DEMAND DRIVEN EVALUATION
WITH EQUATIONS

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Demand Driven Evaluation with Equations

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1. Introduction

Term rewriting systems, such as The $\lambda$-calculus (Barendregt, 1981), combinatory logic (Curry and Feys, 1958), and the FP system (Backus, 1978), are fundamental to applicative programming. A recent trend is to permit a programmer to describe term rewriting systems using sets of first-order equations (Burstall, et al, 1980, Goguen and Tardo, 1979, Hoffmann and O'Donnell, 1982, Turner, 1976). These systems are the focus of this paper, and we refer to them as First-Order Equational Systems (FOES). FOES expand the expressive power of applicative programming in many ways. They can be used to directly define primitive functions on tree-like data-structures (Burstall, et al, 1980). Hoffmann and O'Donnell (1982) describe their use in succinct definitions of interpreters for languages such as LISP and LUCID (Ashcroft and Wadge, 1977). Their reliance on pattern matching in defining functions largely eliminates the need for the explicit use of selectors and predicates such as LISP's CAR, CDR and NULL, and greatly improves clarity and readability. FOES readily admit a many-sorted interpretation (Burstall, et al, 1980, Goguen and Tardo, 1979), and polymorphism (MacQueen, 1981, Milner, 1978), and hence a very sophisticated form of strict static type-
checking. A lazy interpretation of FOES extends the notion of computation with infinite structures to all tree-like structures, and indirectly permits applicative definitions of circular data structures such as doubly-linked lists, which can be viewed as cyclically infinite binary trees.

An examination of the operational semantics of existing term rewriting systems reveals that two approaches are possible. The evaluation of an expression in any rewriting system amounts to repeated application of the rewrite rules until an irreducible (normal) form is reached. Most expressions admit many alternative routes for the rewriting process, not all of which may terminate. Those that do terminate may not all reach the same normal form. Some systems determine the correct route, or computation rule, by decree. For instance, HOPE (Burstall, et al., 1980) uses innermost first evaluation except for a special variant of the cons constructor. FP (Backus, 1978) and OBJ (Goguen and Tardo, 1979) uniformly use the innermost first rule. The advantage in choosing a rule based on operational considerations is that efficiency and simplicity of implementation can be built into the choice. Often, restrictions on the permissible combinations of equations are also avoided.

The other commonly used approach is to treat the rewrite rules as the basis for a congruence relation and declare congruence classes to be the meaning of the terms they contain. The correctness or safeness of a computation rule
is judged by the requirement that it must reduce each term to its unique normal form equivalent if one exists. This more "mathematical" approach has been used for the \( \lambda \)-calculus, combinatory logic, and some FOES (see, e.g., Hoffmann and O'Donnell, 1982). Apart from the obvious appeal of semantic elegance, the mathematical approach coincides with lazy evaluation, which has been shown to be an attractive way to deal with many programming problems (Friedman and Wise, 1976, Henderson and Morris, 1976, Kahn and MacQueen, 1977). One clear limitation of the approach is that it can only be applied to systems which satisfy the confluence property (Huet, 1977). More troublesome is the need to find a safe and efficient computation rule for each system.

The standard example of a safe and optimal computation rule is the "normal order" rule for the \( \lambda \)-calculus (Vuillemin, 1974). The trouble with FOES is that they are meant to be user-defined, therefore it is impossible to look for an appropriate computation rule for each individual system. The ideal solution would be a rule that is safe and optimal for all confluent FOES. The "full substitution" rule of Vuillemin (1974) is safe but not optimal, since it requires all outermost redices in an expression to be reduced, including, in general, many unnecessary ones. One way to approach optimality is to look for a safe sequential rule, i.e., a rule that selects exactly one redex at each step. Such a strategy turns out to be optimal in number of
reduction steps in the so-called noncopying implementations, in which expressions are represented by directed acyclic graphs, and all residuals of a given subexpression share the same representation (O'Donnell, 1977, Vuillemin, 1974). Safe sequential evaluation is identical in spirit to the less specific notion of demand driven evaluation, and we use the two terms interchangeably throughout this paper. Unfortunately, not only is a safe sequential rule for all confluent FOES not known, it almost certainly does not exist, since taking account of the right-hand sides of equations amounts to unbounded look-ahead. The reported work on the subject therefore confines itself to considering sequentiality based only on the left-hand sides of equations (Hoffmann and O'Donnell, 1984, Huet and Levy, 1979).

Our goal is to extend the notion of demand driven evaluation to confluent FOES, i.e., to find a computation rule for FOES that is both safe and sequential, even when the right-hand sides of equations are disregarded. The most commonly studied class of confluent FOES is the class called regular by Hoffmann and O'Donnell (1984), and nonambiguous linear by Huet and Levy (1979). We shall adopt the former term. In this paper, we study a subclass of regular FOES that is practically important and substantially easier to deal with. The defining characteristic of members of this class is that constructors and defined functions are represented by disjoint sets of symbols. This restriction is used in several actual languages incorporating FOES.
(Burstall, et al, 1980, Subrahmanyan and You, 1983, Turner, 1976), and is analogous to the separation of function and predicate symbols in logic programming (Kowalski, 1979). We call this subclass constructor FOES to emphasize that argument patterns in equations of this class contain only constructor symbols and variables. It turns out that only a subset of the class of constructor FOES admits the kind of safe sequential rule we need. Along with the rule, we therefore need a decision procedure to identify members of this subset.

In Section 2 we informally describe our approach to the problem, Section 3 contains the mathematical results, in Section 4 we discuss possible applications and extensions of the results, Section 5 describes related work, and Section 6 contains the conclusions.

2. Informal Description

Let us start with an example to show that the leftmost outermost first rule is not safe for all FOES.

\begin{verbatim}
Example 1.
pair_int(X,nil) = nil
pair_int(X,A.L) = <X,A>.pair_int(X,L)
append(nil,L) = L
append(A.L1,L2) = A.append(L1,L2)
f = f
evaluate: pair_int(f,append(nil,nil))
\end{verbatim}

The syntactic convention in Example 1 and subsequent
examples is that identifiers beginning with upper case letters are variables and the rest are function symbols. The symbols cons is represented by the infix operator "." for brevity. It is clear that although both arguments of pair_int in the expression are redices, choosing the leftmost first leads to divergence, while choosing the rightmost one quickly leads to the normal form nil for the entire expression. The example suggests that, first, it is necessary to infer the strictness or otherwise of a defined function with respect to each of its arguments, and second, a function is nonstrict with respect to an argument when that argument is represented by a variable in some equation.

Actually, the example is too simple to bring out the full complexity of the situation. The distinction between divergent and nondivergent arguments is not sharp. For instance, the second argument of pair_int needs only as much constructor structure as is necessary to match one of the two patterns available for it, as Example 2 illustrates.

Example 2.

\[
\text{head(nil)} = \text{error}, \quad \text{head(A.L)} = A \\
\text{evaluate: head(pair_int(1,append(1.nil,f)))}
\]

Although an attempt to completely evaluate the second argument of pair_int in the expression would diverge, any outermost first strategy yields the normal form \(<1,1>\) for the entire expression. Example 3 shows that even this degree of strictness may be conditional.
Example 3.

\[
\begin{align*}
\text{pair\_list}(X, \text{nil}) &= \text{nil} \\
\text{pair\_list}(A, L1, B, L2) &= \langle A, B \rangle . \text{pair\_list}(L1, L2) \\
\text{pair\_list}(\text{nil}, B, L2) &= \langle -1, B \rangle . \text{pair\_list}(\text{nil}, L2) \\
\text{evaluate}: (1) \text{head}(\text{pair\_list}(f, \text{nil})) \\
& \quad (2) \text{head}(\text{pair\_list}(f, 1. \text{nil}))
\end{align*}
\]

Expression (1) leads to normal form "error" and (2) is divergent, although both contain exactly the same divergent subexpression in the same argument position. In other words, pair_list tolerates a divergent first argument only if the second argument is nil. Once the second argument reveals sufficient constructor structure to preclude this possibility, the structure of the first argument is needed to make further progress.

The outlines of the overall evaluation strategy we need have now emerged. As in any other demand driven strategy, applications of defined functions are evaluated from the outside inwards, as needed. Only the outermost applications are needed a priori. The reason for selecting any other application is that it stands in such an argument position for a needed outer application that its evaluated (constructor) structure must be made manifest in some degree, and the only way to reveal constructor structure is to reduce the application. In order to turn the application into a redex, we must decide which equation is applicable to it. We therefore choose an argument whose structure is
critical to that decision. When sufficient structure has been revealed to narrow the choice down as far as possible, we choose another argument to narrow it down still further, and so on, until a single equation remains. Note that we concern ourselves only with the left-hand sides of equations. The structure of these must be such that we are able to choose a "critical argument" at each step. If we cannot, the strategy fails. Fortunately, it turns out that in such a situation no strategy can succeed if it relies only on the structure of the left-hand sides.

This somewhat simplified description has glossed over two important details. First, in all our examples so far, the argument patterns have consisted of at most one constructor. The approach generalizes to deeper patterns with a little care, as illustrated by Example 4. The example uses two equations for dual cases of AVL tree rebalancing by single rotations (see Knuth, 1973, p.454).

Example 4.

\[
\text{rotate(node(Alpha,d_plus,node(Beta,plus,Gamma)))} = \text{node(node(Alpha,balance,Beta),balance,Gamma)}
\]
\[
\text{rotate(node(node(Alpha,minus,Beta),d_minus,Gamma))} = \text{node(Alpha,balance,node(Beta,balance,Gamma))}
\]

Let us ignore the real significance of the balance factors d_plus, etc., and consider an arbitrary application of rotate. After ensuring that the outermost constructor of the argument is "node", neither the left nor the right
subtree of the node can be safely evaluated since one of the equations has a "don't care" attitude in each case. The next critical structure is the balance factor for the node. If this is d_plus, then the second equation does not apply while the first may, and vice versa for d_minus.

The reader will have recognized that the selection process taking place here is commonly done manually in languages like LISP, where the outermost structure of an argument is first queried using a predicate such as NULL. The argument may then be taken apart using selectors, and the pieces further queried selectively, etc. It is instructive to rewrite the rotate function using conditionals, selectors, and predicates, in order to appreciate the gain in clarity brought about by using equations. The safe sequential computation rule we are seeking can be interpreted as an automatic version of the selection process based on the implicit logical necessity described by equations.

The second detail to note is that all functions may not be exhaustively defined, i.e., none of the equations may be applicable for certain combinations of arguments. In such an expression, which we call "root-stable" following Hoffmann and O'Donnell (1984), the outer function symbol is playing the role of a constructor. All applications immediately subordinate to a root-stable expression are therefore needed a priori, as in Example 5.

The outermost application of rotate cannot be dealt with
Example 5.

evaluate: rotate(node(head(e1),balance,rotate(e1,e2,e3)))

using either of the equations in Example 4, hence the applications of head and rotate immediately subordinate to it become "outermost". Another way to describe the situation is that the outermost rotate will be a part of any eventual normal form for the expression.

We conclude this section with two further examples.
Example 6 shows that all confluent FOES do not safely admit a sequential evaluation rule. The example uses the "parallel" logical or function provided as a primitive in certain applicative languages.

Example 6.

or(X,true) = true, \[ \text{or(true,X)} = \text{true} \]
or(false,false) = false

evaluate: or(g,h)

Case 1: g=true, h=h. Case 2: g=g, h=true.

It is impossible to choose either argument for evaluation. Choosing the first argument is unsafe if g and h are defined by the equations labeled Case 2, and the second one is unsafe if Case 1 is used. Finally, we would like to emphasize that FOES for which our approach fails may actually turn out to be sequential, if the right-hand sides are taken into account, as in the case of the "funny append" in Example 7:
Example 7.

\[
\text{append}(L,\text{nil}) = L, \quad \text{append}(\text{nil},L) = L \\
\text{append}(A.L1,L2) = A.\text{append}(L1,L2)
\]

Although the first equation appears to add generality, in fact it is redundant, since lack of structure for the first argument still means lack of structure for an application of append. Therefore, it is still safe to evaluate the first argument first.

3. Mathematical Results

3.1 Preliminaries and Definitions

3.1.1 Preliminaries

Most of the material in this section is a review of terminology and constructions commonly used in the literature on term rewriting systems.

A reduction system operates in a non-empty \textit{ranked alphabet} $\Sigma$ which contains all symbols in the system. $T_\Sigma$ denotes the set of all \textit{ground} $\Sigma$-terms for the alphabet. $\Sigma$-terms, in general, may contain \textit{variables} drawn from a countable set $X$. A $\Sigma$-term is said to be \textit{linear} iff no variable occurs more than once in it. We drop the prefix signifying the alphabet and speak of terms whenever possible without confusion.

A \textit{path} $p$ is a possibly empty string of integers. We say that $p$ \textit{reaches} subterm $t/p$ in term $t$. The empty string $\Lambda$ reaches the term itself, the string 'k' reaches the $k^{th}$ argument, 'km' reaches the $m^{th}$ argument of the $k^{th}$ argument,
etc. Given paths p and q, p.q denotes their concatenation. The symbols \( \leq \) and \( < \) denote the prefix and proper prefix relations on paths respectively. Paths p and q are said to be independent iff neither \( p \leq q \) nor \( q \leq p \). Paths\( (t) \) denotes the set of all paths that reach some subterm in \( t \). Paths\( (t) \) is partitioned into XPaths\( (t) \) and ΣPaths\( (t) \), where XPaths\( (t) \) is the subset that reaches variables in \( t \). We frequently refer to a path \( p \in Paths\( (t) \) \) as an occurrence of the subterm \( t/p \) in \( t \). The expression \( t[p\leftarrow w] \) denotes the term obtained by replacing \( t/p \) at \( p \) by \( w \).

A substitution is a map from variables to terms. The meaning of a substitution can be extended naturally to a map from terms to terms. The application of a substitution \( \alpha \) to a term \( t \) is conventionally denoted by \( t\alpha \), where \( t\alpha \) is the instance of \( t \) produced by simultaneously substituting \( \alpha(x) \) for every variable \( x \) in \( t \). We use the notation \( t\leq u \) to denote that \( u \) is an instance of \( t \); \( t\leq u \) means \( t\leq u \) and \( t\neq u \). If neither \( t\leq u \) nor \( u\leq t \) then \( t \) and \( u \) are said to be independent. The relation \( \leq \) is clearly a partial order.

The usual first-order unification algorithm of Robinson (1965) is denoted by UNIFY. If two terms \( t \) and \( u \) have no common instance, then UNIFY\( (t,u) \) fails, otherwise it succeeds and returns a substitution \( \alpha \) such that \( t\alpha = u\alpha \) is the least common instance of \( t \) and \( u \), i.e., in order-theoretic terminology, their join.

Since we shall consider safety of computations independent of the right-hand sides of equations, it is
natural for us to treat the collection of left-hand sides as an independent entity with its own properties. We refer to this collection as a base for a reduction system.

**Definition 1**: A base $L$ for a reduction system in the alphabet $\Sigma$ is a finite set \{l_i, m\geq i\geq 1\} of linear $\Sigma$-terms such that:

1. $i \neq j$ implies $l_i$ and $l_j$ are independent.
2. $\exists t \in T_{\Sigma}$ such that $l_i \neq t$, $m \geq i \geq 1$.
3. If $l_1, l_2 \in L$ and $p \in \SigmaPaths(l_1)$, then $\text{UNIFY}(l_1/p, l_2)$ fails unless $p = \Lambda$ and $l_1 = l_2$.

Each member of $L$ is called a (redex) pattern.

Condition (1) states that there are no redundant left-hand sides, (2) states that there are normal forms, and (3) states that there are no "critical pairs" (Knuth and Bendix, 1970), which is used to ensure that any reduction system based on $L$ will be confluent (Huet, 1977).

**Definition 2**: A reduction system $S$ in alphabet $\Sigma$ is a finite set \{<l_i, r_i>, m \geq i \geq 1\} of pairs of $\Sigma$-terms such that:

1. $L = \{l_i, m \geq i \geq 1\}$ is a valid base.
2. Each variable in $r_i$ appears also in $l_i$, $m \geq i \geq 1$.

Throughout the following, we shall deal with a fixed set $\Sigma$ of function symbols, a fixed base $L$, and a fixed reduction system $S$ based on $L$, unless mentioned otherwise. In many definitions, the entities being defined are qualified with the subscript $L$ or $S$ signifying the context. These subscripts are dropped whenever the base or system concerned
is the fixed one.

Any $u \in T_\Sigma$ such that $u \geq 1$, for some $1 \in L$, is called a redex. If $t/p$ is a redex for some $p \in \Sigma \text{Paths}(t)$, then $p$ is called a redex occurrence in $t$. The set of all redex occurrences in $t$ is denoted by $RO_L(t)$.

A simple reduction $t \xrightarrow{S} u$ occurs in $S$ iff $t/s=1 \alpha$ for some $1 \in L$, $\langle 1, r \rangle \in S$, and $u=t[s \cdot r \alpha]$. We write $t \rightarrow u$ iff $t \xrightarrow{S} u$ for some $s \in RO(t)$. Finally, $t \xrightarrow{*} u$ is called a reduction sequence where $\xrightarrow{*}$ is the reflexive transitive closure of $\rightarrow$. We use the notation $A: t \rightarrow u$ in order to attach a name (in this case $A$) to a simple reduction, and similarly also $B: t \xrightarrow{*} u$. $\lambda$ ambiguously denotes all empty reduction sequences. As with paths, we use the notation $A \cdot B$ to denote the concatenation of sequences $A$ and $B$. Such concatenation is meaningful only if the last term in $A$ is identical to the first term in $B$.

The set $NF_L \subseteq T_\Sigma$ is the set of normal forms, and $t \in NF_L$ iff $RO_L(t) = \phi$. If $t \xrightarrow{*} u$ in $S$ and $u \in NF_L$, then $t$ is said to normalize to $u$ in $S$, written as $u=\text{NORM}_S(t)$.

$P \subseteq \text{Paths}(t)$ is said to be an independent set of paths in $t$ iff all pairs of distinct paths in $P$ are independent. If $P \subseteq RO_L(t)$ then $P$ is called an independent set of redex occurrences, denoted by $P \in ISRO_L(t)$. The significance of independent sets of redices is that the order in which they are reduced is immaterial to the result obtained by reducing them all. This permits us to introduce the notion of a multireduction.
Definition 3: Let \( Q_{\epsilon} ISR_{\epsilon}(t) \), and let \( \{ q_1, \ldots, q_m \} \) be an arbitrary enumeration of \( Q \), where \( t/q_i = l_i \alpha_i, <1_i, r_i> \in S \), \( m \geq i \geq 1 \). Let \( t_1 = t \), and \( t_{i+1} = t_i[q_i+r_i \alpha_i] \), \( m \geq i \geq 1 \). We say that the multireduction \( t \xrightarrow{Q} t_{m+1} \) occurs in \( S \). The term \( t_{m+1} \) is clearly independent of the enumeration of \( Q \).

We treat a multireduction as a sequence of simple reductions whenever convenient. Although the notion of reduction sequence generalizes naturally to that of a multisequence, we do not need the generalization. In fact, our only serious use of multireductions occurs in Proposition 35, where it is necessary to deal with the forward migration of residual redices in rearranging a reduction sequence.

The notion of residuals, which is introduced next, is crucial in analysing the behavior of sequences of reductions. Residuals describe the way in which different parts of an expression are affected by a simple reduction. The left-hand side pattern in the redex is destroyed. Parts of the expression that are reached by extensions of the redex occurrence going beyond the pattern are rearranged according to the right-hand side. A possible rearrangement is disappearance, which is how outermost reductions sometimes make divergent inner expressions disappear. The rest of the expression is essentially unaffected.

Definition 4: Suppose \( A : t \xrightarrow{a} u \), where \( t/a = l \alpha, <1, r> \in S \). Let \( q_{\epsilon} Paths(t) \). The set \( q \setminus A \), called the residuals of \( q \) after \( A \), is defined as follows:
(1) if \( a \neq q \) then \( q \backslash A = \{ q \} \)

(2) if \( a = q \) then \( q \backslash A = \emptyset \)

(3) otherwise, let \( a = q.s. \) if \( s \in \Sigma\text{Paths}(1) \) then \( q \backslash A = \emptyset \).

Otherwise, \( \exists s' \in \chi\text{Paths}(1) \) such that \( s' < s \). Let \( s = s'.w, \) \( x = l/s' \), and \( R = \{ v \mid r/v = x \} \). Then \( q \backslash A = \{ a.v.w \mid v \in R \} \)

**Proposition 5:** If \( A:t \xrightarrow{R} t' \) and \( q \in \text{Paths}(t) \), then \( q \backslash A \) is an independent set of paths.

**Proof:** Obvious from the definition of \( q \backslash A \).

The following Proposition is a succinct statement of the reason for excluding "critical pairs" in Definition 1.

**Proposition 6:** \( q \in \text{RO}(t) \), \( A:t \rightarrow u \), implies that \( q \backslash A \in \text{RO}(u) \).

**Proof:** Straightforward by condition (3) in Definition 1.

Suppose \( A:t \rightarrow u \) and \( Q \subseteq \text{Paths}(t) \). Then the set \( Q \backslash A \) is simply \( u \{ q \backslash A \mid q \in Q \} \). This can be extended to sequences of reductions by composition, i.e.,

\[
Q \backslash A = Q
\]

\[
Q \backslash B.C = (Q \backslash B) \backslash C
\]

If \( q \in \text{Paths}(t) \), we use the notation \( q \backslash B \) instead of \( \{q\} \backslash B \).

**Proposition 7:** If \( P \in \text{ISRO}(t) \), and \( A:t \rightarrow t' \), then \( P \backslash A \in \text{ISRO}(t') \).

**Proof:** Simple consequence of Propositions 5 and 6.

We conclude this section by stating a fundamental property of our class of confluent FOES. This property is called "closure" by O'Donnell (1977), and the "parallel moves lemma" by Huet and Levy (1979). Another appropriate name would be the "one step confluence" property.
\textbf{Lemma 8}: Let $A : t \xrightarrow{P} u$ and $B : t \xrightarrow{Q} v$ be multireductions. Let $R = P \setminus B$, $S = Q \setminus A$, and $u \xrightarrow{S} w$. Then, $v \xrightarrow{R} w$.

\textbf{Proof}: By Proposition 7, $S \in \text{ISR}(u)$, and $R \in \text{ISR}(v)$. For the rest, see the proof of Lemma 11 in Huet (1977) or the proofs of Lemma 12 and Theorem 17 in O'Donnell (1977). 

3.1.2 Definitions

The first concept in need of a definition is the concept of sequential evaluation itself. We are interested in sequential evaluation because we believe it embodies the prerequisites for efficient evaluation strategies for FOES. As such, our definition excludes the possibility of using look-ahead and/or memory in choosing redices.

A \textit{(redex) selection} algorithm $A$ is a function of a base $L$ and a ground term $t$ such that, $A(L, t)$ either fails or returns some $g \in \text{RO}_L(t)$. If $L$ is fixed then the specialized or "curried" version of $A$ is denoted by $A_L$.

\textbf{Definition 9}: Given a selection algorithm $A$, and $t \in \Sigma$, the \textit{evaluation sequence} for $t$ produced by $A$ is the reduction sequence $t_0 \rightarrow \ldots \rightarrow t_n \rightarrow \ldots$ such that:

1. $t_0 = t$
2. if $A_L(t_n)$ fails then the sequence terminates in $t_n$, else $A(L)(t_n)$ returns $r$ and $t_n \xrightarrow{E} t_{n+1}$.

The sequence is denoted by $\sigma_{A,S}(t)$. If the sequence terminates in $t_n$ for some $n$, then we write $\sigma_{A,S}(t) = t_n$, else we write $\sigma_{A,S}(t)$. 

\textbf{Definition 10}: A selection algorithm $A$ is said to be \textit{safe} for a system $S$ based on $L$ iff given any $u \in \text{NF}_L$ and any $t$ such
that \( u = \text{NORM}_S(t) \), \( \sigma_A, S(t)^+ = u \). 

**Definition 11:** A base \( L \) is said to be **sequential** iff there is a selection algorithm \( A \) which is safe for any \( S \) based on \( L \). In this case, \( A \) is said to **sequentialize** \( L \). 

So far, we have been speaking of confluent FOES in general. We now introduce the class of constructor FOES which forms the subject of our study in this paper.

**Definition 12:** Suppose \( L = \{ l_i, \ m \geq i \geq 1 \} \). Let \( l_i = f_i(u_{i1}, \ldots, u_{in_i}) \), \( m \geq i \geq 1 \), \( \Gamma_L = \{ f_i, \ m \geq i \geq 1 \} \), and \( \Delta_L = \Sigma - \Gamma_L \). \( \Delta_L \) is called the set of **constructors** of \( L \). A term \( f(u_1, \ldots, u_m) \) is said to be a \( \Delta_L \)-**pattern** iff \( f \in \Gamma_L \) and each \( u_i \) is a \( \Delta_L \)-term, \( m \geq i \geq 1 \). \( L \) is said to be a **constructor base** iff all \( l_i \in L \) are \( \Delta_L \)-patterns. A system based on such an \( L \) is called a **constructor system**. 

Notice that, by definition, applications of constructors are always root-stable. In the following discussion, we assume that our fixed base \( L \) is a constructor base, and the alphabet \( \Sigma \) is accordingly partitioned into \( \Gamma \) and \( \Delta \), where \( \Delta \) is the set of constructors.

We now have a precise definition of our problem. We need

1. A decision procedure to identify sequential constructor bases, and
2. A redex selection algorithm ("computation rule") that sequentializes all sequential constructor bases.

We still need a precise formulation of the demand driven approach outlined in section 2. The critical notion is that
of a necessary redex, which is intuitively a redex that helps narrow down the set of equations compatible with the given term. The identification of compatible equations is based on the known structure of a given term, which is inferred using the fact that applications of constructors are root-stable.

Given a ground term $t$, $u = KNOWN(t)$ is the largest linear $\Delta$-term or $\Delta$-pattern such that $u \leq t$. Note that the set of all linear $\Delta$-terms or $\Delta$-patterns $\pi$ such that $\pi \leq t$ is nonempty and finite, and hence contains a largest member by the properties of $UNIFY$. The variables in $KNOWN(t)$ are assumed to be new. For a ground term $t$, define $L_t = \{l \in L \mid UNIFY(1, KNOWN(t)) \text{ succeeds} \}$, where $L_t$ is the set of equations compatible with $t$.

Let $PDNO(t) = Paths(KNOWN(t))$. Each $p \in PDNO(t)$ is said to be potentially directly necessary for $t$. If, in addition, $p \in Paths(1)$ for each $l \in L_t$, then $p$ is said to be directly necessary for $t$, denoted by $p \in PDNO_L(t)$. The notion of necessary redex occurrences essentially iterates directly necessary occurrences until a redex is reached. The exception is the situation noted in section 2, when root-stable outermost applications of defined functions must be taken into account. The structure of the definition below caters more to the needs of later proofs by structural induction than to intuitive transparency.

**Definition 13:** $p \in RO_L(t)$ is said to be necessary for $t$, written as $p \in NRO_L(t)$, iff one of the following holds:
(1) \( p = \Lambda \) or \( p \in DNO_L(t) \)
(2) \( q \in DNO_L(t) \), \( p = q \cdot r \), and \( r \in NRO_L(t/q) \)
(3) \( \exists q \in DNO_L(t) \) such that \( ROL(t/q) = \phi \), \( r \in PDNO(t) \), \( p = r \cdot s \), and \( s \in NRO_L(t/r) \)

*Example 8 illustrates these definitions.*

**Example 8.**

\[
L = \{ f(x,a,b), f(b,x,a), f(a,b,x) \}
\]

\[
t = f(a,f(b,a,f(a,a,b)),f(b,b,a))
\]

\[
u = t/2 = f(b,a,f(a,a,b))
\]

\[
\text{KNOWN}(t) = f(a,x1,y2) \quad \text{PDNO}(t) = \{2,3\}
\]

\[
DNO(t) = \{2\} \quad ROL(t) = \{2,3,3\}
\]

\[
\text{KNOWN}(u) = f(b,a,y3) \quad \text{PDNO}(u) = \{3\}
\]

\[
DNO(u) = \{3\} \quad ROL(u) = \{3\}
\]

\[
NRO(u) = \{3\} \quad NRO(t) = \{2,3\}
\]

Our whole approach rests on the assumption that a necessary redex can always be found in a reducible expression. This is clearly not the case for all FOES, and our conjecture is that only those FOES which satisfy this assumption are sequential in the sense of Definition 11. We say that the bases of such FOES are strictly sequential.

**Definition 14:** \( L \) is said to be strictly sequential iff whenever \( ROL(t) \) is nonempty, \( NRO_L(t) \) is also nonempty. *

It is clearly straightforward to write an algorithm that, given a strictly sequential base and a reducible expression, will always return a necessary redex occurrence.
It remains to show that:

(1) Strict sequentiaity is decidable, and
(2) Ordinary and strict sequentiaity coincide for constructor bases.

3.2 The Decidability of Strict Sequentiality

Our decision procedure Check is given in Algorithm 1. Check uses the notation that for each \( f \in \Sigma \), with arity \( k \),
\[
z^f = f(x_1, \ldots, x_k), \text{ all } x_i \in X \text{ are new.}
\]
\[
L^f = \{ l \in L \mid z^f \leq l \} \text{ and } p^f = \{ 1, \ldots, k \}.
\]

As the structure of Check indicates, strict sequentiality is a property of the group of equations defining each individual function. In fact, fcheck is simply an abstract version of a translator that transforms each such group into a single equation involving nested conditionals, selectors, and predicates. The path \( p \) chosen in each call of fcheck corresponds to the argument or part of argument whose structure needs to be queried next, given that the structure of the application of \( f \) discovered so far corresponds to \( z \). The argument \( z \) plays no part in the decision procedure. Its presence is solely a device to facilitate statements and proofs of the properties of Check. An example of a base rejected by Check is found in Example 9 below.

We first prove a few simple technical facts about Check. A call \( \text{fcheck}(L, P, z) \) is said to be legitimate iff it results from the call \( \text{Check}(L) \).

**Proposition 15:** In any legitimate call \( \text{fcheck}(L, P, z) \) the
Function Check(L)
Let \( s_f = f\text{check}(L_f, P_f, z_f) \)
Return the conjunction of \( s_f, f \in T \)

Function f\text{check}(L, P, z)
If \( |L| \leq 1 \) Then Return True
Else Let \( Q = \{ q \in P \mid q \notin XPaths(l), l \in L \} \)
If \( Q = \emptyset \) Then Return False
Else choose any \( p \in Q \)
partition \( L \) into \( L_c, c \in \Delta \)
where \( l \in L_c \) iff \( l/p \geq c(x_1, \ldots, x_k), k \geq 0, \)
and each \( x_i \) is a new variable
for each \( c \),
\( P_c = P - \{ p \} \cup \{ p \cdot j, 1 \leq j \leq k \} \)
\( z_c = z[p \cdot c(y_1, \ldots, y_k)] \)
where each \( y_i \) is a new variable
Return the conjunction of f\text{check}(L_c, P_c, z_c), c \in \Delta

Algorithm 1

following statements hold:
(1) \( P \subseteq XPaths(l) \) for each \( l \in L \).
(2) \( z \leq l \) for every \( l \in L \).
(3) for each \( l \in L-L, UNIFY(l, z) \) fails.
(4) \( P=XPaths(z) \).
(5) \( |L| > 1 \) implies \( P \neq \emptyset \).
(6) if f\text{check}(L', P', z') is also a legitimate call, then
either UNIFY(z,z') fails, or z and z' are not independent.

Proof: Assertion (5) follows from (2) and (4). The proofs of the rest proceed by induction on the length of the calling chain resulting in the call fcheck(L,P,z).

Basis: The length is 1, and the call is fcheck(L^f,P^f,z^f) for some f. All assertions except (6) are obviously true. For (6), it is sufficient to note that if a call fcheck(L_1,P_1,z_1) results eventually from a call fcheck(L_2,P_2,z_2), then z_1 > z_2, and also that UNIFY(z^f,z^g) fails whenever f ≠ g.

Induction: The inductive assumption is that the assertions hold for all calls with calling chains of length n≥1. Suppose the call fcheck(L,P,z) has a chain of length n+1. Clearly, the call results immediately from another call with chain length n, and all assertions hold for the latter by the inductive assumption. The inductive step is now straightforward for all assertions except (6). For (6), the same observations as in the basis case suffice. ■

Proposition 16: Check(L) terminates.

Proof: Let zlength(z) = ∫ length(p), p ∈ Paths(z). By (2) in Proposition 15, zlength(z) has a finite upper bound in any legitimate call fcheck(L,P,z) unless L = φ. Moreover, zlength(z_c) > zlength(z) by (4) in Proposition 15. Hence, each legitimate call fcheck(L,P,z) terminates, either because L = φ or because each chain of recursive calls issuing from it is bounded by the limit on the monotonic increasing
quantity zlength(\(z\)). ■

The reason Check guarantees strict sequentiality is that, whenever we have a term \(t\) that is neither irreducible nor a redex, the path \(p\) chosen in the call fcheck(\(L, P, z\)) with the largest \(z\) compatible with \(t\) reaches a directly necessary subterm of \(t\). This is the intuitive idea in the next Proposition.

**Proposition 17:** Check(\(L\)), \(t \not\in \text{NF}_L\), and \(\Lambda \not\in \text{RO}(t)\) implies \(\text{DN0}(t) \neq \phi\).

**Proof:** Suppose the antecedents hold. If \(L_t = \phi\), then condition (2) in the definition of \(\text{DN0}(t)\) is vacuous, and it is easy to see that in that case, \(\text{RO}(t) \neq \phi\) implies \(\text{DN0}(t) \neq \phi\).

Suppose \(L_t \neq \phi\). Let \(Z\) be the set of \(\Lambda\)-patterns \(u\) such that \(u \preceq t\) and fcheck(\(M, R, u\)) is a legitimate call for some \(M\) and \(R\). Since \(L_t \neq \phi\), \(Z\) is clearly nonempty, and also finite. By (6) in Proposition 15, \(Z\) is linearly ordered, and hence contains a maximal element \(z\), which occurs in some call fcheck(\(L, P, z\)).

Let \(Q = \{q \in P \mid q \in \text{XPaths}(1)\ \text{for any } l \in L\}\). By (3) in Proposition 15, \(L \geq L_t\). By the maximality of \(z\), \(Q \subseteq \text{XPaths}(\text{KNOWN}(t))\) = \(\text{PDNO}(t)\). By (1) in Proposition 15, and the above considerations, we have \(Q \subseteq \text{DN0}(t)\). It remains to show that \(Q \neq \phi\). If \(|L| > 1\) then since Check(\(L\)) we have \(Q \neq \phi\).

Since \(L_t \neq \phi\), the only other possibility is that \(L = \{1\}\). Since \(\Lambda \not\in \text{RO}(t)\), \(1 \not\in \text{KNOWN}(t)\). Since \(P \subseteq \text{Paths}(1)\) and \(P = \text{XPaths}(z)\) by Proposition 15, if \(Q = \phi\) then the only way \(z \leq 1\) is if \(z = 1\). However, by (2) in Proposition 15, \(z \leq 1\), hence \(Q = \phi\) implies
z=1 and l≤t, contradicting the assumption that λ∉RO(t). ■

It only remains to iterate this fact by structural induction.

**Lemma 18**: Check(L) implies L is strictly sequential.

**Proof**: We must show that whenever RO(t) is nonempty, NRO(t) is also nonempty. The proof proceeds by induction on the structure of t.

**Basis**: t∈T, where RO(t)=NRO(t)={λ}.

**Induction**: The inductive assumption is that the required property holds for all proper subterms of t. If t is a redex then λ∉NRO(t). Suppose it is not, and r∉RO(t). By Proposition 17 above we have DNO(t)≠φ. Suppose p∉DNO(t). If RO(t/p)≠φ then by the inductive assumption ∃q∉NRO(t/p), and p.q∉NRO(t). If RO(t/p)=φ then let s∈Paths(t) be the nonempty path such that r=s.w and s∉PDNO(t). By the inductive assumption NRO(t/s)≠φ, and therefore by (3) in Definition 13, NRO(t)≠φ. ■

We show that Check is a complete decision procedure by defining a function "strange" which constructs a counter example to prove that L is not strictly sequential, whenever Check(L) fails.

**Definition 19**: If a legitimate call fcheck(L,P,z) directly returns false, being unable to find a suitable p∈P, then strange(L,P,z) is a term defined as follows. Let L = {l₀,...,lₘ}, m≥1. Partition P into P₁,...,Pₘ such that Pᵢ ⊆ XPaths(lᵢ). This is possible by the failure condition in fcheck. Let Vᵢ={z/p | p∈Pᵢ}. By (4) in Proposition 15,
each $V_i \subseteq X$. Let $n(x) = (i+1) \mod (m+1)$. Define a substitution $\alpha$ such that:

$$\alpha(x) = \begin{cases} x \in V_i & \text{then a simple ground instance of } l_{n(x)} \\ \text{else } x & \end{cases}$$

where a simple ground instance of a term is an instance in which each variable is replaced by a normal form. Note that $\alpha$ is well-defined since $V_i$ are disjoint by the construction of $z$. strange($L,P,z$) \text{ def } z\alpha.

**Example 9.**

Let $L$ be as in Example 8. Clearly, $f$-check($L^f,P^f,z^f$) fails directly, and thus strange($L^f,P^f,z^f$) is well defined. We have,

$$P = P^f = \{1,2,3\}, \quad z = z^f = f(z0,z1,z2)$$

$$l_0 = f(x,a,b), \quad l_1 = f(b,x,a), \quad l_2 = f(a,b,x)$$

Therefore, $P_0 = \{1\}, \quad P_1 = \{2\}, \quad P_2 = \{3\}$

$$V_0 = \{z0\}, \quad V_1 = \{z1\}, \quad V_2 = \{z2\}$$

With straightforward choices for simple ground instances,

$$\text{strange}(L,P,z) = f(f(b,a,a),f(a,b,a),f(a,a,b)).$$

**Proposition 20:** Whenever $w = \text{strange}(L,P,z)$ is well-defined,

1. $w$ is a ground term.
2. $P = RO(w) \neq \emptyset$
3. $DNO(w) = \emptyset$
4. $NRO(w) = \emptyset$

**Proof:** (1) is a consequence of (4) in Proposition 15. The $P=RO(w)$ part of (2) merely asserts that $\forall x \in RO(w)$. To see this, note that since $w$ is well-defined $|L| > 1$. Moreover,
$\exists l \in L$ such that $l \leq z$, and by (2) in Proposition 15, $l = z$. This would imply, again by (2) in Proposition 15, that $L$ is not an independent set, contradicting (1) in Definition 1. $P \neq \emptyset$ follows from (5) in Proposition 15 since $|L| > 1$. By the construction of $w$, $\text{KNOWN}(w) = z$, hence by (2) and (3) in Proposition 15, $L = L_w$. We know that each $q \in \text{PDNO}(w) = \text{PATH}(z)$ reaches a variable in some $l \in L = L_w$ by the construction of $w$, hence $\text{DNNO}(w) = \emptyset$. Since $\text{PDNO}(w) = \text{PATH}(z) = P = \text{RO}(w)$ by (4) in Proposition 15 and (2) above, $\text{NRO}(w) = \emptyset$ is obvious. ■

Lemma 21: Check($L$) = false implies that $L$ is not strictly sequential.

Proof: Check($L$) = false implies that some legitimate call $f_{\text{check}}(L, P, z)$ fails directly, and therefore $w = \text{strange}(L, P, z)$ is well-defined. Assertions (2) and (4) in Proposition 20 then imply that $L$ is not strictly sequential. ■

This concludes the demonstration of the decidability of strict sequentiality.

Theorem 22: Check($L$) iff $L$ is strictly sequential, i.e., Check is a decision procedure for strict sequentiality of constructor bases.

Proof: Immediate consequence of Lemmas 18 and 21. ■

3.3 The Equivalence of Ordinary and Strict Sequentiality

The first part of this proof shows that every strictly sequential system is sequential. The proof proceeds in two steps. The first step is to show that all demand driven computations for normalizable terms are essentially the
same. The second step is to show that the normal form equivalent of each normalizable term can be obtained by some demand driven computation. Together, the two steps show that any demand driven computation obtains the normal form equivalent of a normalizable term.

We first need to establish certain important properties of necessary redex occurrences:

(1) A necessary redex is always an outermost redex (Proposition 25),

(2) Necessary redex occurrences persist as such until they are reduced (Lemma 29), and

(3) In a strictly sequential system, the last redex to be reduced before reaching normal form is always a necessary one (Lemma 31).

Note that (3) is really a corollary of (2). In proving (1) and (2), we start with properties of DNO, and extend them to NRO by structural induction.

**Proposition 23:** $\lambda \in R(t)$ implies $DNO(t) = \phi$.

**Proof:** $\lambda \in R(t)$ implies $KNOWNT = \{l_1\}$ for some $1 \in L_t$, therefore $PDNO(t) \cap \Sigma Paths(l_1)$ is clearly empty. ■

**Corollary 24:** $\lambda \in R(t)$ implies $NRO(t) = \{\lambda\}$. ■

**Proposition 25:** If $p \in NRO(t)$ and $s \in R(t)$ then $s \notin p$.

**Proof:** By induction on the structure of $t$. The basis case is that $t \in \Sigma$, which is trivial. For the inductive step, assume that the proposition holds for all proper subterms of $t$. There are three cases, according to Definition 13.

**Case 1:** $p = \lambda$ or $p \in DNO(t)$. Recall that if $p \in DNO(t)$ then by
Proposition 23, $\Lambda \not\in \text{RO}(t)$, hence $s \not\in \Lambda$. The rest follows from the fact that $p \in \text{PDNO}(t)$.

Case 2: $q \in \text{DNO}(t)$, $p = q \cdot r$, and $r \in \text{NRO}(t/q)$. We have $\Lambda \not\in \text{RO}(t)$ as before. By Corollary 24, if $q \in \text{RO}(t)$ then $p = q$. Therefore, $s \not\in \Lambda$ and $s \not\in q$. The rest follows by the inductive assumption.

Case 3: Similar to Case 2.

Corollary 26: $\text{NRO}(t) \epsilon \text{ISRO}(t)$.

Proposition 27: Suppose $p \in \text{DNO}(t)$, $A : t \rightarrow u$, and $p \not\models r$, then $p \setminus A = \{p\}$, and $p \in \text{DNO}(u)$.

Proof: Since $\text{DNO}(t) \not\models \phi$, by Proposition 23, $r \not\models \Lambda$. By the antecedents and the definition of DNO, $r \not\models p$, hence $p \setminus A = \{p\}$. Moreover, since $r \not\models \Lambda$, $L_u \subseteq L_t$, and $p \in \text{DNO}(u)$.

Proposition 28: $p \in \text{NRO}(t)$, $A : t \rightarrow^s u$, and $p \models s$ implies that $p \setminus A = \{p\}$ and $p \in \text{NRO}(u)$.

Proof: By Proposition 25, $s \not\models p$, therefore $p \setminus A = \{p\}$, and moreover, $s \not\models \Lambda$. $p \in \text{NRO}(u)$ is then straightforward by analysing the cases of Definition 13, using Proposition 27.

A sequence $B : t \rightarrow u$ is said to preserve $p \in \text{RO}(t)$ iff $p \setminus B = \{p\}$.

Lemma 29: Suppose $A : t \rightarrow u$ and $r \in \text{NRO}(t)$. Then either $A$ preserves $r$ and $r \in \text{NRO}(u)$, or $A = A_1 \cdot A_2 \cdot A_3$, where $A_1 : t \rightarrow v$ preserves $r$, $A_2 : v \rightarrow^r w$, and $A_3 : w \rightarrow^s u$.

Proof: By induction on $|A|$. The basis $|A| = 0$ is trivial. Suppose $|A| = n + 1$. Let $A = A_1 \cdot A_2$, where $A_1 : t \rightarrow^s t_1$ is the first reduction in the sequence. By Proposition 28, either $r = s$, or $r \setminus A_1 = \{r\}$ and $r \in \text{NRO}(t_1)$. The rest follows by the
inductive assumption. ■

**Corollary 30:** If \( q \in NRO(t) \), \( A : t \rightarrow^O u \) is a multireduction, and \( q \notin O \), then \( q^\downarrow A \{ q \} \) and \( q \in NRO(u) \). ■

**Lemma 31:** In a strictly sequential system, if \( A : t \rightarrow^r u \) and \( u \in NF_L \) then \( r \in NRO(t) \).

**Proof:** By strict sequentiality, \( NRO(t) \) is not empty. By Lemma 29, each member of \( NRO(t) \) is either reduced or preserved, and since \( u \) is irreducible, it must be the former. ■

A reduction sequence \( t_0 \rightarrow t_1 \rightarrow \ldots \rightarrow t_i \rightarrow \ldots \) is said to be **normal** iff each simple reduction is a necessary one, i.e., iff in each reduction \( t_i \rightarrow^S t_{i+1} \) in the sequence, \( s \in NRO(t_i) \). A is normal by convention. Normal sequences are simply the formal version of demand driven computations in our context. Using the properties of necessary sequences, we now establish the properties of normal sequences.

**Proposition 32:** If \( A : t \rightarrow^* u \) is normal, \( u \in NF_L \), \( r \in NRO(t) \), and \( t \rightarrow^r t' \), then there is a normal sequence \( B : t' \rightarrow^* u \) such that \( |B| = |A| - 1 \).

**Proof:** Since \( u \in NF_L \), \( A \) does not preserve \( r \). Therefore, by Lemma 29, \( A = A_1 . A_2 . A_3 \) such that \( A_1 : t \rightarrow^* v \) preserves \( r \), and \( A_2 : v \rightarrow^r w \). Proceed by induction on \( |A_1| \).

**Basis:** \( |A_1| = 0 \). Trivial.

**Induction:** Assume for \( |A_1| = n \). Suppose \( |A_1| = n + 1 \). Let \( A_1 = A_{11} . A_{12} \) where \( A_{11} : t \rightarrow^* v' \), \( A_{12} : v' \rightarrow^S v \). Clearly, \( |A_{11}| = n \). Since \( A_1 \) preserves \( r \), \( r \in NRO(v') \), and since \( A_1 \) is normal \( s \in NRO(v') \), therefore \( r \) and \( s \) are independent. The order of
reductions using $r$ and $s$ may therefore be reversed without affecting the result, i.e., $v' \xrightarrow{r} v'' \xrightarrow{s} w$. The rest follows by applying the inductive assumption to $A_{11}$. ■

**Proposition 33:** If $A : t \rightarrow^* u$ and $B : t \rightarrow^* w$ are normal sequences, $u \in NF_L$, and $|A| = |B|$, then $u = w$.

**Proof:** Straightforward by induction on $|A|$ using Proposition 32. ■

**Lemma 34:** If $A : t \rightarrow^* u$ and $B : t \rightarrow^* w$ are normal sequences in a strictly sequential system, and $u \in NF_L$, then there is a normal sequence $D = B . C$ such that $|D| = |A|$, and $D : t \rightarrow^* u$.

**Proof:** By Proposition 33, $|B| > |A|$ is impossible, and whenever $|B| < |A|$, $w \notin NF_L$, hence $NRO(w) \neq \emptyset$. $B$ can therefore be extended as a normal sequence to the same length as $A$. Let this extension be $C$, and $D = B . C$. By Proposition 33, $D : t \rightarrow^* u$. ■

Lemma 34 shows that all normal sequences for normalizable terms are essentially the same. The next Proposition contains our only real use of multireductions and the so-called parallel moves lemma (Lemma 8). The Proposition states that in a strictly sequential system, any sequence which starts with an arbitrary multireduction, and reaches normal form by a demand driven computation thereafter, can be rearranged to an entirely demand driven computation. The essential idea is to migrate the redices in the initial multireduction forward as residuals until they either become necessary or disappear. The parallel moves lemma guarantees that the migration can take place
without changing the final result. The upper bound on the length of terminating normal sequences is crucial in ensuring that the migration process terminates.

**Proposition 35:** In a strictly sequential system, if \( A_1 : t \xrightarrow{S} u \) is a multireduction, \( A_2 : u \xrightarrow{*} y \) is normal, and \( y \in NF_L \), then there is a normal sequence \( B : t \xrightarrow{*} y \).

**Proof:** The proof proceeds by induction on \( |A_2| \).

**Basis:** \( |A_2| = 0 \). In this case, since \( S \) is an independent set, and hence may be reduced in any order, Lemma 31 implies that \( S \subseteq NRO(t) \), and \( B = A_1 \).

**Induction:** Suppose \( |A_2| = n + 1 \). We may assume without loss of generality that \( S \cap NRO(t) = \emptyset \), since normal reductions in \( S \) could be included in \( A_2 \). By strict sequentiality, \( NRO(t) \neq \emptyset \). Let \( r \in NRO(t) \). By Corollary 30, \( r \setminus A_1 = \{r\} \), and \( r \in NRO(u) \). By Proposition 32, if \( A_3 : u \xrightarrow{R} u' \), there is \( A_4 : u' \xrightarrow{*} y \) such that \( A_4 \) is normal and \( |A_4| = n \). Let \( A_5 : t \xrightarrow{P} t' \), \( R = S \setminus A_5 \), \( P = R \cap NRO(t') \), and \( Q = R \setminus P \). By Lemma 8, \( t' \xrightarrow{R} u' \). Therefore \( t' \xrightarrow{P} t'' \) and \( A_6 : t'' \xrightarrow{Q} u' \). The rest follows by applying the inductive assumption to \( A_6 \) and \( A_4 \).

**Lemma 36:** In a strictly sequential constructor system, if \( u = NORM(t) \) then there is a normal sequence \( B : t \xrightarrow{*} u \).

**Proof:** Suppose \( A : t \xrightarrow{*} u \). If \( |A| = 0 \) then \( B = A \). Suppose \( |A| > 0 \). Let \( A_2 \) be the longest normal suffix of \( A \), and \( A = A_1 \cdot A_2 \). By Lemma 31, \( |A_2| > 0 \). Proceed by induction on \( |A_1| \).

**Basis:** \( |A_1| = 0 \), \( B = A_2 = A \).

**Induction:** Assume the result for \( |A_1| = n \). Suppose \( |A_1| = n + 1 \). Let \( A_1 = A_{11} \cdot A_{12} \), \( A_{11} : t \xrightarrow{*} v \), \( A_{12} : v \xrightarrow{R} w \). By Proposition 35,
there is a normal sequence $A_3 : v \rightarrow^* u$. Applying the inductive assumption to $A_{11}$ yields the rest.

$N$ is said to be a normal selection algorithm iff whenever $NRO_L(t) \neq \emptyset$, $N(L,t) \in NRO_L(t)$, and $N(L,t)$ fails otherwise.

**Lemma 37:** Any normal selection algorithm $N$ sequentializes all strictly sequential constructor bases.

**Proof:** Suppose $L$ is strictly sequential, $S$ is based on $L$ and $u = NORM_S(t)$. By Lemma 36, there is a normal sequence $B : t \rightarrow^* u$. From Lemma 34, it follows that $|\sigma_{N,S}(t)| = |B|$, and $\sigma_{N,S}(t) \uparrow = u$.

Finally, if the decision procedure for strict sequentiality rejects a base, that base is necessarily nonsequential. The function strange is again used in constructing the necessary counter example.

**Lemma 38:** If $L$ is a constructor base and $Check(L) =$ false, then $L$ is not sequential.

**Proof (sketch):** $Check(L) =$ false implies that $w = strange(L,P,z)$ is well-defined for some $L = \{l_0, \ldots, l_m\}$, $P = P_0 u \ldots u P_m$, and $z$, where $m \geq 1$. Now suppose $A$ sequentializes $L$. Clearly, $w$ can be normalized in some system based on $L$, hence $A(L,w)$ must succeed and return some path $q \in P$. Suppose $q \in P_i$. We now construct a system $S$ based on $L$ for which $A$ is not safe. Let $k = (i+1) \mod (m+1)$, and let $<l_k, l_k> \in S$. Since $t/q$ is an instance of $l_k$, $\sigma_{A,S}(w) \uparrow$. By the construction of $w$, $q \in XPaths(l_i)$. Let $<l_i, r> \in S$, where $r$ is any normal form. Clearly, $w$ can be made an instance of $l_i$ by suitably
"filling in" $S$ (including a suitable choice for $r$), since all instances of $l_k$ in $w$ are reached by paths that reach variables in $l_i$, and the right-hand sides corresponding to left-hand sides other than $l_i$ and $l_k$ are unconstrained given (3) in Definition 1. Thus, $r = \text{NORM}_S(w)$, and for $A$ to be safe for $S$, we must have $\sigma_{A,S}(w) \vdash r$. Example 10 illustrates the construction suggested in the proof above.

**Example 10.**

Let $w = \text{strange}(L,P,z)$ as constructed in Example 9. 

$R0(w) = \{1,2,3\}$. Suppose $A_L(w) = 1 \epsilon P_0$. Then $i = 0$ and $k = 1$.

The required counterexample $S$ is:

$f(X,a,b) = b$, $f(b,X,a) = f(b,X,a)$, $f(a,b,X) = a$

**Theorem 39:** A constructor base is sequential iff it is strictly sequential.

**Proof:** Straightforward consequence of Theorem 22, and Lemmas 37 and 38. ■

**Corollary 40:** Check is a decision procedure for the sequentiality of constructor bases. ■

**Corollary 41:** Any normal selection algorithm sequentializes all sequential constructor bases. ■

4. Applications

The obvious application of the results of the previous section is an efficient evaluator for sequential FOES. A first effort in that direction is the (nondeterministic) normal selection algorithm Select given in Algorithm 2.

Although Select can be used to produce normal sequences
Function `Select(L,t)`

Return `get(φ,∪εL,t)`

Function `get(L,P,t)`

Let \( Q = \{ q ∈ P \mid q ∈ \text{Paths}(l), l ∈ L \} \)

If \( Q = φ \) Then Fail, Else Choose any \( p ∈ Q \)

Let \( u = t/p \)

If \( u \) is a redex then return \( p \)

Else If \( u = g(u_1,\ldots,u_k), k ≥ 0, g ∈ Γ \)

Then Let \( q = \text{get}(L^g,P^g,u) \)

If \( \text{get} \) is successful Then Return \( p.q \)

Else Return \( \text{get}(φ,P−{p},t) \)

Else Let \( u = c(u_1,\ldots,u_k), k ≥ 0, c ∈ Δ \)

Return \( \text{get}(L_c,P_c,t) \)

where \( L_c, P_c \) are defined as in fccheck

**Algorithm 2**

which are optimal in the abstract world where only the reduction steps in a noncopying reducer count, its actual use is obviously very expensive since the cost of finding a necessary redex far outweighs the cost of the reduction step. The search cost can be reduced substantially if the search does not begin at the root of the entire expression each time. Instead, once a directly necessary subexpression is identified, it should be worked on until it is reduced to a root-stable form. Such a strategy reduces search cost by
increasingly localizing the search for the next redex. The function Evaluate in Algorithm 3 realizes this modification. Note that Evaluate is not a selection algorithm, it is an actual evaluator that returns the normal form equivalent of the given expression. The sequential system involved is S.

Algorithm 3

Function Evaluate(t)

Let \( w = \text{Root\_Stabilize}(t) \)
Let \( Q = PDNO(w) \)

For \( q \in Q \) Do

\[ w := w[q\_\text{Evaluate}(w/q)] \]

Return \( w \)

Function \( \text{Spin}(L,P,t) \)

Let \( Q = \{ q \in P \mid q \not\in \text{XPath}\_\alpha(l), l \in L \} \)

If \( Q = \emptyset \) Then \( t = 1\alpha, \langle l, r \rangle \in S \) (see Proposition 17)

Return \( \text{Root\_Stabilize}(r\alpha) \)

Else Choose some \( q \in Q \)

Let \( w = \text{Root\_Stabilize}(t/q) \)

Let \( v = t[q\_w] \)

If \( c \in T \) Then Return \( v \)

Else Return \( \text{Spin}(L_C, P_C, v) \)

where \( L_C, P_C \) are defined as in fcheck
The nondeterministic choice of an occurrence from set $Q$ in both Root_Stabilize and Evaluate suggests the possibility of safe parallel evaluation. Safe parallelism is usually associated with the notion of strictness. Conventionally, the strictness of a function with respect to an argument is understood to mean that it is safe to reduce that argument to normal form before applying the function. In our case, a finer-grained notion of strictness is appropriate. A function is strict with respect to an argument in this sense if it is safe to reduce that argument to root-stable form before attempting to apply the function. This notion of strictness is applicable to parts of arguments as well. We believe that our notion of strictness and the parallelism it offers may be practically significant, since root-stabilization is clearly a much more substantial operation than single step reduction, coinciding with complete evaluation in the case of scalar types.

Other possible applications include translation of groups of equations to a single logically equivalent one, in preparation for optimization as described by Hudak (1984), and safe translation of lazy equational programs to logic programming languages such as PROLOG (Subrahmanyan and You, 1983). Our results can also be used to derive a complete "semantic unification algorithm" for use in integrating logic and functional programming as suggested by Subrahmanyan and You (1984).

Although the formal results presented in section 3 are
applicable to constructor FOES only, they can be extended to cover all (regular) bases, using the simulation technique described by Thatte (1984). The basic idea of the technique is to treat the outermost function symbol in every root-stable term as a constructor. It is clearly possible to extend our Evaluate function to accommodate this change.

5. Related Work

Safe and optimal computations with recursive equations have been extensively investigated (Vuillemin, 1974, Berry and Levy, 1979, Cadiou, 1972, Wadsworth, 1971). Confluence and other properties of FOES are discussed by Huet (1977), O'Donnell (1977), and Rosen (1973). Much of this work has been influenced by the seminal study of confluence by Knuth and Bendix (1970). The issue of sequential evaluation with confluent FOES has been addressed by Hoffmann and O'Donnell (1979, 1984), and by Huet and Levy (1979). Hoffmann and O'Donnell's methods are applicable to a restricted class of FOES for which a preorder examination of redices without backtracking is safe. Huet and Levy have independently arrived at results similar to ours for regular FOES using a much more abstract approach. In contrast to their work, we have chosen to tackle the simpler problem of constructor FOES using a direct constructive approach which we feel is more easily amenable to efficient implementation.

6. Conclusions

We have presented abstract algorithms for checking sequentiality, and for safe sequential, i.e., demand driven
evaluation of expressions, in the context of constructor FOES. Clearly, much work remains to be done on methods for efficient implementation before the approach becomes a serious practical alternative. We believe that a suitable concrete implementation of sequential FOES must also be amenable to smooth integration with an implementation of a language with higher-order functions, since the two paradigms bring complementary strengths to applicative programming. We are currently investigating the possibility of using a generalization of the implementation technique based on combinators due to Turner (1979).

More work is also needed to explore the applications discussed in Section 4, especially the possibility of using our refined notion of strictness in parallel evaluation. Finally, our results need to be extended to cover all regular FOES.

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References


