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Technical Note

NOTES ON MATHEMATICAL AUTOMATA THEORY

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INTRODUCTION

The formulation of the concept of finite automaton in terms of monadic algebras is due to J. R. Büchi ([3], [5]). The author of these notes was first introduced to this notion in a seminar conducted jointly by Büchi and J. B. Wright in the fall of 1960. The equivalence with the usual definitions (see, for example, Rabin and Scott [14]) is clear. But, in the author's opinion, the present definition has definite advantages. In considering the theory of the structure of finite automata (as opposed to the behavior), the set of final states can be dropped from the definition. Here, results from the study of abstract algebras yield directly, for example, the decomposition theorems for finite automata to be found in Hartmanis [7].

But at least as important as the direct application of known results from abstract algebra, is the insight and motivation that derives from such a formulation within the domain of a well developed branch of mathematics.

The purpose of these notes is to present some of the basic concepts and results of finite automata theory within this algebraic framework. Much of this material is already available in the notes of the seminar [3] referred to above.

I. MONADIC ALGEBRAS

Definition 1: A k-monadic algebra is an abstract algebra,

$$\mathcal{A} = \langle A, a_0, \alpha_1, \ldots, \alpha_k \rangle$$

with one distinguished element and k monadic operators. A pure monadic algebra has no distinguished element.

The modifier "k-monadic" will often be omitted. Also the letter $\mathcal{A}$ will denote an arbitrary k-monadic algebra with components as in Definition 1.

The following concepts are concepts of abstract algebra and are reviewed here for the special case of monadic algebras.[1]

Let $\mathcal{A}$ be a k-monadic algebra. The algebra,

$$\mathcal{A}' = \langle A', a_0', \alpha_1', \ldots, \alpha_k' \rangle$$

is a subalgebra of $\mathcal{A}$ if $A' \subseteq A$, α_i' is the restriction of α_i to A' ($\alpha_i' = \alpha_i | A'$), and $a_0 = a_0'$.

An element $a \in A$ is a generator of $\mathcal{A}$ if for every $a' \in A$ there exists an operator α which is a composition of the α_i such that $a\alpha = a'$. (Suffix notation will be used for the operators of a monadic algebra.) An algebra is said to be monogenic if there exists an element $a \in A$ such that a is a generator of $\mathcal{A}$.

Let $\mathcal{B}$ be a second monadic algebra,

$$\mathcal{B} = \langle B, b_0, \beta_1, \ldots, \beta_k \rangle$$

A function h from A onto B is a <u>homomorphism</u> of $\mathcal{A}$ onto $\mathcal{B}$ if:

i) $h(a_0) = b_0$

ii) $h(a\alpha_j) = h(a)\beta_j \qquad a \in A,\ j = 1,2,\ldots,k.$

We will write $\mathcal{A} \leq \mathcal{B}$ if there exists a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. If h is a one-to-one homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ then $\mathcal{A}$ is <u>isomorphic</u> to $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$.[2]

A <u>congruence relation</u> on a monadic algebra $\mathcal{A}$ is an equivalence relation, π, on A with the property:

$$a\ \pi\ a' \rightarrow a\alpha_j\ \pi\ a'\alpha_j \qquad j = 1,2,\ldots,k.$$

Every homomorphism h on a monadic algebra induces a unique congruence relation denoted π_h:

$$a\ \pi_h\ a' \leftrightarrow h(a) = h(a')$$

Further, given the conguence relation π on $\mathcal{A}$, the natural homomorphism h_π is defined

$$h_\pi(a) = \pi_a$$

where π_a is the congruence class to which a belongs. This homomorphism maps $\mathcal{A}$ onto the <u>quotient algebra</u> $\mathcal{A}/\pi$ where,

$$\mathcal{A}/\pi = \langle A/\pi, \pi_{a_0}, \alpha_1/\pi, \ldots, \alpha_k/\pi \rangle$$

and,

$$A/\pi = \{\pi_a : a \in A\}$$

$$(\pi_a)\alpha_j/\pi = \pi_{a\alpha_j} \qquad j = 1,2,\ldots,k.$$

Definition 2: $\mathcal{A}$ is a reduced k-monadic algebra if a_0 is a generator of $\mathcal{A}$. For an arbitrary k-monadic algebra, $\mathcal{A}$, the reduced monadic algebra, rd($\mathcal{A}$), is the subalgebra of $\mathcal{A}$ generated by a_0. The domain of rd($\mathcal{A}$) will denoted rd(A).[3]

Alternatively $\mathcal{A}$ is reduced if $\mathcal{A}$ contains no proper subalgebra; rd($\mathcal{A}$) is the smallest subalgebra of $\mathcal{A}$.

In the following discussion all monadic algebras will be assumed to be reduced. The term monadic algebra will mean reduced monadic algebra unless stated explicitly to the contrary.

2. THE FREE MONOID - LATTICES OF CONGRUENCE RELATIONS

Let N_k be the set of all words on k distinct letters, $\Sigma_k = \{1,2,\ldots,k\}$ including the word λ of zero length. The length of a word x is the number of letters in x. Let $\mathcal{N}_k$ be the free monoid on k generators;

$$\mathcal{N}_k = \langle N_k, \text{concatenation}, \lambda \rangle .$$

Throughout this paper the variables, $x,y,z,w,x_1,y_1,\ldots$, will range over N_k.

Every homomorphism of $\mathcal{N}_k$ determines a congruence relation on $\mathcal{N}_k$. Explicitly an equivalence relation π on N_k is a congruence relation if:

$$x \; \pi \; y \rightarrow \forall z \forall w [zxw \; \pi \; zyw].$$

We will also be interested in right congruence relations which are equivalence relations which satisfy

$$x \; \pi \; y \rightarrow \forall z [xz \; \pi \; yz].$$

Similarly a left congruence on $\mathcal{N}_k$ is an equivalence relation π for which

$$x \; \pi \; y \rightarrow \forall z [zx \; \pi \; zy].$$

For equivalence relations π_1,π_2 on any domain we will use the notation $\pi_1 \subseteq \pi_2$ for set theoretic containment of ordered pairs (π_1 refines π_2). The lattice operations will be denoted

$$\pi_1 \cup \pi_2 = \text{lub}(\pi_1,\pi_2)$$

$$\pi_1 \cap \pi_2 = \text{glb}(\pi_1,\pi_2).$$

The meet, $\pi_1 \cap \pi_2$, is the set theoretic intersection of π_1 and π_2 as sets of ordered pairs. The join, $\pi_1 \cup \pi_2$ is the smallest equivalence relation containing π_1 and π_2. Alternatively, $\pi_1 \cup \pi_2$ is the transitive closure of the set theoretic union of π_1 and π_2.

For two relations π_1,π_2, the product of π_1 and π_2 (composite, relative product or Peirce product) is the relation $\pi_1\pi_2$ defined by,

$$x\ \pi_1\pi_2\ y \leftrightarrow \exists z[x\ \pi_1\ z \wedge z\ \pi_2\ y].$$

Two relations π_1,π_2 commute if $\pi_1\pi_2 = \pi_2\pi_1$. Note that if this is the case then $\pi_1\pi_2 = \pi_1 \cup \pi_2$.[4]

The universal relation on a domain will be denoted $\bigwedge$, the identity will be denoted ϕ. Let R_k and $\overline{R}_k$ be the sets of right congruences and congruences on $\mathcal{N}_k$ respectively. Then $\cap$ and $\cup$ are lattice operations on R_k and on $\overline{R}_k$.[5] We will denote the lattices by $\mathcal{R}_k$ and $\overline{\mathcal{R}}_k$:

$$\mathcal{R}_k = \langle R_k, \cap, \cup, \bigwedge, \phi \rangle$$

$$\overline{\mathcal{R}}_k = \langle \overline{R}_k, \cap, \cup, \bigwedge, \phi \rangle.$$

Definition 3: Let π be an arbitrary equivalence relation on N_k. The relations $\overline{\pi}$, $\underline{\pi}$, and $\overline{\underline{\pi}}$ are defined as follows:

i) $x\ \overline{\pi}\ y \leftrightarrow \forall z[zx\ \pi\ zy]$

$$\text{ii)}\quad x\ \underline{\pi}\ y \leftrightarrow \forall z[xz\ \pi\ yz]$$

$$\text{iii)}\quad x\ \overline{\underline{\pi}}\ y \leftrightarrow \forall z \forall w[zxw\ \pi\ zyw].$$

$\overline{\pi}$ is the *induced left congruence*, $\underline{\pi}$ the *induced right congruence*, and $\overline{\underline{\pi}}$ the *induced congruence* of π.

Theorem 1: $\underline{\pi}$ ($\overline{\pi}$, $\overline{\underline{\pi}}$) is the maximal right congruence (left congruence, congruence) which refines π.

This theorem in the form presented here is due to Buchi. It is the fundamental part of the minimality theorem for (finite) automata.

3. DERIVED MONADIC ALGEBRAS

Let $\mathcal{S} = \langle S, \cdot, e \rangle$ be an arbitrary monoid with identity e and generators $s_1, \ldots, s_k$. Define for $s \in S$, the right translation, R_s of $\mathcal{S}$ by s by:

$$\forall s'_{s' \in S} (s')R_s = s' \cdot s.$$

The dot will usually be omitted in the product operation.

Let

$$\mathcal{R}_{\mathcal{S}} = \langle \{R_s : s \in S\}, \text{composition}, R_e \rangle$$

It can easily be verified that $\mathcal{S} \cong \mathcal{R}_{\mathcal{S}}$ (Caley's theorem for monoids).

From the monoid $\mathcal{S}$ with k generators we define the k-monadic algebra $\bar{\mathcal{S}}$ derived from $\mathcal{S}$: [6]

$$\bar{\mathcal{S}} = \langle S, e, R_{s_1}, \ldots, R_{s_k} \rangle$$

Note that since $s_1, \ldots, s_k$ are generators of the monoid $\mathcal{S}$, e will be a generator of the monadic algebra $\bar{\mathcal{S}}$.

The free monoid $\mathcal{N}_k$ was defined above. The associated monadic algebra is

$$\bar{\mathcal{N}}_k = \langle N_k, \lambda, r_1, \ldots, r_k \rangle,$$

the free k-monadic algebra. In the notation for $\bar{\mathcal{N}}_k$, r_j is the right translation of $\mathcal{N}_k$ by the generator j:

$$x\, r_j = xj, \qquad j \in \Sigma_k.$$

4. CONGRUENCES AND TRANSITION ALGEBRAS

In the last section the free monadic algebra was derived from the free monoid. Alternatively the standard method of constructing the free algebra could have been employed, the result being isomorphic to $\overline{\mathcal{N}}_k$.

Suppose that $\mathcal{A}$ is an arbitrary (reduced) k-monadic algebra. Since $\mathcal{N}_k$ is free on the generator λ, the mapping of λ onto the generator a_0 of $\mathcal{A}$ can be extended uniquely to a homomorphism $\overline{\mathcal{N}}_k$ onto $\mathcal{A}$. This relationship is important in both the theory of structure and behavior. Deriving $\overline{\mathcal{N}}_k$ in the manner above and associating a word in N_k with every composition of functions in a monadic algebra (Definition 4) we are able to use this property of the free algebra conveniently.

<u>Definition 4</u>: Let $\mathcal{A}$ be a k-monadic algebra. For $x \in N_k$, the <u>transition operator of x</u> is the operator α_x on A defined by the recursion:

$$a\alpha_\lambda = a$$

$$a\alpha_{xj} = (a\alpha_x)\alpha_j.$$

Thus if x is the word, $j_1j_2\ldots j_n$, then α_x is the composition of functions $\alpha_{j_1},\alpha_{j_2},\ldots,\alpha_{j_n}$:

$$\alpha_{j_1j_2\ldots j_n} = \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_n}.$$

This correspondence indicates the motivation for the choice of suffix notation for the monadic operators.

The condition that an equivalence relation π be a congruence relation on $\mathcal{a}$ was stated in Section 1. A simple induction shows that this is equivalent to the condition:

$$a \ \pi \ a' \rightarrow \forall z[a\alpha_z \ \pi \ a'\alpha_z].$$

Similarly, the second condition for a homomorphism can be stated:

$$\text{(ii)} \quad \forall x[h(a\alpha_x) = h(a)\beta_x]$$

<u>Proposition 1</u>: If π is a congruence on $\overline{\mathcal{N}}_k$ then π is a right congruence on $\mathcal{N}_k$ and conversely.

<u>Proof</u>: If π is congruence we have from the observation above that for all x,y,z in N_k;

$$x \ \pi \ y \rightarrow [x \ r_z \ \pi \ y \ r_z \leftrightarrow xz \ \pi \ yz]$$

Proposition 1 establishes the fact that the congruences on $\overline{\mathcal{N}}_k$ are precisely the right congruences on $\mathcal{N}_k$.

Let $\mathcal{a}$ be a k-monadic algebra. The mapping of λ onto the generator a_o of $\mathcal{a}$ is extended to the homomorphism $h_{\mathcal{a}} : \overline{\mathcal{N}}_k \rightarrow \mathcal{a}$ by

$$h_{\mathcal{a}}(x) = a_o\alpha_x.$$

The congruence relation $\pi_{h_{\mathcal{a}}}$ on $\overline{\mathcal{N}}_k$ will be denoted $\pi_{\mathcal{a}}$:

$$x \ \pi_{\mathcal{a}} \ y \leftrightarrow h_{\mathcal{a}}(x) = h_{\mathcal{a}}(y) \leftrightarrow a_o\alpha_x = a_o\alpha_y.$$

From Proposition 1 and the definition of $h_{\mathcal{A}}$ we have:

Proposition 2: Let $\mathcal{A}$ be a reduced k-monadic algebra.

i) $\overline{\mathcal{N}}_k \leq \mathcal{A}$ (the homomorphism is $h_{\mathcal{A}}$)

ii) $\pi_{\mathcal{A}}$ is a right congruence on $\mathcal{N}_k$.

Proposition 3: Let $\mathcal{A}$ and $\mathcal{B}$ be reduced k-monadic algebras, then

$$\mathcal{A} \leq \mathcal{B} \leftrightarrow \pi_{\mathcal{A}} \subseteq \pi_{\mathcal{B}},$$

with isomorphism if and only if $\pi_{\mathcal{A}} = \pi_{\mathcal{B}}$.

Proof: Let g be the homomorphism $\mathcal{A}$ onto $\mathcal{B}$. The homomorphism $h_{\mathcal{B}}$ is unique and $gh_{\mathcal{A}} = h_{\mathcal{B}}$ since $gh_{\mathcal{A}}$ is a homomorphism $\overline{\mathcal{N}}_k$ onto $\mathcal{B}$. Hence,

$$x\ \pi_a\ y \leftrightarrow h_{\mathcal{A}}(x) = h_{\mathcal{A}}(y)$$

$$\rightarrow gh_{\mathcal{A}}(x) = gh_{\mathcal{A}}(y) \leftrightarrow h_{\mathcal{B}}(x) = h_{\mathcal{B}}(y)$$

$$\rightarrow x\ \pi_{\mathcal{B}}\ y.$$

If g is an isomorphism then $\pi_{\mathcal{A}} = \pi_{\mathcal{B}}$. Conversely, assume $\pi_{\mathcal{A}} \subseteq \pi_{\mathcal{B}}$. We define the homomorphism $g: \mathcal{A} \rightarrow \mathcal{B}$ as follows:

$$g(h_{\mathcal{A}}(x)) = h_{\mathcal{B}}(x).$$

g is well defined since $h_{\mathcal{A}}(x)$ is a homomorphism onto $\mathcal{A}$. We show that g is a homomorphism by,

$$g(h_{\mathcal{A}}(x)\alpha_y) = g(h_{\mathcal{A}}(xr_y)) = h_{\mathcal{B}}(xr_y) = h_{\mathcal{B}}(x)\beta_y,$$

since $h_{\mathcal{A}}$ and $h_{\mathcal{B}}$ are homomorphisms. The mapping is onto since $h_{\mathcal{B}}$ is onto. If $\pi_{\mathcal{A}} = \pi_{\mathcal{B}}$ then,

$$[h_{\mathcal{A}}(x) = h_{\mathcal{A}}(y)] \leftrightarrow [h_{\mathcal{B}}(x) = h_{\mathcal{B}}(y)],$$

and g is an isomorphism.

<u>Proposition 4</u>: Let $\mathcal{A}$ be a reduced k-monadic algebra and let σ be a congruence on $\overline{\mathcal{N}}_k$.

$$\text{(i)} \quad \pi_{\overline{\mathcal{N}}_k/\sigma} = \sigma$$

$$\text{(ii)} \quad \mathcal{A} \cong \overline{\mathcal{N}}_k/\pi_{\mathcal{A}}.$$

Proof: (i) the homomorphism h_σ from $\overline{\mathcal{N}}_k$ onto $\overline{\mathcal{N}}_k/\sigma$ was defined in Section 1 with $h_\sigma(x) = \sigma_x$. But,

$$x \; \pi_{\overline{\mathcal{N}}_k/\sigma} \; y \leftrightarrow \sigma_x = \sigma_y \leftrightarrow x \; \sigma \; y.$$

(ii) From (i) with $\sigma = \pi_{\mathcal{A}}$ we have $\pi_{\overline{\mathcal{N}}_k/\pi_{\mathcal{A}}} = \pi_{\mathcal{A}}$. From Proposition 3 the equivalent congruence relations yield the isomorphism (ii).

Let $\mathcal{M}_k$ be the partially ordered set of isomorphism types of reduced k-monadic algebras. Proposition 4 establishes a one-to-one correspondence between $\mathcal{M}_k$ and $\mathcal{R}_k$. Let ψ be this correspondence. Then

$$\psi(\mathcal{A}) = \pi_{\mathcal{A}}$$

$$\psi^{-1}(\pi) = \overline{\mathcal{N}}_k/\pi.$$

ψ and ψ^{-1} are well defined by Propositions 1 and 2. From Proposition 4 we have:

$$\psi^{-1}\psi(\mathcal{A}) \cong \mathcal{A}$$

$$\psi\psi^{-1}(\pi) = \pi.$$

Further, ψ is order preserving by Proposition 3. This one-to-one correspondence induces lattice operations on $\mathcal{M}_k$. In particular:

<u>Proposition 5</u>: $\mathcal{M}_k$ is a lattice with meet $\wedge$ and join $\vee$:

$$\mathcal{A}\wedge\mathcal{B} = \mathrm{glb}(\mathcal{A},\mathcal{B}) = \psi^{-1}(\psi(\mathcal{A})\cap\psi(\mathcal{B}))$$

$$\mathcal{A}\vee\mathcal{B} = \mathrm{lub}(\mathcal{A},\mathcal{B}) = \psi^{-1}(\psi(\mathcal{A})\cup\psi(\mathcal{B})).$$

Propositions 4 and 5 may be interpreted to say that the theory of reduced k-monadic algebras reduces to the theory of right congruences on $\mathcal{N}_k$.

The set of functions α_x for $x \in N_k$ forms a monoid $\mathcal{F}_{\mathcal{A}}$ which will be called the transition monoid of $\mathcal{A}$:[7]

$$F_{\mathcal{A}} = \{\alpha_x : x \in N_k\}$$

$$\mathcal{F}_{\mathcal{A}} = \langle F_{\mathcal{A}}, \text{composition}, \alpha_\lambda \rangle.$$

Note that the set $F_{\mathcal{A}}$ of transition operators is a subset of the set of all functions from A into A. If A is finite $F_{\mathcal{A}}$ is finite. Since it is assumed that $\mathcal{A}$ is reduced, $|F_{\mathcal{A}}| \geq |A|$.

Now the derived algebra of $\mathcal{F}_{\mathcal{A}}$ can be formed. We will denote this by $\overline{\mathcal{A}}$:

$$\overline{\mathcal{A}} = \langle F_{\mathcal{A}}, \alpha_\lambda, R_{\alpha_1}, \ldots, R_{\alpha_k} \rangle$$

Definition 5: A k-monadic algebra, $\mathcal{A}$, is a transition algebra if $\cong \overline{\mathcal{A}}$. For an arbitrary k-monadic algebra $\mathcal{A}$, $\overline{\mathcal{A}}$ is called the transition algebra of $\mathcal{A}$.[8]

The second part of this definition is justified by observing that $\overline{\mathcal{A}} \cong \overline{\overline{\mathcal{A}}}$. This follows from Caley's theorem. Note that since a transition algebra is isomorphic to a derived algebra of a monoid it follows that a transition algebra is always reduced.

Proposition 6: Let $\mathcal{A}$ be a reduced k-monadic algebra.

i) $\pi_{\overline{\mathcal{A}}}$ is a congruence on $\mathcal{N}_k$.

ii) $\pi_{\overline{\mathcal{A}}} \subseteq \pi_{\mathcal{A}}$

iii) $\overline{\mathcal{N}}_k \leq \overline{\mathcal{A}} \leq \mathcal{A}$

iv) $\pi_{\overline{\mathcal{A}}}$ is the maximal congruence of $\mathcal{N}_k$ refining the right congruence $\pi_{\mathcal{A}}$.

Proof: (i) The homomorphism $h_{\overline{\mathcal{A}}} : \overline{\mathcal{N}}_k \to \overline{\mathcal{A}}$ is defined by $h_{\overline{\mathcal{A}}}(x) = \alpha_x$. Since $h_{\overline{\mathcal{A}}}$ is a mapping of N_k onto $F_{\mathcal{A}}$ it is a candidate for a monoid homomorphism $\mathcal{N}_k$ onto $\mathcal{F}_{\mathcal{A}}$. Indeed,

$$h_{\overline{\mathcal{A}}}(xy) = \alpha_{xy} = \alpha_x\alpha_y = h_{\overline{\mathcal{A}}}(x)h_{\overline{\mathcal{A}}}(y).$$

With $h_{\overline{\mathcal{A}}}$ a homomorphism of $\mathcal{N}_k$ we have (i). (ii) Let x,y be elements of N_k.

$$x\ \pi_{\overline{\mathcal{A}}}\ y \leftrightarrow \alpha_x = \alpha_y \rightarrow a_0\alpha_x = a_0\alpha_y \leftrightarrow x\ \pi_{\mathcal{A}}\ y.$$

Hence $\pi_{\overline{\mathcal{A}}} \subseteq \pi_{\mathcal{A}}$. (iii) From (ii) we have $\overline{\mathcal{A}} \leq \mathcal{A}$ by Proposition 3. (iv) Assume σ is a congruence on $\mathcal{N}_k$ refining $\pi_{\mathcal{A}}$. Then for any x,y in N_k,

$$\begin{aligned}
x\ \sigma\ y &\rightarrow \forall z\ zx\ \sigma\ zy \\
&\rightarrow \forall z\ zx\ \pi_{\mathcal{A}}\ zy \\
&\rightarrow \forall z\ a_0\alpha_z\alpha_x = a_0\alpha_z\alpha_y \\
&\rightarrow \forall a_{a \in A}\ a\alpha x = a\alpha y \\
&\rightarrow \alpha_x = \alpha_y \rightarrow x\ \pi_{\overline{\mathcal{A}}}\ y.
\end{aligned}$$

Hence σ refines $\pi_{\overline{\mathcal{A}}}$ and (iv is established).

From the homomorphism $h_{\overline{\mathcal{A}}}: \mathcal{N}_k \rightarrow \mathcal{F}_{\mathcal{A}}$ we have the

<u>Corollary</u>: For a k-monadic algebra, $\mathcal{A}$, $\mathcal{F}_{\mathcal{A}} \cong \mathcal{N}_k/\pi_{\overline{\mathcal{A}}}$.

<u>Proposition 7</u>: Let $\mathcal{A}$ be a reduced k-monadic algebra. The following statements are equivalent:

i) $\mathcal{A}$ is a transition algebra.

ii) $\pi_{\mathcal{A}} = \pi_{\overline{\mathcal{A}}}$

iii) $\pi_{\mathcal{A}}$ is a congruence on $\mathcal{N}_k$.

iv) $a_0\alpha_x = a_0\alpha_y \rightarrow \alpha_x = \alpha_y$

Proof: We will show (i) → (ii) → (iii) → (iv) → (i). By Definition 5, if $\mathcal{A}$ is a transition algebra then $\mathcal{A} \cong \overline{\mathcal{A}}$ and by Proposition 3 $\pi_{\mathcal{A}} = \pi_{\overline{\mathcal{A}}}$. Since $\pi_{\overline{\mathcal{A}}}$ is a congruence (Proposition 6, (i)) $\pi_{\mathcal{A}}$ is also congruence. If $\pi_{\mathcal{A}}$ is a congruence then $\pi_{\mathcal{A}} = \pi_{\overline{\mathcal{A}}}$ by Proposition 6 (iv) and

$$a_o\alpha_x = a_o\alpha_y \leftrightarrow x\ \pi_{\mathcal{A}}\ y \leftrightarrow x\ \pi_{\overline{\mathcal{A}}}\ y \leftrightarrow \alpha_x = \alpha_y.$$

Statement (iv) in turn implies $\pi_{\mathcal{A}} = \pi_{\overline{\mathcal{A}}}$ and by Propositions 3 and 4 $\mathcal{A} \cong \overline{\mathcal{A}}$.

We obtain a further characterization of finite transition algebras by the following:

Proposition 8: If $\mathcal{A}$ is a finite reduced k-monadic algebra then $\mathcal{A}$ is a transition algebra if and only if $|F_{\mathcal{A}}| = |A|$.

Proof: If $|F_{\mathcal{A}}| = |A|$ then the homomorphism $\overline{\mathcal{A}}$ onto $\mathcal{A}$ must be one-to-one. Conversely if the homomorphism is one-to-one and A is finite, $|F_{\mathcal{A}}| = |A|$.

Let $\overline{\mathcal{M}}_k$ be the partially ordered set of isomorphism types of k-transition algebras. From Proposition 6 (i) $\psi|\overline{\mathcal{M}}_k$ is a partial order isomorphism between $\overline{\mathcal{M}}_k$ and $\overline{\mathcal{R}}_k$. Hence with $\wedge$ and $\vee$ as specified above $\overline{\mathcal{M}}_k$ is a sublattice of $\mathcal{M}_k$.

5. THE DIRECT PRODUCT OF MONADIC ALGEBRAS

The <u>direct</u> <u>product</u> of two k-monadic algebras $\mathcal{A}$ and $\mathcal{B}$ is the monadic algebra

$$\mathcal{A} \times \mathcal{B} = \langle A \times B, \langle a_0, b_0 \rangle, \alpha_1 \times \beta_1, \ldots, \alpha_k \times \beta_k \rangle,$$

where A x B is the cartesian product of the sets A and B and for any pair $\langle a,b \rangle \in A \times B$ the operations $\alpha_i \times \beta_i$ are defined by,

$$\langle a,b \rangle \alpha_i \times \beta_i = \langle a\alpha_i, b\beta_i \rangle$$

Let γ_i be the operator $\alpha_i \times \beta_i$. From the recursive definition of γ_x we note that:

$$\gamma_\lambda = \alpha_\lambda \times \beta_\lambda .$$

By induction assume that $\gamma_x = \alpha_x \times \beta_x$. Then

$$\langle \alpha,\beta \rangle (\alpha_x \times \beta_x)(\alpha_i \times \beta_i) = \langle a\alpha_x, b\beta_x \rangle (\alpha_i \times \beta_i) = \langle a\alpha_x\alpha_i, b\beta_x\beta_i \rangle .$$

Hence $\gamma_{xi} = \alpha_{xi} \times \beta_{xi}$. And we have that for every $x \in N_k$

$$\gamma_x = \alpha_x \times \beta_x .$$

Although both $\mathcal{A}$ and $\mathcal{B}$ may be reduced (as is our general assumption) it is not necessarily the case that $\mathcal{A} \times \mathcal{B}$ is reduced. We will use the notation

$$\mathcal{A} \underline{\times} \mathcal{B} = rd(\mathcal{A} \times \mathcal{B}).$$

Hence $\mathcal{A} \underline{\times} \mathcal{B}$ (the reduced direct product) is the smallest subdirect product of $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2: (A generalization of a theorem of Birkhoff [1] applied to the special case of monadic algebras.) Let $\mathcal{A}$ be a reduced k-monadic algebra and let π, σ be two congruence relations on $\mathcal{A}$. Then

$$\mathcal{A}/\pi \underline{\times} \mathcal{A}/\sigma \cong \mathcal{A}/\pi \cap \sigma.$$

Proof: We first observe that the pair $\langle \pi_a, \sigma_{a'} \rangle$ is in the domain of $\mathcal{A}/\pi \underline{\times} \mathcal{B}/\sigma$ if and only if $\pi_a \cap \sigma_{a'} \neq \phi$. Suppose $a'' \in \pi_a \cap \sigma_{a'}$. Then assume $a_0 \alpha_x = a''$. We have

$$\langle \pi_{a_0}, \sigma_{a_0} \rangle (\alpha_x/\pi \times \alpha_x/\sigma) = \langle \pi_{a''}, \sigma_{a''} \rangle = \langle \pi_a, \sigma_{a'} \rangle.$$

Conversely if the above equation holds $a'' \in \pi_a \cap \sigma_{a'}$ and $\pi_a \cap \sigma_{a'} \neq \phi$.

We define the natural mappings:

$$\phi : rd(A/\pi \times A/\sigma) \to A/\pi \cap \sigma$$

by

$$\phi(\pi_a, \sigma_a) = \pi_a \cap \sigma_a ,$$

This is well defined and onto by the observation above. That ϕ is a homomorphism follows from the fact that $(\pi_a \cap \sigma_a)\alpha_x/\pi\cap\sigma = \pi_{a\alpha_x} \cap \sigma_{a\alpha_x}$. The mapping is one-to-one since $\pi_a \cap \sigma_a = \pi_{a'} \cap \sigma_{a'}$ if and only if $\pi_a = \pi_{a'}$ and $\sigma_a = \sigma_{a'}$.

The following corollaries follow directly from Theorem 2:

<u>Corollary</u> <u>2.1</u>: If $\mathcal{A}$ is a reduced k-monadic algebra and π and σ are congruences on $\mathcal{A}$ then:

$$\pi \cap \sigma = \phi \leftrightarrow \mathcal{A}/\pi \underline{x} \mathcal{A}/\sigma \cong \mathcal{A}$$

<u>Corollary</u> <u>2.2</u>: If $\mathcal{A}$ is a reduced k-monadic algebra and π and σ are congruences on $\mathcal{A}$ then:

$$\pi \sigma = \bigwedge \leftrightarrow \mathcal{A}/\pi \times \mathcal{A}/\sigma \cong \mathcal{A}/\pi \cap \sigma$$

The hypotheses of Corollary 2.2 may be restated by saying that π and σ commute and $\pi \cup \sigma = \bigwedge$. The following corollary is essentially the theorem of Birkhoff ([2] Theorem 4, page 87) restricted to two factors which are monadic algebras. Hartmanis [7] states this result as a necessary and sufficient condition for the decomposition of a machine into the direct product of simpler machines.

<u>Corollary</u> <u>2.3</u>: For a reduced k-monadic algebra $\mathcal{A}$, with congruences and π,σ,

$$\mathcal{A}/\pi \times \mathcal{A}/\sigma \cong \mathcal{A}$$

if and only if the following three conditions hold:

i) π,σ commute

ii) $\pi \cup \sigma = \bigwedge$

iii) $\pi \cap \sigma = \phi$.

Corollary 2.4: Let $\mathcal{A}$ and $\mathcal{B}$ be two k-monadic algebras. The induced congruence $\pi_{\mathcal{A} \times \mathcal{B}}$ is precisely $\pi_{\mathcal{A}} \cap \pi_{\mathcal{B}}$.

Corollary 2.5: The operation $\curlywedge$ in the lattice of isomorphism types of reduced k-monadic algebras is precisely the operation $\underline{x}$ (reduced direct product).

6. MONADIC ALGEBRAS - BEHAVIOR

Definition 6: An interpreted k-monadic algebra is a system denoted $\mathcal{A}[A']$ were

$$\mathcal{A} = \langle A, a_0, \alpha_1, \ldots, \alpha_k \rangle$$

is a monadic algebra and A' is a subset of A.

For any monadic algebra $\mathcal{A}$ and any subset $A' \subseteq A$ there is the corresponding interpreted monadic algebra $\mathcal{A}[A']$.

Let $\mathcal{A}[A']$ and $\mathcal{B}[B']$ be two interpreted monadic algebras. A homomorphism h from $\mathcal{A}$ onto $\mathcal{B}$ is a homomorphism $\mathcal{A}[A']$ onto $\mathcal{B}[B']$ if in addition

$$\text{iii)} \quad a \in A' \rightarrow h(a) \in B' .$$

We say h is a strong homomorphism if we replace (iii) by:

$$\text{iii')} \quad a \in A' \leftrightarrow h(a) \in B' .$$

$\mathcal{A}[A']$ is isomorphic to $\mathcal{B}[B']$ just in case there is a strong homomorphism from $\mathcal{A}[A']$ onto $\mathcal{B}[B']$ which is one-to-one. We will use the notation $\mathcal{A}[A'] \leq \mathcal{B}[B']$ for the existence of a homomorphism from $\mathcal{A}[A']$ onto $\mathcal{B}[B']$ (not necessarily a strong homomorphism).

Definition 7: Let $\mathcal{A}[A']$ be an interpreted monadic algebra. The behavior of $\mathcal{A}[A']$ is a subset of N_k:

$$bh_{\mathcal{A}}[A'] = \{x: a_0 \alpha_x \in A'\}$$

A subset Γ of N_k is <u>definable</u> by a monadic algebra $\mathcal{A}$ if there exists an $A' \subseteq A$ such that $bh_{\mathcal{A}}(A') = \Gamma$.[9]

Note that for a fixed algebra $\mathcal{A}$, $bh_{\mathcal{A}}$ can be viewed as a function from pA into pN_k. In fact the range of $bh_{\mathcal{A}}$ is a boolean algebra of subsets in pN_k with

$$bh_{\mathcal{A}}(A') \cap bh_{\mathcal{A}}(A'') = bh_{\mathcal{A}}(A' \cap A'')$$

$$N_k - bh_{\mathcal{A}}(A') = bh_{\mathcal{A}}(A - A').$$

It should also be observed that every subset Γ of N_k is definable by some monadic algebra. In particular Γ is defined by $\overline{\mathcal{N}}_k[\Gamma]$.

Definition 7 corresponds to the usual definition of behavior; the set of all words which cause transition from the initial state to one of a set of designated "final" states. Alternatively we obtain:

<u>Proposition 9</u>: For any monadic algebra $\mathcal{A}$ and any subset $A' \subseteq A$,

$$bh_{\mathcal{A}}(A') = h_{\mathcal{A}}^{-1}(A').$$

With this characterization of definability we can indicate the correspondence between definability and homomorphisms.

<u>Proposition 10</u>: (Buchi) let $\mathcal{A}$ and $\mathcal{B}$ be two reduced k-monadic algebras with $\mathcal{A} \leq \mathcal{B}$. Then if $\Gamma \subseteq N_k$ is definable by $\mathcal{B}$, Γ is definable by $\mathcal{A}$.

<u>Proof</u>: Let g be the homomorphism $\mathcal{A}$ onto $\mathcal{B}$ and let $\Gamma = h_{\mathcal{B}}^{-1}(B')$. Since $h_{\mathcal{B}} = gh_{\mathcal{A}}$ we have $\Gamma = h_{\mathcal{A}}^{-1}\, g^{-1}(B')$ and hence Γ is definable by $\mathcal{A}$ with $A' = g^{-1}(B')$.

Proposition 11: Let $\mathcal{A}[A']$ and $\mathcal{B}[B']$ be two interpreted reduced k-monadic algebras.

$$\mathcal{A}[A'] \leq \mathcal{B}[B'] \rightarrow bh_{\mathcal{A}}(A') \subseteq bh_{\mathcal{B}}(B')$$

with equivalence of behavior just in case the homomorphism is a strong homomorphism.

Proof: Let g be the homomorphism if $\mathcal{A}$ onto $\mathcal{B}$. Then,

$$bh_{\mathcal{B}}(B') = h_{\mathcal{B}}^{-1}(B') = h_{\mathcal{A}}^{-1} g^{-1}(B') \supseteq h_{\mathcal{A}}^{-1}(A') = bh_{\mathcal{A}}(A'),$$

since $g^{-1}(B') \subseteq A'$, where g is an interpreted monadic algebra homomorphism. And $g^{-1}(B') = A'$ if g is a strong homomorphism.

The transition algebra of $\mathcal{A}$ was defined above. We can find a corresponding interpreted transition algebra as follows. Let

$$\overline{A}' = \{\alpha_x : a_o\alpha_x \in A'\}$$

The homomorphism from $\overline{\mathcal{A}}$ onto $\mathcal{A}$ is not a strong homomorphism from $\overline{\mathcal{A}}[\overline{A}']$ onto $\mathcal{A}[A']$ and hence $bh_{\overline{\mathcal{A}}}(\overline{A}') = bh_{\mathcal{A}}(A')$ from Proposition 11. Hence we have the

Proposition 12: Any subset $\Gamma \subseteq N_k$ definable by a monadic algebra $\mathcal{A}$ is definable by its transition algebra $\overline{\mathcal{A}}$.

For a subset $\Gamma \subseteq N_k$, the relation of Γ is the dichotomy γ:

$$x \, \gamma \, y \leftrightarrow (x \in \Gamma \leftrightarrow y \in \Gamma).$$

Proposition 13: If $\mathcal{A}$ is a k-monadic and $\Gamma \subseteq N_k$ is definable by $\mathcal{A}$ then

$\pi_{\mathcal{A}} \subseteq \gamma$.

Proof: Let $A' \subseteq A$ such that $\Gamma = bh_{\mathcal{A}}(A')$. If $x \; \pi_{\mathcal{A}} \; y$ then $a_0\alpha_x = a_0\alpha_y$ and hence $a_0\alpha_x \in A' \rightarrow a_0\alpha_y \in A'$. And by definition $x \in bh_{\mathcal{A}}(A') \rightarrow y \in bh_{\mathcal{A}}(A')$.

Proposition 14: If π is a right congruence on $\mathcal{N}_k$ refining the dichotomy γ then $\Gamma = bh_{\overline{\mathcal{N}}_k/\pi}(\Gamma/\pi)$.

Proof: Γ/π is well defined since $\pi \subseteq \gamma$. $\overline{\mathcal{N}}_k/\pi$ is a monadic algebra since π is a congruence on $\overline{\mathcal{N}}_k$. h_π is the homomorphism $\overline{\mathcal{N}}_k \rightarrow \overline{\mathcal{N}}_k/\pi$ and indeed h_π is a strong homomorphism $\overline{\mathcal{N}}_k[\Gamma]$ onto $\overline{\mathcal{N}}_k/\pi[\Gamma/\pi]$. Hence Proposition 14 follows from Proposition 11.

From Propositions 13 and 14 we obtain the necessary and sufficient conditions for a subset $\Gamma \subseteq N_k$ to be definable by a k-monadic algebra $\mathcal{A}$.

Corollary: Γ is definable by $\mathcal{A}$ if and only if $\pi_{\mathcal{A}} \subseteq \gamma$.

The existance of a maximal algebra (with respect $\leqq$), for which Γ is definable is insured by this Corollary and Theorem 1.

Theorem 3: Γ is definable by $\overline{\mathcal{N}}_k/\underline{\gamma}$ where $\underline{\gamma}$ is the induced right congruence of γ. If Γ definable by $\mathcal{A}$ then $\mathcal{A} \leqq \overline{\mathcal{N}}_k/\underline{\gamma}$.

Proof: From Theorem 1, $\underline{\gamma}$ is a right congruence on $\mathcal{N}_k$ hence $\overline{\mathcal{N}}_k/\underline{\gamma}$ is well defined and $\underline{\gamma} \leq \gamma$. Hence by the Corollary (and Proposition 4) Γ is definable by $\overline{\mathcal{N}}_k/\underline{\gamma}$ $\underline{\gamma}$ is the maximal right congruence refining γ. By the Corollary again and the partial order isomorphism of $\mathcal{R}_k$ and $\mathcal{M}_k$, $\overline{\mathcal{N}}_k/\underline{\gamma}$ is maximal among machines for which Γ is definable.

7. NON-DETERMINISTIC AUTOMATA - TRANSITION SYSTEMS

Definition 8: A pure k-transition system $\mathcal{T}$ is a relational system:

$$\mathcal{T} = \langle T, \tau_1, \ldots, \tau_k \rangle$$

where $\tau_1, \ldots, \tau_k$ are binary relations on T.

Note that the study of pure transition systems includes the study of pure monadic algebras by restricting the relations to be functional.

Definition 9: For any $x \in N_k$, the transition relation of x, τ_x, of a pure k-transition system $\mathcal{T}$ is defined by the recursion:

$$\tau_\lambda = \{<t,t>: t \in T\}$$

$$\tau_{xj} = \tau_x \tau_j \qquad j = 1, \ldots, k.$$

The relation τ_λ is the identity relation on T. In general the relation τ_{xy} is the product $\tau_x \tau_y$.

Analogous to the case for monadic algebras, the set of transition relations, τ_x for $x \in N_k$, under product forms a monoid with identity, τ_λ. Explicitly, we define the transition monoid $\mathcal{G}_\mathcal{T}$ of a transition system $\mathcal{T}$ as follows:

$$F_\mathcal{T} = \{\tau_x: x \in N_k\}$$

$$\mathcal{G}_\mathcal{T} = \langle F_\mathcal{T}, \quad \text{product}, \tau_\lambda \rangle.$$

Hence from the transition system $\mathcal{T}$ we can form the transition algebra associated with $\mathcal{T}$:

$$\overline{\mathcal{T}} = \langle F_{\mathcal{T}}, \tau_\lambda, R_{\tau_1}, \ldots, R_{\tau_k} \rangle.$$

The direct product of two transition systems $\mathcal{S}$ and $\mathcal{T}$ is the system

$$\mathcal{S} \times \mathcal{T} = \langle S \times T, \sigma_1 \times \tau_1, \ldots, \sigma_k \times \tau_k \rangle,$$

where, analogous to the case for algebras, the relations are analysed by components:

$$\langle s,t \rangle \; \sigma_i \times \tau_i \; \langle s',t' \rangle \Leftrightarrow s \; \sigma_i \; s' \wedge t \; \tau_i \; t'.$$

S x T is the cartesian product of the domain of $\mathcal{S}$ and $\mathcal{T}$.

8. BEHAVIOR OF TRANSITION SYSTEMS

<u>Definition 10</u>: <u>An interpreted transition system</u> denoted $\mathcal{T}[T°,T']$ is a transition system

$$\mathcal{T} = \langle T,\tau_1,\ldots,\tau_k\rangle$$

where $T°$, T' and subsets of T.

<u>Definition 11</u>: The <u>behavior</u> of an interpreted transition system $[T°,T']$ is a subset of N_k defined as follows

$$bh_{\mathcal{T}}(T°,T') = \{x: \exists t_{t \in T°} \exists u_{u \in T'}\ t\ \tau_x\ u\}.$$

A subset Γ of N_k is <u>definable</u> by a transition system $\mathcal{T}$ if there exist subsets $T°$, T' of T with $bh_{\mathcal{T}}(T°,T') = \Gamma$.[10]

Every monadic algebra $\mathcal{A}[A']$ can be formulated as a transition system $\mathcal{T}[T°,T']$ where $T°$ is chosen as the singleton set $\{a_0\}$ and $T' = A'$. Then every behavior of a monadic algebra is the behavior of a transition system. We will now obtain the converse.

Let $\bar{\mathcal{T}}$ be the transition algebra associated with a pure transition system $\mathcal{T}$. Define the set $\bar{T}'$ as follows:

$$\bar{T}' = \{\tau_x: \exists t_{t \in T°} \exists u_{u \in T'}\ t\ \tau_x\ u\}.$$

The subset $\bar{T}'$ of the set $F_{\mathcal{T}}$ is the set of all transition relations which relate an initial state with one of the final states.

Theorem 4: Let $\mathcal{T}[T^\circ, T']$ be any interpreted transition system and let $[\bar{\mathcal{T}}']$ be the interpreted transition algebra defined above. Then

$$bh_{\mathcal{T}}(T^\circ, T') = bh_{\bar{\mathcal{T}}}(\bar{T}').$$

Proof: Let x be any word in N_k. We have by the definitions above:

$$x \in bh_{\mathcal{T}}(T^\circ, T') \leftrightarrow \tau_x \in \bar{T}' \leftrightarrow x \in bh_{\bar{\mathcal{T}}}[\bar{T}']$$

where we recall that:

$$x \in bh_{\bar{\mathcal{T}}}[\bar{T}'] \leftrightarrow \tau_\lambda \tau_x \in \bar{T}' \leftrightarrow \tau_x \in \bar{T}'$$

Corollary: (Myhill) any subset Γ of N_k definable by a transition system, $\mathcal{T}$, is definable by the transition algebra $\bar{\mathcal{T}}$. Further if $\mathcal{T}$ is finite then $\bar{\mathcal{T}}$ is finite.

The second part of this Corollary follows from the fact that $F_{\mathcal{T}}$ is finite if $\mathcal{T}$ is finite. In particular,

$$|F_{\mathcal{T}}| \leq 2^{|T|^2},$$

since the number of relations in $F_{\mathcal{T}}$ is no greater than the total number of relations definable over the set T.

FOOTNOTES

1. See, in particular, Birkhoff [2], Forward on Algebra and Chapter VI and Birkhoff [1].

2. When the binary transition function is used, as Weeg [16] observed, functions which we would like to call homomorphisms are not quite such since the alphabet is not effected by the mapping (this, of course, is not always the case, for example, see Ritchie [15]). The "operation preserving functions" are precisely homomorphisms in the formulation being presented.

3. A reduced algebra is one in which every state is accessable. If $\mathcal{A}$ is a pure monadic algebra then the property of being reduced corresponds to that of strong connectedness (Moore [11]). In a strongly connected automaton every state is a generator. That homomorphisms preserve strong connectedness follows from the fact that in any algebra the homomorphic image of a generator is a generator.

4. Congruence relations which commute play an important role in the decomposition theorems for algebras. The pioneering paper discussing the importance and properties of such relations is Dubreil [6]. See also Birkhoff [1,2].

5. $\overline{R}_k$ is the set of congruences on $\mathcal{R}_k$ and by Theorem 5, page 24, Birkhoff [2], $\overline{\mathcal{R}}_k$ is a lattice. If π, σ are right congruences on $\mathcal{R}_k$ than one can

verify that $\pi \cap \sigma$ is a right congruence as is $\pi \cup \sigma$. Hence $\mathcal{R}_k$ is a lattice. It will be shown below that $\mathcal{R}_k$ is precisely the set of congruence relations on the free monadic algebra and the lattice property of $\mathcal{R}_k$ again follows from Birkhoff's theorem.

6. This construction was used by Medvedev [10] in specifying the automaton defined by a semigroup.

7. This semigroup is called the semigroup of $\mathcal{A}$ by Myhill [13]. In this paper some of the relationships between $\mathcal{F}_{\mathcal{A}}$ and the structure of $\mathcal{A}$ were studied.

8. If $\mathcal{A}$ is a transition algebra, $\mathcal{F}_{\mathcal{A}}$ is isomorphic to the monoid of endomorphism of $\mathcal{A}$. Let $\mathcal{F}_{\mathcal{A}}^*$ be the largest submonoid which is a group (see Kimura [8]), then $\mathcal{F}_{\mathcal{A}}^*$ is the group of automorphism of $\mathcal{A}$. This is called the group of $\mathcal{A}$ by Weeg [16].

9. In the terminology of [14] and [15], $bh_{\mathcal{A}}[A']$ is the set of tapes accepted by $\mathcal{A}[A']$. The term behavior is used by Kleene [9]. Also, "recognizable" and "representable" are to be found in the literature in place of "definable."

10. An interesting generalization of Definitions 10 and 11 suggested by Wright is as follows: Let ρ be an arbitrary relation on the domain T of the transition system $\mathcal{T}$. An interpreted k-transition system is the system $\mathcal{T}[\rho]$ (alternatively a relational system with k+1 binary relations) and

$$bh_{\tau}(\rho) = \{x \mid \tau_X \quad \rho \neq \emptyset\}$$

We obtain 10 and 11 as special cases with $\rho = T^{\circ} \times T'$.

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