# Components of Algebraic Sets of Commuting and Nearly Commuting Matrices

by

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To Mom and Dad

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# CHAPTER I

# Introduction

## 1.1 The Objects We Study and Statement of Our Results

In this thesis we study algebraic sets of tuples whose entries are mutually commuting matrices possibly satisfying some additional condition, such as nilpotence, or which satisfy some weakening of the condition of commutativity. For example, we study pairs of matrices whose commutator is nilpotent, and also those pairs whose commutator is a diagonal matrix. In particular, we study several instances in which such algebraic sets are complete intersections, and also investigate the behavior of the components of these algebraic sets. In some instances, it is remarkably difficult to prove irreducibility, or to determine explicitly how many components there are.

We also study some classes of jet schemes that are closely related to these algebraic sets of tuples of matrices.

Some of our results are valid over fields of arbitrary characteristic. Frequently, we need to restrict to characteristic diffferent from 2, and occasionally we must restrict to characteristic 0.

The main result of chapter II on commuting nilpotent tuples is Theorem 2.2.10, which gives a new proof of the irreducibility of commuting pairs of nilpotent matrices and also slightly generalizes it. Let $\Omega_n$ be the set of quadruples $(A, B, u, v)$ where $A$ and $B$ are commuting nilpotent $n \times n$ matrices, $u$ and $v$ are $n-$vectors such that $Au = Bv$. Then we have:

**Theorem** 2.2.10. $\Omega_n$ is irreducible for all $n$ (For all characteristics except 2.)

The above theorem implies the irreducibility of commuting pairs of nilpotent $n \times n$ matrices. In addition, we settle the first open case for commuting nilpotent triples by proving the set of commuting triples of $4 \times 4$ matrices is irreducible. We denote such triples by $N(3, 4)$.

**Theorem** 2.2.1. $N(3, 4)$ is irreducible (for all characteristics).

In Chapter III, we study the jet schemes of determinantal varieties. We prove that in general these jet schemes are reducible and give explicitly the number of components. Let $L_{r,s}^{m,n}$ be the vanishing set of all $r \times r$ minors in $m \times n$ matrices over $k[t]/(t^s)$, where $n \geq m \geq r$, and $n, m, r$ and $s$ are all positive integers $\geq 2$. To calculate the number of components in a minimal decomposition, we can always reduce to the case where $r \leq s$. If $r = m$, then the algebraic set defined by maximal minor determinants is irreducible, this is **Proposition** 3.2.1. The main result of this chapter is that the number of components actually has a very simple form:

**Theorem** 3.2.9 When $n \geq m > r$, $N\left(L_{r,s}^{m,n}\right)$, the number of components of $L_{r,s}^{m,n}$ is equal to

$$s + 1 - \lceil \frac{s}{r} \rceil = \lfloor \frac{r-1}{r} s + 1 \rfloor.$$

Note that this theorem had been obtained before our work (see the introduction at the beginning of chapter III), but our proof is new and shorter. Actually, although the theorem above is the culmination of the results on calculating the numbers of components, technically (in our context) it is just a corollary of **Theorem** 3.2.2, which gives a recursive formula. Our formulation also allows direct calculation for the dimension of a given component.

In chapter IV, we study the matrix pairs whose commutators are nilpotent. When the size of the matrices is $2 \times 2$, a commutator is nilpotent if and only if it is of rank 1. The pairs whose commutators are of rank 1 were first studied by Hulek, and his results state that the set of such pairs is an irreducible complete intersection ([HUL]). In higher dimensions, however, a nilpotent commutator is not the same thing as a rank 1 commutator, and we believe that the study of nilpotent commutators is initiated here. Let $Z_n$ be the set of pairs of $n \times n$ matrices whose commutator is a nilpotent matrix. Our main results of chapter IV are:

**Theorem** 4.2.3. $Z_n$ is a complete intersection of dimension $2n^2 - n + 1$ (for all characteristics).

**Theorem** 4.2.5. $Z_n$ is an irreducible algebraic set when the characteristic does not divide $n$.

Also:

**Theorem** 4.2.7. If the characteristic of our base field is 0, then $Z_n$ is a reduced scheme. Thus, the coordinate ring of $Z_n$ is a complete intersection domain.

In chapter V, we first study the so-called diagonal commutator scheme, $S_n$, which consists of pairs of $n \times n$ matrices whose commutator is a diagonal matrix. In the paper [KNU], it is proven that this scheme is a reduced complete intersection when the base field is $\boldsymbol{C}$, using a flat degeneration argument. We have worked out a completely different proof for $S_n$ being a complete intersection that applies to prime characteristic too. In addition, in [KNU], it was noted that there is a proof, using Lie group theory, showing that the diagonal commutator scheme has only two components, one of them being the variety of commuting pairs, when the base field is $\boldsymbol{C}$. However, that proof has never been published, nor recorded in any form, and can not be found or recalled now, and we were feeling uncomfortable about this situation. Therefore, we have worked out a proof using only elementary algebraic geometry, and our proof applies to all characteristics. Utilizing our proof for the theorem that $S_n$ has only two components, we also get a simpler proof that $S_n$ is reduced as a scheme when the characteristic of the base field is zero.

The facts mentioned above are the first main results of chapter V:

**Theorem** 5.2.5. $S_n$ is a complete intersection of dimension $n^2 + n$ (for all characteristics).

**Theorem** 5.2.9. $S_n$ has two components, one of which being the variety of commuting matrix pairs (for all characteristics).

**Theorem** 5.2.10. $S_n$ is a reduced scheme when the base field is of characteristic zero.

The last section of chapter V contains results upon the algebraic set consisting of pairs of $n \times n$ matrices whose commutator has vanishing diagonal entries, henceforth denoted by $U_n$. Our results on $U_n$ are similar to those we have described:

**Theorem** 5.3.1. $U_n$ is a complete intersection of dimension

$2n^2 - n + 1$ (for all characteristics).

**Theorem** 5.3.2. $U_n$ is a reduced scheme when the base field is of characteristic zero.

**Theorem** 5.3.3. $U_n$ is irreducible when $n$ equals 2 or 3 (for arbitrary characteristic).

A more detailed introduction is given at the beginning of each chapter.

## 1.2 Common Definitions and Notation

Throughout our article, a field $k$ is always assumed to be algebraically closed, no matter what its characteristic is.

$\boldsymbol{R}(A)$, where $A$ is a matrix, means the range (image) of $A$.

We say a nilpotent matrix $A$ is stable if there is a dense open set in the irreducible set of the nilpotent matrices that commute with $A$ such that the matrices are all similar to $A$ (having the same Jordan form).

In general, a matrix is called regular or cyclic if all matrices that commute with it can be written as a polynomial in it.

A nilpotent matrix $P$, or equivalently, its Jordan form $J$, will be called stable if on the irreducible set of nilpotent matrices that commute with it, there is a dense open set consisting of matrices conjugate to $P$ (having the same Jordan form).

Let $A$ be an $n \times n$ matrix, then the discriminant of $A$ is denoted by $disc(A)$, the determinant by $det(A)$.

Let $R$ be a commutative ring.

$M_R(m, n)$ denotes the set of $m \times n$ matrices over $R$.

$GL(n, R)$ or $GL_R(n)$ will denote the group of $n \times n$ invertible matrices over $R$.

$C_R(m,n)$ denotes the set of mutually commuting $m$-tuples of $n \times n$ matrices over $R$. $N_R(m,n)$ denotes the set of mutually commuting $m$-tuples of $n \times n$ nilpotent matrices over $R$.

When $R$ is omitted then it means we are taking matrices over an algebraically closed field. Both $M_n$ and $M(n)$ will mean the $n \times n$ matrices, and both $N_n$ and $N(n)$ will mean the $n \times n$ nilpotent matrices.

In all of the above cases we usually omit $R$ or $k$ if the ring is a field. The field $k$ will always be assumed to be algebraically closed, but there is no restriction for its characteristic unless otherwise specified. $k^n$ will mean the $n$-dimensional affine space over $k$ and $k^*$ is the multiplicative group of $k$.

*Grass* $(n,m)$ is the Grassmannian of $m$-dimensional subspaces in an $n$-dimensional vector space.

$\boldsymbol{P}^n$ is $n$-dimensional projective space.

$L_{r,s}^{m,n}$ is the determinantal set where all $r \times r$ minors vanish in an $m \times n$ matrix over the ring $k[t]/(t^s)$.

If $\tau \in L_{r,s}^{m,n}$, we write $\Gamma_\tau$ for the orbit of

$$\{\alpha\tau\beta : \alpha \in GL(m, k[t]/(t^s)), \beta \in GL(n, k[t]/(t^s))\}$$

in $L_{r,s}^{m,n}$, and we say $\overline{\Gamma_\tau}$ is the potential component generated by $\tau$, although it may not be an actual component in the minimal decomposition but only an irreducible closed set really.

If a component $C$ is generated by

$$\tau = \begin{bmatrix} t^{p} & 0 & 0 & \ldots \\ 0 & t^{q_2} & 0 & \ldots \\ 0 & 0 & t^{q_3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is, $C = \overline{\Gamma_\tau}$, then we also say $\tau$ is the canonical matrix type of the component $C$. $End\,(n,k) = gl\,(n,k)$ means the algebra of $n \times n$ matrices over the field $k$, and $sl\,(n,k)$ the special linear (trace-zero) Lie algebra. $sl\,(n,k)$ is only a Lie subalgebra of $gl\,(n,k)$, it is not an algebra. When the field is the complex number field $\boldsymbol{C}$, we usually just write $gl\,(n)$ and $sl\,(n)$.

# CHAPTER II

# Commuting Tuples of Nilpotent Matrices

## 2.1 Introduction

Let $C(m,n)$ be the algebraic set of commuting $m$-tuples of $n \times n$ matrices over the field $k$, where we usually only assume that the characteristic is not 2, but some of our proofs only work when the characteristic is 0. Let $N(m,n)$ be the analogous algebraic set where the commuting matrices are nilpotent.

**Theorem 2.1.1.** *$N(m,n)$ is irreducible if $m = 2$ for all $n$ and, if $n \leq 3$, for all $m$, while if both $m$ and $n$ are greater than or equal to 4, it is reducible.*

(See [GUR] and [GS] )

Now, when $m = 2$ (commuting pairs) both are known to be irreducible for all $n$, but the proofs for the nilpotent case are relatively recent ([BAR], [BAS] and [PRE] independently.) For $n \leq 3$, $C(m,n)$ is known to be irreducible for all $m$ while if both $m$ and $n$ are $\geq 4$ it is known to be reducible ([GER] and [GS]). Note that both of the corresponding statements hold for nilpotent matrices, although neither the statement nor the proof is found in the current literature. The proof of the irreducibility of nilpotent commuting pairs in [BAR] depends on some highly sophisticated facts from algebraic geometry and the theory of punctual Hilbert schemes, and is also more restricted concerning the characteristic of the field. While the proof of [BAS] is very elementary (requires only basic facts from algebraic geometry and linear algebra), it is also significantly longer and more complicated. The path taken by [PRE] is very different in that almost all facts are formulated and proven in the setting of Lie algebra and algebraic group theory (both of characteristic zero and prime characteristic), therefore the reading and understanding of [PRE] are somewhat cumbersome to people who do not work in representation theory. We will give a slightly generalized fact and obtain a new proof for the irreducibility of nilpotent commuting pairs. Our proof is more geometric and conceptual, and also slightly more general concerning prime characteristics, while being more elementary compared with [BAR] and [PRE] . However, the apparent conciseness of our proof may be somewhat deceptive, because we have quoted a fact from [HW]. The proof of the quoted fact is elementary and relatively easy and straightforward, however, if written out in complete details it would take at least several pages. In addition to the new proof and generalization of the irreducibility of nilpotent commuting pairs, we will also prove the irreducibility of nilpotent commuting triples of $4 \times 4$ matrices, the proof of which is not hard, but already much more technical compared with the case of general commuting triples, and indicates the complexity of proving irreducibility for nilpotent commuting triples of larger sizes. Of course, for sizes larger than 29, the nilpotent commuting triples are expected to be reducible, just as general commuting triples, but again there does not seem to be a proof in the literature. Finally in this chapter we will prove a fact concerning finite dimensional algebras whose commuting pairs are irreducible. Although the proposition had been known before our proof (the author was unaware of this when working out the proof), we decide to include its proof because it is difficult to locate both the statement and the proof in the literature.

Having the general linear group acting on nilpotent matrices by conjugation, we can represent each orbit by its Jordan form, which would be unique if we also require the sizes of the blocks have been ordered from largest to the smallest. Following [GER] , we can define a partial order ($\succeq$) on the set of nilpotent matrices, and $A \succeq B$ if and only if $B$ is in the closure of the orbit of $A$. The actual definition of the partial order is not important to us, because we will not be using it.

We will need the following facts:

**Theorem 2.1.2.** *(Theorem 6. of [HW]) Given a nilpotent matrix $M$, with kernel dimension $K$ and the dimension of the centralizer algebra $C$, the nilpotent elements of the centralizer algebra form an irreducible algebraic set with dimension $C - K$. (Note this is only a part of the theorem in [HW] , as we only need this part.)*

## 2.2 Nilpotent Commuting Matrices

As a reminder, a matrix is called regular or cyclic if all the matrices that commute with it are polynomials in it. This condition is equivalent to having only one Jordan block for each eigenvalue.

**Theorem 2.2.1.** *$N(3, 4)$ is irreducible.*

*Proof.* There always is a component where on a dense open set all the matrices that appear in those commuting triples are nilpotent with rank 3, and hence are regular. This component will be referred to as the cyclic component.We only have to prove that all commuting triples of similar (conjugate) nilpotent matrices are in the closure of the cyclic component, since in every component there would be a dense open set where this holds with a unique maximal (with our partial order) Jordan form. We begin with the rank 1 case:

$$(1)\quad \left( \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

Now, using

$$(2)\quad \left( \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),$$

we see the triples (1) are in the closure of the rank 2 square-zero triples. Thus, if we prove that the triples in (2) are in the cyclic component, we are done with the rank 1 case as well. Note the transposed triples can be dealt with similarly.

For the rank 2 and square zero case, we can reduce to a standard form:

$$\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right),$$

where $a \neq b$, and neither of them is zero, because such triples will generate a dense open set of the irreducible closed set. If we look at

$$\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ \epsilon & 0 & 0 & 1 \\ 0 & 0 & 0 & \epsilon\frac{a}{b} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right),$$

we see that it is indeed in the cyclic component.

For the rank 2 but square non-zero case, all such triples are in the closure of the set where the third matrix is a linear combination of polynomials (respectively) of the first two, and this set in turn is in the cyclic component. Therefore, all cases of Jordan forms are done and the theorem is proven.

□

**Remark 2.2.2.** *This theorem appears to be a new result, since it was stated as (part of) an open problem in some notes from a workshop in June 2008 ([PAN]), and I searched for it in the literature and on internet but neither the statement nor the proof* *seems to exist elsewhere.*

The general case of commuting $m$-tuples of $n \times n$ matrices is much more complicated. When both $m$ and $n$ are greater than or equal to 4, it is known the set of such commuting tuples is reducible ([GER]), but no results concerning how many components there actually are have been found in the literature. It is also known that $C(3, n)$ is irreducible for $n \leq 8$ ([GS], [HO], [OML], [HAN] and [SIV]) and reducible for $n \geq 30$ ([GUR] and [HO]), while the remaining cases are open. As far as the author knows the question of the irreducibility of commuting nilpotent triples of general size is not in the literature. Note we can prove that the commuting nilpotent triples will be reducible for $n \geq 48$ by simply adding the nilpotent condition to the proof for general matrices ([GUR]). Also, by utilizing the irreducibility of general commuting triples we know the set of commuting nilpotent triples will become reducible for some $n \leq 30$. In addition, in characteristic 0, if $C(m, n)$ is reducible, then $C(m, n+1)$ is reducible ([HO]). However, the author does not know whether this is known in prime characteristic or for nilpotent matrices.

**Theorem 2.2.3.** *$N(3, n)$ is reducible for at least one $n$ such that $5 \leq n \leq 30$.*

*Proof.* Let $p$ be the least integer where $C(3, p)$ becomes reducible. We know $9 \leq p \leq 30$. If $N(3, n)$ is irreducible, then by simultaneous block decompositions of commuting matrices and the fact that $C(3, m)$ is irreducible for all $m$ such that $1 \leq m < n$, we deduce the irreducibility of $C(3, n)$, which, of course, is a contradiction. Hence, there must be at least one $n$ between 9 and 30 such that $N(3, n)$ is reducible, since $9 \leq n \leq 30$, therefore there must be at least one such $n$ between 5 and 30. $\square$

**Definition 2.2.4.** $\Omega_n$=$(A, B, u, v)$, *where* $A$ *and* $B$ *are commuting* $n \times n$ *matrices and* $u$ *and* $v$ *are vectors in* $k^n$ *such that* $Au = Bv$.

First, our aim is to prove that the set $\Omega_n$ is irreducible for all $n$. Because once we prove the irreducibility of these sets, we can use the actions of the irreducible algebraic groups $GL(n+1)$ to prove the irreducibility of commuting pairs of $(n+1) \times (n+1)$ matrices (general or nilpotent). And of course, the irreducibility of all these in low dimensions such as $n$ =1 or 2 are trivial.

Notice, however, the analogous sets for $m \geq 3$ become reducible quickly, in fact, the corresponding set for commuting triples of nilpotent $3 \times 3$ matrices is reducible, although the commuting $m-$tuples of $3 \times 3$ nilpotent matrices are always irreducible.

**Definition 2.2.5.** *The set* $\Omega_3^3 = \{(A, B, C, u, v, w)\}$, *where* $A, B$ *and* $C$ *commuting nilpotent* $3 \times 3$ *matrices and* $u, v, w \in k^3$, $Au = Bv = Cw$.

**Proposition 2.2.6.** *The set* $\Omega_3^3$ *is reducible (for all characteristics).*

*Proof.* There is always a component where on a dense open set, all $A, B$ and $C$ appearing are nilpotent regular. This component has dimension = 6+2+2+3+1+1 = 15, the 3 and $1's$ come from the fiber dimension (as the fibers of the projection onto commuting nilpotent triples of $3 \times 3$ matrices). If the algebraic set is irreducible, then this component, again called the cyclic component, is the whole set. However, consider the following element of $\Omega$:

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u, \quad v, \quad w \right).$$

The closure of the set generated by such elements and their obits under the group conjugation action will have dimension $= 4+2+2+3+2+2 = 15$. Since no matrix appearing in this closure will be regular, if it is contained in the cyclic component, its dimension has to be strictly less than 15, which is not the case as we have just seen. Therefore $\Omega$ is reducible. □

**Theorem 2.2.7.** *$\Omega_n$ is irreducible for all $n$, and this also establishes the irreducibility of $C(2, n)$ for all $n$ (for all characteristics).*

*Proof.* By induction on $n$, we can assume the set of commuting pairs of $n \times n$ matrices is irreducible (because this is implied by the irreducibility of $\Omega_{n-1}$) and that will make it have a dense open set, on which both projections are regular and invertible. This will give us the dimension of the set $\Omega_n$ if it is irreducible, and the dimension will be $n^2+2n$. To prove it is irreducible, remember that there is always a component in which over a dense open set, the matrices that appear are all regular. Henceforth we will call this component the cyclic component. Notice that $\Omega_n$ is defined by $n$ homogeneous equations over the polynomial ring with $2n$ variables over the commuting pairs of $n \times n$ matrices, which is assumed to be irreducible with dimension $n^2 + n$, therefore any component has to have dimension at least $n^2 + 2n$. We will actually prove that if there is any component other than the cyclic component, then the dimension of this new component is actually strictly less than $n^2 + 2n$, which leads to a contradiction and the irreducibility follows. (The irreducibility of $\Omega_n$ will imply the irreducibility of $C(2, n+1)$, thus by induction implies the irreducibility of commuting pairs of all sizes. This process is specified in detail below.) Now the set of commuting pairs is irreducible and has dimension $n^2 + n$, and the dimension we get from appending $(u, v)$ is actually $2n$ since we are taking the polynomial ring with $2n$ variables over the irreducible coordinate ring. Then our condition for the set $\Omega_n$ is actually given by $n$ equations in the affine coordinates, and so any component will have dimension at least $n^2 + n + 2n - n = n^2 + 2n$.

The proof then will proceed as follows: take a component $B$, and consider the first projection onto the the space of $n \times n$ matrices. If the image $\Theta$ intersects the set of invertible matrices, then on a dense open set of $B$, the fiber dimension will be less than or equal to $n$, while the image of the component in the set of commuting pairs will have dimension strictly less than $n^2+n$ (otherwise it will be the cyclic component $C$). Therefore, $B$ has dimension strictly less than $n^2 + 2n$. On the other hand, if $\Theta$ intersects the set where the matrix has at least 2 different eigenvalues, then this holds on a dense open set of $B$. Therefore every point in this dense open set will be in the closure of the cyclic component $C$ by induction and block decomposition. Thus, the only possible case left is that the image $Q$ of projection of the component $B$ onto the set of commuting pairs is contained in the closed subset of nilpotent commuting pairs. However, the dimension of the set of commuting pairs of $n \times n$ nilpotent matrices is less than or equal to $n^2-1$ (this fact can be proven without assuming the irreducibility of commuting nilpotent pairs), and so the component will have dimension less than or equal to $n^2 - 1 + 2n$, which is also less than the lower bound. Thus, the set $\Omega_n$ is irreducible. Hence the commuting pairs of $(n+1) \times (n+1)$ matrices with vanishing first column will be irreducible. By the action of $k^2$ as additions of scalar matrices, we also get the irreducibility for the set of commuting pairs of $(n+1) \times (n+1)$ matrices where $a_{1j} = 0$ for all $j \geq 2$. We then have developed the irreducibility for the set of commuting pairs of $(n+1) \times (n+1)$ matrices by group action. Of course, we have used here the facts that the components will be closed under permutations and the action of the general linear group, which are easy to prove (but not totally trivial). First taking a minimal decomposition of the algebraic set. Then the image of a component under general matrix acting and permutations is irreducible and contains the component itself. Since we started with a minimal decomposition, the image of the component must coincide with the component. □

**Corollary 2.2.8.** *The set $(A, B, v) \in C\,(2, n) \times (k^n - \{0\}) \mid [A, B] = 0$ and $v$ is a common eigenvector of $A$ and $B$ is irreducible.*

**Corollary 2.2.9.** *The set $[A, B] = 0$ where $A$ and $B$ are $n \times n$ matrices, and $a_{j1} = 0, b_{j1} = 0, j \geq 2$ is irreducible. So is the subset where also $a_{11} = b_{11} = 0$.*

The proof for the nilpotent case is similar but more technical.

**Theorem 2.2.10.** *$\Omega_n$=$(A, B, u, v)$ where $A$ and $B$ are commuting nilpotent matrices, is irreducible for all $n$, and this also establishes the irreducibility of $N\,(2, n)$ for all $n$ (for all characteristics).*

*Proof.* The naive dimension estimate will give a lower bound of $n^2 + n - 1$, but we really need it to be $n^2 + n$, since this will be the dimension of the cyclic component. What we have to do is to notice that the sum of the respective ranges of two commuting nilpotent matrices will always be contained in some $(n-1)$-dimensional subspace. This fact is shown by applying Nakayama's lemma to the finite length module $k^n$ over the formal power series ring of two variables. Therefore, for any commuting nilpotent pair $(A, B)$ we can always find a $g \in \mathit{Grass}\,(n, n-\mathit{1}) \cong \boldsymbol{P}^{n-\mathit{1}}$ such that $\boldsymbol{R}(A) + \boldsymbol{R}(B) \subseteq g$. This subspace is not unique in general. The set $\{(A, B, g) \in N\,(\mathit{2}, n) \times G\,(n, n-\mathit{1}) \mid \boldsymbol{R}(A) + \boldsymbol{R}(B) \subseteq g\}$ is irreducible by the induction hypothesis for smaller size $\Omega_n$=$(A, B, u, v)$. Because by taking transposition (adjoint map) between the vector space and its dual, this set is isomorphic to that which consists of a pair of commuting nilpotent matrices and a vector in the intersection of their kernels. Now take the affine open cover of the projective space by identifying $(n-1)$-dimensional spaces as kernels of 1-forms, so now a point is defined by $\Sigma a_i x_i = 0$, where $a_i$'s are coefficients of the one form. Over the preimage of each of the open set where $a_i \neq 0$, the number of independent equations of $Au = Bv$ is reduced to $n-1$ because the in the $n-1$-dimensional subspace one of the coordinates is controlled by the others, so the open set will have components with dimension at least $n^2 - 1 + 2n - (n-1) = n^2 + n$, since they also form an open cover of $(A, B, g)$, and since it is irreducible each open set is dense. Now the projection from $(A, B, g)$ onto $N\,(\mathit{2}, n)$ is a surjection, and hence, the image of such an open set is dense in $N\,(\mathit{2}, n)$, and over it the number of equations defining $\Omega_n$ is reduced to $n-1$. Thus every component has dimension at least $n^2 + n$. Since the irreducible sets together cover $\Omega_n$, the components of $\Omega_n$ will all have dimensions greater than or equal to $n^2 + n$.

Now, take a component $X$ other than the one whose image dominating $N(2, n)$, assuming such a component exists. Then its image in $N(2, n)$ will be the closure of an open set where the 2 matrices of all the commuting pairs are similar (conjugate) to each other and the Jordan form is unique. The dimension (as a vector space) of the kernel of the Jordan form will be denoted by $K$, and the Jordan form itself by $P$, the dimension of the subspace of $M_n(k)$ that commutes with $P$ by $C$

Recall the meaning of stable nilpotent matrices ( see [BI] and the section "Definitions and Notation"): a nilpotent matrix $M$ is called stable if the nilpotent elements in its centralizer algebra are generically conjugate to the same Jordan canonical form as $M$, and being stable is equivalent to the condition that the differences of sizes of the Jordan blocks are at least 2. Note we are only using the definition of stable nilpotent matrices, but our proof does not depend on or require any related results and proofs in [BI]. Now if $P$ is not stable, then the dimension of this component will be at most $n^2 - C + C - K - 1 + K + n = n^2 + n - 1$ (this formula is obtained from [HW, Theorem 6.], where since $P$ is not stable we have subtracted 1 from the dimension of nilpotent elements of the centralizer algebra and the fiber dimension $K + n$ is appended because of our vectors $u$ and $v$.) Thus, such a component actually can not exist.

If, on the other hand, $P$ is stable, then according to [BAR, Theorem 4.], we can take a smaller but still dense open set of $X$ where the matrices $A$ and $B$ are not only commuting and conjugate but also having the vector space $k^n$ as a cyclic module over

$k\left[A,B\right]$. However, this also means the two matrices can not have the same range, otherwise they both have to have cyclic Jordan form (the biggest in our partial order on the partitions), in which case $X$ will be the cyclic component. In turn, this means for some nonzero $u$ , there is no $v$ such that $Au = Bv$ Therefore, the dimension of $X$ is at most $n^2 - C + C - K + K + n - 1 = n^2 + n - 1$ (notice this time the 1 is subtracted at a different place, meaning it is subtracted from the fiber dimension). Again, this means our $X$ can not be a true component in the irreducible decomposition.

Thus, $\Omega_n$ is irreducible, and by action of $GL\left(n+1\right)$ on $\Omega_n$ we get the irreducibility of $N\left(2, n+1\right)$. By induction $N\left(2,p\right)$ and $\Omega_p$ are irreducible for all positive integers $p$.

□

**Corollary 2.2.11.** *The set*

$\{(A,B,v) \in N\left(2,n\right) \times \left(k^n - \{0\}\right) \mid A^n = B^n = 0, [A,B] = 0, Av = Bv = 0\}$ *is irreducible (for all characteristics).*

**Definition 2.2.12.** $\Theta\left(2,n\right)$ *is the set*

$$\left\{(A,B) = \left(\left(a_{ij}\right),\left(b_{ij}\right)\right) \in N\left(2,n\right), a_{i1} = 0, b_{i1} = 0, i \geq 1\right\}.$$

**Corollary 2.2.13.** $\Theta\left(2,n\right)$*is irreducible (for all characteristics).*

**Proposition 2.2.14.** *Let $A$ be a finite dimensional associative algebra over the complex numbers* $\boldsymbol{C}$*. By taking the natural commutator $[a,b] = ab - ba$ we make $R$ a Lie algebra too. Then $R$ is a reductive Lie algebra if and only if as an algebra it is a* *direct product of a commutative Artin algebra and a semisimple algebra.*

*Proof.* Let $\tau$ be the Jacobson radical of $A$. From the general theory of finite dimensional algebras (see, for example, [PIE]), we know $\tau$ is a nilpotent two-sided ideal, so it is also an ideal in the Lie algebra sense. As a Lie algebra, $A \cong s \oplus g_1 \oplus \ldots \oplus g_n$, where $s$ is the center and every $g_i$ is a (non-trivial) simple ideal. Since $\tau$ is nilpotent and $[g_i, g_i] = g_i$, $\tau$ can not contain any simple Lie ideal. Thus $\tau$ is contained in the center. Also, for any element $\alpha \in g_i$, there are $b_j$ and $c_j \in g_i$ such that $\alpha = \Sigma b_j c_j - c_j b_j$. Therefore, for a $\beta \in \tau$, $\alpha\beta = \Sigma\,(b_j c_j \beta - c_j b_j \beta) = \Sigma\,(b_j c_j \beta - b_j \beta c_j) = \Sigma\,(b_j c_j \beta - b_j c_j \beta) = 0$. Now, recall the Wedderburn-Malcev Principal Theorem ([PIE, p.209-211]), there is a semisimple subalgebra $B \subseteq A$, such that $B \cong A/\tau$ and $A = B \bigoplus \tau$ as a vector space. A semisimple algebra over $\boldsymbol{C}$ is isomorphic to a direct product of $gl\,(n)$ for various $n$ (if $n = 1$ then it is a copy of $C$ and is called a trivial semisimple algebra), and this fact is the Wedderburn Structure Theorem for semisimple algebras ([PIE]). A non-trivial $gl\,(n)$ can be generated as an algebra by its first derived Lie ideal $gl\,(n)^{(1)} = [gl\,(n)\,, gl\,(n)] \cong sl\,(n)$, so any element $d$ in the direct product (in $B$) of those non-trivial simple algebras must have $d\tau = 0$. Let $\mathbf{1}_n$ be the idempotent that acts as the identity in a copy of $gl\,(n)$ and $\gamma$ be the sum of all such idempotents, and lastly let $t = \mathbf{1} - \gamma$. Then $t$, $\tau$ and those copies of trivial simple algebra ($\boldsymbol{C}$) generate a commutative ideal $J$ of $A$, and $J$ is a commutative finite dimensional algebra itself. Furthermore, $A/J$ is a semisimple algebra and $A \cong J \times A/J$ as an algebra. □

**Remark 2.2.15.** *After I formed and proved this statement, I consulted several experts if they had known this fact. Most of them did not appear to know about it. However,* *one of them said it had been known but did not point out a reference to me. Since it is not very well known and I have not been able to locate it in the literature, I decide to include this result here, as it is an interesting fact.*

# CHAPTER III

# Jet Schemes of Determinantal Varieties

## 3.1 Introduction

Let $L_{r,s}^{m,n}$ be the vanishing set of all $r \times r$ minors in $m \times n$ matrices over $k[t]/(t^s)$, where $n \geq m \geq r$, and $n, m, r$ and $s$ are all positive integers $\geq 2$.

Our study of the determinantal sets over truncated polynomial rings, or, equivalently, the jet schemes of determinantal sets, is motivated by our study of commuting pairs: the irreducibility of the commuting pairs of $2 \times 2$ matrices over $k[t]/(t^s)$ can be deduced from the irreducibility of $L_{2,s}^{2,3}$, which is essentially the approach taken by [NS], although they did not make it explicit. The first systematic study of such algebraic sets seems to occur in [YUE], [KS] and [KS2]. The results of [YUE] are relatively partial and limited (but of course those results are only a part of her thesis), basically only proving that the jet schemes of general determinantal sets are reducible and not Cohen-Macaulay, with no calculation on how many components there are or what their actual dimensions are. Independently from [YUE], in [KS] and [KS2], the authors not only proved those qualitative results, but also determined the number of components and their dimensions for small $s$ ($2 \leq s \leq 4$) or small $r$ ($2 \leq r \leq 4$). However, their method is highly technical and computational, and does not indicate a way to generalize to larger $s$ and $r$ immediately. The approach we take here is more geometric and elementary, and results in much shorter proofs. In addition, we can completely decide the number of components and their dimensions. The work in this chapter was mainly done in early 2009, between February and April. At that time, we thought there had been no results on these jet schemes except the works we mentioned above. However, as we discovered in early 2010, there is a Ph.D. dissertation, [ALV], which appeared around summer 2009 that basically include the same results. According to one of the experts, the results in [ALV] had been obtained in 2007 or 2008, but were never published elsewhere except in the dissertation, and the dissertation appeared available on internet only after April 2009, when we had done our work.

Although we have now found that the results in this chapter had been obtained before, we still think our proof is worth being recorded for the following reasons: first, it is more elementary and second, it is much shorter. In [ALV], although their proof is elementary in spirit and shares the same guideline as ours (using the orbits of column and row operations), the results are formulated with a lot of terminology from algebraic geometry and combinatorics, for example, arc spaces, contact loci, directed graphs and Young tableaux, etc, thus also making the proofs much longer than ours. Of course, the formality is perhaps necessary in their case, because in later parts of [ALV] they went on to calculate some motivic integrals and also proved some analogous results for toric varieties, neither of which appears in our context.

The main novelty of our approach is that we used only very basic definitions and facts from algebraic geometry and algebraic groups, and reduced everything to simple calculations. We did not even need any terms from combinatorics.

## 3.2 The Theorems and Their Proofs

By row and column operations a generic element of any component can be brought into the following form and all components will be generated using row and column operations on this special form (as the closure of such a set will be irreducible and the union of all such irreducible closed sets is the whole determinantal set)

$$\begin{bmatrix} t^p & 0 & \dots & 0 & 0 & \dots \\ 0 & t^{q_2} & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & t^{q_r} & 0 & \dots \\ \dots & \dots & \dots & 0 & t^{q_r} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $0 \leq p = q_1 \leq q_2 \leq \ldots \leq q_r \leq s$ (of course $t^s = 0$, we still list it for combinatorial convenience,) $p (= q_1) + q_2 + \cdots + q_r = s$ and if any of $q_i$ is reduced by 1, then some $r$ terms of $q_i$ will have a sum less than $s$. This matrix will be referred to as the canonical matrix type of the component it generates. Now if $s < r$, then $p = 0$ ($t^p = 1$) and $L^{m,n}_{r,s}$ is isomorphic to a product of an affine space and a determinantal set of smaller sizes (for $m, n$ and $s$)

$$L^{m,n}_{r,s} \cong A^{s(n+m-1)} \times L^{m-1,n-1}_{r-1,s}$$

hence they have the same number of components, thus we can always reduce to the case where $s \geq r$.

If $\tau \in L^{m,n}_{r,s}$, we write $\Gamma_\tau$ for the orbit of and $\overline{\Gamma_\tau}$ for the closure of the orbit of

$$\{\alpha\tau\beta : \alpha \in GL\left(m, k\left[t\right] / \left(t^s\right)\right), \beta \in GL\left(n, k\left[t\right] / \left(t^s\right)\right)\}$$

in $L^{m,n}_{r,s}$, and we say $\overline{\Gamma_\tau}$ *is the potential component generated by* $\tau$ , although it may not be a component in the minimal decomposition but only an irreducible closed set really.

**Proposition 3.2.1.** *The maximal minor determinantal sets* $L^{r,n}_{r,s}$ *will all be irreducible, and if* $s < r$, $L^{m,n}_{r,s}$ *will have as many components as* $L^{m-1,n-1}_{r-1,s}$.

*Proof.* For the first statement, note that we can use an approximation matrix

$$\sigma = \begin{bmatrix} t^p & 0 & 0 & \dots & \epsilon t^{p-1} & \dots \\ 0 & t^{q_2} & 0 & \dots & \vdots & \dots \\ 0 & 0 & t^{q_3} & \dots & \vdots & \dots \\ \vdots & \vdots & \vdots & \ddots & t^{q_i} & \dots \end{bmatrix},$$

where $q_i$ is the largest index that is strictly less than $s$. Doing this shows that the orbit is actually in the orbit closure of the element obtained by increasing $q_i$ by 1 and decreasing $p$ by 1. Keep doing this until $p$ becomes 0 (if this happens before $q_i$ hits $s$), and then do the same thing with $q_2$ and $q_i$. Eventually we see everything is in the closure of the orbit where the largest $q_i$ is $s$, and all smaller $q_i$ are 0.

For the second statement, if $s < r$, then the least index $p$ must be 0, so $L_{r,s}^{m,n} \cong L_{r-1,s}^{m-1,n-1} \times \boldsymbol{A}^{m+n-1}$.

□

Given an algebraic set $A$, we denote the number of irreducible components of $A$ in a minimal decomposition of it by $N(A)$. If $B$ is a finite set, we denote by $N(B)$ the number of elements of $B$. In addition, given real numbers $r, a$ and $b$, where $a \leq b$, the number of integral multiples of $r$ between $a$ and $b$ will be denoted by $N(r, a, b)$.

**Theorem 3.2.2.** *$L_{r,s}^{m,n}$, where $n \geq m > r$ and $s \geq r$, is reducible (having more than one component). $N\left(L_{r,s}^{m,n}\right)$ is equal to*

$$N\left(L_{r-1,s}^{m-1,n-1}\right) + N\left(r-1, s - \lfloor \frac{s}{r} \rfloor, s-1\right).$$

*(For all characteristics.)*

*Proof.* Again, the components will have the form $\overline{\Gamma_\tau}$, where

$$\tau = \begin{bmatrix} t^p & 0 & 0 & 0 & \dots \\ 0 & t^{q_2} & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots & \\ \dots & \vdots & \vdots & t^{q_r} & \dots \\ \vdots & \vdots & \vdots & \vdots & t^{q_r} \end{bmatrix}$$

and $0 \leq p \leq q_2 \leq q_3 \leq \cdots \leq q_r \leq s$, and $p + q_2 + \cdots + q_r = s$, and decreasing any of $q_i$ (recall $p \equiv q_1$) will produce a point not in the determinantal set.

The first step is to prove that for an actual component that has $p \geq 1$, $q_2 = q_3 = \cdots = q_r$ (and then we add the components from $L^{m-1,n-1}_{r-1,s}$ and get all components by induction). Suppose not, take the first $q_i$, $i \geq 2$, where $p \leq q_i < q_{i+1}$, , then put the term with $\epsilon t^{p-1}$ at $(1, i)$. Then it is obvious that this potential component can not be an actual component unless its canonical matrix type is in the form we stated.

Next, we want to prove that all such sets are actual components. Suppose:

$$\eta = \begin{bmatrix} t^{p_1} & 0 & 0 & 0 & \dots \\ 0 & t^{q_1} & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots & \\ \dots & \vdots & \vdots & t^{q_1} & \dots \\ \vdots & \vdots & \vdots & \vdots & t^{q_1} \end{bmatrix},$$

where $q_1 = \frac{s-p_1}{r-1} \geq p_1$, and

$$\delta = \begin{bmatrix} t^{p_2} & 0 & 0 & 0 & \dots \\ 0 & t^{q_2} & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots & \\ \dots & \vdots & \vdots & t^{q_2} & \dots \\ \vdots & \vdots & \vdots & \vdots & t^{q_2} \end{bmatrix},$$

where $q_2 = \frac{s-p_2}{r-1} \geq p_2$. Without loss of generality, assume $p_1 < p_2$, then it is obvious that $\overline{\Gamma_\delta}$ does not contain $\overline{\Gamma_\eta}$, and we need to prove that $\overline{\Gamma_\delta}$ is not contained in $\overline{\Gamma_\eta}$. Consider the inverse images of $\Gamma_\eta$ and $\Gamma_\delta$ under the projection map:

$$\pi : M_{m\times n}\left(\frac{k\left[t\right]}{\left(t^{p_1+(m-1)\frac{s-p_1}{r-1}}\right)}\right) \to M_{m\times n}\left(\frac{k\left[t\right]}{\left(t^s\right)}\right)$$

. Notice by definition $p_1 \leq \frac{s-p_1}{r-1}$ and $p_2 \leq \frac{s-p_2}{r-1}$. Denote $\left[p_1 + (m-1)\frac{s-p_1}{r-1}\right]$ by $s'$. Now,

$$\overline{\Gamma_\delta} \subset \overline{\Gamma_\eta} \Rightarrow \pi^{-1}\left(\overline{\Gamma_\delta}\right) \subset \pi^{-1}\left(\overline{\Gamma_\eta}\right) \Rightarrow \overline{\pi^{-1}\left(\Gamma_\delta\right)} \subset \overline{\pi^{-1}\left(\Gamma_\eta\right)}$$

(because the projection morphism is faithfully flat so

$$\pi^{-1}\left(\overline{\Gamma_\delta}\right) = \overline{\pi^{-1}\left(\Gamma_\delta\right)}),$$

but

$$\pi^{-1}\left(\overline{\Gamma_\eta}\right) \subset L^{m,n}_{m,s'},\ \pi^{-1}\left(\overline{\Gamma_\delta}\right) \not\subseteq L^{m,n}_{m,s'},$$

since

$$\left\lceil p_2 + (m-1)\,\frac{s-p_2}{r-1} \right\rceil < \left\lceil p_1 + (m-1)\,\frac{s-p_1}{r-1} \right\rceil .$$

Notice the inverse images are no longer in the determinantal set of $(r \times r)$ minors, and we were checking if they are in the determinantal set of $(m \times m)$ minors. Now we should look at the cases where $t^{p_1}$ are preceded by some 1's and deal with the cases $p_1 \leq p_2$ and $p_1 > p_2$ separately, but the proofs are really exactly the same: take the inverse image under projection maps from matrices of higher order truncated polynomials and show

$$\left\lceil p_2 + (m-1)\,\frac{s-p_2}{r-1} \right\rceil < \left\lceil p_1 + (m-f-1)\,\frac{s-p_1}{r-f-1} \right\rceil$$

where $f$ is the number of 1's preceding $t^{p_1}$, and all this is straightforward calculation. The final statement about the number of components is a trivial induction, and gives a recursive formula to calculate the numbers of components of all general determinantal sets of non-maximal minors over truncated polynomial rings.

□

Now look at the $2 \times 2$ minor case.

**Corollary 3.2.3.** *The non-maximal $2 \times 2$ minor determinantal sets $L^{m,n}_{2,s}$ where $n \geq m \geq 3$ all have $1 + \lfloor \frac{s}{2} \rfloor$ components, hence are reducible.*

*Proof.* Let

$$\gamma = \begin{bmatrix} t^p & 0 & 0 & \dots \\ 0 & t^q & 0 & \dots \\ 0 & 0 & t^q & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then the possible components are all in the form of $\Gamma_\gamma$. Since $p$ can range from 0 to $\lfloor \frac{s}{2} \rfloor$, therefore there are $1 + \lfloor \frac{s}{2} \rfloor$ components. □

Now, for the $3 \times 3$ minor case where $r = 3$.

**Corollary 3.2.4.** *$L^{m,n}_{3,s}$, where $n \geq m \geq 4$, and $s \geq 3$, will have $\lfloor 1 + \frac{s}{2} + \frac{s}{6} \rfloor$ components.*

*Proof.* Again, the components will have the form $\Gamma_\tau$, where

$$\tau = \begin{bmatrix} t^p & 0 & 0 & \dots \\ 0 & t^{q_2} & 0 & \dots \\ 0 & 0 & t^{q_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $0 \leq p \leq q_2 \leq q_3 \leq s$, and $p + q_2 + q_3 = s$.

If $p = 0$, we get $1 + \lfloor \frac{s}{2} \rfloor$ components from $L^{m-1,n-1}_{2,s}$ that can not be contained in the closure of other components. Therefore we can now concentrate on $p \geq 1$. However, remember that for an actual component, $q_2 = q_3$.

Now if we do a case by case analysis according to $s \equiv 0, 1, 2, 3, 4$ or $5 \pmod 6$, we get the number of components in the statement. If $s = 6k$. First of all we get $1 + \lfloor \frac{6k}{2} \rfloor = 1 + 3k$ components from $L_{2,s}^{m-1,n-1}$. Since $\frac{6k-p}{2} = q$ and there are $k$ even numbers between 1 and $2k = \lfloor \frac{6k}{3} \rfloor$, we have $1 + 4k$ components in total. Since $1+4k = 1+\lfloor \frac{s}{2} + \frac{s}{2\times 3} \rfloor$, we have verified our statement in this case. If $s = 6k+1$. We have $1 + \lfloor \frac{6k+1}{2} \rfloor = 1 + 3k$ components from $L_{2,s}^{m-1,n-1}$ . Then, since $\frac{6k+1-p}{2} = q$ and there are $k$ odd numbers between 1 and $2k = \lfloor \frac{6k+1}{3} \rfloor$, we have $1 + 4k$ components. Again, $1 + 4k = 1 + \lfloor \frac{s}{2} + \frac{s}{2\times 3} \rfloor$, our statement is verified. The remaining cases are totally similar.

□

**Corollary 3.2.5.** *$L_{r,2}^{m,n}$, that is when $s = 2$, has 2 components for all $n \geq m > r \geq 2$.*

*Proof.* Reduce $r$ until it equals 2. □

**Corollary 3.2.6.** *$L_{r,3}^{m,n}$, that is when $s = 3$, has 3 components if $r \geq 3$ . $L_{2,3}^{m,n}$ has 2 components.*

**Corollary 3.2.7.** *$L_{4,4}^{m,n}$, that is when $s = 4$, has 4 components.*

**Remark 3.2.8.** *Notice the previous two corollaries are actually stronger than the corresponding results in [KS] and [KS2]. Thus, our method is not only simpler, but also covers more for the low $s$ cases as well as for general $m, n, r$ and $s$.*

**Theorem 3.2.9.** *When $n \geq m > r$, $N\left(L_{r,s}^{m,n}\right)$, the number of components of $L_{r,s}^{m,n}$ is equal to*

$$s + 1 - \lceil \frac{s}{r} \rceil = \lfloor \frac{r-1}{r} s + 1 \rfloor.$$

*(For all characteristics.)*

*Proof.* We use our recursive formula and induction. Fix any $s \geq 2$, and do induction for $r$. Notice for $r = 2, 3$ the statement holds as shown in our corollaries. We will use $p$ to denote $N\left(r-1, s-\lfloor\frac{s}{r}\rfloor, s-1\right)$, so what we really have to prove is

$$\lfloor\frac{r-1}{r}s\rfloor - \lfloor\frac{r-2}{r-1}s\rfloor = p.$$

Now if $s = a \times r$, where $a$ is a positive integer, then the left hand side is equal to $a - \lfloor\frac{r-2}{r-1}a\rfloor$, while $p$ will be $1+\lfloor\frac{a-1}{r-1}\rfloor = \lceil\frac{a}{r-1}\rceil$, but $a-\lfloor\frac{r-2}{r-1}a\rfloor = a-\lfloor a-\frac{a}{r-1}\rfloor = \lceil\frac{a}{r-1}\rceil$, so the equality holds when $r$ is a divisor of $s$.

If $s = ar+b$, where $a$ is a non-negative integer, and $0 < b \leq r-1$ is a positive integer, then

$$\lfloor\frac{r-1}{r}s\rfloor - \lfloor\frac{r-2}{r-1}s\rfloor = a(r-1) + \lfloor\frac{r-1}{r}b\rfloor - a(r-2) - \lfloor\frac{r-2}{r-1}(a+b)\rfloor$$

$$= a + \lfloor\frac{r-1}{r}b\rfloor - \lfloor\frac{r-2}{r-1}(a+b)\rfloor$$

$$= a + b - 1 - \lfloor\frac{r-2}{r-1}(a+b)\rfloor.$$

While $p$ will be $\lfloor\frac{a+b-1}{r-1}\rfloor$. Set $r-1 = u$ and $a+b = v$, then what we need to prove is that

$$v - 1 - \lfloor\frac{u-1}{u}v\rfloor = \lfloor\frac{v-1}{u}\rfloor.$$

If $v = tu$, where $t$ is a positive integer, then both sides are $t-1$. On the other hand, if $v = tu + w$, where $t$ is a non-negative integer and $w$ is a positive integer such that

$1 \leq w \leq u-1$, then $\lfloor \frac{tu+w-1}{u} \rfloor = t$, while

$$tu+w-1-\lfloor \frac{u-1}{u}(tu+w) \rfloor$$

$$= tu+w-1-\left(t(u-1)+\lfloor \frac{u-1}{u} w \rfloor\right)$$

$$= tu+w-1-tu+t-w+1 = t.$$

Therefore the equality still holds and we have proven our theorem.

□

**Remark 3.2.10.** *Originally, we did not obtain this excat formula for the number of components but only gave the recursive formula as in the previous theorem. In April 2010, when we found out the dissertation [ALV], we immediately realized that this closed exact formula, which is included in [ALV], can be deduced from our recursive formula and induction.*

# CHAPTER IV

# Nilpotent Commutators

## 4.1 Introduction

The results in this chapter seem to be completely original: no one appears to have asked the same questions as we did. However, of course the questions did not pop out randomly: the author was inspired by two papers, [KNU] and [GG]. The paper [KNU] studied matrix pairs whose commutators are diagonal, and in [GG] the essential point (of the first part of that paper) is to study matrix pairs whose commutators have ranks $\leq 1$. The author then wanted to explore matrix pairs whose commutators are nilpotent, and they turn out to be interesting, being complete intersections, irreducible and reduced (the reducedness has only been proved for characteristic zero so far).

## 4.2 The Properties of Matrix Pairs Having Nilpotent Commutators

**Definition 4.2.1.** *Denote by* $Z_n$ *the set of pairs of* $n \times n$ *matrices* $(A, B)$ *such that* $[A, B] = AB - BA$ *is nilpotent.* $LZ_n$ *is the subset of* $Z_n$ *where* $A$ *has pairwise distinct eigenvalues.*

**Definition 4.2.2.** *Let* $D_n$ *be the set of* $n \times n$ *nilpotent matrices such that the diagonal entries are all zero.* $Dm_n$ *is the subset of* $D_n$ *of rank* $n - 1$ *nilpotent zero diagonal matrices.*

Notice that in the following we always assume $n \geq 3$, because our proof will not work for $n = 2$, and $D_2$ is not irreducible (but is still a complete intersection with the expected dimension), but $Z_2$ is still irreducible by a result of Hulek, [HUL] .

First facts we know about $Z_n$:

**Theorem 4.2.3.**

1. *$Z_n$ is defined by $n - 1$ equations (for all characteristics).*
2. *$Z_n$ is a complete intersection of dimension $2n^2 - n + 1$ (for all characteristics).*
3. *$LZ_n$ is dense open in $Z_n$, if $p$, the characteristic of the field, does not divide $n$.*

*Proof.* The condition that the commutator is nilpotent consists of $n-1$ homogeneous equations in the polynomial ring of $2n^2$ variables, because the commutator will automatically have trace zero. To prove that $Z_n$ is a complete intersection, we will find $2n^2 - n + 1$ other homogeneous equations such that when putting together with our nilpotent commutator condition the solution set is of zero dimension, therefore we have a regular sequence of $2n^2$ elements, hence the $n-1$ equations defining $Z_n$ are a regular sequence (since all equations we have killed are homogeneous, any part of a permutation of the sequence is still regular). Now, let $(A, B)$ be

$$\left(\begin{pmatrix} 0 & 0 & \dots & a_{1,n} \\ a_{2,1} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & a_{n,n-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{2,1} & \dots & 0 \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & \dots & a_{n,n-1} \\ a_{1,n} & 0 & \dots & 0 \end{pmatrix}\right),$$

then the commutator $AB - BA$ is

$$\begin{bmatrix} a_{1,n}^2 - a_{2,1}^2 & 0 & \dots & 0 \\ 0 & a_{2,1}^2 - a_{3,2}^2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{n,n-1}^2 - a_{1,n}^2 \end{bmatrix}.$$

If this diagonal matrix is nilpotent, then all of its diagonal entries have to be zero (note we have killed $2n^2 - 1$ polynomials by now). Now, the homogeneous equations $a_{1,n} = 0$, $det\,(A) = 0$ or $disc\,(A) = 0$ will all reduce our algebraic solution set to a single point, where both $A$ and $B$ are the zero matrix. Therefore, $Z_n$ is a complete intersection and $LZ_n$ is an open dense subset of $Z_n$, since the discriminant of $A$ is a non-zero divisor, if the characteristic of the base field does not divide $n$.

□

Denote the set of diagonal matrices where all entries are pairwise distinct by $\Lambda$, and call the set generated by the conjugation action of the general linear group on the set of such matrices $P$, then we can diagonalize any $A \in P$ into

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

then the condition that $[A, B]$ being nilpotent will force $B$ to be a sum of a diagonal matrix and a matrix $D_{ij}$ having zero diagonal, such that $\{(\lambda_i - \lambda_j)\, D_{ij}\}$ is nilpotent. Consider the projection from $Z_n$ to the first matrix (the first $n^2$ coordinates), then over the dense open set $P$ in the affine space, the fiber of the projection map is isomorphic to $k^n \oplus D_n$, where the affine $n-$space will be mapped onto the diagonal. In addition, we then have a surjective map from $GL\,(n) \times \Lambda \times k^n \times D_n$ to $LZ_n$, thus if we prove the irreducibility of $D_n$, the irreducibility of $LZ_n$ and $Z_n$ will follow. First of all, we need to develop the fact that $D_n$ is a complete intersection.

**Theorem 4.2.4.** *$D_n$ is a complete intersection of dimension $n^2 - 2n + 1$ (for all characteristics).*

*Proof.* $D_n$ is defined by $2n - 1$ homogeneous equations, so its dimension is at least $n^2 - 2n + 1$, and if $dim\,(D_n) = n^2 - 2n + 1$, $D_n$ is a complete intersection. Now we have a map $F$ from $GL(n) \times \Lambda \times k^n \times D_n$ to $LZ_n$:

$$F : (\gamma, \lambda, v, D) \rightarrow \left( \gamma\lambda\gamma^{-1}, \gamma \left( \frac{1}{\lambda_i - \lambda_j} D_{ij} + \begin{bmatrix} v_1 & 0 & 0 & \dots \\ 0 & v_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & v_n \end{bmatrix} \right) \gamma^{-1} \right),$$

and since the centralizer of every such $\lambda$ has dimension $n$, the dimension of $D_n$ satisfies

$$dim(D_n) \leq 2n^2 - n + 1 - n - n - \left(n^2 - n\right) = n^2 - 2n + 1,$$

hence $D_n$ is a complete intersection.

□

Any nilpotent $n \times n$ matrix has rank at most $n-1$, and any rank $n-1$ matrix can be written as $\alpha \circ \beta$, where $\alpha$ is an $n \times (n-1)$ matrix and $\beta$ is an $(n-1) \times n$ matrix. Without loss of generality, $\beta$ can be assumed to have rank $n-1$, so there is at least one $n-1$ minor non-vanishing. Define

$$\Xi \equiv \{(\alpha, \beta) \,\|\, \alpha \in M_{n,n-1},\ \beta \in M_{n-1,n},\ \alpha \circ \beta \in D_n,\ rank(\beta) = n-1\}$$

. An example

$$\left( \begin{pmatrix} N \\ 0 \end{pmatrix} \circ \begin{pmatrix} Id_{n-1} & \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{pmatrix} \right),$$

where $N \in D_{n-1}$, shows that the open sets where the $n-1$ minors do not vanish have non-empty intersection. Now, every pair of such $(\alpha, \beta)$ in the open set $\Gamma_n$ where the first $n-1$ columns of $\beta$ forms an invertible $(n-1)\times(n-1)$ matrix (the subscript $n$ means that the matrix is the result of throwing out the $n$-th column) can be transformed into

$$\left(\begin{pmatrix} D \\ \\ v \end{pmatrix} \circ \begin{pmatrix} Id_{n-1} & & u \end{pmatrix}\right),$$

where $D$ is an $(n-1)\times(n-1)$ matrix having zero diagonal entries, $v$ is a $1\times(n-1)$ matrix and $u$ is an $(n-1)\times 1$ matrix such that $D + u * v$ is nilpotent. Then, this set is easily seen to be isomorphic to the set

$$\Gamma^{'} \equiv \{(N, w, z) \parallel N \in N(n-1), w \in k^{n-1}, z \in k^{n-1}, N_{i,i} = w_i z_i\}$$

Therefore, we only have to prove that $\Gamma^{'}$ is irreducible, then we will develop the irreducibility of $\Gamma_n$. Since the $\Gamma_i$ have non-empty intersection, this in turn will develop the irreducibility of $\Xi$.

**Theorem 4.2.5.**

$$\Gamma^{'} \equiv \{(N, w, z) \parallel N \in N(n-1), w \in k^{n-1}, z \in k^{n-1}, N_{i,i} = w_i z_i\}$$

*is irreducible, and this implies the irreducibility of $D_n$ and $Z_n$. (For the irreducibility of $Z_n$, we need to assume that $p$, the characteristic of our base field, does not divide $n$.)*

*Proof.* Let $A$ be the open set where all diagonal entries of $N$ are non-zero and $N'_{n-1}$ to be the subset of $N_{n-1}$ where all the diagonal entries are non-zero. First we show that $A$ is dense in $\Gamma'$, that is, $\Gamma' \subseteq \overline{A}$. If not, there is a component $X$ where , say, $N_{1,1}$ is identically zero. Now, take the subset $X'$ of $X$ where all diagonal entries are identically zero, and $X'$ is non-empty because we can make $k^*$ act on $\Gamma'$ by multiplying on $N$ and $w$, and X as a component will contain the closure of the image of $k^* \times X$, thus containing a point where $N = 0$. Notice that the dimension of $N_{n-1}$ is

$$n^2 - 3n + 2,$$

thus every component of $\Gamma'$ has to have its dimension

$$\geq n^2 - 3n + 2 + 2(n-1) - (n-1) = n^2 - 2n + 1.$$

However, $X'$ is the result of killing $n-3$ equations from $X$, and is contained in a set of dimension

$$(n-1)^2 - 2(n-1) + 1 + (n-1) = n^2 - 3n + 3,$$

because we already know $dim(D_{n-1}) = n^2 - 4n + 4$. Thus, $X$ can only have dimension

$$\leq n^2 - 3n + 3 + (n-3) = n^2 - 2n,$$

which is a contradiction. Therefore, $\Gamma' = \overline{A}$. Now, $A$ is irreducible because we have a surjective morphism from $N'_{n-1} \times \prod\{z_i w_i = 1\}$ to it, and the latter is irreducible. □

**Theorem 4.2.6.** *In the case of characteristic* $0$*,* $D_n$ *as a scheme is reduced.*

*Proof.* For $n \geq 3$, $D_n$ is a complete intersection and irreducible. Being Cohen-Macaulay, reducedness will be implied by generic reducedness. Since $D_n$ is irreducible, we only need to find a smooth point. Now let the set of $n \times n$ matrices having zero diagonal entries be $H_n$. The map $f$ from $H_n$ to the $(n-1)$-dimenional affine space defined by the last $n-1$ coefficients of the characteristic polynomial is algebraic (note the first coefficient of order $n$ is 1, and the second coefficient of order $n-1$ is 0 because we are looking at a set of matrices where the trace is identically 0), and $D_n$ is the zero fiber of $f$. To find a smooth point, we have to find a point where the induced map $df$ is a submersion onto the target tangent space. However,

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ \vdots & \vdots & 0 & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

is such a smooth point. For $n = 2$, we find a smooth point in each component:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

thus $D_2$ is also reduced.

□

**Theorem 4.2.7.** *In the case of characteristic* $0$*,* $Z_n$ *is reduced.*

*Proof.* Again, we need to find a smooth point since $Z_n$ is Cohen-Macaulay and irreducible. Let $\Lambda$ be a diagonal matrix with pairwise distinct diagonal entries, and let $J$ be the nilpotent matrix as in the proof of the previous theorem, then

$$(\Lambda, J)$$

is such a smooth point. □

# CHAPTER V

# The Diagonal of the Commutator

## 5.1 Introduction

In this chapter, we study the diagonal of the commutator of a pair of $n \times n$ matrices. Our study is inspired, again, by the paper [KNU]. In [KNU], Knutson defined the diagonal commutator scheme over $\boldsymbol{C}$, which consists of pairs whose commutator is diagonal, and this scheme will be denoted by $S_n$ in the following. He proved that $S_n$ is a reduced complete intersection using a flat degeneration to a scheme called "Upper-Upper scheme" in [KNU], and the proof involves with quite an amount of non-trivial calculation. We will give a more direct and more geometric proof for these facts (but of course, their study actually has some applications to the so-called Brauer loop scheme, while ours is only about the diagonal commutator scheme itself), which are Theorem 5.2.5 and Theorem 5.2.10. [KNU] also conjectured that $S_n$ has only two components, one of them being the commuting variety. Later, it was noted by Knutson that Joel Rosenberg proved this when the base field is $\boldsymbol{C}$ using methods from Lie group theory. However, this proof has never been published nor recorded, and can not even be found or recalled by the author of [KNU] . Besides filling this gap in the literature, our proof has the advantage that it applies to all characteristics (zero or prime), since we use purely algebraic geometric arguments. In the opposite direction, we study in a later section the matrix pairs whose commutator has zero diagonal. Here, we again prove that it is a complete intersection, reduced as a scheme when the characteristic is zero, and irreducible when the matrices are of smaller sizes ($2 \times 2$ and $3 \times 3$).

## 5.2 The Diagonal Commutator Scheme

Again, $S_n$ is the set of pairs of $n \times n$ matrices $(A, B)$ such that their commutator $K = [A, B] = AB - BA$ is a diagonal matrix. Note $S_n$ is defined by $n^2 - n$ homogeneous equations, and evidently contains the $n$-dimensional commuting variety. Given an arbitrary $n \times n$ matrix $X$, we now define a matrix $D(X)$ which will be important to our development.

**Definition 5.2.1.** *$D(X)$ is the $n \times n$ matrix whose $j$-th column consists of the diagonal entries of $X^{j-1}$, ordered from upper left to lower right . In particular, the entries in the first column of $D(X)$ are all equal to 1.*

Notice that if $\Lambda$ is a diagonal matrix, then $D(\Lambda)$ is the Vandermonde matrix of $\Lambda_{ii}$. We should begin with some useful technical lemmas:

**Lemma 5.2.2.** *Given an $n \times n$ matrix $A$, if there exists a matrix $B$ such that $K = [A, B]$ is a non-zero diagonal matrix, then $Det(D(A)) = 0$.*

*Proof.* Whether $K$ is diagonal or not, $tr\left(A^{i}K\right) = 0$ for all $i$. Now if $K$ is diagonal, this says that the row vector $K^{'}$ whose $i$-th entry is $K_{ii}$ is in the kernel of $D\left(A\right)$ (the matrix acting from the right of the row vectors). Thus, the kernel of $D\left(A\right)$ is non-trivial, so the determinant has to be 0. □

**Lemma 5.2.3.** *The polynomial $P = Det\left(D\left(X\right)\right)$ is irreducible in the polynomial ring in $n^2$ variables, so that the algebraic set $L$ defined by $P$ is irreducible with the coordinate ring $R$ being a domain.*

*Proof.* First of all, $P$ is homogeneous. Suppose that we kill some homogeneous elements in the polynomial ring and let $\overline{P}$ be the image of $P$ in the resulting ring, then the irreducibility of $P$ would be implied if $\overline{P}$ is irreducible. First, we kill all entries in the first column or the first row except the first diagonal entry ($X_{11}$). We denote the lower right block matrix by $X_0$, and its characteristic polynomial by $C_{X_0}$. Then, the determinant of the $D$ matrix becomes the product of $det\left(D\left(X_0\right)\right)$ (irreducible by induction) and $C_{X_0}\left(X_{11}\right)$ (irreducible as shown Lemma 5.2.8, the proof there does not depend the irreducibility of $P$). Next, we kill all entries in the last column or the last row except the last diagonal entry ($X_{nn}$). We denote the upper left block matrix by $X_1$, and its characteristic polynomial by $C_{X_1}$. Then, the determinant of the $D$ matrix becomes the product of $det\left(D\left(X_1\right)\right)$ (irreducible by induction) and $C_{X_1}\left(X_{nn}\right)$ (irreducible). If $P$ is reducible, then there has to be a homogeneous factor polynomial with degree $n-1$ that turns into $C_{X_0}\left(X_{11}\right)$ under the first specialization and $C_{X_1}\left(X_{nn}\right)$ under the second specialization. However, this is impossible. For, if we kill all entries except $X_{11}$ and $X_{nn}$, then $C_{X_0}(X_{11})$ becomes

$$(X_{11} - X_{nn}) X_{11}^{n-2},$$

and $C_{X_1}(X_{nn})$ becomes

$$(X_{nn} - X_{11}) X_{nn}^{n-2},$$

and they are not the same. Therefore, $P$ is irreducible.

□

**Lemma 5.2.4.** *$T_n$, the subset of $S_n$, where both $A$ and $B$ are cyclic is dense in $S_n$.*

*Proof.* Given a pair $(A, B)$ in $S_n$, the pair $(A, B + \epsilon C)$, where $\epsilon$ is in our base field $k$ and $C$ is any matrix in the centralizer of $A$, is also in $S_n$. However, every matrix commutes with some cyclic matrix (see [GUR] or [GER]), hence the lemma holds by symmetry of $A$ and $B$. □

Now, the first notable property of $S_n$ is being a complete intersection:

**Theorem 5.2.5.** *$S_n$ is a complete intersection of dimension $n^2 + n$, and the commuting variety $C(2, n)$ is one of its components (for all characteristics).*

*Proof.* $S_n$ is defined by $n^2 - n$ homogeneous equations in the $2n^2$-dimensional affine space, therefore, the dimension of each component of $S_n$ is at least $n^2 + n$. If we prove that the dimension of any component is actually less than or equal to $n^2 + n$ then we are done. Now, let $\pi$ be the projection map onto the first matrix (the first $n^2$ coordinates). In the image $V$ of a component $U$ under $\pi$ there is a dense set where the matrix is cyclic. If $V$ has non-empty intersection with the set where $Det\left(D\left(A\right)\right) \neq 0$, then this intersection $V^{'}$ is dense in $V$. The fiber over any cyclic point (again dense in $V$) in $V^{'}$ is just the centralizer, which is an $n$-dimensional affine space, therefore it will give us a component which is in fact the commuting variety. The fiber over any cyclic point in the set $A_d$ where $rank\left(D\left(A\right)\right) \leq n-d$ has dimension $\leq d+n$. If we prove that dimension of the set $A_d$ is less than or equal to $n^2-d$, then we are done showing that any component of $S_n$ other than the commuting variety has dimension $\leq n^2+n$. However, by setting all off-diagonal entries to 0 (killing $n^2-n$ homogeneous equations) in $A_d$, we see the resulting set is contained in the set of diagonal matrices of which at most $n-d$ entries can be distinct, and that has dimension $n-d$ (notice that since all the equations we are killing are homogeneous, the resulting set is non-empty). Therefore, $A_d$ has dimension $\leq n^2-d$, and we are done showing $S_n$ being a complete intersection. □

Our goal is to prove that $S_n$ has only two components, one being the commuting variety. We will first explain the strategy of our proof, because it does appear to be quite technical at some points. We will begin by showing that there is a component of $S_n$ generated by points of a dense subset $Q$ of L, where $rank\left(D\left(X\right)\right) = n-1$ for all $X \in Q$, and fibers over $Q$, and we will denote the generated component by $Q^{'}$. Suppose that there are other components other than $C\left(2,n\right)$ and $Q^{'}$, and look at the image of one such component $\Omega$ under the projection $\pi$. The image must be contained in some $A_d$ where $d \geq 2$ (otherwise $\Omega$ would be contained in $C\left(2,n\right)$ or $Q^{'}$), but then we prove $dim\left(A_d\right) \leq n^2-d-1$ for all $d \geq 2$ (we proved $dim\left(A_d\right) \leq n^2-d$

in the previous theorem), therefore for dimensional reason there can not exist more components other than $C(2,n)$ and $Q'$. Let us establish the existence of $Q'$ now:

**Lemma 5.2.6.** *There is a dense open set $Q$ in the algebraic set $L$ (defined by $P$) where for every point $A$ in $Q$, there exists a matrix $B$ such that $[A,B]$ is a non-zero diagonal matrix.*

*Proof.* First, we look at some special points in $L$, say,

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ \vdots & \vdots & 0 & 1 \\ x_1 & \dots & x_{n-1} & 0 \end{bmatrix}.$$

Now, points like this prove that the set of matrices having pairwise distinct eigenvalues is dense in $L$. Therefore, $L$ has a dense open subset $Z$ where every matrix is of the form $A = U\Lambda U^{-1}$, where $\Lambda$ is a diagonal matrix with pairwise distinct entries and $U$ is a invertible matrix. The condition of the existence of the $B$ we want is then implied by the vanishing of a polynomial in entries of $U$, because of the following reason: the condition for a non-zero diagonal $[A,B]$ is equal to some linear map not having full rank. Let $K$ be the vector space of matrices having zero diagonal, and let $h$ be the projection map from $M_n$ to $K$. The map:

$$f : \alpha \in K \to h\left(U\alpha U^{-1}\right)$$

is a linear map from $K$ to itself. If there does not exist such a wanted $B$, then $f$ must have full rank because the direct sum of $UKU^{-1}$ and the space of diagonal matrices will be the full $M_n$, so $det\,(f) \neq 0$. Therefore, there is a subset $O$ of dimension $n^2 - 1$ in $GL\,(n)$ such that all points in $O$ satisfies the existence condition for $B$, and the map

$$g : U \rightarrow U\Lambda U^{-1}$$

($\Lambda$ is, again, a diagonal matrix with pairwise distinct entries ) will take $O$ into $L$. If we now take the image of the product of $O$ and the set of all $\Lambda$, whose dimension is $n$, the dimension of the image will be $n^2 - 1$ (the centralizer of every $\Lambda$ is of dimension $n$), hence dense in $L$ ($L$ being irreducible), thus, containing a dense open subset of $L$. Thus, we have established the existence of the component $Q'$. □

**Theorem 5.2.7.** *For all $d \geq 2$, $dim\,(A_d) \leq n^2 - d - 1$ (for all characteristics).*

*Proof.* Actually, the most important case is when $d = 2$, so we will first concentrate on the proof of this case. For $n = 2$ or 3 this is trivial. Therefore we will use induction and assume the statement holds for $n - 1$. By restricting to subset where both the last column and last row are all zero except the $n$-th diagonal entry, $u$, we will be able to use the induction condition. Let us call the upper left $n - 1 \times n - 1$ matrix $X_0$. We will discuss two separate cases, one where $u$ is not an eigenvalue of $X_0$, the other case being that $u$ is always an eigenvalue of $X_0$. When $u$ is not an eigenvalue of $X_0$, but canceling the last column except the last entry of $D(X)$, where

$$X = \begin{pmatrix} X_0 & 0 \\ 0 & u \end{pmatrix},$$

we see that $rank\,(D(X_0)) \leq n-3 = (n-1)-2$. By induction, the subset of $A_2$ where $u$ is not an eigenvalue of $X_0$ has dimension $\leq (n-1)^2 - 3 + 1 + 2(n-1) = n^2 - 3$, and we are done with this case. On the other hand, when $u$ is an eigenvalue of $X_0$, we can only deduce that $rank\,(D(X_0)) \leq n-2 = (n-1)-1$ at first. However, denote by $F$ the polynomial in $u$ and entries of $X_0$ which is the result of $u$ substituted into the characteristic polynomial of $X_0$, and note that $F$ is monic in $u$ where the coefficients are polynomials of entries of $X_0$. Even with the vanishing of both $P(X_0)$ and $F$, the resulting set will not be contained in $A_2$. As we showed, the set where matrices having pairwise distinct eigenvalues is dense in $L$. If the condition defining $A_2$ holds identically, it will follow that the Vandermonde matrix of the $n-1$ distinct eigenvalues has rank $\leq n-1$, since all $n-1$ minors vanishing means the row vectors from different eigenvalues will always be in the span of some $n-2$ rows of $D(X_0)$, which is of course not true. Therefore we can prove that $P$ and $F$ generate a prime ideal in the polynomial ring $k[X_0, u]$, then we will be done, since then the set would have dimension

$$\leq (n-1)^2 + 1 - 3 + 2n - 2 = n^2 - 3.$$

This statement will be dealt with as a lemma after we finish the induction proof for all $d$, using the result of $d = 2$. Since now we are assuming the theorem for $A_2$, we take induction on $d$ and let $d$ now be greater than or equal to 3. Again, by killing both the last row and the last column except the entry on the diagonal, we separate two cases by whether $u$ is an eigenvalue of $X_0$. When $u$ is not an eigenvalue of $X_0$, we proceed exactly as we did for $A_2$. Now restricting to the set where $u$ is an eigenvalue of $X_0$, the dimension is $\leq (n-1)^2 - (d-1) - 1 + 2n - 2 = n^2 - d - 1$, because $u$ being an eigenvalue of $X_0$, can only be chosen from a finite set for each $X_0$, thus not providing an extra dimension. And we are doe with general $d \geq 3$. □

Now, we have to go back to the technical lemma and finish the proof for $A_2$:

**Lemma 5.2.8.** *The ideal generated by $P$ and $F$ is prime in the polynomial ring*

$$k\left[x_{1,1}, \ldots, x_{n,n}, u\right].$$

*Proof.* Denote by $R$ the ring resulting from killing $P$ in the polynomial ring in $n^2$ variables. We already know $R$ is a domain. Let $V$ be the fraction field of $R$. Since $F$ is monic in $u$, $\frac{R[u]}{(F)}$ is a free module, hence is embedded in $\frac{V[u]}{(F)}$, and we only really need to prove that $F$ is irreducible over $V$. We know $F$ is irreducible over $R$, because we can map $R$ onto the polynomial ring where the diagonal entries are killed, and by further specialization to matrices in the following form

$$\begin{pmatrix} 0 & a_{1,2} & \ldots & 0 \\ 0 & 0 & \ddots & \ldots \\ 0 & 0 & \ldots & a_{n-1,n} \\ a_{n,1} & 0 & \ldots & 0 \end{pmatrix},$$

we see that $F$ is irreducible there. If monic polynomial over a normal domain is reducible over its fraction field, then it is reducible over the domain itself. Our $R$ might not be normal, but the ring $S = k[x_{ij}, i \neq j]$ certainly is (it is a polynomial ring). Therefore, we will utilize the map from $R$ to $S$. Consider $t_i = \frac{\partial P}{\partial_{x_{ii}}}$, they are generically non-zero in $R$ (this can be proven by taking a diagonal matrix with two entries equal to zero). The localized ring $R_{t_i}$ is by definition regular, hence normal. If $F$ factors in $V[u]$, then it factors in every $R_{t_i}$, hence there is a $p$ such that $t_i^p F$ factors in $R[u]$. If we prove that the image of some $t_i$, denoted by $\bar{t_i}$, is non-zero in $S$, then we know $\bar{t_i}^p \bar{F}$ factors over the fraction field of $S$. However, then $\bar{F}$ factors over $S$, since $S$ is normal, which is a contradiction. Therefore, it eventually comes down to whether we can prove some $\bar{t_i}$ is non-zero in $S$, which is equivalent to find a matrix $C$ with all diagonal entries zero (hence $det(D(C)) = 0$), but $rank(D(C)) = n - 1$. For $n = 2$ this is trivial, and when $n = 3$, it is easy, just look at the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we use induction on $n$. Since the rank condition is a generic condition, and we also know that a generic zero-diagonal matrix is invertible, we can find an $n-1 \times n-1$ matrix $E$ such that all diagonal entries of $E$ are zero, $rank(D(E)) = n - 2$, and $E$

being invertible. Then, take the matrix

$$E' = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

and since $E$ is invertible, 0 is not an eigenvalue of $E$. Therefore, $rank\left(D\left(E'\right)\right) = n-1$, and we are done.

□

Finally, we have established the main result of this section:

**Theorem 5.2.9.** *For all $n \geq 2$ and all characteristics, $S_n$, the diagonal commutator scheme, is a complete intersection with only two components, one of them being the $n$-dimensional commuting variety. We will call the other component the skew component, and denote it by $\zeta$.*

Next, we restrict to the case of characteristic 0, and prove the reducedness there.

**Theorem 5.2.10.** *In characteristic 0, $S_n$, as a scheme, is reduced.*

*Proof.* First, we have the map $g$ from $M_n \times M_n$ to $K$, the space of matrices with zero-diagonal:

$$g : (\alpha, \beta) \to h\left([\alpha, \beta]\right),$$

where $h$ is the projection map from $M_n$ to $K$. $S_n$ is the zero fiber of $g$, and we need only to find a point in each component of $S_n$ such that $h$ is a submersion at the point. (Again, $S_n$ is Cohen-Macaulay, so we only have to prove the generic reducedness.) For the commuting variety component, we can take a point where $\alpha$ is a diagonal matrix where all entries are distinct. For the skew component, let us look at the point

$$(\beta, \alpha) = \left( \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \right),$$

and prove the differential tangent map $dg$ is surjective there. Notice that since the domain of $dg$ is a $2n^2$ dimensional vector space and the target space is of dimension $n^2 - n$, we only need to prove the kernel of $dg$ is of dimension $n^2 + n$. The kernel, as an algebraic variety is defined by $n^2 - n$ homogeneous (linear) equations in the affine space $M_n \times M_n$, so its dimension is at least $n^2 + n$. Therefore, we only need to find $n^2 + n$ homogeneous (linear) equations to reduce the kernel to a zero-dimensional algebraic set (in fact, a single point), and we will be done. Let $(A, B)$ be in the kernel of $dg$, which means $[\alpha, A] - [\beta, B] = 0$. Now, kill the strictly lower triangle and the last column of $A$, and kill the strictly upper triangle and the first column of $B$, meaning we are setting

$$A_{ij} = 0, \ if \ i > j \ or \ j = n; \ B_{kl} = 0, \ if \ k < l \ or \ l = 1.$$

The condition $[\alpha, A] - [\beta, B] = 0$ will force both $A$ and $B$ to be zero, and we are done reducing the kernel to one point by killing $n^2 + n$ homogeneous (linear) equations.

$\square$

## 5.3 When the Diagonal of the Commutator Vanishes

In contrast to the previous section, we will study the condition of the diagonal entries of the commutator being zero. Let $U_n$ be the algebraic set in $M_n \times M_n$ consisting of such points:

$$\{(A, B)\,,\ diag\,([A, B]) = 0,\}$$

where *diag* is the projection map from $M_n$ to the space of diagonal matrices. Notice that $U_n$ is defined by $n-1$ homogeneous equations, because we always have the trace equal to zero for a commutator. The first theorem we are going to prove about $U_n$ is that it is a complete intersection (again!!):

**Theorem 5.3.1.** *$U_n$ is a complete intersection of dimension*

$$2n^2 - n + 1$$

*for all $n$ (for all characteristics).*

*Proof.* Actually, the proof is essentially identical to the proof we gave for the case of nilpotent commutators. Now, let $(A, B)$ be

$$\left( \begin{pmatrix} 0 & 0 & \dots & a_{1,n} \\ a_{2,1} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & a_{n,n-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{2,1} & \dots & 0 \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & \dots & a_{n,n-1} \\ a_{1,n} & 0 & \dots & 0 \end{pmatrix} \right),$$

then the commutator $AB - BA$ is

$$\begin{bmatrix} a_{1,n}^2 - a_{2,1}^2 & 0 & \dots & 0 \\ 0 & a_{2,1}^2 - a_{3,2}^2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{n,n-1}^2 - a_{1,n}^2 \end{bmatrix}.$$

Now throw in the condition that the $n-1$ homogeneous equations from the diagonal of the commutator vanish, and also one extra condition $a_{1,n} = 0$, and we have forced the resulting set to be reduced to one single point $(0,0)$. Since all the equations we kill are homogeneous, we are done proving the equations forming a regular sequence and $U_n$ being a complete intersection. □

If the characteristic is 0, since the equations of $U_n$ are all defined over the rational number field $Q$, we can assume the base field is the complex field $C$. All other algebraically closed characteristic 0 fields can either be embedded in $\boldsymbol{C}$ or produce a faithfully flat base extension without changing the reducedness (if proven for $\boldsymbol{C}$). Therefore, we should now introduce some concepts and notations of complex differential and symplectic geometry that will be used in later development.

Let $g$ be the Lie algebra $M_n(\boldsymbol{C})$, we will identify the dual space $g^*$ with $g$ via the pairing $g \otimes g \to \boldsymbol{C}$, $X \otimes Y \to tr(XY)$. Therefore, $g \times g$ can be identified with the cotangent bundle of $g$, thus having a complex symplectic structure. Let $\Delta$ be the group of diagonal matrices with determinant 1 and $\delta$ be its Lie algebra (the diagonal matrices with trace zero, note $\delta^* \simeq \delta$). $\Delta$ will act on $g \times g$ diagonally, as given by the formula

$$u \cdot (X, Y) = (uXu^{-1}, uYu^{-1}) \quad \text{where } u \in \Delta.$$

This action is symplectic with the canonical symplectic form on a cotangent bundle (in fact, Hamiltonian), and the corresponding moment map is

$$\mu : g \times g \to \delta^* \simeq \delta, \qquad (X, Y) \longmapsto diag\,([X, Y])\,. \tag{5.3.1}$$

(For the facts above, see [GG], especially section 2 of that paper.) Thus, $U_n$ is the zero fiber of the moment map. The next theorem will somehow look familiar:

**Theorem 5.3.2.** *When the characteristic of the base field is* 0*,* $U_n$ *is a reduced scheme.*

*Proof.* As stated above, we can assume the base field is $\boldsymbol{C}$ and $U_n$ is the zero fiber of the moment map. Now, with $U_n$ being Cohen-Macaulay, we only need to find one smooth point in each of its component (we do not know yet how many components there are) to prove the reducedness. We will do this by showing that the moment map is a submersion at generic points of every component. Note that in the proof of Theorem 5.3.1, we actually also proved that all off-diagonal entries of matrices $A$ and $B$ are regular elements in the coordinate ring of $U_n$, namely, non-zero divisors (in fact, all entries are regular, not just the off-diagonal ones, but we only need them in our proof). Now, being regular means generically non-zero, and so there will be a dense open set in every component where all off-diagonal entries are non-zero. However, this will imply the $\Delta$-action on all those open sets is free. The action being free at the generic points will imply that the moment map is a submersion at those generic points (for this statement, see [GG], Theorem 1.1.2 and section 2). □

Although we have shown that $U_n$ is a complete intersection, and that it is reduced in the case of characteristic zero, we have not acquired any information concerning how many components of $U_n$ there are, or even whether it is irreducible in general. We have not been able to answer this question in general yet, but for small sizes of the matrices, specifically, for $n = 2$ *or* 3, we have found that $U_n$ is irreducible.

**Theorem 5.3.3.** *For $n = 2$ or 3, $U_n$ is irreducible for all characteristics, and this also means the coordinate ring will be a domain in case of characteristic zero.*

*Proof.* For $n = 2$, $U_2$ is an irreducible hypersurface, and the proof is trivial. We will now show the irreducibility for $U_3$. Writing the matrix pair as

$$\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix}\right),$$

the condition that the commutator has zero diagonal consists of two equations:

$$(1)\ \ bd' + cg' - b'd - c'g = 0,$$

and

$$(2)\ \ db' + fh' - d'b - f'h = 0.$$

Recall that all entries will be generically non-zero, then there exists a dense open set $\Omega_3$ in $U_3$ such that $\Omega_3$ is defined by equation (2) over the irreducible (actually smooth) algebraic set $k^{16} \times k^*$ where $b \neq 0$ and $d' = \frac{1}{b}\left(b'd + c'g - cg'\right)$. Now, by substituting this expression in (2), the equation becomes

$$fh' - f'h - cg' + c'g,$$

which defines an irreducible algebraic set $H$ in $k^{16}$. Thus, we see that $\Omega_3$ is isomorphic to $H \times k^*$, hence irreducible. Since $\Omega_3$ is dense in $U_3$, this in turn means $U_3$ is irreducible.

□

## 5.4 Directions for Future Research

Things that can be studied in the future:

1. Determining the number of components of $C(m,n)$ or $N(m,n)$, and their dimensions, for general $m,\ n \geq 4$, nor any results on their dimensions remains an open question.

2. Components of commuting pairs of general size matrices over truncated polynomial rings, especially the first order case, are not understood. It is not known if they are irreducible in general, although this seems to have been settled in the negative (reducible in general) sense in the preprint [SS]. But we still do not know which sizes of matrices will make the commuting pairs irreducible. In [SS] it proved that it is irreducible for size $3 \times 3$ over general truncated polynomial rings. Also, the question of whether the set of nilpotent commuting pairs is irreducible is still open even for $3 \times 3$.

3. If the coordinate rings of commuting pairs are Cohen–Macaulay (which is true for small sizes), then the associated sets $\Omega_n$ defined by us in Definition 2.2.4 are also Cohen–Macaulay (being complete intersections in the polynomial rings over the coordinate rings). This may be used to get results concerning commuting matrices with larger sizes.

4. If the set of commuting triples of size $(n+1) \times (n+1)$ matrices is known (or assumed) to be irreducible, we may try to prove the irreducibility must hold for $n \times n$ matrices. There is a proof for general matrices in characteristic 0 using

analytic arguments, but no algebraic proof is known to us. In addition, the same statement should, in the author's opinion, hold for the nilpotent matrices, and in this case no proof seems to exist.

5. We defined $A_d$ when studying the diagonal commutator scheme, and found an upper bound for its dimension. However, in general, we do not know whether $A_d$ is irreducible, nor its exact dimension.

6. The skew component $\zeta$ of the diagonal commutator scheme is still quite mysterious to us. If it is Cohen-Macaulay, then the commuting variety will be Cohen-Macaulay too, hence reduced (by the theory of direct linkage). Of course, we currently have no idea how to prove this statement. In fact, we do not even know the defining equations for the skew component yet. There is a conjecture concerning the defining equations in [KNU], but no progress has been made in verifying the conjecture.

7. With supporting evidence in lower dimensions (2 and 3), we conjecture that $U_n$, the pairs with zero diagonal commutator, is irreducible in general.

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