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COLLEGE OF ENGINEERING
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Tire and Suspension Systems Research Group

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PLANE VIBRATION CHARACTERISTICS OF A PNEUMATIC TIRE MODEL

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NOMENCLATURE

ENGLISH LETTERS

a	-	Shell undeformed radius
b	-	Shell width
E	-	Shell modulus of elasticity
e	-	Shell extensional strain
h	-	Shell thickness
H, K	-	Coefficients
I_o	-	A time integral
I	-	Cross-sectional moment of inertia
k	-	Shell foundation modulus
p_o	-	Internal pressure
p	-	Circular frequency
$\bar{p}$	-	Oscillatory frequency
R	-	Radius vector from a fixed point to origin of moving co-ordinate system
r	-	Radius
s	-	Wave number
T	-	Kinetic energy
V	-	Potential energy
v, w	-	Tangential and radial shell displacements, respectively
W	-	Work done

NOMENCLATURE (Concluded)

GREEK LETTERS

α, β	-	General symbols for angles
ϵ	-	Moving co-ordinate directions
θ	-	Angular position on shell
ψ	-	v/a, a dimensionless variable
ϕ	-	$\theta + \psi$
Ω	-	Shell angular velocity
$\bar{\omega}$	-	Angular velocity vector of the origin of a moving co-ordinate system

SUBSCRIPTS

Subscripts indicate differentation with respect to the subscripted variable.

I. ABSTRACT

An elastically supported cylindrical shell is used to represent the motion of a pneumatic tire in the plane of the wheel. This is an attempt to utilize shell motion as an analog to the plane motion of the pneumatic tire tread. This idea is suggested by the constructional features of a pneumatic tire, both from the point of view of mass distribution and the distribution of elastic stiffness.

The equations of motion for such a model are derived by reference to conventional energy methods. In this derivation, the influence of internal pressure and elastic support of the shell is taken into account. The frequencies are determined as functions of the mode shape, and it is shown that nodes, as well as antinodes, rotate with an angular velocity somewhat less than the angular velocity of the rotating pneumatic tire, to an extent determined by the particular mode shape in question. It is hoped that this phenomena may be useful in explaining or understanding some vibratory or acoustic interactions between a tire and suspension system.

II. FOREWORD

The problem of the vibration of a two dimensional cylindrical shell has been treated by Rayleigh, Ref. 1. A study of this type of shell under rotation has been published by Bryan, Ref. 2. Both of these earlier papers contain considerable information which may be used directly for the study of this problem, and a great deal of the background material in this paper is drawn from these sources.

In this particular paper, an attempt is made to utilize the circular cylindrical shell supported on an elastic foundation as a model for the motion of the tread of a pneumatic tire in the plane of the wheel. Such a shell model is shown in Figure 1.

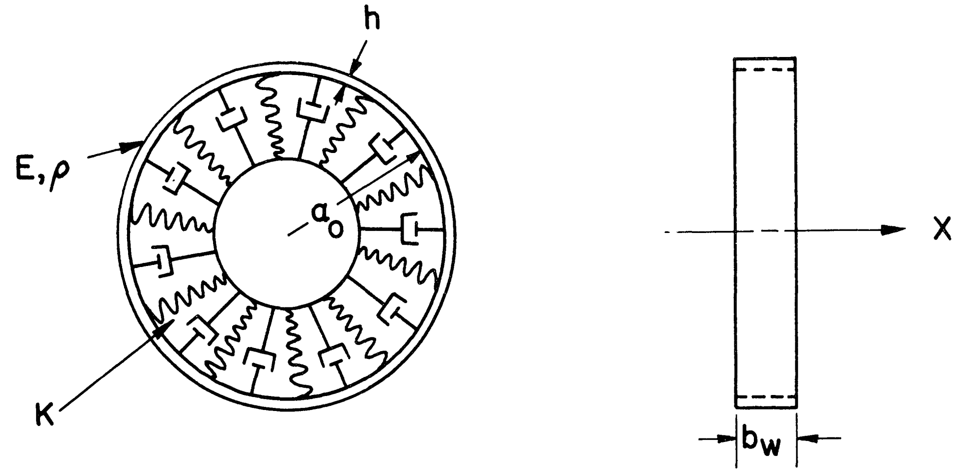

Figure 1. Basic cylindrical shell model for a pneumatic tire.

Previous studies have shown that such a model may be used for the prediction of contact patch lengths (Ref. 3), and of loads carried by a real pneumatic tire (Ref. 4). Thus, there is some reason to believe that such a cylindrical shell model might be used to represent motion of the tread of a tire in the plane of the wheel.

It should be pointed out here that this represents one path for the idealization of the problem of the vibration of a pneumatic tire. In effect, this point of view says that we will consider the tread to be a ring, or shell, elastically supported and capable of motion in its plane. An alternate approach might be to attempt to study the vibration characteristics of a isotropic uniform torus, and eventually to generalize this to a problem involving arbitrary mass and stiffness distribution similar to a pneumatic tire. There is little question but what this second approach would in the long run prove more fruitful. The justification for the present method lies in the fact that observations seem to indicate that a great many pneumatic tire problems can be attributed to motion in the plane of the wheel. The particular mass and stiffness distribution associated with the usual pneumatic tire, and in particular with the radial or belted tire, causes such a model as shown in Figure 1 to be a reasonably good approximation to the true structure. Since cylindrical shells have been studied extensively, it seems reasonable to attempt to utilize these results in gaining some general insight into the form of the vibratory motion of such a model.

III. KINEMATICS OF MOTION

In order to define the displacements accurately, we consider a rotating cylindrical shell such as shown in Figure 2.

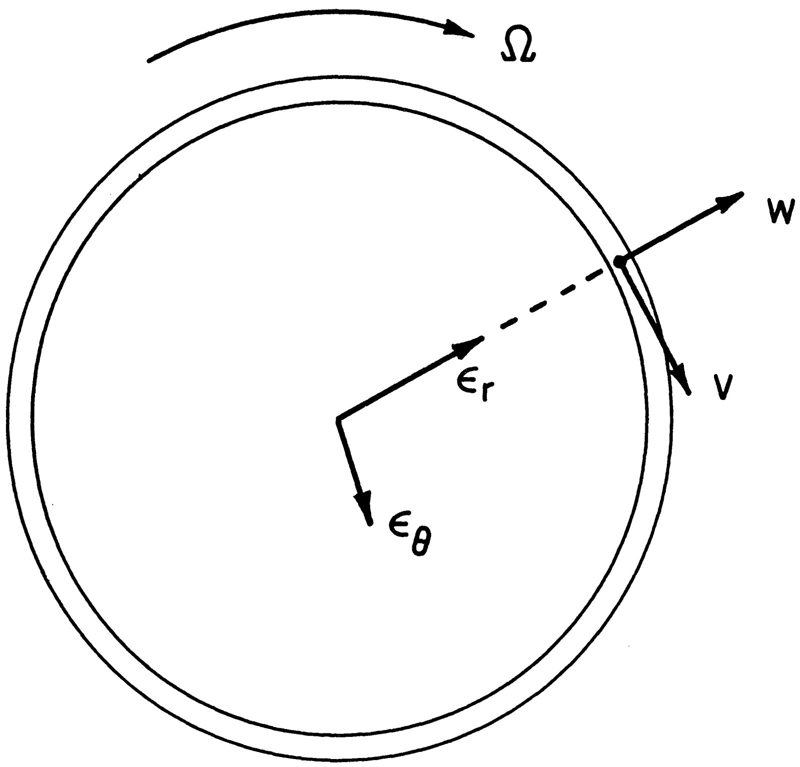

Figure 2. Coordinate nomenclature.

We will first assume that all motion is independent of dimension z along the width of the shell. Thus, no quantity will depend upon the dimension z nor will any motion in the z direction be considered.

A moving coordinate system, designated by the unit vectors ϵ_r, ϵ_ϕ, ϵ_z, is attached to the shell at its axis. The shell is allowed to rotate with angular velocity Ω. A fixed coordinate system has its origin at point O, the center of rotation of the shell, as shown in Figure 2. Displacements w in the radial direction and v in the tangential direction are presumed to take place away from the equilibrium position of the shell. Thus, these quantities are small elastic displacements. The general radius vector to any point on the midline surface of the shell in the deformed condition is given by Eq. (1).

$$\bar{\rho} = (a + w)\,\bar{\epsilon}_r + v\bar{\epsilon}_\phi \tag{1}$$

In Eq. (1), the radius vector $\bar{\rho}$ represents the total displacement vector of a point on the shell neutral surface as measured in the rotating coordinate system using the unit vector directions ϵ_r and ϵ_ϕ. By conventional methods, the radius vector in a fixed, or inertial, reference frame is given by

$$\ddot{\bar{r}} = \ddot{\bar{R}} + \ddot{\bar{\rho}} + 2\bar{\omega} \times \dot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) \tag{2}$$

where

$\omega = \Omega_z$.

This leads immediately to expressions for the velocity and acceleration of this material particle in the form

$$\dot{\bar{r}} = [\dot{w} - \Omega v]\bar{\epsilon}_r + [\dot{v} + \Omega(a + w)]\bar{\epsilon}_\phi \tag{3}$$

$$\ddot{\bar{r}} = [\ddot{w} - \Omega^2(a + w) - 2\Omega\dot{v}]\bar{\epsilon}_r + [\ddot{v} - \Omega^2 v + 2\Omega\dot{w}]\bar{\epsilon}_\phi \tag{4}$$

Equation (3) is of considerable use since it gives the absolute velocity vector of a particle on the rotating cylindrical shell, this particle moving with velocities $\dot{w}$ and $\dot{v}$ with respect to the original shell configuration. This may be immediately used to generate the total kinetic energy of the shell in the form

$$T = \frac{1}{2} \rho bha \int_0^{2\pi} \{[\dot{w} - \Omega v]^2 + [\dot{v} + \Omega(a + w)]^2\} \, d\theta \tag{5}$$

and by defining the quantity ψ in the form

$$v = a\psi \tag{6}$$

then one may expand Eq. (5) and write

$$T = \frac{1}{2} \rho bha \int_0^{2\pi} \{ w_t^2 - 2a\Omega\psi w_t + \Omega^2 a^2 \psi^2 + a^2 \psi_t^2 + 2a^2 \Omega\psi_t + 2a\Omega w\psi_t + \Omega^2 a^2 + 2\Omega^2 aw + \Omega^2 w^2\} \, d\theta \tag{7}$$

IV. STRESS STRAIN GEOMETRY AND POTENTIAL ENERGY

First order approximations for the potential energy stored in the shell due to bending and membrane effects are well known. For example, the potential energy storage due to bending may be obtained by considering the integral of the form

$$V = \frac{1}{2} EIa \int_0^{2\pi} (\delta \frac{1}{\rho})^2 \, d\theta \tag{8}$$

and this has been shown by Bryan to be expressible in the form

$$V_B = \frac{1}{2} \frac{EI}{a^3} \int_0^{2\pi} [w^2 + 2w \frac{\partial^2 w}{\partial \theta^2} + (\frac{\partial^2 w}{\partial \theta^2})^2] \, d\theta \tag{9}$$

The extensional strain in a cylindrical shell is known to be of the form

$$e = \frac{1}{a} (w + \frac{\partial v}{\partial \theta}) \tag{10}$$

This immediately allows the strain energy of extension to be written in the form

$$V_E = \frac{1}{2} Ebha \int_0^{2\pi} [(\frac{\partial \psi}{\partial \theta})^2 + 2 \frac{w}{a} \frac{\partial \psi}{\partial \theta} + (\frac{w}{a})^2] \, d\theta \tag{11}$$

The basic model considered here uses an elastic foundation for the cylindrical shell in order to approximate the support of the preloaded side walls of a real pneumatic tire. Here, for simplicity of analytical manipulation, a purely elastic distributed foundation is introduced whose spring rate is given as k units of pressure per unit deflection. Accordingly, arbitrary displacements w around the circumference of the shell store potential energy in this distributed elastic foundation in the form

$$V_s = \int_0^{2\pi} \frac{1}{2} kw^2 ba \, d\theta \tag{12}$$

Finally, the work done by internal pressure should be included since the shell in question may be inflated. Considering the internal pressure to be positive outward, the work done by the pressure forces may be integrated in the form

$$W_p = P_o\left(\frac{1}{2} b \int_0^{2\pi} r^2 d\phi - \pi a^2 b\right) .$$

A point on the deformed midsurface is located by $r = a + w$ and $\phi = \psi + \theta$. Writing $d\phi = (\psi_\theta + 1)\, d\theta$, the work integral consistent to second order terms becomes

$$W_p = \frac{1}{2} bP_o \int_0^{2\pi} (a^2\psi_\theta + 2aw\psi_\theta + 2aw + w^2) \, d\theta \tag{13}$$

V. EQUATIONS OF MOTION

The various integrations forming kinetic, potential energy and work terms may be combined to form the total Lagrangian density of the system in the form

$$L = \frac{1}{2} \rho bha \, [w_t{}^2 - 2a\Omega\psi w_t + \Omega^2 a^2 \psi^2 + a^2 \psi_t{}^2 + 2a^2 \Omega \psi_t + 2a\Omega w \psi_t + \Omega^2 a^2$$

$$+ 2\Omega^2 aw + \Omega^2 w^2] - \frac{EI}{2a^3} [w^2 + 2ww_{\theta\theta} + w_{\theta\theta}^2] - \frac{Ebha}{2} [\psi_\theta{}^2 + \frac{2}{a} w\psi_\theta \qquad (14)$$

$$+ \frac{1}{a^2} w^2] - \frac{b}{2} kaw^2 + \frac{1}{2} p_o b \, [a^2 \psi_\theta + 2aw\psi_\theta + 2aw + w^2]$$

We now desire to minimize this functional according to Hamilton's principle. Specifically, we desire to minimize

$$I_o = \int_{t_1}^{t_2} L \, dt \qquad (15)$$

It may be shown by conventional means that the appropriate Euler-Lagrange differential equations for this case of two dependent variables, w and ψ, expressed in terms of two independent variables Θ and t, are

$$L,_w - \frac{\partial}{\partial t} L,_{w_t} + \frac{\partial^2}{\partial \theta^2} L,_{w_{\theta\theta}} = 0 \, , \qquad (16)$$

$$L,_\psi - \frac{\partial}{\partial t} L,_{\psi_t} - \frac{\partial}{\partial \theta} L,_{\psi_\theta} = 0 \quad , \qquad (17)$$

Applying these Euler-Lagrange equations to the Lagrangian functional, one obtains two equations of motion for the cylindrical shell.

$$\ddot{w} - \Omega[2a\,\dot{\psi} + \Omega(w + a)] + \frac{Eh^2}{12\rho a^4}(w^{IV} + 2w'' + w) + \frac{E}{\rho a}(\psi' + \frac{w}{a})$$

$$- \frac{p_o}{\rho h}(\psi' + \frac{w}{a} + 1) - \frac{kw}{\rho h} = 0 \qquad (18)$$

$$\ddot{\psi} - \Omega(\Omega\psi - \frac{2}{a}\dot{w}) - \frac{E}{\rho a^2}(\psi'' + \frac{1}{a}w') + \frac{p_o}{\rho h a^2}w' = 0 \qquad (19)$$

Rather than dealing with both of the equations of motion just developed, an attempt will be made here to simplify the development using the concept that the outer elastic band of the cylindrical shell model is nearly inextensible. Equation (19) may be written in the form

$$\frac{E}{\rho a}(\psi'' + \frac{1}{a}w')' = a\ddot{\psi}' - \Omega(a\Omega\psi' - 2\dot{w}') + \frac{p_o}{\rho h a}w'' \qquad (20)$$

which is obtained by one more differentiation of Eq. (19) with respect to θ. Differentiating Eq. (18) twice with respect to θ and inserting Eq. (20) results in

$$\ddot{w}'' - \Omega[2(a\dot{\psi}'' - \dot{w}') + \Omega(a\psi' + w'')] + \frac{Eh^2}{12\rho a^4}[w^{VI} + 2w^{IV} + w'']$$

$$+ a\ddot{\psi}' - \frac{p_o}{\rho h}\psi''' - \frac{kw}{\rho h} = 0 \qquad (21)$$

We now draw upon the assumption of inextensibility. It would be possible to carry on a somewhat lengthy argument concerning the true degree of extensibility experienced in a real pneumatic tire tread, but as a first approximation it appears possible to consider many modes of motion as primarily bending in form. This would be particularly true of radial or belted tires. Even in the conventional bias ply tire, the distribution of cord angle is such that the primary longitudinal stiffness occurs in the tread region indicating that this is an area of high circumferential stiffness. For these reasons, it is felt that this assumption is warranted as an approximation of some validity to the real problem.

Mathematically, this may be accomplished by requiring that the extentional strain defined earlier vanish, yielding the expression

$$a\psi' = -w \tag{22}$$

from which one may obtain by direct differentiation

$$a\ddot{\psi}' = -\ddot{w} \tag{23}$$

$$a\dot{\psi}'' = -\dot{w}' \tag{24}$$

These latter equations may be substituted into Eq. (21) from which one may finally obtain an equation of motion involving the radial displacement w only in the form

$$\ddot{w} - \ddot{w}'' - 4\Omega\dot{w}' + \Omega^2(w'' - w) - \frac{Eh^2}{12\rho a^4}(w^{VI} + 2w^{IV} + w'') - \left(\frac{p_o}{\rho h a} + \frac{k}{\rho h}\right)w'' = 0 \tag{25}$$

VI. OSCILLATORY MOTION

Oscillatory motion may be studied by use of Eq. (25). For simplicity, one may denote two of the coefficients occurring in Eq. (25) by the symbols H and K in the form

$$H = \frac{p_o}{\rho h a} + \frac{k}{\rho h}, \quad K = \frac{Eh^2}{12\rho a^4} \tag{26}$$

and assume that the radial motion w is harmonic in the form

$$w = A \sin(s\theta + pt) \tag{27}$$

Substituting this into Eq. (25), one obtains

$$- p^2 - s^2p^2 + 4\Omega sp - \Omega^2(s^2 + 1) - K(- s^6 + 2s^4 - s^2) + Hs^2 = 0 \tag{28}$$

from which one may solve for a frequency in the form

$$\left(p - \frac{2\Omega s}{s^2 + 1}\right)^2 = \frac{4\Omega^2 s}{(s^2 + 1)^2} + \frac{Ks^2(s^2 - 1)^2}{s^2 + 1} + \frac{Hs^2}{s^2 + 1} - \Omega^2 \tag{29}$$

or

$$\overline{p^2} = \frac{4\Omega^2 s^2}{(s^2 + 1)^2} + K \frac{s^2(s^2 - 1)^2}{s^2 + 1} + H \frac{s^2}{s^2 + 1} - \Omega^2 \tag{30}$$

where the vibration frequencies are of the form

$$p = \frac{2\Omega s}{s^2 + 1} \pm \overline{p} \tag{31}$$

From these solutions one sees that two possible frequencies are available, one involving the use of the positive sign of Eq. (31), and the other the use of the negative sign. These are given as Eqs. (32).

$$w_1 = A \sin[s\theta + \frac{2\Omega st}{s^2 + 1} + \bar{p}t] \tag{32a}$$

$$w_2 = A \sin[s\theta + \frac{2\Omega st}{s^2 + 1} - \bar{p}t] \tag{32b}$$

Following Bryan (Ref. 2) we will refer this motion back to a stationary coordinate system. This may be done by letting the angle θ represent the difference between a position coordinate and the rotating angular position, in the form

$$\theta = \phi - \Omega t \tag{33}$$

where ϕ is measured from a fixed line in space. Using Eq. (33) in Eqs. (32a) and (32b), one obtains expressions for the radial displacement w in terms of position coordinates measured from a line fixed in space, and this gives

$$w_1 = A \sin[s\phi - \frac{(s^2 - 1)}{(s^2 + 1)} s\Omega t + \bar{p}t] \tag{34a}$$

$$w_2 = A \sin[s\phi - \frac{(s^2 - 1)}{(s^2 + 1)} s\Omega t - \bar{p}t] \tag{34b}$$

One method of studying the seemingly dissimilar motions given in Eqs. (34a) and (34b) is to consider the consequences of both occurring with equal amplitude. If this is the case, then Eqs. (34) take the form of

$$\begin{aligned} w_1 &= A \sin(\alpha + \beta) \\ w_2 &= A \sin(\alpha - \beta) \end{aligned} \tag{35}$$

where

$$\alpha = s[\phi - \frac{(s^2 - 1)}{(s^2 + 1)} \Omega t] \qquad (36)$$

$$\beta = \overline{p}t \qquad (37)$$

The result of adding algebraically the motion generated by w_1 and w_2 separately is given by

$$w = A[\sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta] \qquad (38)$$

This may be immediately interpreted as

$$w(\phi,t) = 2A \sin s[\phi - \frac{(s^2 - 1)}{(s^2 + 1)} \Omega t] \cos \overline{p}t \qquad (39)$$

where the frequency is given by

$$\overline{p}^2 = \frac{Eh^2}{12\rho a^4} \frac{s^2(s^2 - 1)^2}{(s^2 + 1)} + \frac{1}{\rho h} (\frac{p_o}{a} + k) \frac{s^2}{(s^2 + 1)} - \Omega^2(\frac{s^2 - 1}{s^2 + 1})^2 \qquad (40)$$

VII. DISCUSSION OF RESULTS

Equations (39) and (40) give expressions for the radial motion and frequency of oscillatory movement of the cylindrical shell model representing the tread of a pneumatic tire. Equation (40) represents perhaps the simplest ideas conceptually and will be discussed first.

In commenting upon Eq. (40), it may be seen that it represents the square of the circular frequency, and is proportional to the usual bending stiffness Eh^3 which enters into most bending vibration problems. Membrane effects do not appear directly due to the assumption of inextensibility. Secondly, the frequency of oscillation is also directly proportional to both internal pressure and to the elastic foundation modulus k. This is reasonable physically since the inclusion of the elastic foundation modulus can do nothing other than increase the potential energy of any possible movement, hence increasing frequency. Internal pressure, on the other hand, introduces membrane forces similar to the tension in a string, where it is well known that frequency rises as tension increases. Hence, both of these quantities should in their positive sense tend to increase the frequency of oscillation, as is observed. Finally, frequency appears to be decreased by the presence of rotating speed. This may be explained in part by the fact that the positive displacement of a body in a rotating field introduces additional centrifugal forces in the same direction as that of displacement, thus providing what is in effect a negative spring rate associated with displacement from some equilibrium position. This effect is indeed observed in Eq. (40).

Finally, in discussion of Eq. (40), the wave number s is seen to correspond exactly with the order of the harmonic involved in the oscillation. Sketches of several mode shapes for small values of s are given in Figure 3, where it should be noted that s = 1 corresponds to a rigid body translation which is not of great particular interest here since neither the bending nor rotational effects influence that frequency.

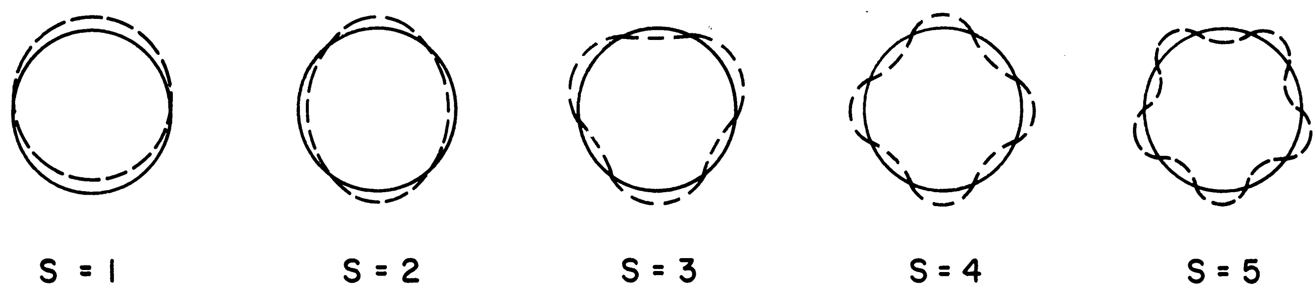

Figure 3. Mode shapes.

Equation (39) represents the radial motion of the shell as a function of the frequency $\bar{p}$ and mode shape s. This is a very interesting equation since it indicates that nodal points, or antinodes, actually rotate around the perimeter of the shell at an angular velocity different than the rotating speed Ω. This was first pointed out by Bryan, and is again borne out here for this particular case of the elastically restrained shell with internal pressure. It may be seen that neither the elastic restraints nor the internal pressure influence this velocity of node rotation, which is purely a function of the mode number. First imagine that the cylindrical shell oscillates with the frequency $\bar{p}$ and in one of the specific mode shapes shown in Figure 3, in particular any involving s = 2 or greater. Associated with this oscillatory motion will be nodal points, and it may be seen from the form of the Eq. (39) that these will rotate with angular velocity given by the symbol Ω'.

$$\Omega' = \frac{s^2 - 1}{s^2 + 1} \Omega \tag{41}$$

It may be seen that this is some fraction of the angular velocity Ω, and again one may refer to Bryan to note that these may be expressed conveniently in a small table, given here as Table I.

TABLE I

s	Number of Nodes	Ω'/Ω
2	4	0.6
3	6	0.8
4	8	0.882
5	10	0.923

It would seem reasonable on this basis to predict that low frequency vibratory or acoustic phenomena should be observed at fractions of the angular velocity of the rotating pneumatic tire, these fractions becoming closer to the angular velocity as the mode shape becomes higher. So far as is known, no experimental verification of these phenomena exists. It can be imagined that the antinodes, representing the points of maximum vibratory amplitude, also rotate around the moving shell with the same angular velocity as the nodes.

Imagine this idea applied to a pneumatic tire, and imagine a pure second mode of oscillation. The points of maximum oscillatory motion will pass through the contact patch of this tire at an angular velocity equal to 0.6 times the angular velocity of the tire. Thus, one might predict that there exists a relatively low frequency oscillatory or acoustic phenomena associated with the normal vibration states of a pneumatic tire, such phenomena being associated with this backward, or retrograde, rotation of the antinodes.

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