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Technical Report

GENERALIZED PRANDTL-MEYER FLOW

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1. INTRODUCTION

The class of fluid motions to be studied here is characterized by possessing one family of ∞^1 plane characteristic surfaces tangent to a cylindrical surface. Our considerations will be limited to the case where the bicharacteristics and their orthogonal trajectories are two families of parallel straight lines in each characteristic plane. An additional assumption is also made that the stagnation enthalpy and the specific entropy are both constant.

The well-known two-dimensional Prandtl-Meyer flows[1] are flows that possess straight line bicharacteristics. We shall show that the class of fluid motions under consideration are a generalization of the Prandtl-Meyer flows in the sense that: (1) the present flows are space flows; (2) the angle $\overline{\phi}$, which the bicharacteristics make with the z-axis varies from one characteristic plane to another (in the Prandtl-Meyer flows, $\overline{\phi} = \frac{\pi}{2}$); and (3) the flows are irrotational.

2. BASIC EQUATIONS

The characteristic relations for the steady, supersonic, three-dimensional flows of a polytropic gas in intrinsic form have already been derived by Coburn.[2] Using his notation, let x^j, $j = 1, 2, 3$, denote a Cartesian orthogonal coordinate system in Euclidean three space and let:

γ = ratio of specific heats

c = local speed of sound

q^2 = square of the magnitude of the velocity vector

b^2 = $q^2 - c^2$

t_k = unit vector along a bicharacteristic

n_k = unit vector orthogonal to one family of characteristic surfaces

p_k = unit vector orthogonal to both t_k and n_k such that p_k, t_k, and n_k form an orthogonal right-hand ordered triple at each point

w_j = $p_k \dfrac{\partial p_j}{\partial x^k}$ curvature vector of the p_j congruence of curves

m_j = $t_k \dfrac{\partial t_j}{\partial x^k}$ curvature vector of the t_j congruence of curves

u_j = $n_k \dfrac{\partial n_j}{\partial x^k}$ curvature vector of the n_j congruence of curves

S_{jk} = second fundamental tensor of the ∞^1 characteristic surfaces orthogonal to n_j

M^* = $g^{jk}S_{jk}$ mean curvature of the characteristic surfaces

K = $n^j p^k \left(\dfrac{\partial t_k}{\partial x^j} - \dfrac{\partial t_j}{\partial x^k}\right)$ the curvature term and $\dfrac{\partial}{\partial t}$, $\dfrac{\partial}{\partial n}$, $\dfrac{\partial}{\partial p}$ are directional derivatives along t_j, n_j, p_j directions, respectively.

Because the characteristic surfaces are planes, the second fundamental tensor of any one of these planes is $S_{ij} = 0$; also, the mean curvature of these planes, M^*, vanishes. Since we are assuming the stagnation enthalpy and the specific entropy are both constant, the basic equations become:

$$b^2 \frac{\partial}{\partial t} \frac{q^2}{b} = c^2 b \, t_k w^k \tag{2.1}$$

$$b[(\gamma - 3)q^2 + 4c^2] \frac{\partial b}{\partial n} = - cb[(\gamma - 3)q^2 + 4c^2]\, t_k u^k - (\gamma - 1)b^3 c t_k w^k \quad (2.2)$$

$$q \frac{\partial q}{\partial p} = (b^2 m_k + c^2 u_k)p^k + cbK \quad (2.3)$$

3. GEOMETRY

In order to reduce the basic equations to a workable form, we must first consider the geometry of the problem. The terms that are required to reduce the basic equations will be obtained in this section. To do this we employ vector analysis;[3] accordingly, vector notation shall be used throughout this section.

The characteristic planes are the tangent planes, D, to an arbitray cylindrical surface E. We assume E to be generated by lines parallel to the z-axis. The traces of D in the x-y plane are lines, L, tangent to the curve C_o, which is the trace of E in the x-y plane. Let C^* be any curve orthogonal to the lines L, and let P denote the point of intersection of C^* and a line L in question. Now if $\vec{\rho}$ is the vector from the origin to the point P, and θ is the angle between the vector $\vec{\rho}$ and x-axis, then $\vec{\rho} = \vec{\rho}(\theta)$. Again, let $\vec{s}_1 = \vec{s}_1(\theta)$ denote a unit vector along line L and let s be measured from point P along line L; then any point in the characteristic plane is given by the position vector $\vec{R}$:

$$\vec{R} = \vec{\rho}(\theta) + s\,\vec{s}_1(\theta) + z\vec{k} \quad , \quad (3.1)$$

where $\vec{k}$ is a unit vector in the z-direction.

In any characteristic plane we introduce two families of parametric curves that are parallel straight lines: α = variable along $\vec{t}$ and β = variable along $\vec{p}$; α is measured along $\vec{t}$ from the line parallel to this direction which passes through the point P; β is measured along $\vec{p}$ from the line parallel to this direction and passes through point P. Arc length will be considered positive when measured along $\vec{t}$, $\vec{p}$ respectively. Thus

$$\alpha\vec{t} + \beta\vec{p} = s\,\vec{s}_1(\theta) + z\vec{k} \quad ,$$

and the position vector $\vec{R}$ becomes

$$\vec{R} = \vec{\rho}(\theta) + \alpha\vec{t} + \beta\vec{p} \quad . \tag{3.2}$$

If we select the three families of curves consisting of α = variable, β = variable, and θ = variable as our new coordinate curves, we need to prove that these three families of curves form an <u>orthogonal</u> coordinate system. Since $\vec{t}$ is perpendicular to $\vec{p}$, the lines α = variable are perpendicular to the lines β = variable. It remains to be shown that the curves θ = variable are orthogonal to both $\vec{t}$ and $\vec{p}$. We also note that these last curves, θ = variable, lie in the $\vec{n}$ direction. The ordered triad $\vec{k}$, $\vec{s}_1$, $\vec{n}$, are assumed to form a right-hand system.

Differentiating (3.1) with respect to θ, we have

$$\frac{\partial\vec{R}}{\partial\theta} = \frac{d\vec{\rho}}{d\theta} + s\,\frac{d\vec{s}_1}{d\theta} \tag{3.3}$$

Taking the scalar product of (3.3) with $\vec{s}_1$, we get

$$\frac{\partial \vec{R}}{\partial \Theta} \cdot \vec{s}_1 = \frac{d\vec{\rho}}{d\Theta} \cdot \vec{s}_1 + s \frac{d\vec{s}_1}{d\Theta} \cdot \vec{s}_1 \tag{3.4}$$

Since $\vec{s}_1$ is a unit vector, it follows that $d\vec{s}_1/d\Theta \cdot \vec{s}_1 = 0$. Also $\vec{s}_1$ is orthogonal to C^* whose tangent vector is $d\vec{\rho}/d\Theta$, hence $d\vec{\rho}/d\Theta \cdot \vec{s}_1 = 0$. We note that the terms in the right-hand side of Eq. (3.4) all vanish; therefore

$$\frac{\partial \vec{R}}{\partial \Theta} \cdot \vec{s}_1 = 0 \tag{3.5}$$

By taking the scalar product of (3.3) with $\vec{k}$, we find

$$\frac{\partial \vec{R}}{\partial \Theta} \cdot \vec{k} = \frac{d\vec{\rho}}{d\Theta} \cdot \vec{k} + s \frac{d\vec{s}_1}{d\Theta} \cdot \vec{k} \tag{3.6}$$

Since the vectors $\vec{s}_1$ and $\vec{k}$ are mutually orthogonal, it follows that $d\vec{s}_1/d\Theta \cdot \vec{k} = 0$. Also $d\vec{\rho}/d\Theta$ is a vector in the x-y plane and therefore it is orthogonal to $\vec{k}$. Hence $d\vec{\rho}/d\Theta \cdot \vec{k} = 0$. Again we note that the terms in the right-hand side of Eq. (3.6) vanish, or

$$\frac{\partial \vec{R}}{\partial \Theta} \cdot \vec{k} = 0 \tag{3.7}$$

Equation (3.5) states that the curves Θ = variable are orthogonal to $\vec{s}_1$, and Eq. (3.7) states that they are orthogonal to $\vec{k}$. Therefore we conclude that the curves Θ = variable are perpendicular to the characteristic planes and hence orthogonal to both $\vec{t}$ and $\vec{p}$.

The arc length in our new orthogonal coordinate system is

$$ds^2 = (Ad\alpha)^2 + (Bd\beta)^2 + (Cd\Theta)^2 \tag{3.8}$$

where A, B, C are the metric coefficients. From (3.2), it follows that $\partial\vec{R}/\partial\alpha = \vec{t}$, $\partial\vec{R}/\partial\beta = \vec{p}$. Therefore

$$A^2 = \frac{\partial\vec{R}}{\partial\alpha}\cdot\frac{\partial\vec{R}}{\partial\alpha} = 1 \qquad (3.9)$$

$$B^2 = \frac{\partial\vec{R}}{\partial\beta}\cdot\frac{\partial\vec{R}}{\partial\beta} = 1 \qquad (3.10)$$

From (3.3), we find

$$C^2 = \frac{\partial\vec{R}}{\partial\theta}\cdot\frac{\partial\vec{R}}{\partial\theta}$$

$$= \left(\frac{d\vec{\rho}}{d\theta} + s\,\frac{d\vec{s}_1}{d\theta}\right)\cdot\left(\frac{d\vec{\rho}}{d\theta} + s\,\frac{d\vec{s}_1}{d\theta}\right) \qquad (3.11)$$

But $\vec{s}_1$ lies along L, therefore $d\vec{s}_1/d\theta$ lies along the normal to the characteristic planes: $d\vec{s}_1/d\theta = \vec{n}$. Similarly, the tangent vector to C^* lies along $\vec{n}$, or

$$\frac{d\vec{\rho}}{d\theta} = \lambda\vec{n} \;,\quad \lambda = ds^*/d\theta \qquad (3.12a)$$

where λ is a known scalar function of θ to be determined from the curve C^* and $s^*$is the arc length parameter of C^*. From (3.11), finally we have

$$C^2 = (\lambda + s)^2 \qquad (3.12b)$$

The curvature vectors $\vec{w}$ and $\vec{m}$ of the $\vec{p}$, $\vec{t}$ congruence of curves are both zero since we are assuming that the families of curves α = variable, β = variable are parallel straight lines:

$$\vec{w} = \vec{m} = 0 \qquad (3.13)$$

The curvature vector $\vec{u}$ of the $\vec{n}$ congruence of curves is found by use of Frenet formula for curves:

$$\vec{u} = \frac{d\vec{n}}{ds} = -\frac{d\theta}{ds}\vec{s}_1 \qquad (3.14)$$

From (3.8), the arc length element along α = constant and β = constant is $Cd\theta = ds$ and $d\vec{n}/d\theta = -\vec{s}_1$. It follows that

$$\vec{u} = -\frac{1}{\lambda + s}\vec{s}_1 \qquad (3.15)$$

To find the projections of $\vec{u}$ onto $\vec{p}$, $\vec{t}$, we denote by ϕ the angle made by the unit vector $\vec{t}$ with line L. It follows that $\bar{\phi} = \frac{\pi}{2} - \phi$ and

$$\vec{t} = \cos\phi\,\vec{s}_1 + \sin\phi\,\vec{k} \qquad (3.16)$$

$$\vec{p} = -\sin\phi\,\vec{s}_1 + \cos\phi\,\vec{k} \qquad (3.17)$$

Taking the scalar product of (3.15) and (3.16), (3.15) and (3.17), we find

$$\vec{u}\cdot\vec{t} = -\frac{\cos\phi}{\lambda + s} \qquad (3.18)$$

$$\vec{u}\cdot\vec{p} = \frac{\sin\phi}{\lambda + s} \qquad (3.19)$$

The curvature term K of (2.3) is obtained by use of (3.10).

$$K = \frac{1}{C}\frac{\partial\vec{t}}{\partial\theta}\circ\vec{p} - \frac{\partial\vec{t}}{\partial\beta}\cdot\vec{n}$$

Since $\vec{t}$ varies only with θ

$$K = \frac{1}{C} \frac{\partial \vec{t}}{\partial \theta} \cdot \vec{p} \qquad (3.20)$$

Differentiating (3.16) with respect to θ and using the fact that $ds_1/d\theta$ is $\vec{n}$, we find that

$$\frac{\partial \vec{t}}{\partial \theta} = \cos \phi \, \vec{n} - \sin \phi \frac{d\phi}{d\theta} \vec{s}_1 + \cos \phi \frac{d\phi}{d\theta} \vec{k} \qquad (3.21)$$

Taking the scalar product of (3.21) with $\vec{p}$, we find by use of (3.17) that

$$\frac{\partial \vec{t}}{\partial \theta} \cdot \vec{p} = \frac{d\phi}{d\theta} \qquad (3.22)$$

Therefore from (3.20), (3.12) we find that

$$K = \frac{1}{\lambda + s} \frac{d\phi}{d\theta} \qquad (3.23)$$

4. THE FLOW QUANTITIES

We are now in a position to obtain the basic quantities b, ϕ, and the vorticity. From (3.9), (3.10), (3.12), we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} , \quad \frac{\partial}{\partial p} = \frac{\partial}{\partial \beta} , \quad \frac{\partial}{\partial n} = \frac{1}{\lambda + s} \frac{\partial}{\partial \theta} \qquad (4.1)$$

By use of (4.1) and (3.13), we find that (2.1) becomes

$$b^2 \frac{\partial}{\partial \alpha} \frac{q^2}{b} = 0 \qquad (4.2)$$

By use of (4.1), (3.13), (3.18), we find that (2.2) reduces to

$$\frac{\partial b}{\partial \theta} = c \cos \phi \tag{4.3}$$

Finally, by use of (4.1), (3.13), (3.19), (3.23), we find that (2.3) reduces to

$$q \frac{\partial q}{\partial \beta} = \frac{c}{\lambda + s} \left(c \sin \phi + b \frac{d\phi}{d\theta} \right) \tag{4.4}$$

Equations (4.2), (4.3), and (4.4) are the basic equations resulting from the assumptions that have been made. If we assume q to be only a function of θ, then Eq. (4.2) is automatically satisfied, and Eqs. (4.3) and (4.4) become:

$$\frac{db}{d\theta} = c \cos \phi \tag{4.5}$$

$$\frac{d\phi}{d\theta} = \frac{-c}{b} \sin \phi \tag{4.6}$$

Note that <u>if $\phi = 0$, then (4.5) is the Prandtl-Meyer equation for plane flows with straight line bicharacteristics.</u>[4] From (4.5) and (4.6) we have

$$-\frac{db}{b} = \cot \phi \, d\phi \tag{4.7}$$

Integrating (4.7), we obtain

$$b = \frac{C_1}{\sin \phi} , \tag{4.8}$$

where C_1 is an arbitrary constant of integration. Hence

$$\cos \phi = [1 - (C_1/b)^2]^{1/2} \tag{4.9}$$

From the Bernoulli relation for a polytropic gas,[4] we have

$$C_*^2 - c^2 = \mu^2 b^2 \quad (4.10)$$

where μ^2 is γ-1/γ+1 and C_* is the critical speed of sound. Inserting (4.9) and (4.10) into Eq. (4.5), we find:

$$2b \frac{db}{d\theta} = 2[- \mu^2 b^4 + (C_*^2 + \mu^2 C_1^2) b^2 - C_1^2 C_*^2]^{1/2} \quad (4.11)$$

Upon integrating, we find:

$$b^2 = \frac{1}{2\mu^2} [(C_*^2 - \mu^2 C_1^2) \sin 2\mu(\theta - \theta_*) + (C_*^2 + \mu^2 C_1^2)] \quad (4.12)$$

Equation (4.12) gives us an explicit relation between b and θ and hence q and θ, and θ_* is an arbitrary constant of integration. If we substitute (4.12) into (4.8), the angle ϕ can be determined explicitly as a function of θ.

We shall now discuss the streamlines. From the known theory of characteristic manifolds,[5] the velocity vector is

$$v^j = cn^j + bt^j \quad (4.13)$$

In terms of our coordinate unit vectors, it is [see (3.16)]

$$\vec{v} = c\vec{n} + b(\cos \phi \, \vec{s}_1 + \sin \phi \, \vec{k}) \quad (4.14)$$

By differentiating (3.1) with respect to time and using (3.12a), (3.12b), and the fact that $d\vec{s}_1/d\theta$ is $\vec{n}$, the velocity vector is

$$\vec{v} = \frac{ds}{dt}\vec{s}_1 + (\lambda + s)\frac{d\theta}{dt}\vec{n} + \frac{dz}{dt}\vec{k} \tag{4.15}$$

Equating the components of Eqs. (4.14) and (4.15), it follows that

$$\begin{aligned} \frac{ds}{dt} &= b \cos \phi \\ (\lambda + s)\frac{d\theta}{dt} &= c \\ \frac{dz}{dt} &= b \sin \phi \end{aligned} \tag{4.16}$$

From which, we find that the differential equations of the streamlines are

$$\frac{ds}{(\lambda + s)\, d\theta} = \cos \phi \,\frac{b}{c} \tag{4.17}$$

$$\frac{ds}{dz} = \cot \phi \tag{4.18}$$

Once the curve C^* is known, the differential equations for the streamlines (4.17) and (4.18) can be integrated and the streamlines can be determined with the aid of (4.6), (4.12), (3.12a).

To obtain the vorticity vector, we calculate its projections on t_j, n_j, p_j. They are:[6]

$$\omega^j t_j = cp^k u_k - \frac{\partial c}{\partial p} + bK \tag{4.19}$$

$$\omega^j n_j = \frac{\partial b}{\partial p} - b\, p^k m_k \tag{4.20}$$

$$\omega^j p_j = \frac{\partial c}{\partial t} - \frac{\partial b}{\partial n} - c\, t_k u^k \tag{4.21}$$

By using the fact that $q = q(\Theta)$ and (3.13), (3.15), (3.17), (3.18), (3.19), (3.23), (4.5), (4.6), we find that the components of the vorticity vector all vanish. Hence

$$\omega^j = 0$$

The flows under consideration are irrotational.

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REFERENCES

1. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, N. Y., p. 267 (1948).

2. N. Coburn, "Intrinsic Form of the Characteristic Relations in the Steady Supersonic Flow of a Compressible Fluid," Quarterly of Applied Mathematics, 3, 237-248 (1957).

3. N. Coburn, Vector and Tensor Analysis, MacMillan Co., N. Y., 1955.

4. Courant and Friedrichs, op. cit., p. 265.

5. N. Coburn and C. L. Dolph, "The Methods of Characteristics in the Three-Dimensional Stationary Flow of a Compressible Gas," Proc. of the First Symposium of Appl. Math., 1947, Am. Math. Soc., 55-66 (1949).

6. Coburn, "Intrinsic Form of the Characteristic Relations in the Steady Supersonic Flow of a Compressible Fluid," loc. cit., 240.