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COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Communication Sciences

Technical Report

DECISION PROBLEMS FOR MULTIPLE SUCCESSOR ARITHMETICS(1)

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Background: The Logic of Computers Group of the Communication Sciences Department of The University of Michigan is investigating the application of logic and mathematics to the theory of the design of computing automata. A study of the decision problem for various formal languages is a part of this investigation.

Condensed Report Contents:

Let N_k denote the set of words over the alphabet $\Sigma_k = \{1, \ldots, k\}$. N_k contains the null word which is denoted λ. We consider decision problems for various first-order interpreted predicate languages in which the variables range over $N_k (k \geq 2)$. Our main result is that there is no decision procedure for truth in the interpreted language which has the subword relation as its only non-logical primitive. This, together with known results summarized in the report, settles the decision problem for any language constructed on the basis of a large number of relations and functions.

For further information: The complete report is available in the major Navy technical libraries and can be obtained from the Defense Documenation Center. A few copies are available for distribution by the author.

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1. Introduction. Let N_k denote the set of words over the alphabet $\Sigma_k = \{1, \ldots, k\}$. N_k contains the null word which is denoted λ. We consider decision problems for various first-order interpreted predicate languages in which the variables range over $N_k (k \geq 2)$. Our main result is that there is no decision procedure for truth in the interpreted language which has the subword relation as its only non-logical primitive. This, together with known results summarized in Section 4, settles the decision problem for any language constructed on the basis of the relations and functions listed below.

Concatenation	$u \frown v = uv$	
Subword	$u \leq v \longleftrightarrow \exists x \exists y\, [v = xuy]$	
Prefix	$u \preceq v \longleftrightarrow \exists x\, [ux = v]$	
Suffix	$v \succeq u \longleftrightarrow \exists x\, [xu = v]$	
Reflection	$c(\sigma_1 \ldots \sigma_n) = \sigma_n \ldots \sigma_1$	$(\sigma_i \in \Sigma_k)$
Right Successors	$r_\sigma(u) = u\sigma$	$(\sigma \in \Sigma_k)$
Left Successors	$\ell_\sigma(u) = \sigma u$	$(\sigma \in \Sigma_k)$
Equal length	$L(u, v) \longleftrightarrow u$ and v have the same number of symbols.	

$N_k(\leq)$ is the structure $\langle N_k, \leq \rangle$. The language of $N_k(\leq)$, $L_k(\leq)$, is the first-order applied predicate calculus with equality having individual variables $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}, \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}, \underset{\sim}{u}_1 \ldots$ ranging over N_k and having a single non-logical constant $\underset{\sim}{\leq}$ interpreted as the corresponding relation on N_k. The theory of $N_k(\leq)$(or of $L_k(\leq)$), $T_k(\leq)$, is the set of sentences of $L_k(\leq)$ which are true in $N_k(\leq)$. A relation $R \subseteq N_k^m$ is definable over $N_k(\leq)$(or definable in $L_k(\leq)$) if there exists a formula $F(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m)$ in $L_k(\leq)$ such that $R(x_1, \ldots, x_m)$ holds iff $<x_1, \ldots, x_m>$ satisfies $F(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m)$ in $N_k(\leq)$ for every m-tuple $<x_1, \ldots, x_m> \in N_k^m$. A function $f: N_k^m \to N_k$ is definable over $N_k(\leq)$ iff its graph (the relation $f(x_1, \ldots, x_m) = y$) is definable.

By analogy, the meanings of the notations $L_k(\preceq, \succeq)$, $T_k(r_1, \ldots, r_k, \preceq, L)$ (written $T_k(r, \preceq, L)$, for convenience) and $N_k(1, \ldots, k, {}^\frown)$(written $N_k(\sigma, {}^\frown)$) should be clear. We will say that L is an interpreted language over N_k if L is the language of some structure with domain N_k.

2. Method of Proof. The schema for defining a function φ(on N_k) by primitive recursion from functions ψ and $\chi_1, \ldots, \chi_k$ is given by:

(1) $$\varphi(x_1, \ldots, x_n, \lambda) = \psi(x_1, \ldots, x_n)$$

$$\varphi(x_1, \ldots, x_n, r_\sigma(y)) = \chi_\sigma(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n, y), y) \quad \sigma = 1, \ldots, k.$$

The class of k-primitive recursive functions is defined to be the least class of functions containing the constant function, $q(x) = \lambda$, the right successor functions and the projection functions $(\pi_n^i(x_1, \ldots, x_n) = x_i)$ and closed under composition and primitive recursion according to the schema above. Generalizing the concept for $k = 1$, we will define the k-arithmetic relations to be the closure of the k-primitive recursive functions under first-order definability.(2)

Our undecidability proofs are based on:

Proposition 1. Let L be an interpreted language with theory T. If all arithmetic relations are definable in L then T is undecidable.

The most direct argument for Proposition 1 would be Tarski's [7]: under any effective numbering of L in N_1, it can be shown that the set of numbers correlated with T is not definable in L. But all recursive subsets of N_1 are arithmetic and therefore the set of numbers correlated with T is not recursive.

Let U, V be arbitrary sets. For a relation $B \subseteq U^n$ and $w \in U$, we will write B_w to denote the (n-1)-ary relation obtained from B by fixing the first argument. $B \subseteq V \times U^n$ is universal with respect to V for a class $\mathcal{R}$ of relations in U^n if $\mathcal{R} = \{B_w \mid w \in V\}$. A relation $B \subseteq U \times U$ is strongly universal for finite subsets of $V \subseteq U$ if there exists a sequence $V_0 = V, V_1, \ldots, V_n, \ldots$ of subsets of U such that B is universal for finite subsets of V_i with respect to

V_{i+1} $(i = 0, 1, \ldots)$.

It is well known that quantification over finite monadic functions permits the conversion of recursive definitions into explicit ones. This second-order quantification can be reduced to first-order quantification if one has available a relation which is universal for finite monadic functions.[3] The extension of these ideas to the use of finite binary relations offers no difficulty. Thus, the following proposition provides a method of applying Proposition 1. (All (1-) arithmetic relations are m-arithmetic for any m.)

Proposition 2. Let L be an interpreted language over N_k. All m-arithmetic ($m \leq k$) relations are definable in L if (i) $r_\sigma \upharpoonright N_m$ is definable in L ($\sigma = 1, \ldots, m$) and (ii) there is a relation $B \subseteq N_k \times N_m^2$ which is definable in L and universal for finite binary relations on N_m with respect to N_k.

To expand on the comments preceeding the statement of Proposition 2, we observe that the following formula (with quantification over finite binary relations) is an explicit definition of $\varphi(x, y) = z$ (on N_m) defined by primitive recursion from ψ and $\chi_1, \ldots, \chi_m$ according to schema (1).

$$(2) \qquad \exists S[\forall w[S(\lambda, w) \longleftrightarrow w = \psi(x)] \wedge S(y, z)$$

$$\wedge \bigwedge_{\sigma=1}^{m} \forall u \forall w[S(r_\sigma(u), w) \rightarrow \exists v[S(u, v) \wedge w = \chi_\sigma(x, v, u)]]] .$$

If χ_σ and ψ are definable in L and if L satisfies the hypotheses of Proposition 2 then (2) can be expressed in L. Thus the functions on N_m which are definable in L are closed under primitive recursion and we can immediately conclude that all m-arithmetic relations are definable in L.

To simplify the specific undecidability proofs, we give the following:

Lemma 1. If B is strongly universal for finite subsets of $V \subseteq U$ then a universal relation for finite relations in $V \times V$ is definable over U(B).

Proof: Consider the following sequence of definitions:

$$D(z, x, y) =_{df} B(z, x) \wedge B(z, y) \wedge \forall u[B(z, u) \rightarrow u = x \vee u = y] ,$$

$$T(z, x, y) =_{df} \exists u \exists w[D(z, u, w) \wedge D(u, x, y) \wedge D(w, y, y)] ,$$

$$R(z, x, y) =_{df} \exists v [B(z, v) \wedge T(v, x, y)] .$$

It is clear that $<z, x, y>$ satisfies D iff $B_z = \{x, y\}$. T defines a pairing relation on V. For any ordered pair $<x, y>$ there exist u, w (ϵV_1) with $B_u = \{x, y\}$ and $B_w = \{y\}$. Also there exists a z (ϵV_2)

with $B_z = \{u, w\}$ and therefore $<z, x, y>$ satisfies T. But for any x', y' in V, if w, x', y' satisfy T then $x' = x$ and $y' = y$ because, in effect, the definition, $\{\{x, y\}, \{y\}\}$, of the ordered pair $<x, y>$ is a good one. For any finite binary relation S it is now clear that there exists a w (ϵV_3) such that $<w, x, y>$ satisfies R iff $S(x, y)$. Hence R is a universal relation for finite binary relations on V.

The final form for the application of Proposition 1 can now be stated as:

Proposition 3. Let L be an interpreted language over N_k with theory T. If $r_\sigma \upharpoonright N_m$ are definable $(\sigma = 1, \ldots, m)$ and if a strongly universal relation for finite subsets of N_m is definable in L then all m-arithmetic relations are definable in L and T is undecidable.

3. Undecidability Results. The following definitions in $L_k(\sigma, \leq)$ will be useful $(\sigma = 1, \ldots, k)$:

$$T_\sigma(x) =_{df} \forall z[z \leq x \rightarrow z \leq \sigma \vee \sigma \leq z],$$

$$t_\sigma(x) = y =_{df} T_\sigma(y) \wedge y \leq x \wedge \forall z[T_\sigma(z) \wedge z \leq x \rightarrow z \leq y],$$

$$s_\sigma(x) = y =_{df} T_\sigma(x) \wedge T_\sigma(y) \wedge x \subset y \wedge \forall z[x \leq z \wedge z \leq y \rightarrow x = z \vee z = y].$$

T_σ defines the set of σ-tallys $\{\lambda, \sigma, \sigma\sigma, \ldots\}$; $t_\sigma(x) = y$ is the graph of the maximum σ-tally function - $t_\sigma(x)$ is the largest σ-tally contained as a subword in x; $s_\sigma(x) = y$ is the graph of the successor function on T_σ.

The concept of maximum tally which Quine [6] used to prove the undecidability of $T_2(\sigma, \frown)$ plays the crucial role in defining a strongly universal relation for finite sets.(4)

Lemma 2. The relation,

$$B(w, u) \longleftrightarrow t_2(u) \neq t_2(w) \wedge t_2(w)1u1t_2(w) \leq w,$$

is universal for finite subsets of N_k with respect to N_k and is therefore strongly universal for finite subsets of N_k.

Proof: Let $S = \{u_1, \ldots, u_n\}$ be a finite subset of N_k and take v to be any 2-tally larger than every one of the $t_2(u_i)$, $i = 1, \ldots, n$. Then for

$$w = v1u_1 1v1u_2 1v1 \ldots 1v1u_n 1v,$$

we claim that $B_w = S$. Certainly $B_w \supseteq S$ because, by construction, $t_2(w) = v \neq t_2(u_i)$ and for each i, $v1u_i1v$ is a subword of w. But the occurrences of v in w are uniquely determined (w is uniquely decomposable in the form written above) and thus, for any subword of the form $v1u1v$, either $v \leq u$ (and $u \notin B_w$) or u is one of the u_i. Therefore $B_w = S$ and B is indeed universal for finite subsets of N_k.

From the definition it is clear that B is definable over $N_k(\sigma, ^\frown)$. Since concatenation is k-primitive recursive it follows that all definable relations of $L_k(\sigma, \subseteq)$ are k-arithmetic. Thus as a consequence of the previous Lemma and Proposition 3 we obtain a slightly modified form of Quine's result:

Theorem 1. (Quine) A relation is definable over $N_k(\sigma, ^\frown)$ iff it is k-arithmetic and $T_k(\sigma, ^\frown)$ is undecidable.

Lemma 3. The relation B as given in Lemma 2 is definable over $N_k(r, \ell, \subseteq)$.

Proof: The following formula is claimed to be a definition of B in $L_2(r, \ell, \leq)$.[5] The expression lyl is used to abbreviate $\ell_1 r_1(y)$.

$$
\begin{aligned}
(3) \qquad & t_2(w) \neq t_2(u) \wedge \exists z[z \leq w \wedge lul \leq z \wedge \forall y[lyl \leq z \rightarrow y \leq u] \\
& \wedge r_1 t_2(w) \leq z \wedge \ell_1 t_2(w) \leq z] .
\end{aligned}
$$

It is clear that if $B(w, u)$ holds then z can be taken to be $t_2(w)lult_2(w)$ and $\langle w, u\rangle$ satisfies (3). On the other hand, if $\langle w, u\rangle$ satisfies (3) then $z = z_1 lulz_2 \leq w$ for some z_1 and z_2. By the maximality condition on subwords of the form lyl, z_1 and z_2 must be 2-tallys. z_1, z_2 must be at least as large as $t_2(w)$ by the second line of the definition and no larger than $t_2(w)$ since $z \leq w$. Hence $z = t_2(w)lult_2(w)$, $z \leq w$ and $t_2(w) \neq t_2(u)$, i.e., $B(w, u)$ holds.

With Lemmas 2 and 3, we can apply Proposition 3 to obtain:

Theorem 2. A relation is definable in $L_k(r, \ell, \subseteq)$ iff it is k-arithmetic and thus $T_k(r, \ell, \subseteq)$ is undecidable; $L_k(r, \ell, \subseteq)$ and $L_k(\sigma, {}^\frown)$ are equivalent as to definability.

The subword relation is definable in terms of the prefix and suffix relations:

$$x \subseteq y \longleftrightarrow \exists z\, [y \succeq z \wedge x \preceq z] \,.$$

Also it is easy to verify that the left and right successor functions are definable in $L_k(\sigma, \preceq, \succeq)$. Thus we obtain the following corollary to Theorem 2, the first part of which was conjectured by Büchi in [2] .

Corollary 1. $T_k(\sigma, \preceq, \succeq)$ is undecidable. The languages $L_k(\sigma, {}^\frown)$ and $L_k(\sigma, \preceq, \succeq)$ are equivalent as to definability.

The reflection function c is an automorphism of the structure $N_k(\sigma, \subseteq)$ in the sense that $c(\sigma) = \sigma$ and $u \subseteq v \longleftrightarrow c(u) \subseteq c(v)$. It is evident that this automorphism property carries over to all relations definable over the structure. Thus, if $R(u, v)$ is a relation definable in $L_k(\sigma, \subseteq)$, then

$$R(u, v) \longleftrightarrow R(c(u), c(v)).$$

Because the graph of concatenation does not have this property, it is not definable over $N_k(\sigma, \subseteq)$ and therefore $L_k(\sigma, \subseteq)$ is weaker than

$L_k(r, \ell, \leq)$ with respect to definability. Any relation on N_1, on the other hand, is invariant under c since $c \restriction N_1$ is the identity. Therefore there is no reason to suspect that the arithmetic relations are not definable over $N_k(\sigma, \leq)$. We will show in the sequel that these relations are indeed definable.

Again we begin by giving a mathematical definition of a relation on N_k which has the required universal properties and subsequently show that it is definable over $N_k(\sigma, \leq)$.

First we describe a set $S \subseteq N_2$ from which will be chosen the codings for finite sets and ultimately, finite binary relations.(6) The definition is given in levels:

$$S_n = \{2^n 1u1 2^n \mid t_2(u) \leq 2^{n-1} \text{ and } 11 \leq 1u1\},$$

and S is taken to be

$$\bigcup_{n \geq 1} S_n .$$

Before defining a universal relation for finite subsets of N_1 (as is required), we observe:

Lemma 4. The relation,

$$U'(w, v) \longleftrightarrow w \in S \wedge v \in S \wedge t_2(w) = r_2 t_2(v) \wedge v \leq w,$$

is strongly universal for finite subsets of S_1.

Proof: The sequence $S_1, \ldots, S_n, \ldots$ satisfies the definition of strong universality for the relation U'. To verify this we must show that U' is universal for finite subsets of S_n with respect to S_{n+1}. Consider the subset $\{u_1, \ldots, u_m\}$ of S_n. Each u_i is of the form $2^n v_i 2^n$ where the uniquely determined v_i begins and ends with 1 and $t_2(v_i) \leq 2^{n-1}$. Then taking

$$w = 22^n v_1 2^n v_2 \ldots 2^n v_m 2^n 2,$$

it is clear that $U'(w, u_i)$ for each i and $w \in S_{n+1}$. But again, the uniqueness of decomposition in the form above insures us that if $U'(w, u)$ then u is indeed one of the u_i's.

In effect, what we have in U' is a strongly universal relation for encodings of finite subsets of N_1. Any element of S_1 is of the form $2v2$ where v is a 1-tally of length greater than one. Clearly we can associate with each element of S_1 an element of N_1 ($21^m 2$ corresponds to 1^{m-2}) to obtain from U' a strongly universal relation for finite subsets of N_1.

$$U(w, v) \longleftrightarrow \exists z[\, U'(w, z) \wedge t_2(z) = 2 \wedge r_1 r_1(v) = t_1(z)] \vee U'(w, v) \wedge t_2(v) \neq 2.$$

Lemma 5. U is strongly universal for finite subsets of N_1.

Proof. Now the sequence $N_1, S_2, \ldots, S_n, \ldots$ works in the definition of strong universality.

Lemma 6. The relation U is definable over $N_k(\sigma, \subseteq)$.

Proof: In consideration of the definitions of T_σ, t_σ, and s_σ and of U and U' it is sufficient for us to show that S is definable over $N_k(\sigma, \subseteq)$. First we introduce the definition:

$$M(z, x) =_{df} x \subset z \wedge \forall y[x \subseteq y \wedge y \subseteq z \rightarrow x = y \vee y = z] .$$

$M(z, x)$ holds when x is a maximal proper subword of z. Since λ has no proper subwords, $M(\lambda, x)$ holds for no x but for $z \neq \lambda$, it is easily seen that M_z is exactly the set consisting of the right and left predecessors of z. We will prove the following formula is a definition of S.

$$\begin{aligned}(4)\quad & s_1(1) \subseteq w \wedge \exists x_1 \exists x_2 \exists x[M(w, x_1) \wedge M(w, x_2) \wedge x_1 \neq x_2 \wedge M(x_1, x) \wedge M(x_2, x) \\ & \wedge x \neq \lambda \wedge t_2(x_1) = t_2(x_2) = t_2(w) = s_2t_2(x)] .\end{aligned}$$

We must first show that every element of S satisfies the formula above. Let $w = 2^m1u12^m \in S_m (m \geq 1)$. The words $x_1 = 2^m1u12^{m-1}$ and $x_2 = 2^{m-1}1u12^m$ are distinct maximal subwords of w and with $x = 2^{m-1}1u12^{m-1}$ it is clear that w satisfies (4). For the converse, assume that $w = \sigma_1\sigma_2\ldots\sigma_n$ satisfies (4). We know that n is greater than two since $x \neq \lambda$. Then $x_1 = \sigma_1\ldots\sigma_{n-1}$ and $x_2 = \sigma_2\ldots\sigma_n$ (possibly interchanging x_1, x_2). If x were not the "middle" of w then we would have $x = \sigma_1\ldots\sigma_{n-2} = \sigma_3\ldots\sigma_n$ and this entails $\sigma_1 = \sigma_3 = \sigma_5\ldots$ and $\sigma_2 = \sigma_4 = \sigma_6\ldots$. But with $11 \subseteq w$ we know $1 = \sigma_i = \sigma_{i+1}$ for some i and thus $1 = \sigma_1 = \sigma_2 = \sigma_3\ldots$ which contradicts $x_1 \neq x_2$. Therefore $x = \sigma_2\ldots\sigma_{n-1}$ as is desired. Let $t_2(w) = 2^k$ and write $w = 2^m 1y$. If $m \neq k$ then $t_2(x_1) = 2^k$ implies that $t_2(x) = 2^k$ which is impossible. Thus, we must have $m = k$ and with the same argument for the other end of the word we get $w = 2^k 1w'12^k$ and with $11 \subseteq w$ we have $w \in S$.

From Lemmas 5 and 6 we know that a strongly universal relation for finite subsets of N_1 is definable in $L_k(\sigma, \subseteq)$. Also, the graph of r_1 restricted to N_1 is definable as was indicated at the beginning of this section. Therefore, applying Proposition 3 again we have:

Theorem 3. All arithmetic relations are definable in $L_k(\sigma, \subseteq)$ and $T_k(\sigma, \subseteq)$ is undecidable.

Let δ be any permutation of the symbols Σ_k and let $\hat{\delta}$ be the extension of δ to a concatenation automorphism. Any such mapping is an automorphism of $N_k(\subseteq)$. Thus definability over this structure is even weaker than that over $N_k(\sigma, \leq)$. Of course, it is impossible to define in $N_k(\leq)$ any specific symbol $(\sigma \in \Sigma_k)$ but the following formula defines the set Σ_k of symbols:

$$\forall z[z \leq x \rightarrow z = x \vee \forall y[z \leq y]] \wedge \exists z \neg [x \leq z] .$$

The first part of the definition is satisfied by elements of $\Sigma_k \cup \{\lambda\}$ whereas the second part excludes λ. We will use $\Sigma_k(x)$ to abbreviate the formula above.

Theorem 4. $T_k(\leq)$ is undecidable.

Proof: Let A be any formula in $L_k(\sigma, \leq)$. We associate with A the following formula of $L_k(\leq)$:

$$A^*(z_1, \ldots, z_k) = \bigwedge_{i=1}^{k} \Sigma_k(z_i) \wedge \bigwedge_{i \neq j} z_i \neq z_j \wedge S^{\sigma}_{z_\sigma} A,$$

where $S^{\sigma}_{z_\sigma} A$ is the formula obtained from A by substituting z_σ for every occurrence of σ $(\sigma = 1, \ldots, k)$. It is assumed that $z_1, \ldots, z_k$ do not occur in A. If A is a sentence true in $N_k(\sigma, \leq)$ then clearly $A^*(1, \ldots, k)$ is true in $N_k(\leq)$ and, in particular $\exists z_1 \ldots \exists z_k A^*$ is true in $N_k(\leq)$. Conversely, if this last sentence is true in $N_k(\leq)$, then by the construction of A^*, $A^*(\sigma_1, \ldots, \sigma_k)$ is true where $\{\sigma_i\} = \Sigma_k$. Because permuting the symbols produces an automorphism of $N_k(\leq)$, we also know that $A^*(1, \ldots, k)$ is true in $N_k(\leq)$ and thus A is true in $N_k(\sigma, \leq)$. In this way we have a transformation $*$ with the property that $A \in T_k(\sigma, \leq)$ if and only if $\exists \underset{\sim}{z}_1 \ldots \exists \underset{\sim}{z}_k A^* \in T_k(\leq)$ and a decision procedure for the latter theory would yield a procedure for the former. By Theorem 4, $T_k(\leq)$ is undecidable.

The proof applies equally well to the two other theories with constants which were considered above.

Corollary 2. $T_k(\frown)$ and $T_k(\precsim, \succsim)$ are undecidable.

4. Conclusion. The results of the previous section all apply to N_k for $k \geq 2$. The analogous problems for the special case, $k = 1$, corresponding to the natural numbers, have long been solved. The structure $N_1(\frown)$ is simply the natural numbers under addition and $T_1(\frown)$ is known to be decidable (Presburger [5] and Hilbert and Bernays [4]). $N_1(r, \ell, \leq)$, $N_1(\sigma, \precsim, \succsim)$ and $N_1(\leq)$ are even weaker being equivalent to the natural numbers under successor and $\leq$. Indicative of the power of the additional generator is the fact that only finite and cofinite sets are definable in $L_1(r, \ell, \leq)$(Hilbert and Bernays [3]) whereas all arithmetic sets (in fact 2-arithmetic) are definable in $L_2(r, \ell, \leq)$.

Applying the methods of Elgot and Büchi [3, 2] , J. C. Shepherdson noted that the language $L_k(r, \precsim, L)$ is one in which definability corresponds to acceptance by finite automata, that is, to regularity.[7] As a consequence of this correspondence it follows that the decision problem for this language has a positive solution. Since $N_k(r, \preceq, L)$ and $N_k(\ell, \succeq, L)$ are isomorphic under c, we also know that $T_k(\ell, \succeq, L)$ is decidable. In addition, it is easy to verify that $L_k(r, \ell, \preceq, L)$ and $L_k(r, \preceq, L)$ are equivalent with respect to definability and thus $T_k(r, \ell, \preceq, L)$ and $T_k(r, \ell, \succeq, L)$ are both decidable.

By extending an elimination of quantifiers method which the author applied to $L_k(r)$, J. H. Bennett (personal communication) has been able to characterize the definable relations in $L_k(r, \ell, c, L)$ and in particular he has shown that $T_k(r, \ell, c, L)$ is decidable.

In summary, we now know that the following structures (and all reducts) have decidable theories:

(a) (Shepherdson, Elgot and Bϋchi) $N_k(r, \ell, \preceq, L)$ and $N_k(r, \ell, \succeq, L)$

(b) (Bennett) $N_k(r, \ell, c, L)$,

and the following structures (and all expansions) have undecidable theories;

(c) (Quine) $N_k(\frown)$

(d) (Theorem 4) $N_k(\subseteq)$

(e) (Corollary 2) $N_k(\preceq, \succeq)$.

Since every structure based on the objects listed in Section 1 is equivalent to either a reduct of (a) or (b) or an expansion of (c), (d), or (e), it follows that the decision problem for the theory of any such structure can be settled by reference to these cases.

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BIBLIOGRAPHY

[1] Asser, G. "Rekursive Wortfunktionen", Fund. Math. 6 (1960)258-278.

[2] Büchi, J. R. "Weak second-order arithmetic and finite automata", Z. Math. Logic. Grundlagen Math. 6 (1960)66-92.

[3] Elgot, C. C. "Decision problems of finite automata design and related arithmetics" Doctoral dissertation, The University of Michigan, Ann Arbor, Michigan, 1959.

[4] Hilbert, D. and P. Bernays Grundlagen der Mathematik, Berlin, 1939.

[5] Presburger, M. "Uber die Vollstandigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritte", Comptes Rendus du I Congress des Mathematiciens des Pays Slavs (arszawa 1929), pp. 29-101 and 395.

[6] Quine, W. V. "Concatenation as a basis of arithmetic", J. Symb. Logic, 11 (1946) 105-114.

[7] Tarski, A. "Der Wahreitsbegriff in den formalisierten Sprachen" Studia Philosophica 1 (1930) 361-404.

[8] Thatcher, J. W. "Decision problems and definability for generalized arithmetic", Doctoral dissertation, The University of Michigan, Ann Arbor, Michigan, 1964.

FOOTNOTES

(1) This paper is part of a Ph. D thesis submitted to the Program in Communication Sciences at the University of Michigan and was presented to the American Mathematical Society, NAMS 11, Abstract 64T-359 (1964) 582.

(2) Interpreting the set N_k as the set of k-adic notations for the natural numbers ($\sigma_0 \ldots \sigma_n$ is the notation for $\Sigma\sigma_i k^i$) Asser [1] has shown that the k-primitive recursive functions correspond to the (1-)primitive recursive functions and thus, under this interpretation, the k-arithmetic relations are simply notational variants of the arithmetic relations.

(3) The author is grateful to J. R. Büchi for pointing out that there is a concise history of this method of converting recursive definitions into explicit ones to be found in Hilbert and Bernays[4] .

(4) J. R. Büchi obtained a universal relation for finite monadic functions using similar techniques (personal communication). This led to his statement in [2] of the undecidability of $T_k(\sigma, \preceq, \succ)$. (See Corollary 1).

(5) We will give the definitions here and below for the case $k = 2$. For undecidability results this is actually sufficient; for definability it should be clear how to extend the definitions to L_k for arbitrary $k \geq 2$. In (3), for example, $1y1 \leq z$ is replaced by the disjunction of $\sigma_1 y \sigma_2 \leq z$ for $\sigma_1, \sigma_2 \neq 2$.

(6) See footnote 5.

(7) In a letter to C. C. Elgot (1959) Shepherdson described the equal length theory and the theorem stated here.

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