) < \eta - 4^{-1}\epsilon \text{ meas } S_\sigma \\
&= \eta - \epsilon\sigma^2
\end{aligned}
$$

where $|\eta| < \epsilon\sigma^2/2$ from no. 5. Thus, $I[v_\epsilon] - I[v_o] < -\epsilon\sigma^2/2 < 0$. This is contrary to the assumption that v_o is optimal. The contradiction arose because of (ii). It follows that for any $(x, y) \in B$, we have $H(x, y, v_o(x, y)) \leq H(x, y, u)$ for all $u \in U$. This concludes the proof of the theorem.

7. DISCUSSION AND EXAMPLES

In this section, we shall discuss the Pontryagin-type necessary condition given in theorem (6.i) in relation to the results of Cesari [2] and A. I. Egorov [3 a]. We shall first show that our results yield those of A. I. Egorov under conditions of smoothness. We shall also give examples where our necessary condition applies. In particular, example 3 of (D) below will show that our results are actually more general than those of A. I. Egorov.

A. The Linear Case

If the state equations are linear, i.e., of the form $z_{xy} = Az + Bz_x + Cz_y + D(x, y, u)$ where A, B, C are matrix-valued functions on G, then we have seen that the increment formula reduces to $I(v_\epsilon) - I(v_o) = \iint_G H(v_\epsilon(x, y)) - H(v_o(x, y))dxdy$. Now, if a control $v_o \in \Gamma$ satisfies the minimum condition, then in particular $H(v_o(x, y)) \leq H(v_\epsilon(x, y))$ a.e. in G and hence $I(v_o) \leq I(v_\epsilon)$ for all v_ϵ in Γ; i.e., v_o is optimal for I. Thus, the necessary condition is also sufficient in the linear case. For the existence of solutions for the Goursat problem (1.1, 1.2) [as well as the conjugate problem (4.3, 4.4)] in this case, we may require that the matrix-valued functions A(x, y), B(x, y), C(x, y) be in $L_\infty(G)$ and that D(x, y, u) be continuous in u. Further, we shall require D(x, y, v(x, y)) to be in $[L_p(G)]^n$ for $v \in \Gamma$.

B. Various Types of Cost Functional

B1. It is clear that the cost functional (1.4) or $I[z,v] = \sum_{i=1}^n A_i z^i(a+h, b+k)$ can be written in the Lagrange form

$$J[z , v] = \iint_G f_o(x, y, z(x, y), z_x(x, y), z_y(x, y), v(x, y))dxdy$$

with $f_o = \sum_{i=1}^n A_i f_i$ and the f_i as in (1.1).

However, the Lagrange problem of the minimum of J[z, v] with f_o not necessarily equal to $\sum A_i f_i$, and z, v satisfying (1.1-3) can always be formulated as the Mayer problem (1.1-4) by suitable transformations. This is done, as usual, by introducing a new variable z^o with $z^o_{xy} = f_o(x, y, z, z_x, z_y, v)$,

$z^o(a, y) = 0$, $z^o(x, b) = 0$. Then, the functional J can be written in the form $J = z^o(a + h, b + k)$ (cfr. Cesari [2] or the author [6 a]).

B2. The general Mayer problem with cost functional $I'[z, v] = \phi(z(a + h, b + k))$ with z, v satisfying (1.1-3) and an arbitrary $\phi(\zeta)$, $\zeta \in E_n$ (twice continuously differentiable) can be reduced to problem (1.1-4). As above, this is usually done by introducing a new variable z^o satisfying

$$z^o_{xy} = \Sigma_{i,j=1}^n (\partial^2\phi/\partial z^i \partial z^j) z^i_x z^i_y + \Sigma_{i=1}^n (\partial\phi/\partial z^i) f_i(x, y, z, z_x, z_y, v)$$

and $$z^o(a, y) = \phi(\psi^1(y), \ldots \psi^n(y)); \quad z^o(x, b) = \phi(\varphi^1(x), \ldots \varphi^n(x))$$

where $\varphi(x) = (\varphi^1, \ldots, \varphi^n)$, $\psi(y) = (\psi^1, \ldots, \psi^n)$, are the initial data as in (1.2). (See A. I. Egorov [3 a] and the author [6 a]).

B3. The problem of minimum of $J[z,v] = \int_a^{a+h} F(x, z(x, b+k), z_x(x, b+k))dx$ can be reduced to that of $z^o(a+h, b+k)$ where z^o is defined by $z^o_{xy} = \Sigma_{i=1}^n [(\partial F/\partial z^i) z^i_y + (\partial F/\partial z^i_x) f_i(x, y, z, z_x, z_y, v)]$ and $z^o(a,y) = 0$; $z^o(x,b) = \int_a^x F(\alpha, \varphi(\alpha), \varphi'(\alpha))d\alpha$ (see A. I. Egorov [3a]).

B4. Let J denote any linear combination of the functionals mentioned above in B1, B2, B3. It is clear that the functional J can be reduced to the form (1.4) by suitable addition of an auxiliary variable z^o.

C. Comparison With A. I. Egorov's Results

The optimization problem (1.1-4) was studied by A. I. Egorov [3 a] where he proposed a necessary condition in terms of the Hamiltonian

$$H(x, y, z, z_x, z_y, v, \Theta) = \Sigma_{i=1}^n \Theta^i f_i(x, y, z, z_x, z_y, v)$$

and multipliers $\Theta(x, y) = (\Theta^1,\ldots,\Theta^n)$ satisfying the Goursat-type problem

$$\Theta^i_{xy} = \partial H/\partial z^i - (\partial/\partial x)(\partial H/\partial z^i_x)) - (\partial/\partial y)(\partial H/\partial z^i_y), \tag{7.1}$$

$(x, y) \in G$, $i = 1,\ldots,n$,

with boundary conditions

$$\Theta^i_x = -(\partial H/\partial z^i_y) \text{ for } y = b+k, \quad \Theta^i_y = -(\partial H/\partial z^i_x) \text{ for } x = a+h, \tag{7.2}$$

$$\Theta^i(a+h, b+k) = A_i, \quad i = 1,\ldots,n. \tag{7.3}$$

In (7.1) the derivatives are evaluated at (z, z_x, z_y, v, Θ) and the numbers A_i in (7.3) are those in (1.4).

In [3 a] the control variables v_i are assumed to be piecewise continuous. In the present paper the controls v_i are assumed to be only measurable, and we proved, under our general assumption (H_1) - (H_5) that the functions z^i belong to a Sobolev class $W^1_p(G)$ and are continuous in G. In any case, the derivatives $(\partial/\partial x)(\partial H/\partial z^i_x)$, $(\partial/\partial y)(\partial H/\partial z^i_y)$ which appear in (7.1) need not in general exist, as example 3 below in D will show.

On the other hand, under suitable regularity conditions, equations (7.1-3) can be derived from the conjugate problem (4.3-4), by defining $\Theta = (\Theta^1,\ldots,\Theta^n)$ in terms of the multipliers λ_1, λ_3, μ_1, μ_2.

We need the following assumptions:

*For a given optimal pair (z, v) and corresponding multipliers λ_1, λ_3, μ_1, μ_2, let us assume that the following partial derivatives exist as generalized derivatives

$$(\partial/\partial x)(\Sigma_{j=1}^{n}(\lambda_3^j + \mu_2^j)(\partial f_j/\partial z_x^i)), \quad (\partial/\partial y)(\Sigma_{j=1}^{n} (\lambda_3^j + \mu_2^j)(\partial f_i/\partial z_y^i),$$

$i = 1,\dots,n$

From (4.5) and (*) it follows that λ_{3xy}^i and μ_{2xy}^i, $i = 1,\dots,n$, exist as generalized derivatives, and in view of (4.3) we have

$$\lambda_{3xy}^i = -\mu_{1y}^i - (\partial/\partial y)(\Sigma_{j=1}^{n}(\lambda_3^j + \mu_2^j)(\partial f_j/\partial z_y^i),$$

$$\mu_{2xy}^i = -\lambda_{1x}^i - (\partial/\partial x)(\Sigma_{j=1}^{n}(\lambda_3^j + \mu_2^j)(\partial f_i/\partial z_x^i),$$

a.e., in G, $i = 1,\dots,n$. Thus, if $\Theta = \lambda_3 + \mu_2$, that is, $\Theta^i = \lambda_3^i + \mu_2^i$, $i = 1,\dots,n$, then Θ_{xy} exists a.e. in G as a generalized derivative and, in view of (4.3) we get (7.1) with $H + \Sigma_i \Theta^i f_i$. Again, from (4.5) we get

$$\Theta_x^i = \lambda_{3x}^i = -\mu_1^i - (\partial H/\partial z_y^i) = -(\partial H/\partial z_y^i) \text{ for } y = b+k, \tag{7.4}$$

and analogously

$$\Theta_y^i = -(\partial H/\partial z_x^i) \text{ for } x = a+h,\ i = 1,\dots,n. \tag{7.5}$$

Finally,

$$\Theta(a+h,\ b+k) = (\lambda_3 + \mu_2)(a+h,\ b+k) = A.$$

Thus, under assumption (*), the sums $\Theta^i = \lambda_3^i + \mu_2^i$, $i = 1,\dots,n$, act as multipliers Θ^i described by (7.1-2). It is of interest to note that $\Theta = \lambda_3 + \mu_2$ is obtained as the fixed point of the contraction operator T (see remark (a) in no. 4), and thus Θ is unique, in harmony with the uniqueness of Egorov's solution Θ of (7.1-2).

D. Examples

(1) ([3 a], p. 560). Let $G = \{(x, y) | 0 \leq x \leq 1;\ 0 \leq y \leq 1\}$ and consider the problem of minimum of $S = \int_0^1 \int_0^1 (1 - 2y)z(x, y)dxdy$ with side condition

$$z_{xy} = -2z - 2z_x - 2z_y - v,$$

boundary conditions $z(x, o) = z(o, y) = 0$; here z is a scalar and v is a control variable with values [0, 1]. To obtain the multipliers (and use theorem (6.i)), we first introduce a new set of variables z_i, $i=1,\ldots,6$ defined by $z_1 = z$; $z_2 = z_x$; $z_3 = z_y$; $z_4 = \int_0^x \int_0^y (1 - 2\beta)z_1(\alpha, \beta)d\alpha d\beta$; $z_5 = z_{4x}$; $z_6 = z_{4y}$. Then S attains minimum together with $z_4(1, 1)$ and the side conditions are now

$$z_{2y} = z_{3x} = z_{1xy} = -2z_1 - 2z_2 - z_3 - v;\ z_{5y} = z_{6x} = z_{4xy} = (1 - 2y)z_1.$$

The corresponding conjugate problem is

$$\lambda_{1x} + \mu_{1y} = -\partial H/\partial z_1 = 2(\lambda_3 + \mu_2) - (1 - 2y)(\lambda_6 + \mu_5);$$

$$\mu_{2y} = -\partial H/\partial z_2 = -\lambda_1 + 2(\lambda_3 + \mu_2);$$

$$\lambda_{3x} = -\partial H/\partial z_3 = -\mu_1 + (\lambda_3 + \mu_2);$$

$$\lambda_{4x} + \mu_{4y} = -\partial H/\partial z_4 = 0;$$

$$\mu_{5y} = -\partial H/\partial z_5 = -\lambda_4;\quad \lambda_{6x} = -\partial H/\partial z_6 = -\mu_4;$$

with boundary conditions, $\lambda_i(1, y) = 0 = \mu_j(x, 1)$ for $i \neq 6$ and $j \neq 5$; $\lambda_6(1, y) = \mu_5(x, 1) = 1/2$. Here $H = \lambda_1 z_2 + \mu_1 z_3 + (\lambda_3 + \mu_2)(-2z_1 - 2z_2 - z_3 - v) + \lambda_4 z_5 + \mu_4 z_6 + (\lambda_6 + \mu_5)(1-2y)z_1$. It is seen from the boundary conditions that one can take $\lambda_4 = \mu_4 = 0$; $\mu_5 = \lambda_6 = 1/2$ on G. Further, if $\Theta = \mu_2 + \lambda_3$ and $\zeta = \Theta_x - \Theta$, then by formal differentiation of the above equations, we get $\Theta_{xy} = \Theta_y + 2(\Theta_x - \Theta) + (1 - 2y)$ or $\zeta_y = 2\zeta + 1 - 2y$; $\zeta(x, 1) = 0$. Solving for ζ as a function of y, we get $\zeta(x, y) = y - e^{2(y-1)}$. Thus, $\Theta_x - \Theta = y - e^{2(y-1)}$. Solving for Θ as a function of x, we get $\Theta(x, y) = (e^{x-1}-1)(y - e^{2(y-1)})$. It is clear that for $(x, y) \in G$, $e^{x-1} \leq 1$ and hence $\Theta(x, y) \geq 0$ or ≤ 0 according as $y \leq y_o$ or $y \geq y_o$ where $y_o = e^{2y_o-2}$. But then, H as a function of v is minimum for $v_o(x, y)$ where v_o is a function on G defined by: $v_o(x, y) = 1$ for $y \leq y_o$; $= -1$ for $y \geq y_o$. Now, to obtain the value of the functional, we first solve the following for z: $z_{xy} = -2z - 2z_x - z_y - v_o$ with $z(x, o) = z(o, y) = 0$. It is seen that z is given by

$$z(x, y) = 2^{-1}(1 - e^{-x})(e^{-2y} - 1) \text{ for } y \leq y_o$$

$$= 2^{-1}(1 - e^{-x})(1 + e^{-2y} - 2e^{2y_o - 2y}) \text{ for } y \geq y_o,$$

and the functional takes the value

$$S = e^{-1}(y_o^2 + \frac{1}{2} e^{-2} - y_o) \text{ where } y_o = e^{2y_o-2}.$$

This is the optimum value obtained in [3a] also.

(2) ([3a], p. 561): To find the minimum of the functional $S = \int_0^1 z(x,1)dx - \int_0^1 z(1, y)dy$ where the side conditions on z are given by

$$z_{xy} = v; \; |v| \leq 1, \; 0 \leq x \leq 1; \; 0 \leq y \leq 1; \; z(x,o) = z(o, y) = 0 \quad (7.6)$$

If z_o is a variable satisfying the relations

$$z_{oxy} = z_y - z_x \text{ and } z_o(o, y) = z_o(x, o) = 0 \quad (7.7)$$

then the above optimization problem reduces to the problem of minimum of $z_o(1,1)$ with (7.6, 7.7) as side conditions. The conjugate problem is described in terms of the multipliers λ_1, μ_1, λ_3, μ_2, λ_4, μ_4, λ_6, μ_5; $\lambda_{1x} + \mu_{1y} = 0$; $\mu_{2y} = -\lambda_1 + (\lambda_6 + \mu_5)$; $\lambda_{3x} = -\mu_1 - (\lambda_6 + \mu_5)$; $\lambda_{4x} + \mu_{4y} = 0$; $\mu_{5y} = -\lambda_4$; $\lambda_{6x} = -\mu_4$; with boundary conditions $\lambda_i(1, y) = 0$; $\mu_j(x, 1) = 0$ for $i \neq 6$, and $j \neq 5$; $\lambda_6(1, y) = 1/2$; $\mu_5(x, 1) = 1/2$. In order to obtain a set of solutions, we may introduce the auxiliary equations $\lambda_1 = 0$ and $\lambda_4 = 0$ on G. Then, we obtain $\mu_1 = 0$; $\mu_4 = 0$; $\mu_5 = 1/2$ and $\lambda_6 = 1/2$ on G. Also, $\mu_2 = (y - 1)$ and $\lambda_3 = -(x - 1)$. Thus, the Hamiltonian reduces to $H = (y - x)v + (z_y - z_x)$. It follows that as a function of v alone H is minimum at $v_o(x, y)$ where v_o is defined on G as follows:

$$v_o(x, y) = -1 \text{ for } 0 \leq x < y \leq 1; \; = 1 \text{ for } 0 \leq y < x \leq 1.$$

Substituting in (7.6), and integrating we

$$z(x, y) = -xy + \phi(x) \text{ for } 0 \leq x < y \leq 1; \; = xy + \psi(y) \text{ for } 0 \leq y < x \leq 1 \quad (7.8)$$

where ϕ and ψ are absolutely continuous functions defined on [0, 1] with $\phi(0) = \psi(0) = 0$. Now ϕ and ψ are to be chosen so that the two expressions of (7.8) coincide for $x = y$. Thus $\phi(y) - y^2 = y^2 + \psi(y)$, i.e., $\psi(y) = \phi(y) - 2y^2$.

Hence $z(x, y) = -xy + \phi(x)$ for $0 \leq x < y \leq 1$; $= xy - 2y^2 + \phi(y)$ for $0 \leq y < x \leq 1$; where ϕ is some arbitrary absolutely continuous function defined on [0, 1] with $\phi(0) = 0$. The corresponding value of the functional is

$$S = \int_0^1 z(x, 1)dx - \int_0^1 z(1, y)dy = \int_0^1 [\varphi(x) - x]dx - \int_0^1 [\varphi(y) + y - 2y^2]dy = -1/3$$

This again is in harmony with the optimum obtained in [3 a].

(3) ([6 a], p. 207). Let G be the rectangle [0, 1] x [0, 1]. Let us consider the problem of minimum of the functional $S = z(1, 1)$ with side considerations and constraints

$$z_{xy} = (1 + z_x)v;\ z(x, o) = 0,\ z(o, y) = 0;\ -1 \leq v \leq 1 \qquad (7.9)$$

Let us first observe that for any $v(x, y)$ in $L_1(G)$, the solution of (7.9) is given by

$$z(x, y) = \int_0^x [-1 + \exp(\int_0^y v(\alpha, \beta)d\beta)]d\alpha \qquad (7.10)$$

Now, since $v(\alpha,\beta) \geq -1$ for all (α,β), $\int_0^1 v(\alpha,\beta)d\beta \geq -1$ and hence $\int_0^1 \exp(\int_0^1 v(\alpha,\beta)d\beta)d\alpha \geq 1/e$. Thus, $S = z(1, 1) \geq -1 + (1/e)$ for any admissible pair (z, v) satisfying (7.9). It follows that the function $v_o(x, y)$ defined by $v_o(x, y) = -1$ for almost all $(x,y) \in G$, is optimal for S, and vice versa.

In order to verify that v_o satisfies the minimum condition, we formulate the conjugate problem $\lambda_{1x} + \mu_{1y} = -\partial H/\partial z_1 = 0$; $\mu_{2y} = -\partial H/\partial z_2 = -\lambda_1 - v(\lambda_3 + \mu_2)$; $\lambda_{3x} = -\partial H/\partial z_3 = -\mu_1$; with boundary conditions, $\lambda_1(1, y) = \mu_1(x, 1) = 0$;

$\lambda_3(1, y) = \mu_2(x, 1) = 1/2$. Here the Hamiltonian H is given by $H = \lambda_1 z_2 + \mu_1 z_3 + (\lambda_3 + \mu_2)f$ where $z_1 = z$; $z_2 = z_x$; $z_3 = z_y$; $f = (1+z_x)v$. The multipliers corresponding to v_o are obtained as solutions of the above system of equations with v replaced by $v_o(x, y)$. Clearly, we may chose $\lambda_1 = \mu_1 = 0$; and $\lambda_3 = 1/2$ on G. But then $\mu_{2y} = -v_o(\mu_2 + 2^{-1})$; $\mu_2(x, 1) = 1/2$. A solution of this equation is given by $\mu_2(x, y) = \exp(\int_y^1 v_o(x, \beta)d\beta) - (1/2)$. Substituting in the Hamiltonian, we get $H(x, y, z, u, \lambda, \mu) = u(1 + z_x) \exp(\int_y^1 v_o(x, \beta)d\beta)$ where z corresponds to v_o. Now, if $v_o(x, y) = -1$ on G, then $H = u(1 + z_x) \exp(y - 1) = u(\exp \int_o^y v_o(x, \beta)d\beta) \exp(y - 1) = u/e$. Clearly, $H(u) \geq H(v_o(x, y))$ for $(x, y) \in G$ and $u \in U = [-1, 1]$.

<u>Remarks:</u> It is to be observed that in the above example, the optimal solution $v_o(x, y) = -1$ happens to be smooth, and as mentioned in no. 7 (c), Egorov's condition also holds. However, this example (3) can be easily modified into another one for which Egorov's necessary condition cannot be applied. Indeed, if $w(x)$, $0 \leq x \leq 1$ is a fixed continuous, positive nowhere differentiable function (such a function exists), we consider instead of (7.9) the equation $z_{xy} = (1 + z_x) v \cdot w$ with the same boundary conditions and constraints as above. Then (7.10) is replaced by

$$z(x, y) = \int_o^x [-1 + \exp(w(\alpha) \int_o^y v(\alpha, \beta)d\beta)]d\alpha$$

and the optimal control is still $v_o(x, y) = -1$ a.e. in G. Here, Egorov's Hamiltonian H [3a] is given by $\Theta(1 + z_x)vw$ and the second order derivative $(H_{z_x})_x$ required in (7.1) does not exist.

Also of interest, in the above example is the fact that the multiplier $\mu_2(x, y)$ need not have partial derivative with respect to x. Thus, in general, the multipliers need not have partial derivatives with respect to both the variables; as such they may not belong to a Sobolev class.

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