Technical Report

DECISION PROBLEMS FOR MULTIPLE SUCCESSOR ARITHMETICS(1)

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Background: The Logic of Computers Group of the Communication Sciences Department of The University of Michigan is investigating the application of logic and mathematics to the theory of the design of computing automata. A study of the decision problem for various formal languages is a part of this investigation.

Condensed Report Contents:

Let $N_k$ denote the set of words over the alphabet $\Sigma_k = \{1, \ldots, k\}$. $N_k$ contains the null word which is denoted $\lambda$. We consider decision problems for various first-order interpreted predicate languages in which the variables range over $N_k (k \geq 2)$. Our main result is that there is no decision procedure for truth in the interpreted language which has the subword relation as its only non-logical primitive. This, together with known results summarized in the report, settles the decision problem for any language constructed on the basis of a large number of relations and functions.

For further information: The complete report is available in the major Navy technical libraries and can be obtained from the Defense Document Center. A few copies are available for distribution by the author.
TABLE OF CONTENTS

1. Introduction ........................................ 1
2. Method of Proof ...................................... 2
3. Undecidability Results .............................. 6
4. Conclusion .............................................. 15
Acknowledgment .......................................... 18
Bibliography ................................................ 19
Footnotes ................................................... 20
1. Introduction. Let $\Sigma_k$ denote the set of words over the alphabet $\Sigma = \{1, \ldots, k\}$. $N_k$ contains the null word which is denoted $\lambda$. We consider decision problems for various first-order interpreted predicate languages in which the variables range over $N_k$ ($k \geq 2$). Our main result is that there is no decision procedure for truth in the interpreted language which has the subword relation as its only non-logical primitive. This, together with known results summarized in Section 4, settles the decision problem for any language constructed on the basis of the relations and functions listed below.

Concatenation $u \circ v = uv$

Subword $u \not\subseteq v \iff \exists x \exists y [v = xuy]$

Prefix $u \not\prefix v \iff \exists x [ux = v]$

Suffix $v \not\suffix u \iff \exists x [xu = v]$

Reflection $c(\sigma_1 \ldots \sigma_n) = \sigma_n \ldots \sigma_1$ ($\sigma_i \in \Sigma_k$)

Right Successors $r_\sigma(u) = u\sigma$ ($\sigma \in \Sigma_k$)

Left Successors $l_\sigma(u) = \sigma u$ ($\sigma \in \Sigma_k$)

Equal length $L(u, v) \iff u$ and $v$ have the same number of symbols.

$N_k(\epsilon)$ is the structure $\langle N_k, \epsilon \rangle$. The language of $N_k(\epsilon)$, $L_k(\epsilon)$, is the first-order applied predicate calculus with equality
having individual variables \( u, v, w, x, y, z, u_1, \ldots \) ranging over \( N_k \)
and having a single non-logical constant \( \varepsilon \) interpreted as the
corresponding relation on \( N_k \). The theory of \( N_k(\varepsilon) \)(or of \( L_k(\varepsilon) \)),
\( T_k(\varepsilon) \), is the set of sentences of \( L_k(\varepsilon) \) which are true in \( N_k(\varepsilon) \).
A relation \( R \subseteq N_k^m \) is definable over \( N_k(\varepsilon) \)(or definable in \( L_k(\varepsilon) \))
if there exists a formula \( F(x_1, \ldots, x_m) \) in \( L_k(\varepsilon) \) such that
\( R(x_1, \ldots, x_m) \) holds iff \( \langle x_1, \ldots, x_m \rangle \) satisfies \( F(x_1, \ldots, x_m) \) in
\( N_k(\varepsilon) \) for every \( m \)-tuple \( \langle x_1, \ldots, x_m \rangle \in N_k^m \). A function \( f: N_k \rightarrow N_k \)
is definable over \( N_k(\varepsilon) \) iff its graph (the relation \( f(x_1, \ldots, x_m) = y \))
is definable.

By analogy, the meanings of the notations \( L_k(\leq, \leq), T_k(r_1, \ldots, r_k, \neq, L) \)
(written \( T_k(r, \neq, L) \), for convenience) and \( N_k(1, \ldots, k, \overline{\cdot}) \)(written \( N_k(\sigma, \overline{\cdot}) \))
should be clear. We will say that \( L \) is an interpreted language over \( N_k \) if
\( L \) is the language of some structure with domain \( N_k \).

2. Method of Proof. The schema for defining a function
\( \varphi \) on \( N_k \) by primitive recursion from functions \( \psi \) and \( \chi_1, \ldots, \chi_k \)
is given by:

\[
\begin{align*}
\varphi(x_1, \ldots, x_n, \lambda) &= \psi(x_1, \ldots, x_n) \\
\varphi(x_1, \ldots, x_n, r_\sigma(y)) &= \chi_\sigma(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n, y), y) \quad \sigma = 1, \ldots, k.
\end{align*}
\]
The class of $k$-primitive recursive functions is defined to be the least class of functions containing the constant function, $q(x) = \lambda$, the right successor functions and the projection functions $(\pi_n^1(x_1, \ldots, x_n) = x_i)$ and closed under composition and primitive recursion according to the schema above. Generalizing the concept for $k = 1$, we will define the $k$-arithmetic relations to be the closure of the $k$-primitive recursive functions under first-order definability. \(^{(2)}\)

Our undecidability proofs are based on:

Proposition 1. Let $L$ be an interpreted language with theory $T$. If all arithmetic relations are definable in $L$ then $T$ is undecidable.

The most direct argument for Proposition 1 would be Tarski's \([7]\): under any effective numbering of $L$ in $N_1$, it can be shown that the set of numbers correlated with $T$ is not definable in $L$. But all recursive subsets of $N_1$ are arithmetic and therefore the set of numbers correlated with $T$ is not recursive.

Let $U, V$ be arbitrary sets. For a relation $B \subseteq U^n$ and $w \in U$, we will write $B_w$ to denote the $(n-1)$-ary relation obtained from $B$ by fixing the first argument. $B \subseteq V \times U^n$ is universal with respect to $V$ for a class $\mathcal{R}$ of relations in $U^n$ if $\mathcal{R} = \{B_w \mid w \in V\}$. A relation $B \subseteq U \times U$ is strongly universal for finite subsets of $V \subseteq U$ if there exists a sequence $V_0 = V, V_1, \ldots, V_n, \ldots$ of subsets of $U$ such that $B$ is universal for finite subsets of $V_i$ with respect to
\( V_{i+1} (i = 0, 1, \ldots) \).

It is well known that quantification over finite monadic functions permits the conversion of recursive definitions into explicit ones. This second-order quantification can be reduced to first-order quantification if one has available a relation which is universal for finite monadic functions. The extension of these ideas to the use of finite binary relations offers no difficulty. Thus, the following proposition provides a method of applying Proposition 1. (All (1-) arithmetic relations are m-arithmetic for any m.)

Proposition 2. Let \( L \) be an interpreted language over \( N_k \). All m-arithmetic (\( m \leq k \)) relations are definable in \( L \) if (i) \( r_\sigma^1 N_m \) is definable in \( L (\sigma = 1, \ldots, m) \) and (ii) there is a relation \( B \subseteq N_k \times N_m^2 \) which is definable in \( L \) and universal for finite binary relations on \( N_m \) with respect to \( N_k \).

To expand on the comments preceding the statement of Proposition 2, we observe that the following formula (with quantification over finite binary relations) is an explicit definition of \( \varphi(x, y) = z \) (on \( N_m \)) defined by primitive recursion from \( \psi \) and \( \chi_1, \ldots, \chi_m \) according to schema (1).

\[
(2) \quad \exists S[\forall w[ S(\lambda, w) \leftrightarrow w = \psi(x)] \land S(y, z)]
\]
\[
\land \bigwedge_{\sigma = 1}^{m} \forall u \forall w[ S(r_{\sigma} (u), w) \leftrightarrow \exists v[ S(u, v) \land w = \chi_{\sigma} (x, v, u)] ]].
\]
If $\chi_\sigma$ and $\psi$ are definable in $L$ and if $L$ satisfies the hypotheses of Proposition 2 then (2) can be expressed in $L$. Thus the functions on $N_m$ which are definable in $L$ are closed under primitive recursion and we can immediately conclude that all $m$-arithmetic relations are definable in $L$.

To simplify the specific undecidability proofs, we give the following:

**Lemma 1.** If $B$ is strongly universal for finite subsets of $V \subseteq U$ then a universal relation for finite relations in $V \times V$ is definable over $U(B)$.

**Proof:** Consider the following sequence of definitions:

$$D(z, x, y) = \text{df} \ B(z, x) \land B(z, y) \land \forall u \ [B(z, u) \rightarrow u = x \lor u = y],$$

$$T(z, x, y) = \text{df} \ \exists u \forall w \ [D(z, u, w) \land D(u, x, y) \land D(w, y, y)],$$

$$R(z, x, y) = \text{df} \ \exists v \ [B(z, v) \land T(v, x, y)].$$

It is clear that $<z, x, y>$ satisfies $D$ iff $B_z = \{x, y\}$. $T$ defines a pairing relation on $V$. For any ordered pair $<x, y>$ there exist $u, w$ ($\epsilon V_1$) with $B_u = \{x, y\}$ and $B_w = \{y\}$. Also there exists a $z$ ($\epsilon V_2$)
with \( B_z = \{u, w\} \) and therefore \( <z, x, y> \) satisfies \( T \). But for any \( x', y' \) in \( V \), if \( w, x', y' \) satisfy \( T \) then \( x' = x \) and \( y' = y \) because, in effect, the definition, \( \{\{x, y\}, \{y\}\} \), of the ordered pair \( <x, y> \) is a good one. For any finite binary relation \( S \) it is now clear that there exists a \( w (\in V_3) \) such that \( <w, x, y> \) satisfies \( R \) iff \( S(x, y) \). Hence \( R \) is a universal relation for finite binary relations on \( V \).

The final form for the application of Proposition 1 can now be stated as:

**Proposition 3.** Let \( L \) be an interpreted language over \( N_k \) with theory \( T \). If \( r = 1, \ldots, m \) are definable \( (r = 1, \ldots, m) \) and if a strongly universal relation for finite subsets of \( N_m \) is definable in \( L \) then all \( m \)-arithmetic relations are definable in \( L \) and \( T \) is undecidable.

3. Undecidability Results. The following definitions in \( L_k(\sigma, \leq) \) will be useful \( (\sigma = 1, \ldots, k) \):

\[
\begin{align*}
T_{\sigma}(x) &= \text{df} \forall z[z \leq x \rightarrow z \leq \sigma \vee \sigma \leq z], \\
T_{\sigma}(y) &= \text{df} \forall z[z \leq y \rightarrow z \leq x \leq y], \\
T_{\sigma}(z) &= \text{df} \forall x \in x \leq y \rightarrow x \leq z \leq y].
\end{align*}
\]

\( T_{\sigma} \) defines the set of \( \sigma \)-tallys \( \{\lambda, \sigma, \sigma \sigma, \ldots\} \); \( t_{\sigma}(x) = y \) is the graph of the maximum \( \sigma \)-tally function \( - t_{\sigma}(x) \) is the largest \( \sigma \)-tally contained as a subword in \( x \); \( s_{\sigma}(x) = y \) is the graph of the successor.
function on \( T_o \).

The concept of maximum tally which Quine [6] used to prove the undecidability of \( T_2(\sigma, \triangleright) \) plays the crucial role in defining a strongly universal relation for finite sets. \(^{(4)}\)

Lemma 2. The relation,

\[
B(w, u) \leftrightarrow t_2(u) \neq t_2(w) \land t_2(w) \not\leq t_2(w) \leq w,
\]

is universal for finite subsets of \( N_k \) with respect to \( N_k \) and is therefore strongly universal for finite subsets of \( N_k \).

Proof: Let \( S = \{u_1, \ldots, u_n\} \) be a finite subset of \( N_k \) and take \( v \) to be any 2-tally larger than every one of the \( t_2(u_i), i = 1, \ldots, n \).

Then for

\[
w = vl_{u_1}lv_{u_2}lv_{1} \ldots lv_{u_n} lv,
\]

we claim that \( B_w = S \). Certainly \( B_w \supseteq S \) because, by construction, \( t_2(w) = v \neq t_2(u_i) \) and for each \( i \), \( vl_{u_i} lv \) is a subword of \( w \). But the occurrences of \( v \) in \( w \) are uniquely determined (\( w \) is uniquely decomposable in the form written above) and thus, for any subword of the form \( vlulv \), either \( v \in u \) (and \( u \notin B_w \)) or \( u \) is one of the \( u_i \). Therefore \( B_w = S \) and \( B \) is indeed universal for finite subsets of \( N_k \).
From the definition it is clear that \( B \) is definable over \( N_k(\sigma, \preceq) \). Since concatenation is \( k \)-primitive recursive it follows that all definable relations of \( L_k(\sigma, \preceq) \) are \( k \)-arithmetic. Thus as a consequence of the previous Lemma and Proposition 3 we obtain a slightly modified form of Quine's result:

**Theorem 1.** (Quine) A relation is definable over \( N_k(\sigma, \preceq) \) iff it is \( k \)-arithmetic and \( T_k(\sigma, \preceq) \) is undecidable.

**Lemma 3.** The relation \( B \) as given in Lemma 2 is definable over \( N_k(r, t, \preceq), \).

**Proof:** The following formula is claimed to be a definition of \( B \) in \( L_2(r, t, \preceq). \) (5) The expression \( \underline{lyl} \) is used to abbreviate \( \underline{l}_1 r_1(y). \)

\[
(3) \quad t_2(w) \neq t_2(u) \land \exists z [ z \preceq w \land \text{lul} z \wedge \forall y [ \text{lyl} \preceq \text{z} \rightarrow y \preceq u ] \]
\[
\land [ \underline{l}_1 t_2(w) \preceq z \wedge \underline{r}_1 t_2(w) \preceq z ] .
\]

It is clear that if \( B(w, u) \) holds then \( z \) can be taken to be \( t_2(w) \text{lul} t_2(w) \) and \( <w, u> \) satisfies (3). On the other hand, if \( <w, u> \) satisfies (3) then \( z = z_1 \text{lul} z_2 \preceq w \) for some \( z_1 \) and \( z_2 \). By the maximality condition on subwords of the form \( \text{lyl}, z_1 \) and \( z_2 \) must be \( 2 \)-tallys. \( z_1, z_2 \) must be at least as large as \( t_2(w) \) by the second line of the definition and no larger than \( t_2(w) \) since \( z \preceq w \). Hence \( z = t_2(w) \text{lul} t_2(w), \) \( z \preceq w \) and \( t_2(w) \neq t_2(u), \) i.e., \( B(w, u) \) holds.
With Lemmas 2 and 3, we can apply Proposition 3 to obtain:

**Theorem 2.** A relation is definable in $L_k(r, l, \leq)$ iff it is $k$-arithmetic and thus \(T_k(r, l, \leq)\) is undecidable; \(L_k(r, l, \leq)\) and \(L_k(\sigma, \prec)\) are equivalent as to definability.

The subword relation is definable in terms of the prefix and suffix relations:

\[ x \preceq y \iff \exists z \ [y \mathrel{\sharp} z \land x \mathrel{\sharp} z]. \]

Also it is easy to verify that the left and right successor functions are definable in \(L_k(\sigma, \leq, \mathrel{\sharp})\). Thus we obtain the following corollary to Theorem 2, the first part of which was conjectured by Büchi in [2].

**Corollary 1.** \(T_k(\sigma, \leq, \mathrel{\sharp})\) is undecidable. The languages \(L_k(\sigma, \prec)\) and \(L_k(\sigma, \mathrel{\sharp}, \mathrel{\sharp})\) are equivalent as to definability.

The reflection function \(c\) is an automorphism of the structure \(N_k(\sigma, \leq)\) in the sense that \(c(\sigma) = \sigma\) and \(u \preceq v \iff c(u) \preceq c(v)\). It is evident that this automorphism property carries over to all relations definable over the structure. Thus, if \(R(u, v)\) is a relation definable in \(L_k(\sigma, \leq)\), then

\[ R(u, v) \iff R(c(u), c(v)). \]

Because the graph of concatenation does not have this property, it is not definable over \(N_k(\sigma, \leq)\) and therefore \(L_k(\sigma, \leq)\) is weaker than
$L_k(r, l, \leq)$ with respect to definability. Any relation on $N_1$, on the other hand, is invariant under $c$ since $c \upharpoonright N_1$ is the identity. Therefore there is no reason to suspect that the arithmetic relations are not definable over $N_k(\sigma, \leq)$. We will show in the sequel that these relations are indeed definable.

Again we begin by giving a mathematical definition of a relation on $N_k$ which has the required universal properties and subsequently show that it is definable over $N_k(\sigma, \leq)$.

First we describe a set $S \subseteq N_2$ from which will be chosen the codings for finite sets and ultimately, finite binary relations.\(^{(6)}\) The definition is given in levels:

$$S_n = \{2^n u \downarrow 2^n | t_2(u) \leq 2^{n-1} \text{ and } 11 \leq u \downarrow 1\},$$

and $S$ is taken to be

$$\bigcup_{n \geq 1} S_n.$$

Before defining a universal relation for finite subsets of $N_1$ (as is required), we observe:

Lemma 4. The relation,

$$U(w, v) \iff w \epsilon S \land v \epsilon S \land t_2(w) = r_2 t_2(v) \land v \leq w,$$
is strongly universal for finite subsets of $S_1$.

Proof: The sequence $S_1, \ldots, S_n, \ldots$ satisfies the definition of strong universality for the relation $U'$. To verify this we must show that $U'$ is universal for finite subsets of $S_n$ with respect to $S_{n+1}$. Consider the subset $\{u_1, \ldots, u_m\}$ of $S_n$. Each $u_i$ is of the form $2^n v_i 2^n$ where the uniquely determined $v_i$ begins and ends with 1 and $t_2(v_i) \leq 2^{n-1}$. Then taking

$$w = 2^n v_1 2^n v_2 \ldots 2^n v_m 2^n 2,$$

it is clear that $U'(w, u_i)$ for each $i$ and $w \in S_{n+1}$. But again, the uniqueness of decomposition in the form above insures us that if $U'(w, u)$ then $u$ is indeed one of the $u_i$'s.

In effect, what we have in $U'$ is a strongly universal relation for encodings of finite subsets of $N_1$. Any element of $S_1$ is of the form $2v2$ where $v$ is a 1-tally of length greater than one. Clearly we can associate with each element of $S_1$ an element of $N_1(2^m 2$ corresponds to $1^{m-2})$ to obtain from $U'$ a strongly universal relation for finite subsets of $N_1$.

$$U(w, v) \iff \exists z[U'(w, z) \land t_2(z) = 2 \land r_1 r_1(v) = t_1(z) \land U'(w, v) \land t_2(v) \neq 2].$$

Lemma 5. $U$ is strongly universal for finite subsets of $N_1$. 
Proof. Now the sequence $N_1, S_2, \ldots, S_n, \ldots$ works in the definition of strong universality.

Lemma 6. The relation $U$ is definable over $N_k(\sigma, \leq)$.

Proof: In consideration of the definitions of $T_\sigma, t_\sigma, \text{ and } S_\sigma$, and of $U$ and $U'$, it is sufficient for us to show that $S$ is definable over $N_k(\sigma, \leq)$. First we introduce the definition:

$$M(x, y) = \exists x \leq y \forall x [x \leq y \land y \leq z \iff x = y \lor y = z].$$

$M(x, y)$ holds when $x$ is a maximal proper subword of $z$. Since $\lambda$ has no proper subwords, $M(\lambda, x)$ holds for no $x$ but for $z \neq \lambda$, it is easily seen that $M_z$ is exactly the set consisting of the right and left predecessors of $z$. We will prove the following formula is a definition of $S$.

$$s_1(x_1, x_2) = \exists x_1 \exists x_2 [M(x_1, x_2) \land x_1 \neq x_2 \land M(x_1, x) \land M(x_2, x) \land t_2(x_1) = t_2(x_2) = t_2(x) = s_2 t_2(x)].$$

We must first show that every element of $S$ satisfies the formula above.

Let $w = 2^m 1 u \cdot \cdot \cdot _m^m (m \geq 1)$. The words $x_1 = 2^m 1 u \cdot \cdot \cdot _m^{m-1}$ and $x_2 = 2^{m-1} 1 u \cdot \cdot \cdot _m^m$ are distinct maximal subwords of $w$ and with $x = 2^{m-1} 1 u \cdot \cdot \cdot _m^{m-1}$ it is clear that $w$ satisfies (4). For the converse,
assume that \( w = \sigma_1 \sigma_2 \ldots \sigma_n \) satisfies (4). We know that \( n \) is greater than two since \( x \neq \lambda \). Then \( x_1 = \sigma_1 \ldots \sigma_{n-1} \) and \( x_2 = \sigma_2 \ldots \sigma_n \) (possibly interchanging \( x_1, x_2 \)). If \( x \) were not the "middle" of \( w \) then we would have \( x = \sigma_1 \ldots \sigma_{n-2} = \sigma_3 \ldots \sigma_n \) and this entails 
\[ \sigma_1 = \sigma_3 = \sigma_5 \ldots \text{ and } \sigma_2 = \sigma_4 = \sigma_6 \ldots . \] But with \( 11 \notin w \) we know
\[ 1 = \sigma_i = \sigma_{i+1} \] for some \( i \) and thus \( 1 = \sigma_1 = \sigma_2 = \sigma_3 \ldots \) which contradicts \( x_1 \neq x_2 \). Therefore \( x = \sigma_2 \ldots \sigma_{n-1} \) as is desired. Let \( t_2(w) = 2^k \) and write \( w = 2^m 1y \). If \( m \neq k \) then \( t_2(x_1) = 2^k \) implies that \( t_2(x) = 2^k \) which is impossible. Thus, we must have \( m = k \) and with the same argument for the other end of the word we get \( w = 2^k 1w' 12^k \) and with \( 11 \notin w \) we have \( w \in S \).

From Lemmas 5 and 6 we know that a strongly universal relation for finite subsets of \( N_1 \) is definable in \( L_k(\sigma, \varepsilon) \). Also, the graph of \( r_1 \) restricted to \( N_1 \) is definable as was indicated at the beginning of this section. Therefore, applying Proposition 3 again we have:

**Theorem 3.** All arithmetic relations are definable in \( L_k(\sigma, \varepsilon) \) and \( T_k(\sigma, \varepsilon) \) is undecidable.

Let \( \delta \) be any permutation of the symbols \( \Sigma_k \) and let \( \hat{\delta} \) be the extension of \( \delta \) to a concatenation automorphism. Any such mapping is an automorphism of \( N_k(\varepsilon) \). Thus definability over this structure is
even weaker than that over \( N_k(\sigma, \varepsilon) \). Of course, it is impossible to define in \( N_k(\varepsilon) \) any specific symbol \((\sigma \in \Sigma_k)\) but the following formula defines the set \( \Sigma_k \) of symbols:

\[
\forall z \exists x \neg \exists z = \exists x \forall y [z \varepsilon y] \land \exists z \neg [x \varepsilon z].
\]

The first part of the definition is satisfied by elements of \( \Sigma_k \cup \{\lambda\} \) whereas the second part excludes \( \lambda \). We will use \( \Sigma_k(x) \) to abbreviate the formula above.

**Theorem 4.** \( T_k(\varepsilon) \) is undecidable.

**Proof:** Let \( A \) be any formula in \( L_k(\sigma, \varepsilon) \). We associate with \( A \) the following formula of \( L_k(\varepsilon) \):

\[
A^*(z_1, \ldots, z_k) = \bigwedge_{i=1}^{k} \Sigma_k(z_i) \land \bigwedge_{i \neq j} z_i \neq z_j \land S_{\varepsilon}^A,
\]

where \( S_{\varepsilon}^A \) is the formula obtained from \( A \) by substituting \( z_{\sigma} \) for every occurrence of \( \varepsilon \) (\( \varepsilon = 1, \ldots, k \)). It is assumed that \( z_1, \ldots, z_k \)
do not occur in \( A \). If \( A \) is a sentence true in \( N_k(\sigma, \varepsilon) \) then clearly \( A^*(1, \ldots, k) \) is true in \( N_k(\varepsilon) \) and, in particular \( \exists z_1 \ldots \exists z_k A^* \) is true in \( N_k(\varepsilon) \). Conversely, if this last sentence is true in \( N_k(\varepsilon) \), then by the construction of \( A^* \), \( A^*(\sigma_1, \ldots, \sigma_k) \) is true where \( \{\sigma_i\} = \Sigma_k \). Because permuting the symbols produces an automorphism of \( N_k(\varepsilon) \),
we also know that $A^k(1, \ldots, k)$ is true in $N_k(\bar{\varepsilon})$ and thus $A$ is true in $N_k(\sigma, \bar{\varepsilon})$. In this way we have a transformation $* \equiv$ with the property that $A \in T_k(\sigma, \bar{\varepsilon})$ if and only if $\exists \bar{x}_1 \ldots \exists \bar{x}_k A^* \in T_k(\varepsilon)$ and a decision procedure for the latter theory would yield a procedure for the former. By Theorem 4, $T_k(\varepsilon)$ is undecidable.

The proof applies equally well to the two other theories with constants which were considered above.

Corollary 2. $T_k(\bar{\wedge})$ and $T_k(\bar{\leq}, \bar{\leq})$ are undecidable.

4. Conclusion. The results of the previous section all apply to $N_k$ for $k \geq 2$. The analogous problems for the special case, $k = 1$, corresponding to the natural numbers, have long been solved. The structure $N_1(\bar{\wedge})$ is simply the natural numbers under addition and $T_1(\bar{\wedge})$ is known to be decidable (Presburger [5] and Hilbert and Bernays [4]). $N_1(\sigma, \bar{\varepsilon})$, $N_1(\sigma, \bar{\leq}, \bar{\leq})$ and $N_1(\varepsilon)$ are even weaker being equivalent to the natural numbers under successor and $\leq$. Indicative of the power of the additional generator is the fact that only finite and cofinite sets are definable in $L_1(\sigma, \sigma, \varepsilon)(\text{Hilbert and Bernays [3]})$ whereas all arithmetic sets (in fact 2-arithmetic) are definable in $L_2(\sigma, \sigma, \varepsilon)$.

Applying the methods of Elgot and Büchi [3, 2], J. C. Shepherdson noted that the language $L_k(\sigma, \bar{\leq}, L)$ is one in which definability corresponds
to acceptance by finite automata, that is, to regularity.\(^{(7)}\) As a consequence of this correspondence it follows that the decision problem for this language has a positive solution. Since \(N_k(r, \leq, L)\) and \(N_k(\ell, \leq, L)\) are isomorphic under \(c\), we also know that \(T_k(\ell, \leq, L)\) is decidable. In addition, it is easy to verify that \(L_k(r, \ell, \leq, L)\) and \(L_k(r, \leq, L)\) are equivalent with respect to definability and thus \(T_k(r, \ell, \leq, L)\) and \(T_k(r, \ell, \leq, L)\) are both decidable.

By extending an elimination of quantifiers method which the author applied to \(L_k(r)\), J. H. Bennett (personal communication) has been able to characterize the definable relations in \(L_k(r, \ell, c, L)\) and in particular he has shown that \(T_k(r, \ell, c, L)\) is decidable.

In summary, we now know that the following structures (and all reducts) have decidable theories:

(a) (Shepherdson, Elgot and Büchi) \(N_k(r, \ell, \leq, L)\) and \(N_k(\ell, \leq, L)\)

(b) (Bennett) \(N_k(r, \ell, c, L)\),

and the following structures (and all expansions) have undecidable theories:

(c) (Quine) \(N_k(\sim)\)

(d) (Theorem 4) \(N_k(\leq)\)

(e) (Corollary 2) \(N_k(\leq, \leq)\).
Since every structure based on the objects listed in Section 1 is equivalent to either a reduct of (a) or (b) or an expansion of (c), (d), or (e), it follows that the decision problem for the theory of any such structure can be settled by reference to these cases.
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FOOTNOTES

(1) This paper is part of a Ph.D thesis submitted to the Program in Communication Sciences at the University of Michigan and was presented to the American Mathematical Society, NAMS 11, Abstract 64T-359 (1964) 582.

(2) Interpreting the set $\mathbb{N}_k$ as the set of $k$-adic notations for the natural numbers $(\sigma^0, \ldots, \sigma^n)$ is the notation for $\Sigma^1_k$ Asser [1] has shown that the $k$-primitive recursive functions correspond to the (1-)primitive recursive functions and thus, under this interpretation, the $k$-arithmetic relations are simply notational variants of the arithmetic relations.

(3) The author is grateful to J. R. Büchi for pointing out that there is a concise history of this method of converting recursive definitions into explicit ones to be found in Hilbert and Bernays[4].

(4) J. R. Büchi obtained a universal relation for finite monadic functions using similar techniques (personal communication). This led to his statement in [2] of the undecidability of $T_k(\pi, \exists, \forall)$. (See Corollary 1).

(5) We will give the definitions here and below for the case $k = 2$. For undecidability results this is actually sufficient; for definability it should be clear how to extend the definitions to $L_k$ for arbitrary $k \geq 2$. In (3), for example, $\forall y \exists z$ is replaced by the disjunction of $\Sigma^1_1 \forall y z \forall y z$ for $\sigma_1, \sigma_2 \neq 2$.

(6) See footnote 5.

(7) In a letter to C. C. Elgot (1959) Shepherdson described the equal length theory and the theorem stated here.
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