GEOGRAPHICAL COORDINATE COMPUTATIONS

Part I

General Considerations

Technical Report No. 2

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W. R. Tobler
Department of Geography
University of Michigan
Ann Arbor, Michigan

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ABSTRACT

Part I provides a discussion of the usefulness of coordinate models for studies of geographically distributed phenomena with comments on specific coordinate systems and their relevance for the analysis and inventorying of geographical information. Appendices include equations for conversion from the Public Land Survey system into latitude and longitude and to rectangular map projection coordinates. Part II considers map projections in greater detail, including estimates of the errors introduced by the substitution of map projection coordinates for spherical coordinates. Statistical computations of finite distortion are related to Tissot's Indicatrix as a general contribution to the analysis of map projections.

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INTRODUCTION

In recent years there has been a rapid increase in the use of formal mathematical and statistical methods for the analysis of terrestrial distributions. Such procedures have been found to be of considerable assistance in fields such as city and regional planning, demography, ecology, geography, geology, and regional science. The present study is concerned with only one of the several mathematical strategies which have been utilized for such analyses; the "coordinate model". This term is taken to include that class of studies which specifically refers to the location of observational phenomena by some system of coordinates.

As an example, a technique associated with contemporary theories in geology consists of estimating the departures of empirical geological observations from a "regional trend". Here one has a collection of numerical observations (z_i) at specific terrestrial locations (x_i, y_i), $i = 1, 2, \ldots, n$. The procedure begins by estimating a specific portion of the locational trend of the observations as a least squares equation $\hat{z} = f(x, y)$. The trend is then subtracted from the observations to obtain the residual, which is subsequently interpreted in terms of geological theory. In this instance the recording of locations in some system of coordinates is an essential prerequisite for the analysis. There are many other such examples. The coordinate model is widely used, and is enjoying increasing popularity since the advent of electronic computers, which permit facile manipulation of the metricized locations. The researcher now also has available to him a rapidly increasing amount of information recorded in terms of coordinates; the U.S. Bureau of the Census, for example, now provides population statistics in terms of latitude and longitude coordinates. It is difficult to over-estimate the usefulness of this manner of recording information since a large number of the analytical methods designed for the analysis of distributions assume the existence of a system of coordinates.

As a system of locational labels the specific coordinates employed for the recording of observations are not of direct or inherent interest, but rather only what they enable one to deduce regarding interrelations among the phenomena observed. In this sense the particular coordinate system utilized is irrelevant. On the other hand, computations may often be simplified by the choice of a convenient coordinate notation. From a scientific point of view descriptions of phenomena and their interrelations are often simplified by appropriate formalizations involving a "natural" coordinate system for that phenomena; the use of geomagnetic coordinates in the study of terrestrial magnetism, for example. The present study, however, considers only systems which appear to be of practical utility for the large scale recording of terrestrial observations, with some emphasis on systems available in the United States.

The actual surface of the earth can be referred to as the topographic surface. This bumpy two-dimensional surface is difficult to describe in all its detail. Theoretically it is possible to introduce a system of coordinates on this surface such that ground distances, etc., between all points can be calculated. In practice this is not attempted. As an approximation to the topographic surface the geodesist utilizes a surface of constant gravitational potential, the geoid. This bumpy but rather smooth surface is still too complicated for practical computations. A further simplification is made by assuming the earth to have the shape of an ellipsoid, generally an ellipsoid of revolution. Geodetic coordinates are then defined for positions on this ellipsoid. An even simpler model of the topographic surface is to consider the earth to be a perfect sphere. One can continue thusly, finally arriving at the assumption that the earth is a flat plane. Each of these assumed models of the earth has its advantages and disadvantages since realism may lead to extreme cumbrousness. In practical terms, the following (somewhat contradictory) criteria seem appropriate:

a) The coordinates should permit accurate and economical formulae for computation. The highest level of precision available today can be achieved only through the use of geodetic formulae. These formulae are fairly complicated. Computational simplicity can be obtained, with a consequent reduction in accuracy, by employing spherical formulae. Further computational simplicity can be achieved by the use of plane coordinates based on an appropriate map projection, again with some loss of accuracy.

Computational simplicity is important for two reasons. The cost and time required for computation, particularly when large amounts of information are to be processed, can be reduced by significant amounts through the use of simplified formulae. In addition, the number of persons and agencies who can effectively make use of the information is significantly increased by the use of simplified formulae.

It is more difficult to discuss accuracy, upon which the level of computational simplificty depends, since the degree of precision required in particular studies varies considerably. There is no reason to employ a method which results in accuracies greater than required or greater than those with which the information was recorded. An objective of this study has been to estimate the accuracy obtainable by employing several alternate methods. This allows the individual researcher to choose the simplest computational method which yields the requisite level of accuracy.

b) A rapid and accurate method of determining the coordinates of a position should be available. In general a system of coordinates which requires a carefully executed geodetic survey can be considered highly accurate, but relatively slow. A system which enables one to read coordinates from an aerial photograph or map (either manually, mechanically, or electronically) is more rapid but the accuracy is dependent on the map scale. The convenience of this method is also dependent on the availability of maps or photographs to which the coordinate system has been affixed. Between the extremes of geodetic surveying and map scaling are a number of intermediate systems, including automatic navigation devices which permit virtually instantaneous, in situ position determination, with fair accuracy. Emphasis in this study is on map scaling procedures.

c) The coordinates should be widely available and should be equally convenient for use at a local, national, and international level. This objective arises since most types of information collected at a national level are used both nationally and locally. Census records provide a good example. National use of local records also is increasing. The accuracy requirements at these two levels generally differ, however. At the local level an accuracy of fifty feet may be insufficient, whereas at the national level an accuracy of five miles may suffice.

TERRESTRIAL COORDINATE SYSTEMS

There are many locational coordinate systems in use throughout the world. Emphasis in the current discussion is on systems available in the United States.

Geodetic Coordinates:

Geodetic latitude and longitude provide the traditional method of identifying locations on the surface of the earth. The earth is assumed to be an oblate spheroid and the geodetic coordinates are based on actual measurement (triangulation) between sets of locations on the topographic surface. These values are then adjusted to fit an ellipsoid representative of the region in question. Different ellipsoids are employed for the several continents of the world, with an International Ellipsoid in use for world-wide computations. Geodetic coordinates, based on the Clarke Ellipsoid of 1866 (1927 adjustment), are indicated on maps published by the U.S. Geological Survey and by the U.S. Coast and Geodetic Survey. Latitude and longitude scaled from such maps are geodetic coordinates, but such scaling will not yield the same accuracy as when the positions are established in the field by an expensive first order geodetic survey.

Computations employing geodetic coordinates usually take into account the ellipsoidal shape assumed for the earth. The relevant formulae are fairly complicated. For precise geodetic work it is necessary to carry approximately fifteen significant digits. Experience with a digital computer, however, indicates that, once programmed, the ellipsoidal formulae do not require appreciably more effort than the simpler spherical formulae. A floating point program with seven significant digits (as employed) for this study) yielded values which differed less than 100 meters from more precise values over a range of 6000 kilometers.

Assumptions required to apply geodetic computations to the surface of the earth are that (a) the geodetic latitudes and longitudes are known without error, (b) the ellipsoid chosen is representative of the region in question, and (c) the points involved lie on the surface of the ellipsoid (roughly, at sea level). On the other hand, this is the most accurate system available. The actual proportional error in distance, based on misclosures of the U.S. continental triangulation network, appears to be on the order of

$$\frac{1}{20000} D^{1/3}$$

where D is the computed distance in miles on the Clarke Ellipsoid of

1866 (1927 adjustment). The differences between the several ellipsoids in use throughout the world are small; on the order of three kilometers per 6000 miles. Connections of this length on one ellipsoidal datum are rare and the figure given does not take into account the fact that the relation between the several datums in actual use are not yet known in detail; in other words, distances between positions whose geodetic coordinates are referred to different ellipsoids may be in error by a larger figure.

Astronomic Coordinates:

Astronomic latitude and longitude are based on celestial observations and may depart from geodetic coordinates by as much as two kilometers at any point, due to departure of the geoid from the ellipsoid. Astronomical observations are usually available only for isolated points, and will not be considered in this report.

The U.S. Public Land Survey

The Public Land Survey system is based on a set of six mile squares numbered as townships north and south of a base parallel, and as ranges east and west of a base meridian. These six mile squares are then subdivided into 36 sections, each one mile square, and numbered in serpentine fashion. Each section can be further subdivided into quarter-sections, each one sixteenth of a mile in area. Several systems similar to the Public Land Survey exist in various parts of the United States; these are not considered here.

Strictly speaking, the Public Land Survey is an areal identification scheme and not a metrical coordinate system, though it is often regarded as such. As a partitioning of areas the system does not differ from county or census units, except that the elemental areas are roughly of equal size and are labeled in a more convenient fashion. The system is not complete, in the sense that it is defined only for certain portions of the western United States. In these areas large amounts of information have been (and continue to be) collected and recorded in terms of Sections, Townships, and Ranges. These collections of information provide a valuable source of raw data for research workers. The Public Land Survey, however, was not designed for the analytical manipulations usually required in research work. For example, statistical analyses of spatial distributions may require calculation of the average location (and its variance, and so on) of phenomena. For such computations the distances between observed locations may be needed. The distance between the SW¼, Sec. 25, T5S, R7W, Willamette Meridian, and the NE¼, Sec. 2, T6N, R8E, Black Hills Meridian, is not immediately apparent, nor is there any simple formula which can be employed to obtain this difference. Observations recorded in the Public Land Survey System, however, can be convered to a coordinate system having the requisite metrical properties. This is because the Public Land Survey has many of the topological ordering properties of a coordinate system. The most direct and convenient conversion is to latitude and longitude. This can be effected in several ways. The system of Townships and Ranges is shown on U.S. Geological Survey topographic maps and approximate coordinates could be scaled from there maps. A more convenient procedure is to attempt a direct calculation. The equations for such a conversion are given in the Appendix, along with an estimate of their validity. The errors are fairly small so that they might be of little consequence when working with observations from the entire United States. The urban researcher working within one city, on the other hand, might find these errors intolerable.

The GEOREF and Marsden Squares Systems

The GEOREF System is used by the U.S. Air Force to identify locations. It is a modification of latitude and longitude in which letters are substituted for the numerical values. Every combination of letters is taken to represent a quadrilateral bounded by latitude and longitude. In this sense the system is a partitioning of area rather than a true coordinate system. The same results can be achieved by using latitude and longitude with a convention regarding the quadrant in which the quadrilateral of area lies. The system has certain advantages in applications which require error-free rapid verbal communications (e.g. radio). The system is shown on maps published by the U.S. Air Force.

The Marsden Squares system employed by the National Oceanographic Data Center is similar to the GEOREF System in that a numbering of latitude and longitude quadrilaterals is substituted for the geodetic coordinates. There are many other such systems available, including the World Aeronautical Chart designations and the International Millionth Map of the World system. The advantage of these systems is largely one of bookkeeping. Such systems are not further considered in this report.

THE SPHERICAL ASSUMPTION

The various computations are simplified if it is assumed that the earth is a sphere. The results of such computations do not differ by large amounts from the corresponding ellipsoidal values - the polar flattening of the earth, after all, is quite small. On the basis of a number of computations it appears that a reasonable and convenient rule of thumb is that the flattening of the earth can be taken as an approximate upper bound on the percentage error of measures calculated on a spherical as compared to an ellipsoidal assumption. This is about one part in 300. An even safer estimate is that the error will be less than one percent. For some purposes this is intolerably large, but for the majority of requirements it is far more accurate than are the data or theories now available. Detailed numerical differences between an ellipsoid and sphere for distances and angles also have recently been published. Computation of the differences for a random sample of 200 pairs of points within the continental United States resulted in the following values:

Distance Differences (miles)	Angular differences (degrees)
Average: 0.046	Average: 0.006
Standard Deviation: 1.896	Standard Deviation: 0.083
Minimum: -3.788	Minimum: -0.150
Maximum: 4.871	Maximum: 0.159

A second sample might yield somewhat different results, but the sample is probably representative for the country as a whole. As expected, the differences depend on both the distance and on the direction of the point pairs. A comparison of surface areas is given in the accompanying table.

In performing these computations it has been assumed that the earth is a sphere whose radius is equal to the equatorial radius of the Clarke Ellipsoid of 1866, and that the geodetic latitude and longitude can, without modification of their numerical values, be considered to be spherical coordinates. These assumptions have the advantage of extreme simplicity. A slight improvement in accuracy can be obtained if they are not retained. For example, the spherical radius employed might be the average radius of terrestrial ellipsoid in the vicinity of the area of interest, rather than the equatorial radius. It can be proven that the average radius at any latitude is the geometric mean of the radii of curvature along the meridian and normal to the meridian. This average radius is given in the accompanying tables. Conversion of geodetic latitude to spherical latitude can also be accomplished in a large number of ways. Four of the simpler methods are illustrated in the figure. Mathematical treatments can be found in works on geodesy and map projections.

THE PLANE ASSUMPTION

It often is convenient to employ plane coordinates for the inventorying of analysis of terrestrially distributed phenomena. In particular, many of the numerous statistical and analytical methods which have been devised for the analysis of two dimensional distributions assume the existence of a system of Cartesian coordinates. As a very simple example, suppose that an objective is to compute the average location and the locational variance of a set of discrete phenomena on the surface of a sphere. One can proceed in several ways:

a) Record the observations in latitude and longitude and then perform the calculations using the spherical formulae for average and variance.

b) Plot the distribution on a map, assign arbitrary rectangular coordinates to the map, record the observations in these coordinates, and then perform the calculations using the plane formulae for the average and variance.

COMPARISON OF AREAS FOR A ONE DEGREE ZONE
OF LONGITUDE WITHIN THE UNITED STATES

(Values in square miles, rounded to the nearest square mile)

Latitude	Ellipsoidal Area	Spherical Area*
26N to 27N	4265	4282
27N to 28N	4228	4244
28N to 29N	4189	4205
29N to 30N	4150	4164
30N to 31N	4109	4123
31N to 32N	4067	4080
32N to 33N	4024	4035
33N to 34N	3979	3990
34N to 35N	3934	3943
35N to 36N	3887	3895
36N to 37N	3839	3846
37N to 38N	3789	3796
38N to 39N	3739	3744
39N to 40N	3687	3692
40N to 41N	3634	3638
41N to 42N	3581	3583
42N to 43N	3526	3528
43N to 44N	3469	3471
44N to 45N	3412	3413
45N to 46N	3354	3354
46N to 47N	3295	3294
47N to 48N	3234	3232

*Radius equal to equatorial radius of Clarke ellipsoid of 1866.

CLARKE ELLIPSOID OF 1866

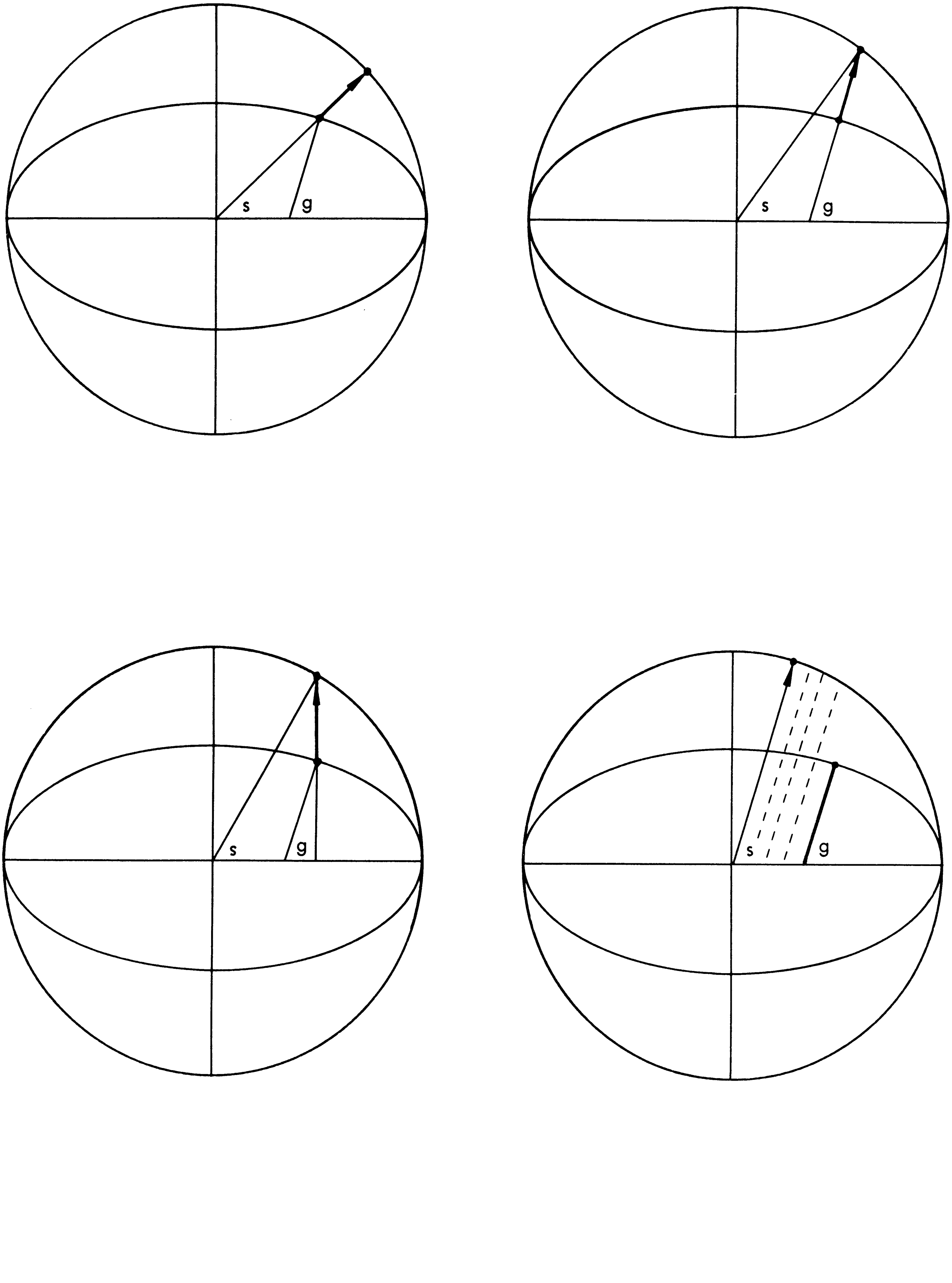
LATITUDE	RADII IN MILES: RADIUS OF THE MERICIAN	RADII IN MILES: RADIUS NORMAL TO THE MERIDIAN	RADII IN MILES: MEAN RADIUS	RADII IN MILES: RADIUS OF THE PARALLEL	RADII IN KILOMETERS: RADIUS OF THE MERIDIAN	RADII IN KILOMETERS: RADIUS NORMAL TO THE MERIDIAN	RADII IN KILOMETERS: MEAN RADIUS	RADII IN KILOMETERS: RADIUS OF THE PARALLEL
.0	3936.4000	3963.2258	3949.7901	3963.2258	6335.0344	6378.2064	6356.5837	6378.2064
2.5	3936.4761	3963.2513	3949.8410	3959.4791	6335.1569	6378.2474	6356.6656	6372.1767
5.0	3936.7036	3963.3276	3949.9932	3948.2459	6335.5231	6378.3703	6356.9106	6354.0986
7.5	3937.0810	3963.4543	3950.2456	3929.5463	6336.1304	6378.5741	6357.3168	6324.0045
10.0	3937.6055	3963.6303	3950.5964	3903.4138	6336.9745	6378.8574	6357.8814	6281.9481
12.5	3938.2731	3963.8542	3951.0429	3869.8950	6338.0489	6379.2178	6358.6000	6228.0048
15.0	3939.0788	3964.1245	3951.5818	3829.0502	6339.3456	6379.6528	6359.4672	6162.2714
17.5	3940.0167	3964.4391	3952.2090	3780.9528	6340.8549	6380.1591	6360.4766	6084.8658
20.0	3941.0799	3964.7957	3952.9200	3725.6892	6342.5659	6380.7330	6361.6208	5995.9276
22.5	3942.2603	3965.1915	3953.7092	3663.3592	6344.4656	6381.3699	6362.8909	5895.6169
25.0	3943.5491	3965.6236	3954.5709	3594.0755	6346.5398	6382.0652	6364.2777	5784.1154
27.5	3944.9368	3966.0886	3955.4985	3517.9636	6348.7730	6382.8137	6365.7706	5661.6249
30.0	3946.4128	3966.5832	3956.4852	3435.1618	6351.1485	6383.6097	6367.3584	5528.3680
32.5	3947.9662	3967.1036	3957.5233	3345.8212	6353.6484	6384.4472	6369.0292	5384.5881
35.0	3949.5852	3967.6458	3958.6052	3250.1051	6356.2540	6385.3198	6370.7703	5230.5477
37.5	3951.2578	3968.2058	3959.7227	3148.1893	6358.9457	6386.2209	6372.5687	5066.5297
40.0	3952.9710	3968.7792	3960.8672	3040.2613	6361.7029	6387.1439	6374.4106	4892.8360
42.5	3954.7122	3969.3619	3962.0303	2926.5205	6364.5051	6388.0815	6376.2824	4709.7877
45.0	3956.4680	3969.9492	3963.2029	2807.1780	6367.3308	6389.0268	6378.1696	4517.7242
47.5	3958.2252	3970.5369	3964.3763	2682.4558	6370.1588	6389.9725	6380.0579	4317.0029
50.0	3959.9702	3971.1203	3965.5413	2552.5869	6372.9670	6390.9114	6381.9329	4107.9987
52.5	3961.6898	3971.6950	3966.6893	2417.8147	6375.7345	6391.8364	6383.7803	3891.1034
55.0	3963.3709	3972.2567	3967.8113	2278.3928	6378.4399	6392.7403	6385.5861	3666.7252
57.5	3965.0004	3972.8010	3968.8988	2134.5844	6381.0624	6393.6163	6387.3362	3435.2876
60.0	3966.5660	3973.3239	3969.9435	1986.6619	6383.5820	6394.4577	6389.0175	3197.2289
62.5	3968.0557	3973.8212	3970.9373	1834.9064	6385.9794	6395.2581	6390.6170	2953.0015
65.0	3969.4577	3974.2891	3971.8727	1679.6072	6388.2357	6396.0112	6392.1223	2703.0712
67.5	3970.7614	3974.7242	3972.7423	1521.0611	6390.3339	6396.7114	6393.5218	2447.9155
70.0	3971.9566	3975.1230	3973.5395	1359.5722	6392.2574	6397.3531	6394.8047	2188.0237
72.5	3973.0342	3975.4824	3974.2581	1195.4507	6393.9916	6397.9316	6395.9612	1923.8952
75.0	3973.9856	3975.7997	3974.8925	1029.0127	6395.5226	6398.4422	6396.9822	1656.0388
77.5	3974.8036	3976.0724	3975.4380	860.5796	6396.8391	6398.8812	6397.8600	1384.9715
80.0	3975.4817	3976.2986	3975.8901	690.4770	6397.9304	6399.2451	6398.5876	1111.2173
82.5	3976.0147	3976.4763	3976.2454	519.0344	6398.7882	6399.5310	6399.1595	835.3065
85.0	3976.3984	3976.6042	3976.5013	346.5839	6399.4057	6399.7369	6399.5712	557.7739
87.5	3976.6297	3976.6813	3976.6555	173.4605	6399.7780	6399.8610	6399.8195	279.1581
90.0	3976.7071	3976.7071	3976.7070	.0001	6399.9025	6399.9025	6399.9024	.0001

INTERNATIONAL ELLIPSOID

LATITUDE	RADII IN MILES: RADIUS OF THE MERICIAN	RADII IN MILES: RADIUS NORMAL TO THE MERIDIAN	RADII IN MILES: MEAN RADIUS	RADII IN MILES: RADIUS OF THE PARALLEL	RADII IN KILOMETERS: RADIUS OF THE MERIDIAN	RADII IN KILOMETERS: RADIUS NORMAL TO THE MERIDIAN	RADII IN KILOMETERS: MEAN RADIUS	RADII IN KILOMETERS: RADIUS OF THE PARALLEL
.0	3936.6635	3963.3075	3949.9630	3963.3075	6335.4584	6378.3380	6356.8619	6378.3380
2.5	3936.7391	3963.3329	3950.0136	3959.5606	6335.5801	6378.3787	6356.9434	6372.3079
5.0	3936.9650	3963.4087	3950.1647	3948.3267	6335.9437	6378.5008	6357.1866	6354.2286
7.5	3937.3400	3963.5345	3950.4155	3929.6259	6336.5471	6378.7032	6357.5902	6324.1325
10.0	3937.8609	3963.7093	3950.7639	3903.4916	6337.3854	6378.9846	6358.1509	6282.0734
12.5	3938.5240	3963.9317	3951.2074	3869.9707	6338.4526	6379.3425	6358.8647	6228.1266
15.0	3939.3242	3964.2002	3951.7426	3829.1233	6339.7405	6379.7746	6359.7260	6162.3890
17.5	3940.2558	3964.5127	3952.3656	3781.0229	6341.2397	6380.2775	6360.7286	6084.9787
20.0	3941.3118	3964.8668	3953.0717	3725.7560	6342.9391	6380.8474	6361.8650	5996.0350
22.5	3942.4842	3965.2599	3953.8557	3663.4225	6344.8260	6381.4800	6363.1266	5895.7188
25.0	3943.7644	3965.6891	3954.7115	3594.1348	6346.8862	6382.1707	6364.5040	5784.2109
27.5	3945.1426	3966.1510	3955.6329	3518.0189	6349.1043	6382.9141	6365.9868	5661.7139
30.0	3946.6087	3966.6422	3956.6127	3435.2129	6351.4638	6383.7046	6367.5637	5528.4503
32.5	3948.1517	3967.1591	3957.6439	3345.8680	6353.9470	6384.5364	6369.2233	5384.6633
35.0	3949.7598	3967.6976	3958.7185	3250.1476	6356.5349	6385.4031	6370.9526	5230.6160
37.5	3951.4209	3968.2538	3959.8284	3148.2274	6359.2083	6386.2982	6372.7388	5066.5909
40.0	3953.1227	3968.8234	3960.9652	3040.2950	6361.9470	6387.2148	6374.5684	4892.8904
42.5	3954.8520	3969.4020	3962.1203	2926.5501	6364.7301	6388.1461	6376.4273	4709.8353
45.0	3956.5960	3969.9854	3963.2849	2807.2036	6367.5367	6389.0849	6378.3016	4517.7653
47.5	3958.3411	3970.5689	3964.4503	2682.4775	6370.3453	6390.0241	6380.1771	4317.0377
50.0	3960.0743	3971.1484	3965.6075	2552.6050	6373.1346	6390.9567	6382.0394	4108.0278
52.5	3961.7822	3971.7192	3966.7476	2417.8295	6375.8832	6391.8753	6383.8742	3891.1271
55.0	3963.4518	3972.2771	3967.8619	2278.4045	6378.5701	6392.7731	6385.6676	3666.7440
57.5	3965.0704	3972.8177	3968.9421	2134.5934	6381.1750	6393.6431	6387.4059	3435.3020
60.0	3966.6253	3973.3370	3969.9797	1986.6685	6383.6775	6394.4788	6389.0758	3197.2394
62.5	3968.1047	3973.8309	3970.9668	1834.9109	6386.0583	6395.2737	6390.6644	2953.0088
65.0	3969.4972	3974.2957	3971.8957	1679.6100	6388.2993	6396.0217	6392.1592	2703.0756
67.5	3970.7920	3974.7277	3972.7594	1521.0625	6390.3831	6396.7170	6393.5493	2447.9177
70.0	3971.9791	3975.1238	3973.5511	1359.5725	6392.2936	6397.3544	6394.8234	2188.0242
72.5	3973.0492	3975.4807	3974.2647	1195.4501	6394.0157	6397.9289	6395.9719	1923.8944
75.0	3973.9942	3975.7959	3974.8949	1029.0117	6395.5365	6398.4361	6396.9861	1656.0372
77.5	3974.8066	3976.0668	3975.4366	860.5784	6396.8439	6398.8721	6397.8578	1384.9695
80.0	3975.4800	3976.2913	3975.8856	690.4758	6397.9277	6399.2334	6398.5804	1111.2153
82.5	3976.0093	3976.4678	3976.2385	519.0333	6398.7796	6399.5175	6399.1484	835.3047
85.0	3976.3904	3976.5948	3976.4926	346.5831	6399.3928	6399.7219	6399.5573	557.7726
87.5	3976.6203	3976.6715	3976.6458	173.4601	6399.7629	6399.8452	6399.8040	279.1575
90.0	3976.6971	3976.6971	3976.6971	.0001	6399.8864	6399.8864	6399.8864	.0001

Four simple methods for the

CONVERSION OF GEODETIC LATITUDE TO SPHERICAL LATITUDE



c) Record the observations in latitude and longitude, apply a transformation to obtain rectangular coordinates, and then perform the claculations using the plane formulae for the average and variance.

Procedure (a) has the disadvantage of being more complicated. A sufficiently small portion of the earth's surface can be considered a plane and the additional complication introduced by the use of spherical versions of the statistical formulae may not be warrented. Somewhat similar problems have been investigated in the field of land surveying and are reported in most works on geodesy. Procedures (b) and (c), above, are mathematically equivalent since maps are made by transforming latitude and longitude to plane coordinates via a map projection. Hence a study of the numerical differences between computations on a plane and on the earth becomes a study of map projection distortions.

The official map producing agencies of the various countries of the world have recognized the advantages of rectangular coordinates for local purposes and save the map user the trouble of assigning his own system of rectangular coordinates. They do this by publishing maps which have the official plane coordinates printed directly on the maps. Two map projection systems of this type are available in the United States, and comparable systems exist in most other countries of the world. The use of these systems is not restricted to calculations; they might also be used to record and index information in terms of the plane coordinates, perhaps scaled from topographic maps. The systems now available have several features in common. The coordinates are usually given as rectangular coordinates, often chosen so that all values are positive. More importantly, the errors in computing as though the earth were a plane disk can be evaluated. This implies that the region within which one can perform plane computations with a specified level of precision can be defined on an a priori basis. If the allowable error is small, the region must be small or several map projection systems (called zones) must be used within the region. In the latter event conversion between zones may be required. This conversion may be directly from zone to zone or may involve reconversion to geodetic coordinates as an intermediate step. There are certain advantages in using a conformal map projection for such a system since the scale errors are then independant of direction and a scale factor can be applied to improve the accuracy of short lengths . The two systems employed in the United States are:

1) The State Plane Coordinate System: This system comprises approximately 120 zones covering the entire United States, with the orientation toward individual states. The accuracy within each zone is one part in 10,000. The larger states therefore require several zones. The zones overlap,with boundaries between zones lying along minor civil divisions (usually counties). The Lambert Conformal Conical projection and the Transverse Mercator projection are employed (with only one exception),

depending on the shape of the individual states. This system is admirably suited to the needs of the local land surveyor and has been officially adopted by many local governmental units. In many states it has legal status, is used for land ownership, and appears on large scale maps. Conversion tables are available and simple to use for any particular zone. Conversion between zones, and especially between states, is somewhat more inconvenient. The location of the zones occasionally is awkward. In Washington state, for example, the two merging metropolitan areas of Seattle and Tacoma each lie in a separate zone. The system of State Plane Coordinates appears on all recent U.S. geological Survey topographic maps.

2) The Transverse Mercator System:

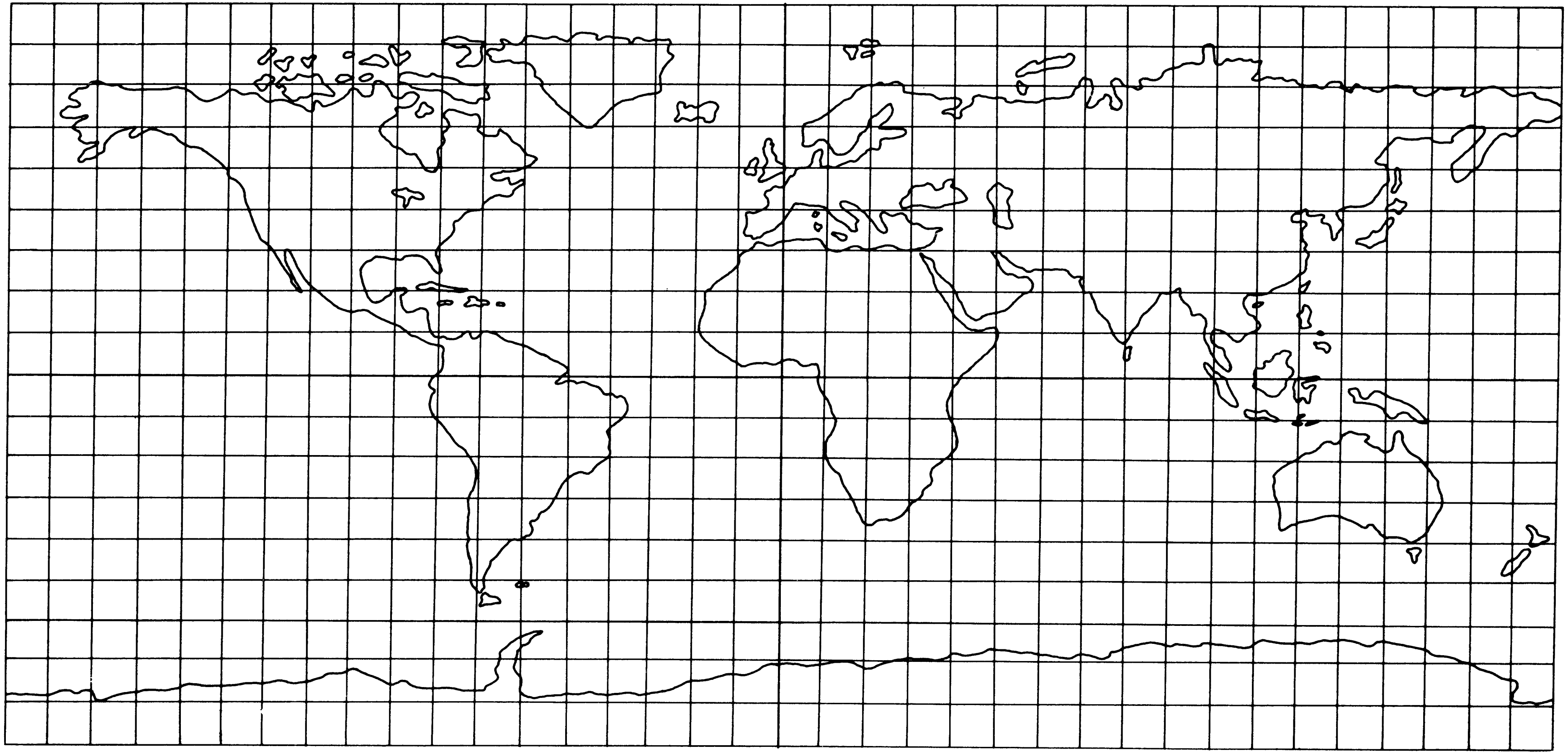
Known as the Universal Transverse Mercator grid system (UTM) this system is employed by the U.S. Army. The UTM grid extends to eighty degrees north and south latitude, beyond which a Polar Stereographic grid is employed. The UTM grid extends around the world in sixty north-south zones, each covering six degrees of longitude with an overlap of one half degree. The accuracy within each zone is one part in 2500. Since different areas of the world are based on distinct ellipsoidal datums, separate tables are required for various parts of the world. Procedures are available for converting directly from one zone to adjacent zones. The zonal nature of the system is occasionally inconvenient. An alphanumeric partitioning of areas is available in the system. The UTM grid appears on all Army Map Service topographic maps, on some foreign maps, and on all recent U.S. Geological Survey topographic maps.

Both of the foregoing systems have several advantages. They can be employed for virtually all computations without serious error. Further, any information recorded in either of these systems can be related to geodetic coordinates and hence to information collected anywhere else in the world. Also, these coordinates are already shown on published maps, and most photogrammetric firms are sufficiently familiar with these systems to add them to aerial photographic or maps compiled by photogrammetric methods. The disadvantages of these systems stem largely from their advantages. The very refinement required to provide coordinates of high accuracy restrict these systems to relatively small portions of the earth's surface and the transformation equations, either between zones, or to and from geodetic coordinates, are relatively complicated. These difficulties can be circumvented in several ways.

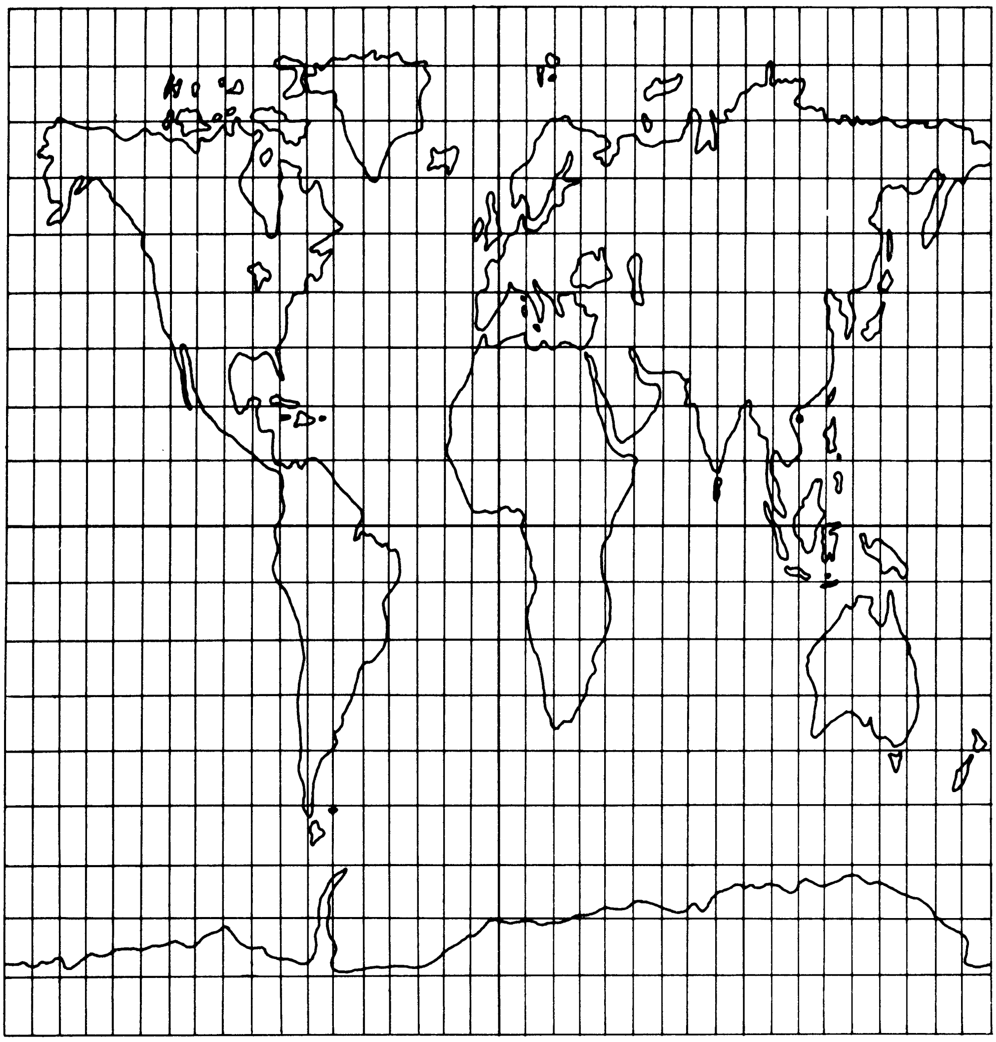
When a map projection system is to be used soley for computational purposes, and not necessarily to be indicated on published maps, the choice of a particular projection depends on the type of computation contemplated. The systems cited above are so refined that they yield a stated level of accuracy for virtually all computations. This is a restriction which narrows the range of suitable projections and results in projection which require fairly involved computations. For a given problem there may be a specific projection which is computationally much simpler but which yields results which are of equal accuracy. For example, a problem which requires interpolation between two points on a sphere might be attacked by using the gnomonic projection (see Appendix) since all great circles are straight lines on this projection; linear interpolation in gnomonic coordinates will yield a point lying on the arc connecting the two given points. Similarly, problems involving circles on a sphere may be attacked using the stereographic projection. In other situations computational simplicity and speed may be more important than a few tens of meters of accuracy. Kao, for example, has recently shown that the geometric (perspective) projections are especially well suited for calculation by digital computer, particularly when large amounts of locational information are required within fractions of a second (i.e., in real time problems). Clearly the choice depends on the nature of the problems and the volume of the information to be processed. Computer calclation of distance and direction on a sphere (or ellipsoid) may, in many instances, be easier than attempting to convert to plane co-ordinates. On the other hand a more complicated problem, as for example, occurs in weather prediction, may advantageously be solved by the use of an appropriate map projection. In this instance the problem is to construct contour-type maps of the entire northern hemisphere from information received from locations scattered within this region. Rather than attempting to solve the contour interpolation problem on a sphere, the Weather Bureau employs stereographic map projection coordinates with a local correction for the projection distortion and solves the problem in plane coordinates.

If one has information recorded in latitude and longitude simple conversions to map projection coordiantes are available. For example, one can pretend that these are already the ordinate and abscissa of a plane coordinate system. The resulting projection is known as the square projection. Computations performed in this manner will differ from the true values by amounts which depend on the size of the region and on the latitude. Another simple, but slightly better, conversion is to multiply all the abscissas (longitudes) by a constant equal to the cosine of the average latitude of the region in question (the square projection with a standard parallel; also known as the rectangular projection). Such a procedure might for example, by employed in urban analysis, depending on the size of the area. Another alternative would be to transform to rectangular coordinates by converting all values into distances north and east (that is, measured along a parallel) from some arbitrary point within the region. This yields the sinusoidal projection. The equations for all of the above projections are extremely simple. Somewhat more refined, but also more complicated, solutions take into cosideration the shape of the area of concern. Albers' equal area conical projection with standard parallels at 29^{o} 30'N and 45^{o} 30'N, and Lambert's conformal conical projection with standard parallels at 33^{o}N and 45^{o}N, for example, are two systems which might be suitable

SQUARE PROJECTION



RECTANGULAR PROJECTION



Standard parallels at 60N and 60S

for the continental United States. The distance error in computing with these latter systems is not likely to exceed fifty miles.

The use of latitude and longitude, while advantageous from the point of view of long run national needs, entails some local difficulties. In the process of recording it may be necessary to interpolate between curved lines, and the system of minutes and seconds is awkward (Decimal degrees are more convenient). The complete number of digits required to specify a given location in its world context is excessively large for local use, and the north-south and east-west designation is often superflouous (a mathematical convenience is obtained if south latitudes and west longitudes are considered negative). Finally, and perhaps most important, it is often difficult to determine the latitude and longitude of a particular spot.

The direct recording and storage of geographical information in terms of rectangular coordinates circumvents some of these difficulties, but introduces others. The majority of the electro - mechanical data reduction devices (specifically, coordinate readers) which are now on the market utilize rectangular coordinates. These instruments reduce the teduim of coordinate reading, even when the desired result is latitude and longitude coordinates. In this case the inverse map projection equations are required. Curiously, these are not widely available in the literature on the subject of map projections (with a few exceptions) since the previous technology prohibited their extensive use.

If the objective of the study does not include subsequent conversion to latitude and longitude, a convenient procedure is to draw arbitrary rectangular (or polar) coordinates on whatever maps or aerial photographs are available. One advantage is that this can be done by persons with no training and with virtually no intellectual effort or financial expenditure. When the map used is accurate and at a "sufficiently large" scale these arbitrary coordinates may be employed as are the map projection coordinates discussed above. If the information collected has no permanent value, this procedure is perfectly satisfactory.

A disadvantage is that the errors introduced are not known. The limits within which a certain level of accuracy obtains is uncertain an one never knows whether the system can be extended to include a neighboring territory. A second major disadvantage is that it may not be possible to use information collected for one study in a second study which either (a) encompasses a larger area than the original study, or (b) which is subsequent in time to the original study, especially if the original map has been lost, or (c) which requires a higher level of accuracy than the

the original study. One can imagine the difficulty of analyzing the greater metropolitan area of Kansas City if Kansas City, Missouri and Kansas City, Kansas, used two different and unrelated grid systems. Or if each bureau of a city government employed a distinct system of coordinates. The actual occurance of situations of this very nature in the field of civil engineering is what gave impetus to the establishment of the system of State Plane Coordinates by the U.S. Coast and Geodetic Survey in the 1930's.

Conversion between arbitrary map coordinates can be effected with relative ease if the relation between the two systems is known, or if both systems are related to latitude and longitude by known inverse equations. If the relation between systems is not known it is theoretically possible to estimate the relation if the coordinates of a sufficient number of points are accurately known in both systems (see Appendix). Such conversions may occasionally be required but are expensive.

A final distinction should be made between coordinates and areas. Coordinates describe points, not areas, and one must distinguish between an areal recording unit such as a census tract and between the coordinate system used to pinpoint some centroid taken to represent that areal unit. Areal information recording units are extremely numerous and differ widely in size and shape. As a consequence it is often necessary to convert from one areal unit (e.g. census tract) to other areal units (school district, political precint, and so on). These areal conversions differ somewhat from the coordinate conversions discussed in this report. In general, specification of the areal boundaries must be included in the mathematical conversion statements. There are then again several procedures, of varying accuracy and complexity, which may be employed for the conversions.

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APPENDIX I

CONVERSION FROM THE PUBLIC LAND SURVEY SYSTEM TO LATITUDE AND LONGITUDE

The simplest conversion begins with a procedure which assumes that the Public Land Survey conforms to the exact specifications upon which it is based. The system, as is well known, does not conform to these specifications, for a number of reasons including measurement errors unavoidable in any empirical work and a certain laxity of supervision during the establishment of the system. For conversion into latitude and longitude the following notation is convenient:

i is an index to indicate the initial point of the survey. It is necessary to distinguish at least 37 initial points in the Western United States.

ϕ_i is the latitude of the i^{th} base parallel.

λ_i is the longitude of the i^{th} base meridian, with west longitudes negative.

a is the equatorial radius of the ellipsoid taken to represent the earth. For the Clarke Ellipsoid of 1866, $a=3963.2257$ miles.

e is the eccentricity of the ellipsoid taken to represent the earth. For the Clarke Ellipsoid of 1866 $e = 0.0822718542$.

M_i is the radius of the meridian at the i^{th} initial point. M_i is given by

$$M_i = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi_i)^{3/2}}$$

T is the township number of the location in question, with north townships taken as positive and south townships taken as negative.

R is the range number of the location in question, with east ranges taken as positive and west ranges taken as negative.

S_n is the northing of the section in question, with the sign convention as above.

S_e is the section easting, with the sign convention as above.

Q_n is the quarter section northing, with signs as above.

Q_e is the quarter section easting.

ϕ is the latitude (to be found) of the location in question.

N_ϕ is the radius of curvature perpendicular to the meridian at latitude ϕ:

$$N_\phi = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}}$$

λ is the longitude (to be found) of the location in question.

The necessary equations are then:

$$\phi = \phi_i + \frac{6 T \mp 3 + S_n + Q_n}{M_i},$$

and

$$\lambda = \lambda_i + \frac{6 R \mp 3 + S_e + Q_e}{N_\phi \cos \phi}$$

The formulae are established by observing that the center of the township in question should be six miles times the number of the township north (south) of the base parallel, minus three miles to obtain the center of the township. The section northing and easting give the distance of the center of the section from the center of the township, and the quarter-section northing and easting give the distance of the center of the quarter-section from the center of the section. For the SW 1/4, Sec. 25, T 5 N, R 17 E, one should have for example, that the center of the township is 27 miles (5 x 6 - 3) north of the initial point. S_n is -1.5 miles and Q_n is -0.25 miles. The total distance in the north-south direction from the initial point should therefore be + 25.25 miles. This distance must then be converted to the appropriate number of degrees and added to the latitude of the initial point. A further refinement, though hardly necessary, would be to iterate on the latitude obtained in the first step in order to adjust the meridional radius employed in the computation. Determination of the longitude is similar but slightly more difficult since the distances are measured along a parallel (a loxodrome, not a great circle or geodesic), whose radius varies with the latitude.

In programming the outlined procedure for a digital computer it is simplest to employ radians instead of degrees and to store a table of S_n, S_e, Q_n, Q_e, ϕ_i, and λ_i. The computer can perform the assignment to the correct initial point by letter for letter examination of the name of the principal meridian. A convention is necessary to distinguish the two different initial points employed for the Fourth Principal Meridian. The method detailed assigns latitude and longitude (to about the nearest 1/4 mile) on the assumption that the Public Land Survey designations are where they should be. Of course they are not exactly there: the legal strategy is to assign to the actual locations a status of incontestable correctness, irrespective of any errors which may have been introduced during the survey. To adjust the calculated values to conform to their legal positions requires detailed historical and empirical corrections, and can be quite tedious. For many research purposes, however, such a refinement may not be necessary. To obtain an order-of-magnitude estimate of the discrepancies, the actual latitude and longitude (as recorded on large scale topographic maps) of a scattered set of locations have been compared with the computed values. For a selection of 74 points within the State of Michigan the errors are as follows:

Distribution of Errors: (N = 74)

Mean: 2.849 miles

Standard deviation: 1.827 miles

Maximum: 9.339 miles

60% of the errors are less than 2 miles

93% of the errors are less than 5.5 miles

The directional errors appear evenly distributed in all directions. A random selection of points (N = 25) from other states indicates that the errors are quite comparable and of the same order of magnitude. A sample computation is as follows:

observed location:	SW 1/4, Sec. 28, T 2 S, R 6 E, Michigan Meridian	
calculated Lat/Lon:	42^{o} 16'07" N,	83^{o} 44'08" W
observed Lat/Lon:	42^{o} 17'10" N,	83^{o} 44'49" W
difference:	1'03"	41"
difference in miles:	2.38	
direction of difference:	154.34^{o} (E of N)	

The method given above does not include an adjustment for the convergence of the meridians. Since the edges of the ranges run due north and south, the ranges become narrower as the meridians converge. To adjust for this, standard parallels are established every twenty-four miles north and south of the base parallel. The ranges are again made six miles wide at these standard parallels. The system thus is self correcting every twenty-four miles. The order of magnitude of the difference in width of ranges, separated by twenty-four miles in a north-south direction, can be established as follows: The radius of the parallel at 45^{o}N latitude is 2807.178 miles. At a distance of 24 miles north of 45^{o}N it is approximately 2789.834 miles. The east-west width of the northern edge of the range 24 miles north of the 45th parallel is therefore not six miles but 0.037104 miles (195.9 feet) less than six miles. On this basis the error at R 50 E, an extreme value, would be 1.86 miles. Another slight error is introduced by the topographic elevation, since the radii employed apply to a mean sea level ellipsoid.

Empirical corrections for Michigan would need to include the fact that the standard parallels are 60 (not 24) miles apart (in accord with the surveying instructions in force at the time), and that R 1 E is consistently too narrow from T 1 N to T 20 N. An adjustment for these, and other, systematic departures could be incorporated into the computer program. Conversion of the Section, Township, and Range information to latitude and longitude can be followed by conversion to map projection coordinates for map plotting or computational purposes. Direct conversion to Cartesian map coordinates also is possible but is less convenient for the entire Western United States. This is more appropriate for operations restricted to a limited area, e.g., one individual state.

APPENDIX II

MAP PROJECTION EQUATIONS

The following list gives the mathematical rules for the most common map projections of a sphere. The following notation is standard.

φ Latitude of a point whose projection coordinates are desired.

λ Longitude of a point whose projection coordinates are desired.

x Abscissa of a plane cartesian coordinate system.

y Ordinate of a plane cartesian coordinate system.

r Radial distance of a plane polar coordinate system.

θ Angular direction of a plane polar coordinate system.

φ_0 Latitude of the center of the map; either the point of "tangency", or a single standard parallel.

φ_1 Southerly standard parallel for projections having two standard parallels.

φ_2 Northerly standard parallel for projections having two standard parallels.

λ_0 Longitude of the center of the map; either the point of "tangency", or the central meridian.

c The constant of the cone for conic projections.

r_1 The radial distance from the origin to the image of the southerly standard parallel in plane polar coordinates.

All equations are given for a sphere of unit radius (R = 1) and all values are assumed to be in radians. Conversion to scale can be achieved by multiplying all distances by the appropriate scale factor. North latitudes and east longitudes are taken to be positive, i.e.

$$-\frac{\pi}{2} \leq \varphi \leq +\frac{\pi}{2}$$

$$-\pi \leq \lambda \leq +\pi$$

The equations are given in their most commonly applied form. The conical projections, for example, are not given in their oblique cases.

(1) Albers' equal area conic projection with two standard parallels:

$$C = \frac{\sin\varphi_1 + \sin\varphi_2}{2}$$

$$r = \left[\frac{4}{C^2}\left(\sin^2\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)\sin^2\left(\frac{\pi}{4}-\frac{\varphi_2}{2}\right)\right) + \frac{4}{C}\sin^2\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right]^{1/2}$$

$$\theta = C(\lambda - \lambda_0)$$

This puts the origin of the coordinates somewhat beyond the north pole, which is rather inconvenient. The origin can be shifted to the intersection of the southern standard parallel with the central meridian by using

$$X = r\sin\theta$$

$$y = r_1 - r\cos\theta$$

(2) Azimuthal equidistant projection :

$$r = \arccos\left[\sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos(\lambda-\lambda_0)\right]$$

$$\theta = \arcsin\left[\frac{\cos\varphi\sin(\lambda-\lambda_0)}{\sin r}\right]$$

The origin of the coordinates is at φ_0, λ_0

(3) Bonne's Equal Area projection :

$$r = \varphi_0 - \varphi + \tan\left(\frac{\pi}{2} - \varphi_0\right)$$

$$\theta = \frac{(\lambda-\lambda_0)\sin\left(\frac{\pi}{2}-\varphi\right)}{r}$$

To place the origin at the intersection of the standard parallel and the central meridian use:

$$r_1 = \tan\left(\frac{\pi}{2} - \varphi_0\right)$$

$$X = r \sin\theta$$

$$Y = r_1 - r\cos\theta$$

(4) Cassini Projection

$$X = \arc\sin\left[\cos\varphi \sin(\lambda - \lambda_0)\right]$$

$$Y = -\varphi_0 + \arctan\left[\frac{\tan\varphi}{\cos(\lambda - \lambda_0)}\right].$$

(5) Gnomonic Projection:

$$X = \frac{\cos\varphi \sin(\lambda - \lambda_0)}{\sin\varphi \sin\varphi_0 + \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0)}$$

$$Y = \frac{\sin\varphi \cos\varphi_0 - \sin\varphi_0 \cos\varphi \cos(\lambda - \lambda_0)}{\sin\varphi \sin\varphi_0 + \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0)}$$

(6) Lambert's azimuthal equal area projection:

$$c = \arccos\left[\sin\varphi \sin\varphi_0 + \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0)\right]$$

$$r = 2\sin\left(\frac{c}{2}\right)$$

$$\theta = \arcsin\left[\frac{\cos\varphi \sin(\lambda - \lambda_0)}{\sin c}\right]$$

(7) Lambert's cylindrical equal area projection:

$$X = \lambda - \lambda_0$$

$$Y = \sin\varphi$$

Or, with a standard parallel:

$$X = (\lambda - \lambda_0) \cos \varphi_0$$

$$Y = \sin \varphi$$

(8) Lambert's conformal conic with two standard parallels:

$$C = \frac{\ln \cos \varphi_1 - \ln \cos \varphi_2}{\ln \tan\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right) - \ln \tan\left(\frac{\pi}{4} - \frac{\varphi_2}{2}\right)}$$

$$C_1 = \frac{\cos \varphi_1}{C \tan^C\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)}$$

$$r = C_1 \tan^C\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$$

$$\theta = C(\lambda - \lambda_0)$$

$$X = r \sin \theta$$

$$Y = r_1 - r \cos \theta$$

(9) Mercator's conformal cylindrical projection:

$$X = \lambda - \lambda_0$$

$$Y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

(10) Miller's Cylindrical projection:

$$X = \lambda - \lambda_0$$

$$Y = 1.25 \ln \tan\left(\frac{\pi}{4} + \frac{2}{5}\varphi\right)$$

(11) Mollweide's equal area elliptical projection:

Define chi by $2\psi + 2\sin\psi = \pi \sin\varphi$, then

$$X = 2\sqrt{2}\,(\lambda - \lambda_0)\cos\psi$$

$$Y = \sqrt{2}\,\sin\psi$$

(12) Orthographic projection:

$$X = \sin\varphi\cos\varphi_0 - \cos\varphi\sin\varphi_0\cos(\lambda - \lambda_0)$$

$$Y = \cos\varphi\sin(\lambda - \lambda_0)$$

(13) Polyconic projection (American polyconic):

$$r = \mathrm{ctg}\,\varphi$$

$$\theta = (\lambda - \lambda_0)\sin\varphi$$

$$X = r\sin\theta$$

$$Y = r + \varphi - r\cos\theta$$

Which puts the origin at the equator.

(14) Sinusoidal equal area projection:

$$X = (\lambda - \lambda_0)\cos\varphi$$

$$Y = \varphi$$

(15) Square projection:

$$X = \lambda - \lambda_0$$

$$Y = \varphi$$

or, with a standard parallel (also known as the rectangular projection):

$$X = (\lambda - \lambda_0)\cos\varphi_0$$

$$Y = \varphi$$

(16) Stereographic projection:

$$X = \frac{\cos\varphi \sin(\lambda - \lambda_0)}{1 + \sin\varphi \sin\varphi_0 + \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0)}$$

$$Y = \frac{\sin\varphi \cos\varphi_0 - \sin\varphi_0 \cos\varphi \cos(\lambda - \lambda_0)}{1 + \sin\varphi \sin\varphi_0 + \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0)}$$

(17) Transverse Mercator projection.

$$X = \frac{1}{2} \ln \left[\frac{1 + \cos\varphi \sin(\lambda - \lambda_0)}{1 - \cos\varphi \sin(\lambda - \lambda_0)} \right]$$

$$Y = \arctan\left[\tan\varphi \sec(\lambda - \lambda_0) \right]$$

APPENDIX III

LEAST SQUARES CONVERSION FROM ONE SYSTEM OF RECTANGULAR COORDINATES TO ANOTHER

Given two sets of coordinates on the same map, with a minimum of five points identified in both systems of coordinates, it is possible to convert the coordinates of one set to the other by a two-dimensional version of a least-squares "line". The procedure is most easily effected using complex numbers.

Let x,y be one set of coordinates and u,v be the other set, and let $W_j = x + iy$ and $Z_j = u + iv$, where $i^2 = -1$, be the complex numbers representing the i^{th} point. The objective is then to find the complex constants $A = a_1 + ia_2$ and $B = b_1 + ib_2$ in the equation $\hat{W} = A + BZ$ such that the squared residual

$$\sum_{j=1}^{N} \left| \hat{W}_j - W_j \right|^2$$

is a minimum. The normal equations are readily obtained by differentiation. The equation can be rewritten as a pair of transformation equations by separating the real and imaginary parts, viz:

$Re(\hat{W}) = \hat{x} = a_1 + b_1 u + b_2 v$

$Im(\hat{W}) = \hat{y} = a_2 + b_2 u - b_1 v$

where $\hat{x}$ and $\hat{y}$ are the estimates of the x,y coordinates as obtained from the known u,v coordinates. The standard error, etc., of the estimate can be obtained in a manner analogous to that employed for ordinary least squares procedures.

A similar, but considerably more complicated, procedure must be employed if the two sets of coordinates do not come from the same map, or if the relation to latitude and longitude is to be estimated, or if an attempt is made to determine the map projection of an arbitrary map.

APPENDIX IV

```
           R CLARKE ELLIPSOID OF 1866
           R DISTANCE AND DIRECTION / SODANO METHOD
           R  W. R. TOBLER / UNIVERSITY OF MICHIGAN / GEOGRAPHY
$COMPILE MAD, PUNCH OBJECT
            EXTERNAL FUNCTION (LT1,LG1,LT2,LG2,DIS,DIRD)
           R ENTRY IN RADIANS
           R RETURNS DISTANCE IN KILOMETERS
           R RETURNS DIRECTION IN DECIMAL DEGREES
           R ACIC TR 80, PAGES 41-47.
           R NECESSARY CONSTANTS
            VECTOR VALUES PI=314159265E-8
            VECTOR VALUES TPI=628318531E-8
            VECTOR VALUES ARAD=63782064E-4
            VECTOR VALUES BRAD=63565838E-4
            VECTOR VALUES BOVRA=9966099247E-10
            VECTOR VALUES VK1=2356218428E-7
            VECTOR VALUES VK2=6956258069E-5
            VECTOR VALUES VK3=4986428206E-9
            VECTOR VALUES VK4=-4010886986E-10
            VECTOR VALUES VK5=-7994556507E-10
            VECTOR VALUES VK6=3986428206E-9
            VECTOR VALUES E1=17036962E-10
            VECTOR VALUES E2=21769E-10
            VECTOR VALUES E3=29026E-10
            VECTOR VALUES E4=3628E-10
            VECTOR VALUES RAD=174532925E-10
           R BEGIN COMPUTATION
            ENTRY TO CLARKE.
            INDEX=1.
            TANB1=BOVRA*(SIN.(LT1)/COS.(LT1))
            TANB2=BOVRA*(SIN.(LT2)/COS.(LT2))
            COSB1=1./SQRT.(1.+(TANB1*TANB1))
            COSB2=1./SQRT.(1.+(TANB2*TANB2))
            SINB1=TANB1*COSB1
            SINB2=TANB2*COSB2
            C1=SINB1*SINB2
            D1=COSB1*COSB2
            DIFLON=LG2-LG1
            WHENEVER DIFLON.L.0.,INDEX=-1.
            DIFLON=.ABS.DIFLON
            CDIF=COS.(DIFLON)
            CDIS=C1+D1*CDIF
            SDIS=SQRT.(1.-CDIS*CDIS)
            CA=D1*SIN.(DIFLON)/SDIS
            CB=CA*CA
            CC=CDIS*(1.-CB)/VK3
            CD=VK4*C1
            CE=VK5*C1
            CF=VK6*CC
            CG1=2.*ATAN.(SQRT.((1.-CDIS)/(1.+CDIS)))
            CG=CG1/SDIS
            CX=CA*((CG1*(VK1+CB)+SDIS*(CC+CD)+CG*(CE+CF))/VK2)
            DELTAL=CX+DIFLON
            SDELTL=SIN.(DELTAL)
            CDELTL=COS.(DELTAL)
            DEN=TANB2*COSB1-SINB1*CDELTL
```

```
DIR=ATN1.(SDELTL,DEN)
WHENEVER DIR.G.PI,DIR=DIR-TPI
DIRD=DIR/RAD
DIRD=DIRD*INDEX
CPHO=C1+D1*CDELTL
SPHO=SQRT.(1.-CPHO*CPHO)
CBO=D1*SDELTL/SPHO
APHO=2.*ATAN.(SQRT.((1.-CPHO)/(1.+CPHO)))
SBO2=1.-CBO*CBO
C2DEL=(2.*C1/SBO2)-CPHO
C4DEL=(2.*C2DEL*C2DEL)-1.
SBO4=SBO2*SBO2
S2PHO=SIN.(2.*APHO)
AO=1.+E1*SBO2-E2*SBO4
BO=E1*SBO2-E3*SBO4
CO=E4*SBO4
DIS=BRAD*(AO*APHO+BO*SPHO*C2DEL-CO*S2PHO*C4DEL)
FUNCTION RETURN
END OF FUNCTION
```

APPENDIX V

```
          R CONVERSION OF PUBLIC LAND SURVEY INFORMATION
          R INTO LATITUDE AND LONGITUDE
          R SUBROUTINES NEEDED ARE DEGRAD, RADEG, SPHERE, AVERAD
          R W. R. TOBLER /UNIVERSITY OF MICHIGAN / GEOGRAPHY
$COMPILE MAD,PRINT OBJECT,PUNCH OBJECT,EXECUTE                    TRC
           INTEGER MER,C1,C2,C3,C4,C5
           INTEGER COMPAR,N,TWP,RNG,Q1,Q2,S,PRINC,N1
           INTEGER R,S,T
           D'N SECE(37),SECN(37),PMERID(36),BLINE(36),RMER(36)
           V'S DLT(1)=43.0,43.0,35.0,31.0,36.0,61.0,64.0,34.0,40.0,40.0,
          1 42.0,33.0,40.0,34.0,34.0,31.0,42.0,37.0,35.0,34.0,45.0,40.0,
          2 34.0,38.0,60.0,40.0,30.0,30.0,30.0,38.0,40.0,39.0,30.0,45.0,
          3 43.0
           V'S MLT(1)=59.0,22.0,1.0,52.0,30.0,49.0,51.0,38.0,59.0,0.0,
          1 30.0,22.0,25.0,59.0,29.0,0.0,25.0,52.0,44.0,15.0,47.0,46.0,
          2 7.0,28.0,7.0,0.0,59.0,59.0,26.0,28.0,25.0,6.0,59.0,31.0,0.0
           V'S SLT(1)=44.0,21.0,58.0,32.0,5.0,21.0,50.0,45.0,22.0,50.0,
          1 27.0,38.0,2.0,27.0,32.0,31.0,28.0,54.0,56.0,35.0,13.0,11.0,
          2 20.0,14.0,36.0,7.0,56.0,51.0,3.0,27.0,59.0,23.0,56.0,11.0,
          3 41.0
           V'S DLG(1)=104.0,116.0,89.0,90.0,103.0,145.0,147.0,91.0,84.0,
          1 90.0,90.0,112.0,124.0,86.0,97.0,92.0,84.0,121.0,108.0,106.0,
          2 111.0,111.0,116.0,86.0,149.0,97.0,91.0,88.0,84.0,89.0,109.0,
          3 108.0,91.0,122.0,108.0
           V'S MLG(1)=3.0,23.0,14.0,14.0,0.0,18.0,38.0,3.0,48.0,27.0,
          1 25.0,18.0,7.0,34.0,14.0,24.0,21.0,54.0,31.0,53.0,39.0,53.0,
          2 55.0,27.0,21.0,22.0,9.0,1.0,16.0,8.0,56.0,31.0,9.0,44.0,48.0
           V'S SLG(1)=16.0,35.0,47.0,41.0,7.0,13.0,26.0,7.0,11.0,11.0,
          1 37.0,19.0,10.0,16.0,49.0,55.0,53.0,47.0,59.0,12.0,33.0,27.0,
          2 17.0,21.0,24.0,8.0,36.0,20.0,38.0,54.0,6.0,59.0,36.0,34.0,
          3 49.0
           SECE(0)=0.0
           V'S SECE(1)=2.5,1.5,0.5,-0.5,-1.5,-2.5,-2.5,-1.5,-0.5,0.5,
          11.5,2.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-2.5,-1.5,-0.5,0.5,1.5,
          22.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-2.5,-1.5,-0.5,0.5,1.5,2.5
           SECN(0)=0.0
           V'S SECN(1)=2.5,2.5,2.5,2.5,2.5,2.5,1.5,1.5,1.5,1.5,1.5,1.5,
          10.5,0.5,0.5,0.5,0.5,0.5,-0.5,-0.5,-0.5,-0.5,-0.5,-0.5,-1.5,
          2-1.5,-1.5,-1.5,-1.5,-1.5,-2.5,-2.5,-2.5,-2.5,-2.5,-2.5
           R1=63782064E-04
           MILE=0.62136994
           ESQR=6768658E-09
           RAD=174532925E-10
           T'H INITAL, FOR I=1,1,I.G.35
           DLG(I)=-DLG(I)
           EXECUTE DEGRAD.(DLT(I),MLT(I),SLT(I),BLINE(I))
           EXECUTE DEGRAD.(DLG(I),MLG(I),SLG(I),PMERID(I))
           SMLT=SIN.(BLINE(I))
           DUM=SQRT.(1.-ESQR*SMLT*SMLT)
           DUMCUB=DUM.P.3
           DUMMY=(1.-ESQR)*R1
           RMER(I)=DUMMY*MILE/DUMCUB
 INITAL    CONTINUE
           R'T CONS,COMPAR
           V'S CONS=$S3,I1*$
           N=0
```

```
            N1=0
            P'T SKIP
            V'S SKIP=$1H1*$
START       W'R COMPAR.GE.1
            R'T LATLON,Q1,Q2,S,T,TWP,R,RNG,C1,C2,C3,C4,C5,
           1DLAT,MLAT,SLAT,DLON,MLON,SLON
            V'SLATLON=$2C1,S10,I2,S3,I2,C1,S3,I2,C1,S2,5C1,S16,F3.0,
           12F2.0,S1,F4.0,2F2.0*$
            EXECUTE DEGRAD.(DLAT,MLAT,SLAT,RLAT)
            EXECUTE DEGRAD.(DLON,MLON,SLON,RLON)
            O'E
            R'T INDAT,Q1,Q2,S,T,TWP,R,RNG,C1,C2,C3,C4,C5
            V'SINDAT =$2C1,S10,I2,S3,I2,C1,S3,I2,C1,S2,5C1   *$
            E'L
            N=N+1
            N1=N1+1
            W'R TWP.E.$N$
            A=T*6.0-3.0
            O'R TWP.E.$S$
            A=-(T*6.0-3.0)
            O'E
            T'O ERR
            E'L
            W'R RNG.E.$E$
            B=R*6.0-3.0
            O'R RNG.E.$W$
            B=-R*6.0+3.0
            O'E
            T'O ERR
            E'L
            W'R Q1.E.$N$
            QN=0.25
            O'R Q1.E.$S$
            QN=-0.25
            O'R Q1.E.$ $
            QN=0.0
            O'E
            T'O ERR
            E'L
            W'R Q2.E.$E$
            QE=0.25
            O'R Q2.E.$W$
            QE=-0.25
            O'R Q2.E.$ $
            QE=0.0
            O'E
            T'O ERR
            E'L
            W'R C1.E.$B$
              W'R C2.E.$O$
                MER=2
              O'E
                MER=1
              E'L
            O'R C1.E.$C$
              W'R C2.E.$I$
                MER=5
            O'R C2.E.$O$
              MER=6
```

```
  O'E
    W'R C3.E.$I$
      MER=3
    O'E
      MER=4
    E'L
  E'L
O'R C1.E.$F$
  MER=7
O'R C1.E.$5$
  MER=8
O'R C1.E.$1$
  MER=9
O'R C1.E.$4$
  W'R (C4.E.$A$).OR.(C5.E.$A$)
    MER=10
O'E
  MER=11
E'L
O'R C1.E.$G$
  MER=12
O'R C1.E.$H$
  W'R C3.E.$M$
    MER=13
  O'E
    MER=14
  E'L
O'R C1.E.$I$
  MER=15
O'R C1.E.$L$
  MER=16
O'R C1.E.$M$
  W'R C2.E.$I$
    MER=17
  O'E
    MER=18
E'L
O'R C1.E.$N$
  W'R C2.E.$A$
    MER=19
  O'E
    MER=20
  E'L
O'R C1.E.$P$
  MER=21
O'R C1.E.$S$
  W'R C3.E.$L$
    MER=22
  O'R C3.E.$N$
    MER=23
  O'R C3.E.$W$
    MER=25
O'R(C3.E.$H$).OR.(C4.E.$H$).OR.(C5.E.$H$)
    MER=27
  O'E
    MER=28
  E'L
O'R C1.E.$2$
    MER=24
```

```
            O'R C1.E.$6$
              MER=26
            O'R C1.E.$T$
             MER=29
            O'R C1.E.$U$
             W'R C2.E.$I$
               MER=31
             O'E
               MER=32
             E'L
            O'R C1.E.$W$
             W'R C3.E.$N$
               MER=35
             O'E
               MER=34
             W'R C2.E.$A$,MER=33
             E'L
            O'R C1.E.$3$
             MER=30
            O'E
             T'O ERR
            E'L
            A=A+SECN(S)+QN
            A=A/RMER(MER)
            LATIT=BLINE(MER)+A
            CLAT=COS.(LATIT)
            SMLT=SIN.(LATIT)
            DUM=SQRT.(1.-ESQR*SMLT*SMLT)
            RPAR=R1*MILE*CLAT/DUM
            B=(B+SECE(S)+QE)/RPAR
            LONGIT=PMERID(MER)+B
            EXECUTE RADEG.(LATIT,LTD,LTM,LTS)
            EXECUTE RADEG.(LONGIT,LGD,LGM,LGS)
            P'T ONE,N1
            V'S ONE=$1H ////S1,I4*$
            P'TRI,Q1,Q2,S,T,TWP,R,RNG,C1,C2,C3,C4,C5,
           1LTD,LTM,LTS,LGD,LGM,LGS
            V'SRI=$1H ,2C1,10H 1/4, SEC ,I2,3H, T,I2,C1,3H, R,I2,C1,
           12H, ,S2,5C1,10H. MERIDIAN //S1,2(F5.0,F3.0,F3.0,S5),
           219HCALCULATED LAT/LONG   *$
            W'R COMPAR.GE.1
            DIFLON=LONGIT-RLON
            DIFLAT=LATIT-RLAT
            EXECUTE RADEG.(DIFLAT,DLTD,DLTM,DLTS)
            EXECUTE RADEG.(DIFLON,DLGD,DLGM,DLGS)
            EXECUTE SPHERE.(RLAT,RLON,LATIT,LONGIT,RHO,ALPHA)
            EXECUTE AVERAD.(RLAT,LATIT,R1,ESQR,MRAD  )
            RHO=RHO*MRAD
            ALPHA=ALPHA/RAD
            P'TR2,DLAT,MLAT,SLAT,DLON,MLON,SLON,DLTD,DLTM,DLTS,DLGD,
           1DLGM,DLGS,RHO,ALPHA
            V'S R2=$1H ,2(F5.0,F3.0,F3.0,S5),17HOBSERVED LAT/LONG /S1,2(F
           15.0,F3.0,F3.0,S5),10HDIFFERENCE /S1,F11.4,S6,F9.4,S6,
           222HMILES AND DIRECTION      *$
            PUNCH FORMAT OUT,N1,RHO,ALPHA
            V'SOUT=$I5,S2,F11.4,S2,F9.4*$
            E'L
            TRANSFER TO START
ERR         PRINT FORMAT ONE,N
```

```
            PRINT COMMENT$ THIS OBSERVATION IS INCORRECTLY RECORDED$
            N1=N1-1
            TRANSFER TO START
            E'M
```

```
$COMPILEMAD,PUNCHOBJECT                                           AVERAD
            EXTERNAL FUNCTION (LLT,ULT,R1,ESQR,R3)
           R MEAN RADIUS ON ELLIPSOID
           R LATITUDES IN RADIANS
            ENTRY TO AVERAD.
            SMLT=SIN.((LLT+ULT)/2.)
            DUM=SQRT.(1.-ESQR*SMLT*SMLT)
            DUMCUB=DUM*DUM*DUM
            DUMMY=(1.-ESQR)*R1
            RMER=DUMMY/DUMCUB
            RPAR=R1/DUM
            R3=SQRT.(RMER*RPAR)
            FUNCTION RETURN
            END OF FUNCTION
```

```
$COMPILEMAD,PUNCHOBJECT                                           DEGRAD
            EXTERNAL FUNCTION (DEG,MIN,SEC,RAD)
           R SUBROUTINE TO CONVERT DEGREES TO RADIANS
            ENTRY TO DEGRAD.
            VECTOR VALUES RADIAN=174532925E-10
            SIGN=RADIAN
            WHENEVER DEG.L.0., SIGN=-RADIAN
            RAD=SIGN*(.ABS.(DEG)+(MIN/60.)+(SEC/3600.))
            FUNCTION RETURN
            END OF FUNCTION
```

```
$COMPILEMAD,PUNCHOBJECT                                           RADEG
            EXTERNAL FUNCTION (RAD,DEG,MIN,SEC)
           R  CONVERTS RADIANS TO DEGREES,MINUTES, AND DECIMAL SECONDS
            INTEGER I
            ENTRY TO RADEG.
            VECTOR VALUES CONS=206264806E-3
            SEC=.ABS.(RAD)*CONS
            I=SEC/3600.
            REMAIN=SEC-(I*3600.)
            DEG=I*1.
            I=REMAIN/60.
            MIN=I*1.
            SEC=REMAIN-(I*60.)
            WHENEVER RAD.L.0.,DEG=-DEG
            FUNCTION RETURN
            END OF FUNCTION
```

```
$COMPILE MAD,PRINT OBJECT,PUNCH OBJECT                            SPHERE
           R COMPUTES OBLIQUE SPHERICAL COORDINATES
            EXTERNAL FUNCTION(NLT,NLG,LAT,LON,RHO2,GA)
            VECTORVALUESPI=314159265E-8
            VECTORVALUESTPI=628318531E-8
            VECTORVALUESPIOVR2=157079633E-8
            VECTORVALUESEPS=0.0000001
            VECTORVALUESRAD=174532925E-10
```

```
ENTRY TO SPHERE.
WHENEVERNLT.E.(90.*RAD)
GA=LON-NLG
RHO2=PIOVR2-LAT
OTHERWISE
WHENEVER(LT.NE.NLT).OR.(LG.NE.NLG)
PI=314159265E-8
TPI=2.*PI
PIOVR2=PI/2.
EPS=0.0000001
CNLT=COS.(NLT)
SNLT=SIN.(NLT)
END OF CONDITIONAL
WHENEVER LON.NE.LON1
DIF=LON-NLG
CDIF=COS.(DIF)
SDIF=SIN.(DIF)
END OF CONDITIONAL
CLT=COS.(LAT)
SLT=SIN.(LAT)
Q=SLT*SNLT+CLT*CNLT*CDIF
WHENEVER Q.GE.1.
RHO2=0.
ORWHENEVER Q.LE.-1.
RHO2=PI
OTHERWISE
RHO2=ARCCOS.(Q)
END OF CONDITIONAL
NUM=CLT*SDIF
DEN=CNLT*SLT-SNLT*CLT*CDIF
WHENEVER.ABS.DEN.L.EPS
WHENEVER.ABS.NUM.L.EPS
GA=0.
OTHERWISE
GA=PIOVR2
WHENEVER NUM.L.0.,GA=-GA
END OF CONDITIONAL
ORWHENEVER.ABS.NUM.L.EPS
GA=0.
WHENEVER DEN.L.0.,GA=PI
OTHERWISE
GA=ATN1.(NUM,DEN)
WHENEVER GA.G.PI,GA=GA-TPI
END OF CONDITIONAL
LON1=LON
LT=NLT
LG=NLG
END OF CONDITIONAL
FUNCTION RETURN
END OF FUNCTION
```

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Part I provides a discussion of the usefulness of coordinate models for studies of geographically distributed phenomena with comments on specific coordinate systems and their relevance for the analysis and inventorying of geographical information. Appendices include equations for conversion from the Public Land Survey system into latitude and longitude and to rectangular map projection coordinates. Part II considers map projections in greater detail, including estimates of the errors introduced by the substitution of map projection coordinates for spherical coordinates. Statistical computations of finite distortion are related to Tissot's Indicatrix as a general contribution to the analysis of map projections.

14. KEY WORDS	LINK A		LINK B		LINK C	
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