T H E U N I V E R S I T Y O F M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Philosophy

Technical Note

INSEPARABLE SETS AND REDUCIBILITY

S. Tennenbaum

ORA Project 03105

under contract with:
DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr 1224(21)
WASHINGTON, D.C.

administered through:
OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

July 1961

(Reprinted November 1961)

Inseparable Sets and Reducibility*

Stanley Tennenbaum

Recursively inseparable sets have a number of interesting applications to mathematical logic. For example, the existence of undecidable formulas in all merely consistent extensions of classical arithmetic is an immediate consequence of the representability of such sets in that formalism. This note contains a construction of inseparable sets of a new type and a rather curious application of these to the theory of machine reducibility initiated by E. Post.

Two disjoint sets A, B of non-negative integers (in the sequel, simply "numbers") are <u>inseparable</u> if every pair of recursively enumerable sets A', B', with $A \subset A'$ and $B \subset B'$, fails to be complementary. The most direct construction of recursively enumerable sets with this property is as follows: Let D_n be an effective enumeration of all possible partial recursive definitions of 0, 1 functions of one argument. Let $F_n(x)$ be the partial function defined by D_n. Then $F_n(x)$ is a partial recursive 0, 1 function of the numbers n and x. Taking addition mod 2, let $A = x|F_x(x) + 1 = 0$ and $B = x|F_x(x) + 1 = 1$. A and B are clearly recursively enumerable and disjoint. In fact, they are inseparable. For every pair of disjoint recursively enumerable sets A', B', with $A \subset A'$ and $B \subset B'$, determines in a natural way a partial recursive 0, 1 function, $F_k(x)$ such that $A' = x|F_k(x) = 0$ and $B' = x|F_k(x) = 1$. Now $k \notin A$. For if $k \in A'$ then

*Support for this research was given through grants from the National Science Foundation and the U. S. Army Signal Corps and through contract funds from the Office of Naval Research.

$F_k(k) = 0$ and thus $F_k(k) + 1 = 1$. But then $k \in B$ and $A' \cap B \neq \Lambda$, contradicting the choice of A'. Similarly $k \notin B'$. Thus $k \in \overline{A' \cup B'}$, i.e. A' and B' are not complementary.

The two sets A and B enjoy a further property implicit in the above argument. If $F_{a'}(x)$ and $F_{b'}(x)$ are partial recursive characteristic functions for A' and B' (i.e. if $F_{a'}(x)$ and $F_{b'}(y)$ are defined precisely when $x \in A'$ and $y \in B'$ and are equal to 1 when they are defined) then an "index" k of the partial function, $F_k(x)$ used above, can effectively be obtained from a' and b'. But k is "witness" to the fact that A' and B' are not complementary. More precisely, there is associated with the inseparable sets A and B a recursive function h, of two arguments. This function attaches to any pair of numbers a', b' which happen to be indices of partial characteristic functions of disjoint sets A', B' with $A \subset A'$ and $B \subset B'$ a witness number $h(a', b') = k$ such that $k \in \overline{A' \cup B'}$. Because of the existence of such a function, A and B are called "effectively" inseparable.

The question whether there exist inseparable recursively enumerable sets which are not effectively inseparable was raised by Uspenskii[4]. Schoenfield[3] has given examples of such "non-effectively" inseparable sets. The sets which we construct are also examples. However, they are non-effectively inseparable in a most extreme sense. We shall call them S and T. They are recursively enumerable, disjoint, and $\overline{S \cup T}$ is infinite. Their special feature is the following: if S', T' are disjoint recursively enumerable sets with $S \subset S'$ and $T \subset T'$ then $S' - S$ and $T' - T$ are both finite! Thus S and T are obviously inseparable. To see that they are not effectively inseparable, observe that $\overline{S \cup T}$ though infinite, contains no infinite recursively enumerable subset. (Infinite sets devoid of infinite recursively enumerable subsets are called immune. Recursively enumerable sets, such as $S \cup T$, with immune complements were first discovered by Post[2] and are called simple.) For if an infinite recursively enumerable set X were contained in $\overline{S \cup T}$, $S \cup X$ would be recursively enumerable, disjoint from T, and $(S \cup X) - S$ would be infinite, contradicting the special feature of S. However if S and T were effectively inseparable the "witnessing' function $h(x_1, x_2)$ could trivially be used to produce an infinite recursively enumerable subset of $\overline{S \cup T}$.

To construct S and T, we set $E_n = x|F_n(x) = 1$ and thus obtain in E_n an effective enumeration of all recursively enumerable sets. We imagine E_n as being generated in the course of some fixed routine devised to compute $F_n(x)$ for all possible n and s. As $F_n(x)$ is computed we arrange for each n to have all members of E_n greater than 4n recorded in a list L_n in the order in which they are generated. Next we settle on a schedule of repeatedly inspecting these lists so that as the computation of $F_n(x)$ goes on, we will sooner or later see any number which is recorded in any list. Finally, we carry with us a sheet divided into two columns in which we copy numbers from L_n according to the following three rules:

(1) If L_n bears numbers and none of them appears in our columns, copy the smallest number of L_n into first column.

(2) If L_n bears numbers and some but not all of them appear in one of our columns, but none in the other, copy the smallest number in L_n not appearing in the one column into the other.

(3) Otherwise, pass on to the next list in the schedule.

Now let S = set of numbers copied into our first and T = set of numbers copied into our second column. S and T are recursively enumerable. It is an exercise to construct partial recursive characteristic functions for them from the preceding intuitive sketch. (Unfortunately the formal definition of these functions does not expose the simple idea which guides the construction of S and T. In fact our construction is just a modification of Post's construction of a simple set[2]. They are clearly disjoint. Now suppose S' is recursively enumerable and disjoint from T while containing S. S' would be generated by some $E_{S'}$. If S' — S were infinite sooner or later $L_{S'}$ would contain a number $x \epsilon S' - S$. Rule (2) would apply and $S' \cap T \neq \Lambda$. Therefore S' — S is finite. The same argument shows that if T' contains T and is disjoint from S then T' — T is finite. Finally $\overline{S \cup T}$ is infinite since at most 2(n-1) of the numbers up to 4n can be placed in $S \cup T$.

One set of numbers is said to be reducible to another set of numbers if the characteristic function of the first is recursive in the characteristic function of the second. This means that there exists a machine so programmed that if an outside source supplies it successively with the correct values of the characteristic function of the first set, the machine is able to compute successively the correct values of the characteristic function of the second set. It is readily shown that for recursively enumerable sets A and B, this condition can be simplified so that A is reducible to B just in case there exists a partial recursive function $\ell_x(n)$ defined for all n when $x \epsilon \bar{A}$ such that

$$(\forall x) \quad \left\{ x \in \overline{A} \longleftrightarrow (\exists n)(\rho_{\ell_x}(n) \subset \overline{B}) \right\}$$

Here ρ_m is ary effective enumeration of all finite sets of numbers. Intuitively, this means that for a machine to discover whether $x \in \overline{A}$ it computes one by one a sequence of finite sets attached to each x. For every finite set it "asks" the outside source whether all numbers belong to $\overline{B}$. If the answer is yes, the machine declares that $x \in \overline{A}$. If no, it computes the next "round of questions." In this manner, if A is reducible to B, for each x the machine will receive the answer "yes" to some round of questions in which case $x \in \overline{A}$ or else x will turn up in the auxillary process of generating the recursively enumerable set A. It is important to recognize that, in general, if $\ell_x(n)$ supplies a "reduction" of A to B there may be no a priori effective upper bound to the number n of rounds, $\rho_{\ell_x}(1)$, $\rho_{\ell_x}(2)$, $\ldots \rho_{\ell_x}(n)$, the machine must compute before either it receives the answer "yes" to some round or x turns up in A.

In case each round of a reduction contains just one question, i.e., if $\rho_{\ell_x}(n)$ is a single number for every x and each n, we call $\ell_x(n)$ a Q-reduction of A to B. If B is the set of "indices" in some effective enumeration of those diophantine equations, which have solutions, and if A is an arbitrary recursively enumerable set, then A is reducible to B just in case there is a Q-reduction of A to B. This follows immediately from that fact that there is a trivial mapping which assigns to each finite set of diophantine equations a single equation such that none of the finite set has a solution if, and only if, the single equation has no solution. It seems plausible that more generally for any pair A and B of recursively enumerable sets A is reducible to B just in case A is

Q-reducible to B. Indeed, since the number of rounds of single questions in a Q-reduction has no a priori bound, what possible advantage accrues to rounds with more than one question?

Nevertheless, there are recursively enumerable sets A and B with A reducible to B but not Q-reducible to B. We first show that no simple set is Q-reducible to S. To this end, let B be an arbitrary such set. To sya that B is Q-reducible to S is equivalent (by birtue of the fact that in Q-reductions each round contains a single question) to asserting the existence of a recursive function $\ell(x)$ such that

$$(\forall x)(x \in \overline{B} \longleftrightarrow E_{\ell(x)} \cap \overline{S} \neq \Lambda)$$

where E_m is, as before, an effective enumeration of all recursively enumerable sets.

Now to each $x \in \overline{B}$ there belongs a recursively enumerable set $E_{\ell(x)}$ (all the rounds of single questions for x). Let $B_1 = x | E_{\ell(x)} \cap T \neq \Lambda$. B_1 is obviously recursively enumerable, and by the definition of a Q-reduction, B_1 $\overline{B}$. But since B is simple, $\overline{B}$ contains no infinite recursively enumerable subset. Therefore B_1 is finite, and thus its complement $\overline{B}_1$ is recursively enumerable. Let $C = \bigcup_{x \in \overline{B}_1} E_{\ell(x)}$. C is recursively enumerable and disjoint from T. Therefore considering the special feature of s, $C - S$ if finite. It follows that $B_2 = x | E_{\ell(x)} \cap C - S \neq \Lambda$ is a recursively enumerable subset of $\overline{B}$ and accordingly finite. Yet $B_1 \cup B_2 = \overline{B}$. In other words $\overline{B}$ is finite, so we have a contradiction. Conclusion: There is no simple set Q-reducible to S.

On the other hand, Dekker[1] has found a natural method of assigning to any recursively enumerable and non-recursive set, a simple set which is reducible to the given set: If A is the given set, let f be any 1-1 recursive function ranging over A. Define

$$A' = n | (\exists k)(k > n \,\&\, f(k) < f(n))$$

It is easily shown that A' is simple and reducible to A. Consequently there is a simple set which is reducible to S. If follows that Q is not the most general type of reducibility.

Bibliography

1. J. C. E. Dekker, "A Theorem on Hyper-simple sets", Proceedings of the American Mathematical Society, 5, 5, 791-796, 1954.

2. E. L. Post, "Recursively Enumerable Sets of Positive Integers and Their Decision Problems", Bulletin of the American Mathematical Society, 50, 281-316, 1944.

3. J. R. Shoenfield, "Quasicreative Sets," Proceedings of the American Mathematical Society, 8, 5, 964-967, 1957.

4. V. A. Uspensky, "Godel's Theorem and the Theory of Algorithms," Doklady Akademii Nauk S.S.S.P. 91, 4, 737-740, 1953.