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DYNAMIC RESPONSE OF A THIN-WALLED CYLINDRICAL TUBE UNDER INTERNAL MOVING PRESSURE

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TABLE OF CONTENTS

Page

TABLE OF CONTENTS (CONT'D)

TABLE OF CONTENTS (CONT'D)

LIST OF FIGURES

LIST OF FIGURES (CONT'D)

LIST OF FIGURES (CONT'D)

NOMENCLATURE

A, A_1, A_2	dimensionless amplitudes, (amplitude)/h
B_1, B_2	constants of integration
C	dimensionless constant proportional to the coefficient of the viscous damping
C_1, C_2	paths of integration
D	flexural modulus of a plate, $\frac{Eh^3}{12(1-\nu^2)}$
F, F_1, F_2	functions
G	shear modulus
$H_i (i=1,\ldots,8)$	functions
I_1, I_b, I_s	improper integrals
$I_2, I_{b1}, I_{b2}, I_{s1}, I_{s2}$	improper integrals
L	dimensionless length of the finite tube, $\frac{\sqrt{12}\,l}{h}$
M	dimensionless bending moment, $\frac{1-\nu^2}{Eh^2} M_{xx}$
M_{xx}	transverse bending moment per unit length of undeformed mean circumference
N	dimensionless wave number, (wave number) x $h/\sqrt{12}$, or parameter in the Fourier transform
N_{xx}	resultant normal force per unit length of undeformed mean circumference on the plane perpendicular to the axis of the tube
$N_{\theta\theta}$	resultant normal force per unit length of the undeformed tube on the radial plane
P	dimensionless internal pressure, $\frac{p}{12\kappa G}$
Q	dimensionless resultant shearing force, $\frac{Q_x}{\sqrt{12}h\kappa G}$
Q_x	resultant shearing force per unit length of undeformed mean circumference on the plane perpendicular to the axis of the tube in the radial direction
R	mean radius of the tube

S	dimensionless distance from the pressure front (distance from the pressure front) x $\sqrt{12}/h$
T, T_o	dimensionless time coordinates, $\frac{\sqrt{12}\ v_d t}{h}$
$\Delta T, \Delta T'$	dimensionless time increments, $\frac{\sqrt{12}\ v_d \Delta t}{h}$
V	dimensionless phase velocity, or dimensionless velocity of the pressure front, v/v_d
V_{c0}	dimensionless first critical velocity, $V_{c0} = \sqrt{2\delta g}$, from elementary theory, V_{c0} defined on p. 40 from more exact theory
V_{c1}	dimensionless modified shear wave velocity in a plate, $\delta = v_s/v_d$, or dimensionless second critical velocity
V_{c2}	dimensionless dilatational wave velocity in a plate, 1, or dimensionless third critical velocity
W	total dimensionless deflection, w/h
W_b	dimensionless deflection due to bending, w_b/h
W_s	dimensionless deflection due to shear, w_s/h
W_o	dimensionless maximum radial static deflection for $P=1$, $1/g^2$
W_{T_o}	dimensionless radial deflection at T_o, (radial deflection)/h
$\overline{W}, \overline{W}_s, \overline{W}_b$	functions transformed from W, W_s, W_b, respectively
X	dimensionless coordinate along the axis of a tube, $\frac{\sqrt{12}x}{h}$
Y	symbol for unknown value
a	radius of the circular contour in contour integration on the complex plan
$e_{\alpha\beta}$	strain component of β-plane in α-direction
$f_i (i=1,\ldots,6)$	functions
g	dimensionless parameter depending on the ratio of the thickness to the mean radius of a tube, $g^2 = \frac{E}{12\kappa G}(\frac{h}{R})^2$

h	thickness of the tube wall
k	slope of the characteristic lines, $\frac{dT}{dX}$
l	length of the finite tube
m	imaginary part of the dimensionless wave number, N, or imaginary wave number (spatial attenuation number)
n	real part of the dimensionless wave number N, or real wave number
p	internal pressure per unit area
r	radial coordinate
t	time coordinate
u	displacement of a point on the middle surface of a cylinder in x-direction
u_x	displacement in x-direction
u_r	displacement in radial direction
u_Θ	displacement in Θ-direction
u_z	displacement in z-direction
v	velocity of the moving pressure front
v_1	velocity in radial direction
$\bar{v}$	dimensionless velocity in radial direction, $\frac{v_1}{\sqrt{12}\, v_d}$
v_d	dilatational wave velocity in a plate, $\sqrt{\frac{E}{\rho(1-\nu^2)}}$
v_s	modified shear wave velocity in a plate, $\sqrt{\frac{\kappa G}{\rho}}$
w, w_b, w_s	total deflection, deflection due to bending, and deflection due to shear in radial direction along the tube respectively
x	coordinate along the axis of a tube
y, y_1	positive constants
z	local Cartesian coordinate in radial direction

$\gamma_{\alpha\beta}$	shearing strain on β-plane in α-direction in engineering notation
δ	dimensionless parameter, ratio of the modified shear wave velocity to the dilatational wave velocity in a plate, $\delta^2 = \frac{(1-\nu^2)\kappa G}{E}$
Θ	angular coordinate
κ	shear correction factor
λ	dimensionless wave number due to static pressure, $\sqrt{\frac{\delta g}{2}}$
ν	Poisson's ratio
ρ	mass of the tube per unit volume
$\tau_{\alpha\beta}$	stress on β-plane in α-direction
$\bar{\tau}_{\alpha\beta}(\alpha\neq\beta)$	average sharing stress on β-plane in α-direction
ψ_x	angular rotation of the radial line
Ω	dimensionless frequency (frequency) x $h/\sqrt{12}\ v_d$, or parameter in the Fourier transform
Ω^*	maximum dimensionless frequency with complex wave number
ω	angular velocity of the radial line
$\bar{\omega}$	dimensionless angular velocity of the radial line, $\frac{h}{12\ v_d}\ \omega$

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ABSTRACT

The dynamic response of a thin-walled cylindrical tube under internal moving pressure is analysed. Two kinds of approximate equations of motion are used, corresponding to the elementary Euler-Bernoulli theory and the more exact Timoshenko theory of beam vibration. The work contains three major parts which are as follows:

1. Steady state wave motions in the tube wall. Frequency spectra (frequency as a function of wave number) and velocity spectra (phase velocity as a function of wave number) are plotted based upon two kinds of approximate equations of motion. The frequency spectra are useful for studying the transient response of the problem; and the velocity spectra are useful for the steady state response, since in steady state response the velocity of the moving pressure front is identical to the phase velocity of the wave propagation in the tube wall.

2. A study of the steady state response for a tube with infinite length under moving pressure. This is analysed by means of the Fourier transform. In the case of the pressure front moving with velocity greater than critical, the solution is obtained by introducing a viscous damping term in the equation of motion and setting it equal to zero in the limit.

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2. A study of the steady state response for a tube with infinite length under moving pressure. This is analysed by means of the Fourier transform. In the case of the pressure front moving with velocity greater than critical, the solution is obtained by introducing a viscous damping term in the equation of motion and setting it equal to zero in the limit.

3. <u>A study of the transient response for a tube with semi-infinite length under moving pressure.</u> This is analysed by means of the Fourier sine transform in the case of the pressure front moving with velocity less than critical. For over critical velocity, the transient response based upon the equations of motion from the more exact theory is analysed numerically by means of the method of characteristics. Numerical results for different velocities of the moving pressure front are computed.

INTRODUCTION

It is known that the dynamic effect on the stresses in the tube wall is very large when a pressure front with high velocity moves down a tube. Static analysis of the stresses in the tube wall is valid only when the pressure front moves with low velocity. If the velocity is supersonic, dynamic analysis of stresses has to be employed. For instance, when a shock wave is transmitted in a tube, static analysis of stresses in the tube wall will be inaccurate. Shock tubes (i.e., tubes in which shock waves are generated) have been widely used in testing models. A method for analysis of the dynamic stresses in the tube wall will be presented. Exact solutions based upon actual properties of the pressure front and complete three-dimensional theory of elasticity are too complicated to admit analytical treatment. Idealized forcing function due to pressure front and approximate equations of motion must be employed. These simplifications are adequate for the purpose of calculating the stresses in the tube wall.

The pressure front is assumed to be moving with constant velocity parallel to the axis of the tube and the intensity of the pressure is assumed to be uniform. Approximate equations for axially symmetrical motion of a thin-walled cylindrical tube due to Lin and Morgan,[1] and Herrmann and Mirsky[2] are valid for a wide range of frequencies, particularly for higher frequencies. These equations involve terms due to rotatory inertia and shear deformation that are significant for higher frequencies. For low frequencies, neglect of these terms will not cause any serious error. Equations omitting these terms are due to Love.[3]

The difference between equations containing these terms and equations omitting these terms is analogous to the difference between equations of transverse vibration of a beam by Timoshenko theory[4] and by Euler-Bernoulli theory. For higher frequencies, the equation based on the Timoshenko theory should be used.

In this work, the two approximate equations are employed to analyse the stresses in the tube wall. Further simplifications are introduced, so that additional assumptions have to be made, i.e., the inertia force parallel to the axis of the tube is neglected and the resultant longitudinal stress across the thickness of the wall on the plane perpendicular to the axis of the tube vanishes. These simplifications make sense if the strain energy due to radial motion is large compared with that due to axial motion. This work presents the case in which radial motion predominates. After the simplifications mentioned above, these two kinds of approximate equations become identical to the equations of Euler-Bernoulli beam and Timoshenko beam both on an elastic foundation.

Euler-Bernoullie beams on an elastic foundation under a concentrated force moving with constant horizontal velocity have been investigated by many authors. In the investigation of dynamic stress in rails under the wheel of a locomotive, Timoshenko[5,6] formulated this problem. Using Fourier series to solve the problem of a beam with finite length, he found that the dynamic effects was insignificant because the horizontal velocity of the wheel was small compared with so-called critical velocity which depended on the flexure rigidity of the beam as well as the foundation stiffness. Ludwig[7] solved a similar problem, but the beam was infinite in length and the force moved with velocity either less, equal, or greater than the critical. His interest was in the steady state response, so he assumed that the moving force had already acted on the beam for a long time. Mathews[8] used the Fourier transform to solve the rail problem for the steady state response, but his moving force was such that the magnitude of the force varied sinusoidally in time. Dörr[9] formulated a problem -- a semi-infinite Euler-Bernoulli beam on an elastic foundation with one end simply supported, under a concentrated force moving from that end with constant horizontal velocity. He used the Laplace transform to solve this problem. He calculated the inverse transform by the asymptotic method, so the duration of the moving force on the semi-infinite beam had to be infinite and this then was a steady state solution too. This steady state solution with the velocity of the moving force greater than the critical is physically meaningful. He also derived a formula in terms of Fourier integrals for a moving force with velocity less than the critical suddenly applied at the middle of an infinite beam. For a moving force with velocity greater than the critical, he used a power series to solve the transient problem, but no numerical results were given. The effect of viscous damping on this problem was first studied by Kenney.[10] In the case of velocity greater than the critical and with no damping, he took as the physically meaningful solution that which was approached in the limit by a system whose damping approached zero. Crandall[11] used this idea to solve a Timoshenko beam on an elastic foundation under a concentrated force moving with constant horizontal velocity. In a paper -- Transmission of Shock Waves in Thin-Walled Cylindrical Tubes -- Niordson,[12] using the equation identical to that of Euler-Bernoulli beam on an elastic foundation, found the radial deflection of the wall for the steady state response under a moving pressure front.

Supplementing Niordson's contribution, the present work gives a solution based on equations taking into consideration the rotatory inertia and shear deformation under the same forcing function as Niordson's. The Fourier transform is used. Both the forcing function and the method of solution are different from those of Crandall. In this work the transient response of a semi-infinite tube under an internal pressure front moving with constant velocity parallel to the axis of the tube is investigated here for the first time. The result obtained is the main contribution of this thesis. The standard method of solution is to use the Laplace transform, but the inverse integral involves too many branch points to be dealt with. The Fourier sine transform is used to solve the transient problem when the velocity of the pressure front is less than the critical. A numerical method to solve finite difference equations is used when the velocity is greater than the critical, since the Fourier sine transform fails in that case.

For investigation of vibrations, first of all, the wave propagation in the tube wall has to be understood. The frequency spectrum: wave number (complex, real, and imaginary) versus real frequency of the radial vibration of the wall, and the velocity spectrum: wave number versus real phase velocity of the wave propagation are plotted. A minimum phase velocity under which a wave with a real wave number can be propagated along the wall is found. This phase velocity is identically the same as the critical velocity defined by Timoshenko, Mathews, Ludwig, Dorr and Kenney for Euler-Bernoulli beam and by Crandall for Timoshenko beam.

I. EQUATIONS OF MOTION OF A THIN-WALLED CYLINDRICAL TUBE

A. Assumptions and Approximations

1. All assumptions in the linear theory of elasticity are employed.
2. Thickness h of the tube wall is small compared with the mean radius R of the tube. h is also small compared with the wave length of the disturbance.
3. Motion of the tube wall is axially symmetric so that no displacement occurs along the circumference. Displacements, strains, and stresses in the wall are independent of the angular coordinate.
4. Radial lines remain straight after deformation.
5. Displacement of the tube wall in the radial direction is uniform through the thickness of the wall.
6. Inertia force in axial direction is neglected,[12] and the resultant longitudinal stress across the thickness of the wall on the plane perpendicular to the axis of the tube is set equal to zero.[13]
7. In the elementary theory, terms due to shear deformation and rotatory inertia are neglected.

B. Basic Equations of Motion for Free Vibration in Cylindrical Coordinates

Let (r, θ, x) be the cylindrical coordinates of a point within the tube wall, u_r, u_θ, u_x be the corresponding displacement components.

Let the stress tensor be

$$\left\|\tau_{\alpha\beta}\right\| = \left\|\begin{matrix} \tau_{rr} & \tau_{r\theta} & \tau_{rx} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta x} \\ \tau_{xr} & \tau_{x\theta} & \tau_{xx} \end{matrix}\right\| \tag{1.1}$$

where $\tau_{r\Theta} = \tau_{\Theta r}$, $\tau_{rx} = \tau_{xr}$, and $\tau_{\Theta x} = \tau_{x\Theta}$

Equations of motion[14] without body force in r, Θ, and x directions are

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rx}}{\partial x} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2} \tag{1.2}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta x}}{\partial z} + \frac{2}{r}\tau_{r\theta} = \rho \frac{\partial^2 u_\theta}{\partial t^2} \tag{1.3}$$

$$\frac{\partial \tau_{rx}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta x}}{\partial \theta} + \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{r}\tau_{rx} = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{1.4}$$

From the assumption, $\tau_{r\Theta} = \tau_{\Theta x} = 0$, $\frac{\partial \tau_{\Theta\Theta}}{\partial \Theta} = 0$, and $u_\Theta = 0$, Equation (1.3) is automatically satisfied. Equations (1.2) and (1.4) can be simplified to

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rx}}{\partial x} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2} \tag{1.5}$$

$$\frac{\partial \tau_{rx}}{\partial r} + \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{r}\tau_{rx} = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{1.6}$$

C. Approximate Equations of Motion for Free Vibration

1. More Exact Theory Corresponding to Timoshenko Theory in Beam Vibration (Shear and rotatory inertia terms included)

Let $r = R + z$ and $\tau_{rr} = \tau_{zz}$, $\tau_{rx} = \tau_{zx}$, $u_r = u_z$, then the equations of motion become

$$\frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\tau_{zz} - \tau_{\theta\theta}}{R+z} = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{1.7}$$

$$\frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\tau_{zx}}{R+z} = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{1.8}$$

Following Herrmann and Mirsky,[2] we assume that

$$u_x(x, z, t) = u(x,t) - z\,\psi_x(x,t) \tag{1.9}$$

$$u_z(x, z, t) = w(x,t) \tag{1.10}$$

From Equations (1.9) and (1.10) strain components can be computed as

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial \psi_x}{\partial x} \tag{1.11}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{w}{R+z} \tag{1.12}$$

$$\gamma_{zx} = \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} = -\psi_x + \frac{\partial w}{\partial x} \tag{1.13}$$

and all other strain components are zero. With Lloyd,[15] let w be separated into two parts

$$w(x,t) = w_b(x,t) + w_s(x,t) \tag{1.14}$$

where w_b is due to bending and w_s is due to shear.

Then Equation (1.13) becomes

$$\gamma_{zx} = -\psi_x(x,t) + \frac{\partial w_b(x,t)}{\partial x} + \frac{\partial w_s(x,t)}{\partial x}$$

put

$$-\psi_x(x,t) + \frac{\partial w_b(x,t)}{\partial x} = 0$$

or

$$\psi_x(x,t) = \frac{\partial w_b(x,t)}{\partial x} \tag{1.15}$$

Non-zero strain components can be written as

$$e_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} \tag{1.16}$$

$$e_{\theta\theta} = \frac{w_b + w_s}{R + z} \tag{1.17}$$

$$\gamma_{zx} = \frac{\partial w_s}{\partial x} \tag{1.18}$$

By Hooke's law, the non-zero stress components can be computed as

$$\tau_{xx} = \frac{E}{1-\nu^2}[e_{xx} + \nu e_{\theta\theta}]$$

$$= \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} + \nu\frac{w_b + w_s}{R + z}\right] \tag{1.19}$$

$$\tau_{\theta\theta} = \frac{E}{1-\nu^2}[e_{\theta\theta} + \nu e_{xx}]$$

$$= \frac{E}{1-\nu^2}\left[\frac{w_b + w_s}{R + z} + \nu\frac{\partial u}{\partial x} - \nu z\frac{\partial^2 w_b}{\partial x^2}\right] \tag{1.20}$$

$$\bar{\tau}_{zx} = \kappa G \gamma_{zx} = \kappa G \frac{\partial w_s}{\partial x} \tag{1.21}$$

where $\bar{\tau}_{zx}$ is the average shearing stress through the thickness of the tube wall, κ is the shear correction factor which depends upon Poisson's ratio(2) and is approximately equal to that of the rectangular beam.

Let N_{xx} be the resultant normal force per unit length of undeformed mean circumference in the x direction, $N_{\theta\theta}$ be the resultant normal force per unit length of the undeformed tube in the θ direction, Q_x the resultant shearing force per unit length of undeformed mean circumference in the radial direction, and M_{xx} is the transverse bending moment per unit length of undeformed mean circumference. Moment and resultant forces are shown in Figures 1.1 and 1.2. They can be expressed in terms of displacement components and their derivatives as follows

$$N_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx} \left(1 + \frac{z}{R}\right) dz$$

$$\approx \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx}\, dz \approx \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu \frac{w_b + w_s}{R}\right]$$

From the assumption $N_{xx} = 0$, it follows that

$$\frac{\partial u}{\partial x} = -\nu \frac{w_b + w_s}{R} \tag{1.22}$$

$$N_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{\theta\theta}\, dz \approx \frac{Eh}{1-\nu^2}\left[\frac{w_b + w_s}{R} + \nu \frac{\partial u}{\partial x}\right] \tag{1.23}$$

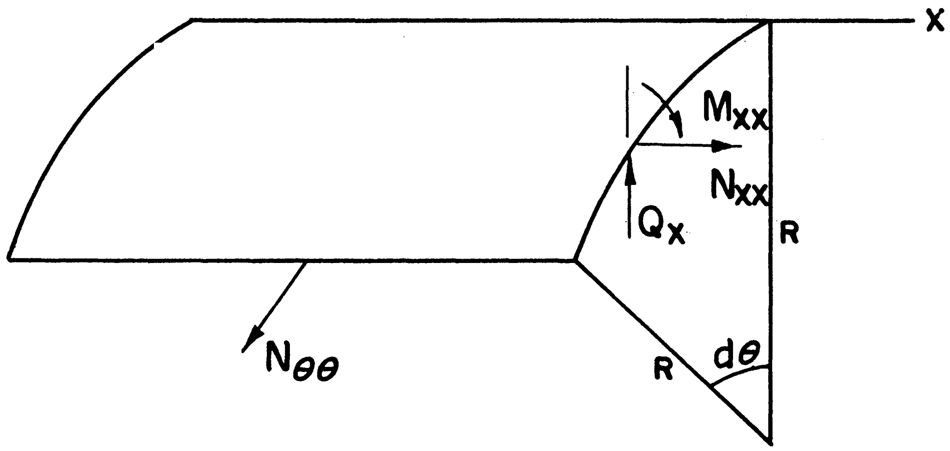

Figure 1.1 Resultant Forces and Moment Acting on an Element.

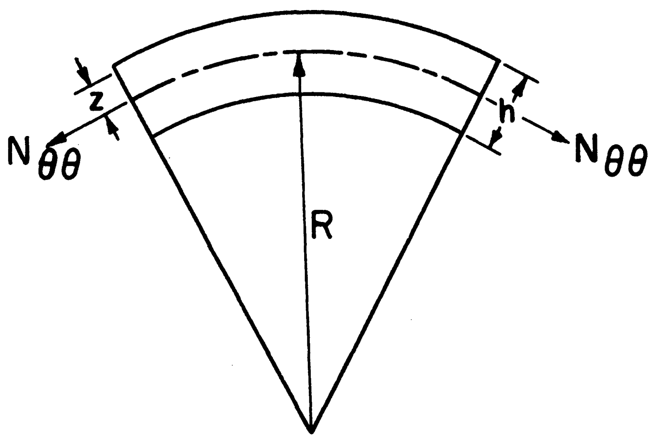

Figure 1.2 Cross Section of an Element.

Using Equation (1.22), $N_{\theta\theta}$ can be expressed in terms of w_b and w_s as follows:

$$N_{\theta\theta} = \frac{Eh}{R}(w_b + w_s) \tag{1.24}$$

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx}\left(1 + \frac{z}{R}\right) dz$$

$$\approx \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx}\, dz$$

$$= h\bar{\tau}_{zx}$$

$$= hkG\frac{\partial w_s}{\partial x} \tag{1.25}$$

$$M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx}\left(1 + \frac{z}{R}\right) z\, dz$$

$$\approx \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx}\, z\, dz$$

$$= -D\frac{\partial^2 w_b}{\partial x^2} \tag{1.26}$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$

Since for thin-walled tube $1 \gg \left|\frac{z}{R}\right|$, where $-\frac{h}{2} \leq z \leq \frac{h}{2}$.

By Equations (1.24), (1.25) and (1.26), the approximate equations of motion can be expressed in terms of w_b, w_s, and their derivatives. Multiplying both sides of Equation (1.7) by $(R + z)$ and integrating with respect to z over the range $-h/2$ to $h/2$, gives the equation of translation in the radial direction:

$$\frac{\partial Q_x}{\partial x} - \frac{N_{\theta\theta}}{R} = \rho h \frac{\partial^2 (w_b + w_s)}{\partial t^2} \tag{1.27}$$

Multiplying both sides of Equation (1.8) by $z(z + R)$ and integrating with respect to z over the range $-h/2$ to $h/2$, gives the equation of rotation:

$$\frac{\partial M_{xx}}{\partial x} - Q_x = -\frac{\rho h^3}{12} \frac{\partial^3 w_b}{\partial x \partial t^2} \tag{1.28}$$

In terms of w_s and w_b only, Equations (1.27) and (1.28) become

$$h \kappa G \frac{\partial^2 w_s}{\partial x^2} - \frac{Eh}{R^2}(w_b + w_s) = \rho h \frac{\partial^2 (w_b + w_s)}{\partial t^2}$$

or

$$\kappa G \frac{\partial^2 w_s}{\partial x^2} - \frac{E}{R^2}(w_b + w_s) = \rho \frac{\partial^2 (w_b + w_s)}{\partial t^2} \tag{1.29}$$

$$-D \frac{\partial^3 w_b}{\partial x^3} - h \kappa G \frac{\partial w_s}{\partial x} = -\frac{\rho h^3}{12} \frac{\partial^3 w_b}{\partial x \partial t^2}$$

or

$$D \frac{\partial^2 w_b}{\partial x^2} + h \kappa G w_s = \frac{\rho h^3}{12} \frac{\partial^2 w_b}{\partial t^2} \tag{1.30}$$

2. Elementary Theory Corresponding to Euler-Bernoulli Theory in Beam Vibration

If the rotatory inertia is neglected, Equation (1.28) becomes

$$\frac{\partial M_{xx}}{\partial x} - Q_x = 0$$

or

$$Q_x = \frac{\partial M_{xx}}{\partial x} \qquad (1.31)$$

Combining Equations (1.27) and (1.31), and setting $w_s = 0$, then

$$\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{N_{\theta\theta}}{R} = \rho h \frac{\partial^2 w_b}{\partial t^2}$$

or

$$D\frac{\partial^4 w_b}{\partial x^4} + \frac{Eh}{R^2} w_b + \rho h \frac{\partial^2 w_b}{\partial t^2} = 0 \qquad (1.32a)$$

Putting $w_b = w$, then

$$D\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{R^2} w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1.32b)$$

which is identical to Niordson's.(12)

D. Approximate Equations of Motion for Forced Vibration

Let $p(x,t)$ be the internal pressure per unit area, and $p(x,t)$ acts outward along the radius of the tube.

1. More Exact Theory

A forcing function $p(x,t)$ must be added to the equation of motion in the radial direction, while the equation due to rotation remains unchanged

$$hkG\frac{\partial^2 w_s}{\partial x^2} - \frac{Eh}{R^2}(w_b + w_s) - \rho h\frac{\partial^2(w_b+w_s)}{\partial t^2} = -p(x,t)$$

or

$$kG\frac{\partial^2 w_s}{\partial x^2} - \frac{E}{R^2}(w_b + w_s) - \rho\frac{\partial^2(w_b+w_s)}{\partial t^2} = -\frac{1}{h}p(x,t) \qquad (1.33)$$

2. <u>Elementary Theory</u>

$$D\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{R^2}w + \rho h\frac{\partial^2 w}{\partial t^2} = p(x,t) \qquad (1.34)$$

E. <u>Approximate Equations of Motion in Dimensionless Form</u>

For convenience of numerical computation, the following dimensionless variables are introduced for x, w, w_b, w_s, t, and p respectively

$$\begin{cases} X = \dfrac{\sqrt{12}\,x}{h} \\ W = \dfrac{w}{h} \\ W_b = \dfrac{w_b}{h} \\ W_s = \dfrac{w_s}{h} \\ T = \dfrac{\sqrt{12}\,v_d t}{h} \\ P = \dfrac{p}{12kG} \end{cases} \qquad (1.35)$$

where $v_d^2 = \dfrac{E}{(1-\nu^2)\rho}$ is the square of the dilatational wave velocity in a plate.

1. Elementary Theory

Replacing x, w, t, and p by X, W, T, and P respectively in Equation (1.34), the equation of motion in dimensionless form is obtained

$$\frac{\partial^4 W}{\partial X^4} + \frac{1-\nu^2}{12}\left(\frac{h}{R}\right)^2 W + \frac{\partial^2 W}{\partial T^2} = \frac{(1-\nu^2)kG}{E} P \qquad (1.36)$$

Let the following dimensionless parameters be defined

$$\begin{cases} g^2 = \dfrac{E}{12kG}\left(\dfrac{h}{R}\right)^2 \\ \delta^2 = \dfrac{(1-\nu^2)kG}{E} = \dfrac{v_s^2}{v_d^2} \end{cases} \qquad (1.37)$$

where v_s^2 is the square of the modified shear wave velocity in a plate,

$$g^2\delta^2 = \frac{1-\nu^2}{12}\left(\frac{h}{R}\right)^2 \qquad (1.38)$$

Equation (1.36) becomes

$$\frac{\partial^4 W}{\partial X^4} + \delta^2 g^2 W + \frac{\partial^2 W}{\partial T^2} = \delta^2 P \qquad (1.39)$$

2. More Exact Theory

Replacing x, w_b, w_s, t, and p by X, W_b, W_s, T, and P respectively and introducing dimensionless parameters g^2, δ^2 into Equations (1.33) and (1.30), the following dimensionless equations are obtained

$$\frac{\partial^2 W_s}{\partial X^2} - g^2(W_b + W_s) - \frac{1}{\delta^2}\frac{\partial^2(W_b + W_s)}{\partial T^2} = -P \qquad (1.40a)$$

$$\frac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \frac{\partial^2 W_b}{\partial T^2} = 0 \qquad (1.40b)$$

II. STEADY STATE WAVE PROPAGATION IN A THIN-WALLED CYLINDRICAL TUBE

A. Elementary Theory (corresponding to Euler-Bernoulli theory in beam vibration)

1. Equation of Motion for Free Vibration

For steady state wave propagation in the wall of a tube with infinite length under free vibration, the dimensionless equation of motion from this theory is

$$\frac{\partial^4 W}{\partial X^4} + \delta^2 g^2 W + \frac{\partial^2 W}{\partial T^2} = 0 \tag{2.1}$$

where dimensionless variables W, X, and T as well as dimensionless parameters δ and g are defined in Chapter I.

2. Frequency Spectrum---Wave Number (real, imaginary, or complex) versus Real Frequency of Vibration

Assume the following solution for Equation (2.1)

$$W = A e^{i(NX - \Omega T)} \tag{2.2}$$

then N and Ω must satisfy the following algebraic equation

$$N^4 + \delta^2 g^2 - \Omega^2 = 0 \tag{2.3}$$

or

$$N^4 = \Omega^2 - \delta^2 g^2 \tag{2.4}$$

Since Ω is always real, the spectrum has three arms, i.e., when $\Omega^2 > \delta^2 g^2$ N is either real or pure imaginary, and this corresponds to the real or imaginary arm; when $\Omega^2 < g^2\delta^2$, N is complex and this corresponds to the complex arm. Let $N = n$ for the real wave number, $N = im$ for the imaginary, and $N = n + im$ for the complex in the frequency spectrum.

The frequency spectrum is plotted only for the first quadrant in Figure 2.1 through 2.3, since the spectrum is symmetrical with respect to all coordinate axes. In those figures, ν is taken to be 0.3, κ to be 0.833 and h/R to be 0.1, 0.06, and 0.03.

3. Velocity Spectrum---Wave Number (real, imaginary, or complex) versus Real Phase Velocity

Assume the solution of Equation (2.1) has the form

$$W = A e^{iN(X-VT)} \tag{2.5}$$

where N is the wave number of the traveling wave, N may be real, imaginary, or complex; V is the phase velocity of the traveling wave, V is always real. If Equation (2.5) is a solution of Equation (2.1), V and N have the relation

$$N^4 - V^2 N^2 + \delta^2 g^2 = 0 \tag{2.6a}$$

or

$$N^2 = \frac{V^2}{2} \pm \left[\left(\frac{V^2}{2}\right)^2 - \delta^2 g^2\right]^{\frac{1}{2}} \tag{2.6b}$$

When $V^4 - 4\delta^2 g^2 < 0$, N is complex. Let V_{co} be $\sqrt{2\delta g}$ which is the minimum phase velocity (critical velocity) with which a sinusoidal wave can be propagated. In this case V is less than V_{co}. When $V^4 - 4\delta^2 g^2 > 0$, or $V > V_{co}$, N is real and N has two real values for a given V. N cannot be imaginary if V is real.

The velocity spectrum is plotted only for the first quadrant in Figures 2.4 through 2.6, since the spectrum is symmetrical with respect

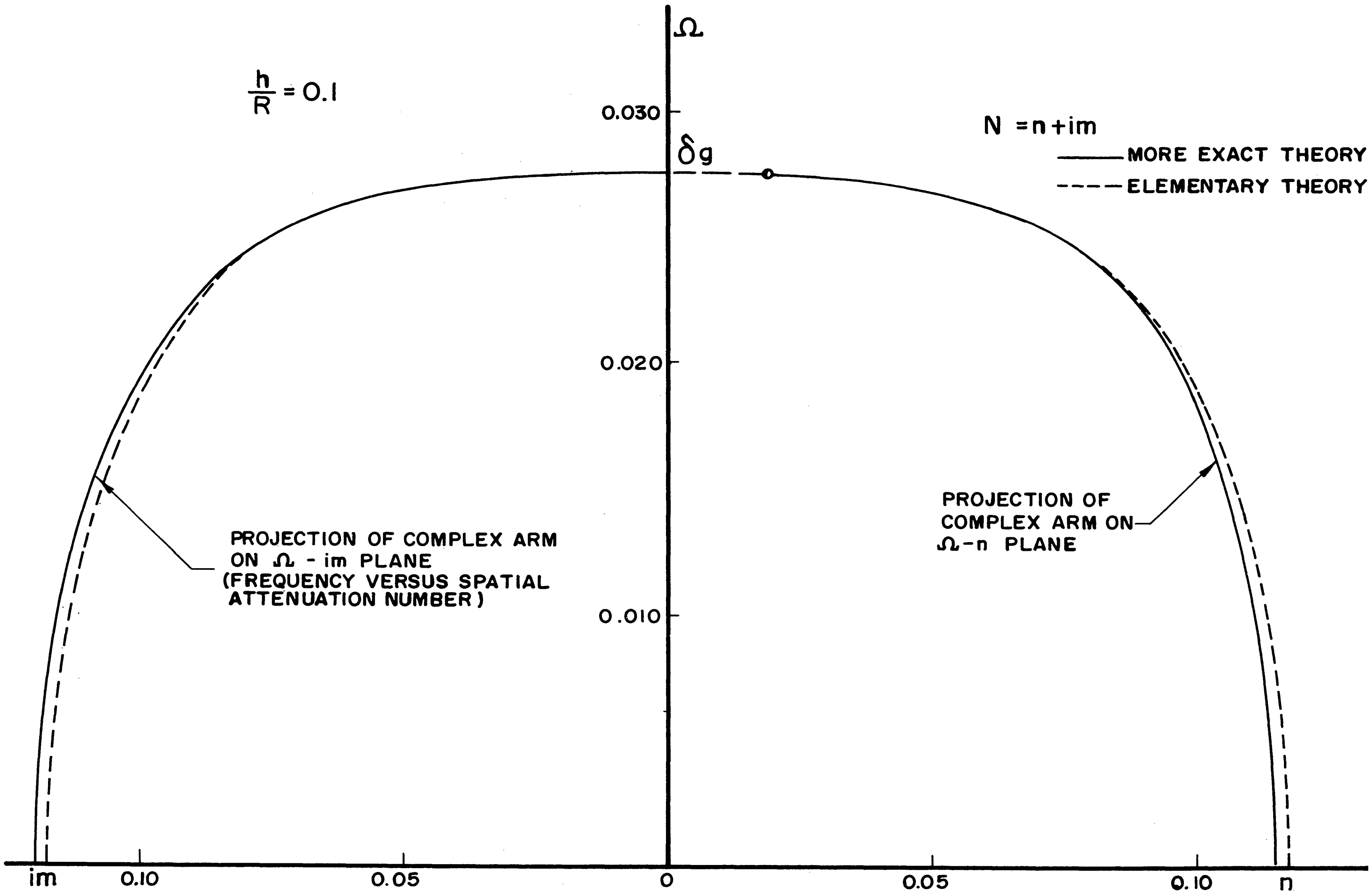

Figure 2.1a Frequency Spectrum for $h/R = 0.1$ (Complex Arm).

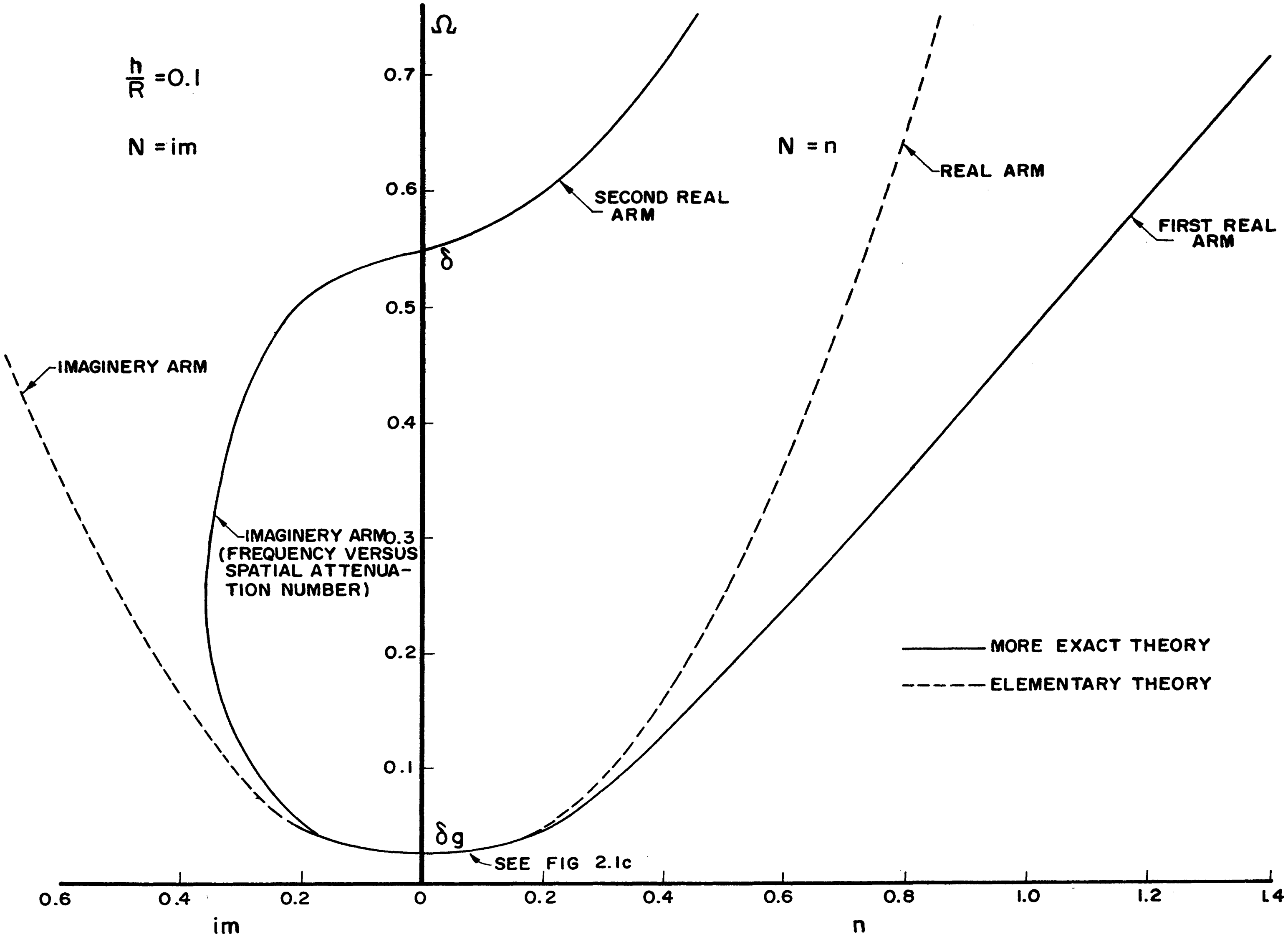

Figure 2.1b Frequency Spectrum for h/R = 0.1 Real and Imaginary Arms.

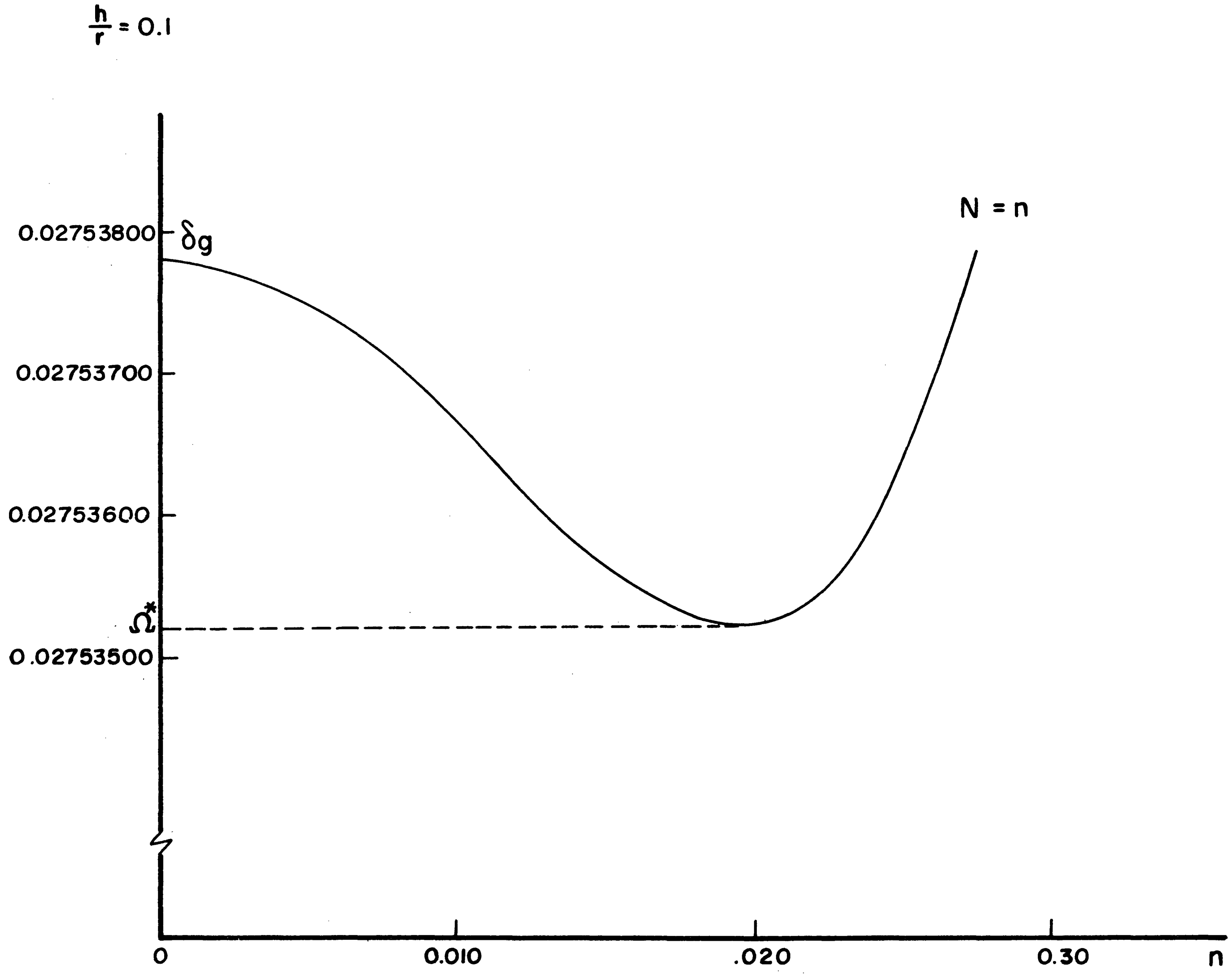

Figure 2.1c Frequency Spectrum for h/R = 0.1 (Real Arm Near $\Omega = \Omega^*$ from More Exact Theory).

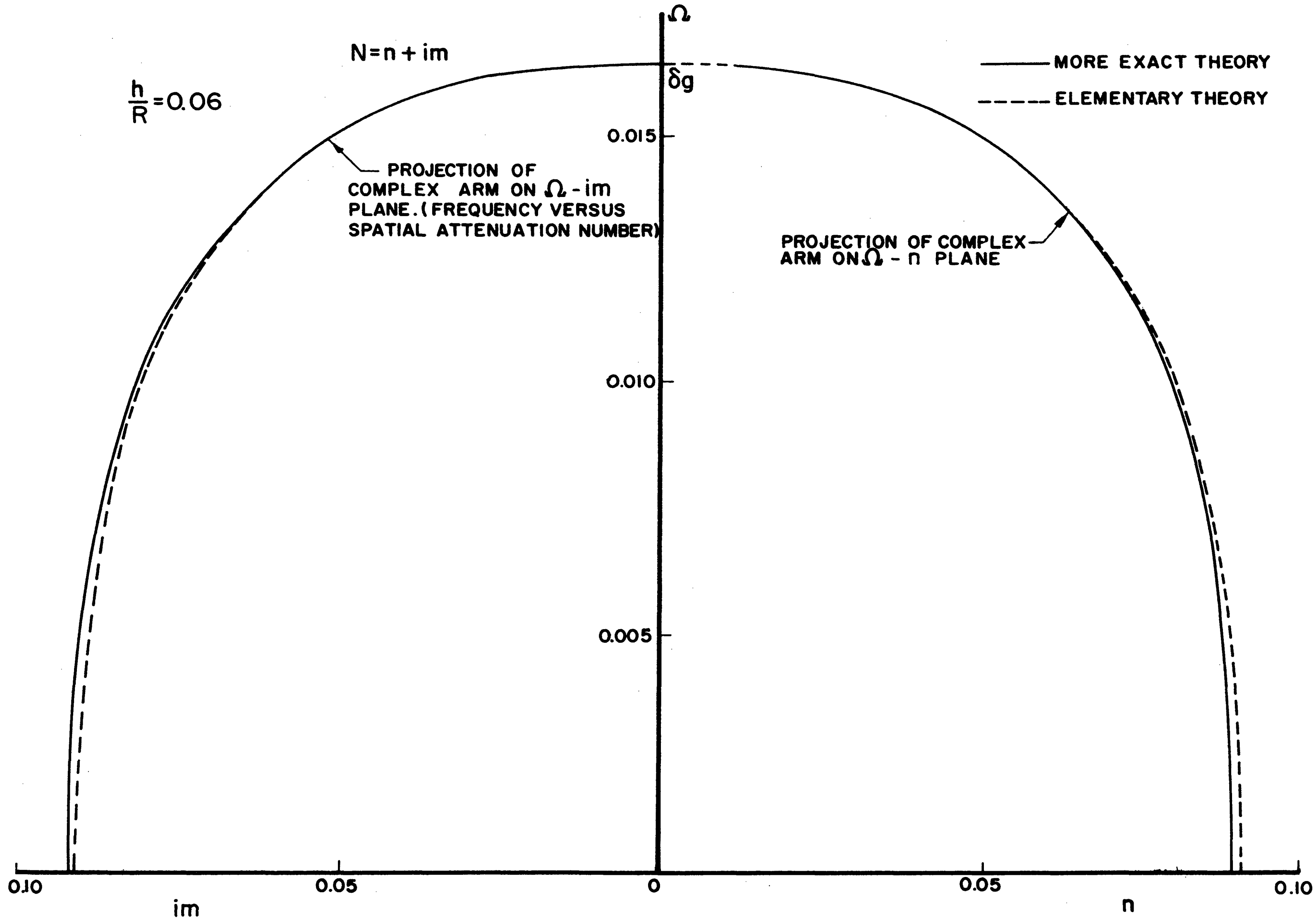

Figure 2.2a Frequency Spectrum for h/R = 0.06 (Complex Arm).

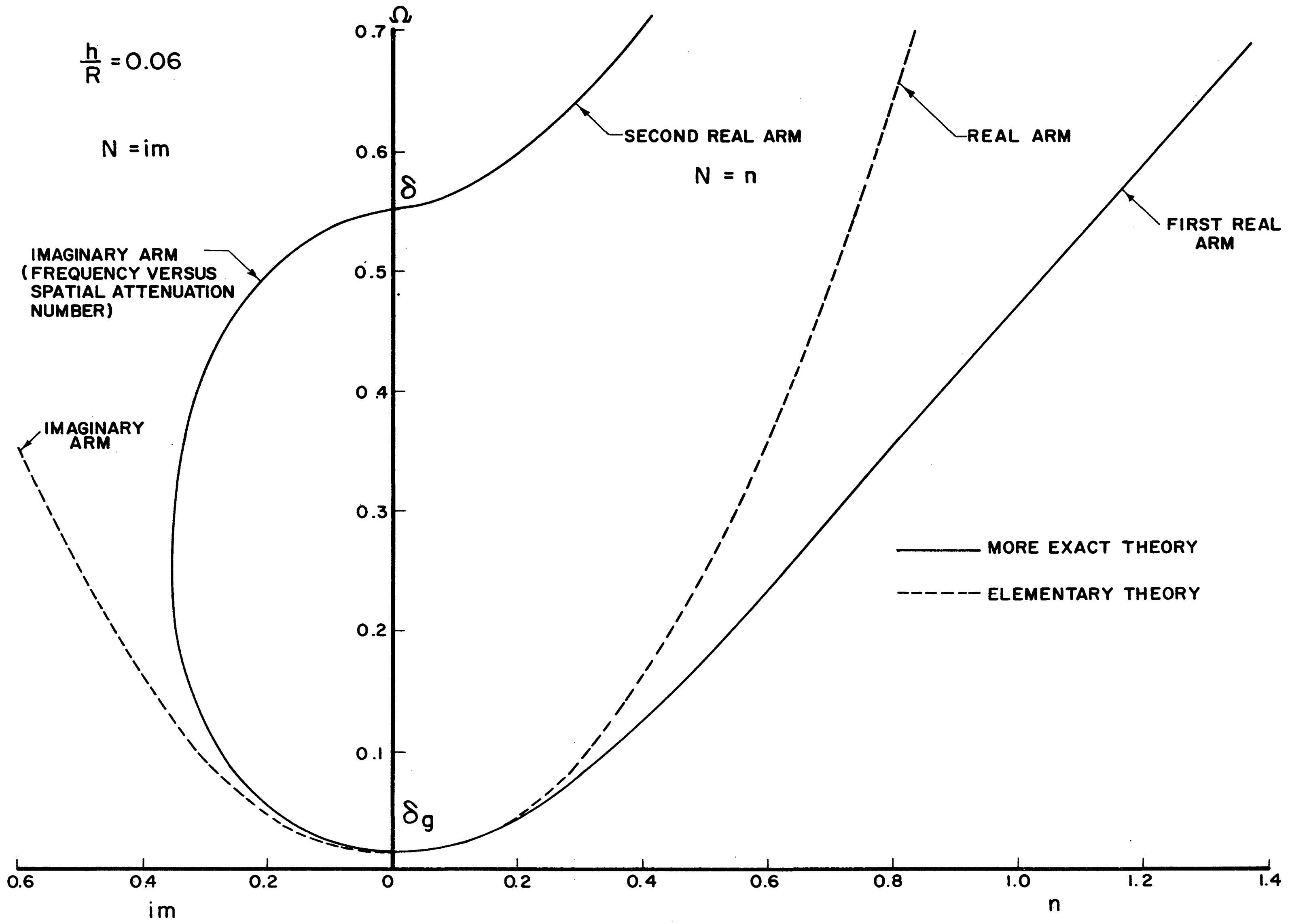

Figure 2.2b Frequency Spectrum for $h/R = 0.06$ (Real and Imaginary Arms).

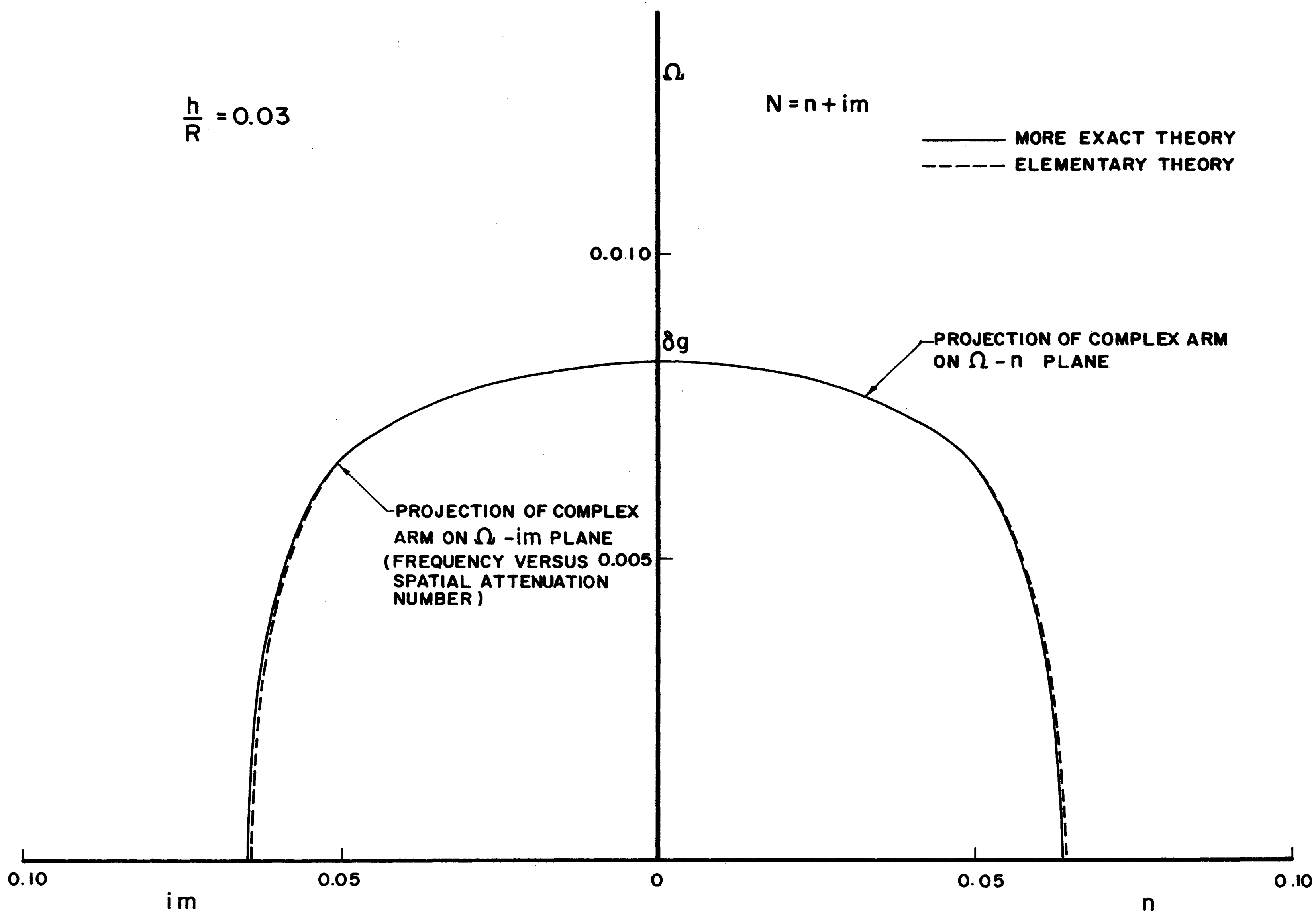

Figure 2.3a Frequency Spectrum for $h/R = 0.03$ (Complex Arm).

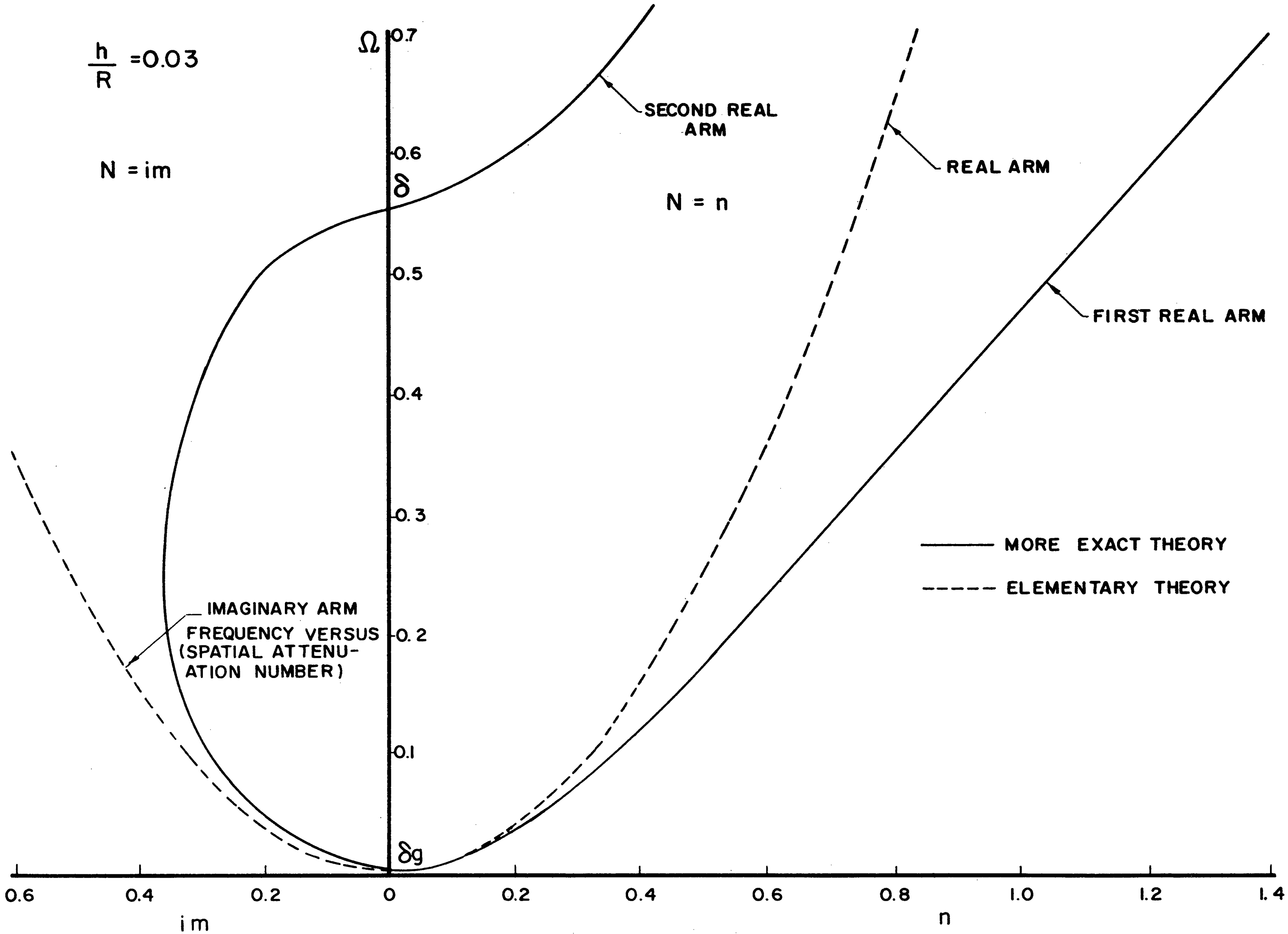

Figure 2.3b Frequency Spectrum for $h/R = 0.03$ (Real and Imaginary Arms).

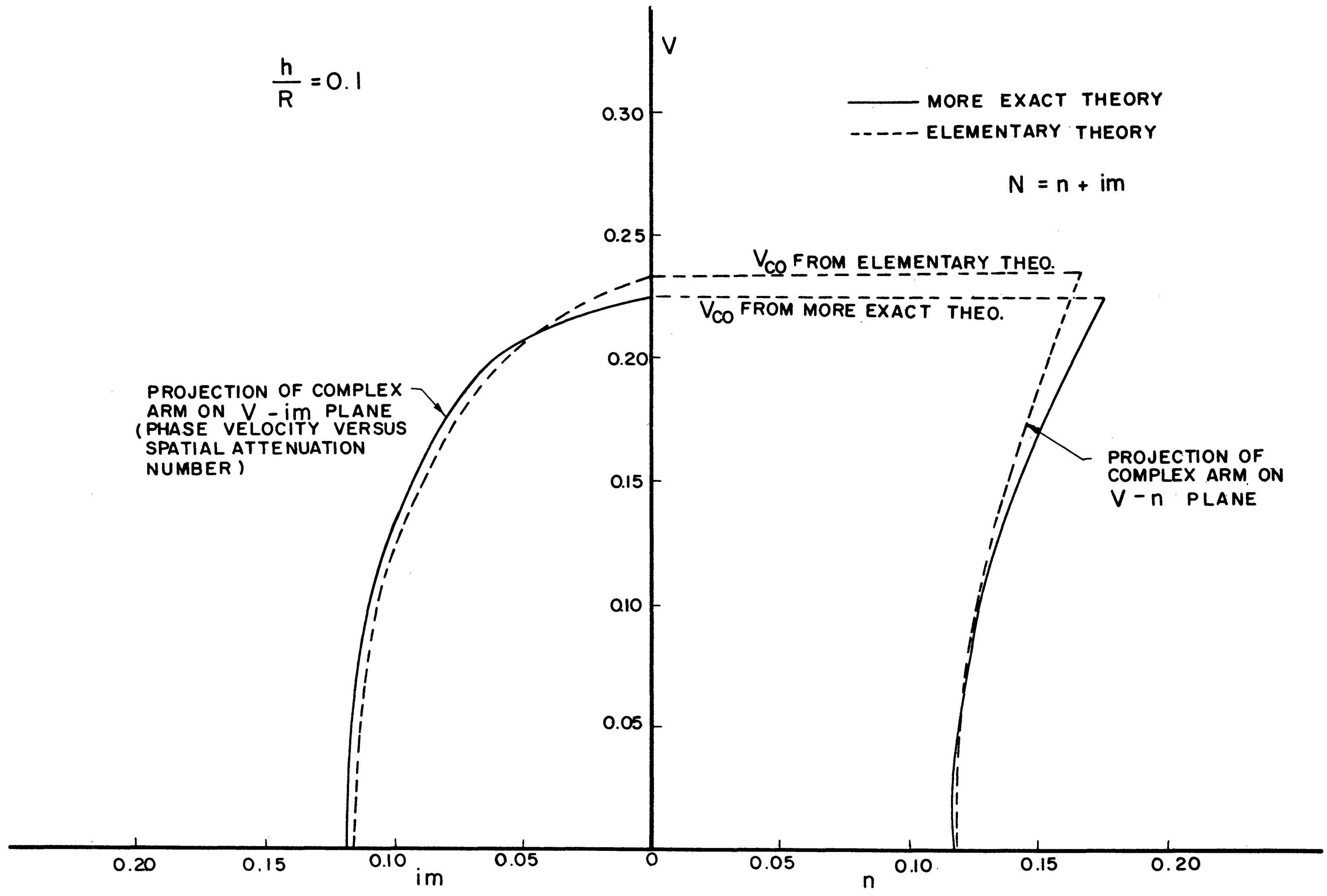

Figure 2.4a Velocity Spectrum for h/R = 0.1 (Complex Arm).

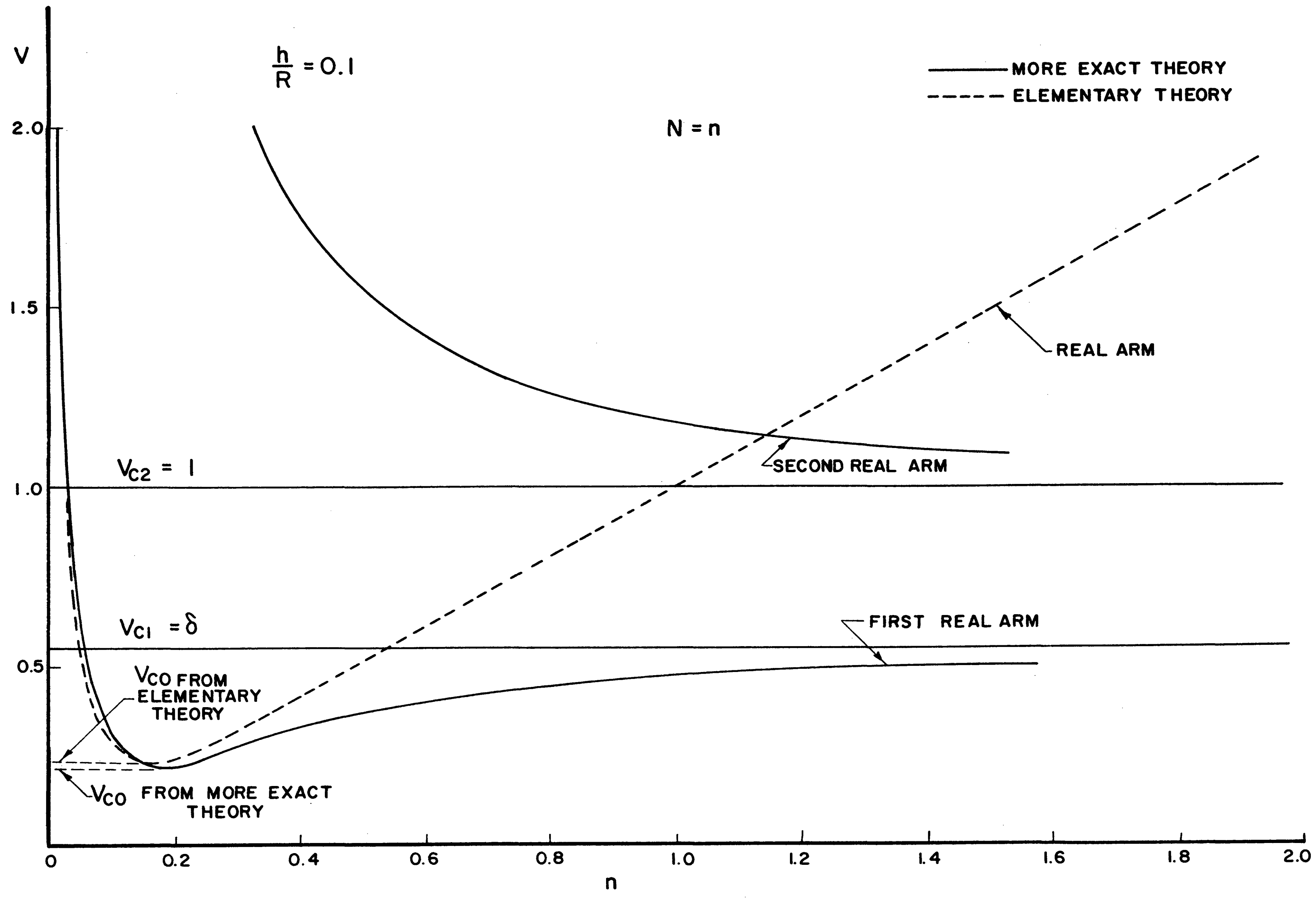

Figure 2.4b Velocity Spectrum for $h/R = 0.1$ (Real Arm).

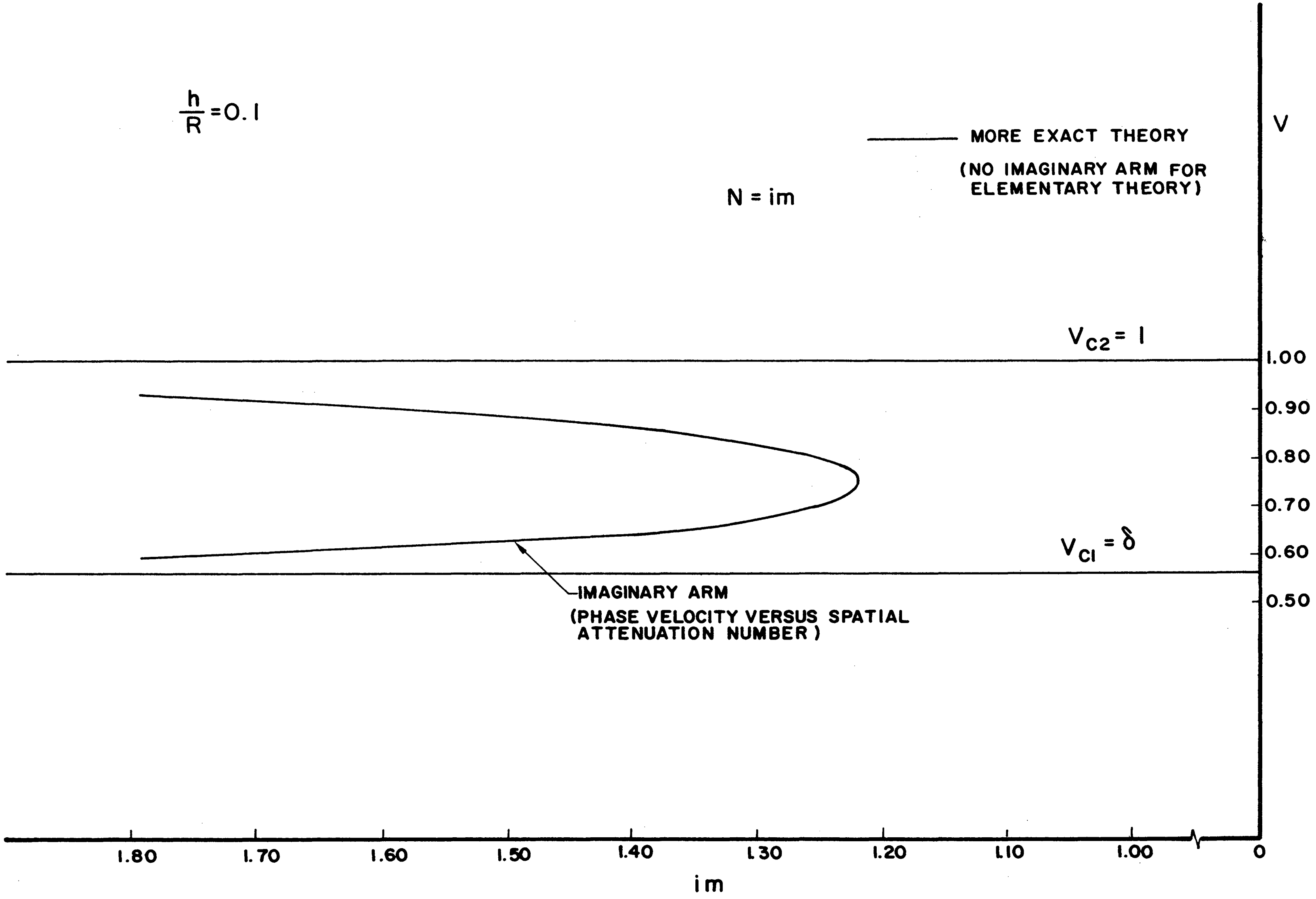

Figure 2.4c Velocity Spectrum for $h/R = 0.1$ (Imaginary Arm).

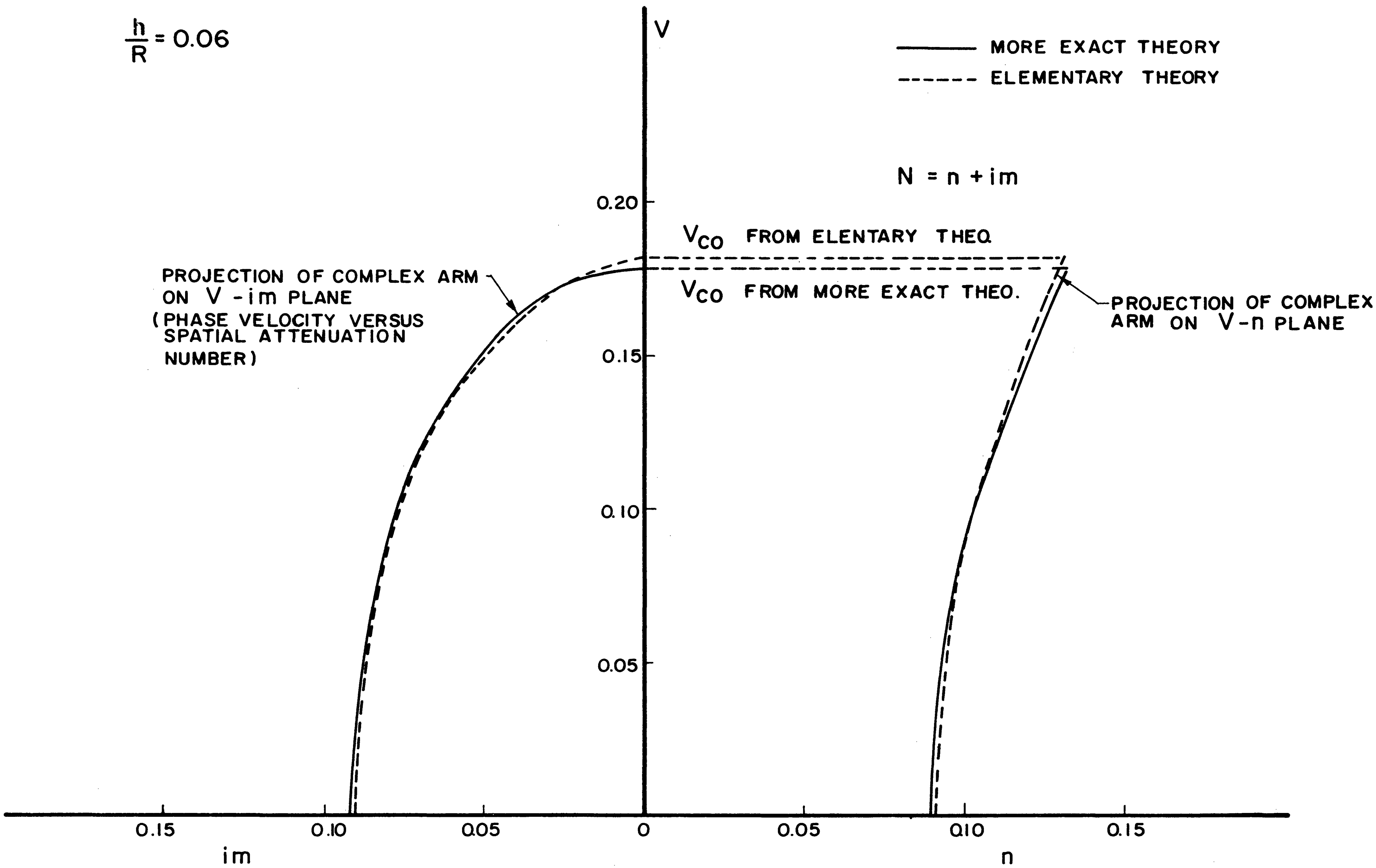

Figure 2.5a Velocity Spectrum for $h/R = 0.06$ (Complex Arm).

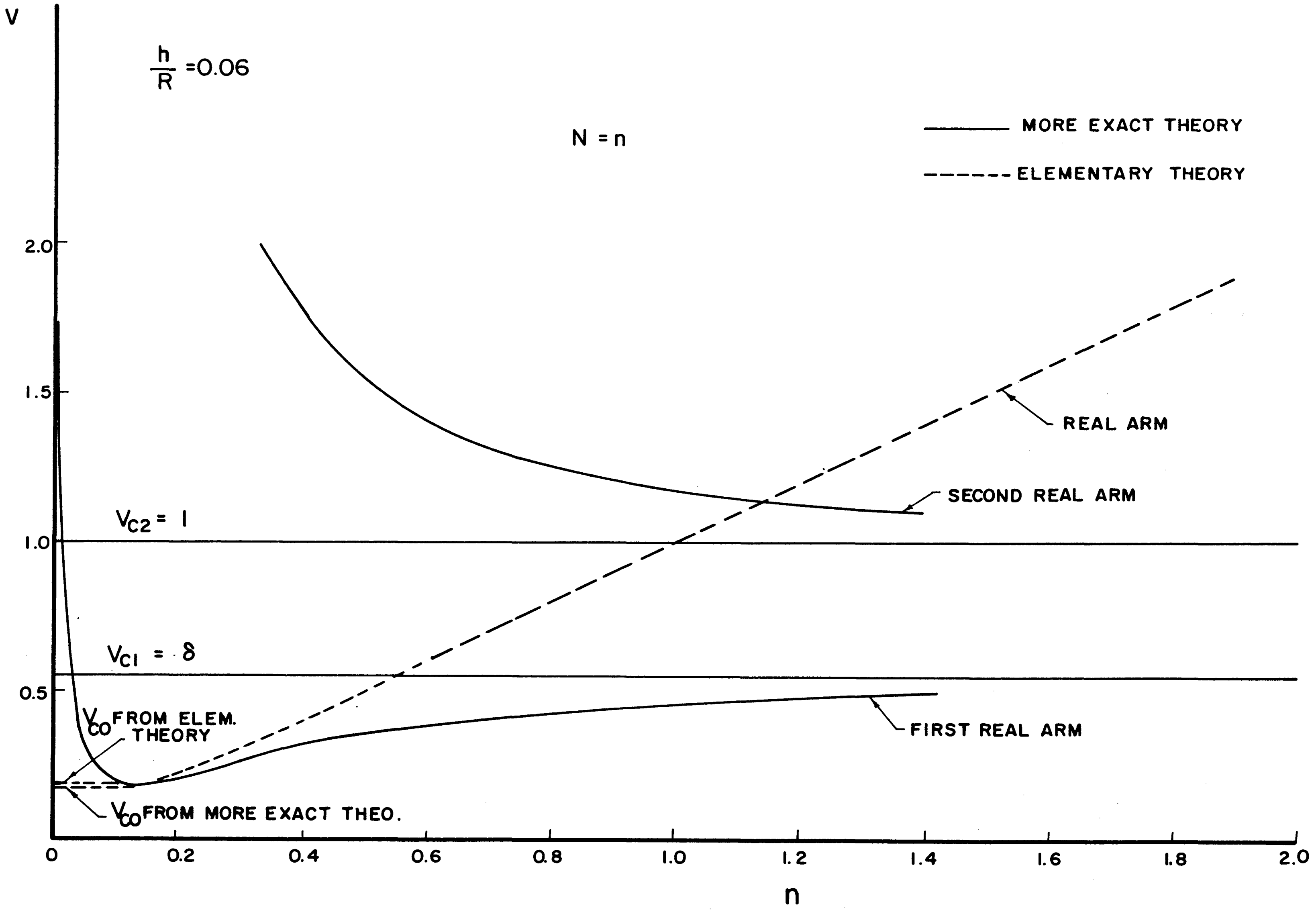

Figure 2.5b Velocity Spectrum for $h/R = 0.06$ (Real Arm).

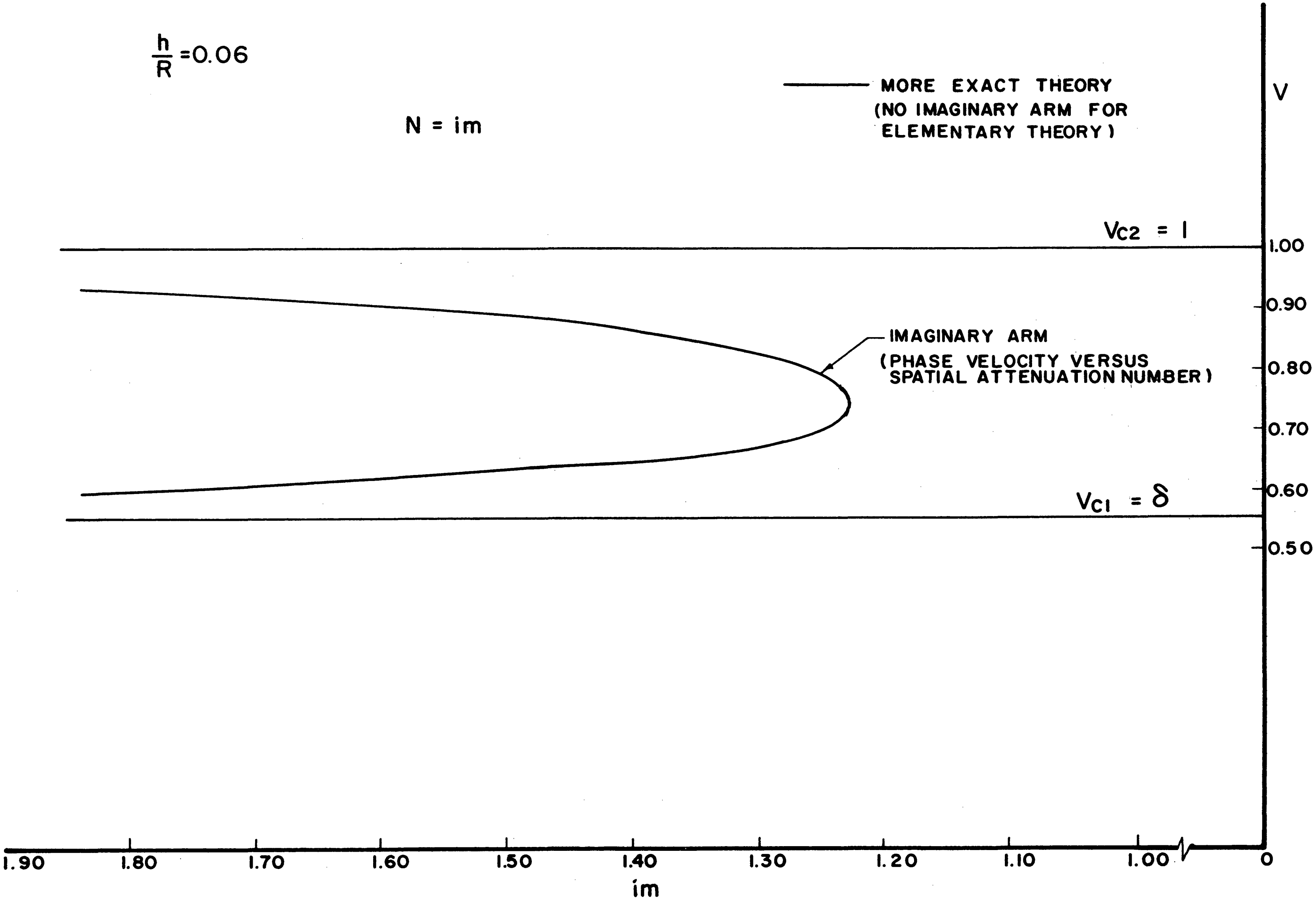

Figure 2.5c Velocity Spectrum for $h/R = 0.06$ (Imaginary Arm).

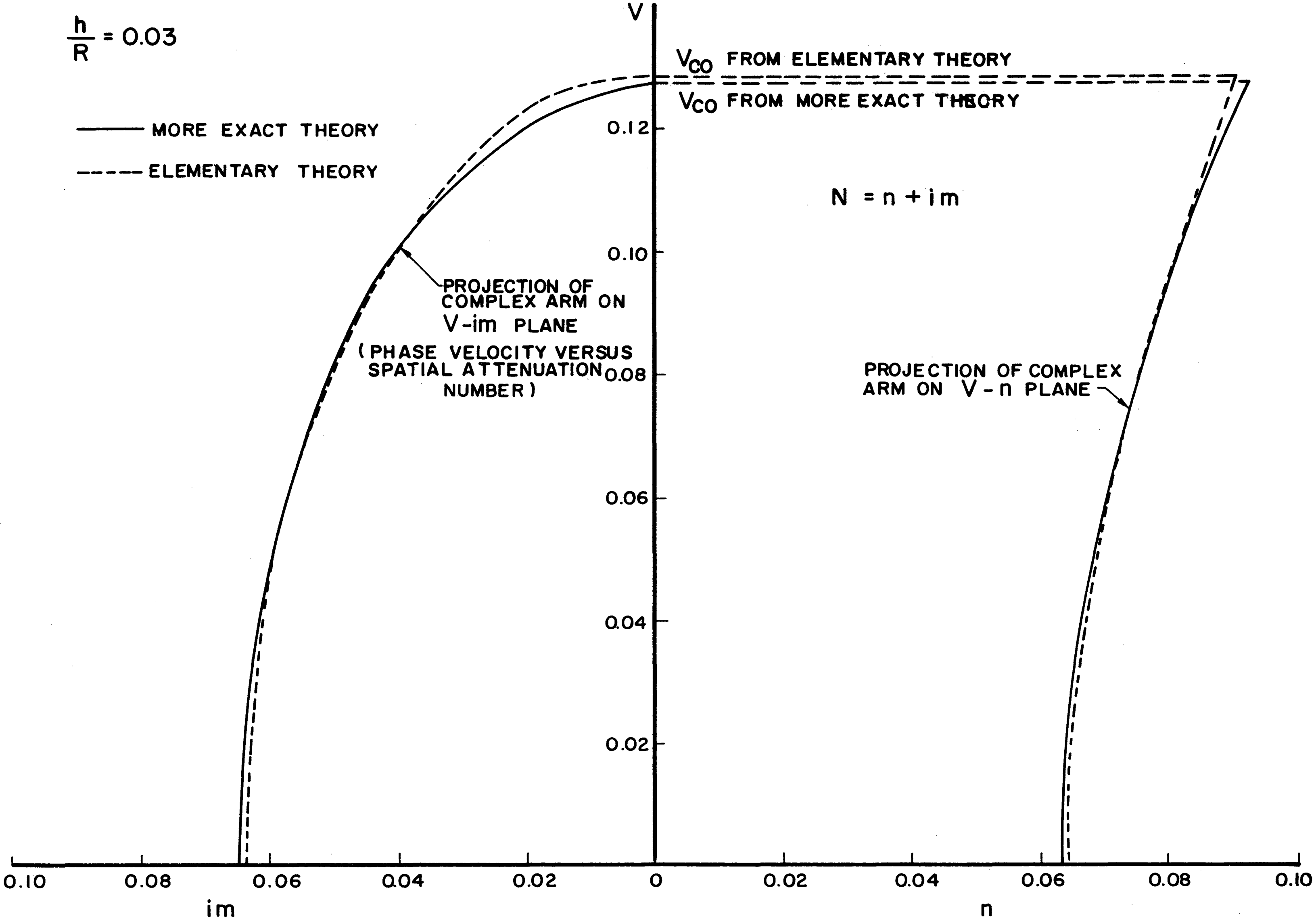

Figure 2.6a Velocity Spectrum for $h/R = 0.03$ (Complex Arm).

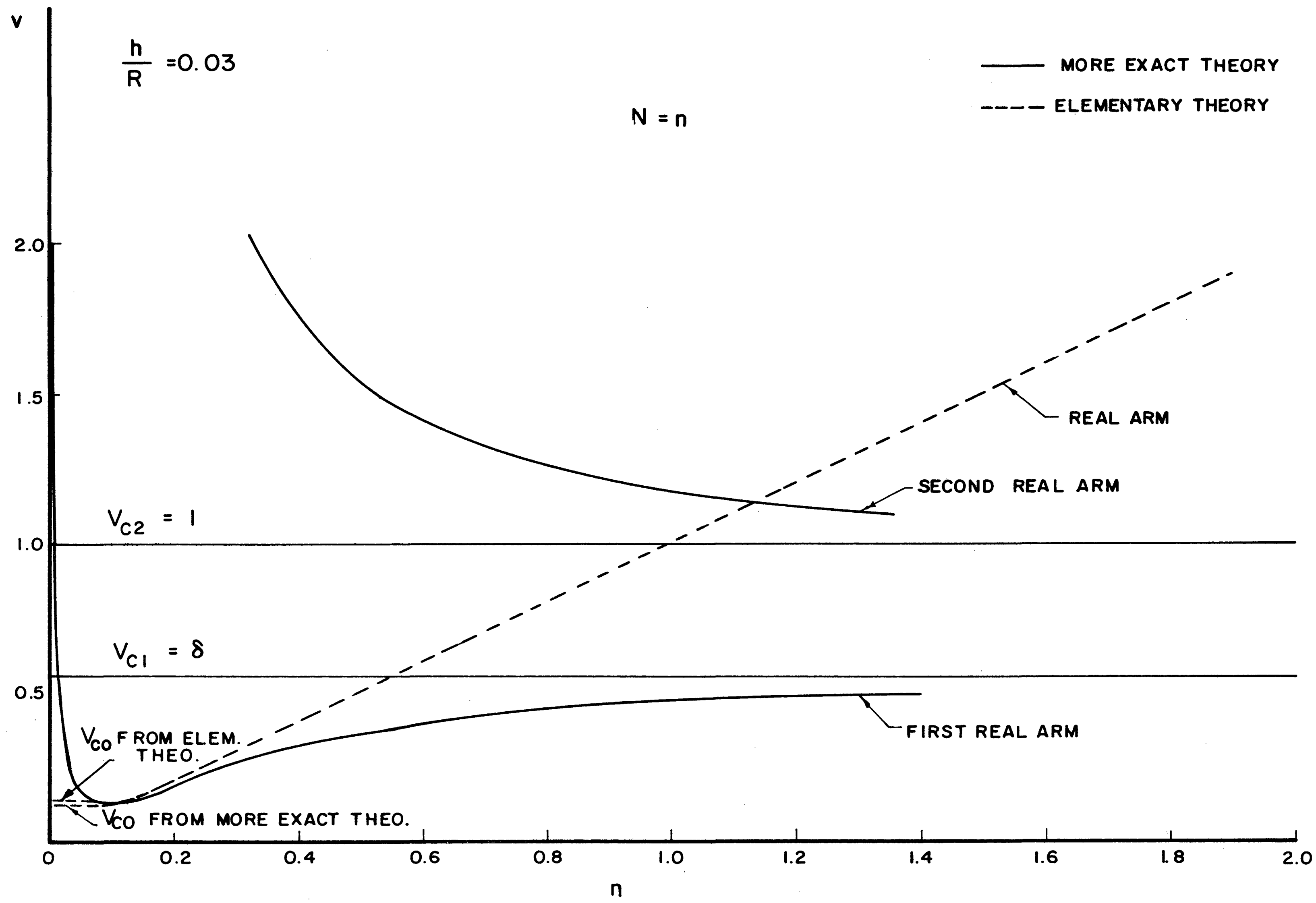

Figure 2.6b Velocity Spectrum for $h/R = 0.03$ (Real Arm).

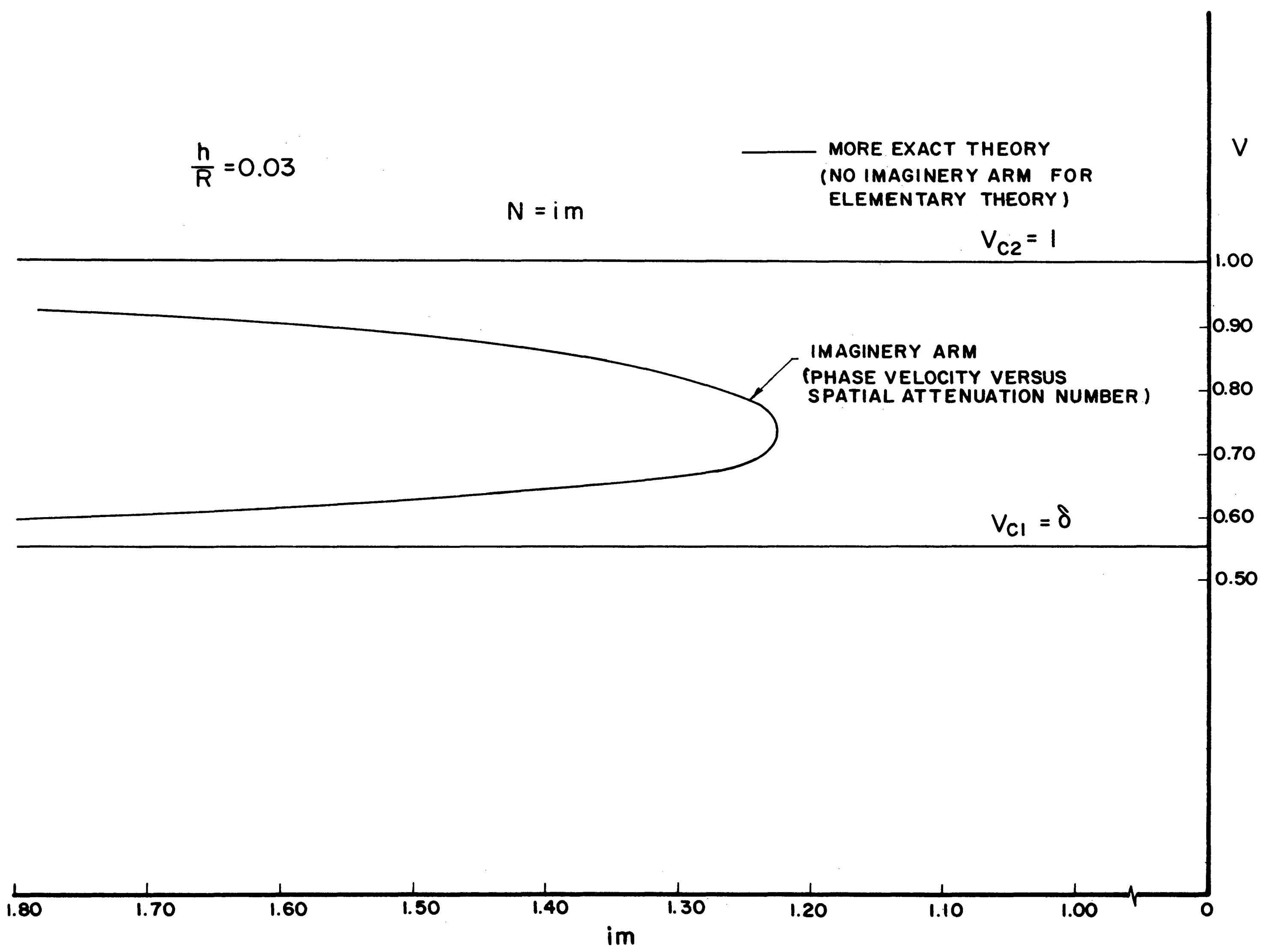

Figure 2.6c Velocity Spectrum for $h/R = 0.03$ (Imaginary Arm).

to all coordinate axes. In those figures, ν is taken to be 0.3, κ to be 0.833 and h/R to be 0.1, 0.06, and 0.03.

B. More Exact Theory (Corresponding to Timoshenko Theory in Beam Vibration)

1. Equations of Motion for Free Vibration

For steady state wave propagation in the wall of a tube with infinite length under free vibration, the dimensionless equations of motion from this theory are

$$\begin{cases} \dfrac{\partial^2 W_s}{\partial X^2} - g^2 (W_b + W_s) - \dfrac{1}{\delta^2} \dfrac{\partial^2 (W_b + W_s)}{\partial T^2} = 0 & (2.7a) \\ \dfrac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \dfrac{\partial^2 W_b}{\partial T^2} = 0 & (2.7b) \end{cases}$$

where dimensionless variables W_b, Ws, X, and T as well as dimensionless parameters δ and g are defined in Chapter I.

2. Frequency Spectrum

Assume the following solution for Equation (2.7)

$$\begin{cases} W_b = A_1 \, e^{i(NX - \Omega T)} & (2.8a) \\ W_s = A_2 \, e^{i(NX - \Omega T)} & (2.8b) \end{cases}$$

where Ω is always real, then A_1 and A_2 must satisfy

$$\left(\frac{\Omega^2}{\delta^2} - g^2\right) A_1 + \left(\frac{\Omega^2}{\delta^2} - g^2 - N^2\right) A_2 = 0 \qquad (2.9a)$$

$$(\Omega^2 - N^2) A_1 + \delta^2 A_2 = 0 \qquad (2.9b)$$

For a non-trivial solution of A_1 and A_2, the determinant of the coefficients must be zero and this then is the frequency equation

$$\begin{vmatrix} (\frac{\Omega^2}{\delta^2} - g^2) & (\frac{\Omega^2}{\delta^2} - g^2 - N^2) \\ (\Omega^2 - N^2) & \delta^2 \end{vmatrix} = 0$$

or

$$N^4 - [\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2] N^2 + \delta^2 (g^2 - \frac{\Omega^2}{\delta^2})(1 - \frac{\Omega^2}{\delta^2}) = 0 \qquad (2.10)$$

$$N^2 = \frac{1}{2}[\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2] \pm [(\frac{\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2}{2})^2 - \delta^2 (g^2 - \frac{\Omega^2}{\delta^2})(1 - \frac{\Omega^2}{\delta^2})]^{\frac{1}{2}} \qquad (2.11)$$

If N is complex, then

$$(\frac{\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2}{2})^2 - \delta^2 (g^2 - \frac{\Omega^2}{\delta^2})(1 - \frac{\Omega^2}{\delta^2}) < 0$$

For Ω to be a maximum, the above expression should be equal to zero, namely,

$$(1-\delta^2)\frac{\Omega^4}{\delta^4} - 2[g^2 - \frac{2\delta^2}{1-\delta^2}]\frac{\Omega^2}{\delta^2} + \frac{g^2(g^2 - 4\delta^2)}{1-\delta^2} = 0$$

Let Ω^* be the root of the above equation, then

$$\Omega^{*2} = \frac{\delta^2}{1-\delta^2}[(g^2 - \frac{2\delta^2}{1-\delta^2}) \pm \frac{2\delta^2}{1-\delta^2}(1 - g^2(1-\delta^2))^{\frac{1}{2}}]$$

Taking the negative sign

$$\Omega^{*2} = \frac{\delta^2}{1-\delta^2}[(g^2 - \frac{2\delta^2}{1-\delta^2}) - \frac{2\delta^2}{1-\delta^2}(1 - g^2(1-\delta^2))^{\frac{1}{2}}]$$

since

$$g^2 = \frac{E}{12KG}(\frac{h}{R})^2 \ll 1, \qquad \delta^2 = \frac{(1-\nu^2)KG}{E} < 1$$

$$g^2 - \frac{2\delta^2}{1-\delta^2} < 0$$

there is no real root.

Taking the positive sign

$$\Omega^{*2} = \frac{\delta^2}{1-\delta^2}\left[\left(g^2 - \frac{2\delta^2}{1-\delta^2}\right) + \frac{2\delta^2}{1-\delta^2}\left(1 - g^2(1-\delta^2)\right)^{\frac{1}{2}}\right] \tag{2.12}$$

$$\approx \frac{\delta^2}{1-\delta^2}\left[\left(g^2 - \frac{2\delta^2}{1-\delta^2}\right) + \frac{2\delta^2}{1-\delta^2}\left(1 - \frac{1}{2}g^2(1-\delta^2) - \cdots\right)\right]$$

$$= \delta^2\left[g^2 - \text{(higher order terms} > 0)\right]$$

so that $\Omega^{*2} < \delta^2 g^2$ and only one real positive root exists.

Taking the positive sign before the radical in Equation (2.11)

$$N^2 = \frac{1}{2}\left[\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2\right] + \left[\left(\frac{\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2}{2}\right)^2 - \delta^2\left(g^2 - \frac{\Omega^2}{\delta^2}\right)\left(1 - \frac{\Omega^2}{\delta^2}\right)\right]^{\frac{1}{2}}$$

If $0 < \Omega < \Omega^*$, N is complex and this is the complex arm in the frequency spectrum. If $\Omega^* < \Omega$, N is real and this is a part of the first real arm in the frequency spectrum.

Taking the negative sign before the radical in Equation (2.11)

$$N^2 = \frac{1}{2}\left[\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2\right] - \left[\left(\frac{\frac{\Omega^2}{\delta^2}(1+\delta^2) - g^2}{2}\right)^2 - \delta^2\left(g^2 - \frac{\Omega^2}{\delta^2}\right)\left(1 - \frac{\Omega^2}{\delta^2}\right)\right]^{\frac{1}{2}}$$

If $\Omega^* > \Omega > 0$, N is complex, and N is conjugate to that in the previous part. If only first quadrant is used, this region need not be plotted due to symmetry of the diagram. If $\Omega^* < \Omega < \delta g$, N is real and this is another part of the first real arm in the frequency spectrum. If $\delta g < \Omega < \delta$, N is imaginary and this is the imaginary arm in the frequency spectrum. If $\Omega > \delta$, N is real again and this is the second real arm in the frequency spectrum.

The frequency spectrum is plotted only for the first quadrant in Figure 2.1 through 2.3, since the spectrum is symmetrical with respect to all coordinate axes. In these figures, ν is taken to be o.3, κ to be 0.833 and h/R to be 0.1, 0.06, and 0.03.

3. Velocity Spectrum

Assume the solution of Equation (2.7) has the form

$$\begin{cases} W_b = A_1 e^{iN(X - VT)} & (2.13a) \\ W_s = A_2 e^{iN(X - VT)} & (2.13b) \end{cases}$$

where V is always real. Substitute into Equation (2.7), relations of A_1 and A_2 are obtained

$$\left(N^2 \frac{V^2}{\delta^2} - g^2\right) A_1 + \left[N^2\left(\frac{V^2}{\delta^2} - 1\right) - g^2\right] A_2 = 0 \qquad (2.14a)$$

$$N^2(V^2 - 1) A_1 + \delta^2 A_2 = 0 \qquad (2.14b)$$

For non-trivial solution of A_1 and A_2, the following determinant must vanish

$$\begin{vmatrix} \left(N^2 \frac{V^2}{\delta^2} - g^2\right) & \left[N^2\left(\frac{V^2}{\delta^2} - 1\right) - g^2\right] \\ N^2(V^2 - 1) & \delta^2 \end{vmatrix} = 0 \qquad (2.15)$$

or

$$(V^2 - 1)\left(\frac{V^2}{\delta^2} - 1\right) N^4 - \left[V^2(1 + g^2) - g^2\right] N^2 + \delta^2 g^2 = 0 \qquad (2.16)$$

$$N^2 = \frac{1}{(V^2 - 1)\left(\frac{V^2}{\delta^2} - 1\right)} \left\{ \frac{V^2(1+g^2) - g^2}{2} \pm \left[\left(\frac{V^2(1+g^2) - g^2}{2}\right)^2 - \delta^2 g^2 (V^2 - 1)\left(\frac{V^2}{\delta^2} - 1\right)\right]^{\frac{1}{2}} \right\} \qquad (2.17)$$

If the positive sign is taken before the radical in Equation (2.17), N becomes infinite at $V = 1$ or $V = \delta$. $V = V_{c1} = \delta$ is the dimensionless modified shear wave velocity in a plate and $V = V_{c2} = 1$ is the dimensionless dilatational wave velocity in a plate.

From Equation (2.17), we see that N becomes complex if the argument under the radical is less than zero and N becomes either real or pure imaginary if the argument is greater than zero. Thus we can define $V = V_{co}$ to be that which makes the argument zero

$$[V^2(1+g^2)-g^2]^2 - 4\delta^2 g^2 (V^2-1)\left(\frac{V^2}{\delta^2}-1\right) = 0$$

or

$$(1-g^2)^2 V^4 + 2g^2(1+2\delta^2-g^2)V^2 + g^2(g^2-4\delta^2) = 0$$

$$V^2 = \frac{1}{(1-g^2)^2}\left\{-g^2(1+2\delta^2-g^2) \pm \left[g^4(1+2\delta^2-g^2)^2 + g^2(1-g^2)^2(4\delta^2-g^2)\right]^{1/2}\right\}$$

Since $4\delta^2 > g^2$ and V is real, a positive sign is taken before the radical, then

$$V_{co}^2 = \frac{1}{(1-g^2)^2}\left\{-g^2(1+2\delta^2-g^2) + \left[g^4(1+2\delta^2-g^2)^2 + g^2(1-g^2)^2(4\delta^2-g^2)\right]^{1/2}\right\} \qquad (2.18)$$

There are three critical velocities V_{co}, V_{c1}, and V_{c2}, which divide the spectrum into four regions. These regions are discussed separately.

Based upon Equation (2.17), the velocity spectrum V versus N can be plotted. There are four regions:

i. $0 < V < V_{co}$

N is complex, and this is the complex arm in the velocity spectrum.

ii. $V_{co} < V < V_{c1}$, where $V_{c1} = \delta$

N is real for both positive and negative signs before the radical in Equation (2.17). This is a part of the first real arm in the velocity spectrum.

iii. $V_{c1} < V < V_{c2}$, where $V_{c2} = 1$

N is real for the negative sign before the radical in Equation (2.17) and this is another part of the first real arm in the velocity spectrum. N is imaginary for the positive sign, and this is the imaginary arm.

iv. $V > V_{c2}$

N is real for both signs before the radical in Equation (2.17). These are still another part of the first real arm and the entire second real arm.

The velocity spectrum is plotted for the first quadrant in Figure 2.4 through 2.6, since the spectrum is symmetrical with respect to all coordinate axes. In these figures, ν is taken to be 0.3, κ to be 0.833 and h/R to be 0.1, 0.06, and 0.03.

C. Comparision of Results from Both Theories

1. Frequency Spectrum

When $\Omega < \delta g$, spectra from both theories are with complex wave numbers, and they almost coincide. The reason is that the frequency is very low. When $\Omega > \delta g$, the first real arm of the spectrum by the more exact theory has the same shape as the real arm of the spectrum by the elementary theory. For frequencies a little bit over δg, they are close. The imaginary arms of the spectra from both theories are quite different except these for low frequencies. There is no second real arm for the elementary theory.

2. <u>Velocity Spectrum</u>

When $V < V_{co}$, the velocity spectra with complex wave numbers for both theories almost coincide. When $V > V_{co}$, the velocity spectrum from elementary theory exists only with real wave numbers. These with smaller real wave numbers lie almost on the corresponding part by the more exact theory. The rest are far from the more exact theory.

III. STEADY STATE RESPONSE OF A THIN-WALLED CYLINDRICAL TUBE WITH INFINITE LENGTH UNDER INTERNAL MOVING PRESSURE

A. Solution of the Equation from Elementary Theory

Let p be the intensity of the pressure whose front moves with velocity v as shown in Figure 3.1. In the ideal case, both p and v are constant. For the steady state response the pressure is assumed to have acted for a long time. Introduce the dimensionless variables P and V for p and v respectively

$$P = \frac{p}{12\,kG} \tag{3.1a}$$

$$V = \frac{v}{v_d} \tag{3.1b}$$

From Equation (1.39), the equation of motion is

$$\frac{\partial^4 W}{\partial X^4} + \delta^2 g^2 W + \frac{\partial^2 W}{\partial T^2} = \delta^2 P, \quad -\infty < X < \infty, \; -\infty < T < \infty \tag{3.2}$$

where

$$P(X,T) = \begin{cases} P & \text{when } X \leq VT \\ 0 & \text{when } X > VT \end{cases}$$

Since this is to find the steady state response, $W(X,T)$ is required to be bounded everywhere instead of assigning any boundary and initial conditions for W.

The partial differential equation given by Equation (3.2) can be transformed into an ordinary one by means of the Fourier transform[16], then it can be solved. Take the Fourier transform with respect to T and the transformed function is

$$\overline{W} = \int_{-\infty}^{\infty} W(X,T)\, e^{i\Omega T}\, dT \tag{3.3}$$

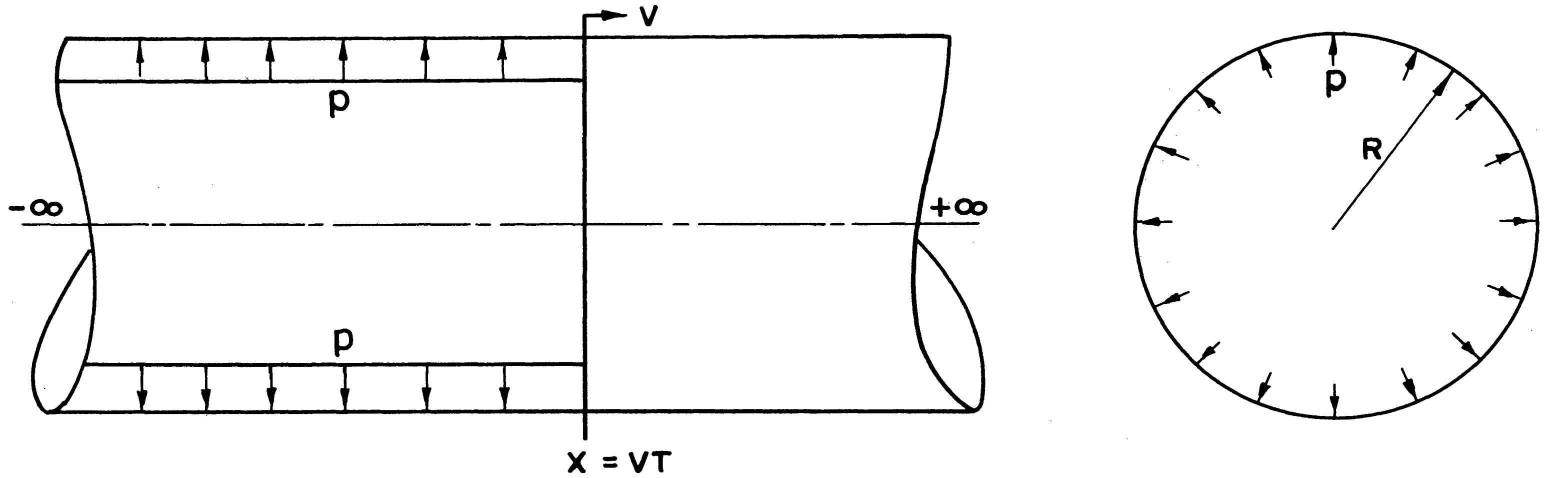

Figure 3.1 Moving Pressure in a Tube with Infinite Length.

Because of the convergence of the transformed forcing function P, imaginary part of Ω must be greater than zero. The inversion formula is given by

$$W(X,T) = \frac{1}{2\pi}\int_{-\infty+iy}^{\infty+iy} \overline{W}(X,\Omega)\, e^{-i\Omega T} d\Omega \tag{3.4}$$

where the constant $y > 0$.

To justify this transform, the integrals given by Equations (3.3) and (3.4) have to be proved convergent. It is assumed this is the case, and when W(X,T) is found one can verify that this W is a solution simply by substituting into Equation (3.2).

After the Fourier transform, Equation (3.2) becomes

$$\frac{d^4\overline{W}}{dX^4} + (\delta^2 g^2 - \Omega^2)\overline{W} = \frac{i\delta^2 P}{\Omega}\, e^{i\frac{\Omega}{V}X} \tag{3.5}$$

with $|\overline{W}(X,\Omega)| < \infty$ for all positive* value of X.

Since we do not need the boundary conditions, the particular solution of Equation (3.5) is enough. Assume it is

$$\overline{W}(X,\Omega) = A e^{i\frac{\Omega}{V}X}$$

To satisfy Equation (3.5), the coefficient must be

$$A = \frac{i\delta^2 P}{\Omega\left(\frac{\Omega^4}{V^4} - \Omega^2 + \delta^2 g^2\right)}$$

---

* For negative values of X, the non-homogeneous term in Equation (3.5) will be changed, but the final solution for $\overline{W}$ is the same. For convenience, use positive values of X to find $\overline{W}$.

The particular solution is

$$\overline{W}(X,\Omega) = \frac{i\delta^2 p}{\Omega\left(\frac{\Omega^4}{V^4} - \Omega^2 + \delta^2 g^2\right)} e^{i\frac{\Omega}{V}X} \tag{3.6}$$

Finally, inversion by Equation (3.4) yields

$$W(X,T) = \frac{i\delta^2 p}{2\pi} \int_{-\infty+iy}^{\infty+iy} \frac{e^{i\frac{\Omega}{V}X} e^{-i\Omega T}}{\Omega\left(\frac{\Omega^4}{V^4} - \Omega^2 + \delta^2 g^2\right)} d\Omega \tag{3.7}$$

Put $N = \frac{\Omega}{V}$ into Equation (3.7) and it becomes

$$W(X,T) = \frac{i\delta^2 p}{2\pi} \int_{-\infty+iy_1}^{\infty+iy_1} \frac{e^{iN(X-VT)}}{N(N^4 - V^2N^2 + \delta^2 g^2)} dN \tag{3.8}$$

where $y_1 = \frac{y}{V}$ and is greater than zero.

From Equation (3.8), it is obvious that $W(X,T)$ depends on (X-VT), the distance from the pressure front, because this is a steady response. To compute the integral given by Equation (3.8), The residue theorem[17] is to be used. Before applying the residue theorem, the poles of the integrand must be investigated. The poles are the roots of the equation

$$N(N^4 - V^2N^2 + \delta^2 g^2) = 0$$

or

$$N = 0$$

and

$$N^4 - V^2N^2 + \delta^2 g^2 = 0$$

The latter is identical to Equation (2.6) which expresses the relation between the phase velocity and the wave number. From Figure 2.4 through

2.6, N's are complex when V is less the critical velocity V_{co} (= $\sqrt{2\delta g}$) and real when V is greater than V_{co}. These two cases are discussed separately.

1. Case for $V < V_{co}$

In this case, N's are complex and let them be

$$N = \pm n \pm im \tag{3.9}$$

a. Deflection for Positions Before the Pressure Front Arriving, i.e., $(X-VT) > 0$

Take the contour as shown in Figure 3.2, then

$$\frac{1}{2\pi i}\left[\int_{-a+iy_1}^{a+iy_1} \frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2g^2)}\,dN - \int_{C_1} \frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2g^2)}\,dN\right]$$

=residues at N_1 and N_2

As "a" approaches infinity, by Jordan's lemma

$$\int_{C_1} \frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2g^2)}\,dN \longrightarrow 0$$

so that

$$\frac{1}{2\pi i}\int_{-\infty+iy_1}^{\infty+iy_1} \frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2g^2)}\,dN \quad = \text{residues at } N_1 \text{ and } N_2$$

$$= e^{-m(X-VT)}\left[\frac{1}{2(n^2+m^2)^2}\cos n(X-VT) - \frac{n^2-m^2}{4mn(n^2+m^2)^2}\sin n(X-VT)\right] \tag{3.10}$$

Since $(n^2 + m^2)^2 = \delta^2 g^2$, the deflection becomes

$$W(X,T) = \frac{P}{2g^2}\, e^{-m(X-VT)}\cdot \left[\cos n(X-VT) - \frac{n^2-m^2}{2mn}\sin n(X-VT)\right] \tag{3.11}$$

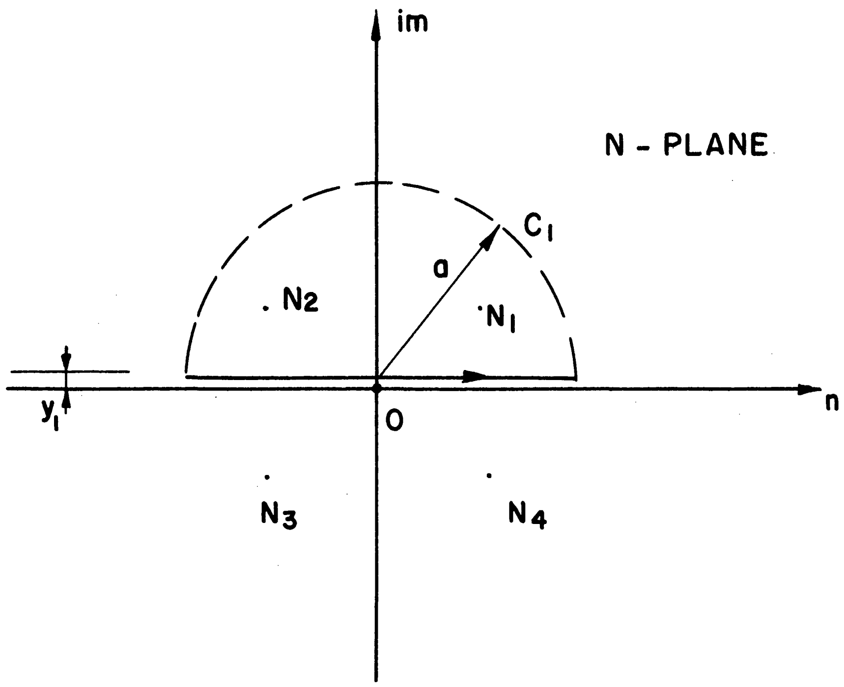

Figure 3.2 Inversion Contour for (X - VT) > 0

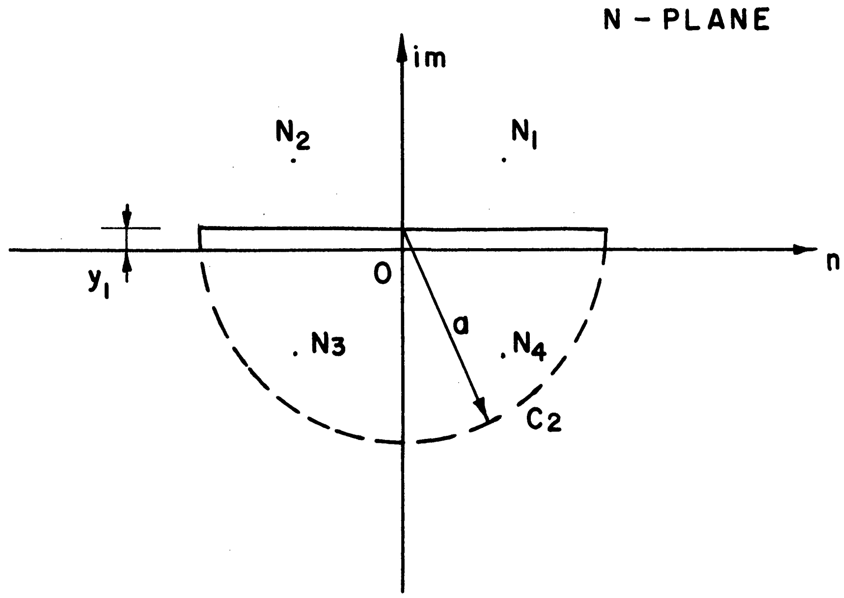

Figure 3.3 Inversion Contour for (X -VT) ≤ 0 .

b. Deflection for Positions After Pressure Front Having Arrived, i.e., (X-VT) ≤ 0

Take the contour as shown in Figure 3.3, then

$$-\frac{1}{2\pi i}\left[\int_{-a+iy_1}^{a+iy_1}\frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2 g^2)}dN-\int_{C_2}\frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2 g^2)}dN\right]$$

= residues at 0, N_3, and N_4

As "a" approaches infinity, by Jordan's lemma

$$\int_{C_2}\frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2 g^2)}dN \longrightarrow 0$$

so that

$$-\frac{1}{2\pi i}\int_{-\infty+iy_1}^{\infty+iy_1}\frac{e^{iN(X-VT)}}{N(N^4-V^2N^2+\delta^2 g^2)}dN \quad = \text{residues at } 0,\ N_3, \text{ and } N_4$$

$$=\frac{1}{\delta^2 g^2}-e^{m(X-VT)}\left[\frac{1}{2(n^2+m^2)^2}\cos n(X-VT)+\frac{n^2-m^2}{4mn(n^2+m^2)^2}\sin n(X-VT)\right] \quad (3.12)$$

The deflection becomes

$$W(X,T)=\frac{P}{g^2}-\frac{P}{2g^2}e^{m(X-VT)}\cdot\left[\cos n(X-VT)+\frac{n^2-m^2}{2mn}\sin n(X-VT)\right] \quad (3.13)$$

c. Conclusion

W(X,T) given by Equations (3.11) and (3.13) satisfies the equation of motion given by Equation (3.2); it is bounded for all values of X and T; and it depends only on the distance from the pressure front. Hence it is a solution of Equation (3.2).

For simplicity, the radial deflection can be expressed in terms of the distance from the pressure front

$$S = X - VT \tag{3.14}$$

Introduce a function of S

$$F(S) = e^{-m|S|}\left[\cos nS - \frac{n^2-m^2}{2mn}\sin n|S|\right] \tag{3.15}$$

The deflection can be expressed in terms of F(S), i.e.,

$$W(S) = \frac{P}{g^2} - \frac{P}{2g^2}F(S) \qquad \text{for } S \leq 0 \tag{3.16a}$$

$$W(S) = \frac{P}{2g^2}F(S) \qquad \text{for } S > 0 \tag{3.16b}$$

2. Case for $V > V_{co}$

In this case, N's are real and let them be

$$N = 0, \quad \pm n_1, \quad \text{and} \ \pm n_2 \tag{3.17}$$

All poles of the integrand in the integral given by Equation (3.8) are on the real axis as shown in Figure 3.4.

By the residue theorem, the deflection can be found as follows

$$W(X,T) = 0 \quad \text{for } (X-VT) > 0 \tag{3.18a}$$

since there is no pole above the real axis.

$$W(X,T) = \frac{P}{g^2} + \frac{P\delta^2}{n_1^2-n_2^2}\left[\frac{\cos n_1(X-VT)}{n_1^2} - \frac{\cos n_2(X-VT)}{n_2^2}\right]$$

$$\text{for } (X-VT) \leq 0 \tag{3.18b}$$

where n_1 and n_2 are positive and $n_1 > n_2$.

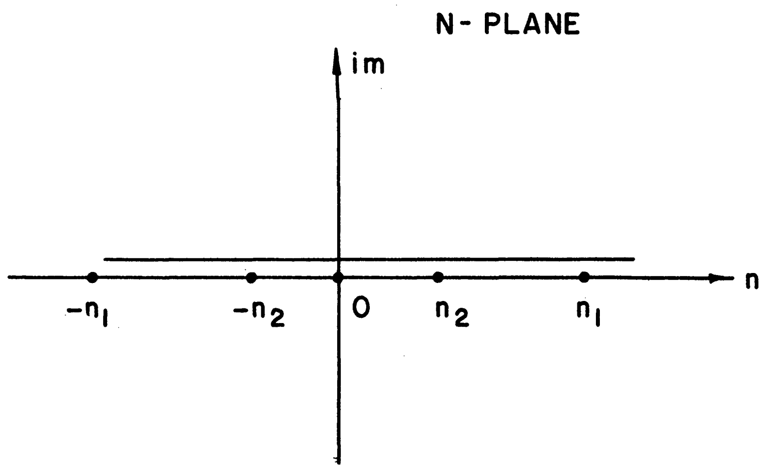

Figure 3.4 Positions of Poles When $V > V_{co}$ Without Damping

It is obvious that the following term can satisfy the homogeneous differential equation given by Equation (3.2):

$$W_1(X,T) = -\frac{p\delta^2}{n_1^2 - n_2^2} \cdot \frac{1}{n_1^2} \cdot \quad \cos n_1 (X-VT) \tag{3.19}$$

If this solution is super-imposed on the solution given by Equation (3.18), the resultant solution

$$W(X,T) = -\frac{p\delta^2}{n_1^2 - n_2^2} \cdot \frac{1}{n_1^2} \cos n_1(X-VT) \text{ for } (X-VT) > 0 \tag{3.20a}$$

$$W(X,T) = \frac{p}{g^2} - \frac{p\delta^2}{n_1^2 - n_2^2} \cdot \frac{1}{n_2^2} \cos n_2(X-VT) \text{ for } (X-VT) \leq 0 \tag{3.20b}$$

still satisfies Equation (3.2) and is bounded every-where. It is difficult to distinguish which of the solutions given by Equations (3.18) and (3.20) is the true one. By means of Kenney's concept of vanishingly small damping, a physically sound solution can be found and it is the true solution.

For taking into consideration the viscous damping, a viscous damping force term is added to Equation (3.2), then it is

$$\frac{\partial^4 W}{\partial X^4} + \delta^2 g^2 W + \frac{\partial^2 W}{\partial T^2} + C\,\frac{\partial W}{\partial T} = \delta^2 P \tag{3.21}$$

where $P(X,T) = \begin{cases} P & \text{when } X \leq VT \\ 0 & \text{when } X > VT \end{cases}$

$|W(X,T)| < \infty$ everywhere, and C is a dimensionless constant proportional to the coefficient of the viscous damping in the tube wall.

By Equation (3.3), Equation (3.21) after transform becomes

$$\frac{d^4 \bar{W}}{d X^4} + (\delta^2 g^2 - \Omega^2 - iC\Omega)\,\bar{W} = \frac{i\delta^2 P}{\Omega}\, e^{i\frac{\Omega}{V}X} \tag{3.22}$$

The particular solution for the above equation is

$$\bar{W}(X,\Omega) = \frac{i\delta^2 p}{\Omega\left(\frac{\Omega^4}{V^4} - \Omega^2 - ic\Omega + \delta^2 g^2\right)} e^{i\frac{\Omega}{V}X} \tag{3.23}$$

The inversion yields

$$\begin{aligned} W(X,T) &= \frac{i\delta^2 p}{2\pi}\int_{-\infty+iy}^{\infty+iy} \frac{e^{i\frac{\Omega}{V}X}\, e^{i\Omega T}}{\Omega\left(\frac{\Omega^4}{V^4} - \Omega^2 - ic\Omega + \delta^2 g^2\right)}\, d\Omega \\ &= \frac{i\delta^2 p}{2\pi}\int_{-\infty+iy_1}^{\infty+iy_1} \frac{e^{iN(X-VT)}}{N(N^4 - V^2N^2 - icVN + \delta^2 g^2)}\, dN \end{aligned} \tag{3.24}$$

where $y_1 > 0$. The poles of the integrand in the integral given by Equation (3.24) are the roots of the equation

$$N(N^4 - V^2N^2 - icVN + \delta^2 g^2) = 0$$

or

$$N = 0$$

and

$$N^4 - V^2N^2 - icVN + \delta^2 g^2 = 0 \tag{3.25}$$

Assume $N = \pm n_1 + im_1,\ \pm n_2 + im_2$ (3.26)

where n_1, n_2, m_1, and m_2 are real, $n_1^2 > n_2^2$; values of m_1 and m_2 are small compared with those of n_1 and n_2 and they approach zero as C becomes vanishingly small. Insert $N = n + im$ into Equation (3.25); the following equations are established

$$n^4 - 6n^2m^2 + m^4 - V^2(n^2 - m^2) + cVm + \delta^2 g^2 = 0 \tag{3.27a}$$

$$4m(n^2 - m^2) - 2mV^2 - cV = 0 \tag{3.27b}$$

Since m is small compared with n, the above equation can be reduced approximately to

$$n^4 - V^2n^2 + \delta^2 g^2 = 0 \tag{3.28}$$

and

$$4mn^2 - 2mV^2 - CV = 0$$

so that

$$m = \frac{CV}{2(2n^2 - V^2)} \tag{3.29}$$

Solutions for Equation (3.28) are

$$n_1^2 = \frac{V^2 + \sqrt{V^4 - 4\delta^2 g^2}}{2} \tag{3.30a}$$

$$n_2^2 = \frac{V^2 - \sqrt{V^4 - 4\delta^2 g^2}}{2} \tag{3.30b}$$

Since $V > V_{co}$ $(V_{co} = \sqrt{2\delta g})$, n_1^2, n_2^2 are real and positive. The corresponding m's are

$$m_1 = \frac{CV}{2\sqrt{V^4 - 4\delta^2 g^2}} > 0 \tag{3.31a}$$

$$m_2 = \frac{CV}{-2\sqrt{V^4 - 4\delta^2 g^2}} < 0 \tag{3.31b}$$

The poles of the integrand in the integral given by Equation (3.24) are shown in Figure 3.5.

a. Case for $(X-VT) > 0$

By the residue theorem, as C approaches zero, the deflection is

$$W(X,T) = -\frac{\delta^2 P}{n_1^2(n_1^2 - n_2^2)} \cos n_1 (X - VT)$$

or

$$W(S) = -\frac{\delta^2 P}{n_1^2(n_1^2 - n_2^2)} \cos n_1 (X - VT) \tag{3.32a}$$

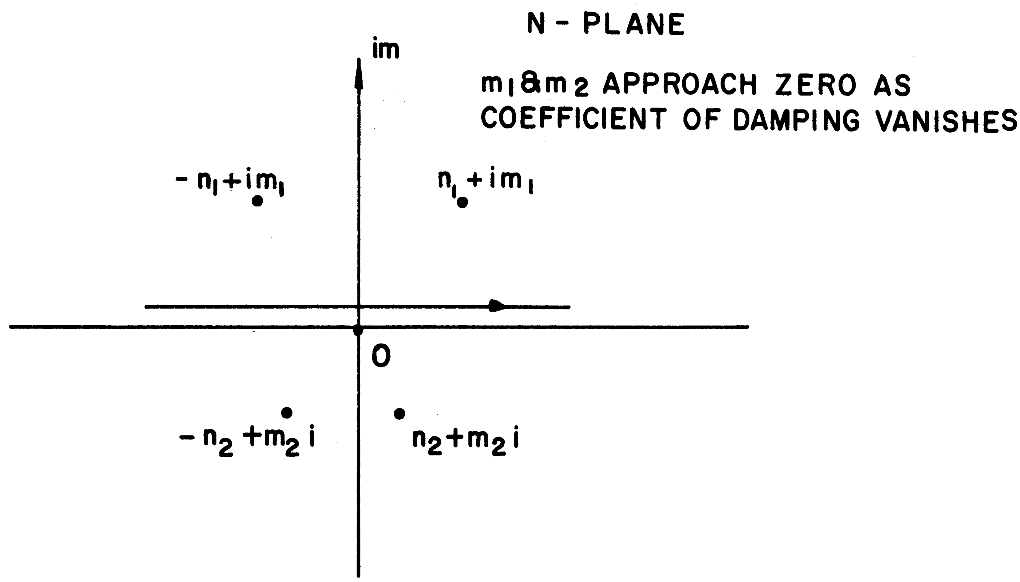

Figure 3.5 Positions of Poles When $V > V_{co}$ with Damping

b. Case for $(X-VT) \leq 0$

By the same procedure, the deflecetion is

$$W(X,T) = \frac{P}{g^2} - \frac{\delta^2 P}{n_2^2(n_1^2 - n_2^2)} \cos n_2 (X - VT)$$

or

$$W(S) = \frac{P}{g^2} - \frac{\delta^2 P}{n_2^2(n_1^2 - n_2^2)} \cos n_2 S \tag{3.32b}$$

B. Solution of the Equation from More Exact Theory

Introduce the same dimensionless variables P and V for p and v respectively as in Equation (3.1), the equations of motion from Equation (1.40) are

$$\begin{cases} \frac{\partial^2 W_s}{\partial X^2} - g^2 (W_b + W_s) - \frac{1}{\delta^2} \frac{\partial^2 (W_b + W_s)}{\partial T^2} = - P(X,T) & (3.33a) \\ \frac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \frac{\partial^2 W_b}{\partial T^2} = 0 & (3.33b) \end{cases}$$

where $P(X,T) = \begin{cases} P & \text{when} \quad X \leq VT \\ 0 & \text{when} \quad X > VT \end{cases}$

Instead of assigning any boundary conditions, $W_b(X,T)$ and $W_s(X,T)$ require bounded values everywhere.

Use Equation (3.3) to transform Equation (3.33) into ordinary ones

$$\begin{cases} \frac{d^2 \bar{W}_s}{dX^2} + \left(\frac{\Omega^2}{\delta^2} - g^2\right)(\bar{W}_b + \bar{W}_s) = - \frac{iP}{\Omega} e^{i\Omega \frac{X}{V}} & (3.34a) \\ \frac{d^2 \bar{W}_b}{dX^2} + \delta^2 \bar{W}_s + \Omega^2 \bar{W}_b = 0 & (3.34b) \end{cases}$$

with $|\bar{W}_b(X,\Omega)| < \infty$ and $|\bar{W}_s(X,\Omega)| < \infty$ for all positive* values of X.

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* See note on Page 45

The particular solution for Equation (3.34) is

$$\begin{cases} \bar{W}_b(X,\Omega) = A_1 e^{i\frac{\Omega}{V}X} & (3.35a) \\ \bar{W}_s(X,\Omega) = A_2 e^{i\frac{\Omega}{V}X} & (3.35b) \end{cases}$$

where

$$A_1 = \frac{iP\delta^2}{\Omega\{(V^2-1)(\frac{V^2}{\delta^2}-1)\frac{\Omega^4}{V^4}-[V^2(1+g^2)-g^2]\frac{\Omega^2}{V^2}+\delta^2 g^2\}}$$

$$A_2 = \frac{-iP(\Omega^2-\frac{\Omega^2}{V^2})}{\Omega\{(V^2-1)(\frac{V^2}{\delta^2}-1)\frac{\Omega^4}{V^4}-[V^2(1+g^2)-g^2]\frac{\Omega^2}{V^2}+\delta^2 g^2\}}$$

The inversion from Equation (3.4) yields

$$W_b(X,T) = \frac{1}{2\pi}\int_{-\infty+iy}^{\infty+iy} A_1 e^{i\frac{\Omega}{V}X} e^{-i\Omega T} d\Omega$$
$$= \frac{i\delta^2 P}{2\pi}\int_{-\infty+iy_1}^{\infty+iy_1} \frac{e^{iN(X-VT)}}{N\{(V^2-1)(\frac{V^2}{\delta^2}-1)N^4-[V^2(1+g^2)-g^2]N^2+\delta^2 g^2\}} dN \qquad (3.36a)$$

$$W_s(X,T) = \frac{1}{2\pi}\int_{-\infty+iy}^{\infty+iy} A_2 e^{i\frac{\Omega}{V}X} e^{-i\Omega T} d\Omega$$
$$= -\frac{iP}{2\pi}\int_{-\infty+iy_1}^{\infty+iy_1} \frac{N(V^2-1)e^{iN(X-VT)}}{\{(V^2-1)(\frac{V^2}{\delta^2}-1)N^4-[V^2(1+g^2)-g^2]N^2+\delta^2 g^2\}} dN \qquad (3.36b)$$

where $N = \frac{\Omega}{V}$ and $y_1 > 0$.

Poles of the integrand in the integrals given by Equation (3.36) are the roots of the equation

$$N\{(V^2-1)(\frac{V^2}{\delta^2}-1)N^4-[V^2(1+g^2)-g^2]N^2+\delta^2 g^2\} = 0$$

or

$$N = 0$$

and

$$(V^2-1)(\frac{V^2}{\delta^2}-1)N^4-[V^2(1+g^2)-g^2]N^2+\delta^2 g^2 = 0$$

The latter is identical to Equation (2.16). From Figure 2.4 through 2.6, N's are complex when $v < V_{co}$, real when $V_{co} < V < V_{c1}$, two real and two imaginary when $V_{c1} < V < V_{c2}$, and real when $V > V_{c2}$. These four cases are discussed separately.

1. <u>Case for $V < V_{co}$</u>

In this case, N's for the latter equation are complex,

$$N = \pm n \pm im \tag{3.37}$$

By the residue theorem, the deflections due to bending and shear are

$$W_b(X,T) = \frac{P}{2g^2}\, e^{-m(X-VT)}\left[\cos n(X-VT) - \frac{n^2-m^2}{2mn}\sin n(X-VT)\right]$$

$$W_s(X,T) = \frac{\delta^2 P}{(V^2-\delta^2)}\cdot\frac{e^{-m(X-VT)}}{4mn}\sin n(X-VT)$$

for $(X - VT) > 0$;

$$W_b(X,T) = \frac{P}{g^2} - \frac{P}{2g^2}\, e^{m(X-VT)}\left[\cos n(X-VT) + \frac{n^2-m^2}{2mn}\sin n(X-VT)\right]$$

$$W_s(X,T) = \frac{\delta^2 P}{(V^2-\delta^2)}\cdot\frac{e^{m(X-VT)}}{4mn}\sin n(X-VT)$$

for $(X - VT) \leq 0$.

Again introduce $S = X - VT$, and define a function of S

$$F(S) = e^{-m|S|}\left[\cos nS - \frac{n^2-m^2}{2mn}\sin n|S|\right]$$

then

$$W_b(S) = \frac{P}{g^2} - \frac{P}{2g^2}F(S) \qquad \text{for } S \leq 0 \tag{3.38a}$$

$$W_b(S) = \frac{P}{2g^2}F(S) \qquad \text{for } S > 0 \tag{3.38b}$$

$$W_s(S) = \frac{\delta^2 P}{(V^2-\delta^2)}\cdot\frac{e^{-m|S|}}{4mn}\sin nS \tag{3.39}$$

for all values of S

2. <u>Case for $V > V_{co}$</u>

In this case, the solution for Equation (3.33) is not unique. In order to find a physically sound solution, a viscous damping term which vanishes in the limit is also introduced into Equation (3.33)

$$\begin{cases} \dfrac{\partial^2 W_s}{\partial X^2} - \gamma^2 (W_b + W_s) - \dfrac{1}{\delta^2}\dfrac{\partial^2 (W_b+W_s)}{\partial T^2} - \dfrac{C}{\delta^2}\dfrac{\partial (W_b+W_s)}{\partial T} = -P(X,T) & (3.40a) \\ \dfrac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \dfrac{\partial^2 W_b}{\partial T^2} = 0 & (3.40b) \end{cases}$$

where $P(X,T) = \begin{cases} P \text{ when } X \leq VT \\ 0 \text{ when } X > VT \end{cases}$

By means of the same technique as used in the elementary theory, the following solutions are found in the limit as C vanishes:

a. <u>$V_{co} < V < V_{c1}$</u>

$$W_b(S) = \frac{P}{\gamma^2} - \frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{\cos n_2 S}{n_2^2(n_1^2 - n_2^2)} \qquad \text{for } S \leq 0 \qquad (3.41a)$$

$$W_b(S) = -\frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{\cos n_1 S}{n_1^2(n_1^2 - n_2^2)} \qquad \text{for } S > 0 \qquad (3.41b)$$

$$W_s(S) = \frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{\cos n_2 S}{(n_1^2 - n_2^2)} \qquad \text{for } S \leq 0 \qquad (3.42a)$$

$$W_s(S) = \frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{\cos n_1 S}{(n_1^2 - n_2^2)} \qquad \text{for } S > 0 \qquad (3.42b)$$

where $S = X - VT$, n_1 and n_2 are the positive real roots of Equation (2.16) and $n_1 > n_2$ is assumed.

b. <u>$V_{c1} < V < V_{c2}$</u>

$$W_b(S) = \frac{P}{\gamma^2} + \frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)} \left[\frac{\cos nS}{n^2} + \frac{e^{mS}}{2m^2}\right] \qquad \text{for } S \leq 0 \qquad (3.43a)$$

$$W_b(S) = -\frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{e^{-mS}}{2m^2(n^2+m^2)} \qquad \text{for } S > 0 \qquad (3.43b)$$

$$W_s(S) = -\frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)}\left[\cos nS - \frac{e^{mS}}{2}\right] \qquad \text{for } S \leq 0 \qquad (3.44a)$$

$$W_s(S) = -\frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{e^{-mS}}{2(m^2+n^2)} \qquad \text{for } S > 0 \qquad (3.44b)$$

Where $\pm n$ are the real roots of Equation (2.16) and $\pm mi$ are the imaginary roots of Equation (2.16).

c. $V > V_{c2}$

$$W_b(S) = \frac{P}{q^2} + \frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{1}{(n_1^2-n_2^2)}\left[\frac{\cos n_1 S}{n_1^2} - \frac{\cos n_2 S}{n_2^2}\right] \qquad \text{for } S \leq 0 \qquad (3.45a)$$

$$W_b(S) = 0 \qquad \text{for } S > 0 \qquad (3.45b)$$

$$W_s(S) = -\frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{1}{(n_1^2-n_2^2)}\left[\cos n_1 S - \cos n_2 S\right] \qquad \text{for } S \leq 0 \qquad (3.46a)$$

$$W_s(S) = 0 \qquad \text{for } S > 0 \qquad (3.46b)$$

where n_1 and n_2 are the positive real roots of Equation (2.16) and $n_1 > n_2$ is assumed. The reason why W_b and W_s are zero before the pressure front arrives lies on the fact that there is no disturbance, since the dilatational wave velocity V_{c2} is less than that of the pressure front in this case.

C. Discussion

1. Soundness of the Sinusoidal Wave Solution for the Case $V > V_{co}$

Since the deflection curve of the intersection of the middle surface and the radial plane is a sinusoidal wave extended to infinite length, the energy in this system becomes infinite. It is doubtful whether this result might violate energy principle. Actually it does not as was pointed out by Dörr(9) and Kenney(10).

2. Comparison of Results from Both Theories

a. $V < V_{co}$

The formula given by Equation (3.16) for W has the identical form as that given by Equation (3.38) for W_b. From the velocity spectra in Figure 2.4 through 2.6, it is obvious that they are almost identical, i.e., n's or m's are nearly the same for both theories. The real part of the wave number is small, so that the frequency of vibration is low. In this case, the total deflection in the elementary theory is nearly equal to that due to the bending in the more exact theory. In calculating the bending stress, the deflection formula given by Equation (3.16) may be used without any serious error.

b. $V > V_{co}$

From the velocity spectra in Figure 2.4 through 2.6, they are quite different for both theories except the part for small wave numbers. In this case, the high frequency of vibration is caused, so the deflection formula given by Equation (3.16) from the elementary theory may not be used. Instead, the formula given by Equation (3.38) from the more exact theory should be used in calculating the bending stress.

IV. TRANSIENT RESPONSE OF A THIN-WALLED CYLINDRICAL TUBE WITH SEMI-INFINITE LENGTH UNDER INTERNAL MOVING PRESSURE

In this chapter, a cylindrical tube with a semi-infinite length is dealt with. The boundary condition at one end of the tube is that the periphery is simply supported in the radial direction. The pressure front starts to move at this end. The assumptions as to the intensity and velocity of the pressure front are the same as those in Chapter III.

A. Solution of the Equation from Elementary Theory

In this section, only the case for the velocity of the pressure front less than the critical, i.e., $V < V_{co}$ is investigated. When $V > V_{co}$, the elementary theory will involve serious error.

1. Formulation of the Problem

From Equation (1.39), the equation of motion which is valid in the quarter plane defined by X and T is

$$\frac{\partial^4 W}{\partial X^4} + \delta^2 q^2 W + \frac{\partial^2 W}{\partial T^2} = \delta^2 P(X,T), \qquad X > 0, \quad T > 0 \qquad (4.1)$$

where $P(X,T) = \begin{cases} P & \text{when } X \leq VT \\ 0 & \text{when } X > VT \end{cases}$

At the near end, the radial deflection and its second partial derivative with respect to X are assumed to vanish, since the periphery is simply supported in the radial direction. At the far end, no disturbance is assumed, so that the radial deflection as well as all its partial derivatives with respect to X vanish. The boundary conditions are

$$W(0,T) = 0, \qquad \frac{\partial^2 W(0,T)}{\partial X^2} = 0 \tag{4.2a}$$

$$\begin{cases} \lim\limits_{X\to\infty} W(X,T) = 0, & \lim\limits_{X\to\infty} \dfrac{\partial W(X,T)}{\partial X} = 0 \\ \lim\limits_{X\to\infty} \dfrac{\partial^2 W(X,T)}{\partial X^2} = 0, & \lim\limits_{X\to\infty} \dfrac{\partial^3 W(X,T)}{\partial X^3} = 0 \end{cases} \tag{4.2b}$$

No initial displacement and velocity are assumed radially, so the initial conditions are

$$W(X,0) = 0, \qquad \frac{\partial W(X,0)}{\partial T} = 0 \tag{4.3}$$

2. Method of Solution

The partial differential equation given by Equation (4.1) can be transformed into an ordinary one by means of the Fourier sine transform.[18] Take the Fourier sine transform with respect to X, then the transformed function is

$$\overline{W}(N,T) = \int_0^\infty W(X,T)\ \mathrm{Sin}NX\ dX \tag{4.4}$$

where N is real. The inversion formula is given by

$$W(X,T) = \frac{2}{\pi}\int_0^\infty \overline{W}(N,T)\ \mathrm{Sin}NX\ dN \tag{4.5}$$

To justify this transform, the integrals in Equations (4.4) and (4.5) have to be proved convergent. It is assumed this is the case, and when W(X,T) is found one can verify that this W is a solution simply by substituting into Equation (4.1).

Due to the boundary conditions given by Equation (4.2), Equation (4.1) after transform becomes

$$\frac{d^2\bar{W}}{dT^2} + (N^4 + \delta^2 g^2)\bar{W} = \frac{\delta^2 P}{N}(1 - \cos VTN) \qquad (4.6)$$

The initial conditions after transform are

$$\bar{W}(N,0) = 0, \qquad \frac{d\bar{W}(N,0)}{dT} = 0 \qquad (4.7)$$

By the convolution theorem, the solution for Equation (4.6) with initial conditions given by Equation (4.7) is

$$\bar{W}(N,T) = \frac{\delta^2 P}{N}\cdot\frac{1}{\Omega}\int_0^T (1 - \cos V\tau N)\sin\Omega(T-\tau)d\tau$$

$$= \frac{\delta^2 P}{N}\left\{\frac{1}{\Omega^2}(1-\cos\Omega T) + \frac{\cos\Omega T - \cos VTN}{\Omega^2 - V^2N^2}\right\} \qquad (4.8)$$

where

$$\Omega^2 = N^4 + \delta^2 g^2 \qquad (4.9)$$

is identical to the frequency Equation (2.3). The inversion yields

$$W(X,T) = \frac{2}{\pi}\int_0^\infty \bar{W}(N,T)\,\sin XN\,dN$$

$$= \delta^2 P\frac{2}{\pi}\left\{\int_0^\infty \frac{\sin XN}{N\Omega^2}dN - \int_0^\infty \frac{\cos VTN \sin XN}{N(\Omega^2 - V^2N^2)}dN\right.$$

$$\left. - \int_0^\infty \left[\frac{1}{\Omega^2} - \frac{1}{\Omega^2 - V^2N^2}\right]\frac{\sin XN}{N}\cos\Omega T\,dN\right\} \qquad (4.10)$$

The first two integrals on the right hand side of Equation (4.10) can be evaluated in closed form by the residue theorem, and they are

$$I_1 = \delta^2 p \frac{2}{\pi} \left\{ \int_0^\infty \frac{\sin XN}{N\Omega^2} dN - \int_0^\infty \frac{\cos VTN \sin XN}{N(\Omega^2 - V^2N^2)} dN \right\}$$

$$= \delta^2 p \frac{2}{\pi} \left\{ \int_0^\infty \frac{\sin XN}{N(N^4+\delta^2 g^2)} dN - \int_0^\infty \frac{\cos VTN \sin XN}{N(N^4 - V^2N^2 + \delta^2 g^2)} dN \right\}$$

$$= \frac{P}{g^2} - \frac{P}{2g^2} e^{m(X-VT)} \left[\cos n(X-VT) + \frac{n^2-m^2}{2mn} \sin n(X-VT) \right]$$

$$+ \frac{P}{2g^2} e^{-m(X+VT)} \left[\cos n(X+VT) - \frac{n^2-m^2}{2mn} \sin n(X+VT) \right]$$

$$- \frac{P}{g^2} e^{-\lambda X} \cos \lambda X \qquad \text{for } (X - VT) \leq 0 \qquad (4.11a)$$

$$I_1 = \frac{P}{2g^2} e^{-m(X-VT)} \left[\cos n(X-VT) - \frac{n^2-m^2}{2mn} \sin n(X-VT) \right]$$

$$+ \frac{P}{2g^2} e^{-m(X+VT)} \left[\cos n(X+VT) - \frac{n^2-m^2}{2mn} \sin n(X+VT) \right]$$

$$- \frac{P}{g^2} e^{-\lambda X} \cos \lambda X \qquad \text{for } (X - VT) > 0 \qquad (4.11b)$$

where $N = n + im$ is the root of Equation (2.6), i.e.,

$$N^4 - V^2N^2 + \delta^2 g^2 = 0$$

n, m are both real and positive and

$$\lambda = \sqrt{\frac{\delta g}{2}} \qquad (4.12)$$

Let the last integral on the right hand side of Equation (4.10) be

$$I_2 = -\frac{2\delta^2 p}{\pi}\int_0^\infty \left[\frac{1}{\Omega^2} - \frac{1}{(\Omega^2 - V^2N^2)}\right] \frac{\sin NX}{N} \cos \Omega T \, dN$$

or

$$I_2 = -\frac{2\delta^2 p}{\pi}\int_0^\infty \left[\frac{1}{(N^4+\delta^2 g^2)} - \frac{1}{(N^4 - V^2N^2 + \delta^2 g^2)}\right] \frac{\sin NX}{N} \cos \sqrt{N^4+\delta^2 g^2}\, T \, dN \qquad (4.13)$$

It is very difficult to evaluate I_2 by means of the residue theorem, since the integrand involves branch points. Numerical integration will be used to compute I_2. The final result is

$$W(X,T) = I_1(X,T) + I_2(X,T) \qquad (4.14)$$

3. Interpretation of the Solution

a. Contribution Due to I_1

The term involving $(X-VT)$ in Equation (4.11) is identical to Equation (3.16) which is the steady state solution with the pressure front moving to the right as shown in Figure 4.1a; the term involving $(X+VT)$ is the steady state solution with the pressure front moving to the left as shown in Figure 4.1b; and the term involving λX is the static solution due to the pressure as shown in Figure 4.1c. The solution super-imposed by these three parts can satisfy Equation (4.1) and boundary conditions in Equation (4.2), but the initial conditions in Equation (4.3) cannot be satisfied. In order to satisfy the initial conditions, I_2 must be super-imposed.

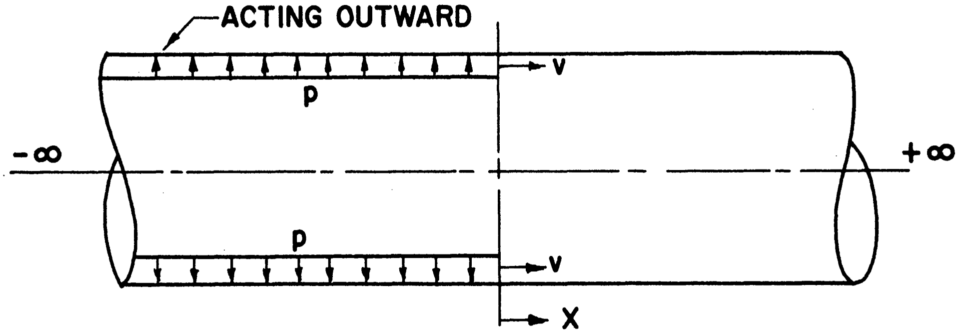

Figure 4.1a Pressure Front Moving to the Right

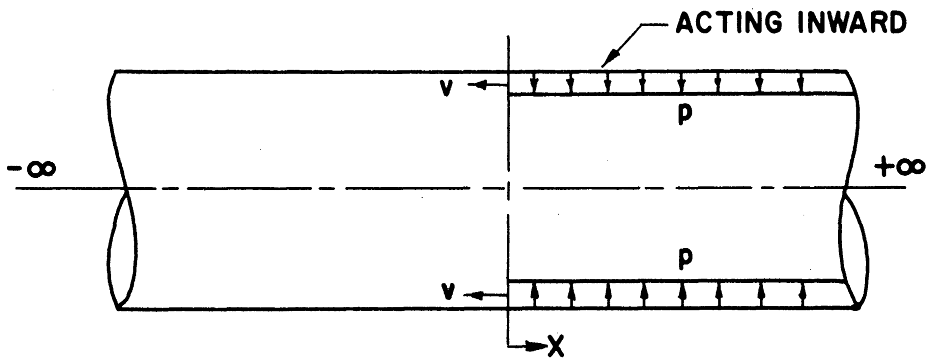

Figure 4.1b Pressure Front Moving to the Left

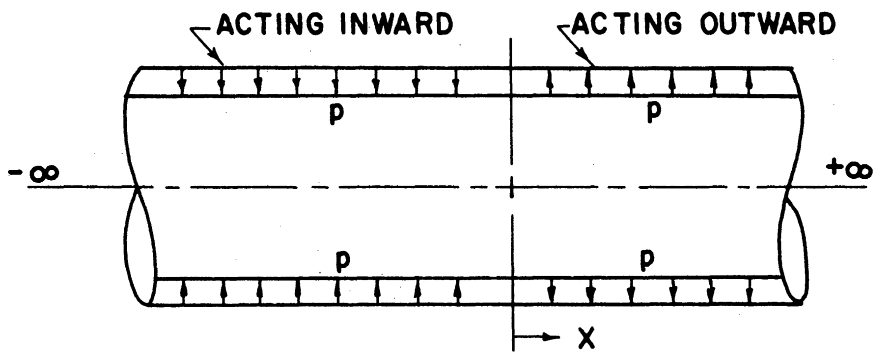

Figure 4.1c Static Pressure

b. Contribution Due to I_2

The solution due to I_2 is to correct the initial conditions given by the solution due to I_1 . This can be interpreted as the transient term and it will disappear as T approaches infinity. This is true from the Riemann-Lebesgue lemma.

B. Solution of the Equation from More Exact Theory

From Equation (1.40), the equation of motion which is valid in the quarter plane defined by X and T is

$$\begin{cases} \dfrac{\partial^2 W_s}{\partial X^2} - g^2 (W_b + W_s) - \dfrac{1}{\delta^2} \dfrac{\partial^2 (W_b + W_s)}{\partial T^2} = - P(X,T) & (4.15a) \\ \\ \dfrac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \dfrac{\partial^2 W_b}{\partial T^2} = 0 & (4.15b) \end{cases}$$

$$X > 0, \quad T > 0$$

$$\text{where} \quad P(X, T) = \begin{cases} P & \text{when} \quad X \leq VT \\ 0 & \text{when} \quad X > VT \end{cases}$$

The boundary condition at the near end is simply suported along the periphery of the tube and no disturbance is assumed to exist at the far end; they are

$$\begin{cases} W_b(0,T) + W_s(0,T) = 0 \\ \dfrac{\partial^2 W_b(0,T)}{\partial X^2} = 0 \end{cases} \qquad (4.16a)$$

$$\begin{cases} \lim\limits_{X \to \infty} W_b(X,T) = 0, & \lim\limits_{X \to \infty} \dfrac{\partial W_b(X,T)}{\partial X} = 0 \\ \lim\limits_{X \to \infty} W_s(X,T) = 0, & \lim\limits_{X \to \infty} \dfrac{\partial W_s(X,T)}{\partial X} = 0 \end{cases} \qquad (4.16b)$$

For convenience in later use, the boundary conditions given by Equation (4.16a) are changed into the form

$$W_b(0,T) = 0 \;, \quad W_s(0,T) = 0 \tag{4.17}$$

These two forms of boundary conditions at the near end are consistant, since $\frac{\partial^2 W_b}{\partial X^2}$ will vanish in the limit of $X = 0$ if $W_s(0,T) = 0$ and $W_b(0,T) = 0$ are inserted into Equation (4.15b). The initial displacement and velocity due to both bending and shear are assumed zero, therefore the initial conditions are

$$\begin{cases} W_b(X,0) = 0 \,, & \frac{\partial W_b(X,0)}{\partial T} = 0 \\ W_s(X,0) = 0 \,, & \frac{\partial W_s(X,0)}{\partial T} = 0 \end{cases} \tag{4.18}$$

1. Case for $V < V_{co}$

The Fourier sine transform is also used to reduce Equation (4.15) to ordinary ones. By Equation (4.4), Equation (4.15) with boundary conditions given by Equation (4.17) becomes

$$-N^2 \bar{W}_s - \gamma^2(\bar{W}_b + \bar{W}_s) - \frac{1}{\delta^2}\frac{d^2(\bar{W}_b + \bar{W}_s)}{dT^2} = -\frac{P}{N}(1 - \cos NVT) \tag{4.19a}$$

$$-N^2 \bar{W}_b + \delta^2 \bar{W}_s - \frac{d^2 \bar{W}_b}{dT^2} = 0 \tag{4.19b}$$

From Equation (4.19b), $\bar{W}_s$ can be expressed in terms of $\bar{W}_b$ and its second derivative

$$\bar{W}_s = \frac{N^2}{\delta^2}\bar{W}_b + \frac{1}{\delta^2}\frac{d^2\bar{W}_b}{dT^2} \tag{4.20}$$

Eliminating $\bar{W}_s$ and $\frac{d^2\bar{W}_s}{dT^2}$ in Equations (4.19a) and (4.20), a fourth order ordinary differential equation is obtained

$$\frac{d^4\bar{W}}{dT^4} + (\delta^2N^2 + \delta^2g^2 + N^2 + \delta^2)\frac{d^2\bar{W}_b}{dT^2} + [\delta^2N^2(g^2+N^2) + \delta^4g^2]\bar{W}_b$$

$$= \frac{\delta^4 P}{N}(1 - \cos NVT) \tag{4.21}$$

The transformed initial conditions are

$$\begin{cases} \bar{W}_b(N,0) = 0, & \frac{d\bar{W}_b(N,0)}{dT} = 0 \\ \bar{W}_s(N,0) = 0, & \frac{d\bar{W}_s(N,0)}{dT} = 0 \end{cases} \tag{4.22}$$

Solutions for Equations (4.21) and (4.20) with initial conditions given by Equation (4.22) are

$$\bar{W}_b(N,T) = B_1\cos\Omega_1 T + B_2\cos\Omega_2 T + \frac{\delta^2 P}{N}\left[\frac{1}{F_1} - \frac{\delta^2\cos NVT}{F_2}\right] \tag{4.23a}$$

$$\bar{W}_s(N,T) = \left(\frac{N^2}{\delta^2} - \frac{\Omega_1^2}{\delta^2}\right)B_1\cos\Omega_1 T + \left(\frac{N^2}{\delta^2} - \frac{\Omega_2^2}{\delta^2}\right)B_2\cos\Omega_2 T$$

$$+ PN\left[\frac{1}{F_1} - \frac{\delta^2(1-V^2)}{F_2}\cos NVT\right] \tag{4.23b}$$

where

$$F_1 = N^4 + g^2N^2 + \delta^2 g^2 \tag{4.24a}$$

$$F_2 = (V^2-1)(V^2-\delta^2)N^4 - [\delta^2 V^2(1+g^2) - \delta^2 g^2]N^2 + \delta^4 g^2 \qquad (4.24b)$$

$$\Omega_1 = \left\{ \frac{N^2(1+\delta^2)+\delta^2(1+g^2)}{2} - \sqrt{\left[\frac{N^2(1+\delta^2)+\delta^2(1+g^2)}{2}\right]^2 - \left[\delta^2 N^2(N^2+g^2)+\delta^4 g^2\right]} \right\}^{1/2} \qquad (4.25a)$$

$$\Omega_2 = \left\{ \frac{N^2(1+\delta^2)+\delta^2(1+g^2)}{2} + \sqrt{\left[\frac{N^2(1+\delta^2)+\delta^2(1+g^2)}{2}\right]^2 - \left[\delta^2 N^2(N^2+g^2)+\delta^4 g^2\right]} \right\}^{1/2} \qquad (4.25b)$$

$$B_1 = \frac{P}{N(\Omega_1^2-\Omega_2^2)} \left\{ \frac{\delta^2\Omega_2^2}{F_1} - \frac{\delta^4(\Omega_2^2 - N^2V^2)}{F_2} \right\} \qquad (4.26a)$$

$$B_2 = -\frac{P}{N(\Omega_1^2-\Omega_2^2)} \left\{ \frac{\delta^2\Omega_1^2}{F_1} - \frac{\delta^4(\Omega_1^2 - N^2V^2)}{F_2} \right\} \qquad (4.26b)$$

In Equation (4.23), the last term on the right hand side is for the particular solution and the first two terms are solutions for the homogeneous equation. Since the particular solution cannot satisfy the initial condition in Equation (4.22), two additional solutions for the homogeneous equation are super-imposed in order to match the homogeneous initial conditions. Equation (4.25) is identical to the frequency Equation (2.10), so that Ω_1 is the frequency in the first real arm and Ω_2 the frequency in the second real arm in the frequency spectrum. All are with real wave number N. By the inversion formula given by Equation (4.5), $W_b(X,T)$

and $W_s(X,T)$ can be found

$$W_b(X,T) = \frac{2}{\pi}\delta^2 P\int_0^\infty \frac{1}{N}\left(\frac{1}{F_1} - \frac{\delta^2 \cos NVT}{F_2}\right)\sin XN\,dN$$

$$+ \frac{2}{\pi}\int_0^\infty B_1 \cos \Omega_1 T \sin XN\,dN$$

$$+ \frac{2}{\pi}\int_0^\infty B_2 \cos \Omega_2 T \sin XN\,dN$$

$$= I_b + I_{b1} + I_{b2} \qquad (4.27a)$$

$$W_s(X,T) = \frac{2}{\pi}P\int_0^\infty N\left[\frac{1}{F_1} - \frac{\delta^2(1-V^2)}{F_2}\cos NVT\right]\sin XN\,dN$$

$$+ \frac{2}{\pi\delta^2}\int_0^\infty B_1 (N^2 - \Omega_1^2)\cos\Omega_1 T \sin XN\,dN$$

$$+ \frac{2}{\pi\delta^2}\int_0^\infty B_2 (N^2 - \Omega_2^2)\cos\Omega_2 T \sin XN\,dN$$

$$= I_s + I_{s1} + I_{s2} \qquad (4.27b)$$

In the above expressions, I_b and I_s are due to forced vibration; I_{b1} and I_{s1} are due to free vibration corresponding to the first real arm; I_{b2} and I_{s2} are due to free vibration corresponding to the second real arm. It will be shown later that in the case $V < V_{co}$, the contribution due to the second real arm of free vibration is small, so that it can be neglected.

The improper integrals in I_b and I_s can be computed without any difficulty be means of the residue theorem, but these integrands in I_{b1}, I_{b2}, I_{s1} and I_{s2} involve too many branch points to be computed by analytic methods. Numerical integration will be used to compute them. By the residue theorem, I_b and I_s are computed as follows:

$$I_b(X,T) = \frac{2}{\pi}\delta^2 P\int_0^\infty \frac{1}{N}\left(\frac{1}{F_1} - \frac{\delta^2 \cos NVT}{F_2}\right)\sin XN\,dN$$

$$= \frac{P}{g^2} - \frac{P}{2g^2}e^{m(X-VT)}\left[\cos n(X-VT) + \frac{n^2-m^2}{2mn}\sin n(X-VT)\right]$$

$$+ \frac{P}{2g^2}e^{-m(X+VT)}\left[\cos n(X+VT) - \frac{n^2-m^2}{2mn}\sin n(X+VT)\right]$$

$$- \frac{P}{g^2}e^{-m_0 X}\left(\cos n_0 X - \frac{n_0^2-m_0^2}{2m_0 n_0}\sin n_0 X\right)$$

for $(X - VT) \leq 0$ (4.28a)

$$I_b(X,T) = \frac{P}{2g^2}e^{-m(X-VT)}\left[\cos n(X-VT) - \frac{n^2-m^2}{2mn}\sin n(X-VT)\right]$$

$$+ \frac{P}{2g^2}e^{-m(X+VT)}\left[\cos n(X+VT) - \frac{n^2-m^2}{2mn}\sin n(X+VT)\right]$$

$$- \frac{P}{g^2}e^{-m_0 X}\left(\cos n_0 X - \frac{n_0^2-m_0^2}{2m_0 n_0}\sin n_0 X\right)$$

for $(X - VT) > 0$ (4.28b)

where $N = n + im$ is the root of the Equation (2.16), i.e.,

$$(V^2-1)(V^2-\delta^2)N^4 - [\delta^2 V^2(1+g^2) - \delta^2 g^2]N^2 + \delta^4 g^2 = 0$$

and n , m are both real and positive; $N_o = n_o + im_o$ is the root of the equation

$$N^4 + q^2 N^2 + \delta^2 q^2 = 0$$

and n_o , m_o are both real and positive.

$$I_s(X,T) = \frac{p\, e^{-m_o X}}{2 m_o n_o} \sin n_o X$$

$$+ \frac{\delta^2 p}{(V^2-\delta^2)} \cdot \frac{e^{-m|X-VT|}}{4mn} \sin n(X-VT)$$

$$+ \frac{\delta^2 p}{(V^2-\delta^2)} \cdot \frac{e^{-m(X+VT)}}{4mn} \sin n(X+VT) \qquad (4.29)$$

The solutions due to I_b and I_s have the similar form as that due to I_1 in the elementary theory, so they have the same physical interpretation. The solutions due to I_{b1} and I_{b2} or I_{s1} and I_{s2} are employed to match the homogeneous initial conditions. I_{b1} or I_{s1} corresponds to I_2 in the elementary theory, but there is an additional term, I_{b2} or I_{s2} , in the more exact theory, since there is a second real arm in the frequency spectrum.

2. <u>Case for $V > V_{co}$</u>

In this case, the Fourier sine transfrom cannot be used, since the transform formula given by Equation (4.4) diverges. The standard method of solution is to use the Laplace transform with respect to time variable T , to reduce the partial differential equation into an ordinary one, then the transformed function is to be determinedby solving the ordinary differential equation. The difficulty lies in the fact that the integrand of the inversion integral involves too many branch points to be dealt with. The method used in this work consists of two parts. First, the particular solution which satisfies the non-homogeneous Equation (4.15) only is found as that in Chapter III; then a solution of the homogeneous equation (set $P(X,T) = 0$ in Equation (4.15)) is super-imposed to the first one in order to match the boundary conditions given by Equation (4.16) and the initial conditions given by Equation (4.18). The solution of the homogeneous equation with specified boundary conditions as well as initial conditions is to be determined numerically by the method of characteristics. Since the forcing function P in Equation (4.15) is a stepped one which has a discontinuity at $X = VT$, the method of characteristics is difficult to apply directly in Equation (4.15). This is the reason why the solution has to be calculated in two parts.

The homogeneous equation is obtained by putting $P = 0$ in Equation (4.15), then

$$\begin{cases} \dfrac{\partial^2 W_s}{\partial X^2} - g^2 (W_b + W_s) - \dfrac{1}{\delta^2} \cdot \dfrac{\partial^2 (W_b + W_s)}{\partial T^2} = 0 & (4.30a) \\ \\ \dfrac{\partial^2 W_b}{\partial X^2} + \delta^2 W_s - \dfrac{\partial^2 W_b}{\partial T^2} = 0 & (4.30b) \end{cases}$$

$$X > 0 , \quad T > 0$$

The boundary conditions are specified at $X = 0$ as

$$\begin{cases} W_b(0,T) + W_s(0,T) = f_1(T) & (4.31a) \\ \\ \dfrac{\partial^2 W_b(0,T)}{\partial X^2} = f_2(T) & (4.31b) \end{cases}$$

The initial conditions are specified at $T = 0$ as

$$\begin{cases} W_b(X,0) = f_3(X) , & \dfrac{\partial W_b(X,0)}{\partial T} = f_4(X) \qquad (4.32a) \\ W_s(X,0) = f_5(X) , & \dfrac{\partial W_s(X,0)}{\partial T} = f_6(X) \qquad (4.32b) \end{cases}$$

Let k be the slope of the characteristic lines of the system in Equation (4.30), k is to be determind by the following Equation[19]

$$\begin{vmatrix} (k^2 - \frac{1}{\delta^2}) & -\frac{1}{\delta^2} \\ 0 & (k^2 - 1) \end{vmatrix} = 0$$

or

$$(k^2 - \frac{1}{\delta^2})(k^2 - 1) = 0$$

There are four real k's , i.e.,

$$k = \pm \frac{1}{\delta} , \pm 1$$

so that there are four sets of real characteristic lines for Equation (4.30) and the system can be reduced to a fourth order hyperbolic differential equation. The characteristic lines are

$$\frac{dT}{dX} = k = \pm \frac{1}{\delta} , \pm 1 \qquad (4.33)$$

Since δ is the dimensionless modified shear wave velocity in a plate and "1" is the dimensionless dilatational wave velocity in a plate, shear waves are propagated along the characteristic lines with slope $\pm \frac{1}{\delta}$, and dilatational waves along the characteristic lines with slope ± 1. There are only two kinds of waves in this elastic system[15]. Each can be propagated with positive or negative velocity, so there are four sets of characteristic lines along which they are propagated from the source as shown in Figure 4.2.

For convenience in writing out the finite difference equations, defined along characteristic lines, four first order simultaneous equations[20] are used instead of two second order simultaneous equations given by Equation (4.30). Five new dimensionless variables are introduced. They are dimensionless total deflection in the radial direction:

$$W = w_b + w_s \tag{4.34a}$$

dimensionless velocity along the radial direction:

$$\bar{v} = \frac{v_1}{\sqrt{12}\, v_d} \tag{4.34b}$$

where

$$v_1 = \frac{\partial(w_b + w_s)}{\partial t}$$

dimensionless angular velocity due to bending:

$$\bar{\omega} = \frac{h}{12\, v_d}\, \omega \tag{4.34c}$$

where

$$\omega = \frac{\partial^2 w_b}{\partial t\, \partial x}$$

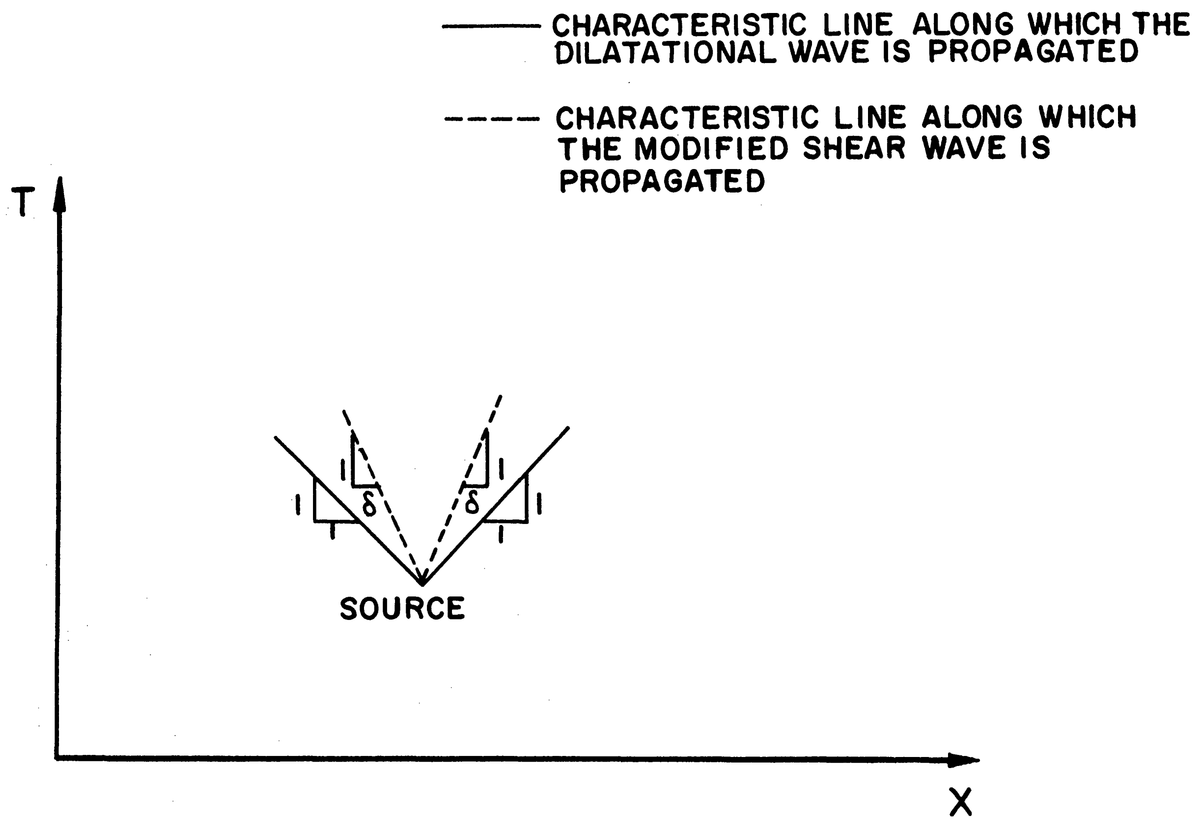

Figure 4.2 Four Characteristic Lines from the Source.

dimensionless resultant shearing force:

$$Q = \frac{Q_x}{\sqrt{12}\,h k G} \tag{4.34d}$$

and dimensionless bending moment:

$$M = \frac{1-\nu^2}{E h^2} M_{xx} \tag{4.34e}$$

The four simultaneous equations are

$$\frac{\partial Q}{\partial X} - g^2 W = \frac{1}{\delta^2}\frac{\partial \bar{v}}{\partial T} \tag{4.35a}$$

$$\frac{\partial Q}{\partial T} = \frac{\partial \bar{v}}{\partial X} - \bar{\omega} \tag{4.35b}$$

$$\frac{\partial M}{\partial X} - \delta^2 Q = -\frac{\partial \bar{\omega}}{\partial T} \tag{4.35c}$$

$$\frac{\partial M}{\partial T} = -\frac{\partial \bar{\omega}}{\partial X} \tag{4.35d}$$

Since

$$W = W_{T_0} + \int_{T_0}^{T} \bar{v}\, dT$$

where W_{T_0} is the radial displacement at $T = T_0$, there are only four unknowns, M, $\bar{\omega}$, Q, and $\bar{v}$, to be determined by the above four equations with boundary conditions specified at $X = 0$ and initial conditions specified at $T = 0$. Along the characteristic lines, these equations are simplified as follows:

along $\frac{dT}{dX} = 1$:

$$dM - \delta^2 Q\, dT + d\bar{\omega} = 0 \tag{4.36a}$$

along $\frac{dT}{dX} = -1$:

$$dM + \delta^2 Q\, dT - d\bar{\omega} = 0 \tag{4.36b}$$

along $\frac{dT}{dX} = \frac{1}{\delta}$:

$$dQ - \delta g^2 W\, dT - \frac{1}{\delta}\, d\bar{v} + \bar{\omega}\, dT = 0 \tag{4.36c}$$

along $\frac{dT}{dX} = -\frac{1}{\delta}$:

$$dQ + \delta g^2 W\, dT + \frac{1}{\delta}\, d\bar{v} + \bar{\omega}\, dT = 0 \tag{4.36d}$$

All derivations in this section are shown in detail in the Appendix.

For solving for M , $\bar{\omega}$, Q and $\bar{v}$, Equation (4.36) is expressed by four finite difference equations for corner points of an element DAPB bounded by two sets of characteristic lines of the dilatational wave family as shown in Figure 4.3a. Equations (4.36c) and (4.36d) are defined on characteristic lines of the shear wave family, so that their finite difference equations are in terms of values at P , A' , and B' . If all values are assumed varied linearly between A and D or B and D [21], then values at A' and B' can be expressed by

$$Y_{A'} = Y_D + \frac{2\delta}{\delta + 1}\,(Y_A - Y_D) \tag{4.37a}$$

$$Y_{B'} = Y_D + \frac{2\delta}{\delta + 1}\,(Y_B - Y_D) \tag{4.37b}$$

where $Y = M$, Q , $\bar{\omega}$, $\bar{v}$, or W .

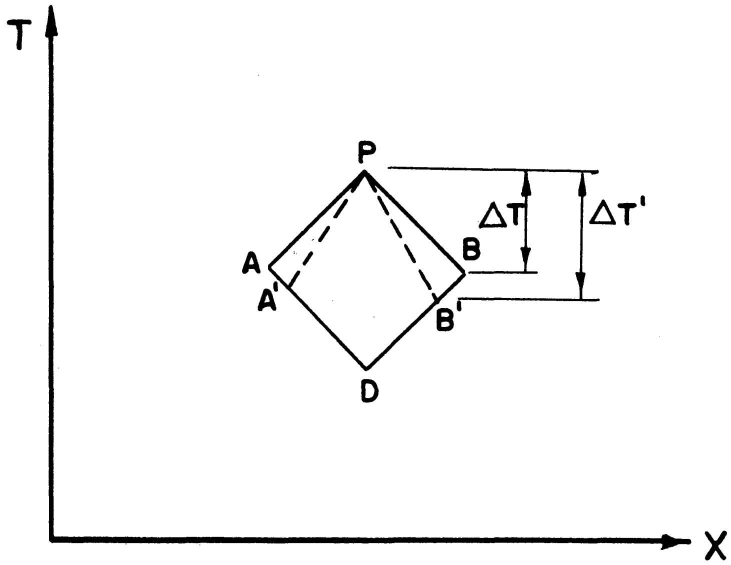

Figure 4.3a Typical Element.

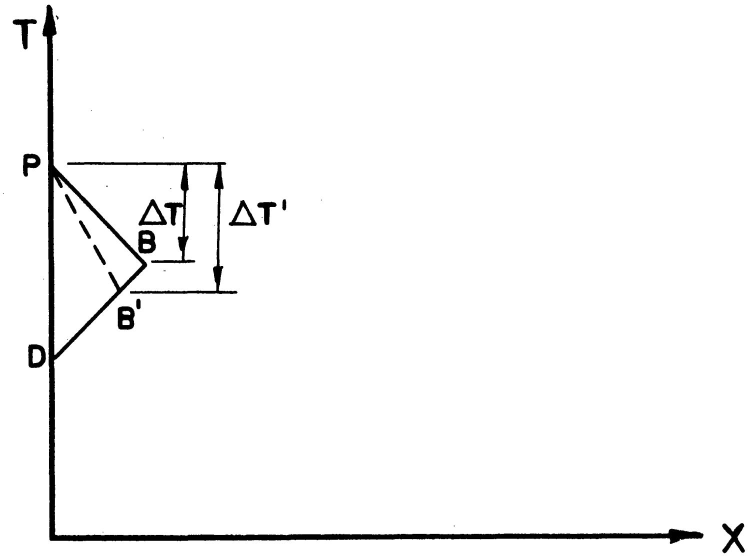

Figure 4.3b Element at Boundary.

$\Delta T'$ can also be expressed in terms of ΔT

$$\Delta T' = \frac{2}{1+\delta}\,\Delta T \tag{4.38}$$

The following difference equations are established

along PA :

$$(M_P - M_A) - \frac{\delta^2 \Delta T}{2}(Q_P + Q_A) + (\bar{\omega}_P - \bar{\omega}_A) = 0 \tag{4.39a}$$

along PB :

$$(M_P - M_B) + \frac{\delta^2 \Delta T}{2}(Q_P + Q_B) - (\bar{\omega}_P - \bar{\omega}_B) = 0 \tag{4.39b}$$

along PA' :

$$(Q_P - Q_{A'}) - \frac{\delta g^2 \Delta T'}{2}(W_P + W_{A'}) - \frac{1}{\delta}(\bar{v}_P - \bar{v}_{A'}) + \frac{\Delta T'}{2}(\bar{\omega}_P + \bar{\omega}_{A'}) = 0 \tag{4.39c}$$

along PB' :

$$(Q_P - Q_{B'}) + \frac{\delta g^2 \Delta T'}{2}(W_P + W_{B'}) + \frac{1}{\delta}(\bar{v}_P - \bar{v}_{B'}) + \frac{\Delta T'}{2}(\bar{\omega}_P + \bar{\omega}_{B'}) = 0 \tag{4.39d}$$

where $W_P = W_D + (\bar{\bar{v}}_P + \bar{\bar{v}}_D)\,\Delta T$ (4.40)

If M , $\bar{\omega}$, Q , and $\bar{\bar{v}}$ are known at A , D , and B , they can be found at P by solving Equation (4.39). Equation (4.39) is written explicitly in the following table:

TABLE 4.1

M_P	Q_P	$\bar{\omega}_P$	$\bar{v}_P$	Given Values
1	$-\frac{\delta^2 \Delta T}{2}$	1	0	$M_A + Q_A \cdot \frac{\delta^2}{2} \Delta T + \bar{\omega}_A$
1	$\frac{\delta^2 \Delta T}{2}$	-1	0	$M_B - Q_B \cdot \frac{\delta^2}{2} \Delta T - \bar{\omega}_B$
0	1	$\frac{\Delta T'}{2}$	$-\frac{\delta g^2}{2} \Delta T \Delta T' - \frac{1}{\delta}$	$Q_{A'} + \frac{\delta g^2 \Delta T \cdot \Delta T'}{2} \bar{v}_D + \frac{\delta g^2 \Delta T'}{2} (W_D + W_{A'}) - \frac{\bar{v}_{A'}}{\delta} - \frac{\Delta T'}{2} \bar{\omega}_{A'}$
0	1	$\frac{\Delta T'}{2}$	$\frac{\delta g^2}{2} \Delta T \Delta T' + \frac{1}{\delta}$	$Q_{B'} - \frac{\delta g^2 \Delta T \cdot \Delta T'}{2} \bar{v}_D - \frac{\delta g^2 \Delta T'}{2} (W_D + W_{B'}) + \frac{\bar{v}_{B'}}{\delta} - \frac{\Delta T'}{2} \bar{\omega}_{B'}$

For finding those values at boundary points, the four equations can be reduced to two, since any two of the four unknown values are specified there. If M and $\bar{v}$ are specified, the other values can be found if all the values at B and D are known. The two equations are shown in the following table:

TABLE 4.2

Q_P	$\bar{\omega}_P$	Given Values
$\frac{\delta^2 \Delta T}{2}$	-1	$-(M_P - M_B) - \frac{\delta^2 \Delta T}{2} Q_B - \bar{\omega}_B$
1	$\frac{\Delta T'}{2}$	$Q_{B'} - \frac{\delta g^2 \Delta T'}{2} (W_P + W_{B'}) - \frac{1}{\delta} (\bar{v}_P - \bar{v}_{B'}) - \frac{\Delta T'}{2} \bar{\omega}_{B'}$

where the triangle PBD is shown in Figure 4.3b .

If M , $\bar{\omega}$, Q and $\bar{v}$ are specified on the line $X = T$ and M and $\bar{v}$ are specified on $X = 0$ as shown in Figure 4.4, $\bar{\omega}$ and Q at A_{11} can be found by expression in Table 4.2. Now M , $\bar{\omega}$, Q and $\bar{v}$ are known at A_{11} , A_{01} and A_{02} , those values can be found at A_{12} by the expression in Table 4.1. By successive use of the expression in Table 4.1, they can be determined at the points through A_{1n} . By repeating the previous procedures, values within the triangular region $OA_{on}A_{nn}$ can be found.

The expressions for boundary conditions on $X = 0$ and $X = T$ are different for the three velocity regions, so that the three cases are discussed separately.

a. <u>Velocity V of the Pressure Front Such that $V_{co} < V < V_{c1}$</u>

From Equations (3.41) and (3.42), the particular solution for Equation (4.15) is

$$W_b(X,T) = \frac{P}{g^2} - \frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \frac{\cos n_2(X-VT)}{n_2^2(n_1^2-n_2^2)} \qquad \text{for } (X-VT) \leq 0$$

$$W_b(X,T) = -\frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{\cos n_1(X-VT)}{n_1^2(n_1^2-n_2^2)} \qquad \text{for } (X-VT) > 0$$

$$W_s(X,T) = \frac{\delta^2 P}{(V^2-\delta^2)} \frac{\cos n_2(X-VT)}{(n_1^2-n_2^2)} \qquad \text{for } (X-VT) \leq 0$$

$$W_s(X,T) = \frac{\delta^2 P}{(V^2-\delta^2)} \frac{\cos n_1(X-VT)}{(n_1^2-n_2^2)} \qquad \text{for } (X-VT) > 0$$

where n_1 and n_2 are real positive roots of the equation

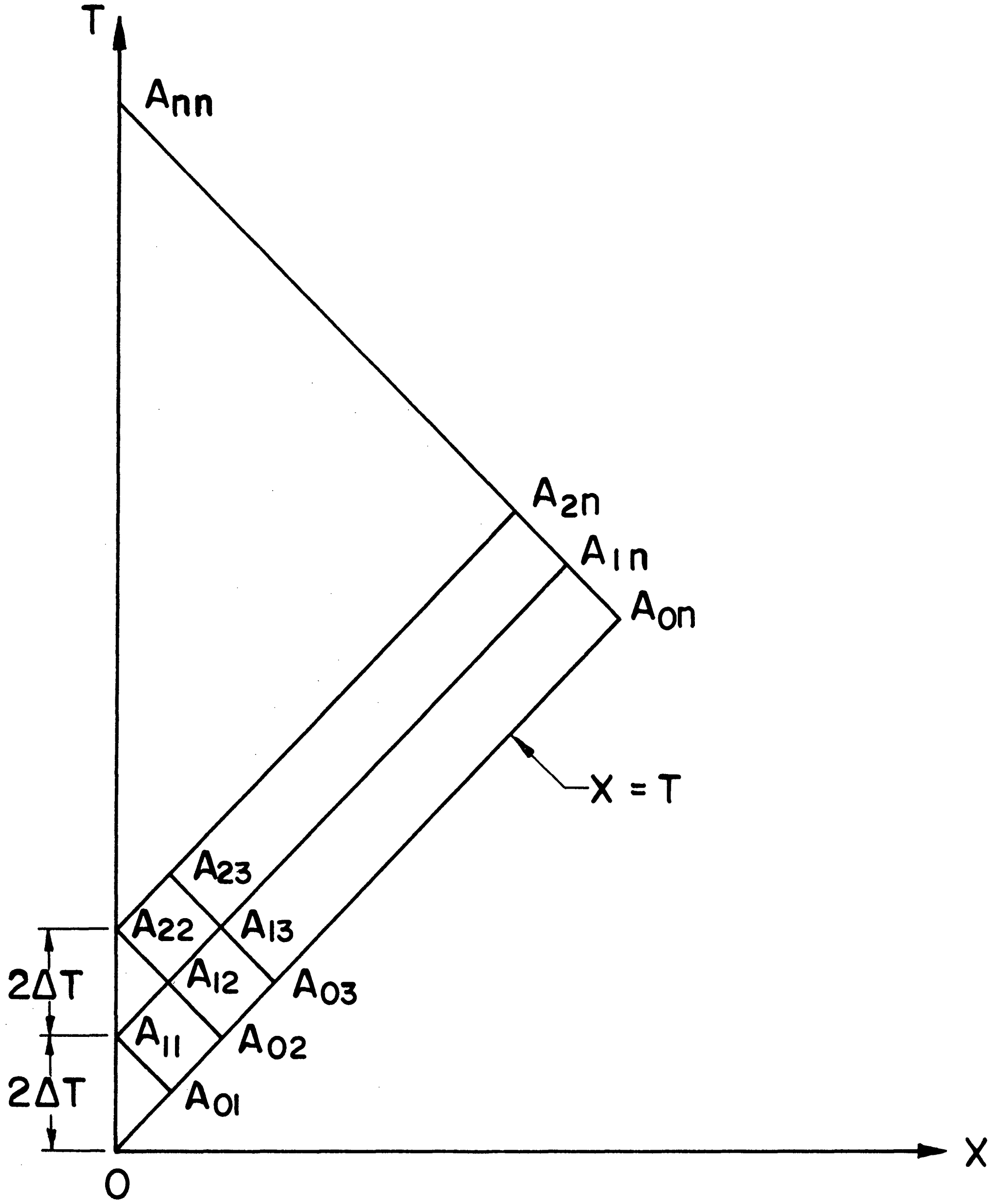

Figure 4.4 Triangular Region Used in Computation

$$(V^2-1)\left(\frac{V^2}{\delta^2}-1\right)N^4-\left[V^2(1+g^2)-g^2\right]N^2+\delta^2 g^2=0$$

and

$$n_1>n_2$$

Since n_1 is the root of the above algebraic equation, the expression

$$\begin{cases} W_b(X,T)=\dfrac{\delta^4 P}{(V^2-1)(V^2-\delta^2)}\dfrac{\cos n_1(X-VT)}{n_1^2(n_1^2-n_2^2)} \\ W_s(X,T)=-\dfrac{\delta^2 P}{(V^2-\delta^2)}\dfrac{\cos n_1(X-VT)}{(n_1^2-n_2^2)} \end{cases} \quad \text{for all values of } (X-VT)$$

is the solution of the Equation (4.30) which is the homogeneous form of Equation (4.15). If these two are added together, a second particular solution of Equation (4.15) is obtained

$$W_b(X,T)=\frac{P}{g^2}+\frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)(n_1^2-n_2^2)}\left\{\frac{\cos n_1(X-VT)}{n_1^2}-\frac{\cos n_2(X-VT)}{n_2^2}\right\} \quad \text{for } (X-VT)\le 0$$

$$W_b(X,T)=0 \qquad \text{for } (X-VT)>0 \qquad (4.41a)$$

$$W_s(X,T)=-\frac{\delta^2 P}{(V^2-\delta^2)(n_1^2-n_2^2)}\left\{\cos n_1(X-VT)-\cos n_2(X-VT)\right\} \quad \text{for } (X-VT)\le 0$$

$$W_s(X,T)=0 \qquad \text{for } (X-VT)>0 \qquad 4.41b)$$

The particular solution given by Equation (4.41) does not satisfy the boundary condition at $X = 0$, if the following calculation is performed

$$W(0,T) = W_b(0,T) + W_s(0,T)$$

$$= \frac{p}{g^2} + \frac{\delta^2 p}{(v^2-\delta^2)(n_1^2-n_2^2)}\left\{\left[\frac{\delta^2}{(v^2-1)n_1^2}-1\right]\cos n_1 vT - \left[\frac{\delta^2}{(v^2-1)n_2^2}-1\right]\cos n_2 vT\right\}$$

$$= H_1(T) \neq 0 \qquad (4.42a)$$

$$\bar{v}(0,T) = \frac{\partial W(0,T)}{\partial T}$$

$$= -\frac{\delta^2 p v}{(v^2-\delta^2)(n_1^2-n_2^2)}\left\{\left[\frac{\delta^2}{(v^2-1)n_1^2}-1\right]n_1 \sin n_1 vT - \left[\frac{\delta^2}{(v^2-1)n_2^2}-1\right]n_2 \sin n_2 vT\right\}$$

$$= H_1'(T) \neq 0 \qquad (4.42b)$$

$$M(0,T) = -\frac{\partial^2 W_b(0,T)}{\partial X^2}$$

$$= \frac{p\delta^4}{(v^2-1)(v^2-\delta^2)}\cdot\frac{1}{(n_1^2-n_2^2)}\left\{\cos n_1 vT - \cos n_2 vT\right\}$$

$$= H_2(T) \neq 0 \qquad (4.42c)$$

The particular solution given by Equation (4.41) does satisfy the homogeneous initial conditions, since W_b and W_s are identically zero for $X \geq VT$ (where $0 \leq X < \infty$ and $V < 1$). For this reason, the values of M , $\bar{\omega}$, Q , $\bar{v}$ are zero on the line $X = T$ in Figure 4.4.

If $\bar{v} = -H_1'(T)$ ($W = -H_1(T)$) and $M = -H_2(T)$ are specified on $X = 0$ and no disturbance is assumed for $X \geq T$, a solution can be found numerically by the previously stated method of characteristics. A final solution which satisfies both the homogeneous boundary and initial conditions is gotten by adding the said solution to the solution given by Equation (4.41).

b. $\underline{V_{c1} < V < V_{c2}}$

From Equations (3.43) and (3.44), the particular solution for Equation (4.15) in this case is

$$W_b(X,T) = \frac{p}{g^2} + \frac{\delta^4 p}{(V^2-1)(V^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)} \left[\frac{\cos n(X-VT)}{n^2} + \frac{e^{m(X-VT)}}{2m^2}\right]$$

for $(X - VT) \leq 0$

$$W_b(X,T) = -\frac{\delta^4 p}{(V^2-1)(V^2-\delta^2)} \cdot \frac{e^{-m(X-VT)}}{2m^2(n^2+m^2)}$$

for $(X - VT) > 0$ (4.43a)

$$W_s(X,T) = -\frac{\delta^2 p}{(V^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)} \left[\cos n(X-VT) - \frac{e^{m(X-VT)}}{2}\right]$$

for $(X - VT) \leq 0$

$$W_s(X,T) = -\frac{\delta^2 p}{(V^2-\delta^2)} \cdot \frac{1}{2(n^2+m^2)} e^{-m(X-VT)}$$

for $(X - VT) > 0$ (4.43b)

where n and mi are the roots of the equation

$$(V^2-1)\left(\frac{V^2}{\delta^2}-1\right)N^4 - \left[V^2(1+g^2)-g^2\right]N^2 + \delta^2 g^2 = 0$$

and n , m are real and positive.

This solution satisfies neither the homogeneous boundary conditions nor the homogeneous initial conditions, if the following check is made

$$W(0,T) = \frac{p}{g^2} + \frac{p\delta^2}{(v^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)} \left\{ \left[\frac{\delta^2}{n(v^2-1)} - 1 \right] \cos n v T + \frac{1}{2} \left[\frac{\delta^2}{m^2(v^2-1)} + 1 \right] e^{-mvT} \right\}$$

$$= H_3(T) \not\equiv 0 \qquad (4.44a)$$

$$\bar{v}(0,T) = \frac{\partial W(0,T)}{\partial T}$$

$$= -\frac{p\delta^2}{(v^2-\delta^2)} \cdot \frac{v}{(n^2+m^2)} \left\{ \left[\frac{\delta^2}{n^2(v^2-1)} - 1 \right] n \sin n v T + \frac{m}{2} \left[\frac{\delta^2}{m^2(v^2-1)} + 1 \right] e^{-mvT} \right\}$$

$$= H_3'(T) \not\equiv 0 \qquad (4.44b)$$

$$M(0,T) = -\frac{\partial^2 W(0,T)}{\partial X^2}$$

$$= \frac{p\delta^4}{(v^2-1)(v^2-\delta^2)(m^2+n^2)} \left(\cos n v T - \frac{e^{-mvT}}{2} \right)$$

$$= H_4(T) \not\equiv 0 \qquad (4.44c)$$

$$W_b(X,0) = -\frac{\delta^4 p}{(v^2-1)(v^2-\delta^2)} \cdot \frac{e^{-mX}}{2m^2(n^2+m^2)} \not\equiv 0$$

$$\frac{\partial W_b(X,0)}{\partial T} = -\frac{\delta^4 p v}{(v^2-1)(v^2-\delta^2)} \cdot \frac{e^{-mX}}{2m(n^2+m^2)} \not\equiv 0$$

$$W_s(X,0) = -\frac{\delta^2 p}{(v^2-\delta^2)} \cdot \frac{e^{-mX}}{2(n^2+m^2)} \not\equiv 0$$

$$\frac{\partial W_s(X,0)}{\partial T} = -\frac{\delta^2 p}{(v^2-\delta^2)} \cdot \frac{m v e^{-mX}}{2(n^2+m^2)} \not\equiv 0$$

Specifying the boundary conditions at $X = 0$ and the initial conditions at $T = 0$, a solution can be obtained numerically. A true solution is gotten by subtracting the said solution from the solution given by Equations (4.43).

To save one half of the computing time, M, $\bar{\omega}$, Q and $\bar{v}$ may be specified at the line $X = T$ in Figure 4.5 instead of the initial conditions at $T = 0$. It is well known that there is no disturbance before the dilatational wave arrives for this velocity range $V < V_{c2}$. M, $\bar{\omega}$, Q and $\bar{v}$ should be zero on the line $X = T$. From the particular solution, M, $\bar{\omega}$, Q and $\bar{v}$ can be computed along the line $X = T$ and they are

$$M = -\frac{\partial^2 W_b}{\partial X^2}\bigg|_{X=T} = \frac{p\delta^4}{(V^2-1)(V^2-\delta^2)} \cdot \frac{e^{-m(1-V)T}}{2(n^2+m^2)} = H_5(T) \tag{4.45a}$$

$$\bar{\omega} = \frac{\partial^2 W_b}{\partial T \partial X}\bigg|_{X=T} = \frac{pV\delta^4}{(V^2-1)(V^2-\delta^2)} \cdot \frac{e^{-m(1-V)T}}{2(n^2+m^2)} = H_6(T) \tag{4.45b}$$

$$Q = \frac{\partial W_s}{\partial X}\bigg|_{X=T} = \frac{p\delta^2}{(V^2-\delta^2)} \cdot \frac{m e^{-m(1-V)T}}{2(n^2+m^2)} = H_7(T) \tag{4.45c}$$

$$\bar{v} = -\frac{p\delta^2}{(V^2-\delta^2)} \cdot \frac{V}{(n^2+m^2)} \cdot \frac{m}{2}\left[\frac{\delta^2}{m^2(V^2-1)} + 1\right] e^{-m(1-V)T} = H_8'(T) \tag{4.45d}$$

$$W = -\frac{p\delta^2}{(V^2-\delta^2)} \cdot \frac{1}{(n^2+m^2)} \cdot \frac{1}{2}\left[\frac{\delta^2}{m^2(V^2-1)} + 1\right] e^{-m(1-V)T} = H_8(T) \tag{4.45e}$$

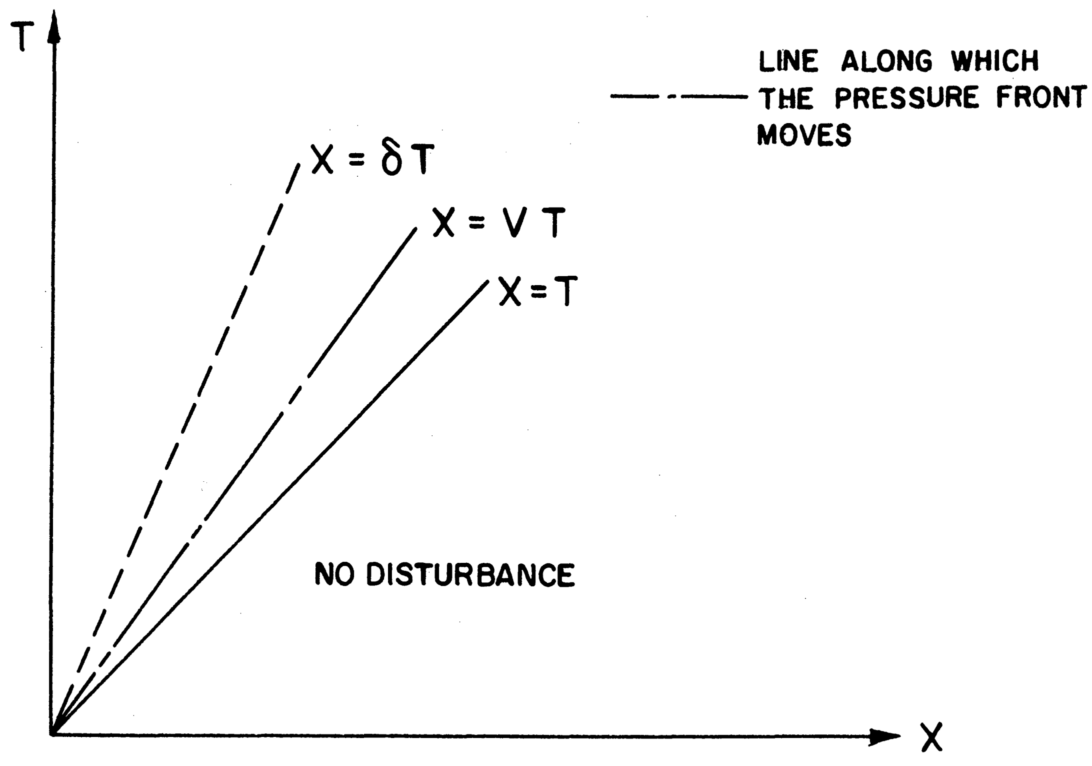

Figure 4.5 Pressure Front in X - T Plane When $V_{c1} < V < V_{c2}$

A solution can be obtained numerically by the method of characteristics, if $\bar{v} = - H_3'(T)$ $(W = - H_3(T))$ and $M = - H_4(T)$ are specified on $X = 0$ and $M = - H_5(T)$, $\bar{\omega} = - H_6(T)$, $Q = - H_7(T)$, $\bar{v} = - H_8'(T)$ $(W = - H_8(T))$ specified on $X = T$. The final solution which satisfies both the homogeneous boundary and initial conditions is gotten by adding this solution to the solution given by Equation (4.43).

c. $V > V_{c2}$

From Equations (3.45) and (3.46), the particular solution for Equation (4.15) in this case is

$$W_b(X,T) = \frac{P}{g^2} + \frac{\delta^4 P}{(V^2-1)(V^2-\delta^2)} \cdot \frac{1}{(n_1^2-n_2^2)}\left[\frac{\cos n_1(X-VT)}{n_1^2} - \frac{\cos n_2(X-VT)}{n_2^2}\right]$$

for $(X - VT) \leq 0$

$$W_b(X,T) = 0$$

for $(X - VT) > 0$

$$W_s(X,T) = -\frac{\delta^2 P}{(V^2-\delta^2)} \cdot \frac{1}{(n_1^2-n_2^2)}\left[\cos n_1(X-VT) - \cos n_2(X-VT)\right]$$

for $(X - VT) \leq 0$

$$W_s(X,T) = 0$$

for $(X - VT) > 0$

These equations are identical to Equation (4.41). The same method as was used for the velocity range $V_{co} < V < V_{c1}$, can be applied here.

In this case, there is a disturbance before the dilatational wave arrives, because the velocity of the pressure front exceeds the velocity of the dilatational wave. Preceeding the arrival of the dilatational wave, the disturbance is caused by the pressure directly. Figure 4.6 shows clearly.

C. Numerical Examples

The constants used in the numerical examples are

$$E = 30 \times 10^6 \text{ lb/sq.in}$$

$$G = 12 \times 10^6 \text{ lb/sq.in}$$

$$\nu = 0.3$$

$$\kappa = 0.8333$$

and $h/R = 1/10$ is used in all examples.

1. Fourier Integral Solution for the Velocity Range $V < V_{co}$

a. Elementary Theory

The radial deflections at sections $X = 20$ and $X = 40$ with $T = 0$ through $T = 460$ are computed based upon Equation (4.14). The velocity of the pressure front $V = 0.1811$ is used. In computing $I_2(X,T)$ in Equation (4.14), numerical integrations by means of Simpson's rule are employed. Due to the fact that there is N^5 in the denominator, the improper integrals converge rather rapidly. W/W_o at $X = 20$ and $X = 40$ versus T is plotted in Figures 4.7 and 4.8 respectively, where $W_o = 1/g^2$ is the maximum radial static deflection of a tube with infinite length under uniformly distributed internal pressure $P = 1$. The results are plotted in Figures 4.7 and 4.8.

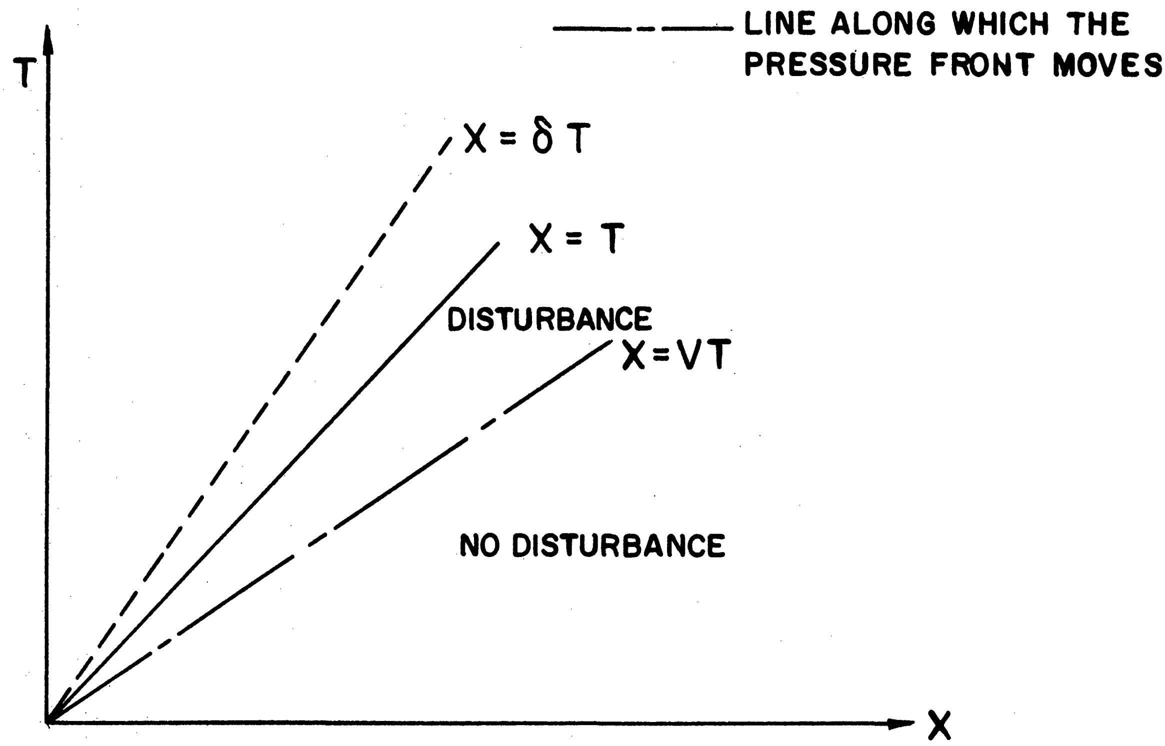

Figure 4.6 Pressure Front in X - T Plane When $V > V_{c2}$

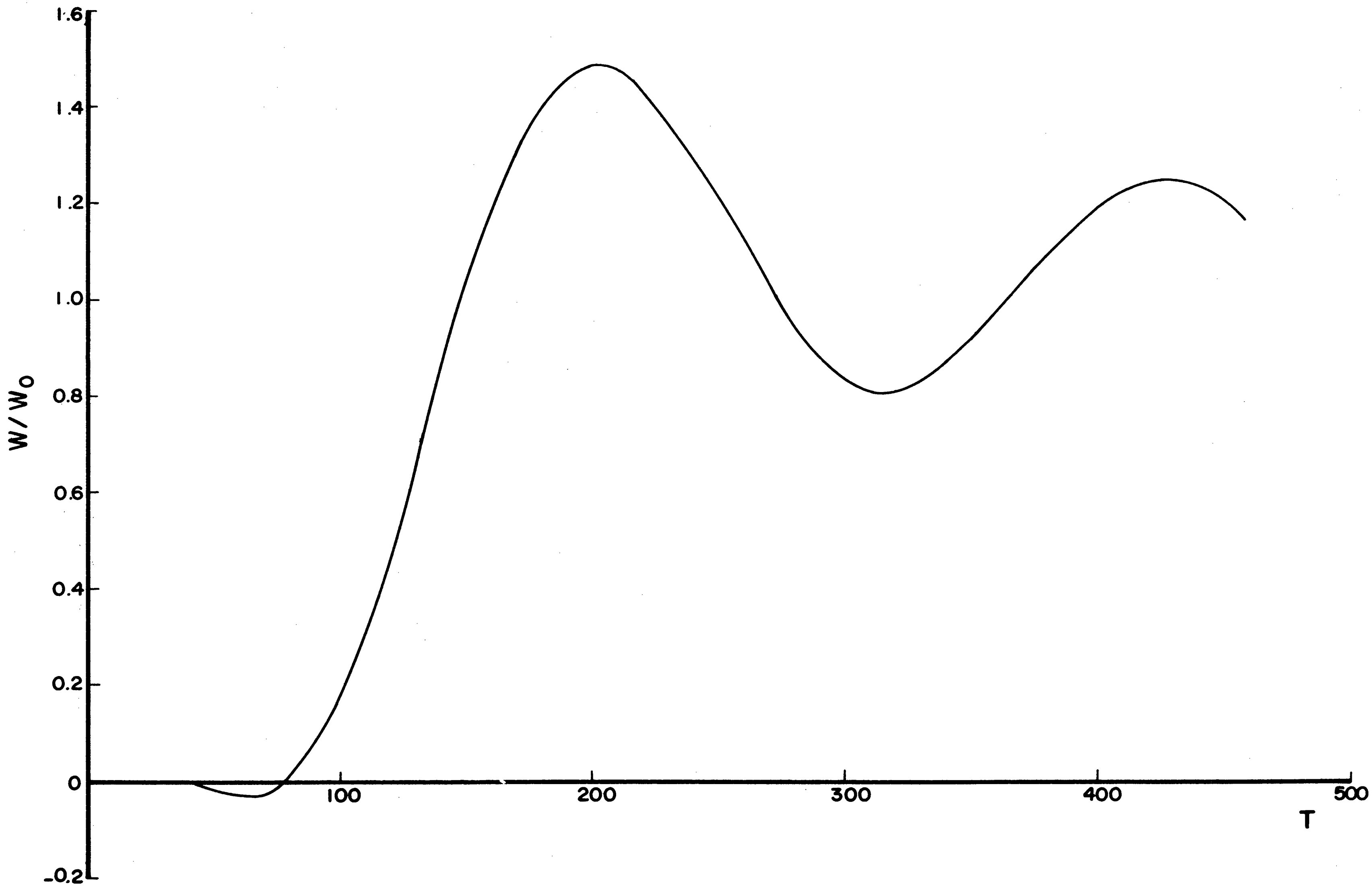

Figure 4.7 Deflection at X = 20 for V = 0.1811 ($V < V_{co}$) from Elementary Theory

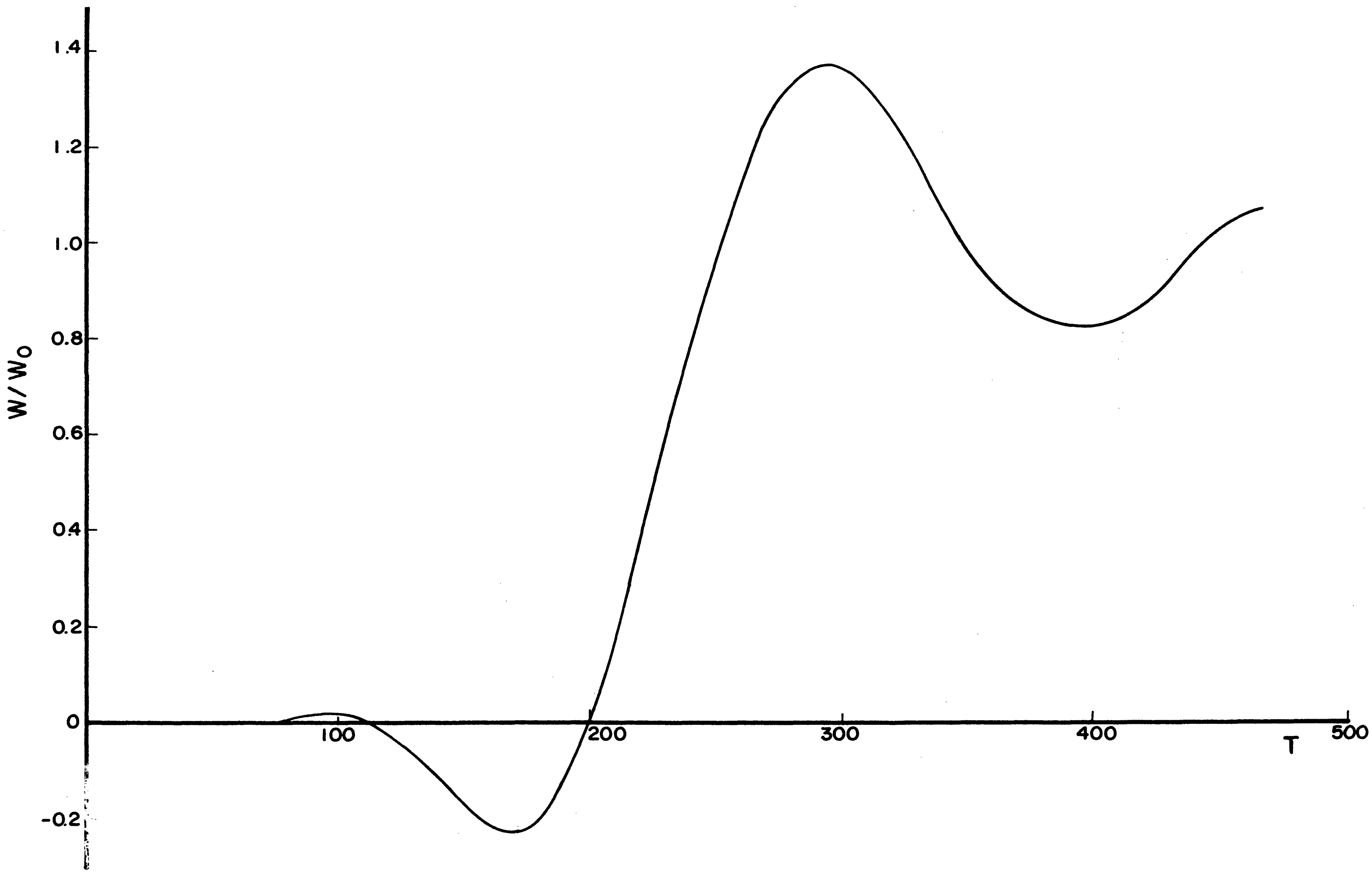

Figure 4.8 Deflection at X = 40 for V = 0.1811 ($V < V_{co}$) from Elementary Theory

b. More Exact Theory

The radial deflections both for bending and shear at sections $X = 20$ and $X = 40$ with $T = 0$ through $T = 363$ are computed based upon Equation (4.27). The velocity of the pressure front $V = 0.1811$ is used. In computing I_{b1}, I_{b2}, I_{s1}, and I_{s2} in Equation (4.27), numerical integrations by means of Simpson's rule are employed. Integrals I_{b1} , I_{b2} converge rapidly due to N^5 in the denominators, but integrals I_{s1} , I_{s2} converge slowly. Total deflection ratio W/W_o versus T is plotted in Figures 4.9 and 4.10 respectively. Shear deflection ratio W_s/W_o versus T is plotted in Figure 4.11 and Figure 4.12 respectively.

2. Numerical Solution for the Velocity Range $V > V_{co}$

Three different velocities are used, i.e., $V_{co} < V < V_{c1}$, $V_{c1} < V < V_{c2}$, and $V < V_{c2}$. They are $V = 0.3561$, 0.7754, and 1.600 in the examples. The radial deflections consist of two parts, one is due to the particular solution, another due to the correction of the boundary conditions. The former is based upon Equation (4.41) and (4.43) as well as Equations (3.45) and (3.46). The latter is based upon the difference equations in Tables 4.1 and 4.2 with $\Delta T = 0.5$. The radial deflections at $X = 20$ and 40 with $T = 0$ through 400 are plotted in Figure 4.13 through Figure 4.18. Each figure consists of three curves, namely, one due to the particular solution, another due to correction of boundary conditions, and still another being the resultant of the preceeding two. The deflection is expressed in terms of the ratio to $W_o = 1/g^2$ which is the maximum radial static deflection of a tube with infinite length under uniformly distributed internal pressure $P = 1$.

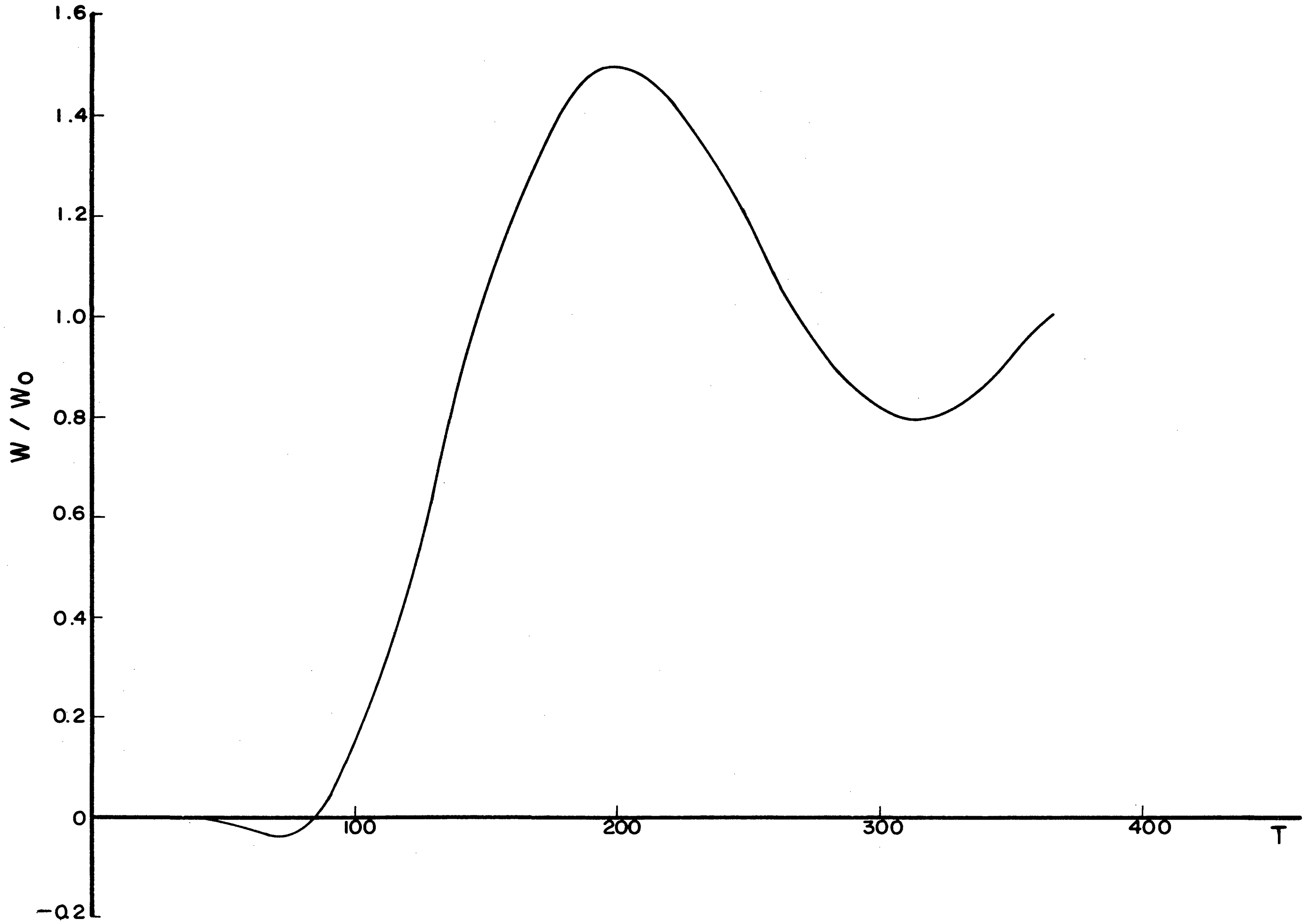

Figure 4.9 Total Deflection at X = 20 for V = 0.1811 ($V < V_{co}$) from More Exact Theory

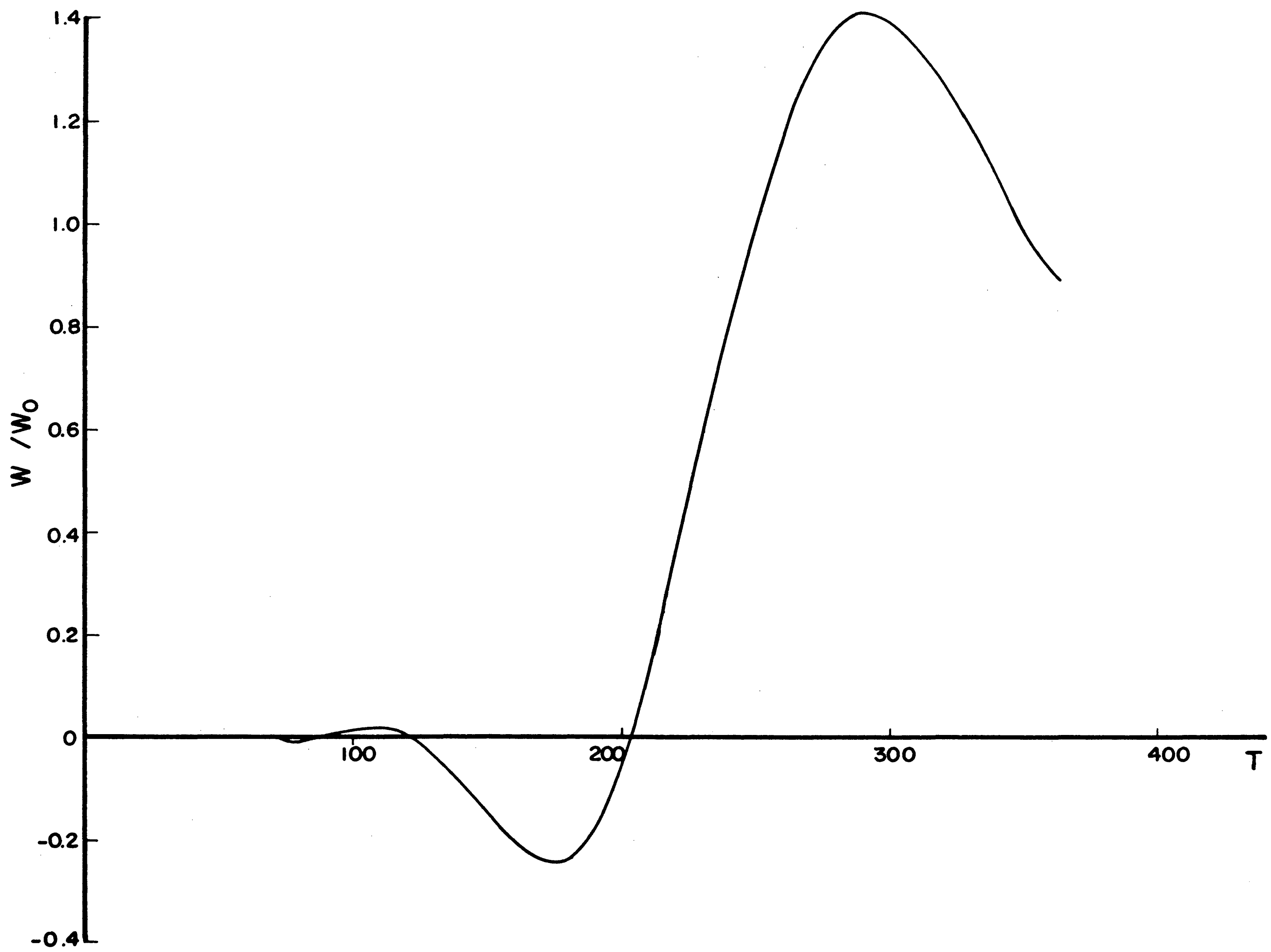

Figure 4.10 Total Deflection at X = 40 for V = 0.1811 ($V < V_{co}$) from More Exact Theory.

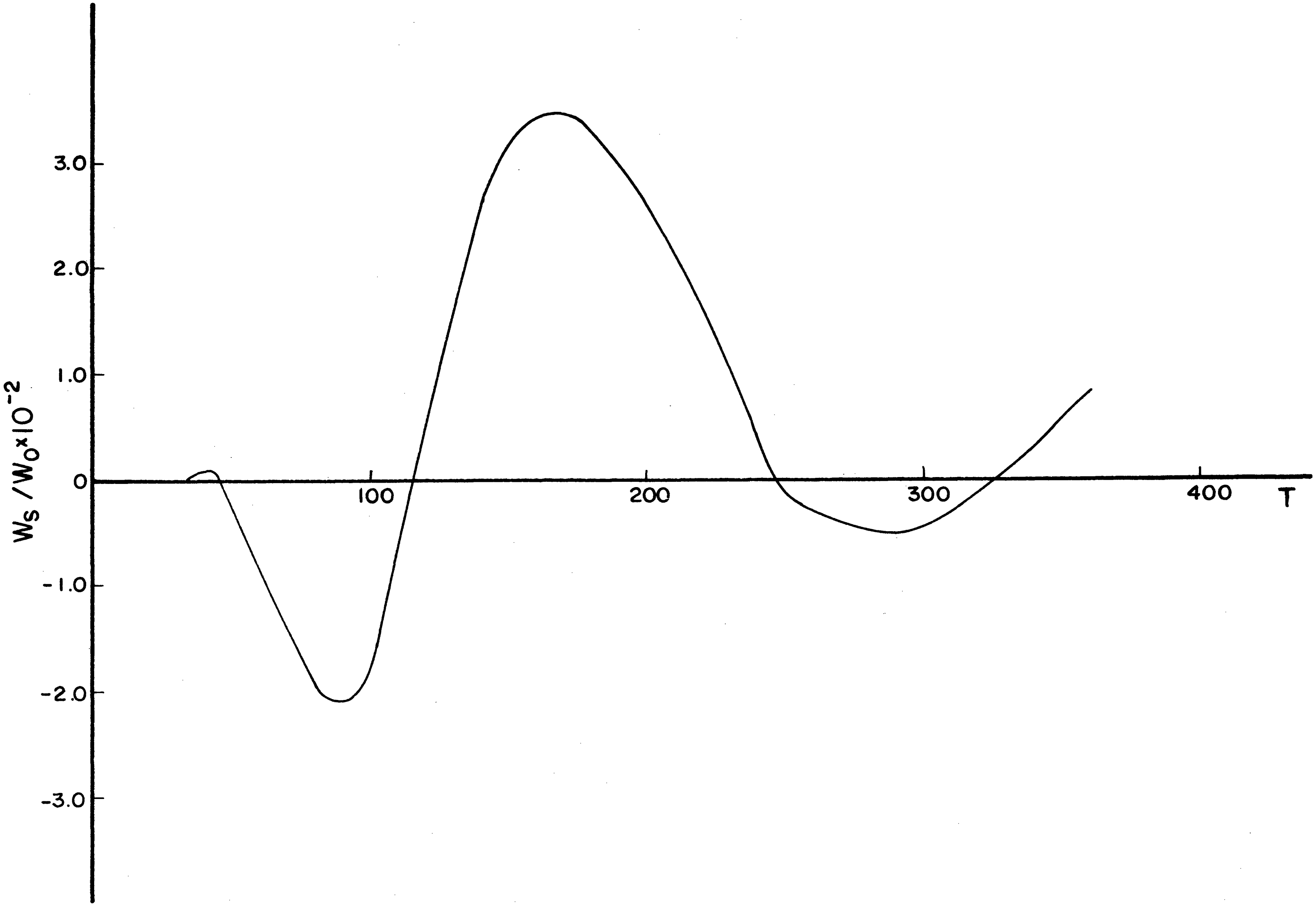

Figure 4.11 Deflection at X = 20 Due to Shear for V = 0.1811 (V < V_{co}) from More Exact Theory

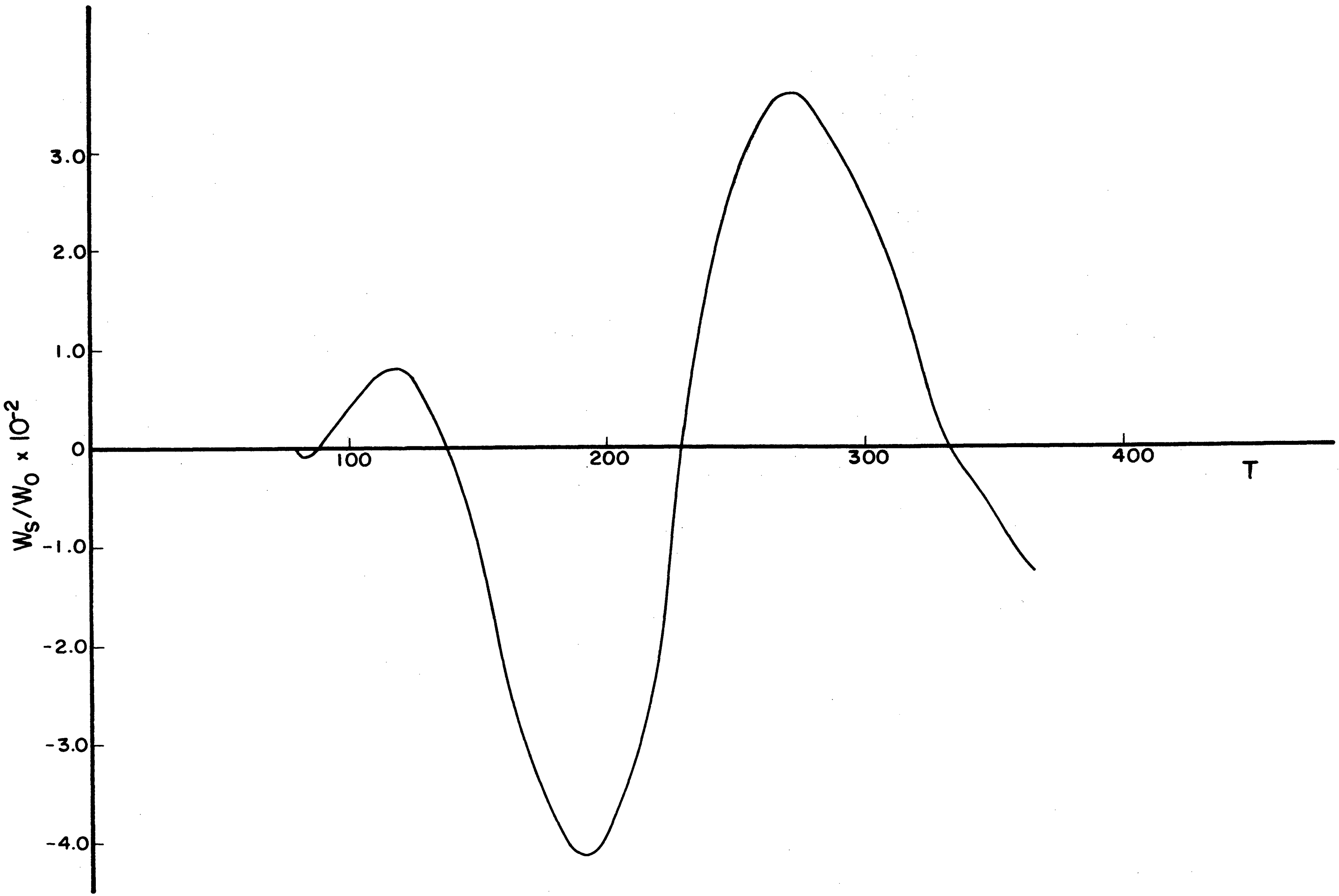

Figure 4.12 Deflection at X = 40 Due to Shear for V = 0.1811 ($V < V_{co}$) from More Exact Theory

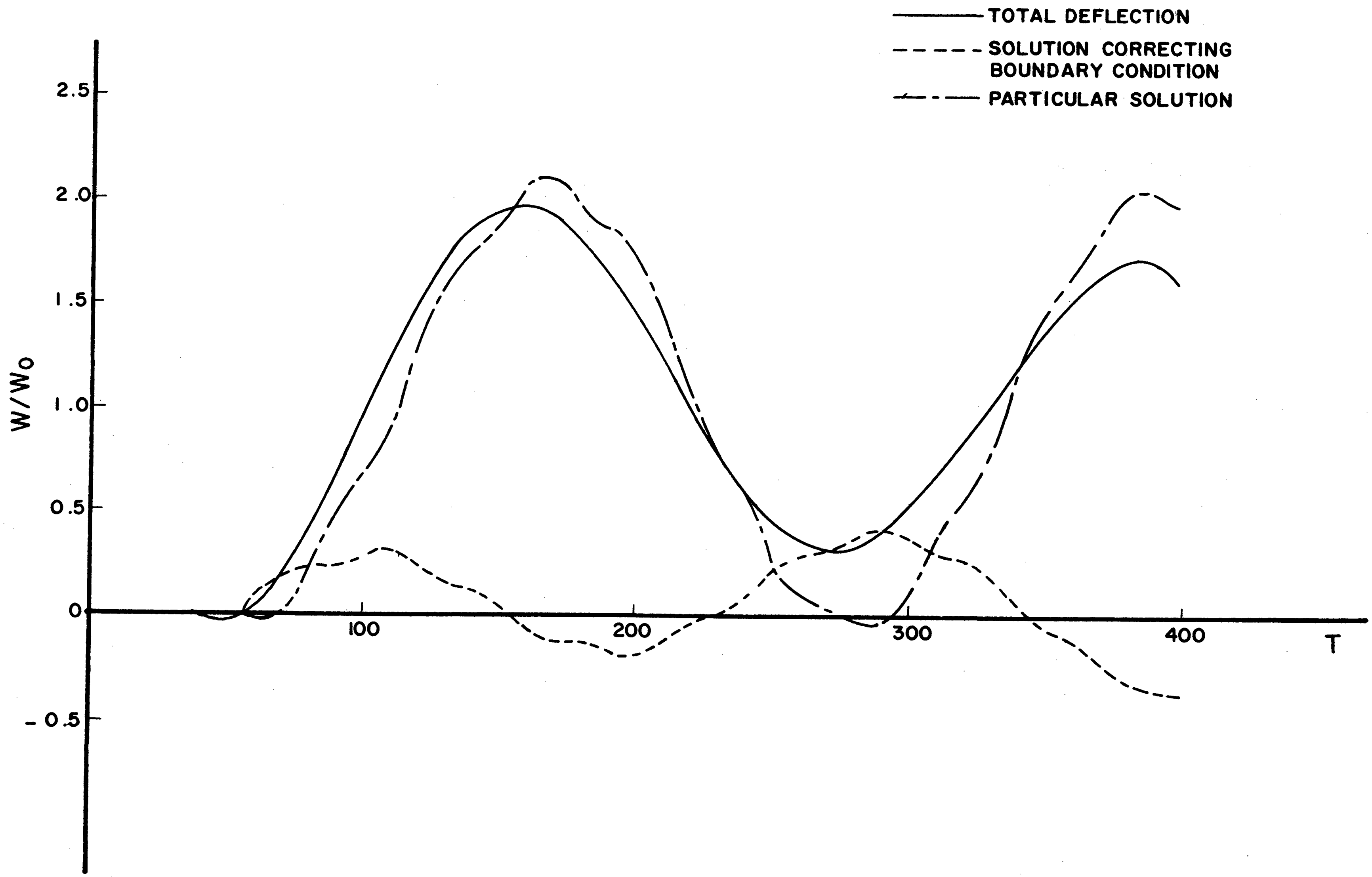

Figure 4.13 Deflection at X = 20 for V = 0.3561 ($V_{co} < V < V_{cl}$) from More Exact Theory.

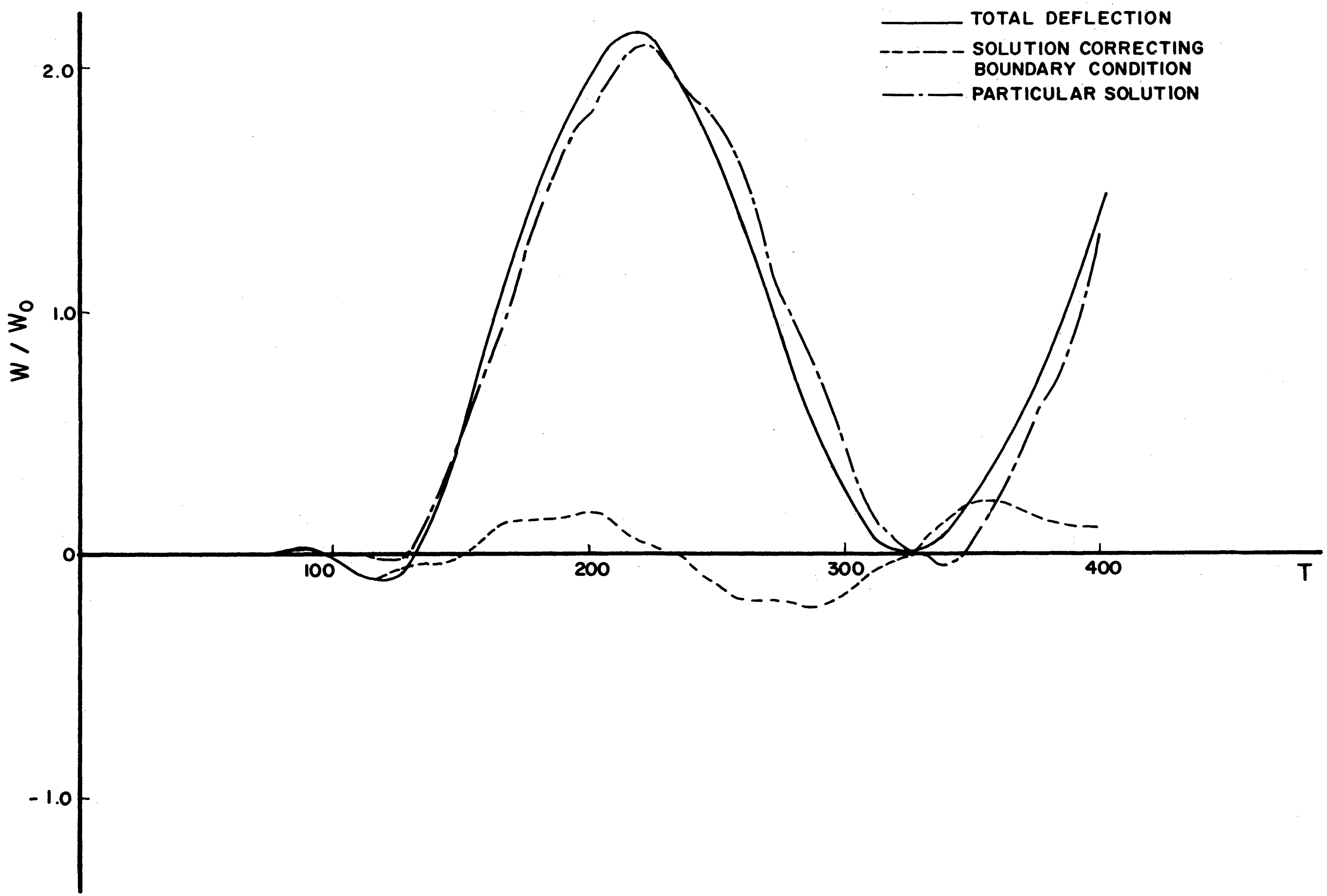

Figure 4.14 Deflection at $X = 40$ for $V = 0.3561$ $(V_{co} < V < V_{cl})$ from More Exact Theory.

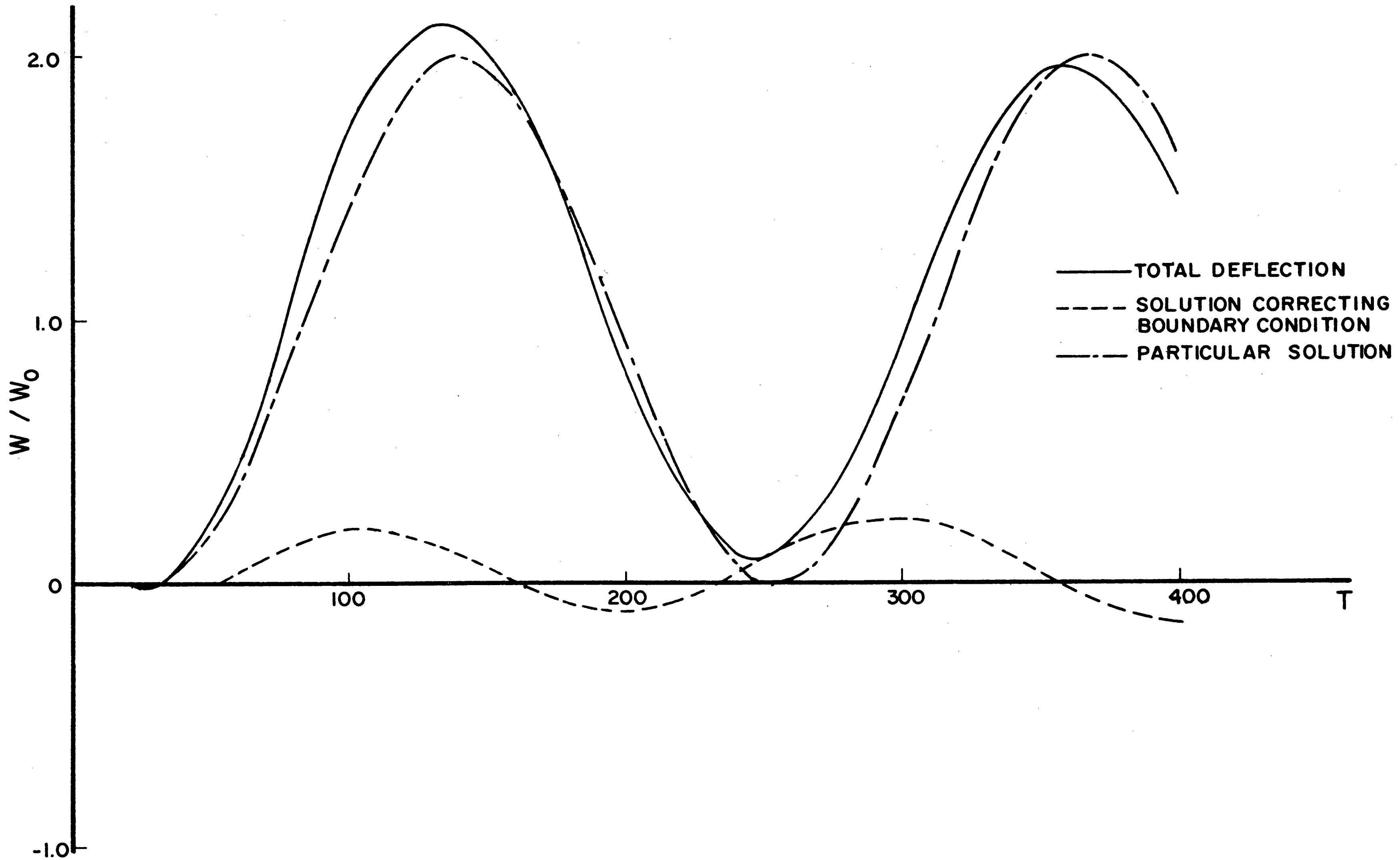

Figure 4.15 Deflection at $X = 20$ for $V = 0.7754$ ($V_{c1} < V < V_{c2}$) from More Exact Theory.

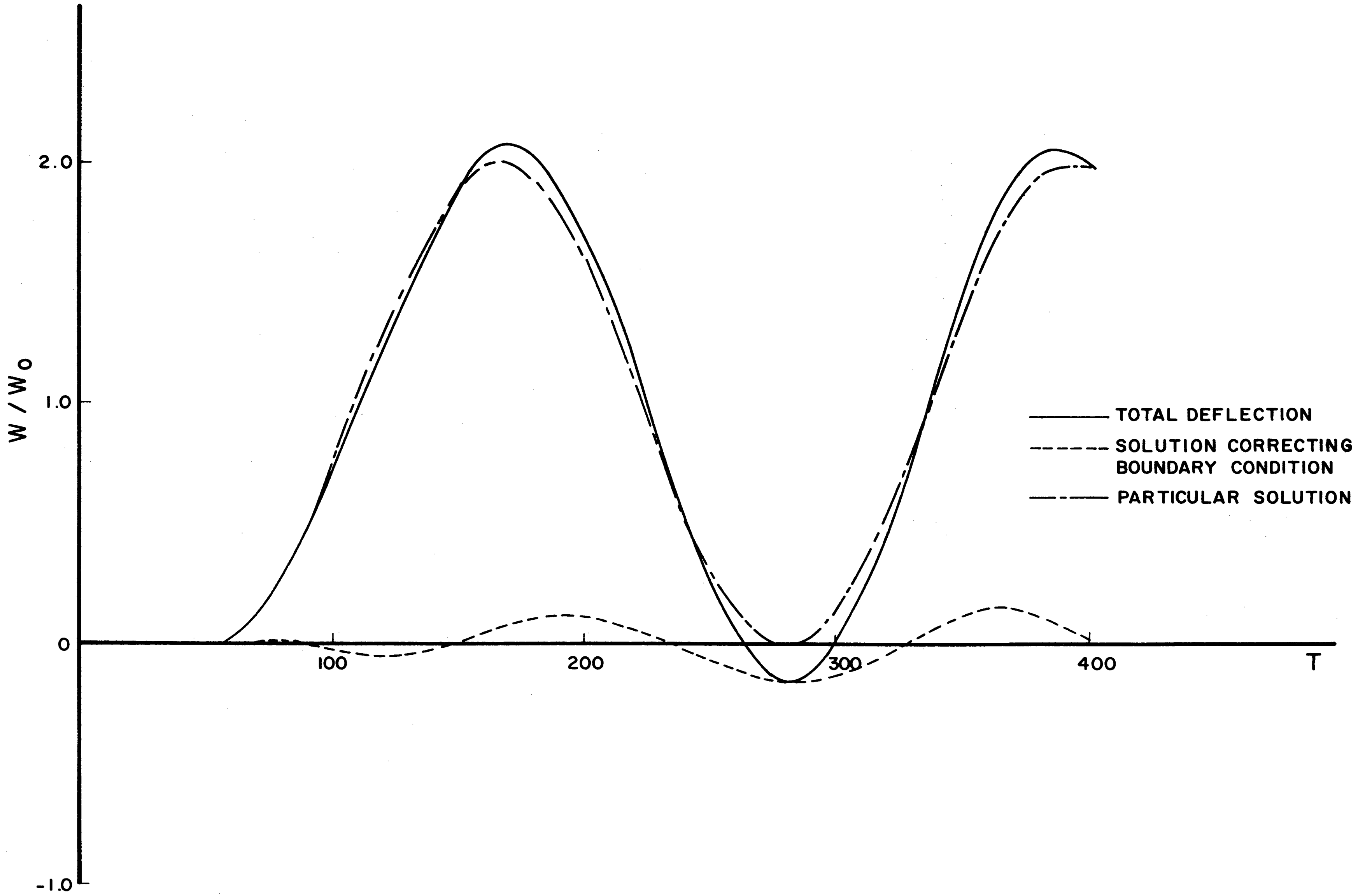

Figure 4.16 Deflection at X = 40 for V = 0.7754 ($V_{co} < V < V_{cl}$) from More Exact Theory

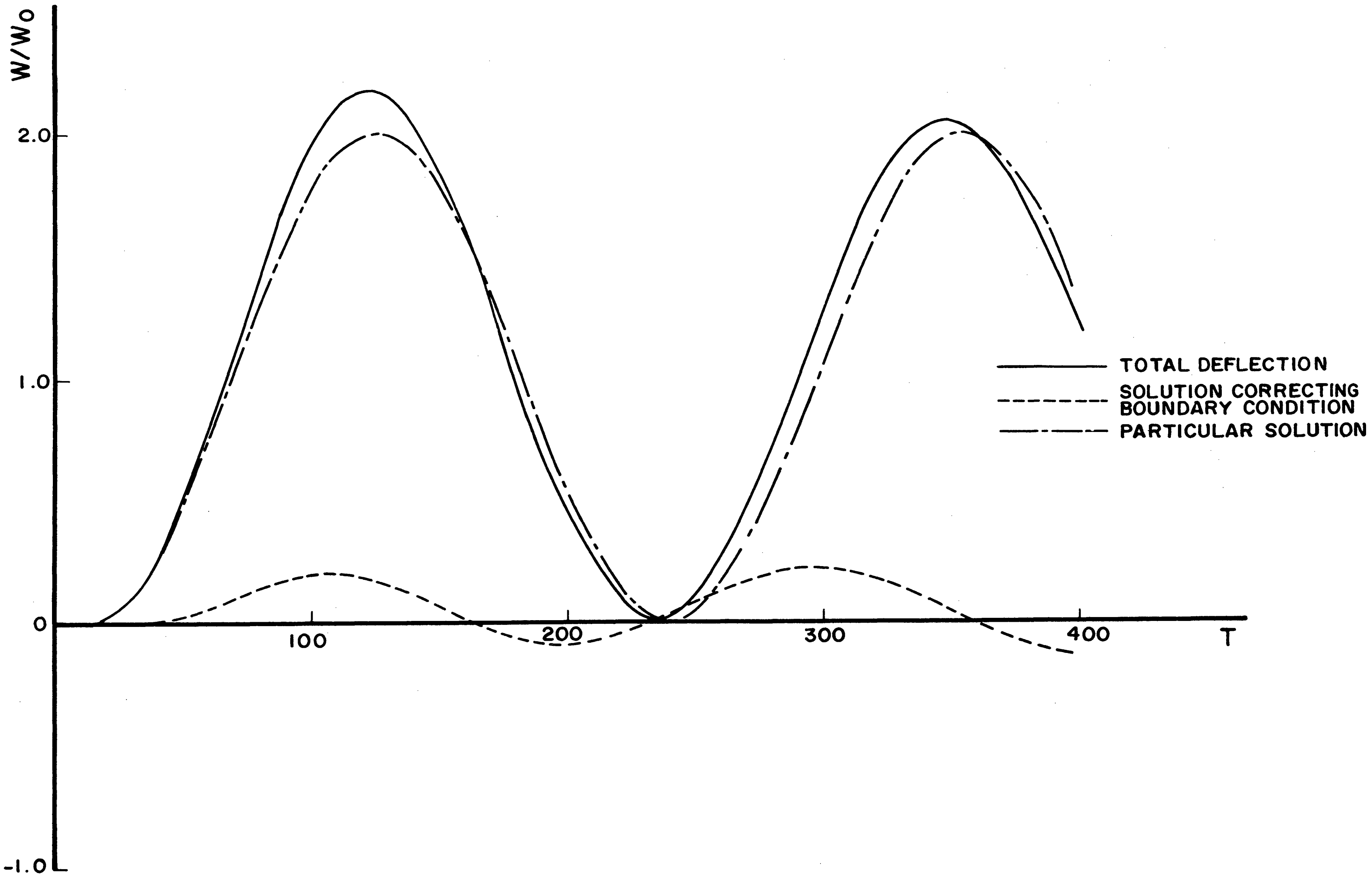

Figure 4.17 Deflection at $X = 20$ for $V = 1.600$ $(V > V_{c2})$ from More Exact Theory.

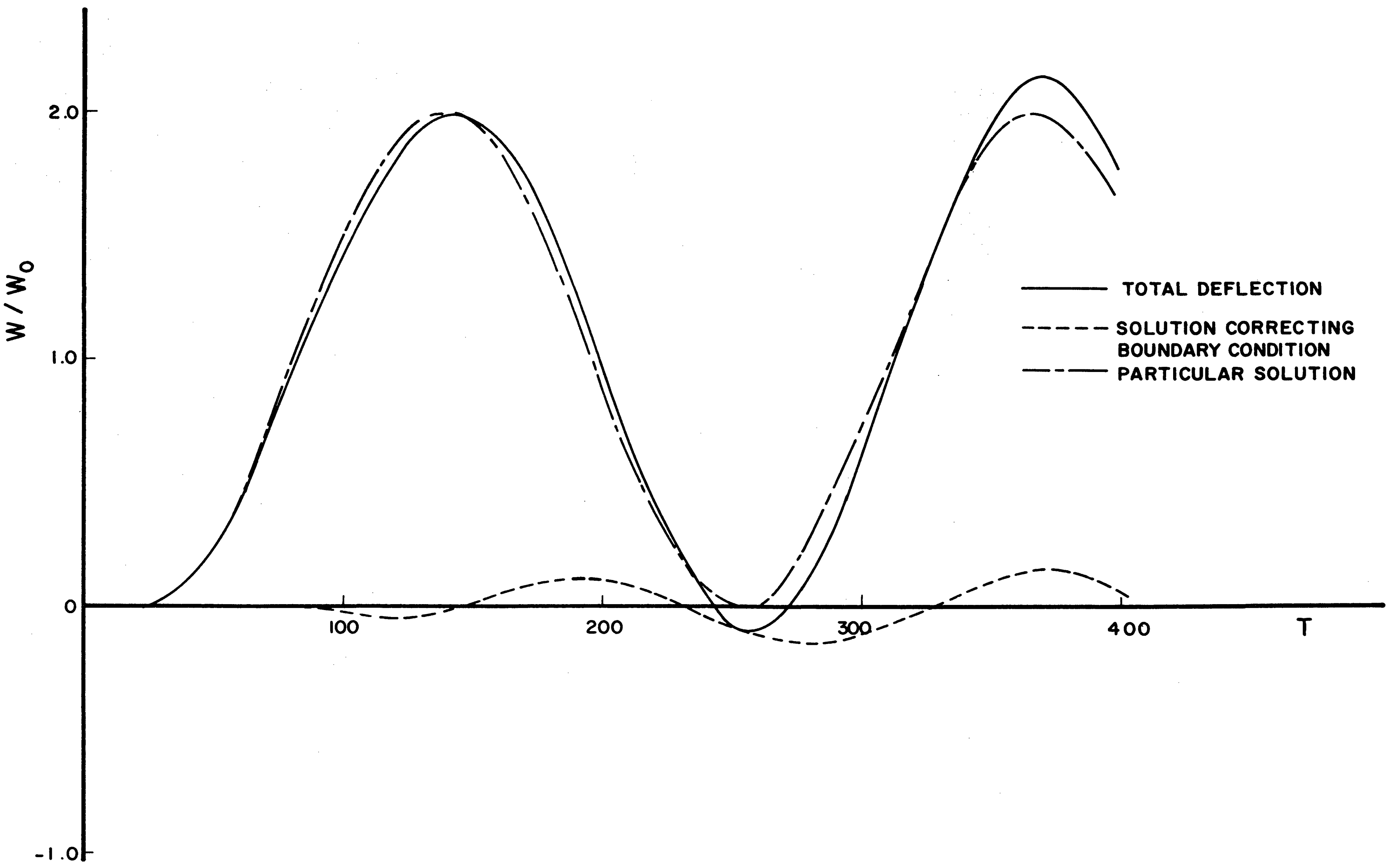

Figure 4.18 Deflection at X = 40 for V = 1.600 (V > V_{c2}) from More Exact Theory

D. Discussion of the Numerical Results

1. Case for $V < V_{co}$

In both elementary and more exact theories, the velocity $V = 0.1811$ of the moving pressure front is taken for numerical computation. Figures 4.8 and 4.10 or Figures 4.9 and 4.11 show that responses at $X = 20$ or $X = 40$ for both theories are almost the same. Before reasoning why, the formulas have to be investigated first. Equation (4.14) expresses the deflection based upon the elementary theory; Equation (4.27a) expresses the deflection due to bending and Equation (4.27b) expresses the deflection due to shear based upon the more exact theory. From the numerical results shown in Figures 4.11 and 4.12, the deflection due to the contribution of shear in the more exact theory is very small, so that it has no significant effect on the total deflection and only the comparison between the total deflection W in the elementary theory and the bending deflection W_b in the more exact theory is given. In this velocity range, namely $V < V_{co}$, the velocity spectra shown in Figure 2.4 through 2.6 are nearly alike for both theories. For a given velocity V, the corresponding complex wave numbers $N = n + im$ are approximately equal for both theories. Formulas for I_b and I_1 have the identical form. That is why $I_1 \approx I_b$. I_2 in Equation (4.14) is due to the contribution of the real arm of the frequency spectrum in the elementary theory. I_{b1} in Equation (4.27a) is due to the contribution of the first real arm of the frequency spectrum in the more exact theory. Those two real arms are very close for small wave numbers. The integrands both in I_2 and I_{b1} have an order of N^{-5}, where N is the real wave number, so that there are no significant contributions due to large wave numbers N or high frequencies. For this reason $I_2 \approx I_{b1}$. There is an additional term in Equation (4.27a), namely I_{b2} which is due to the second real arm of the frequency spectrum. From the numerical results, it has a magnitude of 10^{-5} of the maximum total deflection. In conclusion, the reason why the total deflections are approximately equal lies in the fact that the high frequency modes have no significant contributions in this velocity range.

2. Case for $V > V_{co}$

For the numerical solution part, M , $\bar{\bar{\omega}}$, Q and $\bar{\bar{v}}$ (W) can be found at the same time in the process of calculation, but only the total radial deflection W is plotted in Figure 4.13 through 4.18. In comparison with the particular solution, the deflection due to the particular solution part is relatively important both at X = 20 and X = 40 . Boundary effects are not significant for a section even with moderate distance from the end.

V. EXTENSIONS OF THE NUMERICAL SOLUTION

A. Forcing Function Due to Non-Uniform Pressure or Pressure Front Moving with Non-Uniform Velocity

If P is an arbitrary function of X and T, the numerical solution is still valid, so long as the particular solution for the equation of forced vibration can be found. In this particular solution it is not necessary to satisfy boundary conditions or initial conditions, so it can be easily determined by the integral transform method used in the case of uniform pressure. Once the particular solution has been determined, solutions to the homogeneous equation can be super-imposed on the particular solution to satisfy the prescribed boundary conditions at $X = 0$ and $X = T$. A homogeneous equation with specified boundary conditions can be solved numerically by the method of characteristics.

B. Arbitrary Boundary Conditions at X = 0

There is no difficulty in the application of numerical solutions to the boundary conditions at $X = 0$ other than simply supported. For fixed boundary at $X = 0$, $\bar{v} = 0$ $(W = 0)$ and $\bar{\omega} = 0$, the other two values, M and Q can be found by two simultaneous difference equations defined along the characteristic lines PB and PB' shown in Figure 4.3b, if all values at B and P are known.

C. Tube with Finite Length*

The particular solution for the tube with infinite length is still valid, but one additional boundary condition must be corrected

* Fourier series can be applied in the solution of tubes with finite lengths. This solution is based on orthogonal mode super-position. The frequency spectrum in Fourier series is discrete instead of continuous as in Fourier transform for the infinite length tube.

at $X = L$, where L is the dimensionless length of the tube. In this case the characteristic lines are reflected back and forth between two boundaries as shown in Figure 5.1, and this is equivalent to the dilatational waves as well as the shear waves in the tube wall being reflected back and forth along the characteristic lines.

D. Consideration of the Inertia Force in Axial Direction

If the inertia force in axial direction is included, the numerical method can still be applied. There are two more first order equations due to the longitudinal translation in addition to four previously established, to determine six unknowns. One additional boundary condition, say, longitudinal force or displacement must be specified. There are still four sets of characteristic lines with the same slopes, i.e., ± 1 and $\pm \frac{1}{\delta}$. The only work left is to find the particular solution due to the moving pressure. Once the particular solution is found, the part for the numerical solution will be the same except two more difference equations established along the characteristic lines with slopes ± 1.

For any boundary or initial value problem with homogeneous equations of motion, the numerical method can be applied directly, because there is no forcing function in the equations. Problems in which longitudinal translation predominates such as the problem in which longitudinal displacement is specified at one end or the problem with longitudinal impact, are the examples.

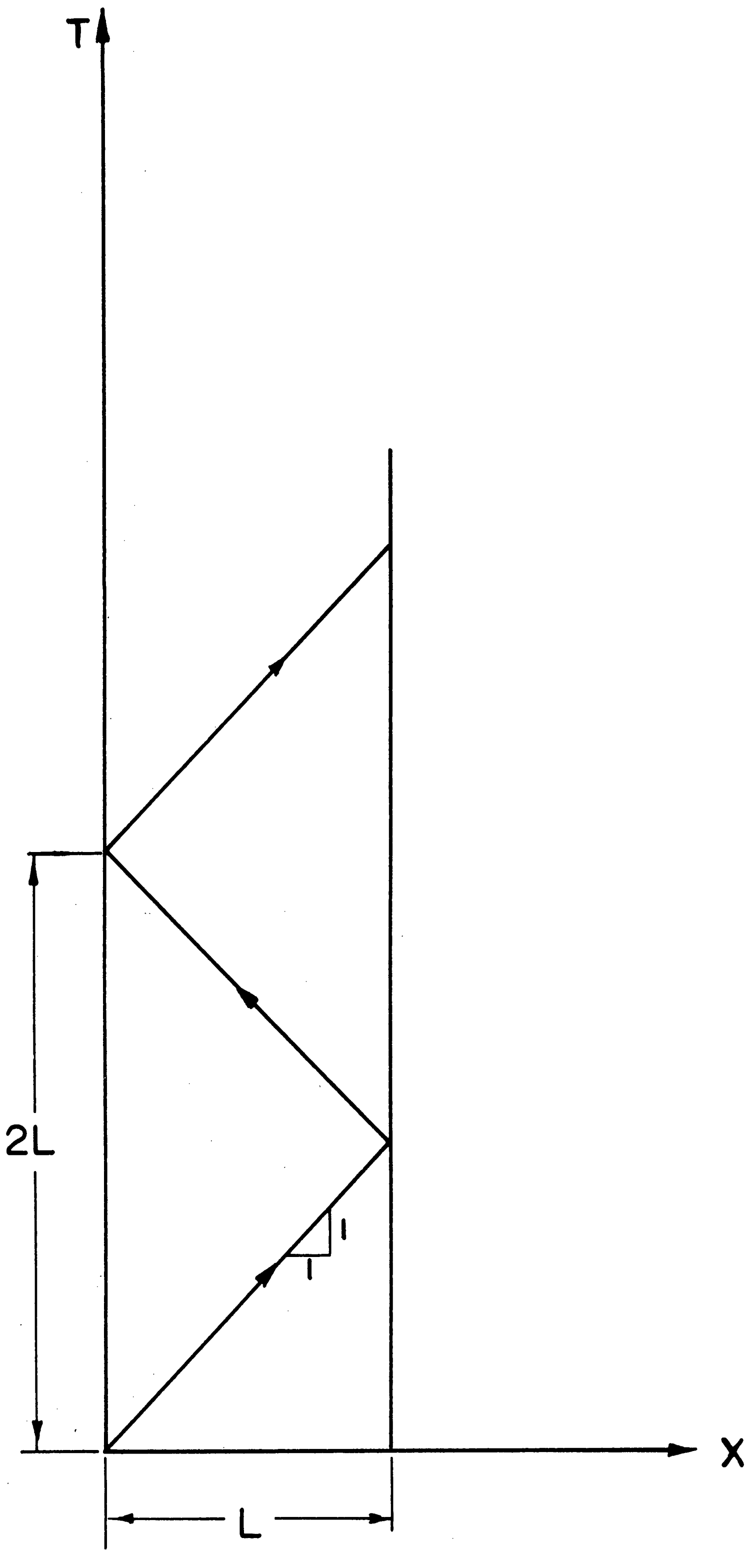

Figure 5.1 Reflection of Characteristic Lines Between Two Boundaries

CONCLUSION

The velocity spectra show that the elementary theory is quite different from the more exact theory for high phase velocities of wave propagation. When the velocity of the moving pressure front is greater than the first critical velocity, the more exact theory must be used in any case.

In the elementary theory for the steady state response, the solution is not unique for the case when the velocity of the moving pressure front is greater than the critical. In the more exact theory, the solution is not unique when the velocity is between the first and second critical velocities or greater than the third critical velocity. The reason for the non-uniqueness of the solution is that no specific boundary conditions are assumed and the only condition is that the radial deflection is bounded everywhere. If the boundary condition is specified at infinity, there will be no solution for the over critical case unless the damping effect is included. Fortunately damping must exist in every physical system. The zero damping response can be obtained by assuming that the damping coefficient approaches zero in the limit.

For the transient response in the semi-infinite tube, the Fourier sine transform is used in the solution of the under critical case. Only the formula for the radial deflection W is derived. If the longitudinal fiber stress in the tube wall is wanted, the bending moment has to be known. It can be obtained by differentiating W twice with respect to X in the elementary theory and W_b in the more exact theory. Both W and W_b contain improper integrals which converge uniformly with respect to X, so it is valid to differentiate twice with respect to X. For simplicity, if the velocity of the moving pressure front is under critical, the simple formula from the elementary theory is allowed. In this velocity range, the response due to modes with large wave numbers or high frequencies is small.

The advantage of the numerical method is that M, $\bar{\omega}$, Q, and $\bar{v}(W)$ can be found at the same time. It can be applied to any boundary conditions at the near end of a semi-infinite tube. If the period of the input forcing function is very long, the longer duration has to be adopted in the calculation. If T_{max} is the duration to be taken in calculation and ΔT is the interval, $N(N + 1)/2$ (where $N = T_{max}/\Delta T$) stations have to be calculated. It will take a long time even for the IBM 7090 computer.

If the velocity of the moving pressure front approaches the first critical, the particular solution is unstable and the amplitude is increasing as time increases. The input forcing function in the second part which corrects the boundary conditions due to the particular solution is also unstable and is increasing as time increases. When the velocity approaches the second or third critical, one of the wave number approaches infinity. In this critical range, the validity of the approximate equations of motion as used herein is doubtful. In the neighborhood of the second or third critical velocity, the frequency of the vibration is extremely high. In these regions, the exact equations of motion from three dimensional theory of elasticity should be used.

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APPENDIX

DERIVATION OF FOUR SIMULTANEOUS FIRST ORDER EQUATIONS USED IN THE METHOD OF CHARACTERISTICS

A. Basic Equations

From Equation (1.27), the equation of translation along the radial direction is

$$\frac{\partial Q_x}{\partial x} - \frac{Eh}{R^2}(w_b + w_s) = \rho h \frac{\partial^2 (w_b + w_s)}{\partial t^2}$$

If $w = w_b + w_s$ and $v_1 = \frac{\partial w}{\partial t}$ are introduced, the equation becomes

$$\frac{\partial Q_x}{\partial x} - \frac{Eh}{R^2} w = \rho h \frac{\partial v_1}{\partial t} \qquad \text{(A.1a)}$$

From Equation (1.25), a relation between resultant shearing force per unit length and the deflection due to shear is obtained

$$Q_x = h k G \frac{\partial w_s}{\partial x}$$

or

$$Q_x = h k G \left(\frac{\partial w}{\partial x} - \frac{\partial w_b}{\partial x} \right)$$

If $\omega = \frac{\partial^2 w_b}{\partial x \partial t}$ is introduced, the equation after being differentiated with respect to t becomes

$$\frac{\partial Q_x}{\partial t} = h k G \left(\frac{\partial v_1}{\partial x} - \omega \right) \qquad \text{(A.1b)}$$

From Equation (1.28), the equation of rotation is

$$\frac{\partial M_{xx}}{\partial x} - Q_x = - \frac{\rho h^3}{12} \frac{\partial^3 w_b}{\partial x \partial t^2}$$

or

$$\frac{\partial M_{xx}}{\partial x} - Q_x = - \frac{\rho h^3}{12} \frac{\partial \omega}{\partial t} \qquad \text{(A.1c)}$$

From Equation (1.26), the relation between moment and the deflection due to bending is

$$M_{xx} = -D \frac{\partial^2 w_b}{\partial x^2}$$

Differentiating both sides with respect to t, it becomes

$$\frac{\partial M_{xx}}{\partial t} = -D \frac{\partial \omega}{\partial x} \qquad \text{(A.1d)}$$

B. Basic Equation in Dimensionless Form

As mentioned previously, the following dimensionless variables are introduced

$$X = \frac{\sqrt{12}\, x}{h}$$

$$T = \frac{\sqrt{12}\, v_d t}{h}$$

$$W = \frac{w}{h}$$

$$\bar{v} = \frac{v_1}{\sqrt{12}\, v_d}$$

$$\bar{\omega} = \frac{h}{12\, v_d}\, \omega$$

$$Q = \frac{Q_x}{\sqrt{12}\, h \kappa G}$$

$$M = \frac{1-\nu^2}{E h^2} M_{xx}$$

Dimensionless parameters are

$$g^2 = \frac{E}{12\, \kappa G} \left(\frac{h}{R}\right)^2$$

$$\delta^2 = \frac{(1-\nu^2)\kappa G}{E}$$

Equation (A.1) becomes

$$\begin{cases} \dfrac{\partial Q}{\partial X} - g^2 \bar{W} = \dfrac{1}{\delta^2} \dfrac{\partial \bar{V}}{\partial T} & \text{(A.2a)} \\[2ex] \dfrac{\partial Q}{\partial T} = \dfrac{\partial \bar{V}}{\partial X} - \bar{\omega} & \text{(A.2b)} \end{cases}$$

$$\begin{cases} \dfrac{\partial M}{\partial X} - \delta^2 Q = - \dfrac{\partial \bar{\omega}}{\partial T} & \text{(A.2c)} \\[2ex] \dfrac{\partial M}{\partial T} = - \dfrac{\partial \bar{\omega}}{\partial X} & \text{(A.2d)} \end{cases}$$

C. <u>Determination of Characteristic Lines and Differential Equations along the Characteristic Lines</u>

The total differential of M along a certain line on X-T plane can be expressed in terms of dX and dT as following:

$$dM = \frac{\partial M}{\partial X} dX + \frac{\partial M}{\partial T} dT$$

Similar relation is hold for $\bar{\omega}$,

$$d\bar{\omega} = \frac{\partial \bar{\omega}}{\partial X} dX + \frac{\partial \bar{\omega}}{\partial T} dT$$

Four simultaneous equations for four unknown partial derivatives, i.e., $\frac{\partial M}{\partial X}, \frac{\partial M}{\partial T}, \frac{\partial \bar{\omega}}{\partial X}, \frac{\partial \bar{\omega}}{\partial T}$, can be obtained by combination of the above two equations with Equations (A.2c) and (A.2d)

$$\frac{\partial M}{\partial X} dX + \frac{\partial M}{\partial T} dT = dM \qquad \text{(A.3a)}$$

$$\frac{\partial \bar{\omega}}{\partial X} dX + \frac{\partial \bar{\omega}}{\partial T} dT = d\bar{\omega} \qquad \text{(A.3b)}$$

$$\frac{\partial M}{\partial X} \qquad\qquad + \frac{\partial \bar{\omega}}{\partial T} = \delta^2 Q \tag{A.3c}$$

$$\frac{\partial M}{\partial T} + \frac{\partial \bar{\omega}}{\partial X} \qquad\qquad = 0 \tag{A.3d}$$

Let the determinant of the coefficients of $\frac{\partial M}{\partial X}$, $\frac{\partial M}{\partial T}$, $\frac{\partial \bar{\omega}}{\partial X}$ and $\frac{\partial \bar{\omega}}{\partial T}$ be zero

$$\begin{vmatrix} dX & dT & 0 & 0 \\ 0 & 0 & dX & dT \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{A.4}$$

A relation between dT and dX can be gotten as follows:

$$\frac{dT}{dX} = \pm 1 \tag{A.5}$$

These are the differential equations of the characteristic lines along which dilatational waves are propagated. If $\frac{\partial M}{\partial X}$, $\frac{\partial M}{\partial T}$, $\frac{\partial \bar{\omega}}{\partial X}$, and $\frac{\partial \bar{\omega}}{\partial T}$ are definite along the characteristic line $\frac{dT}{dX} = 1$, the following determinant must vanish

$$\begin{vmatrix} dM & dT & 0 & 0 \\ d\bar{\omega} & 0 & dX & dT \\ \delta^2 Q & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{A.6}$$

Since $\frac{dT}{dX} = 1$ along the characteristic line, a differential equation is established from Equation (A.6) as follows:

$$dM - \delta^2 Q\, dT + d\bar{\omega} = 0$$

which is Equation (4.36a).

By the same procedure, the differential equation can be established along the characteristic line $\frac{dT}{dX} = -1$ as follows:

$$dM + \delta^2 Q\, dT - d\bar{\omega} = 0$$

which is Equation (4.36b).

Other four simultaneous equations for four unknown partial derivatives, i.e., $\frac{\partial Q}{\partial X}$, $\frac{\partial Q}{\partial T}$, $\frac{\partial \bar{v}}{\partial X}$, and $\frac{\partial \bar{v}}{\partial T}$ can be obtained by combination of the total differentials of Q and $\bar{v}$ along a certain line with Equations (A.2a) and (A.2b)

$$\frac{\partial Q}{\partial X} dX + \frac{\partial Q}{\partial T} dT = dQ \qquad \text{(A.7a)}$$

$$\frac{\partial \bar{v}}{\partial X} dX + \frac{\partial \bar{v}}{\partial T} dT = d\bar{v} \qquad \text{(A.7b)}$$

$$\frac{\partial Q}{\partial X} - \frac{1}{\delta^2}\frac{\partial \bar{v}}{\partial T} = g^2 W \qquad \text{(A.7c)}$$

$$\frac{\partial Q}{\partial T} - \frac{\partial \bar{v}}{\partial X} = -\bar{\omega} \qquad \text{(A.7d)}$$

Let the determinant of the coefficients of $\frac{\partial Q}{\partial X}$, $\frac{\partial Q}{\partial T}$, $\frac{\partial \bar{v}}{\partial X}$ and $\frac{\partial \bar{v}}{\partial T}$ be zero

$$\begin{vmatrix} dX & dT & 0 & 0 \\ 0 & 0 & dX & dT \\ 1 & 0 & 0 & -\frac{1}{\delta^2} \\ 0 & 1 & -1 & 0 \end{vmatrix} = 0 \qquad (A.8)$$

A relation between dT and dX can be gotten

$$\frac{\partial T}{\partial X} = \pm \frac{1}{\delta} \qquad (A.9)$$

These are the characteristic lines along which the modified shear waves are propagated. If $\frac{\partial Q}{\partial X}$, $\frac{\partial Q}{\partial T}$, $\frac{\partial \bar{v}}{\partial X}$ and $\frac{\partial \bar{v}}{\partial T}$ are definite along the characteristic line $\frac{dT}{dX} = \frac{1}{\delta}$ the following determinant must vanish

$$\begin{vmatrix} dQ & dT & 0 & 0 \\ d\bar{v} & 0 & \delta dT & dT \\ g^2 W & 0 & 0 & -\frac{1}{\delta^2} \\ -\bar{\omega} & 1 & -1 & 0 \end{vmatrix} = 0 \qquad (A.10)$$

From Equation (A.10), the following differential equation is established along the characteristic line $\frac{dT}{dX} = \frac{1}{\delta}$

$$dQ - \delta g^2 W dT - \frac{1}{\delta} d\bar{v} + \bar{\omega} dT = 0$$

which is Equation (4.36c).

By the same procedure, the differential equation can be established along the characteristic line $\frac{dT}{dX} = -\frac{1}{\delta}$ as follows:

$$dQ + \delta q^2 W dT + \frac{1}{\delta} d\bar{U} + \bar{\omega} dT = 0$$

which is Equation (4.36d).