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Technical Report

FINITE MOMENT FORMULAE AND PRODUCTS OF GENERALIZED k-STATISTICS WITH A GENERALIZATION OF FISHER'S COMBINATORIAL METHOD

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ABSTRACT

When the parent population is not completely known, a general method for finding an approximate form for the sampling distribution of a statistic is to ascertain its lower moments. Frequently these statistics are symmetric functions of the observations such as moments. Many pioneer workers found formulae for moments of moments, but the algebraic complexity of the results led Fisher (1928) to introduce the k-statistics k_p as symmetric functions of sample observations whose expected values are the parent cumulants κ_p. He developed a combinatorial method to express their cumulants and product cumulants.

Dressel (1940) introduced generalized k-statistics as sample functions whose expected values are products of cumulants. Tukey and Wishart gave some methods and results for products of generalized k-statistics, which were used to obtain moments of moments when sampling from a finite population. Dwyer and Tracy (1962) transformed Tukey's method to a combinatorial method for products of two generalized k-statistics and gave semi-general formulae for $k_{\{\}}\, k_{p_1 p_2 \ldots}$ where {} stands for any set of subscripts and $p_1+p_2+\ldots \leq 4$. Schaeffer and Dwyer (1963) introduced substitution products for unifying expectation and estimation theory.

The aim of this work is to generalize Fisher's combinatorial method to write products of generalized k-statistics as linear functions of the same and to use these to obtain formulae for moments of moments when sampling from a finite population to parallel his formulae for the infinite

case. With this aim, the following is done:

1. Additional rules for the combinatorial method for multiple products of generalized k-statistics are stated and proved, although the rules of Dwyer and Tracy (1962) are found to generalize to the case of multiple products.

2. Just as Fisher's combinatorial approach was based on the determination of a coefficient for certain patterns, so the combinatorial approach for the general case is based on the use of coefficients for patterns generalizing those of Fisher. These coefficients are determined and tabulated for the most common patterns.

3. Semi-general formulae for products of $k_{()}$ and $k_{p_1p_2\ldots}\ k_{q_1q_2\ldots}\ldots$ are provided through weight 4 of the second factor and products of semi-invariant generalized k-statistics are extended to weights 9, 10 and selected ones of weight 12. Checks for these are indicated as well.

4. These results are then applied to obtain moments of multiple products $M(p_1p_2\ldots) = E_N(k_{p_1}-K_{p_1})(k_{p_2}-K_{p_2})\ldots$, where E_N denotes the average over the sample values when sampling from a finite population of size N. Formulae for M(...) or K(...) are needed, for example, in a study of the distribution of ratio-statistics. These are tabulated for cases not involving the sample mean k_1 for weights through 10 and for selected ones of weight 12 and also the corresponding K(...) are given where they differ from the moments M(...). A useful check is provided by the fact that these formulae transform to those of Fisher as $N \to \infty$. Formulae relating moments involving k_1 to those not involving k_1 are also given.

5. Moment formulae can be transformed to estimation formulae rather easily when using generalized k-statistics. This fact is used to obtain estimators of M(...) and K(...), not involving k_1, using substitution products, thus extending the results of Schaeffer and Dwyer (1963).

The work thus generalizes Fisher's paper for the infinite case to finite populations.

INTRODUCTION

After giving a short history of the background material, the aims and objectives of this paper are described.

History

When the parent population is not completely specified, a general method for finding an approximate form for the sampling distribution of a statistic is to ascertain its lower moments. Frequently these statistics are symmetric functions of the observations such as moments and product moments. Thiele, Sheppard, "Student" and Tchouproff were among the pioneers to find formulae for moments of moments, but the algebraic complexity of the results and the amount of work required to reach them led to a search for simpler methods. In fact, Craig (1928)while extending Thiele's results to write semi-invariants of moments and product semi-invariants drew attention to the need of the use of functions other than crude moments if the algebraic formulation was to be made manageable. Fisher (1928) introduced the symmetric functions which provide great simplification for infinite populations. He defined the k-statistics k_p as unbiased estimators of the parent cumulants κ_p. On the basis of the simpler forms so obtained, he developed a combinatorial method to express cumulants and product cumulants of sampling distributions from infinite populations. A further development of this method was given by Fisher and Wishart (1931). Georgescu (1932) extended Craig's results and applied Fisher's idea of a combinatory analysis to the sample moment function. Kendall (1940 a,b,c, 1952) systematized Fisher's combinatorial technique by giving rules for the same and their proofs.

Dressel (1940) introduced symmetric functions whose expected values are products of cumulants $\kappa_{p_1} \kappa_{p_2} \ldots$ and Tukey (1950, 1956), denoting them by $k_{p_1 p_2 \ldots}$ and calling them generalized k-statistics or polykays, showed that these are also unbiased estimators of the corresponding finite population parameters $K_{p_1 p_2 \ldots}$. Wishart (1952) applied combinatorial methods to actually express products of k-statistics as linear combinations of generalized k-statistics. He also obtained products of generalized k-statistics by algebraic manipulation of the above, rather than by combinatorial methods. He applied these results to find moments for finite sampling. Tukey (1956) gave some rules and tables for the direct calculation of the products of two generalized k-statistics. Dwyer (1962) studied the properties of polykays of deviates from the mean. Dwyer and Tracy (1962) developed combinatorial methods for products of two generalized k-statistics and gave many semi-general formulae for $k_{\{\}} \, k_{p_1 p_2 \ldots}$ where $\{\}$ stands for any set of subscripts and $p_1 + p_2 + \ldots \leq 4$. Schaeffer and Dwyer (1963) gave practical methods for computation, introduced substitution products explained on page 75 for unifying expectation and estimation theory and extended the product formulae for generalized k-statistics not including a unit subscript (which are seminvariant in that they are independent of the choice of origin) through weight 8.

Objectives

The main aim of this paper is to generalize Fisher's combinatorial method to obtain multiple products of generalized k-statistics and thus to generalize Fisher's (1928) moment and cumulant formulae to the case of a finite population. With this aim in view, the following has been done:

1. After a review of the basic material, additional rules for the combinatorial method for multiple products are stated and proved, although the rules of Dwyer and Tracy (1962)are found to generalize to the case of multiple products.

2. Just as Fisher's combinatorial approach was based on the determination of a coefficient for certain patterns, so the combinatorial approach for the general case is based on the use of coefficients for patterns generalizing those of Fisher (1928, pp. 223-226). These coefficients are determined and tabulated for the most common patterns.

3. With these coefficients known, semi-general formulae for products of $k_{\{\}}$ and $k_{p_1 p_2 \ldots} \, k_{q_1 q_2 \ldots} \ldots$ are provided through weight 4 of the second factor. With the aid of these, products of seminvariant generalized k-statistics are extended to weights 9 and 10 and selected ones of weight 12 (following Fisher, 1928) and are presented in tabular form. Checks for these are also indicated.

4. Formulae adapted for computation of finite moments $M(p_1 p_2 \ldots) = E_N(k_{p_1} - K_{p_1})(k_{p_2} - K_{p_2})\ldots$ are derived, where E_N denotes the average over the sample values when sampling from a finite population of size N. One needs formulae for M(...) or K(...), for example, in a study of the distributions of ratio-statistics such as $\frac{k_3}{k_2^{3/2}}$ (which is used to measure departure from normality), where the denominator is expanded in the series $K_2^{-3/2}\left(1+\frac{k_2 - K_2}{K_2}\right)^{-3/2}$. These are tabulated for formulae not involving the sample mean k_1 ($p_i \neq 1$) for weights through 10 and for selected cases of weight 12 and also the corresponding K(...) are given where they differ from the moments M(...), generalizing Fisher's (1928)

table of formulae to the finite case. Fisher's formulae can be obtained as $N \rightarrow \infty$, which helps in checking. Formulae relating moments involving k_1 to those not involving k_1 are also given.

5. Estimators of M(...) and K(...) for $p_i \neq 1$ using substitution products are tabulated, extending the results of Schaeffer and Dwyer (1963).

CHAPTER I

BASIC MATERIAL

After defining the terms, some description of the generalized k-statistics is given. The algebraic method given by Tukey (1956) for writing the product of two generalized k-statistics is briefly described with the help of an example and its modification to a combinatorial method by Dwyer and Tracy (1962), following Fisher (1928), is illustrated with the same example. The steps of the combinatorial method are then outlined.

Notation and Definitions

Let a random sample $x_1, x_2, \ldots, x_n$ of size n be drawn from a finite population of size N (whose moments all exist), the sampling being done without replacement. The finite population itself may be looked upon as a random sample of size N from an infinite population. A sample symmetric function is a function of $x_1, x_2, \ldots, x_n$ whose value remains unaltered by any permutation of the x_i's amongst themselves. The sample augmented monomial symmetric functions (Kendall and Stuart, 1958, p.276) or power product sums (Dwyer, 1938, p. 12) are denoted by

$$[p_1\ p_2 \cdots p_s] = \sum_{i \neq j \neq \cdots}^{n} x_i^{p_1} x_j^{p_2} \cdots x_u^{p_s} \quad ,$$

where $p = \sum_{\alpha=1}^{s} p_\alpha$ is the weight and the number of parts s is the order of the symmetric function. If the p_α (distinct) are repeated π_α times,

$$\left[p_1^{\pi_1}\ p_2^{\pi_2} \cdots p_s^{\pi_s}\right] = \sum_{i \neq j \neq \cdots}^{n} x_i^{p_1} x_j^{p_1} \cdots x_q^{p_2} x_r^{p_2} \cdots x_t^{p_s} x_u^{p_s} \quad ,$$

$p = \sum_{\alpha=1}^{s} p_\alpha \pi_\alpha$ being the weight and $\sum_{\alpha=1}^{s} \pi_\alpha$ being the order of the function.

The <u>monomial symmetric function</u> is denoted by

$$(p_1^{\pi_1}\, p_2^{\pi_2} \ldots p_s^{\pi_s}) = \frac{[p_1^{\pi_1}\, p_2^{\pi_2} \ldots p_s^{\pi_s}]}{\pi_1!\,\pi_2! \ldots \pi_s!} \quad *$$

As an example,

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j}^{n} x_i x_j$$

can be expressed as

$$[1]^2 = [2] + [11] \quad ,$$

but also,

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{i<j}^{n} x_i x_j \; ,$$

so

$$(1)^2 = (2) + 2\,(11).$$

In terms of these, power sums s_r are just one-part functions, so that

$$s_r = \sum_{i=1}^{n} x_i^r = [r] = (r).$$

Tables have been provided by David and Kendall (1949) for expressing power sums and augmented monomial symmetric functions in terms of each other through weight 12.

<u>Symmetric means</u> or <u>mean power products</u> are defined as the means of products of powers of different x_i's. Since the sum $[p_1 p_2 \ldots p_\rho] = \sum_{i \neq j \neq \cdots}^{n} x_i^{p_1} x_j^{p_2} \ldots x_u^{p_\rho}$ is over $n(n-1)\ldots(n-\rho+1)$ terms, the symmetric mean $\langle p_1 p_2 \ldots p_\rho \rangle$ (termed angle bracket by Tukey, 1950) can be defined as

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* The notation $(p_1^{\pi_1} \ldots p_s^{\pi_s})$ is later used for partition coefficients.

$$\langle p_1 p_2 \cdots p_\rho \rangle = \frac{[p_1 p_2 \cdots p_\rho]}{n^{(\rho)}} .$$

Then, by a basic theorem of finite sampling theory (Dwyer, 1938, Tukey, 1950 : "inheritance on the average"),

$$E_N \langle p_1 p_2 \cdots p_\rho \rangle = \langle p_1 p_2 \cdots p_\rho \rangle_N$$

where E_N denotes the average over $N^{(n)}$ possible unordered sample values when sampling without replacement and $\langle p_1 p_2 \cdots p_\rho \rangle_N$ is the corresponding population bracket. In case sampling is from an infinite population (or from a finite population with replacement),

$$(1.1) \qquad E \langle p_1 p_2 \cdots p_\rho \rangle = \mu'_{p_1} \mu'_{p_2} \cdots \mu'_{p_\rho} ,$$

where μ' 's denote moments about the origin. (Kendall and Stuart, 1958, p. 276).

A <u>partition coefficient</u> $(p_1 \cdots p_s)$ of p is defined (Dwyer and Tracy, 1962),(Schaeffer and Dwyer, 1963), as the number of ways that the distinct units of p may be collected into distinct parcels described by the specified partition of p. For the ρ-part partition $(p_1^{\pi_1} \cdots p_s^{\pi_s})$, the partition coefficient

$$(p_1^{\pi_1} \cdots p_s^{\pi_s}) = \frac{p\,!}{(p_1\,!)^{\pi_1} \cdots (p_s\,!)^{\pi_s}\, \pi_1\,! \cdots \pi_s\,!} ,$$

where $\sum_{i=1}^{s} p_i \pi_i = p$ (weight), $\sum_{i=1}^{s} \pi_i = \rho$ (order). The multinomial theorem can then be expressed as

$$[1]^p = \sum (p_1 \cdots p_\rho) \, [p_1 \cdots p_\rho] ,$$

where the summation applies to every ρ-part partition of p and $\rho = 1, 2, \ldots, p$.

The p^{th} cumulant κ_p of the infinite population can be expressed in terms of the moments of the same by the formula

$$(1.2) \qquad \kappa_p = \sum_{\rho=1}^{p} \sum_{\rho \text{ fixed}} (-1)^{\rho-1} (\rho-1)! \, (p_1 \cdots p_\rho) \, \mu'_{p_1} \cdots \mu'_{p_\rho} ,$$

where the second summation is over all ρ-part partitions of p.

k-statistics

The p^{th} k-statistic k_p is defined (Fisher, 1928) to be the sample symmetric function such that $E(k_p) = \kappa_p$. In the case of a finite population, $E_N(k_p) = K_p$, where K_p, the K-parameter is the same function of the finite population as k_p is of the sample. Its uniqueness has been shown by Kendall and Stuart (1958) and also by David and Barton (1962) for all distributions. Then, from (1.1) and (1.2),

$$(1.3) \qquad k_p = \sum_{\rho=1}^{p} \sum_{\rho \text{ fixed}} (-1)^{\rho-1} (\rho-1)! \, (p_1 \cdots p_\rho) \langle p_1 \cdots p_\rho \rangle ,$$

as given essentially by Cornish and Fisher (1937, p.5).
The value of k_p, $p > 1$, is independent of origin (seminvariant) as shown by Kendall and Stuart (1958), and k_p, $p > n$, are not defined.

The k_p are homogeneous polynomials of degree p and can be written in terms of power sums. Such expressions for k_1 through k_6 are given by Fisher (1928), for k_7 and k_8 by Dressel (1940) and for k_9 and k_{10} by Zia-ud-Din (1954), who has also given an expression for k_{11} (1959).

Generalized k-statistics

The generalized (multiple) k-statistics $k_{p_1 p_2 \cdots}$ are symmetric functions of sample observations which have the basic property of being estimators of products of cumulants (Dressel, 1940), i.e., $E(k_{p_1 p_2 \cdots}) = \kappa_{p_1} \kappa_{p_2} \cdots$. It is further known that $E_N(k_{p_1 p_2 \cdots}) = K_{p_1 p_2 \cdots}$, where $K_{p_1 p_2 \cdots}$ is the same function of the finite population as $k_{p_1 p_2 \cdots}$ is of the sample. Tukey (1956) calls them polykays and defines them by a symbolic multiplication (o) in which products of brackets are replaced by brackets enclosing the product factors. Thus,

$$k_{p_1 p_2 \cdots} = k_{p_1} o k_{p_2} o \cdots$$

$$(1.4) \qquad = \sum (-1)^{\sum(\rho_i - 1)} \prod (\rho_i - 1)! \left[(p_{11} \cdots p_{1\rho_1})(p_{21} \cdots p_{2\rho_2}) \cdots \right] \langle p_{11} \cdots p_{1\rho_1} p_{21} \cdots p_{2\rho_2} \cdots \rangle ,$$

where the summation extends over all combinations of partitions $p_{i1}, \ldots ,$ $p_{i\rho_i}$ of p_i . It follows from (1.4) that

$$E\left[k_{p_1 p_2 \cdots}\right] = \sum (-1)^{\sum(\rho_i - 1)} \prod (\rho_i - 1)! \left[(p_{11} \cdots p_{1\rho_1})(p_{21} \cdots p_{2\rho_2}) \cdots \right] \mu'_{p_{11}} \cdots \mu'_{p_{1\rho_1}} \mu'_{p_{21}} \cdots \mu'_{p_{2\rho_2}} \cdots$$

$$(1.5) \qquad = \kappa_{p_1} \kappa_{p_2} \cdots .$$

Thus $k_{p_1 p_2 \cdots}$ is an unbiased estimator of $\kappa_{p_1} \kappa_{p_2} \cdots$.

Again $k_{p_1 p_2 \cdots}$ is uniquely defined for monomial symmetric functions by the implication

$$(1.6) \qquad E(k_{p_1 p_2 \cdots}) = \kappa_{p_1} \kappa_{p_2} \cdots$$

for all distributions. For suppose there is another monomial symmetric function $k'_{\cdots}$ of weight p whose expected value is also $\kappa_{p_1} \kappa_{p_2} \cdots$ for all distributions, then $E(k_{p_1 p_2 \cdots} - k'_{\cdots}) = 0$. But $k_{p_1 p_2 \cdots} - k'_{\cdots}$ is a monomial symmetric function being the difference of two such functions and can therefore be expressed as the sum of terms $\sum x^{p_i}$, $\sum x_a^{p_i} x_b^{p_j}$ etc. Thus its expectation is a sum of terms each of which is a moment-product. The vanishing of this series implies a polynomial identity relationship between the moments of x which is impossible except perhaps for a particular subclass of populations. Hence $k_{p_1 p_2 \ldots} - k'_{\ldots}$ vanishes identically and so $k_{p_1 p_2 \ldots} = k'_{\ldots}$. Some work of Halmos (1946) on the uniqueness of estimates is interesting in this connection.

It can also be noted that a generalized k-statistic $k_{p_1 p_2 \ldots}$ is seminvariant when no $p_i = 1$. For, by Taylor's theorem, if we write z for $x_1, \ldots, x_n$,

$$k_{p_1 p_2 \ldots}(z+h) - k_{p_1 p_2 \ldots}(z) = hf(z).$$

Taking expectations and remembering that when $p_i \neq 1$, κ_{p_i} is independent of origin, we have

$$0 = h\, E\, f(z) \tag{1.7}$$

In view of the remarks above, since a polynomial identity relationship among the moments for all distributions is impossible, $f(z) = 0$ and k is seminvariant when $p_i \neq 1$, all i. If one or more p_i is 1 the expectation of the corresponding generalized k-statistic involves $\kappa_1 = \mu_1'$ as a factor which certainly depends upon the origin.

An advantage of the generalized k-statistics derives from the fact that we can express them in terms of augmented symmetric functions once and for all (Wishart 1952) and hence derive non-linear functions of them as linear functions to which the Irwin-Kendall principle (Irwin and

Kendall, 1944, p. 138),(Kendall and Stuart, 1958, p. 301) will apply. This principle says that if for a symmetric function f, $E(f) = \sum a_j \chi_j$, then $E_N(f) = \sum a_j K_j$, for otherwise, $E_N(f)$ could be expressed as some other function of K's whose expectation E would be the same as that of $\sum a_j K_j$. This would imply a polynomial relationship among the K's.

The polykays enable us to write down unbiased estimators of products of cumulants. In fact, the operations of taking expectation and estimation become trivial after the functions of the observations x_i are reduced to linear functions of generalized k-statistics.

Combinatorial Method

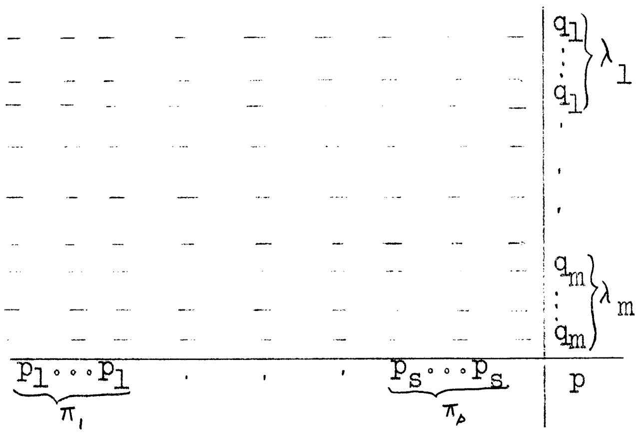
Fisher (1928) tackled the problem of writing the sampling moments and cumulants of k-statistics in terms of parent cumulants by algebraic as well as combinatorial methods. The problem is essentially that of finding mean values of powers and products of these k-statistics. To any number p with partition $p_1^{\pi_1} \ldots p_s^{\pi_s}$, there is a moment

$$\mu'(p_1^{\pi_1} \ldots p_s^{\pi_s}) = E(k_{p_1}^{\pi_1} \ldots k_{p_s}^{\pi_s})$$

and a cumulant $\chi(p_1^{\pi_1} \ldots p_s^{\pi_s})$ related to these moments by the usual identity in t's, (Kendall and Stuart, 1958, p.282)

$$\sum \left\{ \kappa(p_1^{\pi_1} \cdots p_s^{\pi_s}) \prod \frac{t_{p_i}^{\pi_i}}{\pi_i!} \right\} = \log \left\{ \sum \mu'(q_1^{\lambda_1} \cdots q_m^{\lambda_m}) \prod \frac{t_{q_j}^{\lambda_j}}{\lambda_j!} \right\}$$

where p_i, i = 1,2,..., s and q_j, j = 1,2,..., m are column and row totals repeated π_i, λ_j times respectively ($\sum_i p_i \pi_i = \sum_j q_j \lambda_j = p$) for the two-way array



where a row corresponds to every κ in $\kappa_{q_1}^{\lambda_1} \ldots \kappa_{q_m}^{\lambda_m}$ and a column to every part in $\kappa(p_1^{\pi_1} \ \ldots \ p_s^{\pi_s})$ and we consider all the ways in which the body of the array can be completed by the insertion of numbers whose column and row totals are the respective p_i, q_j. To take an example from Kendall and Stuart (1958), when seeking the coefficient of $\kappa_6 \kappa_2^2$ in $\kappa(4^2 2)$, we consider such arrays as

2	2	2	6
1	1	0	2
1	1	0	2
4	4	2	10

2	3	1	6
1	1	0	2
1	0	1	2
4	4	2	10

3	3	0	6
1	0	1	2
0	1	1	2
4	4	2	10

Fisher's (1928) empirical rules were stated more formally by Kendall (Kendall and Stuart, 1958) for writing the sampling cumulants of k-statistics using a combinatorial method. A proof of these rules is also provided by Kendall (Kendall and Stuart, 1958, Chapter 13) by employing an operator.

The algebraic coefficients (p.25) of many useful patterns of arrays have been provided by Fisher (1928). Wishart (1952) modified the Fisher-Kendall rules in order to obtain products of k-statistics and also used combinatorial methods. Then he proceeded to find products of generalized k-statistics algebraically and has listed all such products through weight 6. He has also given products of single subscript k's through weight 8. Products of seminvariant generalized k-statistics (independent of the origin, i.e. not having unit parts) up to weight 8 were given by Schaeffer and Dwyer (1963).

Tukey (1956) gave a method for finding expressions for products of two generalized k-statistics using a table for multiplication of brackets (from which we have adapted Table 1) and using a rule involving the number of unit parts.

Table 1

COEFFICIENTS FOR MULTIPLICATION OF
TWO BRACKETS

Number of parts in multiplying brackets	Number of parts in bracket whose coefficient sought				
	1	2	3	4	5
1 x 1	$\frac{1}{n}$	$1 - \frac{1}{n}$	-	-	-
1 x 2	-	$\frac{1}{n}$	$1 - \frac{2}{n}$	-	-
1 x 3	-	-	$\frac{1}{n}$	$1 - \frac{3}{n}$	-
2 x 2	-	$\frac{1}{n^{(2)}}$	$\frac{1}{n} - \frac{1}{n^{(2)}}$	$1 - \frac{4}{n} + \frac{2}{n^{(2)}}$	-
2 x 3	-	-	$\frac{1}{n^{(2)}}$	$\frac{1}{n} - \frac{2}{n^{(2)}}$	$1 - \frac{6}{n} + \frac{6}{n^{(2)}}$

As an example of bracket multiplication,

$$
\begin{aligned}
\langle a\rangle\langle b\rangle &= \frac{1}{n^2}\sum_{i=1}^{n} x_i^{\,a} \sum_{j=1}^{n} x_j^{\,b} \\
&= \frac{1}{n^2}\left\{ \sum_{i\neq j}^{n} x_i^{\,a} x_j^{\,b} + \sum_{k=1}^{n} x_k^{\,a+b} \right\} \\
&= \frac{1}{n^2}\left\{ [ab] + [a+b] \right\} \\
&= \frac{1}{n^2}\left\{ n(n-1)\langle ab\rangle + n\,\langle a+b\rangle \right\} \\
&= (1-\frac{1}{n})\langle ab\rangle + \frac{1}{n}\langle a+b\rangle \,.
\end{aligned}
$$

The first row of Table 1 expresses this result. In general, one has to obtain "all products which can be obtained by matching some (including none) of the letters in one bracket with letters in the other and then replacing matched letters by their sum." (Tukey, 1956, p.46).

Tukey (1956) also observed that when a bracket with g unit parts is written in terms of polykays, only polykays with at least g unit parts appear and vice versa. He used $O(1^g)$ for any set of terms each of which, when expanded linearly in brackets or polykays, contains at least g unit parts. Also, he used the term <u>unit weight</u> of an expression for the maximum number of unit parts appearing in any term of that expression. In terms of these, the rule is that while expressing a polynomial in polykays as a linear combination of the same, the unit weight on the linear side can not exceed the unit weight on the other side. This implies that the coefficient of every k... having more unit subscripts than the set of original subscripts is zero.

The multiplication of polykays is carried out in three steps:

(a) expressing each $k_{...}$ as a linear function of brackets,

(b) multiplying out the brackets, and

(c) reconverting the resulting brackets to polykays.

For example, Tukey (1956) considers

$$k_{21}\, k_2 = (k_2 \circ k_1)\, k_2$$

$$= \left[\left\{\langle 2\rangle - \langle 11\rangle\right\} \circ \langle 1\rangle\right] \left[\langle 2\rangle - \langle 11\rangle\right]$$

$$= \left[\langle 2\rangle \circ \langle 1\rangle - \langle 11\rangle \circ \langle 1\rangle\right]\left[\langle 2\rangle - \langle 11\rangle\right]$$

$$= \left[\langle 21\rangle - \langle 111\rangle\right]\left[\langle 2\rangle - \langle 11\rangle\right] \text{, using (1.4)}$$

(1.8) $$= \langle 21\rangle\langle 2\rangle - \langle 111\rangle\langle 2\rangle - \langle 21\rangle\langle 11\rangle + \langle 111\rangle\langle 11\rangle$$

$$= (1 - \tfrac{2}{n})\langle 221\rangle + \tfrac{1}{n}\langle 32\rangle + \tfrac{1}{n}\langle 41\rangle - 2(\tfrac{1}{n} - \tfrac{1}{n^{(2)}})\langle 221\rangle$$

$$- \tfrac{2}{n^{(2)}}\langle 32\rangle + \tfrac{6}{n^{(2)}}\langle 221\rangle + 0(1^2)\text{, using Table 1 and the rule of unit parts}$$

$$= (1 - \tfrac{4}{n} + \tfrac{8}{n^{(2)}})\left\{k_{221} \quad 0(1^2)\right\} + (\tfrac{1}{n} - \tfrac{2}{n^{(2)}})\left\{k_{32} + 3k_{221}\right.$$

$$\left. + 0(1^2)\right\} + \tfrac{1}{n}\left\{k_{41} + 3k_{221} + 0(1^2)\right\} + 0(1^2)$$

$$= (\tfrac{1}{n} - \tfrac{2}{n^{(2)}})\, k_{32} + \tfrac{1}{n}\, k_{41} + (1 + \tfrac{2}{n} + \tfrac{2}{n^{(2)}})\, k_{221} + 0(1^2)$$

(1.9) $$= \frac{n-3}{n(n-1)}\, k_{32} + \frac{1}{n}\, k_{41} + \frac{n+1}{n-1}\, k_{221}$$

Tukey, however, believed in "sparing the use of combinatorial techniques as much as we are able" (Tukey, 1956, p.37). Dwyer and Tracy (1962) have modified Tukey's algebraic method to a direct combinatorial method using arrays which consists of the same three steps.

The first step, as in Tukey's method, gives (1.8) which appears at the top of Table 2.

The multiplication of brackets is achieved more directly. To express $\langle 21\rangle\langle 2\rangle$ for example, since

$$(1.10)\qquad \Big(\sum_{i\neq j}^{n} x_i^2 x_j\Big)\Big(\sum_{k=1}^{n} x_k^2\Big) = \sum_{i\neq j}^{n} x_i^4 x_j + \sum_{i\neq j}^{n} x_i^2 x_j^3 + \sum_{i\neq j\neq k}^{n} x_i^2 x_j x_k^2$$

$$\text{or, } [21]\,[2] = [41] + [32] + [221]$$

$$\text{or, } n^2(n-1)\langle 21\rangle\langle 2\rangle = n(n-1)\langle 41\rangle + n(n-1)\langle 32\rangle + n(n-1)(n-2)\langle 221\rangle,$$

we have

$$(1.11)\qquad \langle 21\rangle\langle 2\rangle = \frac{1}{n}\langle 41\rangle + \frac{1}{n}\langle 32\rangle + \frac{n-2}{n}\langle 221\rangle .$$

The coefficients in (1.11) are termed <u>n-coefficients</u> and one need not actually go through all the steps from (1.10) to (1.11) in order to obtain them in a given case. For a bracket having ρ parts, the n-coefficient in the expansion of $k_{p_1 p_2 \ldots} k_{q_1 q_2 \ldots} \ldots$ is simply $\dfrac{n^{(\rho)}}{n^{(r)}\,n^{(s)}}$, where r, s are the total number of parts of the partitions of the p_i and the q_j respectively. In the example, then, the n-coefficient for $\langle 41\rangle$ and $\langle 32\rangle$ is $\dfrac{n^{(2)}}{n^{(2)}n^{(1)}} = \dfrac{1}{n}$ and that for $\langle 221\rangle$ is $\dfrac{n^{(3)}}{n^{(2)}n^{(1)}} = \dfrac{n-2}{n}$.

All these results are indicated in Table 2. Both partitions of 21, i.e. 21 and 111, are matched with both partitions of 2, i.e. 2 and 11, in all possible ways (except for permutations by rows), filling in 0's where needed. The process leads to the 12 arrays which appear in Table 2 and in each of which the first column represents a partition

TABLE 2

ILLUSTRATION FOR $k_{21}k_2$

$k_{21}k_2 = [<21> - <111>][<2> - <11>] = <21><2> - <21><11> - <111><2> + <111><11>$

Array No.		1	2	3	4	5	6	7	8	9	10	11	12
Array		$\begin{array}{c\|c} 22 & 4 \\ 10 & 1 \end{array}$	$\begin{array}{c\|c} 20 & 2 \\ 12 & 3 \end{array}$	$\begin{array}{c\|c} 21 & 3 \\ 11 & 2 \end{array}$	$\begin{array}{c\|c} 21 & 3 \\ 10 & 1 \\ 01 & 1 \end{array}$	$\begin{array}{c\|c} 12 & 3 \\ 10 & 1 \\ 10 & 1 \end{array}$	$\begin{array}{c\|c} 20 & 2 \\ 10 & 1 \\ 02 & 2 \end{array}$	$\begin{array}{c\|c} 20 & 2 \\ 11 & 2 \\ 01 & 1 \end{array}$	$\begin{array}{c\|c} 11 & 2 \\ 11 & 2 \\ 10 & 1 \end{array}$	$\begin{array}{c\|c} 20 & 2 \\ 10 & 1 \\ 01 & 1 \\ 01 & 1 \end{array}$	$\begin{array}{c\|c} 10 & 1 \\ 10 & 1 \\ 10 & 1 \\ 02 & 2 \end{array}$	$\begin{array}{c\|c} 11 & 2 \\ 10 & 1 \\ 10 & 1 \\ 01 & 1 \end{array}$	$\begin{array}{c\|c} 10 & 1 \\ 10 & 1 \\ 10 & 1 \\ 01 & 1 \\ 01 & 1 \end{array}$
Formula Coeff.		1	1	-1	-1	-1	1	-1	1	-1	-1	1	1
Comb. Coeff.		1	1	2	2	3	1	2	6	1	1	6	1
n-Coeff.		$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n(n-1)}$	$\frac{n-2}{n(n-1)}$	$\frac{1}{n}$	$\frac{n-2}{n}$	$\frac{n-2}{n(n-1)}$	$\frac{1}{n(n-1)}$	$\frac{(n-2)(n-3)}{n(n-1)}$	$\frac{n-3}{n}$	$\frac{n-3}{n(n-1)}$	$\frac{(n-3)(n-4)}{n(n-1)}$
μ'-Coeff.		$\frac{1}{n}$	$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	$\frac{-2(n-2)}{n(n-1)}$	$\frac{-3}{n}$	$\frac{n-2}{n}$	$\frac{-2(n-2)}{n(n-1)}$	$\frac{6}{n(n-1)}$	$\frac{-(n-2)(n-3)}{n(n-1)}$	$\frac{-n-3}{n}$	$\frac{6(n-3)}{n(n-1)}$	$\frac{(n-3)(n-4)}{n(n-1)}$
Transformation Coefficient	1	1			2	2	1		2	1	1	4	1
	2		1			1	1	2		1	1	2	1
	3			1	1			1	2	1		3	1
	4				1					1		2	1
	5					1					1	2	1
	6						1			1	1		1
	7							1		1		1	1
	8								1			2	1
	9									1			1
	10										1		1
	11											1	1
	12												1
k-Coeff.		$\frac{1}{n}$	$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	0	0	1	0	$\frac{2}{n-1}$	0	0	0	0

of 21 and the second a partition of 2 while the marginal column represents the resulting bracket. In general the arrays appear in the order of expansion of bracket products, but for convenience in subsequent steps, they are grouped according to the number of rows. The arrays for the product $\langle 21 \rangle \langle 2 \rangle$ are thus numbered 1, 2 and 6 in Table 2. The coefficient obtained from formula (1.8) at the top of the table is termed formula coefficient and shown in the row directly below the arrays. To avoid extensive repetition, equivalent arrays resulting from the permutations of the second column entries have been grouped together and a compensatory combinatorial coefficient supplied in the row so labeled. The n-coefficient is obtained in the manner described above or from Table 1. The product of these three coefficients is the μ'-coefficient or bracket coefficient for the moment product or the bracket indicated by the marginal column. More than one array may lead to the same moment product or bracket.

The result after the first two steps is

$$k_{21}k_2 = \frac{1}{n} \langle 41 \rangle + \frac{1}{n} \langle 32 \rangle - \frac{2}{n(n-1)} \langle 32 \rangle - \frac{2(n-2)}{n(n-1)} \langle 311 \rangle + \dots . \tag{1.12}$$

These results are equivalent to those obtained by Tukey (1956) by the direct algebraic method though, since he does not use arrays, the coefficient of a specified array can not be identified in his result (1.9).

The third step in the derivation requires the expansion of various brackets in terms of generalized k-statistics. Tables are available (Tukey, 1956, p.44), (Abdel-Aty, 1954), (David and Kendall, 1949) for assisting in this. However, the device of introducing the parent cumulants, recommended by Kendall (1952, p.15), and obtaining the final formula by estimation is used. In this example, for instance, from (1.12) or from the μ'-coefficient row in Table 2,

$$(1.13) \qquad E(k_{21}k_2) = \frac{1}{n}\mu_4' \ \mu_1' + \frac{1}{n}\mu_3' \ \mu_2' - \frac{2}{n(n-1)}\mu_3' \ \mu_2' - \frac{2(n-2)}{n(n-1)}\mu_3' \mu_1'^2 + \dots \ .$$

Now, to transform from μ''s to κ's, the two components of the μ''s corresponding to the two columns of the arrays need to be distinguished. This is achieved by using a multipartite notation. Thus the μ_4' of the first row of array number 1 is treated as the bipartite μ_{22}'. Then the expansion in terms of bipartite κ's is

$$(1.14) \quad \mu_{22}' = \kappa_{22} + 2\kappa_{21}\kappa_{01} + 2\kappa_{12}\kappa_{10} + \kappa_{20}\kappa_{02} + 2\kappa_{11}\kappa_{11} + \kappa_{20}\kappa_{01}\kappa_{01} + \kappa_{02}\kappa_{10}\kappa_{10} + 4\kappa_{11}\kappa_{10}\kappa_{01} + \kappa_{10}\kappa_{10}\kappa_{01}\kappa_{01}$$

and the transform of $\mu_{22}'\mu_{10}'$ is the right side of (1.14) multiplied by κ_{10}. The coefficients appear in the first row of "Transformation Co-efficient" of Table 2. Similar transformation coefficients for the μ's indicated by array numbers 2,3, ... , 12 appear in row numbers 2,3, ... , 12 of the "Transformation Coefficient". Coefficients from the unipartite expansions are available for checking.

The calculation of the κ- or <u>k-coefficient</u> is then straightforward. We observe, for example, that array number 4 has non-zero transformation coefficients in rows 1, 3 and 4 (which means that array numbers 1, 3 and 4 yield array number 4 as a separate(page 22). To obtain the k-coefficient for array number 4, then, we multiply the transformation coefficient in the i^{th} row of this column by the μ'-coefficient of array number i and form the sum. For array number 4, the k-coefficient is thus

$$1 \left(\frac{-2(n-2)}{n(n-1)} \right) + 1 \left(\frac{-2}{n(n-1)} \right) + 2 \cdot \frac{1}{n} = 0.$$

A simple way to obtain the k-coefficient of an array is then to multiply each transformation coefficient appearing in that column in the i^{th} row by the μ'-coefficient of array number i and form the sum over i. In practice, the arrays and rows need not be numbered and this step can be achieved by multiplying the transformation coefficients in a column by the μ'-coefficients of the columns indicated by the diagonal terms of "Transformation Coefficient" and forming the sum.

We find in this example that

$$(1.15) \qquad E(k_{21}k_2) = \frac{1}{n}\kappa_4\kappa_1 + \frac{1}{n}\kappa_3\kappa_2 - \frac{2}{n(n-1)}\kappa_3\kappa_2 + \kappa_2\kappa_2\kappa_1 + \frac{2}{n-1}\kappa_2\kappa_2\kappa_1$$

so that, taking estimates, we have

$$(1.16) \qquad k_{21}k_2 = \frac{1}{n}k_{41} + \frac{1}{n}k_{32} - \frac{2}{n(n-1)}k_{32} + k_{221} + \frac{2}{n-1}k_{221}.$$

It can be seen that (1.16) is in good form for approximation with large n and does show the contribution of each array.

Establishment of general rules applicable to the contribution of a given array makes possible further condensation and the development of a true combinatorial method.

We observe that the k-coefficient of every array having a marginal partition with two or more unit parts is zero. This agrees with Tukey's (1956) rule that the coefficient of every $k_{...}$ having more unit subscripts than the set of original subscripts is zero. This eliminates array numbers 4, 5, 9, 10, 11 and 12.

The zero k-coefficient of array number 7 is very noticeable now. This case is not covered by Tukey's rule, but follows from the rule of proper parts,* established later for general products that the coefficient is zero for any array having at least one row with a single non-zero element which is a proper part of some integer subscript. The 1 in the third row of array number 7 is a proper part of the 2. Tukey's rule follows as a corollary since any additional unit subscript must come from a row with a single proper unit part.

For array number 8, although the combinatorial coefficient is 6, the two arrays resulting from the matching of the unit parts of 2 in 2 with unit parts of 21 are the ones contributing $\frac{1}{n-1}$ each to the coefficient, while the other four arrays feature proper parts and hence do not contribute anything. The first two then belong to one <u>array type</u> and the other four to another. The distinction becomes more obvious if we think of the elements of the arrays as composed of distinct units. For example, let e_1, e_2 be the two units comprising the 2 of k_{21} and let e_3 constitute the 1. Again, let e_4, e_5 be the units in the 2 of k_2. Then, array number 8 with a combinatorial coefficient of 6, is actually representing the following six arrays:

$$\begin{matrix} e_1 & e_4 \\ e_2 & e_5 \\ e_3 & 0 \end{matrix}\;,\quad \begin{matrix} e_1 & e_5 \\ e_2 & e_4 \\ e_3 & 0 \end{matrix}\;,\quad \begin{matrix} e_1 & e_4 \\ e_3 & e_5 \\ e_2 & 0 \end{matrix}\;,\quad \begin{matrix} e_1 & e_5 \\ e_3 & e_4 \\ e_2 & 0 \end{matrix}\;,\quad \begin{matrix} e_2 & e_4 \\ e_3 & e_5 \\ e_1 & 0 \end{matrix}\;,\quad \begin{matrix} e_2 & e_5 \\ e_3 & e_4 \\ e_1 & 0 \end{matrix}\;.$$

The first two belong to one array type, the contribution of each being $\frac{1}{n-1}$, whereas the last four have coefficients zero since e_1 and e_2 are proper parts appearing alone in a row.

* A proper part of a partition of an integer is any positive integral value less than the integer.

The need of combinatorial coefficients is eliminated when we consider distinct units and the transformations coefficient may be looked upon as a separations coefficient, indicating the number of ways an "amalgamation" can be separated into a particular "separate" (as used by Fisher, 1928). In Table 2, array number 4 is a separate of array number 1 and array number 1 is an amalgamation of array number 4. With the use of distinct units, of course, each separations coefficient is either 1 or 0, depending upon whether that separate can be obtained from the given amalgamation, and Table 2 transforms to Table 3. Array numbers 3a, 3b originate from array number 3 of Table 2 and similarly array numbers 8a, 8b from array number 8. The vertical lines separate array types from each other. The k-coefficients of arrays of the same array type are the same.

We can observe that array number 2 can be obtained from array number 1 by permuting the entries in the second column, and similarly array number 3b from 3a and 8b from 8a (leaving the row e_3 0 fixed). The contribution to $k_{21}k_2$ from the first two arrays which is

$$\frac{1}{n}\left(k_{(e_1+e_2)+(e_4+e_5),\ e_3}+k_{(e_1+e_2),\ e_3+(e_4+e_5)}\right)$$

can be expressed then, as

$$\frac{1}{n}\,k_{(e_1+e_2),\ e_3+(e_4+e_5),\ 0} = \frac{1}{n}\,k_{2,1+2,0}$$

where $k_{a,b+c,d}$ indicates the sum of the k's with sums of subscripts permuted, i.e. $k_{a,b+c,d} = k_{a+c,\ b+d} + k_{a+d,\ b+c}$. Thus, $k_{21+20} = k_{41} + k_{23}$ (the commas may be dropped when unnecessary). Further, for contribution from array numbers 8a and 8b, where the row e_3 0 is fixed and the entries in the other two rows are permuted, we

TABLE 3

$k_{21}k_2$ WITH USE OF DISTINCT UNITS

ARRAY NO.	1	2	3a	3b	6	8a	8b
Array	e_1+e_2 e_4+e_5 e_3 0	e_1+e_2 0 e_3 e_4+e_5	e_1+e_2 e_4 e_3 e_5	e_1+e_2 e_5 e_3 e_4	e_1+e_2 0 e_3 0 0 e_4+e_5	e_1 e_4 e_2 e_5 e_3 0	e_1 e_5 e_2 e_4 e_3 0
Formula Coeff.	1	1	-1	-1	1	1	1
n-Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$
μ'-Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$-\frac{1}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$
Separations Coefficient	1				1	1	1
		1			1		
			1				
				1			
					1		
						1	
							1
k -Coefficient (distinct units)	$\frac{1}{n}$	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$-\frac{1}{n(n-1)}$	1	$\frac{1}{n-1}$	$\frac{1}{n-1}$

use the notation $k_{e_1\ ,e_2;e_3+e_4,e_5;0}$ to indicate the sum of the k's with sums of permuted subscripts prior to the semi-colons. Then,

$$(1.17)\qquad k_{21}k_2 = \frac{1}{n}\, k_{21+20} - \frac{1}{n(n-1)}\, k_{21+11} + k_{221} + \frac{1}{n-1}\, k_{11;\ 1+11;0}$$

where $k_{21+20} = k_{41} + k_{23}$, $k_{21+11} = 2k_{32}$, $k_{11;\ 1+11;0} = 2k_{221}$.

We can now treat $k_{p_1\ p_2}k_2$ on similar lines. We use p_{i1}, p_{i2}, ... to indicate parts of p_i. Then,

$$\begin{aligned} k_{p_1\ p_2}k_2 &= \Big[\langle p_1\ p_2\rangle - \sum(p_{11}\ p_{12})\langle p_{11}\ p_{12}\ p_2\rangle - \sum(p_{21}\ p_{22})\langle p_1\ p_{21}\ p_{22}\rangle \\ &\qquad + \ldots\Big]\Big[\langle 2\rangle - \langle 11\rangle\Big] \\ &= \langle p_1\ p_2\rangle\langle 2\rangle - \langle p_1\ p_2\rangle\langle 11\rangle + \sum(p_{11}\ p_{12})\langle p_{11}\ p_{12}\ p_2\rangle\langle 11\rangle \\ &\qquad + \sum(p_{21}\ p_{22})\langle p_1\ p_{21}\ p_{22}\rangle\langle 11\rangle + \ldots \end{aligned}$$

$$(1.18)\qquad \begin{aligned} &= \langle p_1\ p_2\rangle\langle 2\rangle - \langle p_1\ p_2\rangle\langle 11\rangle + \sum T\langle p_{11}\ p_{12}\ p_2\rangle\langle 11\rangle \\ &\qquad + \sum T\langle p_1\ p_{21}\ p_{22}\rangle\langle 11\rangle + \ldots , \end{aligned}$$

where the summations are over all 2-part partitions of p_1, p_2 respectively and $T\langle p_{11}\ p_{12}\ p_2\rangle$, called a bracket type, symbolizes $(p_{11}\ p_{12})$ similar brackets. When we consider $E(k_{p_1\ p_2}k_2)$, only the first four terms appearing explicitly in (1.18) contribute non-vanishing coefficients, other bracket products having zero coefficients by the rule of proper parts. Hence, in order to obtain a formula for $k_{p_1\ p_2}k_2$, we need only consider the bracket products corresponding to the four terms explicitly indicated in (1.18).

We now need the concept of a conditional amalgamation. It is an amalgamation in keeping with the conditions of addition of the rows, e.g. p_{i2} can not be added to p_{j1}. The only conditional amalgamation of the array type

$$\begin{matrix} p_{i1} & q_{r1} \\ p_{i2} & q_{r2} \\ p_{i3} & q_{s1} \\ p_j & q_{s2} \end{matrix}$$

is

$$\begin{matrix} p_{i1}+p_{i2} & q_r \\ p_{i3} & q_{s1} \\ p_j & q_{s2} \end{matrix}$$

, $(r = r1 + r2)$, since only parts of the same p_i, ... can be added together. Hence, for two rows to be additive, the non-zero entries in a column should be parts of the same subscript. Thus, $\begin{matrix} 2 & 2 \\ 1 & 0 \end{matrix}$ is a conditional amalgamation of $\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ while considering the products k_3k_2 or $k_{21}k_2$, but not while dealing with $k_{111}k_2$.

For a two-column array type having ρ rows, with $\sum_i r_i = r$ non-zero entries p_{ir_i} in one column and $\sum_j s_j = s$ non-zero entries q_{js_j} in the other, the formula coefficient is $\prod_i (-1)^{r_i-1} (r_i-1)! \prod_j (-1)^{s_j-1} (s_j-1)!$ and the n-coefficient is $\frac{n^{(\rho)}}{n^{(r)}\, n^{(s)}}$. Let us absorb these two in the <u>n'-coefficient</u> =

$$\frac{\prod_i(-1)^{r_i-1} (r_i - 1)! \prod_j (-1)^{s_j-1} (s_j - 1)!\, n^{(\rho)}}{n^{(r)}\, n^{(s)}}$$

Then the <u>algebraic coefficient</u> of the array type is obtained by adding the n'-coefficients of the array type and all its conditional amalgamations

(Rule 1, Chapter II). Thus the algebraic coefficient of $\begin{matrix} p_{11} & 1 \\ p_{12} & 1 \\ p_2 & 0 \end{matrix}$ in considering $k_{p_1p_2}k_2$, with $\begin{matrix} p_1 & 2 \\ p_2 & 0 \end{matrix}$ as the only conditional amalgamation, is

$$\frac{1}{n(n-1)} + \frac{1}{n} = \frac{1}{n-1} .$$

With this setup, a condensed method for obtaining $k_{p_1p_2}k_2$ is presented in Table 4.

Table 4

CONDENSED METHOD FOR THE PRODUCT $k_{p_1p_2}k_2$

Array No.	1-2	6	3	8a	8b
Array type	p_1 2 p_2 0	p_1 0 p_2 0 0 2	p_1 1 p_2 1	p_{11} 1 p_{12} 1 p_2 0	p_1 0 p_{21} 1 p_{22} 1
Alg. Coeff.	$\frac{1}{n}$	1	$\frac{-1}{n(n-1)}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$
Comb. Coeff.	1	1	1	$(p_{11}p_{12})$	$(p_{21}p_{22})$
k-coeff.	$\frac{1}{n}$	1	$\frac{-1}{n(n-1)}$	$\frac{(p_{11}\ p_{12})}{n-1}$	$\frac{(p_{21}\ p_{22})}{n-1}$

A general formula for $k_{p_1p_2}k_2$ is then

$$k_{p_1p_2}k_2 = \frac{1}{n} k_{p_1p_2+20} + k_{p_1p_2 2} - \frac{1}{n(n-1)} k_{p_1p_2+11} + \frac{1}{n-1} \sum (p_{11}p_{12}) k_{p_{11}p_{12};p_2+11;0}$$

$$+ \frac{1}{n-1} \sum (p_{21}p_{22}) k_{p_{21}p_{22};p_1+11;0} . \qquad (1.19)$$

If one subscript, say p_2, is 1, the last term in (1.19) vanishes since 1 has no 2-part partitions. The formula is also applicable to the case with $p_2 = 0$ if we drop the p_2 from all terms containing p_2 as a subscript, and drop all terms containing other functions of p_2 as subscripts. Thus,

$$(1.20) \qquad k_{p_1} \; k_2 = k_{p_1 2} + \frac{1}{n} k_{p_1+2} + \frac{2}{n-1} \sum (p_{11} \; p_{12}) \, k_{p_{11}+1,\; p_{12}+1} \; .$$

Semi-general Formulae

To obtain more general formulae, $\{\}$ is used to represent the set $p_1, p_2, \ldots$ and $\{q_1\}$ indicates the group of array types in which q_1 appears in a row with any element of $\{\}$. Similarly, $\{q_1 q_2\}$ indicates the groups of array types in which q_1, q_2 appear in different rows with elements of $\{\}$. The notation $\{\,\}q_1 q_2$ is used for the array type in which q_1, q_2 appear alone in additional rows with the initial array type. Such an array type is termed an extended array type and is defined as one which consists of an initial array type plus additional rows in which elements are p's (but not proper parts of p's) matched with zeros, or q's (but not proper parts of q's) matched with zeros.

The array types $\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_2 & 0 \end{matrix}$, $\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_2 & 0 \\ 0 & q_2 \end{matrix}$ are some examples of extended array types when the initial array type is $\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \end{matrix}$.

The notation $\{q_1\}\, q_2$ symbolizes the array type with q_1 in a row with some element of $\{\ \}$ and an additional row containing q_2. Double subscripts indicate partitions. Thus $\{q_{11}q_{12}\}\, q_2$ indicates an array type in which the 2-part partitions of q_1 are added concurrently to two elements of $\{\ \}$ and the row containing q_2 is added. Also, $\{p_{11}p_{12} + q_{11}q_{12}:\}$ is used to indicate the array type in which the 2-part partitions of p_1 appear with the 2-part partitions of q_1, the other p's appearing alone. Similarly, $\{p_{11}p_{12} + q_{11}q_{12}:\ q_2\}$ indicates an array type like the previous one with, in addition, q_2 appearing in a row with some p_i, $i \neq 1$. And $\{p_{11}p_{12} + q_{11}q_{12}:\ q_2\}\, q_3 \ldots$ is used to indicate the extended array types with additional rows of q's.

In this notation, we can write the semi-general formula

$$(1.21) \qquad k_{\{\ \}}k_2 = k_{\{\ \}2} + \frac{1}{n}\, k_{\{2\}} - \frac{1}{n(n-1)}\, k_{\{11\}} + \frac{1}{n-1} \sum (p_{i1}\ p_{i2})\, k_{\{p_{i1}p_{i2}+11:\}}\,.$$

Dwyer and Tracy (1962) have provided formulae for $k_{\{\ \}}\, k_{q_1 \cdots q_s}$ for $\sum_j q_j \leq 4$.

From (1.21), to write $k_{p_1p_2}k_2$ for instance, we have

$$k_{p_1p_2}k_2 = k_{p_1p_22} + \frac{1}{n}\, k_{p_1p_2+20} - \frac{1}{n(n-1)}\, k_{p_1p_2+11} + \frac{1}{n-1} \sum (p_{11}\ p_{12}) k_{p_{11}p_{12};p_2+11;0}$$

$$+ \frac{1}{n-1} \sum (p_{21}\ p_{22}) k_{p_{21}p_{22};p_1+11;0},$$

which is the same as (1.19). To get $k_{42}k_2$ now, we can use either (1.19) or (1.21) and obtain

$$k_{42}k_2 = k_{422} + \frac{1}{n}(k_{62} + k_{44}) - \frac{2}{n(n-1)}\, k_{53} + \frac{2}{n-1}\,(4k_{422} + 3k_{332}) + \frac{2}{n-1}\, k_{422}$$

$$= \frac{1}{n}\, k_{62} + \frac{1}{n}\, k_{44} - \frac{2}{n(n-1)}\, k_{53} + \frac{6}{n-1}\, k_{332} + (1 + \frac{10}{n-1})\, k_{422}.$$

<u>Steps of the Combinatorial Method</u>

The steps of the combinatorial method for products of generalized k-statistics can now be stated.

1. Write each generalized k-statistic of the product desired in terms of bracket types.

2. List all possible arrangements of the products of bracket types in which the bracket type components of the first factor are placed in the first column, those of the second factor in the second column, etc., to form the array types. In so doing ignore any array type which has a proper part as a single non-zero element of a row.

3. Compute the combinatorial coefficient for the array type by forming the product of all partition coefficients associated with every partition appearing in the columns of the array type.

4. Compute the algebraic coefficient for each array type as indicated above and using the rules of Chapter II and the results of Chapter III.

5. Multiply the algebraic coefficient by the combinatorial coefficient to obtain the k-coefficient for each array type. The listing of the k-coefficient in the column for the k-term gives the result in combinatorial form.

More explicitly,

6. Write the formula for the sums of the products of the k-coefficients and the k-terms.

7. Expand each of the k-terms to feature explicit k's if more explicit form is desired.

CHAPTER II

GENERAL RULES FOR COMBINATORIAL METHOD

We now consider some rules which are helpful when it is desired to express products of generalized k-statistics as a linear combination of such statistics, (e.g. for purposes of taking expectation and estimation), using a combinatorial method. The first four rules are generalizations of the rules of Dwyer and Tracy (1962) which they used for double products.

Rule 1. Algebraic Coefficient Rule

The algebraic coefficient of an array type is obtained by adding the n'-coefficients of the array type and all its conditional amalgamations.

In order to establish this rule, we need to minimize the effect of the combinatorial coefficient. We need not be concerned with the formula coefficient since it is fixed for each array type. We use distinct units to eliminate the combinatorial coefficients since each of them then becomes unity. Using bracket types (1.18) to indicate all the brackets with distinct units, we have

$$k_{p_1 p_2 \cdots} = (-1)^{\Sigma(\rho_i - 1)} \prod(\rho_i - 1)!\, T\langle p_{11} \cdots p_{1\rho_1} p_{21} \cdots p_{2\rho_2} \cdots \rangle$$

$$(2.1) \qquad = \Sigma f(\rho_i)\, T\langle p_{11} \cdots p_{1\rho_1} p_{21} \cdots p_{2\rho_2} \cdots \rangle$$

with $f(\rho_i) = (-1)^{\Sigma(\rho_i - 1)} \prod(\rho_i - 1)!$, a formula coefficient. Then

$$(2.2) \quad k_{p_1 p_2 \cdots} k_{q_1 q_2 \cdots} k_{\cdots} \cdots = \Sigma f(\rho_i) f(\lambda_j) \cdots T\langle p_{11} \cdots p_{1\rho_1} p_{21} \cdots p_{2\rho_2} \cdots \rangle T\langle q_{11} \cdots q_{1\lambda_1} q_{21} \cdots q_{2\lambda_2} \cdots \rangle T\langle \cdots \rangle \cdots$$

where $f(\rho_i) f(\lambda_j) \cdots$ is the formula coefficient and the combinatorial coefficient of every array is unity. With distinct units, as in Table 3, the separations coefficients for every conditional amalgamation is unity and the algebraic coefficient is simply the sum of the n'-coefficients for the array type and its conditional amalgamations.

Releasing the condition of distinct units so many brackets in (2.2) may be indentical, this quantity is multiplied by the combinatorial coefficient for the collection.

Rule 2. Pattern Rule

Array types with the same pattern have the same algebraic coefficient. Array types are said to have the same pattern when the various groups of the partition parts correspond in location.

Thus, the array types $p_{11} = 3$ $q_{11} = 2$ and $p_{11} = 4$ $q_{11} = 3$

$p_{12} = 2$ $q_{12} = 2$ $\quad$ $p_{12} = 1$ $q_{12} = 1$

$p_2 = 1$ $q_2 = 2$ $\quad$ $p_2 = 1$ $q_2 = 2$

in the expansion of $k_{51}k_{42}$ and the array type $p_{11} = 5$ $q_{11} = 4$

$p_{12} = 4$ $q_{12} = 3$

$p_2 = 8$ $q_2 = 2$

in the expansions of $k_{98}k_{72}$ have the same pattern. For this reason, array types and patterns are used in a synonymous sense.

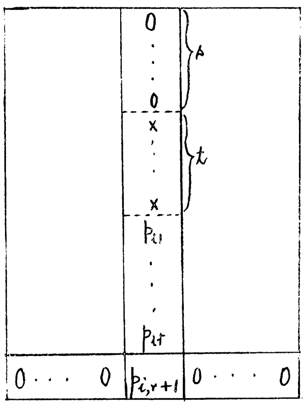
This rule follows from Rule 1 when we notice that array types with the same pattern have the same n'-coefficient and similar conditional amalgamations ("similar" means having the same pattern). In the example considered, the algebraic coefficient is $\frac{1}{n(n-2)}$ in each case since the only conditional amalgamations result from adding the first two rows.

Rule 3. General Rule of Proper Parts

The algebraic coefficient is zero for every array type in which there is at least one row in which a proper part appears alone.

A proof can be given like that for the case of products of two generalized k-statistics as shown by Dwyer and Tracy (1962). Let the proper part appearing along in a row be 1 and let all other non-zero entries be greater than 1. Then each of the $k_{...}$ terms arising from the array type has a unit subscript. Since the product expansion does not have any $k_{...}$ terms with unit subscript, the k-coefficient and hence the algebraic coefficient must be zero (since the combinatorial coefficient can not be zero).

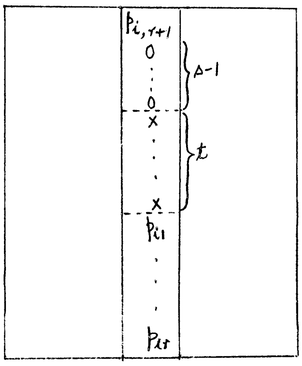
A more formal proof is now presented. Consider an array type with a proper part $p_{\lambda,r+1}$ of p_λ appearing alone in a row. Let the other entries in this column consist of the remaining r parts $p_{\lambda 1}$, ..., $p_{\lambda r}$ of p_λ, s zeros and t other non-zero entries (indicated by crosses in the figure). If we let A absorb the product of $\frac{(-1)^{\rho-1}(\rho-1)!}{n^{(\rho)}}$ for all columns except the one considered (ρ = number of non-zero entries in a column) and the product of all partition coefficients of $p \neq p_\lambda$, then the contribution to the algebraic coefficient from this array is



$$\frac{(-1)^r\, r!}{n^{(r+t+1)}}\, n^{(r+s+t+1)}(p_{\lambda 1}\ \cdots\ p_{\lambda,r+1})\, A = (-1)^r\, r!\ (p_{\lambda 1}\cdots\ p_{\lambda,r+1})\ (n-r-t-1)^{(s)} A.$$

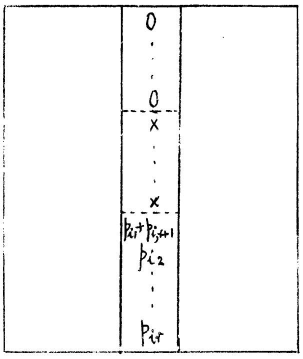
If we consider only additions of the row with the proper part, the resulting conditional amalgamations belong to two array types:

1) s amalgamations of the type when one of the s zero entries of this column is replaced by the proper part $p_{i,r+1}$, resulting by adding that row to one of the s rows having a zero entry in this column. Since the contribution to the coefficient of each such amalgamation is $(-1)^r\, r!\, (p_{i1} \ldots p_{i,r+1})(n-r-t-1)^{(s)} A$, the contribution of s such amalgamations is



$$(-1)^r\, r!\, s\, (p_{i1} \ldots p_{i,r+1})(n-r-t-1)^{(s)} A.$$

2) r amalgamations resulting from the addition of the proper part row to a row having a part of p_i in this column. For example, if we added the proper part row to the row having p_{i1} in this column, the amalgamation will yield a separations coefficient $(p_{i1}, p_{i,r+1})$ to be multiplied by $(-1)^{r-1}(r-1)!\, (p_{i1}+p_{i,r+1}, p_{i2}, \ldots, p_{ir})(n-r-t)^{(s)} A$.



The contribution of such an amalgamation to the coefficient is then $(-1)^{r-1}(r-1)!\, (p_{i1}, \ldots, p_{i,r+1})(n-r-t)^{(s)} A$, since $(p_{i1}, p_{i,r+1})(p_{i1}+p_{i,r+1}, p_{i2}, \ldots, p_{ir})$ $= (p_{i1}, \ldots, p_{i,r+1})$. Thus the contribution of r such amalgamations is $(-1)^{r-1} r!\, (p_{i1}, \ldots, p_{i,r+1})(n-r-t)^{(s)} A$.

Hence the total contribution is

$$(-1)^r\, r!(p_{i1}, \ldots, p_{i,r+1})(n-r-t-1)^{(s)} A \left[(n-r-t-s) + s - (n-r-t)\right],$$

which is zero.

This total contribution is similarly zero for each conditional amalgamation involving additions of sets of the other r + s + t rows among themselves. Thus the coefficient for this array type is $\sum 0 = 0$.

Corollary: Tukey's Rule. When expressing a polynomial in polykays as a linear combination of the same, the unit weight on the linear side can not exceed the unit weight on the other side.

This is so since in the linear expansion of any term of the polynomial, any additional unit subscript must come from a row with a single proper unit part.

Rule 4. Rule for Extended Array Types

The algebraic coefficient of an extended array type is the same as that of the initial array type.

Let us consider an initial array type extended by a row containing a single p_i (not a proper part). Let the column containing this entry have s zero and r other non-zero entries. If C indicates the product of signs and factorials for all columns and $\frac{1}{n^{(\ell)}}$ for all columns but this particular one, the contribution of the array type and its s conditional amalgamations resulting from adding the new row to the initial ones, but not involving any amalgamations of the initial rows, is

$$C.\frac{1}{n^{(r+1)}}\, n^{(r+s+1)} + sC\frac{1}{n^{(r+1)}}\, n^{(r+s)}$$

$$= C.\frac{1}{n^{(r+1)}}\, n^{(r+s)}\Big[(n-r-s) + s\Big] = C.\frac{1}{n^{(r+1)}}\, n^{(r+s)}\,(n-r) = C.\frac{1}{n^{(r)}}\, n^{(r+s)},$$

which is the contribution to the algebraic coefficient for the initial array type. Since this equality holds for every conditional amalgamation of the initial array type and the corresponding contribution of the extended array type, the two algebraic coefficients are the same. The argument applies when more rows of this type are added.

Rule 5. Rule for Augmented Array Types

If the non-zero entries of some columns of an array type are such that they are carried all the way through in all conditional amalgamations and do not impose any further restrictions in the addition of rows, the algebraic coefficient of the array type is the product of the coefficients $\frac{(-1)^{\rho-1}(\rho-1)!}{n^{(\rho)}}$ of these columns (ρ being the number of non-zero entries) and the algebraic coefficient for the rest of the array type. (An array type augmented by these columns is termed an augmented array type when the rest of the array type is looked upon as the initial array type).

The reason for this rule is obvious as each column of this type contributes a factor equal to its own coefficient to the contribution of each conditional amalgamation, and all of these conditional amalgamations are just conditional amalgamations of the initial array type augmented by these columns.

Let us illustrate this rule by considering an augmented array type having one such column. The algebraic coefficient of the array type

$$\begin{matrix} p_{11} & q_{11} & r_{11} \\ p_2 & 0 & r_{12} \\ p_{12} & q_{12} & 0 \\ p_{13} & q_{13} & 0 \end{matrix} \quad \text{is} \quad \frac{-1}{n(n-1)} \quad \text{times the coefficient of} \quad \begin{matrix} p_{11} & q_{11} \\ p_{12} & 0 \\ p_{21} & q_{12} \\ p_{22} & q_{13} \end{matrix}, \text{ i.e.,}$$

$$\frac{-1}{n(n-1)} \cdot \frac{n}{(n-1)(n-2)} = \frac{-1}{(n-1)(n-2)} .$$ (The coefficient of

$$\begin{matrix} p_{11} & q_{11} \\ p_2 & 0 \\ p_{12} & q_{12} \\ p_{13} & q_{13} \end{matrix} \quad \text{is the same as that of} \quad \begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{13} & q_{13} \end{matrix}$$ by Rule 6). Each amalgamation of the two-column array type is just augmented by the extra column

having r_{11} , r_{12} and its effect is only to multiply all contributions by $\frac{-1}{n(n-1)}$. Since no addition of rows in $\begin{matrix} p_{11} & q_{11} \\ p_{2} & 0 \\ p_{12} & q_{12} \\ p_{13} & q_{13} \end{matrix}$ is restricted by the presence of this column, each and every amalgamation of the two-column array type appears in the augmented array type with this column appended. Hence the overall effect is a multiplication of the coefficient for the initial array type by $\frac{-1}{n(n-1)}$.

In contrast, the coefficient of $\begin{matrix} p_{11} & q_{11} & r_{1} \\ p_{12} & 0 & r_{2} \\ p_{21} & q_{12} & 0 \\ p_{22} & q_{13} & 0 \end{matrix}$ is $\frac{1}{n(n-1)(n-2)}$, which is <u>not</u> $\frac{1}{n(n-1)}$ times the coefficient of $\begin{matrix} p_{11} & q_{11} \\ p_{12} & 0 \\ p_{21} & q_{12} \\ p_{22} & q_{13} \end{matrix}$ (which is zero by the rule of proper parts). The reason is that although the last column is carreid in all amalgamations, it imposes restrictions in the addition of the first two rows. These can be added in the two-column array type, but not when the last column is augmented.

In $\begin{matrix} p_{1} & q_{11} & r_{1} \\ p_{21} & 0 & r_{2} \\ p_{22} & q_{12} & 0 \\ p_{23} & q_{13} & 0 \end{matrix}$, however, the first two rows of the two column array type $\begin{matrix} p_{1} & q_{11} \\ p_{21} & 0 \\ p_{22} & q_{12} \\ p_{23} & q_{13} \end{matrix}$ can not be added anyway, so the last column does not impose any additional restrictions in the addition of rows. Also it is carried through in all amalgamations as such, hence the coefficient for the array type is $\frac{1}{n(n-1)}$ times the coefficient of

$\begin{matrix} p_1 & q_{11} \\ p_{21} & 0 \\ p_{22} & q_{12} \\ p_{23} & q_{13} \end{matrix}$, i.e., zero (by the rule of proper parts). All the zero coefficients in Table 5 are attributable to this phenomenon (pages 55-63).

This rule takes a specially simple form when these columns are composed of single non-zero entries. It can then be stated as:

Rule 5a. The algebraic coefficient of an array type having r columns with single non-zero entries is $\frac{1}{n^r}$ times that for the array type with these columns deleted.

For, suppose C is the coefficient of the array type with these r columns deleted. Since no additions for conditional amalgamations depend on these r columns, their only effect is to contribute a factor $\frac{1}{n}$ each, i.e. $\frac{1}{n^r}$, towards the contribution of each amalgamation and hence the algebraic coefficient of the augmented array type is $\frac{1}{n^r}$ C.

Rule 6. Blocks Rule

The algebraic coefficient of an array type which falls into separate blocks is the product of the coefficients for these blocks.

An array type is said to fall into separate blocks if the columns can be divided into two or more classes, each confined to different sets of rows. Fisher (1928) was able to ignore array types consisting of blocks (Kendall and Stuart, 1958, p.283, Rule 3) in the case of single subscript k's as he proceeded straight to cumulants. Wishart (1952) has given this rule for products of single subscript k's.

For a proof, we first consider the case of two blocks A and B, having a and b rows and c and d columns respectively. Let the c columns of A have $a_1, \ldots, a_c$ non-zero entries and the d columns of B have $b_1, \ldots, b_d$ non-zero entries. At this stage, we do not consider any addition within the blocks. Let the symbols A and B absorb the signs and factorials of the blocks A and B respectively.

Let us suppose $a \geq b$ (no loss of generality in doing this due to symmetry). In a typical amalgamation, r rows of B are amalgamated with r rows of A, there being $\frac{a^{(r)}b^{(r)}}{r!}$ such amalgamations. The contribution of each of these to the coefficient is $\frac{AB}{\prod_i n^{(a_i)} \prod_j n^{(b_j)}} n^{(a+b-r)}$. Thus the contribution to the algebraic coefficient from this set of amalgamations, allowing no addition within the blocks, is

$$\frac{AB}{\prod n^{(a_i)} \prod n^{(b_j)}} \sum_{r=0}^{b} \frac{a^{(r)} b^{(r)}}{r!} n^{(a+b-r)} = \frac{AB}{\prod n^{(a_i)} \prod n^{(b_j)}} n^{(a)} \sum_{r=0}^{b} \binom{b}{r} a^{(r)} (n-a)^{(b-r)}$$

$$= \frac{AB}{\prod n^{(a_i)} \prod n^{(b_j)}} n^{(a)} n^{(b)} \quad \text{by Vandermonde's Theorem (Riordan, 1958, p. 9)}$$

$$= \frac{An^{(a)}}{\prod n^{(a_i)}} \cdot \frac{Bn^{(b)}}{\prod n^{(b_j)}},$$

i.e. the product of coefficients for the two blocks.

If we now allow addition within blocks, let the sets of conditional amalgamations be denoted by $\{A_p\}$, $\{B_q\}$. Then, the total algebraic coefficient for the array type $\begin{matrix} A & 0 \\ 0 & B \end{matrix}$ is

$\sum_{p,q}$ Contribution to algebraic coefficient for amalgamation $\begin{matrix} A_p & 0 \\ 0 & B_q \end{matrix}$

$= \sum_{p,q}$ (Contribution of Block A_p)(Contribution of Block B_q)

$= (\sum_p$ Contribution of Block $A_p)(\sum_q$ Contribution of Block $B_q)$

$=$ (Algebraic coefficient of A)(Algebraic coefficient of B).

If now there are three blocks A, B, C, we can treat $\begin{matrix} A & 0 \\ 0 & B \end{matrix}$ as one block and C as the second. Repeating this, we find that the rule holds for any number of blocks.

Rule 7. Rule for Column-bordered Blocks

The algebraic coefficient of an array type consisting of two blocks A and B connected by a common column C, whose portions going with the two blocks are denoted by C_A and C_B, is given by the following rule*:

A	C_A	0
0	C_B	B

(a) When no parts of any p_i occuring in C_A appear in C_B and vice versa, the algebraic coefficient of the array type is the product of the coefficients for the blocks $\boxed{A \mid C_A}$ and $\boxed{C_B \mid B}$.

(b) When there is at least one p_i which has parts in both C_A and C_B, the algebraic coefficient of the array type is zero.

It makes a difference in the proof whether the common column C is solid or not. Solid means that there are no zero entries in the column. The case of a solid column is simpler, so we consider it first.

Case 1. Common column C is solid.

(a) If we just consider the array type

A	C_A	0
0	C_B	B

without considering any conditional amalgamations, it is fairly easy to see that the rule holds. For if no p_i is common to C_A and C_B, the co-

* So far a general proof is not available, but the rule has been established for all cases needed in compiling the products of generalized k-statistics through weight 12.

efficients of the two blocks $\boxed{A \;|\; C_A}$ and $\boxed{C_B \;|\; B}$ are independent of each other. Also, if the contribution of $\boxed{A}$ to the algebraic co-efficient of $\boxed{A \;|\; C_A}$ is denoted by A and the signs and factorials for $\boxed{C_A}$ are absorbed in C_A and similarly for $\boxed{C_B \;|\; B}$, the algebraic coefficient of the array type

$$\begin{array}{|c|c|c|} \hline A & C_A & 0 \\ \hline 0 & C_B & B \\ \hline \end{array}$$

is $AB \dfrac{C_A C_B}{n^{(a+b)}} n^{(a+b)}$ $= ABC_AC_B$. Also the algebraic coefficient of $\boxed{A \;|\; C_A}$ is $A \dfrac{C_A}{n^{(a)}} n^{(a)} = AC_A$ and similarly that for $\boxed{C_B \;|\; B}$ is BC_B. Evidently then, the rule holds when we do not consider amalgamations.

The argument is extendable when amalgamations are allowed. For then, if we denote sets of amalgamations by $\{A_p\}$, $\{B_q\}$, the total algebraic coefficient for the array type

$$\begin{array}{|c|c|c|} \hline A & C_A & 0 \\ \hline 0 & C_B & B \\ \hline \end{array}$$

is

$$\sum_{p,q} \text{Contribution for amalgamation } \begin{array}{|c|c|c|} \hline A_p & C_{A_p} & 0 \\ \hline 0 & C_{B_q} & B_q \\ \hline \end{array}$$

$$= \sum_{p,q} \left(\text{Contribution of } \boxed{A_p \;|\; C_{A_p}}\right)\left(\text{Contribution of } \boxed{C_{B_q} \;|\; B_q}\right)$$

$$= \left(\sum_p \text{Contribution of } \boxed{A_p \;|\; C_{A_p}}\right)\left(\sum_q \text{Contribution of } \boxed{C_{B_q} \;|\; B_q}\right)$$

$$= \left(\text{Algebraic coefficient of } \boxed{A \;|\; C_A}\right)\left(\text{Algebraic coefficient of } \boxed{C_B \;|\; B}\right).$$

(b) Let now a particular p have a parts in C_A and b parts in C_B. Let us again suppose $a \geq b$, without loss of generality.

Not allowing for any other addition of rows except those involving parts of this particular p, a typical amalgamation consists of r rows of B added to r rows of A, $(r \leq b)$, there being $\dfrac{a^{(r)} b^{(r)}}{r!}$ such amalgamations. If the c columns of A have $a_1, \ldots, a_c$ non-zero entries respectively and the d columns of B have $b_1, \ldots, b_d$ non-zero entries, the contribution of this amalgamation to the algebraic coefficient is

$$(-1)^{a+b-r-1} \frac{ABC_AC_B}{\prod n^{(a_i)}\prod n^{(b_j)}} (a+b-r-1)! \quad ,$$

where A and B absorb the signs and factorialsfor blocks A and B respectively and C_A, C_B for entries in C_A, C_B which are not parts of this particular p. Then the algebraic coefficient of the array type is

$$\sum_{r=0}^{b} (-1)^{a+b-r-1} \frac{a^{(r)}b^{(r)}}{r!} \frac{ABC_AC_B}{\prod n^{(a_i)}\prod n^{(b_j)}} (a+b-r-1)!$$

$$= (-1)^{a+b-1} \frac{ABC_AC_B}{\prod n^{(a_i)}\prod n^{(b_j)}} (a-1)! \sum_{r=0}^{b} (-1)^{-r} \frac{a^{(r)}b^{(r)}}{r!} a^{[b-r]}$$

$$= (-1)^{a+b-1} \frac{ABC_AC_B}{\prod n^{(a_i)}\prod n^{(b_j)}} (a-1)! (-1)^{b} \sum_{r=0}^{b} \binom{b}{r} a^{(r)}(-a)^{(b-r)}$$

since $a^{[b-r]} = (-1)^{b-r}(-a)^{(b-r)}$

$= 0$, since $\sum_{r=0}^{b}\binom{b}{r}a^{(r)}(-a)^{(b-r)} = (a-a)^{(b)}$ by Vandermonde's Theorem
$= 0$.

As before, if we now allow for additions which were earlier restricted, the total algebraic coefficient of the array type

A	C_A	0
0	C_B	B

is $\sum_{p,q}$ Contribution for amalgamation

A_p	C	0
0	C	B_q

$= \sum_{p,q} 0 = 0$.

<u>Case 2.</u> Common column C is not solid.

(a) Let the number of rows in A, B be a, B and let C_A, C_B have e, f non-zero and g, h zero entries respectively. It may be noted that $e+g = a$, $f+h = b$. Absorbing signs and factorials in A, B, C_A, C_B, the term contributed by each amalgamation is $\frac{ABC_A C_B}{\prod n^{(a_i)} \prod n^{(b_j)}}$ multiplied by the appropriate factor in the following scheme. We are not allowing for additions within the blocks at this stage.

Number of rows of B added to those of A	Number of Amalgamations	Multiplying Factor
0	1	$(n-e-f)^{(g+h)}$
1	eh + fg + gh	$(n-e-f)^{(g+h-1)}$
2	$g^{(2)}fh + h^{(2)}eg + \frac{f^{(2)}g^{(2)}}{2!} + \frac{g^{(2)}h^{(2)}}{2!} + \frac{e^{(2)}h^{(2)}}{2!} + efgh$	
3	$\frac{e^{(3)}h^{(3)}}{3!} + \frac{f^{(3)}g^{(3)}}{3!} + \frac{g^{(3)}h^{(3)}}{3!} + \frac{g^{(3)}f^{(2)}h}{2!} + \frac{g^{(3)}fh^{(2)}}{2!} + \frac{h^{(3)}e^{(2)}g}{2!} + \frac{h^{(3)}eg^{(2)}}{2!} + \frac{ef^{(2)}g^{(2)}h}{2!} + \frac{e^{(2)}h^{(2)}fg}{2!} + efg^{(2)}h^{(2)}$	$(n-e-f)^{(g+h-3)}$
4	$\frac{f^{(4)}g^{(4)}}{4!} + \frac{e^{(4)}h^{(4)}}{4!} + \frac{g^{(4)}h^{(4)}}{4!} + \frac{e^{(3)}gh^{(4)}}{3!} + \frac{eg^{(3)}h^{(4)}}{3!} + \frac{f^{(3)}g^{(4)}h}{3!} + \frac{fg^{(4)}h^{(3)}}{3!} + \frac{e^{(2)}g^{(2)}h^{(4)}}{4} + \frac{f^{(2)}g^{(4)}h^{(2)}}{4} + \frac{e^{(3)}fgh^{(3)}}{6} + \frac{efg^{(3)}h^{(3)}}{2} + \frac{ef^{(2)}g^{(3)}h^{(2)}}{2} + \frac{e^{(2)}fg^{(2)}h^{(3)}}{2} + \frac{e^{(2)}f^{(2)}g^{(2)}h^{(2)}}{6}$	$(n-e-f)^{(g+h-4)}$
...	...	...

Then, the coefficient

$$= \frac{ABC_A C_B}{\prod n^{(a_i)} \prod n^{(b_j)}} \Big[(n-e-f)^{(g+h)} + (eh + fg + gh)(n-e-f)^{(g+h-1)} + \dots \Big]$$

$$= \frac{ABC_A C_B}{\prod n^{(a_i)} \prod n^{(b_j)}} A_{n,e,f} \;,$$

if we denote the quantity in the square brackets by $A_{n,e,f}$.

Now consider the product $(n-e)^{(g)}(n-f)^{(h)}$. We have

$$(n-e)^{(g)}(n-f)^{(h)} = \frac{1}{n^{(e)}n^{(f)}} n^{(e+g)} n^{(f+h)}$$

$$= \frac{1}{n^{(e)}n^{(f)}} \Big[n^{(e+f+g+h)} + (e+g)(f+h)\, n^{(e+f+g+h-1)} + \frac{(e+g)^{(2)}(f+h)^{(2)}}{2!} n^{(e+f+g+h-2)} + \dots \Big],$$ by Vandermonde's Theorem

$$= \frac{n^{(e+f)}}{n^{(e)}n^{(f)}} \Big[(n-e-f)^{(g+h)} + (e+g)(f+h)(n-e-f)^{(g+h-1)} + \frac{(e+g)^{(2)}(f+h)^{(2)}}{2!} (n-e-f)^{(g+h-2)} + \dots \Big]$$

$$= \frac{n^{(e+f)}}{n^{(e)}n^{(f)}} \Big[(n-e-f)^{(g+h)} + (ef + eh + gf + gh)(n-e-f)^{(g+h-1)} + \Big\{ g^{(2)}fh + h^{(2)}eg + \frac{f^{(2)}g^{(2)}}{2} + \frac{g^{(2)}h^{(2)}}{2} + \frac{e^{(2)}h^{(2)}}{2} + efgh + ef\Big(\frac{e^{(2)}f^{(2)}}{2} + e^{(2)}fh + f^{(2)}eg + efgh \Big) \Big\} (n-e-f)^{(g+h-2)} + \dots \Big]$$

$$= \frac{n^{(e+f)}}{n^{(e)}n^{(f)}} \Big[A_{n,e,f} + ef \Big\{ (n-e-f)^{(g+h-1)} + \big((e-1)h + (f-1)g + gh \big) (n-e-f)^{(g+h-2)} + \dots \Big\} + \frac{e^{(2)}f^{(2)}}{2!} \Big\{ (n-e-f)^{(g+h-2)} + \dots \Big\} + \dots \Big]$$

$$= \frac{n^{(e+f)}}{n^{(e)}n^{(f)}} \Big[A_{n,e,f} + \frac{ef}{n-e-f+1} \Big\{ (n-e-f+1)^{(g+h)} + \big((e-1)h + (f-1)g + gh \big) (n-e-f+1)^{(g+h-1)} + \dots \Big\} + \frac{e^{(2)}f^{(2)}}{2!} \frac{1}{(n-e-f\ 2)^{(2)}} \Big\{ (n-e-f+2)^{(g+h)} + \dots \Big\} + \dots \Big]$$

$$= \frac{n^{(e+f)}}{n^{(e)}n^{(f)}}\left[A_{n,e,f} + \frac{ef}{n-e-f+1}\,A_{n-1,e-1,f-1} + \frac{e^{(2)}f^{(2)}}{2!(n-e-f+2)^{(2)}}\,A_{n-2,e-2,f-2} + \cdots\right].$$

We find that $A_{n,e,f} = A_{n-1,e-1,f-1} = A_{n-2,e-2,f-2} = \cdots$ for all values of e, f, g, h ≤ 3 that we require in order to derive products of seminvariant k-statistics through weight 12. Hence,

$$(n-e)^{(g)}(n-f)^{(h)} = \frac{n^{(e+f)}}{n^{(e)}n^{(f)}}\,A_{n,e,f}\left[1 + \frac{ef}{n-e-f+1} + \frac{e^{(2)}f^{(2)}}{2!(n-e-f\ 2)^{(2)}} + \cdots\right]$$

$$= \frac{A_{n,e,f}}{n^{(e)}n^{(f)}}\left[n^{(e+f)} + efn^{(e+f-1)} + \frac{e^{(2)}f^{(2)}}{2!}\,n^{(e+f-2)} + \cdots\right]$$

$$= \frac{A_{n,e,f}}{n^{(e)}n^{(f)}}\,n^{(e)}\,n^{(f)}$$

$$(2.3) \qquad = A_{n,e,f}\,.$$

Thus, the required coefficient is

$$\frac{ABC_AC_B}{\prod n^{(a_i)}\prod n^{(b_j)}}\,(n-e)^{(g)}(n-f)^{(h)}$$

$$= \frac{AC_A}{\prod n^{(a_i)}}\,\frac{n^{(a)}}{n^{(e)}}\,\frac{BC_B}{\prod n^{(b_j)}}\,\frac{n^{(b)}}{n^{(f)}}$$

$$= (\text{Coefficient of } \begin{array}{|c|c|}\hline A & C_A \\ \hline\end{array})(\text{Coefficient of } \begin{array}{|c|c|}\hline C_B & B \\ \hline\end{array})$$

Hence, as in Case 1, the total algebraic coefficient of the array type $\begin{array}{|c|c|c}\hline A & C_A & 0 \\ \hline 0 & \multicolumn{1}{|c|}{C_B} & \multicolumn{1}{c|}{B} \\ \cline{2-3}\end{array}$ is the product of the algebraic coefficients of $\boxed{\ A \quad \mid C_A\ }$ and $\boxed{\ C_B \mid \quad B\ }$.

(b) Let a particular p have e non-zero parts in C_A and f in C_B. Let the number of zero entries in these portions be g and h respectively and let there be k other non-zero entries in C, so that $e + f + g + h + k = a + b$. Let A, B absorb signs and factorials for blocks A and B and let C_A, C_B do the same for entries in C_A, C_B which are not parts of this particular p. Then the term contributed by each amalgamation is $\dfrac{ABC_AC_B}{\prod n^{(a_i)} \prod n^{(b_j)}}$ multiplied by the corresponding factor in the following scheme. Again, additions within blocks or involving parts of other common p in C are not considered at first.

Number of rows of B added to those of A	Number of Amalgamations	Multiplying Factor
0	1	$(-1)^{e+f-1}(e+f-1)!(n-e-f)^{(g+h)}$
1	ef	$(-1)^{e+f-2}(e+f-2)!(n-e-f+1)^{(g+h)}$
	eh+gf+gh	$(-1)^{e+f-1}(e+f-1)!(n-e-f)^{(g+h-1)}$
2	$\dfrac{e^{(2)}f^{(2)}}{2!}$	$(-1)^{e+f-3}(e+f-3)!(n-e-f+2)^{(g+h)}$
	$e^{(2)}fh + ef^{(2)}g + efgh$	$(-1)^{e+f-2}(e+f-2)!(n-e-f+1)^{(g+h-1)}$
	$fg^{(2)}h + egh^{(2)} + \dfrac{f^{(2)}g^{(2)}}{2!} + \dfrac{g^{(2)}h^{(2)}}{2!} + \dfrac{e^{(2)}h^{(2)}}{2!} + efgh$	$(-1)^{e+f-1}(e+f-1)!(n-e-f)^{(g+h-2)}$
3	$\dfrac{e^{(3)}f^{(3)}}{3!}$	$(-1)^{e+f-4}(e+f-4)!(n-e-f+3)^{(g+h)}$

Number of rows of B added to those of A	Number of Amalgamations	Multiplying Factor
3 (con't.)	$\frac{e^{(3)}f^{(2)}h}{2!} + \frac{e^{(2)}f^{(3)}g}{2!} + \frac{e^{(2)}f^{(2)}gh}{2!}$	$(-1)^{e+f-3}(e+f-3)!(n-e-f+2)^{(g+h-1)}$
	$e^{(3)}fh^{(2)} + ef^{(3)}g^{(2)} + e^{(2)}f^{(2)}gh$	$(-1)^{e+f-2}(e+f-2)!(n-e-f+1)^{(g+h-2)}$
	$+ ef^{(2)}g^{(2)}h + \frac{efg^{(2)}h^{(2)}}{2!} + e^{(2)}fgh^{(2)}$	
	$\frac{e^{(3)}h^{(3)}}{3!} + \frac{f^{(3)}g^{(3)}}{3!} + \frac{g^{(3)}h^{(3)}}{3!} + \frac{f^{(2)}g^{(3)}h}{2!}$	$(-1)^{e+f-1}(e+f-1)!(n-e-f)^{(g+h-3)}$
	$+ \frac{fg^{(3)}h^{(2)}}{2!} + \frac{e^{(2)}gh^{(3)}}{2!} + \frac{eg^{(2)}h^{(3)}}{2!} +$	
	$\frac{ef^{(2)}g^{(2)}h}{2!} + \frac{e^{(2)}fgh^{(2)}}{2!} + efg^{(2)}h^{(2)}$	
4	$\frac{e^{(4)}f^{(4)}}{4!}$	$(-1)^{e+f-5}(e+f-5)!(n-e-f+4)^{(g+h)}$
	$\frac{e^{(4)}f^{(3)}h}{3!} + \frac{f^{(4)}e^{(3)}g}{3!} + \frac{e^{(3)}f^{(3)}gh}{3!}$	$(-1)^{e+f-4}(e+f-4)!(n-e-f+3)^{(g+h-1)}$
	$\frac{e^{(4)}f^{(2)}h^{(2)}}{4} + \frac{f^{(4)}e^{(2)}g^{(2)}}{4} + \frac{e^{(3)}gf^{(2)}h^{(2)}}{2} +$	$(-1)^{e+f-3}(e+f-3)!(n-e-f+2)^{(g+h-2)}$
	$\frac{e^{(2)}g^{(2)}f^{(3)}h}{2} + \frac{e^{(2)}g^{(2)}f^{(2)}h^{(2)}}{4} + \frac{e^{(3)}f^{(3)}gh}{2}$	
	$\frac{eg^{(3)}f^{(4)}}{3!} + \frac{e^{(4)}fh^{(3)}}{3!} + \frac{e^{(3)}h^{(3)}gf}{2} +$	$(-1)^{e+f-2}(e+f-2)!(n-e-f+1)^{(g+h-3)}$
	$\frac{f^{(3)}g^{(3)}eh}{2} + \frac{efg^{(3)}h^{(3)}}{3!} + \frac{e^{(3)}gf^{(2)}h^{(2)}}{2} +$	
	$\frac{e^{(2)}g^{(2)}f^{(3)}h}{2} + \frac{eg^{(3)}f^{(2)}h^{(2)}}{2} + \frac{e^{(2)}g^{(2)}fh^{(3)}}{2}$	
	$+ e^{(2)}g^{(2)}f^{(2)}h^{(2)}$	
	$\frac{g^{(4)}f^{(4)}}{4!} + \frac{e^{(4)}h^{(4)}}{4!} + \frac{g^{(4)}h^{(4)}}{4!} + \frac{e^{(3)}gh^{(4)}}{3!} + \frac{eg^{(3)}h^{(4)}}{3!}$	$(-1)^{e+f-1}(e+f-1)!(n-e-f^{(g+h-4)}$
	$+ \frac{f^{(3)}hg^{(4)}}{3!} + \frac{fh^{(3)}g^{(4)}}{3!} + \frac{e^{(2)}g^{(2)}h^{(4)}}{4} + \frac{f^{(2)}h^{(2)}g^{(4)}}{4} +$	

Number of rows of B added to those of A	Number of Amalgamations	Multiplying Factor
4 (con't.)	$\frac{e^{(3)}h^{(3)}gf}{6} + \frac{f^{(3)}g^{(3)}eh}{6} + \frac{efg^{(3)}h^{(3)}}{2} + \frac{eg^{(3)}f^{(2)}h^{(2)}}{2} + \frac{e^{(2)}g^{(2)}fh^{(3)}}{2} + \frac{e^{(2)}f^{(2)}g^{(2)}h^{(2)}}{6}$	
...	...	...

Then, the coefficient $= (-1)^{e+f-1}(e+f-1)!\left[(n-e-f)^{(g+h)} + (eh+fg+gh)(n-e-f)^{(g+h-1)} + \ldots\right]$

$$+(-1)^{e+f-2}(e+f-2)!\,ef\left[(n-e-f+1)^{(g+h)} + \{(e-1)h+(f-1)g+gh\}(n-e-f+1)^{(g+h-1)} + \ldots\right]$$

$$+(-1)^{e+f-3}(e+f-3)!\,\frac{e^{(2)}f^{(2)}}{2!}\left[(n-e-f+2)^{(g+h)} + \{(e-2)h+(f-2)g+gh\}(n-e-f+2)^{(g+h-1)} + \ldots\right]$$

$$+ \ldots$$

$$= (-1)^{e+f-1}(e+f-1)!(n-e)^{(g)}(n-f)^{(h)} + (-1)^{e+f-2}(e+f-2)!\,ef(n-e)^{(g)}(n-f)^{(h)} + (-1)^{e+f-3}(e+f-3)!\,\frac{e^{(2)}f^{(2)}}{2!}(n-e)^{(g)}(n-f)^{(h)} + \ldots, \text{ by } (2.3)$$

$$= (-1)^{e+f-1}(n-e)^{(g)}(n-f)^{(h)}\left[(e+f-1)! - ef(e+f-2)! + \frac{e^{(2)}f^{(2)}}{2!}(e+f-3)! - \ldots\right]$$

$$= (-1)^{e+f-1}(n-e)^{(g)}(n-f)^{(h)}(0), \text{ by Vandermonde's Theorem}$$

$$= 0.$$

Allowing now the additions which were earlier restricted, it can be shown as in Case 1, that the algebraic coefficient of the array type is zero.

Rule 8. Rule for Row-bordered Blocks

The algebraic coefficient of an array type which falls into t blocks with a common row connecting each two consecutive blocks is $\frac{1}{n^{t-1}}$ times the product of the coefficients for the blocks.

We prove this rule by induction, starting with the first non-trivial case of t = 2, (trivially true for t = 1), in which case the algebraic coefficient of the array type falling into two blocks with a common connecting row is $\frac{1}{n}$ times the product of the coefficients for the blocks.

Let there be a, b rows and c, d columns respectively in the blocks A and B. Again, at first, let us not allow additions within blocks. In a typical amalgamation, r of the (b-1) rows of B can be added to r of the (a-1) rows of A (can assume $a \geq b$). There are $\frac{(a-1)^{(r)}(b-1)^{(r)}}{r!}$ such amalgamations. The contribution of each of these to the coefficient is $\frac{AB}{\prod n^{(a_i)} \prod n^{(b_j)}} n^{(a+b-r-1)}$ where A, B absorb signs and factorials for the two blocks. Then the algebraic coefficient of the array type is

$$\frac{AB}{\prod n^{(a_i)} \prod n^{(b_j)}} \sum_{r=0}^{b-1} \frac{(a-1)^{(r)}(b-1)^{(r)}}{r!} n^{(a+b-r-1)}$$

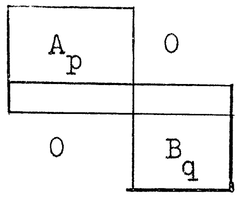
$$= \frac{ABn^{(a)}}{\prod n^{(a_i)} \prod n^{(b_j)}} \sum_{r=0}^{b-1} \frac{(a-1)^{(r)}(b-1)^{(r)}}{r!} (n-a)^{(b-r-1)}$$

$$= \frac{ABn^{(a)}(n-1)^{(b-1)}}{\prod n^{(a_i)} \prod n^{(b_j)}}, \text{ by Vandermonde's Theorem}$$

$$= \frac{1}{n} \frac{An^{(a)}}{\prod n^{(a_i)}} \frac{Bn^{(b)}}{\prod n^{(b_j)}} .$$

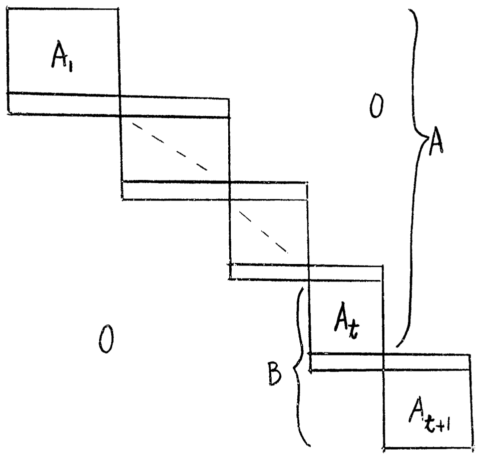
Then, total algebraic coefficient for the array type is

$\sum_{p,q}$ Contribution for amalgamation



$$= \frac{1}{n} \sum_{p,q} (\text{Contribution of } A_p)(\text{Contribution of } B_q)$$
$$= \frac{1}{n} (\sum_{p} \text{Contribution of } A_p)(\sum_{q} \text{Contribution of } B_q)$$
$$= \frac{1}{n} (\text{Algebraic coefficient of A})(\text{Algebraic coefficient of B}).$$

Now, let the array type consist of t+1 row-bordered blocks A_i ,



$i = 1,2, \ldots , t+1$. Let A denote the portion of the array type consisting of the blocks $A_1, \ldots , A_t$ and let B consist of the blocks A_t, A_{t+1}. Suppose the rule is true for t. Then the coefficient of A is $\frac{1}{n^{t-1}} \prod_{i=1}^{t}$ Coefficient of A_i .

If, instead of A_t, we now put B whose coefficient is $\frac{1}{n}$(Coefficient of A_t)(Coefficient of A_{t+1}), we find that the coefficient of the array type is

$$\frac{1}{n^{t-1}} (\prod_{i=1}^{t-1} \text{Coefficient of } A_i)(\text{Coefficient of B})$$
$$= \frac{1}{n^{t-1}} (\prod_{i=1}^{t-1} \text{Coefficient of } A_i) \frac{1}{n} (\text{Coefficient of } A_t)(\text{Coefficient of } A_{t+1})$$
$$= \frac{1}{n^t} \prod_{i=1}^{t+1} \text{Coefficient of } A_i.$$

Thus the rule is true for t+1 if it is true for t. But we have seen that it is true for t = 2, hence it is true for all t.

Most of these rules are the result of a study of coefficients of particular array types which are listed in the next Chapter. On the other hand, once the rules were known, they were very useful in determining coefficients of additional particular array types.

CHAPTER III

COEFFICIENTS OF ARRAY TYPES

Fisher (1928) gave the algebraic coefficients of some commonly occuring patterns when considering the products of single subscript k's. They need to be generalized when we deal with products of generalized k-statistics and each of his patterns gives rise to several array types. These are systematically tabulated for his first twenty-four patterns and coefficients of several general array types are studied.

Generalization of Fisher's Patterns

Fisher (1928), while listing some commonly occuring patterns and indicating their algebraic coefficients, had a simpler situation in that all rows can be added when we consider products of single subscript k-statistics (or their cumulants as Fisher did). When we consider products of generalized k-statistics, addition of rows is restricted as a part p_{ij} of some p_i can only be added to another p_{ik} or to a zero. So the patterns can no longer be indicated by filling in x's for the entries as Fisher did and in fact each of his patterns leads to several patterns distinguishing the location of parts of one p_i from those of another. Fortunately, by the pattern rule, we do not have to distinguish for numerical values of p_{ij} so long as the location is not involved. Also rows and columns of an array type can be permuted at will. We follow the practice of indicating the subscripts going in the first column by p's, those in the second by q's and so on. Where the number of columns is generalized, we change to $p^1, p^2, p^3, \ldots$.

Fisher (1928) has recorded that in the case of two rows and r columns, the algebraic coefficient of the array type $\begin{matrix} xx\ldots x \\ xx\ldots x \end{matrix}$ is $\dfrac{1-\alpha^{r-1}}{n^{r-1}}$,

where $\alpha = -\dfrac{1}{n-1}$. We further observe that the array type

$$\begin{matrix} p_{11}^{1} & p_{11}^{2} & \ldots & p_{11}^{s} & p_{1}^{s+1} & \ldots & p_{1}^{r} \\ p_{12}^{1} & p_{12}^{2} & \ldots & p_{12}^{s} & p_{2}^{s+1} & \ldots & p_{2}^{r} \end{matrix}$$

(encountered, for example, in the generalized product

$k_{p_1^1} \; k_{p_1^2} \ldots k_{p_1^s} \; k_{p_1^{s+1} p_2^{s+1}} \ldots k_{p_1^r p_2^r}$) has an algebraic coefficient =

$$(-1)^{s} \frac{(-\alpha)^{r-1}}{n^{r-1}}, \qquad s < r$$

$$(3.1) \qquad \frac{1}{n^{r-1}}(1-\alpha^{r-1}), \qquad s = r \qquad \text{(Fisher's case).}$$

The distinction emerges from the fact that when $s < r$, no addition of the two rows is possible, whereas $s = r$ makes this possible.

In the case of three rows, when there are r columns, each consisting of three parts of a single p_i, the algebraic coefficient is found to be $\dfrac{1}{n^{r-1}}\left(2\beta^{r-1} - 3\alpha^{r-1} + 1\right)$, where $\alpha = -\dfrac{1}{n-1}$, $\beta = \dfrac{2}{(n-1)(n-2)}$.

We get the special cases listed by Fisher by putting r = 2, 3, 4, the algebraic coefficients being

$\dfrac{n}{(n-1)(n-2)}$ for $\begin{matrix} xx \\ xx \\ xx \end{matrix}$, $\dfrac{n^2-6n+10}{(n-1)^2(n-2)^2}$ for $\begin{matrix} xxx \\ xxx \\ xxx \end{matrix}$ and

$\dfrac{n^4-9n^3+33n^2-60n+48}{n(n-1)^3(n-2)^3}$ for $\begin{matrix} xxxx \\ xxxx \\ xxxx \end{matrix}$.

But now when we consider three rows and two columns, we have to list all the variations such as

$$\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{13} & q_{13} \end{matrix} \; , \; \begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{13} & q_{2} \end{matrix} \; , \; \begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{2} & q_{2} \end{matrix} \; , \; \begin{matrix} p_{11} & q_{1} \\ p_{12} & q_{2} \\ p_{13} & q_{3} \end{matrix} \; , \; \begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{2} \\ p_{2} & q_{12} \end{matrix} \; , \; \begin{matrix} p_{11} & q_{1} \\ p_{12} & q_{2} \\ p_{2} & q_{3} \end{matrix} \; , \; \begin{matrix} p_{1} & q_{1} \\ p_{2} & q_{2} \\ p_{3} & q_{3} \end{matrix} \; ,$$

the algebraic coefficients being

$$\frac{n}{n(n-1)(n-2)} , \; -\frac{1}{(n-1)(n-2)} , \; \frac{1}{n(n-2)} , \; \frac{2}{n(n-1)(n-2)} ,$$

$$\frac{1}{n(n-1)(n-2)} , \; -\frac{1}{n(n-1)(n-2)} , \; \frac{1}{n(n-1)(n-2)} \quad \text{respectively}$$

(Table 5.4). No additions are possible for the last four array types.

Generalizations are, of course, possible in both directions. For example, the algebraic coefficient of the array type $\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{2} & q_{2} \\ p_{3} & q_{3} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ p_{a+1} & q_{a+1} \end{matrix}$, obtained

by adding a row to Fisher's pattern $\begin{smallmatrix} xx \\ xx \end{smallmatrix}$, is

$\frac{n-a}{n^{(a+2)}} = \frac{1}{n^{(a)}(n-a-1)}$ (Dwyer and Tracy, 1962, p.35), which sheds additional light on the coefficient of $\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{2} & q_{2} \end{matrix}$ listed earlier as $\frac{1}{n(n-2)}$ as

actually being $\frac{n-1}{n^{(3)}}$. The formula holds formally when no rows are added, i.e., $a = 0$, and gives the coefficient of Fisher's $\begin{smallmatrix} xx \\ xx \end{smallmatrix}$ as $\frac{n}{n^{(2)}} = \frac{1}{n-1}$.
Similarly, the coefficient of

$\begin{matrix} p_{11} & q_{11} \\ p_{12} & q_{12} \\ p_{13} & q_{13} \\ p_2 & q_2 \\ \vdots & \vdots \\ p_{a+1} & q_{a+1} \end{matrix}$ is $\dfrac{(n-a)^2}{n^{(a+3)}}$ (Dwyer and Tracy, 1962, p. 35), which, for a = 0 gives $\dfrac{n}{(n-1)(n-2)}$.

For $\begin{matrix} p_{11} & q_{11} & r_{11} \\ p_{12} & q_{12} & r_{12} \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ \vdots & \vdots & \vdots \\ p_{a+1} & q_{a+1} & r_{a+1} \end{matrix}$, we find the algebraic coefficient to be

$\dfrac{(n-a)^2 - 2(n-a)}{\left[n^{(a+2)}\right]^2}$, which, when a = 0 gives $\dfrac{n^2 - 2n}{\left[n(n-1)\right]^2} = \dfrac{n-2}{n(n-1)^2}$ as the coefficient for Fisher's $\begin{matrix} xxx \\ xxx \end{matrix}$. Similarly, for $\begin{matrix} p_{11} & q_{11} & r_{11} & s_{11} \\ p_{12} & q_{12} & r_{12} & s_{12} \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \\ \vdots & \vdots & \vdots & \vdots \\ p_{a+1} & q_{a+1} & r_{a+1} & s_{a+1} \end{matrix}$,

the algebraic coefficient is $\dfrac{(n-a)^3 - 3(n-a)^2 + 3(n-a)}{\left[n^{(a+2)}\right]^3}$, which, for a = 0 gives $\dfrac{n^2 - 3n + 3}{n^2(n-1)^3}$ as the coefficient for Fisher's $\begin{matrix} xxxx \\ xxxx \end{matrix}$. For the general case of r columns and a + 1 rows with two rows additive, we get

$\left[\dfrac{1}{n^{(a+1)}}\right]^{r-1}(1 - \alpha^{r-1})$ as the coefficient where $\alpha = -\dfrac{1}{n-a-1}$. This, for a = 0, gives Fisher's result (3.1).

Now generalizing in the other direction, we consider the array type

$$\begin{matrix} p_{11} & q_{11} & r_{11} & \cdots \\ p_{12} & q_{12} & r_{12} & \cdots \\ p_{13} & q_{2} & r_{2} & \cdots \end{matrix}$$

The algebraic coefficient is found to be

$$\left[\frac{1}{n(n-1)}\right]^{r-1}\left[2\ \left(-\ \frac{1}{n-2}\right)^{r-1}\ -1\right]$$, when there are r columns.

Use of Table 5

The more common of Fisher's patterns (first twenty-four, Fisher, 1928, pp. 223-24) are now taken and elaborated into the several cases they yield for products of generalized k-statistics. The results are presented in Tables 5.1 - 5.24. The top left entry in each case denotes Fisher's pattern. Taking Table 5.5 as an example, $\begin{matrix} p_{11} & q_{11} & r_{11} \\ p_{12} & q_{12} & r_{12} \\ p_{13} & q_{13} & r_{13} \end{matrix}$ is successively transformed to all its variations. The top left entry $n^2(n^2-6n+10)$ multiplied by the multiplier $\frac{1}{n^2(n-1)^2\ (n-2)^2}$ at the top of the table indicates that the coefficient of $\begin{matrix} p_{11} & q_{11} & r_{11} \\ p_{12} & q_{12} & r_{12} \\ p_{13} & q_{13} & r_{13} \end{matrix}$ is

$$\frac{n^2(n^2-6n+10)}{n^2(n-1)^2(n-2)^2} = \frac{n^2-6n+10}{(n-1)^2(n-2)^2}.$$

The second entry in the first row is for a pattern where the column $\begin{matrix} q_{11} \\ q_{12} \\ q_{13} \end{matrix}$ of q's is changed to $\begin{matrix} q_{11} \\ q_{12} \\ q_{2} \end{matrix}$ and so on till the last entry in the last row represents the coefficient for $\begin{matrix} p_{1} & q_{1} & r_{1} \\ p_{2} & q_{2} & r_{2} \\ p_{3} & q_{3} & r_{3} \end{matrix}$ (multiplied by $\frac{1}{n^2(n-1)^2(n-2)^2}$). The changes in the column of r's are indicated at the left of the rows and $\begin{matrix} r_{11} \\ r_{12} \\ r_{2} \end{matrix}$ etc. are written as r_{11} r_{12} r_2 just for the convenience of presentation. The choice of indicating changes in p and q columns by columns and those in r columns by rows is purely arbitrary, except for some thought to the shape of the table. Although a pattern like

$$\begin{matrix} p_1 & q_{11} & r_1 \\ p_2 & q_{12} & r_2 \\ p_3 & q_2 & r_3 \end{matrix}$$

does not appear specifically, its coefficient is the same as for

$$\begin{matrix} p_{11} & q_1 & r_1 \\ p_{12} & q_2 & r_2 \\ p_2 & q_3 & r_3 \end{matrix}$$

for reasons of symmetry. In bigger tables like Table 5.18, the saving is considerable due to this as whole rows and columns for certain variations can be ignored.

TABLE 5

PATTERN COEFFICIENTS

Table 5.1

$$\frac{1}{n(n-1)}$$

p_{11} q_{11} p_{12} q_{12}		q_1 q_2
	n	-1
p_1 p_2	-1	1

Table 5.2

$$\frac{1}{n^2(n-1)^2}$$

$p_{11}\ q_{11}\ r_{11}$ $p_{12}\ q_{12}\ r_{12}$		r_1 r_2	$q_1\ r_1$ $q_2\ r_2$
	$n(n-2)$	1	-1
$p_1\ p_2$	1	-1	1

Table 5.3

$$\frac{1}{n^3(n-1)^3}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $p_{12}\ q_{12}\ r_{12}\ s_{12}$		s_1 s_2	$r_1\ s_1$ $r_2\ s_2$
	$n(n^2-3n+3)$	-1	1
$q_1\ q_2$	-1	1	-1
$p_1\ p_2$ $q_1\ q_2$	1	-1	1

Table 5.4

$$\frac{1}{n(n-1)(n-2)}$$

$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_{13}\ q_{13}$		q_{11} q_{12} q_2	q_1 q_2 q_3
	n^2	$-n$	2
$p_{11}p_{12}p_2$	$-n$	$n-1$	-1
$p_{11}p_2p_{12}$	$-n$	1	-1
$p_1\ p_2\ p_3$	2	-1	1

Table 5.5

$$\frac{1}{n^2(n-1)^2(n-2)^2}$$

$p_{11}\ q_{11}\ r_{11}$ $p_{12}\ q_{12}\ r_{12}$ $p_{13}\ q_{13}\ r_{13}$		q_{11} q_{12} q_2	q_{11} q_2 q_{12}	q_1 q_2 q_3	$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_2\ q_2$	$p_2\ q_{11}$ $p_{11}\ q_{12}$ $p_{12}\ q_2$	$p_{11}q_1$ $p_{12}q_2$ $p_2\ q_3$	$p_1\ q_1$ $p_2\ q_2$ $p_3\ q_3$
	$n^2(n^2-6n+10)$	$n(n-4)$	$n(n-4)$	4	$-(n^2-4n+2)$	2	-2	2
$r_{11}\ r_{12}\ r_2$	$n(n-4)$	$-(n^2-4n+2)$	2	-2	$(n-3)(n-1)$	-1	1	-1
$r_{11}\ r_2\ r_{12}$	$n(n-4)$	2	$-(n^2-4n+2)$	-2	-1	-1	1	-1
$r_1\ r_2\ r_3$	4	-2	-2	2	1	1	-1	1

Table 5.6

$$\frac{1}{n^2(n-1)^2(n-2)}$$

$p_{11}\ q_{11}\ r_{11}$ $p_{12}\ q_{12}\ r_{12}$ $p_{13}\ q_{13}\ 0$		q_{11} q_{12} q_2	q_{11} q_2 q_{12}	q_1 q_2 q_3	$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_2\ 2_2$	$p_{11}\ q_{11}$ $p_{12}\ q_2$ $p_2\ q_{12}$	$p_{11}\ q_{11}$ $p_2\ q_2$ $p_{12}\ q_{12}$	$p_{11}\ q_1$ $p_{12}\ q_2$ $p_2\ q_3$	$p_1\ q_1$ $p_2\ q_2$ $p_3\ q_3$
	$n^2(n-3)$	$-n(n-3)$	n	-2	n^2-3n+1	-1	$-(n-1)$	1	-1
$r_1\ r_2\ 0$	$2n$	-2	$-n$	2	1	1	$n-1$	-1	1

Table 5.7

$$\frac{1}{n^2 (n-1)^2}$$

p_{11} q_{11} r_{11} p_{12} 0 r_{12} p_{13} q_{12} 0		q_1 0 q_2	p_{11} p_{12} p_2	r_{11} p_2 p_{12}	p_1 p_2 p_3	p_{11} q_1 p_{12} 0 p_2 q_2	p_1 q_1 p_2 0 p_3 q_2
	n^2	-n	-n	-n	1	n	-1
r_1 r_2 0	-n	1	1	n	-1	-1	1

Table 5.8

$$\frac{1}{n^2 (n-1)^2}$$

0 q_{11} r_{11} p_{11} 0 r_{12} p_{12} q_{12} 0		q_1 0 q_2	0 p_1 p_2	0 q_1 p_1 0 p_2 q_2
	n^2	-n	-n	1
r_1 r_2 0	-n	1	1	n-2

Table 5.9

$$\frac{1}{n^3(n-1)^3(n-2)^3}$$

p_{11} q_{11} r_{11} s_{11} p_{12} q_{12} r_{12} s_{12} p_{13} q_{13} r_{13} s_{13}		s_{11} s_{12} s_2	s_1 s_2 s_3	r_{11} s_{11} r_{12} s_{12} r_2 s_2	r_{11} s_{11} r_{12} s_2 r_2 s_{12}	r_{11} s_2 r_2 s_{11} r_{12} s_{12}	r_{11} s_{11} r_2 s_2 r_{12} s_{12}	r_1 s_{11} r_2 s_2 r_3 s_{12}	r_{11} s_1 r_2 s_2 r_2 s_3	r_1 s_1 r_2 s_2 r_3 s_3
	$n^2(n^4-9n^3+33n^2-60n+43)$	$-n(n^2-6n+12)$	8	$n^3-6n^2+12n-4$	4	4	$n^3-6n^2+12n-4$	-4	-4	4
q_{11} q_{12} q_2	$-n(n^2-6n+12)$	$n^3-6n^2+12n-4$	-4	$-(n^3-6n^2 12n-6)$	-2	-2	-2	2	2	-2
q_1 q_2 q_3	3	-4	4	2	2	2	2	-2	-2	2
p_{11} p_{12} p_2 q_{11} q_{12} q_2	$n^3-6n^2+12n-4$	$-(n^3-6n^2+12n-6)$	2	$(n-1)(n^2-5n+7)$	1	1	1	-1	-1	1
p_{11} p_{12} p_2 q_{11} q_2 q_{12}	4	-2	2	1	1	1	1	-1	-1	1
p_{11} p_{12} p_2 q_1 q_2 q_3	-4	2	-2	-1	-1	-1	-1	1	1	-1
p_1 p_2 p_3 q_1 q_2 q_3	4	-2	2	1	1	1	1	-1	-1	1

Table 5.10

$$\frac{1}{n^3(n-1)^3(n-2)^2}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $p_{12}\ q_{12}\ r_{12}\ s_{12}$ $p_{13}\ q_{13}\ r_{13}\ 0$		s_1 s_2 0	r_{11} r_{12} r_2	r_{11} r_2 r_{12}	r_2 r_{11} r_{12}	r_1 r_2 r_3	$r_{11}\ s_1$ $r_{12}\ s_2$ $r_2\ 0$	$r_{11}\ s_1$ $r_2\ s_2$ $r_{12}\ 0$	$r_2\ s_1$ $r_{11}\ s_2$ $r_{12}\ 0$	$r_1\ s_1$ $r_2\ s_2$ $r_3\ 0$
	$n^2(n-3)(n^2-4n+6)$	$-2n(n-4)$	$n(n^2-5n+8)$	$-n(n-4)$	$-n(n-4)$	-4	-4	$n(n-4)$	$n(n-4)$	4
$q_{11}\ q_{12}\ q_2$	$n(n^2-5n+8)$	-4	$(n-3)(n^2-2n+2)$	-2	-2	2	2	2	2	-2
$q_{11}\ q_2\ q_{12}$	$-n(n-4)$	$n(n-4)$	-2	$(n-3)(n^2-2n+2)$	-2	2	2	$-(n-3)(n^2-2n+2)$	2	-2
$q_1\ q_2\ q_3$	-4	4	2	2	2	-2	-2	-2	-2	2
$p_{11}\ p_{12}\ p_2$ $q_{11}\ q_{12}\ q_2$	$-(n^3-5n^2+8n-2)$	2	n^3-5n^2+8n-3	1	1	-1	-1	-1	-1	1
$p_{11}\ p_2\ p_{12}$ $q_{11}\ q_2\ q_{12}$	n^2-4n+2	$-(n^2-4n+2)$	1	$-(n-3)$	1	-1	-1	$n-3$	-1	1

Table 5.11

$$\frac{1}{n^3(n-1)^3(n-2)}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $p_{12}\ q_{12}\ r_{12}\ s_{12}$ $p_{13}\ q_{13}\ 0\ 0$		q_{11} q_{12} q_2	q_{11} q_2 q_{12}	q_1 q_2 q_3	$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_2\ q_2$	$p_{11}\ q_{11}$ $p_{12}\ q_2$ $p_2\ q_{12}$	$p_{11}\ q_{11}$ $p_2\ q_2$ $p_{12}\ q_{12}$	$p_2\ q_{11}$ $p_{11}\ q_2$ $p_{12}\ q_{12}$	$p_{11}\ q_1$ $p_{12}\ q_2$ $p_2\ q_3$	$p_1\ q_1$ $p_2\ q_2$ $p_3\ q_3$
	$n^2(n^2-4n+5)$	$-n(n^2-4n+5)$	$-n$	2	n^3-4n^2+5n-1	1	$n-1$	1	-1	1
$r_1\ r_2\ 0$	$-2n$	2	n	-2	-1	-1	$-(n-1)$	-1	1	-1
$r_1\ r_2\ 0$ $s_1\ s_2\ 0$	$2n$	-2	$-n$	2	1	1	$n-1$	1	-1	1

Table 5.12

$$\frac{1}{n^3(n-1)^3(n-2)}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $p_{12}\ q_{12}\ 0\ s_{12}$ $p_{13}\ q_{13}\ r_{12}\ 0$		q_{11} q_{12} q_2	q_{11} q_2 q_{12}	q_1 q_2 q_3	$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_2\ q_2$	$p_{11}\ q_{11}$ $p_{12}\ q_2$ $p_2\ q_{12}$	$p_{11}\ q_1$ $p_{12}\ q_2$ $p_2\ q_3$	$p_1 q_1$ $p_2 q_2$ $p_3 q_3$
	$n^2(n^2-5n+7)$	$n(n-3)$	$n(n-3)$	2	$-(n^2-3n+1)$	1	-1	1
$s_1\ s_2\ 0$	$n(n-4)$	2	$-n(n-3)$	-2	-1	-1	1	-1
$r_1\ 0\ r_2$ $s_1\ s_2\ 0$	$n+2$	-2	-2	2	1	1	-1	1

Table 5.13

$$\frac{1}{n^3\ (n-1)^3}$$

$p_{11}\ 0\ r_{11}\ s_{11}$ $p_{12}\ q_{11}\ 0\ s_{12}$ $p_{13}\ q_{12}\ r_{12}\ 0$		0 q_1 q_2	p_{11} p_{12} p_2	p_1 p_2 p_3	$p_{11} 0$ $p_{12} q_1$ $p_2 q_2$	$p_1 0$ $p_2 q_1$ $p_3 q_2$
	$n^2(n-3)$	$2n$	n	-1	$-n$	1
$r_1\ 0\ r_2$	$2n$	$-(n+1)$	$-n$	1	n	-1
$s_1\ s_2\ 0$	$2n$	$-(n+1)$	-1	1	1	-1
$r_1\ 0\ r_2$ $s_1\ s_2\ 0^2$	$-(n+1)$	2	1	-1	-1	1

Table 5.14

$$\frac{1}{n^3(n-1)^3}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $p_{12}\ 0\ r_{12}\ s_{12}$ $p_{13}\ q_{13}\ 0\ 0$		q_1 0 q_2	p_{11} p_{12} p_2	p_{11} p_2 p_{12}	p_2 p_{11} p_{12}	p_1 p_2 p_3	$p_{11}q_1$ $p_{12}0$ $p_2\ q_2$	$p_{11}q_1$ $p_2\ 0$ $p_{12}q_2$	p_2q_1 $p_{11}0$ $p_{12}q_2$	p_1q_1 p_20 p_3q_2
	$n^2(n-2)$	$-n(n-2)$	$-n(n-2)$	n^2-2n+2	0	-1	$n(n-2)$	-1	0	1
$r_1\ r_2\ 0$	n	-1	-1	$-(n^2-2n+2)$	0	1	1	1	0	-1
$r_1\ r_2\ 0$ $s_1\ s_2\ 0$	$-n$	1	1	n^2-2n+2	0	-1	-1	-1	0	1

Table 5.15

$$\frac{1}{n^3(n-1)^3}$$

$0\ q_{11}\ r_{11}\ s_{11}$ $p_{11}0\ r_{12}\ s_{12}$ $p_{12}q_{12}\ 0\ 0$		q_1 0 q_2	0 p_1 p_2	$0\ q_1$ $p_1\ 0$ $p_2\ q_2$
	$n^2(n-2)$	$-n(n-2)$	$-n(n-2)$	n^2-n-1
$r_1\ r_2\ 0$	n	-1	-1	$-(n-2)$
$r_1\ r_2\ 0$ $s_1\ s_2\ 0$	$-n$	1	1	$n-2$

Table 5.16

$$\frac{1}{n^3(n-1)^3}$$

$p_{11}\ q_{11}\ r_{11}\ s_{11}$ $0\ 0\ r_{12}\ s_{12}$ $p_{12}\ q_{12}\ 0\ 0$		q_1 0 q_2	$p_1\ q_1$ $0\ 0$ $p_2\ q_2$
	$n^2(n-1)$	$-n(n-1)$	$n(n-1)$
$r_1\ r_2\ 0$	$-n(n-1)$	$n-1$	$-(n-1)$
$r_1\ r_2\ 0$ $s_1\ s_2\ 0$	$n(n-1)$	$-(n-1)$	$n-1$

Table 5.17

$$\frac{1}{n(n-1)(n-2)(n-3)}$$

$p_{11}\ q_{11}$ $p_{12}\ q_{12}$ $p_{13}\ q_{13}$ $p_{14}\ q_{14}$		q_{11} q_{12} q_{13} q_2	q_2 q_{11} q_{12} q_{13}	q_{11} q_{12} q_{21} q_{22}	q_{11} q_{21} q_{12} q_{22}	q_{11} q_{12} q_2 q_3	q_{11} q_2 q_3 q_{12}	q_{11} q_2 q_{12} q_3	q_2 q_3 q_{11} q_{12}	q_1 q_2 q_3 q_4
	$n^2(n+1)$	$-n(n+1)$	$-n(n+1)$	$-n(n-1)$	$-n(n-1)$	$2n$	$2n$	$2n$	$2n$	6
$p_{11}p_{12}p_{13}p_2$	$-n(n+1)$	$(n-1)^2$	$n+1$	$n-1$	$n-1$	$-(n-1)$	-2	$-(n-1)$	-2	2
$p_{11}p_{12}p_{21}p_{22}$	$-n(n-1)$	$n-1$	$n-1$	n^2-3n+1	1	$-(n-2)$	-1	-1	$-(n-2)$	1
$p_{11}p_{12}p_2p_3$	$2n$	$-(n-1)$	-2	$-(n-2)$	-1	$n-2$	1	1	1	-1
$p_1p_2p_3p_4$	-6	2	2	1	1	-1	-1	-1	-1	1

Table 5.18

$$\frac{1}{n^2(n-1)^2(n-2)^2(n-3)^2}$$

$p_{11}q_{11}r_{11}$ $p_{12}q_{12}r_{12}$ $p_{13}q_{13}r_{13}$ $p_{14}q_{14}r_{14}$		r_{11} r_{12} r_{13} r_2	r_{11} r_{12} r_{21} r_{22}	r_{11} r_{21} r_{12} r_{22}	r_{11} r_{12} r_2 r_3	r_{11} r_2 r_3 r_{12}	r_{11} r_2 r_{12} r_3	r_2 r_3 r_{11} r_{12}	r_1 r_2 r_3 r_4
	$n^2(n^4-12n^3+51n^2-74n-18)$	$n(n^3-10n^2+25n+12)$	$n(n^3-10n^2+29n-12)$	$n(n^3-10n^2+29n-12)$	$4n(n-6)$	$4n(n-6)$	$4n(n-6)$	$4n(n-6)$	36
$q_{11}q_{12}q_{13}q_2$	$n(n^3-10n^2+25n+12)$	$-(n^4-10n^3+31n^2-24n+42)$	$2(n^2-6n+3)$	$-2(n^2-6n+3)$	$-2(n^2-6n+3)$	12	$-2(n^2-6n+3)$	12	-12
$q_{11}q_{12}q_{21}q_{22}$	$n(n^3-10n^2+29n-12)$	$2(n^2-6n+3)$	$n^4-10n^3+41n^2-84n$ 66	-6	$-2(n^2-6n+6)$	6	6	$-2(n^2-6n+6)$	-6
$q_{11}q_{12}q_2q_3$	$4n(n-6)$	$-2(n^2-6n+3)$	$-2(n^2-6n+6)$	6	$2(n^2-6n+6)$	-6	-6	-6	6
$q_1\ q_2\ q_3\ q_4$	36	-12	-6	-6	6	6	6	6	-6
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_{13}q_2$	$-(n^4-10n^3+31n^2-24n+42)$	$n^4-10n^3+34n^2-42n+17$	$-(n-1)(n-5)$	$-(n^2-6n+7)$	n^2-6n+5	-4	n^2-6n+7	-4	4
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_{21}q_{22}$	$2(n^2-6n+3)$	$-(n-1)(n-5)$	$-(n^2-6n+7)$	2	n^2-6n+7	-2	-2	-2	2
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_2\ q_3$	$-2(n^2-6n+3)$	n^2-6n+5	n^2-6n+7	-2	$-(n^2-6n+7)$	2	2	2	-2
$p_{11}p_{12}p_{13}p_2$ $q_1\ q_2\ q_3\ q_4$	-12	4	2	2	-2	-2	-2	-2	2
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{12}q_{21}q_{22}$	$n^4-10n^3+41n^2-84n+66$	$-(n^2-6n+7)$	$n^4-10n^3+35n^2-48n+19$	1	n^2-6n+8	-1	-1	n^2-6n+8	1
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{21}q_{12}q_{22}$	-6	2	1	1	-1	-1	-1	-1	1
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{12}q_2\ q_3$	$-2(n^2-6n+6)$	n^2-6n+7	n^2-6n+8	-1	$-(n^2-6n+8)$	1	1	1	-1
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_2\ q_{12}q_3$	6	-2	-1	-1	1	1	1	–	-1
$p_{11}p_{12}p_{21}p_{22}$ $q_1\ q_2\ q_3\ q_4$	-6	2	1	1	-1	-1	-1	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_{11}q_{12}q_2\ q_3$	$2(n^2-6n+6)$	$-(n^2-6n+7)$	$-(n^2-6n+8)$	1	n^2-6n+8	-1	-1	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_{11}q_2\ q_{12}q_3$	-6	2	1	1	-1	-1	-1	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_2\ q_3\ q_{11}q_{12}$	-6	2	1	1	-1	-1	-1	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_1\ q_2\ q_3\ q_4$	6	-2	-1	-1	1	1	1	1	-1
$p_1\ p_2\ p_3\ p_4$ $q_1\ q_2\ q_3\ q_4$	-6	2	1	1	-1	-1	-1	-1	1

Table 5.19

$$\frac{1}{n^2(n-1)^2(n-2)^2(n-3)}$$

p_{11} q_{11} r_{11}		r_{11}	r_1
p_{12} q_{12} r_{12}		r_{12}	r_2
p_{13} q_{13} r_{13}		r_2	r_3
p_{14} q_{14} 0		0	0
	$n^2(n^3-8n^2+17n+2)$	$2(n^3+11n^2-128n+240)$	36
$q_{11}q_{12}q_{13}q_2$	$-n(n^4-8n^3+22n^2-23n+30)$	$-2n(n-5)$	-12
$q_{11}q_{12}q_2\ q_{13}$	$n(n^2-5n-2)$	$-n(n^2-5n+2)$	$-4n$
$q_{11}q_{12}q_{21}q_{22}$	$n(n^2-5n+2)$	$-n(n^2-5n+4)$	$-2n$
$q_{11}q_{21}q_{12}q_{22}$	$n(n^2-5n+2)$	$2n$	$-2n$
$q_{11}q_{12}q_2\ q_3$	$-2n(n-5)$	$2(n^2-5n+3)$	6
$q_{11}q_2\ q_3\ q_{12}$	$4n$	$-2n$	$2n$
$q_1\ q_2\ q_3\ q_4$	-12	6	-6
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_{13}q_2$	$n^4-8n^3+20n^2-13n+2$	n^2-5n+2	-12
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_2\ q_{13}$	$-(n^2-5n-2)$	$n^2-5n\ 2$	-12
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_{21}q_{22}$	$-(n^2-5n+2)$	n^2-5n+4	2
$p_{11}p_{12}p_{13}p_2$ $q_{11}q_{12}q_2\ q_3$	n^2-5n+2	$-(n^2-5n+4)$	-2
$p_{11}p_{12}p_{13}p_2$ $q_1\ q_2\ q_3\ q_4$	4	-2	2
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{12}q_{21}q_{22}$	$-(n^3-6n^2+9n-2)$	$n^3-6n^2+10n-4$	$n-2$
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{21}q_{12}q_{22}$	2	-1	1
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_{12}q_2\ q_3$	n^2-5n+4	$-(n^2-5n+5)$	-1
$p_{11}p_{12}p_{21}p_{22}$ $q_{11}q_2\ q_3\ q_{12}$	-2	1	-1
$p_{11}p_{12}p_{21}p_{22}$ $q_1\ q_2\ q_3\ q_4$	2	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_{11}q_{12}q_2\ q_3$	$-(n^2-5n+4)$	n^2-5n+5	1
$p_{11}p_{12}p_2\ p_3$ $q_2\ q_3\ q_{11}q_{12}$	2	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_{11}q_2\ q_3\ q_{12}$	2	-1	1
$p_{11}p_{12}p_2\ p_3$ $q_1\ q_2\ q_3\ q_4$	-2	1	-1
$p_1\ p_2\ p_3\ p_4$ $q_1\ q_2\ q_3\ q_4$	2	-1	1

Table 5.20

$$\frac{1}{n^2(n-1)^2(n-2)(n-3)}$$

p_{11} q_{11} r_{11} p_{12} q_{12} r_{12} p_{13} q_{13} 0 p_{14} q_{14} 0		p_{11} p_{12} p_{13} p_2	p_2 p_{11} p_{12} p_{13}	p_{11} p_{12} p_{21} p_{22}	p_1 p_2 p_3 p_4	r_1 r_2 0 0	p_{11} r_1 p_{12} r_2 p_{13} 0 p_2 0	p_2 r_1 $p_{11}r_2$ $p_{12}0$ $p_{13}0$	p_{11} r_1 p_{12} r_2 p_{21} 0 p_{22} 0	p_1 r_1 p_2 r_2 p_3 0 p_4 0
	$n^2(n^2-4n-1)$	$-n(n^2-4n-1)$	$n(n+1)$	$-(n-4)(n^2-2n+3)$	6	$4n^2$	$-4n$	$-n(n+1)$	$-2n$	-6
$q_{11}q_{12}q_{13}q_2$	$-n(n^2-4n-1)$	n^2-4n+1	$-(n+1)$	n^2-4n+1	-2	$-4n$	$n+1$	$n+1$	2	2
$q_{11}q_{12}q_2\ q_{13}$	$-n(n^2-4n-1)$	n^2-4n-1	$-(n+1)$	$-2(n^2-4n+2)$	-2	$-4n$	4	$n+1$	2	2
$q_{11}q_2\ q_{12}q_3$	$-2n$	$n-1$	2	1	1	$2n$	$-(n-1)$	-2	-1	-1
$q_2\ q_{11}q_{12}q_{13}$	$n(n+1)$	$-(n+1)$	$-(n-1)^2$	$-(n-1)$	-2	$-n(n+1)$	$n+1$	$(n-1)^2$	$n-1$	2
$q_{11}q_{12}q_{21}q_{22}$	$-(n-4)(n^2-2n+3)$	n^2-4n+1	$-(n-1)$	n^3-5n^2+6n-1	-1	$-2n$	2	$n-1$	$n-2$	1
$q_{11}q_{21}q_{12}q_{22}$	$n(n-1)$	$-(n-1)$	$-(n-1)$	-1	-1	$-n(n-1)$	$n-1$	$n-1$	1	1
$q_{11}q_{12}q_2\ q_3$	$2n(n-4)$	$-(n^2-4n+1)$	2	$-(n^2-4n+2)$	1	6	-2	-2	-1	-1
$q_2\ q_3\ q_{11}q_{12}$	$-2n$	2	$n-1$	$n-2$	1	$2n$	-2	$-(n-1)$	$-(n-2)$	-1
$q_1\ q_2\ q_3\ q_4$	6	-2	-2	-1	-1	-6	2	2	1	1

Table 5.21

$$\frac{1}{n^2(n-1)^2(n-2)^2}$$

$p_{11}q_{11}r_{11}$ $p_{12}q_{12}r_{12}$ $p_{13}q_{13}0$ $p_{14}0\ \ r_{13}$		r_{11} r_{12} 0 r_2	r_1 r_2 0 r_3	$q_{11}r_{11}$ $q_{12}r_{12}$ q_2 0 0 r_2	$q_{11}r_{11}$ q_2 r_{12} $q_{12}0$ 0 r_2	$q_{11}r_{11}$ q_2 r_2 $q_{12}0$ 0 r_{12}	$q_{11}r_1$ $q_{12}r_2$ q_2 0 0 r_3	$q_{11}r_1$ q_2 r_2 $q_{12}0$ 0 r_3	q_1r_1 q_2r_2 q_30 0 r_3
	$n^3(n-4)$	$n^3-6n^2+16n-2$	-8	$n(n-4)$	$-2n$	$-n^2$	4	$2n$	-4
$p_{11}p_{12}p_{13}p_2$	$-n^2(n-4)$	n^3-4n^2+2n+8	$2n$	$-(n^2-4n+2)$	n	n	-2	$-n$	2
$p_{11}p_2\ p_{12}p_{13}$	n^2	$-n$	n	1	n	n	-1	$n-1$	1
$p_{11}p_{12}p_{21}p_{22}$	0	0	0	0	0	0	0	0	0
$p_{11}p_{21}p_{12}p_{22}$	n^2	$-n$	n	1	$n-1$	$n-1$	-1	$-(n-1)$	1
$p_1\ p_2\ p_3\ p_4$	4	-2	2	1	1	1	-1	-1	1

Table 5.22

$$\frac{1}{n^2(n-1)^2(n-2)^2}$$

p_{11} q_{11} r_{11} p_{12} q_{12} 0 p_{13} 0 r_{12} 0 q_{13} r_{13}		r_{11} 0 r_{12} r_2	r_{11} 0 r_2 r_{12}	r_2 0 r_{11} r_{12}	r_1 0 r_2 r_3
	$n^3(n-3)$	$-(n^3-2n^2-4n+4)$	$-n^2(n-3)$	n^2	$n(n-4)$
$q_{11}q_{12}0\ \ q_2$	$-(n^3-2n^2-4n+4)$	$n(n^2-3n+1)$	$n(n-3)$	$-n$	$-n(n-3)$
$q_{11}q_2\ 0\ \ q_{12}$	$-n^2(n-3)$	$n(n-3)$	$n(n-3)$	$-n$	$-n(n-4)$
$q_2\ q_{11}0\ \ q_{12}$	n^2	$-n$	$-n$	$-n(n-1)$	n
$q_1\ q_2\ 0\ \ q_3$	$n(n-4)$	$-n(n-3)$	$-(n-4)$	n	$n-4$
$p_{11}p_{12}p_2\ 0$ $q_{11}q_{12}0\ \ q_2$	$n(n^2-3n+1)$	$-(n^2-3n+1)$	$-(n^2-3n+1)$	$-(n-1)(n-3)$	n^2-3n+1
$p_{11}p_{12}p_2\ 0$ $q_2\ q_{11}0\ \ q_{12}$	$-n$	1	1	1	-1
$p_2\ p_{11}p_{12}0$ $q_{11}q_{12}0\ \ q_2$	$-n$	1	1	$n-1$	-1
$p_2\ p_{11}p_{12}0$ $q_2\ q_{11}0\ \ q_{12}$	$-n$	$n-1$	$n-1$	$(n-1)^2$	$-(n-1)$
$p_1\ p_2\ p_3\ 0$ $q_1\ q_2\ 0\ \ q_3$	$n-4$	$-(n-3)$	$-(n-3)$	$n-3$	1

Table 5.23

$$\frac{1}{n^2(n-1)^2(n-2)}$$

p_{11} q_{11} r_{11} p_{12} 0 r_{12} p_{13} q_{12} 0 p_{14} q_{13} 0		p_{11} p_{12} p_{13} p_2	p_{11} p_2 p_{12} p_{13}	p_2 p_{11} p_{12} p_{13}	p_{11} p_{12} p_{21} p_{22}	p_{11} p_{21} p_{12} p_{22}	p_{11} p_{12} p_2 p_3	p_{11} p_2 p_{12} p_3	p_2 p_{11} p_{12} p_3	p_2 p_{11} p_3 p_{12}	p_2 p_3 p_{11} p_{12}	p_1 p_2 p_3 p_4
	n^3	$-n^2$	$-n^2$	0	$-n^2$	0	2n	n	0	0	n	-2
q_{11} 0 $q_{12}q_2$	$-n^2$	n(n-2)	n	0	n	0	-n	-(n-1)	0	0	-1	1
q_{11} 0 q_2 q_{12}	$-n^2$	n	n	0	n	0	-n	-1	0	0	-1	1
q_2 0 $q_{11}q_{12}$	$-n^2$	n	n	0	n(n-1)	0	-n	-1	0	0	-(n-1)	1
q_1 0 q_2 q_3	2n	-n	-2	0	-n	0	n	1	0	0	1	-1
r_1 r_2 0 0	$-n^2$	n	n^2	0	n	0	-2	-n	0	0	-n	2
q_{11} 0 $q_{12}q_2$ r_1 r_2 0 0	n	-(n-1)	-n	0	-1	0	1	n-1	0	0	1	-1
q_{11} 0 q_2 q_{12} r_1 r_2 0 0	n	-1	-n	0	-1	0	1	1	0	0	1	-1
q_2 0 q_{11} q_{12} r_1 r_2 0 0	n	-1	-n	0	-(n-1)	0	1	1	0	0	n-1	-1
q_1 0 q_2 q_3 r_1 r_2 0 0	-2	1	2	0	1	0	-1	-1	0	0	-1	1

Table 5.24

$$\frac{1}{n^2(n-1)^2(n-2)}$$

0 $q_{11}r_{11}$ p_{11} 0 r_{12} $p_{12}q_{12}$ 0 $p_{13}q_{13}$ 0		0 p_{11} p_{12} p_2	0 p_{11} p_2 p_{12}	0 p_2 p_{11} p_{12}	0 p_1 p_2 p_3	r_1 r_2 0 0	0 r_1 $p_{11}r_2$ p_{12} 0 p_2 0	0 r_1 $p_{11}r_2$ p_2 0 p_{12} 0	0 r_1 p_2r_2 p_{11} 0 p_{12} 0	0 r_1 p_1r_1 p_2 0 p_3 0
	n^3	$-n^2$	$-(n^2+n-2)$	$-n^2$	2n	$-n^2$	n	2(n-1)	2n-3	-2
q_{11} 0 $q_{12}q_2$	$-n^2$	n(n-1)	2(n-1)	n	-n	n	-(n-1)	-(n-1)	-1	1
q_{11} 0 q_2 q_{12}	$-n^2$	n	n(n-1)	n	-n	n	-1	-(n-1)	1	-1
q_2 0 $q_{11}q_{12}$	$-n^2$	n	n	n	-2	2n-3	-1	-1	n^2-3n+1	-(n-3)
q_1 0 q_2 q_3	2n	-n	-2(n-1)	-2	2	-2	1	1	-(n-3)	n-3

CHAPTER IV

PRODUCTS OF GENERALIZED k-STATISTICS

With the use of machinery developed so far, two types of formulae for the products of generalized k-statistics are worked out. First, formulae for multiplying $k_{\{\}}$, where $\{\}$ is any set of subscripts, by products of $k_{\cdots}$'s up to weight 4 are obtained. Next, these and at times direct combinatorial method are used to write specific formulae for weights 9, 10 and 12 not including unit subscripts. Checks are indicated for both types of formulae.

Semi-general Product Formulae

A generalization of Fisher's combinatorial technique as developed by Dwyer and Tracy (1962) into a combinatorial method for products of two generalized k-statistics (outlined in Chapter I) is used to obtain semi-general formulae for products of more than two generalized k-statistics. The rules obtained in Chapter II are very helpful in determining algebraic coefficients of most array types and, by virtue of the rule of proper parts, quite a few array types need not be considered. Also, for a number of patterns, algebraic coefficients as listed in Table 5 (Chapter III) are already at hand. Products involving k_1 have been further checked by the use of the rule of multiplication by k_1 (Wishart, 1952), expressed by Dwyer and Tracy (1962) as

$$(4.1) \qquad k_{\{\}}\, k_1 = k_{\{\}1} + \frac{1}{n} k_{\{1\}} .$$

Formulae for multiplication of $k_{\{\}}$ by products of $k_{\cdots}$'s up to weight 4 are presented in Table 6. The case of weight 1 simply entails one formula, i.e., (4.1). For weight 2, Table 6.1 gives formulae for $k_{\{\}}\, k_2$, $k_{\{\}}\, k_{11}$ and $k_{\{\}}\, k_1^{\,2}$. The first two have already appeared in Dwyer and Tracy (1962) whereas the last one has been used as a check. It is a special case of the formula

$$(4.2) \qquad k_{\{\}}\, k_1^{r} = \sum_{I=0}^{r} \frac{1}{n^{r-\tau'}} \sum (r_1', \ldots, r_{\tau'}')\, k_{\{\}+I, r_1', \ldots, r_{\tau'}'} ,$$

where the second summation is over all partitions $(r_1', \ldots, r_{\tau'}')$ of $r' = r-I$, not involving unit parts and I is an integer. Formula (4.2) is a generalization of

$$(4.3) \qquad k_p\, k_1^{r} = \sum_I \binom{r}{I} \sum \frac{(r_1', \ldots, r_{\tau'}')}{n^{r-\tau'}}\, k_{p+I, r_1', \ldots, r_{\tau'}'}$$

in Dwyer (1962). Also, in Tables 6.2, 6.3 where the multiplier of $k_{\{\}}$ has weight 3, 4 respectively, the formulae for double products like $k_{\{\}} k_3$, $k_{\{\}} k_{21}$, $k_{\{\}} k_{22}$ have appeared in Dwyer and Tracy (1962) and are included here for completeness To read any formula from Table 6, the coefficients appearing in the appropriate row are multiplied by the k-term at the head of the columns and the sum formed. An illustration is presented to clarify some further abbreviation used at the head of final columns. The formula for $k_{\{\}} k_4$ in Table 6.3 reads

$$\begin{aligned} k_{\{\}} k_4 = {} & k_{\{\}4} - \frac{6}{n^{(4)}} k_{\{1111\}} + \frac{12}{n^{(3)}} k_{\{211\}} - \frac{4}{n(n-1)} k_{\{31\}} - \frac{3}{n(n-1)} k_{\{22\}} \\ & + \frac{1}{n} k_{\{4\}} + \frac{12}{(n-1)^{(3)}} \sum (p_{i1}\ \ p_{i2}) k_{\{p_{i1} p_{i2}+11 : 11\}} \\ & - \frac{6}{(n-1)(n-2)} \sum (p_{i1}\ \ p_{i2})\, k_{\{p_{i1} p_{i2}+11 : 2\}} - \frac{12}{(n-1)(n-2)} \sum (p_{i1}\ \ p_{i2}) k_{\{p_{i1} p_{i2}+21 : 1\}} \\ & + \frac{3}{n-1} \sum (p_{i1}\ \ p_{i2}) k_{\{p_{i1} p_{i2}+22 :\}} + \frac{4}{n-1} \sum (p_{i1}\ \ p_{i2}) k_{\{p_{i1} p_{i2}+31 :\}} \end{aligned}$$

$$- \frac{4(n+1)}{(n-1)^{(3)}} \sum (p_{i1}p_{i2}p_{i3})k_{\{p_{i1}p_{i2}p_{i3}+1111\}} + \frac{6n}{(n-1)^{(2)}} \sum (p_{i1}p_{i2}p_{i3})k_{\{p_{i1}p_{i2}p_{i3}+211\}}$$

$$+ \frac{n(n+1)}{(n-1)^{(3)}} \sum (p_{i1}p_{i2}p_{i3}p_{i4})k_{\{p_{i1}p_{i2}p_{i3}p_{i4}+1111\}}$$

$$- \frac{1}{(n-2)(n-3)} \sum (p_{i1}\ p_{i2})(p_{j1}\ p_{j2})k_{\{p_{i1}p_{i2}p_{j1}p_{j2}+1111\}} .$$

Checks

It may be observed that checks are available. For example, for weight 3, it is known (Wishart, 1952) that

$$k_1^3 = \frac{1}{n^2}k_3 + \frac{3}{n}k_{21} + k_{111} \qquad \text{and also}$$

$$k_1^3 = k_1^2\ k_1 = \frac{1}{n}k_2k_1 + k_{11}k_1 .$$

Hence, it can be checked that

$$k_{\{\}}\ k_1^3 = \frac{1}{n^2}k_{\{\}}k_3 + \frac{3}{n}k_{\{\}}k_{21} + k_{\{\}}\ k_{111} \qquad \text{and}$$

$$k_{\{\}}\ k_1^3 = \frac{1}{n}k_{\{\}}k_2k_1 + k_{\{\}}k_{11}k_1 .$$

Formulae for polykays of deviates (Dwyer, 1962) are also helpful in checking. To take the same example, we should have

$$k_{\{\}}k_3 + 3n\ k_{\{\}}d_{21} + n^2\ k_{\{\}}\ d_{111} = 0,$$

where d... represents the corresponding k... of deviates. This is so since $d_1 = 0$ and

$$\sum \frac{1}{n^{r-\rho}}(r_1, \ldots, r_\rho)d_{r_1 \ldots r_\rho} \equiv 0,$$

where the summation is over all ρ-part partitions $(r_1, \ldots, r_\rho)$ of r, and $\rho = 1, 2, \ldots, r$.

As the weight of the multiplier of $k_{\{\}}$ increases, more and more cross-checks become available, in view of the increasing number of lower weight formulae.

Also the zero coefficients for certain array types for particular products can be obviously expected by some rules of Chapter II.

TABLE 6

Semi-general Product Formulae

Table 6.1

Weight 2

	$\{\}11$	$\{\}2$	$\{1\}\,1$	$\{11\}$	$\{2\}$	$\overline{\Sigma(p_{i1}p_{i2})}$ $+11;\}$
$k_{\{\}}\,k_2$		1		$\frac{-1}{n(n-1)}$	$\frac{1}{n}$	$\frac{1}{n-1}$
$k_{\{\}}\,k_{11}$	1		$\frac{2}{n}$	$\frac{1}{n(n-1)}$		$\frac{-1}{n(n-1)}$
$k_{\{\}}\,k_1^2$	1	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	

Table 6.2

Weight 3

											$\Sigma(p_{i1}p_{i2})$			$\Sigma(p_{i1}p_{i2}p_{i3})$
	$\{\}111$	$\{\}21$	$\{\}3$	$\{1\}11$	$\{1\}2$	$\{11\}1$	$\{2\}$	$\{111\}$	$\{21\}$	$\{3\}$	$+11:\}1$	$+11:1\}$	$+21:\}$	$+111:\}$
$k_{\{\}}\ k_3$			1					$\frac{2}{n(n-1)(n-2)}$	$\frac{-3}{n(n-1)}$	$\frac{1}{n}$		$\frac{-3}{(n-1)(n-2)}$	$\frac{3}{n-1}$	$\frac{n}{(n-1)(n-2)}$
$k_{\{\}}\ k_{21}$		1			$\frac{1}{n}$	$\frac{-1}{n(n-1)}$	$\frac{1}{n}$	$\frac{-1}{n(n-1)(n-2)}$	$\frac{1}{n(n-1)}$		$\frac{1}{n-1}$	$\frac{n+1}{n(n-1)(n-2)}$	$\frac{-1}{n(n-1)}$	$\frac{-1}{(n-1)(n-2)}$
$k_{\{\}}\ k_{111}$	1			$\frac{3}{n}$		$\frac{3}{n(n-1)}$		$\frac{1}{n(n-1)(n-2)}$			$\frac{-3}{n(n-1)}$	$\frac{-3}{n(n-1)(n-2)}$		$\frac{2}{n(n-1)(n-2)}$
$k_{\{\}}\ k_1^3$	1	$\frac{3}{n}$	$\frac{1}{n^2}$	$\frac{3}{n}$	$\frac{3}{n^2}$	$\frac{3}{n^2}$	$\frac{3}{n^2}$	$\frac{1}{n^3}$	$\frac{3}{n^3}$	$\frac{1}{n^3}$				
$k_{\{\}}k_2\ k_1$		1	$\frac{1}{n}$		$\frac{1}{n}$	$\frac{-1}{n(n-1)}$	$\frac{1}{n}$	$\frac{-1}{n^2(n-1)}$	$\frac{n-3}{n^2(n-1)}$	$\frac{1}{n^2}$	$\frac{1}{n-1}$	$\frac{1}{n(n-1)}$	$\frac{2}{n(n-1)}$	
$k_{\{\}}k_{11}k_1$	1	$\frac{2}{n}$		$\frac{3}{n}$	$\frac{2}{n^2}$	$\frac{3n-2}{n^2(n-1)}$	$\frac{2}{n^2}$	$\frac{1}{n^2(n-1)}$	$\frac{2}{n^2(n-1)}$		$\frac{-1}{n(n-1)}$	$\frac{-1}{n^2(n-1)}$	$\frac{-2}{n^2(n-1)}$	

Table 6.3 - WEIGHT 4

	{}1111	{}211	{}31	{}22	{}4	{1}111	{1}21	{1}3	{11}11	{11}2	{2}11	{2}2	{111}1	{21}1	{3}1	{1111}	{211}	{31}	{22}	{4}
$k_{\{\}}k_4$					1											$\frac{-6}{n^{(4)}}$	$\frac{12}{n^{(3)}}$	$\frac{-4}{n(n-1)}$	$\frac{-3}{n(n-1)}$	$\frac{1}{n}$
$k_{\{\}}k_{31}$			1					$\frac{1}{n}$					$\frac{2}{n^{(3)}}$	$\frac{-3}{n(n-1)}$	$\frac{1}{n}$	$\frac{2}{n^{(4)}}$	$\frac{-3}{n^{(3)}}$	$\frac{1}{n(n-1)}$		
$k_{\{\}}k_{22}$				1						$\frac{-2}{n(n-1)}$		$\frac{2}{n}$				$\frac{1}{n^{(4)}}$	$\frac{-2}{n^{(3)}}$		$\frac{1}{n(n-1)}$	
$k_{\{\}}k_{211}$		1					$\frac{2}{n}$		$\frac{-1}{n(n-1)}$	$\frac{1}{n(n-1)}$	$\frac{1}{n}$		$\frac{-2}{n^{(3)}}$	$\frac{2}{n(n-1)}$		$\frac{-1}{n^{(4)}}$	$\frac{1}{n^{(3)}}$			
$k_{\{\}}k_{1111}$	1					$\frac{4}{n}$			$\frac{6}{n(n-1)}$				$\frac{4}{n^{(3)}}$			$\frac{1}{n^{(4)}}$				
$k_{\{\}}k_1^4$	1	$\frac{6}{n}$	$\frac{4}{n^2}$	$\frac{3}{n^2}$	$\frac{1}{n^3}$	$\frac{4}{n}$	$\frac{12}{n^2}$	$\frac{4}{n^3}$	$\frac{6}{n^2}$	$\frac{6}{n^3}$	$\frac{6}{n^2}$	$\frac{6}{n^3}$	$\frac{4}{n^3}$	$\frac{12}{n^3}$	$\frac{4}{n^3}$	$\frac{1}{n^4}$	$\frac{6}{n^4}$	$\frac{4}{n^4}$	$\frac{3}{n^4}$	$\frac{1}{n^4}$
$k_{\{\}}k_3k_1$			1		$\frac{1}{n}$			$\frac{1}{n}$					$\frac{2}{n^{(3)}}$	$\frac{-3}{n(n-1)}$	$\frac{1}{n}$	$\frac{2}{n.n^{(3)}}$	$\frac{-3(n-4)}{n.n^{(3)}}$	$\frac{n-4}{n^2(n-1)}$	$\frac{-3}{n^2(n-1)}$	$\frac{1}{n^2}$
$k_{\{\}}k_{21}k_1$		1	$\frac{1}{n}$	$\frac{1}{n}$			$\frac{2}{n}$	$\frac{1}{n^2}$	$\frac{-1}{n(n-1)}$	$\frac{n-2}{n^2(n-1)}$	$\frac{1}{n}$	$\frac{2}{n^2}$	$\frac{-2}{n^2(n-2)}$	$\frac{2n-3}{n^2(n-1)}$	$\frac{1}{n^2}$	$\frac{-1}{n.n^{(3)}}$	$\frac{n-5}{n.n^{(3)}}$	$\frac{1}{n^2(n-1)}$	$\frac{1}{n^2(n-1)}$	
$k_{\{\}}k_{111}k_1$	1	$\frac{3}{n}$				$\frac{4}{n}$	$\frac{6}{n^2}$		$\frac{3(2n-1)}{n^2(n-1)}$	$\frac{3}{n^2(n-1)}$	$\frac{3}{n^2}$		$\frac{2(2n-3)}{n.n^{(3)}}$	$\frac{6}{n^2(n-1)}$		$\frac{1}{n.n^{(3)}}$	$\frac{3}{n.n^{(3)}}$			
$k_{\{\}}k_2k_1^2$		1	$\frac{2}{n}$	$\frac{1}{n}$	$\frac{1}{n^2}$		$\frac{2}{n}$	$\frac{2}{n^2}$	$\frac{-1}{n(n-1)}$	$\frac{n-2}{n^2(n-1)}$	$\frac{1}{n}$	$\frac{2}{n^2}$	$\frac{-2}{n^2(n-1)}$	$\frac{2(n-3)}{n^2(n-1)}$	$\frac{2}{n^2}$	$\frac{-1}{n^3(n-1)}$	$\frac{n-6}{n^3(n-1)}$	$\frac{2(n-2)}{n^3(n-1)}$	$\frac{n-3}{n^3(n-1)}$	$\frac{1}{n^3}$
$k_{\{\}}k_{11}k_1^2$	1	$\frac{5}{n}$	$\frac{2}{n^2}$	$\frac{2}{n^2}$		$\frac{4}{n}$	$\frac{10}{n^2}$	$\frac{2}{n^3}$	$\frac{6n-5}{n^2(n-1)}$	$\frac{5n-4}{n^3(n-1)}$	$\frac{5}{n^2}$	$\frac{4}{n^3}$	$\frac{2(2n-1)}{n^3(n-1)}$	$\frac{2(5n-3)}{n^3(n-1)}$	$\frac{2}{n^3}$	$\frac{1}{n^3(n-1)}$	$\frac{5}{n^3(n-1)}$	$\frac{2}{n^3(n-1)}$	$\frac{2}{n^3(n-1)}$	
$k_{\{\}}k_2^2$				$\frac{n+1}{n-1}$	$\frac{1}{n}$					$\frac{-2(n+1)}{n(n-1)^2}$		$\frac{2(n+1)}{n(n-1)}$				$\frac{1}{n^2(n-1)^2}$	$\frac{-2(n-3)}{n^2(n-1)^2}$	$\frac{-4}{n^2(n-1)}$	$\frac{n^2-2n+3}{n^2(n-1)^2}$	$\frac{1}{n^2}$
$k_{\{\}}k_2k_{11}$		1	$\frac{2}{n}$	$\frac{-2}{n(n-1)}$			$\frac{2}{n}$	$\frac{2}{n^2}$	$\frac{-1}{n(n-1)}$	$\frac{n^2-n+4}{n^2(n-1)^2}$	$\frac{1}{n}$	$\frac{-4}{n^2(n-1)}$	$\frac{-2}{n^2(n-1)}$	$\frac{2(n-3)}{n^2(n-1)}$	$\frac{2}{n^2}$	$\frac{-1}{n^2(n-1)^2}$	$\frac{n-5}{n^2(n-1)^2}$	$\frac{2}{n^2(n-1)}$	$\frac{-2}{n^2(n-1)^2}$	
$k_{\{\}}k_{11}^2$	1	$\frac{4}{n}$		$\frac{2}{n(n-1)}$		$\frac{4}{n}$	$\frac{8}{n^2}$		$\frac{2(3n-2)}{n^2(n-1)}$	$\frac{4(n-2)}{n^2(n-1)^2}$	$\frac{4}{n^3}$	$\frac{4}{n^2(n-1)}$	$\frac{4}{n^2(n-1)}$	$\frac{8}{n^2(n-1)}$		$\frac{1}{n^2(n-1)^2}$	$\frac{4}{n^2(n-1)^2}$		$\frac{2}{n^2(n-1)^2}$	

	$\Sigma(p_{i_1}p_{i_2})$									$\Sigma(p_{i_1}p_{i_2}p_{i_3})$			$\Sigma(p_{i_1}p_{i_2}p_{i_3}p_{i_4})$	$\Sigma(p_{i_1}p_{i_2})(p_{j_1}p_{j_2})$
	+11:}11	+11:}2	+11:1}1	+2:}1	+11:11}	+11:2}	+21:1}	+22:}	+31:}	+111:}1	+111:1}	+211:}	+1111:}	+1111:}
$k_{\{\}}k_4$					$\frac{12}{(n-1)^{(3)}}$	$\frac{-6}{(n-1)(n-2)}$	$\frac{-12}{(n-1)(n-2)}$	$\frac{3}{n-1}$	$\frac{4}{n-1}$		$\frac{-4(n+1)}{(n-1)^{(3)}}$	$\frac{6n}{(n-1)^{(2)}}$	$\frac{n(n+1)}{(n-1)^{(3)}}$	$\frac{-1}{(n-2)(n-3)}$
$k_{\{\}}k_{31}$			$\frac{-3}{(n-1)(n-2)}$	$\frac{3}{n-1}$	$\frac{-3(n+1)}{n^{(4)}}$	$\frac{3}{n^{(3)}}$	$\frac{3}{(n-1)(n-2)}$		$\frac{-1}{n(n-1)}$	$\frac{n}{(n-1)^{(2)}}$	$\frac{n^2+n+4}{n^{(4)}}$	$\frac{-3}{(n-1)^{(2)}}$	$\frac{-(n+1)}{(n-1)^{(3)}}$	$\frac{1}{n(n-2)(n-3)}$
$k_{\{\}}k_{22}$		$\frac{2}{n-1}$			$\frac{-2}{(n-1)^{(3)}}$	$\frac{2}{n(n-2)}$	$\frac{4}{n^{(3)}}$	$\frac{-1}{n(n-1)}$			$\frac{4}{n(n-2)(n-3)}$	$\frac{-2}{(n-1)^{(2)}}$	$\frac{-1}{(n-2)^{(2)}}$	$\frac{n^2-3n+3}{3n^{(4)}}$
$k_{\{\}}k_{211}$	$\frac{1}{n-1}$	$\frac{-1}{n^{(2)}}$	$\frac{2(n+1)}{n^{(3)}}$	$\frac{-2}{n^{(2)}}$	$\frac{n+3}{n^{(4)}}$	$\frac{-1}{n^{(3)}}$	$\frac{-2}{n^{(3)}}$			$\frac{-2}{(n-1)^{(2)}}$	$\frac{-2(n+1)}{n^{(4)}}$	$\frac{2}{n^{(3)}}$	$\frac{2}{(n-1)^{(3)}}$	$\frac{-1}{3(n-1)^{(3)}}$
$k_{\{\}}k_{1111}$	$\frac{-6}{n(n-1)}$		$\frac{-12}{n^{(3)}}$		$\frac{-6}{n^{(4)}}$					$\frac{8}{n^{(3)}}$	$\frac{8}{n^{(4)}}$		$\frac{-6}{n^{(4)}}$	$\frac{1}{n^{(4)}}$
$k_{\{\}}k_1^4$														
$k_{\{\}}k_3k_1$			$\frac{-3}{(n-1)(n-2)}$	$\frac{3}{n-1}$	$\frac{-3}{n^{(3)}}$	$\frac{-3}{n^{(3)}}$	$\frac{3(n-4)}{n^{(3)}}$	$\frac{2}{n(n-1)}$	$\frac{3}{n(n-1)}$	$\frac{n}{(n-1)^{(2)}}$	$\frac{1}{(n-1)^{(2)}}$	$\frac{3}{(n-1)^{(2)}}$		
$k_{\{\}}k_{21}k_1$	$\frac{1}{n-1}$	$\frac{1}{n(n-1)}$	$\frac{2n-1}{n^{(3)}}$	$\frac{1}{n^{(2)}}$	$\frac{n+1}{n.n^{(3)}}$	$\frac{n+1}{n.n^{(3)}}$	$\frac{n+4}{n.n^{(3)}}$	$\frac{-1}{n^2(n-1)}$	$\frac{-1}{n^2(n-1)}$	$\frac{-1}{(n-1)^{(2)}}$	$\frac{-1}{n^{(3)}}$	$\frac{-3}{n^{(2)}}$		
$k_{\{\}}k_{111}k_1$	$\frac{-3}{n(n-1)}$	$\frac{-3}{n.n^{(2)}}$	$\frac{-6}{n^2(n-2)}$	$\frac{-6}{n.n^{(2)}}$	$\frac{-3}{n.n^{(3)}}$	$\frac{-3}{n.n^{(3)}}$	$\frac{-6}{n.n^{(3)}}$			$\frac{2}{n^{(3)}}$	$\frac{2}{n.n^{(3)}}$	$\frac{6}{n.n^{(3)}}$		
$k_{\{\}}k_2k_1^2$	$\frac{1}{n-1}$	$\frac{1}{n(n-1)}$	$\frac{2}{n(n-1)}$	$\frac{4}{n(n-1)}$	$\frac{1}{n^2(n-1)}$	$\frac{1}{n^2(n-1)}$	$\frac{4}{n^2(n-1)}$	$\frac{2}{n^2(n-1)}$	$\frac{2}{n^2(n-1)}$					
$k_{\{\}}k_{11}k_1^2$	$\frac{-1}{n(n-1)}$	$\frac{-1}{n.n^{(2)}}$	$\frac{-2}{n.n^{(2)}}$	$\frac{-4}{n.n^{(2)}}$	$\frac{-1}{n^3(n-1)}$	$\frac{-1}{n^3(n-1)}$	$\frac{-4}{n^3(n-1)}$	$\frac{-2}{n^3(n-1)}$	$\frac{-2}{n^3(n-1)}$					
$k_{\{\}}k_2^2$		$\frac{2(n+1)}{(n-1)^2}$			$\frac{-2}{n-1}$	$\frac{2}{n(n-1)}$	$\frac{-8}{n(n-1)^2}$	$\frac{2(n-2)}{n(n-1)^2}$	$\frac{4}{n(n-1)}$			$\frac{4}{(n-1)^2}$		$\frac{1}{3(n-1)^2}$
$k_{\{\}}k_2k_{11}$	$\frac{1}{n-1}$	$\frac{-(n+3)}{n(n-1)^2}$	$\frac{2}{n(n-1)}$	$\frac{4}{n(n-1)}$	$\frac{n+1}{n^2(n-1)^2}$	$\frac{-1}{n^2(n-1)}$	$\frac{4(n+1)}{n^2(n-1)^2}$	$\frac{2}{n^2(n-1)^2}$	$\frac{-2}{n^2(n-1)}$			$\frac{4}{n(n-1)}$		$\frac{-1}{3n(n-1)^2}$
$k_{\{\}}k_{11}^2$	$\frac{-2}{n(n-1)}$	$\frac{4}{n^2(n-1)^2}$	$\frac{-4}{n^2(n-1)}$	$\frac{-8}{n^2(n-1)}$	$\frac{-2}{n^2(n-1)^2}$		$\frac{-8}{n^2(n-1)^2}$	$\frac{-2}{n^2(n-1)^2}$				$\frac{4}{n^2(n-1)^2}$		$\frac{1}{3n^2(n-1)^2}$

Particular Product Formulae

The semi-general formulae are next applied to write particular ones by substituting for $\{\}$. Products of seminvariant k-statistics of weights 9 and 10 and powers of these for weight 12 (following Fisher, 1928) are presented in Tables 7.1, 7.2 and 7.3 respectively. The products computed are listed in the first columns of Tables 7.1 and 7.2 and to read a formula for some product desired, the coefficients appearing in the corresponding row are multiplied by the k-terms at the heads of the columns and the sums formed. In Table 7.3, rows and columns have been interchanged for convenience of presentation. Here each product is allotted a column and one reads a formula down the column, multiplying the coefficients by the k-terms in the first column, and forming the sum. Formulae for weight ≤ 6 and for products of single subscript k's for weights 7 and 8 are given by Wishart (1952) while Schaeffer and Dwyer (1963) give formulae for products of seminvariant generalized k-statistics for weights 7 and 8.

Checks

Again several checks are available. For example, since

$$(4.4) \qquad k_2^2 = \frac{1}{n} k_4 + \frac{n+1}{n-1} k_{22} \quad ,$$

we should have, multiplying both sides of (4.4) by $(n-1)\ k_4k_2$ and transposing,

$$(4.5) \qquad (n+1)\ k_4k_{22}k_2 = (n-1)(k_4k_2^3 - \frac{1}{n} k_4^2 k_2).$$

Formula (4.5) checks expansions of $k_4k_{22}k_2$, $k_4k_2^3$, $k_4^2k_2$ simultaneously.

There are further checks possible for products of single subscript seminvariant k's as illustrated for $k_3^2 k_2^2$. In view of (1.14),

$$\mu'(3^2 2^2) = \kappa(3^2 2^2) + 2\kappa(3^2 2)\kappa(2) + 2\kappa(32^2)\kappa(3) + \kappa(3^2)\kappa(2^2) + 2\kappa^2(32) + \kappa(3^2)\kappa^2(2)$$
$$+ \kappa^2(3)\kappa(2^2) + 4\kappa(32)\kappa(3)\kappa(2) + \kappa^2(3)\kappa^2(2)$$

$$= \frac{1}{n^3}\kappa_{10} + \frac{2n+35}{n^2(n-1)}\kappa_8\kappa_2 + \frac{2n^2+98n-160}{n^2(n-1)^2}\kappa_7\kappa_3 + \cdots$$

with the help of formulae given by Fisher (1928)* for cumulants and product cumulants. Hence, taking estimates,

$$k_3^2 k_2^2 = \frac{1}{n^3}k_{10} + \frac{2n+35}{n^2(n-1)} k_{82} + \frac{2n^2+98n-160}{n^2(n-1)^2} k_{73} + \cdots$$

as in Table 7.2.

* Some errors have been corrected by Kendall and Stuart (1958).

TABLE 7

PRODUCTS OF SEMINVARIANT k-STATISTICS

TABLE 7·1 — WEIGHT 9

	k_9	k_{72}	k_{63}	k_{54}	k_{522}	k_{432}	k_{333}	k_{3222}
k_7k_2	$\frac{1}{n}$	$\frac{n+13}{n-1}$	$\frac{42}{n-1}$	$\frac{70}{n-1}$				
k_6k_3	$\frac{1}{n}$	$\frac{18}{n-1}$	$\frac{n+62}{n-1}$	$\frac{105}{n-1}$	$\frac{90n}{(n-1)(n-2)}$	$\frac{360n}{(n-1)(n-2)}$	$\frac{90n}{(n-1)(n-2)}$	
k_5k_4	$\frac{1}{n}$	$\frac{20}{n-1}$	$\frac{70}{n-1}$	$\frac{n+119}{n-1}$	$\frac{120n}{(n-1)(n-2)}$	$\frac{600n}{(n-1)(n-2)}$	$\frac{180n}{(n-1)(n-2)}$	$\frac{240n(n+1)}{(n-1)(n-2)(n-3)}$
$k_5k_2^2$	$\frac{1}{n^2}$	$\frac{2(n+11)}{n(n-1)}$	$\frac{20(3n-4)}{n(n-1)^2}$	$\frac{n^2+98n-139}{n(n-1)^2}$	$\frac{(n+9)(n+11)}{(n-1)^2}$	$\frac{40(n+11)}{(n-1)^2}$	$\frac{120}{(n-1)^2}$	
$k_4k_3k_2$	$\frac{1}{n^2}$	$\frac{n+25}{n(n-1)}$	$\frac{n^2+70n-95}{n(n-1)^2}$	$\frac{n^2+108n-169}{n(n-1)^2}$	$\frac{12(n^2+12n-25)}{(n-1)^2(n-2)}$	$\frac{n^3+40n^2+650n-1450}{(n-1)^2(n-2)}$	$\frac{6(n^2+27n-70)}{(n-1)^2(n-2)}$	$\frac{36n(n+9)}{(n-1)^2(n-2)}$
k_3^3	$\frac{1}{n^2}$	$\frac{27}{n(n-1)}$	$\frac{3(n^2+25n-35)}{n(n-1)^2}$	$\frac{27(4n-7)}{n(n-1)^2}$	$\frac{54(4n-7)}{(n-1)^2(n-2)}$	$\frac{27(n^2+27n-70)}{(n-1)^2(n-2)}$	$\frac{n^4+21n^3+130n^2-876n+1120}{(n-1)^2(n-2)^2}$	$\frac{18n(n^2+27n-70)}{(n-1)^2(n-2)^2}$
$k_3k_2^3$	$\frac{1}{n^3}$	$\frac{3(n+9)}{n^2(n-1)}$	$\frac{n^2+60n-105}{n^2(n-1)^2}$	$\frac{3(n^3+33n^2-89n+63)}{n^2(n-1)^3}$	$\frac{3(n+7)(n+9)}{n(n-1)^2}$	$\frac{6(17n^2+80n-157)}{n(n-1)^3}$	$\frac{4(n^2+27n-70)}{n(n-1)^3}$	$\frac{n^3+15n^2+155n+309}{(n-1)^3}$
$k_{52}k_2$		$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	$\frac{1}{n}$	$\frac{n+11}{n-1}$	$\frac{20}{n-1}$		
$k_{43}k_2$			$\frac{1}{n}$	$\frac{n-3}{n(n-1)}$		$\frac{n+13}{n-1}$	$\frac{6}{n-1}$	
$k_{322}k_2$					$\frac{1}{n}$	$\frac{2(n-2)}{n(n-1)}$	$\frac{-1}{n(n-1)}$	$\frac{n+9}{n-1}$
$k_{42}k_3$		$\frac{1}{n}$	$\frac{-3}{n(n-1)}$	$\frac{n-4}{n(n-1)}$	$\frac{6(2n-5)}{(n-1)(n-2)}$	$\frac{n^2+33n-94}{(n-1)(n-2)}$	$\frac{-18}{(n-1)(n-2)}$	$\frac{36n}{(n-1)(n-2)}$
$k_{33}k_3$			$\frac{2}{n}$	$\frac{-6}{n(n-1)}$		$\frac{18(n-4)}{(n-1)(n-2)}$	$\frac{n+17}{n-1}$	$\frac{12n}{(n-1)(n-2)}$
$k_{222}k_3$					$\frac{3}{n}$	$\frac{-18}{n(n-1)}$	$\frac{12}{n(n-1)(n-2)}$	$\frac{n^2+15n-70}{(n-1)(n-2)}$
$k_{32}k_4$		$\frac{1}{n}$	$\frac{n-5}{n(n-1)}$	$\frac{-10}{n(n-1)}$	$\frac{12(n-3)}{(n-1)(n-2)}$	$\frac{n^2+35n-170}{(n-1)(n-2)}$	$\frac{6(n-8)}{(n-1)(n-2)}$	$\frac{12(3n^2-17n+4)}{(n-1)(n-2)(n-3)}$
$k_{22}k_5$		$\frac{2}{n}$	$\frac{-10}{n(n-1)}$	$\frac{-20}{n(n-1)}$	$\frac{n^2+17n-78}{(n-1)(n-2)}$	$\frac{40(n-7)}{(n-1)(n-2)}$	$\frac{-60}{(n-1)(n-2)}$	$\frac{-240}{(n-2)(n-3)}$
$k_{32}k_{22}$				$\frac{2}{n(n-1)}$	$\frac{2}{n-2}$	$\frac{n^2-n+24}{n(n-1)(n-2)}$	$\frac{-2(n-8)}{n(n-1)(n-2)}$	$\frac{n^4+10n^3-45n^2+30n+84}{n(n-1)(n-2)(n-3)}$
$k_{32}k_2^2$		$\frac{1}{n^2}$	$\frac{n-5}{n^2(n-1)}$	$\frac{n^2-8n+10}{n^2(n-1)^2}$	$\frac{2(n+9)}{n(n-1)}$	$\frac{3n^2+40n-87}{n(n-1)^2}$	$\frac{5(n-5)}{n(n-1)^2}$	$\frac{(n+7)(n+9)}{(n-1)^2}$
$k_{22}k_3k_2$		$\frac{2}{n^2}$	$\frac{-10}{n^2(n-1)}$	$\frac{2(n^2-5n+10)}{n^2(n-1)^2}$	$\frac{3(n^2+9n-26)}{n(n-1)(n-2)}$	$\frac{2(n^3+23n^2-154n+220)}{n(n-1)^2(n-2)}$	$\frac{-2(n^2+27n-70)}{n(n-1)^2(n-2)}$	$\frac{n^3+18n^2+47n-306}{(n-1)^2(n-2)}$

TABLE 7.2 — WEIGHT 10

	k_{10}	k_{82}	k_{73}	k_{64}	k_{55}	k_{622}	k_{532}	k_{442}	k_{433}	k_{4222}	k_{3322}	k_{22222}
k_8k_2	$\frac{1}{n}$	$\frac{n+15}{n-1}$	$\frac{56}{n-1}$	$\frac{112}{n-1}$	$\frac{70}{n-1}$							
k_7k_3	$\frac{1}{n}$	$\frac{21}{n-1}$	$\frac{n+83}{n-1}$	$\frac{168}{n-1}$	$\frac{105}{n-1}$	$\frac{126n}{(n-1)(n-2)}$	$\frac{630n}{(n-1)(n-2)}$	$\frac{420n}{(n-1)(n-2)}$	$\frac{630n}{(n-1)(n-2)}$			
k_6k_4	$\frac{1}{n}$	$\frac{24}{n-1}$	$\frac{96}{n-1}$	$\frac{n+193}{n-1}$	$\frac{120}{n-1}$	$\frac{180n}{(n-1)(n-2)}$	$\frac{1080n}{(n-1)(n-2)}$	$\frac{720n}{(n-1)(n-2)}$	$\frac{1260n}{(n-1)(n-2)}$	$\frac{480n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{1080n(n+1)}{(n-1)(n-2)(n-3)}$	
k_5^2	$\frac{1}{n}$	$\frac{25}{n-1}$	$\frac{100}{n-1}$	$\frac{200}{n-1}$	$\frac{n+124}{n-1}$	$\frac{200n}{(n-1)(n-2)}$	$\frac{1200n}{(n-1)(n-2)}$	$\frac{850n}{(n-1)(n-2)}$	$\frac{1500n}{(n-1)(n-2)}$	$\frac{600n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{1800n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{120n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$
$k_6k_2^2$	$\frac{1}{n^2}$	$\frac{2(n+13)}{n(n-1)}$	$\frac{12(7n-9)}{n(n-1)^2}$	$\frac{n^2+162n-223}{n(n-1)^2}$	$\frac{20(5n-7)}{n(n-1)^2}$	$\frac{n^2+24n+143}{(n-1)^2}$	$\frac{60(n+13)}{(n-1)^2}$	$\frac{40(n+13)}{(n-1)^2}$	$\frac{840}{(n-1)^2}$			
$k_5k_3k_2$	$\frac{1}{n^2}$	$\frac{n+30}{n(n-1)}$	$\frac{n^2+99n-130}{n(n-1)^2}$	$\frac{5(37n-55)}{n(n-1)^2}$	$\frac{n^2+113n-174}{n(n-1)^2}$	$\frac{15(n^2+15n-30)}{(n-1)^2(n-2)}$	$\frac{n^3+57n^2+1172n-2640}{(n-1)^2(n-2)}$	$\frac{30(n^2+27n-60)}{(n-1)^2(n-2)}$	$\frac{10(2n^2+129n-305)}{(n-1)^2(n-2)}$	$\frac{60n(n+11)}{(n-1)^2(n-2)}$	$\frac{90n(n+17)}{(n-1)^2(n-2)}$	
$k_4^2k_2$	$\frac{1}{n^2}$	$\frac{n+31}{n(n-1)}$	$\frac{8(13n-17)}{n(n-1)^2}$	$\frac{2(n^2+96n-145)}{n(n-1)^2}$	$\frac{4(29n-46)}{n(n-1)^2}$	$\frac{8(2n^2+31n-61)}{(n-1)^2(n-2)}$	$\frac{48(n+31)}{(n-1)^2}$	$\frac{n^3+48n^2+857n-2058}{(n-1)^2(n-2)}$	$\frac{12(n^2+123n-298)}{(n-1)^2(n-2)}$	$\frac{24(3n^3+28n^2-99n-4)}{(n-1)^2(n-2)(n-3)}$	$\frac{144(n^3+13n^2-50n-2)}{(n-1)^2(n-2)(n-3)}$	$\frac{24n(n+1)(n+7)}{(n-1)^2(n-2)(n-3)}$
$k_4k_3^2$	$\frac{1}{n^2}$	$\frac{33}{n(n-1)}$	$\frac{2(n^2+55n-74)}{n(n-1)^2}$	$\frac{n^2+193n-320}{n(n-1)^2}$	$\frac{6(19n-34)}{n(n-1)^2}$	$\frac{18(19n-33)}{(n-1)^2(n-2)}$	$\frac{24(n^2+66n-154)}{(n-1)^2(n-2)}$	$\frac{9(n^2+11n-286)}{(n-1)^2(n-2)}$	$\frac{n^4+63n^3+1450n^2-7452n+8916}{(n-1)^2(n-2)^2}$	$\frac{6n(n^2+201n-478)}{(n-1)^2(n-2)^2}$	$\frac{36n(2n^2+75n-206)}{(n-1)^2(n-2)^2}$	$\frac{216n^2}{(n-1)^2(n-2)^2}$
$k_3^2k_2^2$	$\frac{1}{n^3}$	$\frac{2n+35}{n^2(n-1)}$	$\frac{2(n^2+49n-80)}{n^2(n-1)^2}$	$\frac{n^3+180n^2-495n+350}{n^2(n-1)^3}$	$\frac{2(n^3+56n^2-151n+112)}{n^2(n-1)^3}$	$\frac{n^3+40n^2+275n-700}{n(n-1)^2(n-2)}$	$\frac{4(n^4+35n^3+274n^2-1266n+1120)}{n(n-1)^3(n-2)}$	$\frac{9(7n^3+86n^2-365n+350)}{n(n-1)^3(n-2)}$	$\frac{n^4+52n^3+1133n^2-5254n+5616}{n(n-1)^3(n-2)}$	$\frac{3(3n^3+68n^2+125n-412)}{(n-1)^3(n-2)}$	$\frac{n^4+30n^3+379n^2-1062n-5312}{(n-1)^3(n-2)}$	$\frac{6n(n+5)(n+7)}{(n-1)^3(n-2)}$
$k_4k_2^3$	$\frac{1}{n^3}$	$\frac{3(n+11)}{n^2(n-1)}$	$\frac{4(23n-37)}{n^2(n-1)^2}$	$\frac{4(n^3+44n^2-117n+80)}{n^2(n-1)^3}$	$\frac{12(9n^2-24n+17)}{n^2(n-1)^3}$	$\frac{3(n+9)(n+11)}{n(n-1)^2}$	$\frac{24(5n-7)(n+11)}{n(n-1)^3}$	$\frac{12(10n^2+61n-107)}{n(n-1)^3}$	$\frac{2(11n^2+564n-1127)}{n(n-1)^3}$	$\frac{n^3+21n^2+251n+687}{(n-1)^3}$	$\frac{18(n^2+119)}{(n-1)^3}$	
k_2^5	$\frac{1}{n^4}$	$\frac{5(n+7)}{n^3(n-1)}$	$\frac{80(n-2)}{n^3(n-1)^2}$	$\frac{10(n^3+17n^2-45n+35)}{n^3(n-1)^3}$	$\frac{16(n-2)(6n^2-12n+7)}{n^3(n-1)^4}$	$\frac{10(n+5)(n+7)}{n^2(n-1)^2}$	$\frac{160(n-2)(n+7)}{n^2(n-1)^3}$	$\frac{5(n+7)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$	$\frac{40(n^3+20n^2-79n+70)}{n^2(n-1)^4}$	$\frac{10(n+3)(n+5)(n+7)}{n(n-1)^3}$	$\frac{40(n-2)(n+5)(n+7)}{n(n-1)^4}$	$\frac{(n+1)(n+3)(n+5)(n+7)}{(n-1)^4}$
$k_{62}k_2$		$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	$\frac{1}{n}$		$\frac{n+13}{n-1}$	$\frac{30}{n-1}$	$\frac{20}{n-1}$				
$k_{53}k_2$			$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	$\frac{1}{n}$		$\frac{n+15}{n-1}$		$\frac{20}{n-1}$			
$k_{44}k_2$				$\frac{2}{n}$	$\frac{-2}{n(n-1)}$			$\frac{n+15}{n-1}$	$\frac{12}{n-1}$			
$k_{422}k_2$						$\frac{1}{n}$	$\frac{-4}{n(n-1)}$	$\frac{2}{n}$	$\frac{-2}{n(n-1)}$	$\frac{n+11}{n-1}$	$\frac{6}{n-1}$	
$k_{332}k_2$							$\frac{2}{n}$	$\frac{-2}{n(n-1)}$	$\frac{n-5}{n(n-1)}$		$\frac{n+13}{n-1}$	
$k_{2222}k_2$										$\frac{4}{n}$	$\frac{-12}{n(n-1)}$	$\frac{n+7}{n-1}$
$k_{52}k_3$		$\frac{1}{n}$	$\frac{-3}{n(n-1)}$	$\frac{-3}{n(n-1)}$	$\frac{1}{n}$	$\frac{3(5n-12)}{(n-1)(n-2)}$	$\frac{n^2+48n-130}{(n-1)(n-2)}$	$\frac{30}{n-1}$	$\frac{-60}{(n-1)(n-2)}$	$\frac{60n}{(n-1)(n-2)}$	$\frac{90n}{(n-1)(n-2)}$	
$k_{43}k_3$			$\frac{1}{n}$	$\frac{n-4}{n(n-1)}$	$\frac{-3}{n(n-1)}$		$\frac{6(2n-7)}{(n-1)(n-2)}$	$\frac{3(3n-14)}{(n-1)(n-2)}$	$\frac{n^2+36n-94}{(n-1)(n-2)}$	$\frac{6n}{(n-1)(n-2)}$	$\frac{36n}{(n-1)(n-2)}$	
$k_{332}k_3$						$\frac{1}{n}$	$\frac{2(n-4)}{n(n-1)}$	$\frac{-6}{n(n-1)}$	$\frac{-6(n-4)}{n(n-1)(n-2)}$	$\frac{3(3n-10)}{(n-1)(n-2)}$	$\frac{n^2+18n-88}{(n-1)(n-2)}$	$\frac{6n}{(n-1)(n-2)}$
$k_{42}k_4$		$\frac{1}{n}$	$\frac{-4}{n(n-1)}$	$\frac{n-7}{n(n-1)}$	$\frac{-4}{n(n-1)}$	$\frac{4(4n-11)}{(n-1)(n-2)}$	$\frac{24(2n-7)}{(n-1)(n-2)}$	$\frac{n^2+39n-130}{(n-1)(n-2)}$	$\frac{6(n-27)}{(n-1)(n-2)}$	$\frac{24(n-4)(3n-1)}{(n-1)(n-2)(n-3)}$	$\frac{72(2n^2-9n-1)}{(n-1)(n-2)(n-3)}$	$\frac{24n(n+1)}{(n-1)(n-2)(n-3)}$
$k_{33}k_4$				$\frac{-8}{n(n-1)}$	$\frac{-6}{n(n-1)}$		$\frac{24(n-5)}{(n-1)(n-2)}$	$\frac{-72}{(n-1)(n-2)}$	$\frac{n^2+57n-190}{(n-1)(n-2)}$	$\frac{-48(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{72(n^2-6n+3)}{(n-1)(n-2)(n-3)}$	
$k_{222}k_4$						$\frac{3}{n}$	$\frac{-24}{n(n-1)}$	$\frac{-18}{n(n-1)}$	$\frac{72}{n(n-1)(n-2)}$	$\frac{n^2+21n-118}{(n-1)(n-2)}$	$\frac{18(n^2-13n+38)}{(n-1)(n-2)(n-3)}$	$\frac{-72}{(n-2)(n-3)}$
k_6k_{22}		$\frac{2}{n}$	$\frac{-12}{n(n-1)}$	$\frac{-30}{n(n-1)}$	$\frac{-20}{n(n-1)}$	$\frac{n^2+21n-106}{(n-1)(n-2)}$	$\frac{60(n-8)}{(n-1)(n-2)}$	$\frac{40(n-8)}{(n-1)(n-2)}$	$\frac{-420}{(n-1)(n-2)}$	$\frac{-480}{(n-2)(n-3)}$	$\frac{-1080}{(n-2)(n-3)}$	
$k_{42}k_{22}$				$\frac{2}{n(n-1)}$		$\frac{2}{n-2}$	$\frac{-12(n-4)}{n(n-1)(n-2)}$	$\frac{2(n^2-6n+16)}{n(n-1)(n-2)}$	$\frac{2(5n+16)}{n(n-1)(n-2)}$	$\frac{n^3+46n^2-197n+246}{n(n-1)(n-2)(n-3)}$	$\frac{4(3n^3-11n^2+42n-18)}{n(n-1)(n-2)(n-3)}$	$\frac{-24}{(n-2)(n-3)}$
$k_{33}k_{22}$					$\frac{2}{n(n-1)}$		$\frac{4(n+4)}{n(n-2)}$	$\frac{-4(n-8)}{n(n-1)(n-2)}$	$\frac{-12(n-4)}{n(n-1)(n-2)}$	$\frac{48}{n(n-2)(n-3)}$	$\frac{n^4+18n^3-61n^2-6n+216}{n(n-1)(n-2)(n-3)}$	
$k_{222}k_{22}$								$\frac{-6}{n(n-1)}$	$\frac{-12}{n(n-1)(n-2)}$	$\frac{6(n+2)}{n(n-2)}$	$\frac{-6(3n^2-13n+30)}{n(n-1)(n-2)(n-3)}$	$\frac{n^4+6n^3-25n^2-6n+72}{n(n-1)(n-2)(n-3)}$
k_5k_{32}		$\frac{1}{n}$	$\frac{n-6}{n(n-1)}$	$\frac{-15}{n(n-1)}$	$\frac{-10}{n(n-1)}$	$\frac{5(3n-10)}{(n-1)(n-2)}$	$\frac{n^2+5n-288}{(n-1)(n-2)}$	$\frac{10(3n-19)}{(n-1)(n-2)}$	$\frac{10(2n-31)}{(n-1)(n-2)}$	$\frac{60(n^2-7n+2)}{(n-1)(n-2)(n-3)}$	$\frac{90(n^2-11n+4)}{(n-1)(n-2)(n-3)}$	$\frac{-120n}{(n-2)(n-3)(n-4)}$
$k_{42}k_2^2$		$\frac{1}{n^2}$	$\frac{-4}{n^2(n-1)}$	$\frac{3n^2-6n+7}{n^2(n-1)^2}$	$\frac{-4}{n^2(n-1)}$	$\frac{2(n+11)}{n(n-1)}$	$\frac{4(9n-22)}{n(n-1)^2}$	$\frac{2(n^2+17n-40)}{n(n-1)^2}$	$\frac{-2(n^2+41)}{n(n-1)^2}$	$\frac{(n+9)(n+11)}{(n-1)^2}$	$\frac{6(n+11)}{(n-1)^2}$	
$k_{33}k_2^2$			$\frac{2}{n^2}$	$\frac{-8}{n^2(n-1)}$	$\frac{2(n^2-2n+3)}{n^2(n-1)^2}$		$\frac{4(n+13)}{n(n-1)}$	$\frac{-4(n+13)}{n(n-1)^2}$	$\frac{n^2+46n-119}{n(n-1)^2}$		$\frac{(n+11)(n+13)}{(n-1)^2}$	
$k_{222}k_2^2$						$\frac{3}{n^2}$	$\frac{-24}{n^2(n-1)}$	$\frac{6(n^2-2n+3)}{n^2(n-1)^2}$	$\frac{-12(n-3)}{n^2(n-1)^2}$	$\frac{7n+53}{n(n-1)}$	$\frac{-156}{n(n-1)^2}$	$\frac{(n+5)(n+7)}{(n-1)^2}$
$k_{32}k_3k_2$		$\frac{1}{n^2}$	$\frac{n-6}{n^2(n-1)}$	$\frac{n^2-10n+15}{n^2(n-1)^2}$	$\frac{n^2-5n+10}{n^2(n-1)^2}$	$\frac{n^2+20n-50}{n(n-1)(n-2)}$	$\frac{3n^3+61n^2-324n+368}{n(n-1)^2(n-2)}$	$\frac{3(11n^2-63n+80)}{n(n-1)^2(n-2)}$	$\frac{n^3+27n^2-298n+510}{n(n-1)^2(n-2)}$	$\frac{9(n^2+11n-32)}{(n-1)^2(n-2)}$	$\frac{n^3+25n^2+164n-814}{(n-1)^2(n-2)}$	$\frac{6n(n+7)}{(n-1)^2(n-2)}$
$k_{22}k_4k_2$		$\frac{2}{n^2}$	$\frac{-12}{n^2(n-1)}$	$\frac{2(n-3)(n-5)}{n^2(n-1)^2}$	$\frac{-4(2n-5)}{n^2(n-1)^2}$	$\frac{3n^2+35n-106}{n(n-1)(n-2)}$	$\frac{72(n-5)}{n(n-1)^2}$	$\frac{2(n^3+29n^2-178n+256)}{n(n-1)^2(n-2)}$	$\frac{2(5n^2-195n+466)}{n(n-1)^2(n-2)}$	$\frac{n^5+21n^4+17n^3-957n^2+1974n-96}{n(n-1)^2(n-2)(n-3)}$	$\frac{18(n^4+6n^3-97n^2+226n-16)}{n(n-1)^2(n-2)(n-3)}$	$\frac{-24(n+7)}{(n-1)(n-2)(n-3)}$
$k_{22}^2k_2$				$\frac{4}{n^2(n-1)}$	$\frac{-4}{n^2(n-1)^2}$	$\frac{4}{n(n-2)}$	$\frac{-32}{n^2(n-1)}$	$\frac{2(5n^3+5n^2-18n-32)}{n^2(n-1)^2(n-2)}$	$\frac{8(n^2+3n-20)}{n^2(n-1)^2(n-2)}$	$\frac{8(n^4+5n^3-27n^2+9n-12)}{n^2(n-1)(n-2)(n-3)}$	$\frac{4(n^4-42n^3+239n^2-366n+72)}{n^2(n-1)^2(n-2)(n-3)}$	$\frac{(n+7)(n^4+2n^3-21n^2+18n-24)}{n(n-1)^2(n-2)(n-3)}$
$k_{22}k_3^2$		$\frac{2}{n^2}$	$\frac{-12}{n^2(n-1)}$	$\frac{-6(2n-5)}{n^2(n-1)^2}$	$\frac{2(n^2-2n+10)}{n^2(n-1)^2}$	$\frac{n^2+21n-38}{n(n-1)(n-2)}$	$\frac{4(n^3+17n^2-72n+60)}{n(n-1)^2(n-2)}$	$\frac{6(9n^2-30n+28)}{n(n-1)^2(n-2)}$	$\frac{-6(2n^3+41n^2-156n+92)}{n(n-1)^2(n-2)^2}$	$\frac{129n^3-507n^2+888n-1020}{n(n-1)^2(n-2)^2}$	$\frac{n^4+27n^3+108n^2-180n-2992}{(n-1)^2(n-2)^2}$	$\frac{6n(n^2-9n+34)}{(n-1)^2(n-2)^2}$
$k_{22}k_2^3$		$\frac{2}{n^3}$	$\frac{-12}{n^3(n-1)}$	$\frac{6(n^2-2n+5)}{n^3(n-1)^2}$	$\frac{-4(3n^2-6n+5)}{n^3(n-1)^3}$	$\frac{7n+43}{n^2(n-1)}$	$\frac{8(5n-49)}{n^2(n-1)^2}$	$\frac{4(3n^3+25n^2-26n+72)}{n^2(n-1)^3}$	$\frac{6(3n^2-54n+91)}{n^2(n-1)^3}$	$\frac{3(3n^2+38n+119)}{n(n-1)^2}$	$\frac{2(11n^2+18n-509)}{n(n-1)^3}$	$\frac{(n+3)(n+5)(n+7)}{(n-1)^3}$

TABLE 7.3 – WEIGHT 12

	k_6^2	k_4^3	k_3^4	k_2^6
k_{12}	$\frac{1}{n}$	$\frac{1}{n^2}$	$\frac{1}{n^3}$	$\frac{1}{n^5}$
$k_{10,2}$	$\frac{36}{n-1}$	$\frac{48}{n(n-1)}$	$\frac{54}{n^2(n-1)}$	$\frac{6(n+9)}{n^4(n-1)}$
k_{93}	$\frac{180}{n-1}$	$\frac{16(13n-17)}{n(n-1)^2}$	$\frac{4(n^2+52n-80)}{n^2(n-1)^2}$	$\frac{160(n-2)}{n^4(n-1)^2}$
k_{84}	$\frac{465}{n-1}$	$\frac{3(3n^2+158n-257)}{n(n-1)^2}$	$\frac{27(17n^2-49n+35)}{n^2(n-1)^3}$	$\frac{15(n^3+29n^2-77n+63)}{n^4(n-1)^3}$
k_{75}	$\frac{780}{n-1}$	$\frac{48(16n-29)}{n(n-1)^2}$	$\frac{108(7n^2-20n+16)}{n^2(n-1)^3}$	$\frac{96(n-2)(7n^2-14n+9)}{n^4(n-1)^4}$
k_{66}	$\frac{n+460}{n-1}$	$\frac{12(37n-70)}{n(n-1)^2}$	$\frac{3(n^3+150n^2-420n+350)}{n^2(n-1)^3}$	$\frac{2(5n^5+201n^4-990n^3+1850n^2-1575n+525)}{n^4(n-1)^5}$
k_{822}	$\frac{450n}{(n-1)(n-2)}$	$\frac{72(11n-19)}{(n-1)^2(n-2)}$	$\frac{27(37n-70)}{n(n-1)^2(n-2)}$	$\frac{15(n+7)(n+9)}{n^3(n-1)^2}$
k_{732}	$\frac{3600n}{(n-1)(n-2)}$	$\frac{288(19n-41)}{(n-1)^2(n-2)}$	$\frac{108(n^3+53n^2-196n+160)}{n(n-1)^3(n-2)}$	$\frac{480(n-2)(n+9)}{n^3(n-1)^3}$
k_{642}	$\frac{7200n}{(n-1)(n-2)}$	$\frac{48(n^2+200n-521)}{(n-1)^2(n-2)}$	$\frac{54(n^3+191n^2-732n+700)}{n(n-1)^3(n-2)}$	$\frac{60(n+9)(n^3+17n^2-45n+35)}{n^3(n-1)^4}$
k_{633}	$\frac{6300n}{(n-1)(n-2)}$	$\frac{144(56n^2-257n+302)}{(n-1)^2(n-2)^2}$	$\frac{6(n^5+56n^4+1099n^3-7774n^2+16360n-11200)}{n(n-1)^3(n-2)^2}$	$\frac{80(n-2)(n^2+60n-105)}{n^3(n-1)^4}$
k_{552}	$\frac{4500n}{(n-1)(n-2)}$	$\frac{1440(4n-11)}{(n-1)^2(n-2)}$	$\frac{108(59n^2-220n+224)}{n(n-1)^3(n-2)}$	$\frac{96(n-2)(n+9)(6n^2-12n+7)}{n^3(n-1)^5}$
k_{543}	$\frac{21600n}{(n-1)(n-2)}$	$\frac{144(n^3+171n^2-840n+1060)}{(n-1)^2(n-2)^2}$	$\frac{216(2n^4+99n^3-676n^2+1488n-1120)}{n(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n^3+33n^2-89n+63)}{n^3(n-1)^5}$
k_{444}	$\frac{4950n}{(n-1)(n-2)}$	$\frac{n^4+96n^3+5175n^2-26636n+35244}{(n-1)^2(n-2)^2}$	$\frac{27(173n^4-1503n^3+4962n^2-7380n+4200)}{n(n-1)^3(n-2)^3}$	$\frac{15(n^5+27n^4+226n^3-1098n^2+1725n-945)}{n^3(n-1)^5}$
k_{6222}	$\frac{2400n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{288(19n^3-98n^2+125n+2)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{36(n^3+209n^2-784n+700)}{(n-1)^3(n-2)^2}$	$\frac{20(n+5)(n+7)(n+9)}{n^2(n-1)^3}$
k_{5322}	$\frac{21600n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{1728(24n^3-140n^2+200n+4)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{432(2n^3+109n^2-500n+560)}{(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n+7)(n+9)}{n^2(n-1)^4}$
k_{4422}	$\frac{15300n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{216(n^4+116n^3-731n^2+1098n+24)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{243(n^3+121n^2-572n+700)}{(n-1)^3(n-2)^2}$	$\frac{15(n+7)(n+9)(3n^3+23n^2-63n+45)}{n^2(n-1)^5}$
k_{4332}	$\frac{54000n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{432(n^4+200n^3-1247n^2+1890n+48)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{54(n^5+61n^4+1300n^3-11060n^2+27472n-22400)}{(n-1)^3(n-2)^3}$	$\frac{240(n+9)(n^3+20n^2-79n+70)}{n^2(n-1)^5}$
k_{3333}	$\frac{8100n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{288(41n^4-384n^3+1209n^2-1282n-36)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{n^6+45n^5+952n^4+6336n^3-73968n^2+203328n-179200}{(n-1)^3(n-2)^3}$	$\frac{160(n-2)(n^2+27n-70)}{n^2(n-1)^5}$
k_{42222}	$\frac{5400n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$	$\frac{72n(n^3+210n^2-717n-206)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{324n(n^3+64n^2-301n+350)}{(n-1)^3(n-2)^3}$	$\frac{15(n+3)(n+5)(n+7)(n+9)}{n(n-1)^4}$
k_{33222}	$\frac{21600n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$	$\frac{1728n(29n^3-196n^2+317n+62)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{36n(n^4+63n^3+1498n^2-8568n+11200)}{(n-1)^3(n-2)^3}$	$\frac{80(n-2)(n+5)(n+7)(n+9)}{n(n-1)^5}$
k_{222222}	$\frac{720n(n+1)(n^2+15n-4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}$	$\frac{1728n(n+1)(n^2-5n+2)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{108n^2(n^2+27n-70)}{(n-1)^3(n-2)^3}$	$\frac{(n+1)(n+3)(n+5)(n+7)(n+9)}{(n-1)^5}$

CHAPTER V

MOMENT FORMULAE AND THEIR ESTIMATORS

The advantage of formulae expressing products of generalized k-statistics as linear functions of the same shows up in problems involving expectations and estimation. We use $M(rs\ldots) = E_N\ (k_r-K_r)(k_s-K_s)\ldots$ to denote the moment $M(k_rk_s\ldots)$ and $K(rs\ldots)$ for the corresponding K-parameter* of a finite population of size N.

We first study these finite moment formulae where none of r, s, ... is 1. Then we find expressions for finite moment formulae involving 1's (i.e. the sample mean k_1). We also give expressions for the estimators $\hat{M}$, $\hat{K}$ of these moment functions. The fact that these formulae approach the infinite formulae of Fisher (1928) as $N\to\infty$ is used in checking.

One device used is that of substitution products introduced by Schaeffer and Dwyer (1963). $[K_rK_s\ldots]_n$ denotes the expansion of $K_rK_s\ldots$ with N replaced by n and similarly $[k_rk_s\ldots]_N$ denotes the expansion of $k_rk_s\ldots$ with n replaced by N, the expansions being feasible by Table 7. Obviously, $[K_rK_s\ldots]_N = K_rK_s\ldots$ and $[k_rk_s\ldots]_n = k_rk_s\ldots$. In estimation, complex substitution products like $[[k_r^2]_nk_s]_N$ appear.

Finite Moment Formulae

Although we need the linear expansions of products of k...'s for writing moment formulae, the best computational form need not necessarily be linear in K... . Schaeffer and Dwyer (1963) in fact point out that "the completed expansion in terms of K... does not yield a formula which is most desirable for computation since it demands the use of the expanded form for some products when the actual values of the factors of the

* $K(\ldots) = M(\ldots)$ up to triple products. For more factors, $K(\ldots)$ are given in terms of $M(\ldots)$ on page 77for selected values through weight 12.

products are known." For a simple illustration, we consider the variance $M(2^2)$ of k_2.

$$M(2^2) = E_N(k_2 - K_2)^2$$

$$= E_N(k_2^2) - K_2^2$$

$$= E_N\left(\frac{1}{n} k_4 + \frac{n+1}{n-1} k_{22}\right) - \left(\frac{1}{N} K_4 + \frac{N+1}{N-1} K_{22}\right)$$

$$= \left(\frac{1}{n} - \frac{1}{N}\right) K_4 + \left(\frac{n+1}{n-1} - \frac{N+1}{N-1}\right) K_{22}, \text{ using } E_N(k\ldots) = K\ldots$$

$$(5.1) \qquad = \left(\frac{1}{n} - \frac{1}{N}\right) K_4 + \left(\frac{2}{n-1} - \frac{2}{N-1}\right) K_{22}.$$

The expression (5.1) was given by Tukey (1950) and is in good form for estimation, being linear in $K\ldots$, but the variance of k_2 can be better computed as

$$(5.2) \qquad M(2^2) = \left[K_2^2\right]_n - K_2^2 = \frac{1}{n} K_4 + \frac{n+1}{n-1} K_{22} - K_2^2 ,$$

since K_2^2 is better calculated as $K_2 \cdot K_2$ than as $\frac{1}{N} K_4 + \frac{N+1}{N-1} K_{22}$.

We give similar formulae for $M(rs\ldots)$ for weights through 10 and for special cases of weight 12 in Table 8, generalizing Fisher's (1928) table of formulae for the infinite case. The second column headed "Est." for Estimator and the row starting with 0 can be ignored for the present. Use is made of these while dealing with estimators later.* The terms are grouped by the number of factors in the K-products. These formulae can be written directly, like (5.2)(Table 8.1), when the expansions for

* To obtain the expansion for a particular $M(\ldots)$, we multiply each coefficient in that column by the corresponding entry in the "Exp." (expectation) column and formthe sum.

products of k-statistics (Table 7) are known. For a strict generalization of Fisher's (1928) table, we should write the finite K(...) formulae rather than the M(...) formulae. These are, however, equivalent up to triple products. For products having more than three factors, we use

$$K(2^4) = M(2^4) - 3\,M^2(2^2),$$
$$K(32^3) = M(32^3) - 3\,M(2^2)M(32),$$
$$K(3^2 2^2) = M(3^2 2^2) - M(3^2)M(2^2) - 2M^2(32),$$
$$K(42^3) = M(42^3) - 3M(2^2)M(42),$$
$$K(3^4) = M(3^4) - 3M^2(3^2),$$
$$K(2^5) = M(2^5) - 10M(2^3)M(2^2), \text{ and}$$
$$K(2^6) = M(2^6) - 15M(2^4)M(2^2) - 10M^2(2^3) + 30M^3(2^2).$$

Columns at the end of Tables 8.5 to 8.8 give expansions for K(...).

It is useful to note some general results for the finite moment formulae. Schaeffer and Dwyer (1963) give the variance-covariance formulae as

$$(5.3)\qquad M(r^2) = [K_r^2]_n - K_r^2$$

$$(5.4)\qquad M(rs) = [K_r K_s]_n - K_r K_s$$

and also give expressions for $M(r^3)$, $M(r^4)$, $M(r^2 s)$, $M(r^2 s^2)$ in terms of substitution products. In direct generalization of (5.3), we can write

$$M(r^a) = E_N(k_r - K_r)^a$$
$$= E_N\left[\sum_{s=0}^{a} (-1)^s \binom{a}{s} k_r^{a-s} K_r^s\right]$$
$$(5.5)\qquad = \sum_{s=0}^{a-2} (-1)^s \binom{a}{s} \left[K_r^{a-s}\right]_n K_r^s + (-1)^{a-1}(a-1)\, K_r^a \;.$$

This result can be used to write $M(2^2)$, $M(2^3)$, ... , $M(2^6)$, $M(3^3)$, $M(3^4)$, $M(4^3)$ and $M(6^2)$.

TABLE 8

Finite Moment Formulae

TABLE 8.1 --- WEIGHT 4

Exp.	Est.	$M(2^2)$
K_4	0	$\frac{1}{n}$
K_2	0	$\frac{n+1}{n-1}$
0	1	k
K_2^2	$[k_2^2]$	-1

TABLE 8.2 --- WEIGHT 5

Exp.	Est.	M(32)
K_5	0	$\frac{1}{n}$
K_{32}	0	$\frac{n+5}{n-1}$
0	1	k_3k_2
K_3K_2	$[k_3k_2]$	-1

TABLE 8.3 --- WEIGHT 6

Exp.	Est.	M(42)	M(33)	$M(2^3)$
K_6	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^2}$
K_{42}	0	$\frac{n+7}{n-1}$	$\frac{9}{n-1}$	$\frac{3(n+3)}{n(n-1)}$
K_{33}	0	$\frac{6}{n-1}$	$\frac{n+8}{n-1}$	$\frac{4(n-2)}{n(n-1)^2}$
K_{222}	0		$\frac{6n}{(n-1)(n-2)}$	$\frac{(n+1)(n+3)}{(n-1)^2}$
0	1	k_4k_2	k_3^2	k_2^3
K_4K_2	$[k_4k_2]$	-1		$-\frac{3}{n}$
K_3^2	$[k_3^2]$		-1	
$K_{22}K_2$	$[k_{22}k_2]$			$\frac{-3(n+1)}{n-1}$
K_2^3	$[k_2^3]$			2

TABLE 8.4 --- WEIGHT 7

Exp.	Est.	M(52)	M(43)	M(322)
K_7	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
K_{52}	0	$\frac{n+9}{n-1}$	$\frac{12}{n-1}$	$\frac{2(n+7)}{n(n-1)}$
K_{43}	0	$\frac{20}{n-1}$	$\frac{n+29}{n-1}$	$\frac{n^2+22n-35}{n(n-1)^2}$
K_{322}	0		$\frac{36n}{(n-1)(n-2)}$	$\frac{(n+5)(n+7)}{(n-1)^2}$
0	1	k_5k_2	k_4k_3	$k_3k_2^2$
K_5K_2	$[k_5k_2]$	-1		$-\frac{2}{n}$
K_4K_3	$[k_4k_3]$		-1	$-\frac{1}{n}$
$K_{32}K_2$	$[k_{32}k_2]$			$\frac{-2(n+5)}{n-1}$
$K_{22}K_3$	$[k_{22}k_3]$			$-\frac{n+1}{n-1}$
$K_3K_2^2$	$[k_3k_2^2]$			2

TABLE 8·5 — WEIGHT 8

Exp.	Est.	M(62)	M(53)	M(44)	M(422)	M(332)	M(2^4)	K(2^4)
K_8	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^3}$	$\frac{1}{n^3}$
K_{62}	0	$\frac{n+11}{n-1}$	$\frac{15}{n-1}$	$\frac{16}{n-1}$	$\frac{2(n+9)}{n(n-1)}$	$\frac{n+20}{n(n-1)}$	$\frac{4(n+5)}{n^2(n-1)}$	$\frac{4(n+5)}{n^2(n-1)}$
K_{53}	0	$\frac{30}{n-1}$	$\frac{n+44}{n-1}$	$\frac{48}{n-1}$	$\frac{8(5n-7)}{n(n-1)^2}$	$\frac{2(n^2+22n-32)}{n(n-1)^2}$	$\frac{32(n-2)}{n^2(n-1)^2}$	$\frac{32(n-2)}{n^2(n-1)^2}$
K_{44}	0	$\frac{20}{n-1}$	$\frac{30}{n-1}$	$\frac{n+33}{n-1}$	$\frac{n^2+26n-39}{n(n-1)^2}$	$\frac{9(3n-5)}{n(n-1)^2}$	$\frac{3n^3+23n^2-63n+45}{n^2(n-1)^3}$	$\frac{3n^3+23n^2-63n+45}{n^2(n-1)^3}$
K_{422}	0		$\frac{60n}{(n-1)(n-2)}$	$\frac{72n}{(n-1)(n-2)}$	$\frac{(n+7)(n+9)}{(n-1)^2}$	$\frac{9(n^2+9n-20)}{(n-1)^2(n-2)}$	$\frac{6(n+3)(n+5)}{n(n-1)^2}$	$\frac{6(n+3)(n+5)}{n(n-1)^2}$
K_{332}	0		$\frac{90n}{(n-1)(n-2)}$	$\frac{144n}{(n-1)(n-2)}$	$\frac{12(n+9)}{(n-1)^2}$	$\frac{n^3+17n^2+104n-320}{(n-1)^2(n-2)}$	$\frac{16(n-2)(n+5)}{n(n-1)^3}$	$\frac{16(n-2)(n+5)}{n(n-1)^3}$
K_{2222}	0			$\frac{24n(n+1)}{(n-1)(n-2)(n-3)}$		$\frac{6n(n+5)}{(n-1)^2(n-2)}$	$\frac{(n+1)(n+3)(n+5)}{(n-1)^3}$	$\frac{(n+1)(n+3)(n+5)}{(n-1)^3}$
0	1	k_6k_2	k_5k_3	k_4^2	$k_4k_2^2$	$k_3^2k_2$	k_2^4	k_2^4
K_6K_2	$[k_6k_2]$	-1			$-\frac{2}{n}$	$-\frac{1}{n}$	$-\frac{4}{n^2}$	$-\frac{4}{n^2}$
K_5K_3	$[k_5k_3]$		-1			$-\frac{2}{n}$		
K_4^2	$[k_4^2]$			-1	$-\frac{1}{n}$			$-\frac{3}{n^2}$
$K_{42}K_2$	$[k_{42}k_2]$				$-\frac{2(n+7)}{n-1}$	$-\frac{9}{n-1}$	$-\frac{12(n+3)}{n(n-1)}$	$-\frac{12(n+3)}{n(n-1)}$
$K_{22}K_4$	$[k_{22}k_4]$				$-\frac{n+1}{n-1}$			$-\frac{6(n+1)}{n-1}$
$K_{33}K_2$	$[k_{33}k_2]$				$-\frac{12}{n-1}$	$-\frac{n+8}{n-1}$	$-\frac{16(n-2)}{n(n-1)^2}$	$-\frac{16(n-2)}{n(n-1)^2}$
$K_{32}K_3$	$[k_{32}k_3]$				$-\frac{2(n+5)}{n-1}$			
$K_{222}K_2$	$[k_{222}k_2]$					$-\frac{6n}{(n-1)(n-2)}$	$-\frac{4(n+1)(n+3)}{(n-1)^2}$	$-\frac{4(n+1)(n+3)}{(n-1)^2}$
K_{22}^2	$[k_{22}^2]$							$-\frac{3(n+1)^2}{(n-1)^2}$
$K_4K_2^2$	$[k_4k_2^2]$				2		$\frac{6}{n}$	$\frac{12}{n}$
$K_3^2K_2$	$[k_3^2k_2]$					2		
$K_{22}K_2^2$	$[k_{22}k_2^2]$						$\frac{6(n+1)}{n-1}$	$\frac{12(n+1)}{n-1}$
K_2^4	$[k_2^4]$						-3	-6

TABLE 8·6 — WEIGHT 9

Exp.	Est.	M(72)	M(63)	M(54)	$M(52^2)$	M(432)	$M(3^3)$	$M(32^3)$	$K(32^3)$
K_9	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^3}$	$\frac{1}{n^3}$
K_{72}	0	$\frac{n+13}{n-1}$	$\frac{18}{n-1}$	$\frac{20}{n-1}$	$\frac{2(n+11)}{n(n-1)}$	$\frac{n+25}{n(n-1)}$	$\frac{27}{n(n-1)}$	$\frac{3(n+9)}{n^2(n-1)}$	$\frac{3(n+9)}{n^2(n-1)}$
K_{63}	0	$\frac{42}{n-1}$	$\frac{n+62}{n-1}$	$\frac{70}{n-1}$	$\frac{20(3n-4)}{n(n-1)^2}$	$\frac{n^2+70n-95}{n(n-1)^2}$	$\frac{3(n^2+25n-35)}{n(n-1)^2}$	$\frac{n^2+60n-105}{n^2(n-1)^2}$	$\frac{n^2+60n-105}{n^2(n-1)^2}$
K_{54}	0	$\frac{70}{n-1}$	$\frac{105}{n-1}$	$\frac{n+119}{n-1}$	$\frac{n^2+98n-139}{n(n-1)^2}$	$\frac{n^2+108n-169}{n(n-1)^2}$	$\frac{27(4n-7)}{n(n-1)^2}$	$\frac{3(n^3+33n^2-89n+63)}{n^2(n-1)^3}$	$\frac{3(n^3+33n^2-89n+63)}{n^2(n-1)^3}$
K_{522}	0		$\frac{90n}{(n-1)(n-2)}$	$\frac{120n}{(n-1)(n-2)}$	$\frac{(n+9)(n+11)}{(n-1)^2}$	$\frac{12(n^2+12n-25)}{(n-1)^2(n-2)}$	$\frac{54(4n-7)}{(n-1)^2(n-2)}$	$\frac{3(n+7)(n+9)}{n(n-1)^2}$	$\frac{3(n+7)(n+9)}{n(n-1)^2}$
K_{432}	0		$\frac{360n}{(n-1)(n-2)}$	$\frac{600n}{(n-1)(n-2)}$	$\frac{40(n+11)}{(n-1)^2}$	$\frac{n^3+40n^2+605n-1450}{(n-1)^2(n-2)}$	$\frac{27(n^2+27n-70)}{(n-1)^2(n-2)}$	$\frac{6(17n^2+80n-157)}{n(n-1)^3}$	$\frac{6(17n^2+80n-157)}{n(n-1)^3}$
K_{333}	0		$\frac{90n}{(n-1)(n-2)}$	$\frac{180n}{(n-1)(n-2)}$	$\frac{120}{(n-1)^2}$	$\frac{6(n^2+27n-70)}{(n-1)^2(n-2)}$	$\frac{n^4+21n^3+130n^2-876n+1120}{(n-1)^2(n-2)^2}$	$\frac{4(n^2+27n-70)}{n(n-1)^3}$	$\frac{4(n^2+27n-70)}{n(n-1)^3}$
K_{3222}	0			$\frac{240n(n+1)}{(n-1)(n-2)(n-3)}$		$\frac{36n(n+9)}{(n-1)^2(n-2)}$	$\frac{18n(n^2+27n-70)}{(n-1)^2(n-2)^2}$	$\frac{n^3+15n^2+155n+309}{(n-1)^3}$	$\frac{n^3+15n^2+155n+309}{(n-1)^3}$
0	1	k_7k_2	k_6k_3	k_5k_4	$k_5k_2^2$	$k_4k_3k_2$	k_3^3	$k_3k_2^3$	$k_3k_2^3$
K_7K_2	$[k_7k_2]$	-1			$-\frac{2}{n}$	$-\frac{1}{n}$		$-\frac{3}{n^2}$	$-\frac{3}{n^2}$
K_6K_3	$[k_6k_3]$		-1			$-\frac{1}{n}$	$-\frac{3}{n}$	$-\frac{1}{n^2}$	$-\frac{1}{n^2}$
K_5K_4	$[k_5k_4]$			-1	$-\frac{1}{n}$	$-\frac{1}{n}$			$-\frac{3}{n^2}$
$K_{52}K_2$	$[k_{52}k_2]$				$-\frac{2(n+9)}{n-1}$	$-\frac{12}{n-1}$		$-\frac{6(n+7)}{n(n-1)}$	$-\frac{6(n+7)}{n(n-1)}$
$K_{22}K_5$	$[k_{22}k_5]$				$-\frac{n+1}{n-1}$				$-\frac{3(n+1)}{n-1}$
$K_{43}K_2$	$[k_{43}k_2]$				$-\frac{40}{n-1}$	$-\frac{n+29}{n-1}$		$-\frac{3(n^2+22n-35)}{n(n-1)^2}$	$-\frac{3(n^2+22n-35)}{n(n-1)^2}$
$K_{42}K_3$	$[k_{42}k_3]$					$-\frac{n+7}{n-1}$	$-\frac{27}{n-1}$	$-\frac{3(n+3)}{n(n-1)}$	$-\frac{3(n+3)}{n(n-1)}$
$K_{32}K_4$	$[k_{32}k_4]$					$-\frac{n+5}{n-1}$			$-\frac{3(n+5)}{n(n-1)}$
$K_{33}K_3$	$[k_{33}k_3]$					$-\frac{6}{n-1}$	$-\frac{3(n+8)}{n-1}$	$-\frac{4(n-2)}{n(n-1)^2}$	$-\frac{4(n-2)}{n(n-1)^2}$
$K_{322}K_2$	$[k_{322}k_2]$					$-\frac{36n}{(n-1)(n-2)}$		$-\frac{3(n+5)(n+7)}{(n-1)^2}$	$-\frac{3(n+5)(n+7)}{(n-1)^2}$
$K_{32}K_{22}$	$[k_{32}k_{22}]$								$-\frac{3(n+1)(n+5)}{(n-1)^2}$
K_3K_{222}	$[k_3k_{222}]$						$-\frac{18n}{(n-1)(n-2)}$	$-\frac{(n+1)(n+3)}{(n-1)^2}$	$-\frac{(n+1)(n+3)}{(n-1)^2}$
$K_5K_2^2$	$[k_5k_2^2]$				2			$\frac{3}{n}$	$\frac{6}{n}$
$K_4K_3K_2$	$[k_4k_3k_2]$					2		$\frac{3}{n}$	$\frac{6}{n}$
K_3^3	$[k_3^3]$						2		
$K_{32}K_2^2$	$[k_{32}k_2^2]$							$\frac{3(n+5)}{n-1}$	$\frac{6(n+5)}{n-1}$
$K_{22}K_3K_2$	$[k_{22}k_3k_2]$							$\frac{3(n+1)}{n-1}$	$\frac{6(n+1)}{n-1}$
$K_3K_2^3$	$[k_3k_2^3]$							-3	-6

TABLE 8·7 — WEIGHT 10

Exp.	Est.	M(82)	M(73)	M(64)	M(55)	$M(62^2)$
K_{10}	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^2}$
K_{82}	0	$\frac{n+15}{n-1}$	$\frac{21}{n-1}$	$\frac{24}{n-1}$	$\frac{25}{n-1}$	$\frac{2(n+13)}{n(n-1)}$
K_{73}	0	$\frac{56}{n-1}$	$\frac{n+83}{n-1}$	$\frac{96}{n-1}$	$\frac{100}{n-1}$	$\frac{12(7n-9)}{n(n-1)^2}$
K_{64}	0	$\frac{112}{n-1}$	$\frac{168}{n-1}$	$\frac{n+193}{n-1}$	$\frac{200}{n-1}$	$\frac{n^2+162n-223}{n(n-1)^2}$
K_{55}	0	$\frac{70}{n-1}$	$\frac{105}{n-1}$	$\frac{120}{n-1}$	$\frac{n+124}{n-1}$	$\frac{20(5n-7)}{n(n-1)^2}$
K_{622}	0		$\frac{126n}{(n-1)(n-2)}$	$\frac{180n}{(n-1)(n-2)}$	$\frac{200n}{(n-1)(n-2)}$	$\frac{n^2+24n+143}{(n-1)^2}$
K_{532}	0		$\frac{630n}{(n-1)(n-2)}$	$\frac{1080n}{(n-1)(n-2)}$	$\frac{1200n}{(n-1)(n-2)}$	$\frac{60(n+13)}{(n-1)^2}$
K_{442}	0		$\frac{420n}{(n-1)(n-2)}$	$\frac{720n}{(n-1)(n-2)}$	$\frac{850n}{(n-1)(n-2)}$	$\frac{40(n+13)}{(n-1)^2}$
K_{433}	0		$\frac{630n}{(n-1)(n-2)}$	$\frac{1260n}{(n-1)(n-2)}$	$\frac{1500n}{(n-1)(n-2)}$	$\frac{840}{(n-1)^2}$
K_{4222}	0			$\frac{480n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{600n(n+1)}{(n-1)(n-2)(n-3)}$	
K_{3322}	0			$\frac{1080n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{1800n(n+1)}{(n-1)(n-2)(n-3)}$	
K_{22222}	0				$\frac{120n^2(n+5)}{(n-1)^{(4)}}$	
0	1	k_8k_2	k_7k_3	k_6k_4	k_5^2	$k_6k_2^2$
K_8K_2	$[k_8k_2]$	-1				$-\frac{2}{n}$
K_7K_3	$[k_7k_3]$		-1			
K_6K_4	$[k_6k_4]$			-1		$-\frac{1}{n}$
K_5^2	$[k_5^2]$				-1	
$K_{62}K_2$	$[k_{62}k_2]$					$-\frac{2(n+11)}{n-1}$
K_6K_{22}	$[k_6k_{22}]$					$-\frac{n+1}{n-1}$
$K_{53}K_2$	$[k_{53}k_2]$					$-\frac{60}{n-1}$
$K_{52}K_3$	$[k_{52}k_3]$					
K_5K_{32}	$[k_5k_{32}]$					
$K_{44}K_2$	$[k_{44}k_2]$					$-\frac{40}{n-1}$
$K_{42}K_4$	$[k_{42}k_4]$					
$K_{43}K_3$	$[k_{43}k_3]$					
$K_{33}K_4$	$[k_{33}k_4]$					
$K_{422}K_2$	$[k_{422}k_2]$					
$K_{42}K_{22}$	$[k_{42}k_{22}]$					
$K_{222}K_4$	$[k_{222}k_4]$					
$K_{332}K_2$	$[k_{332}k_2]$					
$K_{33}K_{22}$	$[k_{33}k_{22}]$					
K_{32}^2	$[k_{32}^2]$					
$K_{322}K_3$	$[k_{322}k_3]$					
$K_{2222}K_2$	$[k_{2222}k_2]$					
$K_{222}K_{22}$	$[k_{222}k_{22}]$					

Exp.	M(532)	$M(4^22)$	$M(43^2)$	$M(3^22^2)$
K_{10}	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^3}$
K_{82}	$\frac{n+30}{n(n-1)}$	$\frac{n+31}{n(n-1)}$	$\frac{33}{n(n-1)}$	$\frac{2n+35}{n^2(n-1)}$
K_{73}	$\frac{n^2+99n-130}{n(n-1)^2}$	$\frac{8(13n-17)}{n(n-1)^2}$	$\frac{2(n^2+55n-74)}{n(n-1)^2}$	$\frac{2(n^2+49n-80)}{n^2(n-1)^2}$
K_{64}	$\frac{5(37n-55)}{n(n-1)^2}$	$\frac{2(n^2+96n-145)}{n(n-1)^2}$	$\frac{n^2+193n-320}{n(n-1)^2}$	$\frac{n^3+180n^2-495n+350}{n^2(n-1)^3}$
K_{55}	$\frac{n^2+113n-174}{n(n-1)^2}$	$\frac{4(29n-46)}{n(n-1)^2}$	$\frac{6(19n-34)}{n(n-1)^2}$	$\frac{2(n^3+56n^2-151n+112)}{n^2(n-1)^3}$
K_{622}	$\frac{15(n^2+15n-30)}{(n-1)^2(n-2)}$	$\frac{8(2n^2+31n-61)}{(n-1)^2(n-2)}$	$\frac{18(19n-33)}{(n-1)^2(n-2)}$	$\frac{n^3+40n^2+275n-700}{n(n-1)^2(n-2)}$
K_{532}	$\frac{n^3+57n^2+1172n-2640}{(n-1)^2(n-2)}$	$\frac{48(n+31)}{(n-1)^2}$	$\frac{24(n^2+66n-154)}{(n-1)^2(n-2)}$	$\frac{4(n^4+35n^3+274n^2-1266n+1120)}{n(n-1)^3(n-2)}$
K_{442}	$\frac{30(n^2+27n-60)}{(n-1)^2(n-2)}$	$\frac{n^3+48n^2+857n-2058}{(n-1)^2(n-2)}$	$\frac{9(n^2+11n-286)}{(n-1)^2(n-2)}$	$\frac{9(7n^3+86n^2-365n+350)}{n(n-1)^3(n-2)}$
K_{433}	$\frac{10(2n^2+129n-305)}{(n-1)^2(n-2)}$	$\frac{12(n^2+123n-298)}{(n-1)^2(n-2)}$	$\frac{n^4+63n^3+1450n^2-7452n+8816}{(n-1)^2(n-2)^2}$	$\frac{n^4+52n^3+1133n^2-5254n+5616}{n(n-1)^3(n-2)}$
K_{4222}	$\frac{60n(n+11)}{(n-1)^2(n-2)}$	$\frac{24(3n^3+28n^2-99n-4)}{(n-1)^2(n-2)(n-3)}$	$\frac{6n(n^2+201n-478)}{(n-1)^2(n-2)^2}$	$\frac{3(3n^3+68n^2+125n-412)}{(n-1)^3(n-2)}$
K_{3322}	$\frac{90n(n+17)}{(n-1)^2(n-2)}$	$\frac{144(n^3+13n^2-50n-2)}{(n-1)^2(n-2)(n-3)}$	$\frac{36n(2n^2+75n-206)}{(n-1)^2(n-2)^2}$	$\frac{n^4+30n^3+379n^2-1062n-5312}{(n-1)^3(n-2)}$
K_{22222}		$\frac{24n(n+1)(n+7)}{(n-1)^2(n-2)(n-3)}$	$\frac{216n^2}{(n-1)^2(n-2)^2}$	$\frac{6n(n+5)(n+7)}{(n-1)^3(n-2)}$
0	$k_5k_3k_2$	$k_4^2k_2$	$k_4k_3^2$	$k_3^2k_2^2$
K_8K_2	$-\frac{1}{n}$	$-\frac{1}{n}$		$-\frac{2}{n^2}$
K_7K_3	$-\frac{1}{n}$		$-\frac{2}{n}$	$-\frac{2}{n^2}$
K_6K_4		$-\frac{2}{n}$	$-\frac{1}{n}$	
K_5^2				
$K_{62}K_2$	$-\frac{15}{n-1}$	$-\frac{16}{n-1}$		$-\frac{2(n+20)}{n(n-1)}$
K_6K_{22}				
$K_{53}K_2$	$-\frac{n+44}{n-1}$	$-\frac{48}{n-1}$		$-\frac{4(n^2+22n-32)}{n(n-1)^2}$
$K_{52}K_3$	$-\frac{n+9}{n-1}$		$-\frac{24}{n-1}$	$-\frac{4(n+7)}{n(n-1)}$
K_5K_{32}	$-\frac{n+5}{n-1}$			
$K_{44}K_2$	$-\frac{30}{n-1}$	$-\frac{n+33}{n-1}$		$-\frac{18(3n-8)}{n(n-1)^2}$
$K_{42}K_4$		$-\frac{2(n+7)}{n-1}$	$-\frac{9}{n-1}$	
$K_{43}K_3$	$-\frac{20}{n-1}$		$-\frac{2(n+29)}{n-1}$	$-\frac{2(n^2+22n-35)}{n(n-1)^2}$
$K_{33}K_4$		$-\frac{12}{n-1}$	$-\frac{n+8}{n-1}$	
$K_{422}K_2$	$-\frac{60n}{(n-1)^2(n-2)^2}$	$-\frac{72n}{(n-1)^2(n-2)^2}$		$-\frac{18(n^2+9n-20)}{(n-1)^2(n-2)}$
$K_{42}K_{22}$				
$K_{222}K_4$			$-\frac{6n}{(n-1)^2(n-2)^2}$	
$K_{332}K_2$	$-\frac{90n}{(n-1)^2(n-2)^2}$	$-\frac{144n}{(n-1)^2(n-2)^2}$		$-\frac{2(n^3+17n^2+104n-320)}{(n-1)^2(n-2)}$
$K_{33}K_{22}$				
K_{32}^2				
$K_{322}K_3$			$-\frac{72n}{(n-1)^2(n-2)^2}$	$-\frac{2(n+5)(n+7)}{(n-1)^2}$
$K_{2222}K_2$		$-\frac{24n(n+1)}{(n-1)(n-2)(n-3)}$		$-\frac{12n(n+5)}{(n-1)^2(n-2)}$
$K_{222}K_{22}$				

Exp.	$M(42^3)$	$M(2^5)$	$K(3^22^2)$	$K(42^3)$	$K(2^5)$
K_{10}	$\frac{1}{n^3}$	$\frac{1}{n^4}$	$\frac{1}{n^3}$	$\frac{1}{n^3}$	$\frac{1}{n^4}$
K_{82}	$\frac{3(n+11)}{n^2(n-1)}$	$\frac{5(n+7)}{n^3(n-1)}$	$\frac{2n+35}{n^2(n-1)}$	$\frac{3(n+11)}{n^2(n-1)}$	$\frac{5(n+7)}{n^3(n-1)}$
K_{73}	$\frac{4(23n-37)}{n^2(n-1)^2}$	$\frac{80(n-2)}{n^3(n-1)^2}$	$\frac{2(n^2+49n-80)}{n^2(n-1)^2}$	$\frac{4(23n-37)}{n^2(n-1)^2}$	$\frac{80(n-2)}{n^3(n-1)^2}$
K_{64}	$\frac{4(n^3+44n^2-117n+80)}{n^2(n-1)^3}$	$\frac{10(n^3+17n^2-45n+35)}{n^3(n-1)^3}$	$\frac{n^3+180n^2-495n+350}{n^2(n-1)^3}$	$\frac{4(n^3+44n^2-117n+80)}{n^2(n-1)^3}$	$\frac{10(n^3+17n^2-45n+35)}{n^3(n-1)^3}$
K_{55}	$\frac{12(9n^2-24n+17)}{n^2(n-1)^3}$	$\frac{16(n-2)(6n^2-12n+7)}{n^3(n-1)^4}$	$\frac{2(n^3+56n^2-151n+112)}{n^2(n-1)^3}$	$\frac{12(9n^2-24n+17)}{n^2(n-1)^3}$	$\frac{16(n-2)(6n^2-12n+7)}{n^3(n-1)^4}$
K_{622}	$\frac{3(n+9)(n+11)}{n(n-1)^2}$	$\frac{10(n+5)(n+7)}{n^2(n-1)^2}$	$\frac{n^3+40n^2+275n-700}{n(n-1)^2(n-2)}$	$\frac{3(n+9)(n+11)}{n(n-1)^2}$	$\frac{10(n+5)(n+7)}{n^2(n-1)^2}$
K_{532}	$\frac{24(5n-7)(n+11)}{n(n-1)^3}$	$\frac{160(n-2)(n+7)}{n^2(n-1)^3}$	$\frac{4(n^4+35n^3+274n^2-1266n+1120)}{n(n-1)^3(n-2)}$	$\frac{24(5n-7)(n+11)}{n(n-1)^3}$	$\frac{160(n-2)(n+7)}{n^2(n-1)^3}$
K_{442}	$\frac{12(10n^2+61n-107)}{n(n-1)^3}$	$\frac{5(n+7)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$	$\frac{9(7n^3+86n^2-365n+350)}{n(n-1)^3(n-2)}$	$\frac{12(10n^2+61n-107)}{n(n-1)^3}$	$\frac{5(n+7)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$
K_{433}	$\frac{2(11n^2+564n-1127)}{n(n-1)^3}$	$\frac{40(n^3+20n^2-79n+70)}{n^2(n-1)^4}$	$\frac{n^4+52n^3+1133n^2-5254n+5616}{n(n-1)^3(n-2)}$	$\frac{2(11n^2+564n-1127)}{n(n-1)^3}$	$\frac{40(n^3+20n^2-79n+70)}{n^2(n-1)^4}$
K_{4222}	$\frac{n^3+21n^2+251n+687}{(n-1)^3}$	$\frac{10(n+3)(n+5)(n+7)}{n(n-1)^3}$	$\frac{3(3n^3+68n^2+125n-412)}{(n-1)^3(n-2)}$	$\frac{n^3+21n^2+251n+687}{(n-1)^3}$	$\frac{10(n+3)(n+5)(n+7)}{n(n-1)^3}$
K_{3322}	$\frac{18(n^2+119)}{(n-1)^3}$	$\frac{40(n-2)(n+5)(n+7)}{n(n-1)^3}$	$\frac{n^4+30n^3+379n^2-1062n-5312}{(n-1)^3(n-2)}$	$\frac{18(n^2+119)}{(n-1)^3}$	$\frac{40(n-2)(n+5)(n+7)}{n(n-1)^3}$
K_{22222}		$\frac{(n+1)(n+3)(n+5)(n+7)}{(n-1)^4}$	$\frac{6n(n+5)(n+7)}{(n-1)^3(n-2)}$		$\frac{(n+1)(n+3)(n+5)(n+7)}{(n-1)^4}$
0	$k_4k_2^3$	k_2^5	$k_3^2k_2^2$	$k_4k_2^3$	k_2^5
K_8K_2	$-\frac{3}{n^2}$	$-\frac{5}{n^3}$	$-\frac{2}{n^2}$	$-\frac{3}{n^2}$	$-\frac{5}{n^3}$
K_7K_3			$-\frac{2}{n^2}$		
K_6K_4	$-\frac{1}{n^2}$		$-\frac{1}{n^2}$	$-\frac{4}{n^2}$	$-\frac{10}{n^3}$
K_5^2					
$K_{62}K_2$	$-\frac{6(n+9)}{n(n-1)}$	$-\frac{20(n+5)}{n^2(n-1)}$	$-\frac{2(n+20)}{n(n-1)}$	$-\frac{6(n+9)}{n(n-1)}$	$-\frac{20(n+5)}{n^2(n-1)}$
K_6K_{22}			$-\frac{n+1}{n(n-1)}$	$-\frac{3(n+1)}{n(n-1)}$	$-\frac{10(n+1)}{n^2(n-1)}$
$K_{53}K_2$	$-\frac{24(5n-7)}{n(n-1)^2}$	$-\frac{160(n-2)}{n^2(n-1)^2}$	$-\frac{4(n^2+22n-32)}{n(n-1)^2}$	$-\frac{24(5n-7)}{n(n-1)^2}$	$-\frac{160(n-2)}{n^2(n-1)^2}$
$K_{52}K_3$			$-\frac{4(n+7)}{n(n-1)}$		
K_5K_{32}			$-\frac{4(n+5)}{n(n-1)}$		
$K_{44}K_2$	$-\frac{3(n^2+26n-39)}{n(n-1)^2}$	$-\frac{5(3n^3+23n^2-63n+45)}{n^2(n-1)^3}$	$-\frac{18(3n-8)}{n(n-1)^2}$	$-\frac{3(n^2+26n-39)}{n(n-1)^2}$	$-\frac{5(3n^3+23n^2-63n+45)}{n^2(n-1)^3}$
$K_{42}K_4$	$-\frac{3(n+3)}{n(n-1)}$		$-\frac{9}{n(n-1)}$	$-\frac{6(n+5)}{n(n-1)}$	$-\frac{30(n+3)}{n^2(n-1)}$
$K_{43}K_3$			$-\frac{2(n^2+22n-35)}{n(n-1)^2}$		
$K_{33}K_4$	$-\frac{4n(n-2)}{n^2(n-1)^2}$		$-\frac{n+8}{n(n-1)}$	$-\frac{2(11n+13)}{n(n-1)^2}$	$-\frac{40(n-2)}{n^2(n-1)^2}$
$K_{422}K_2$	$-\frac{3(n+7)(n+9)}{(n-1)^2}$	$-\frac{30(n+3)(n+5)}{n(n-1)^2}$	$-\frac{18(n^2+9n-20)}{(n-1)^2(n-2)}$	$-\frac{3(n+7)(n+9)}{(n-1)^2}$	$-\frac{30(n+3)(n+5)}{n(n-1)^2}$
$K_{42}K_{22}$			$-\frac{9(n+1)}{(n-1)^2}$	$-\frac{3(n+1)(n+7)}{(n-1)^2}$	$-\frac{30(n+1)(n+3)}{n(n-1)^2}$
$K_{222}K_4$	$-\frac{(n+1)(n+3)}{(n-1)^2}$		$-\frac{6}{(n-1)(n-2)}$	$-\frac{(n+1)(n+3)}{(n-1)^2}$	$-\frac{10(n+1)(n+3)}{n(n-1)^2}$
$K_{332}K_2$	$-\frac{36(n+9)}{(n-1)^2}$	$-\frac{80(n-2)(n+5)}{n(n-1)^3}$	$-\frac{2(n^3+17n^2+104n-320)}{(n-1)^2(n-2)}$	$-\frac{36(n+9)}{(n-1)^2}$	$-\frac{80(n-2)(n+5)}{n(n-1)^3}$
$K_{33}K_{22}$			$-\frac{(n+1)(n+8)}{(n-1)^2}$	$-\frac{18(n+1)}{(n-1)^2}$	$-\frac{40(n-2)(n+1)}{n(n-1)^3}$
K_{32}^2			$-\frac{2(n+5)^2}{(n-1)^2}$		
$K_{322}K_3$			$-\frac{2(n+5)(n+7)}{(n-1)^2}$		
$K_{2222}K_2$		$-\frac{5(n+1)(n+3)(n+5)}{(n-1)^3}$	$-\frac{12n(n+5)}{(n-1)^2(n-2)}$		$-\frac{5(n+1)(n+3)(n+5)}{(n-1)^3}$
$K_{222}K_{22}$			$-\frac{6n(n+1)}{(n-1)^2(n-2)}$		$-\frac{10(n+1)^2(n+3)}{(n-1)^3}$

TABLE 8·7 (CONCLD.)

Exp.	Est.	M(82)	M(73)	M(64)	M(55)	$M(62^2)$	M(532)	$M(4^22)$	$M(43^2)$	$M(3^22^2)$	$M(42^3)$	$M(2^5)$	$K(3^22^2)$	$K(42^3)$	$K(2^5)$
$K_6K_2^2$	$[k_6k_2^2]$					2				$\frac{1}{n}$	$\frac{3}{n}$	$\frac{10}{n^2}$	$\frac{2}{n}$	$\frac{6}{n}$	$\frac{20}{n^2}$
$K_5K_3K_2$	$[k_5k_3k_2]$						2			$\frac{4}{n}$			$\frac{8}{n}$		
$K_4^2K_2$	$[k_4^2k_2]$							2			$\frac{3}{n}$			$\frac{6}{n}$	$\frac{30}{n^2}$
$K_4K_3^2$	$[k_4k_3^2]$								2	$\frac{1}{n}$			$\frac{2}{n}$		
$K_{42}K_2^2$	$[k_{42}k_2^2]$									$\frac{9}{n-1}$	$\frac{3(n+7)}{n-1}$	$\frac{30(n+3)}{n(n-1)}$	$\frac{18}{n-1}$	$\frac{6(n+7)}{n-1}$	$\frac{60(n+3)}{n(n-1)}$
$K_4K_{22}K_2$	$[k_4k_{22}k_2]$										$\frac{3(n+1)}{n-1}$			$\frac{6(n+1)}{n-1}$	$\frac{60(n+1)}{n(n-1)}$
$K_{33}K_2^2$	$[k_{33}k_2^2]$									$\frac{n+8}{n-1}$	$\frac{18}{n-1}$	$\frac{40(n-2)}{n(n-1)^2}$	$\frac{2(n+8)}{n-1}$	$\frac{36}{n-1}$	$\frac{80(n-2)}{n(n-1)^2}$
$K_{32}K_3K_2$	$[k_{32}k_3k_2]$									$\frac{4(n+5)}{n-1}$			$\frac{8(n+5)}{n-1}$		
$K_{22}K_3^2$	$[k_{22}k_3^2]$									$\frac{n+1}{n-1}$			$\frac{2(n+1)}{n-1}$		
$K_{222}K_2^2$	$[k_{222}k_2^2]$									$\frac{6n}{(n-1)(n-2)}$		$\frac{10(n+1)(n+3)}{(n-1)^2}$	$\frac{12n}{(n-1)(n-2)}$		$\frac{20(n+1)(n+3)}{(n-1)^2}$
$K_{22}^2K_2$	$[k_{22}^2k_2]$														$\frac{30(n+1)^2}{(n-1)^2}$
$K_3^2K_2^2$	$[k_3^2k_2^2]$									-3	$\frac{1}{n}$				
$K_4K_2^3$	$[k_4k_2^3]$										-3	$-\frac{10}{n}$		-6	$-\frac{60}{n}$
$K_{22}K_2^3$	$[k_{22}k_2^3]$											$-\frac{10(n+1)}{n-1}$			$-\frac{60(n+1)}{n-1}$
K_2^5	$[k_2^5]$											4			24

TABLE 8.8 — WEIGHT 12

Exp.	Est.	$M(6^2)$	$M(4^3)$	$M(3^4)$	$M(2^6)$	$K(3^4)$	$K(2^6)$
K_{12}	0	$\frac{1}{n}$	$\frac{1}{n^2}$	$\frac{1}{n^3}$	$\frac{1}{n^5}$	$\frac{1}{n^3}$	$\frac{1}{n^5}$
$K_{10,2}$	0	$\frac{36}{n-1}$	$\frac{48}{n(n-1)}$	$\frac{54}{n^2(n-1)}$	$\frac{6(n+9)}{n^4(n-1)}$	$\frac{54}{n^2(n-1)}$	$\frac{6(n+9)}{n^4(n-1)}$
K_{93}	0	$\frac{180}{n-1}$	$\frac{16(13n-17)}{n(n-1)^2}$	$\frac{4(n^2+52n-80)}{n^2(n-1)^2}$	$\frac{160(n-2)}{n^4(n-1)^2}$	$\frac{4(n^2+52n-80)}{n^2(n-1)^2}$	$\frac{160(n-2)}{n^4(n-1)^2}$
K_{84}	0	$\frac{465}{n-1}$	$\frac{3(3n^2+158n-257)}{n(n-1)^2}$	$\frac{27(17n^2-49n+35)}{n^2(n-1)^3}$	$\frac{15(n^3+29n^2-77n+63)}{n^4(n-1)^3}$	$\frac{27(17n^2-49n+35)}{n^2(n-1)^3}$	$\frac{15(n^3+29n^2-77n+63)}{n^4(n-1)^3}$
K_{75}	0	$\frac{780}{n-1}$	$\frac{48(16n-29)}{n(n-1)^2}$	$\frac{108(7n^2-20n+16)}{n^2(n-1)^3}$	$\frac{96(n-2)(7n^2-14n+9)}{n^4(n-1)^4}$	$\frac{108(7n^2-20n+16)}{n^2(n-1)^3}$	$\frac{96(n-2)(7n^2-14n+9)}{n^4(n-1)^4}$
K_{66}	0	$\frac{n+460}{n-1}$	$\frac{12(37n-70)}{n(n-1)^2}$	$\frac{3(n^3+150n^2-420n+350)}{n^2(n-1)^3}$	$\frac{2(5n^5+20n^4-990n^3+1850n^2-1575n+525)}{n^4(n-1)^5}$	$\frac{3(n^3+150n^2-420n+350)}{n^2(n-1)^3}$	$\frac{2(5n^5+20n^4-990n^3+1850n^2-1575n+525)}{n^4(n-1)^5}$
K_{822}	0	$\frac{450n}{(n-1)(n-2)}$	$\frac{72(11n-19)}{(n-1)^2(n-2)}$	$\frac{27(37n-70)}{n(n-1)^2(n-2)}$	$\frac{15(n+7)(n+9)}{n^3(n-1)^2}$	$\frac{27(37n-70)}{n(n-1)^2(n-2)}$	$\frac{15(n+7)(n+9)}{n^3(n-1)^2}$
K_{732}	0	$\frac{3600n}{(n-1)(n-2)}$	$\frac{288(19n-41)}{(n-1)^2(n-2)}$	$\frac{108(n^3+53n^2-196n+160)}{n(n-1)^3(n-2)}$	$\frac{480(n-2)(n+9)}{n^3(n-1)^3}$	$\frac{108(n^3+53n^2-196n+160)}{n(n-1)^3(n-2)}$	$\frac{480(n-2)(n+9)}{n^3(n-1)^3}$
K_{642}	0	$\frac{7200n}{(n-1)(n-2)}$	$\frac{48(n^2+200n-521)}{(n-1)^2(n-2)}$	$\frac{54(n^3+191n^2-732n+700)}{n(n-1)^3(n-2)}$	$\frac{60(n+9)(n^3+17n^2-45n+35)}{n^3(n-1)^4}$	$\frac{54(n^3+191n^2-732n+700)}{n(n-1)^3(n-2)}$	$\frac{60(n+9)(n^3+17n^2-45n+35)}{n^3(n-1)^4}$
K_{633}	0	$\frac{6300n}{(n-1)(n-2)}$	$\frac{144(56n^2-257n+302)}{(n-1)^2(n-2)^2}$	$\frac{6(n^5+56n^4+1099n^3-7774n^2+16360n-11200)}{n(n-1)^3(n-2)^2}$	$\frac{80(n-2)(n^2+60n-105)}{n^3(n-1)^4}$	$\frac{6(n^5+56n^4+1099n^3-7774n^2+16360n-11200)}{n(n-1)^3(n-2)^2}$	$\frac{80(n-2)(n^2+60n-105)}{n^3(n-1)^4}$
K_{552}	0	$\frac{4500n}{(n-1)(n-2)}$	$\frac{1440(4n-11)}{(n-1)^2(n-2)}$	$\frac{108(59n^2-220n+224)}{n(n-1)^3(n-2)}$	$\frac{96(n-2)(n+9)(6n^2-12n+7)}{n^3(n-1)^5}$	$\frac{108(59n^2-220n+224)}{n(n-1)^3(n-2)}$	$\frac{96(n-2)(n+9)(6n^2-12n+7)}{n^3(n-1)^5}$
K_{543}	0	$\frac{21600n}{(n-1)(n-2)}$	$\frac{144(n^3+171n^2-840n+1060)}{(n-1)^2(n-2)^2}$	$\frac{216(2n^4+99n^3-676n^2+1488n-1120)}{n(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n^3+33n^2-89n+63)}{n^3(n-1)^5}$	$\frac{216(2n^4+99n^3-676n^2+1488n-1120)}{n(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n^3+33n^2-89n+63)}{n^3(n-1)^5}$
K_{444}	0	$\frac{4950n}{(n-1)(n-2)}$	$\frac{n^4+96n^3+5175n^2-26636n+35244}{(n-1)^2(n-2)^2}$	$\frac{27(173n^4-1503n^3+4962n^2-7380n+4200)}{n(n-1)^3(n-2)^3}$	$\frac{15(n^5+27n^4+226n^3-1098n^2+1725n-945)}{n^3(n-1)^5}$	$\frac{27(173n^4-1503n^3+4962n^2-7380n+4200)}{n(n-1)^3(n-2)^3}$	$\frac{15(n^5+27n^4+226n^3-1098n^2+1725n-945)}{n^3(n-1)^5}$
K_{6222}	0	$\frac{2400n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{288(19n^3-98n^2+125n+2)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{36(n^3+209n^2-784n+700)}{(n-1)^3(n-2)^2}$	$\frac{20(n+5)(n+7)(n+9)}{n^2(n-1)^3}$	$\frac{36(n^3+209n^2-784n+700)}{(n-1)^3(n-2)^2}$	$\frac{20(n+5)(n+7)(n+9)}{n^2(n-1)^3}$
K_{5322}	0	$\frac{21600n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{1728(24n^3-140n^2+200n+4)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{432(2n^3+109n^2-500n+560)}{(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n+7)(n+9)}{n^2(n-1)^4}$	$\frac{432(2n^3+109n^2-500n+560)}{(n-1)^3(n-2)^2}$	$\frac{480(n-2)(n+7)(n+9)}{n^2(n-1)^4}$
K_{4422}	0	$\frac{15300n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{216(n^4+116n^3-731n^2+1098n+24)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{243(n^3+121n^2-572n+700)}{(n-1)^3(n-2)^2}$	$\frac{15(n+7)(n+9)(3n^3+23n^2-63n+45)}{n^2(n-1)^5}$	$\frac{243(n^3+121n^2-572n+700)}{(n-1)^3(n-2)^2}$	$\frac{15(n+7)(n+9)(3n^3+23n^2-63n+45)}{n^2(n-1)^5}$
K_{4332}	0	$\frac{54000n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{432(n^4+200n^3-1247n^2+1890n+48)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{54(n^5+61n^4+1300n^3-11060n^2+27472n-22400)}{(n-1)^3(n-2)^3}$	$\frac{240(n+9)(n^3+20n^2-79n+70)}{n^2(n-1)^5}$	$\frac{54(n^5+61n^4+1300n^3-11060n^2+27472n-22400)}{(n-1)^3(n-2)^3}$	$\frac{240(n+9)(n^3+20n^2-79n+70)}{n^2(n-1)^5}$
K_{3333}	0	$\frac{8100n(n+1)}{(n-1)(n-2)(n-3)}$	$\frac{288(41n^4-384n^3+1209n^2-1282n-36)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{n^6+45n^5+952n^4+6336n^3-73968n^2+203328n-179200}{(n-1)^3(n-2)^3}$	$\frac{160(n-2)(n^2+27n-70)}{n^2(n-1)^5}$	$\frac{n^6+45n^5+952n^4+6336n^3-73968n^2+203328n-179200}{(n-1)^3(n-2)^3}$	$\frac{160(n-2)(n^2+27n-70)}{n^2(n-1)^5}$
K_{42222}	0	$\frac{5400n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$	$\frac{72n(n^3+210n^2-717n-206)}{(n-1)^2(n-2)^2(n-3)}$	$\frac{324n(n^3+64n^2-301n+350)}{(n-1)^3(n-2)^3}$	$\frac{15(n+3)(n+5)(n+7)(n+9)}{n(n-1)^4}$	$\frac{324n(n^3+64n^2-301n+350)}{(n-1)^3(n-2)^3}$	$\frac{15(n+3)(n+5)(n+7)(n+9)}{n(n-1)^4}$
K_{33222}	0	$\frac{21600n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$	$\frac{1728n(29n^3-196n^2+317n+62)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{36n(n^4+63n^3+1498n^2-8568n+11200)}{(n-1)^3(n-2)^3}$	$\frac{80(n-2)(n+5)(n+7)(n+9)}{n(n-1)^5}$	$\frac{36n(n^4+63n^3+1498n^2-8568n+11200)}{(n-1)^3(n-2)^3}$	$\frac{80(n-2)(n+5)(n+7)(n+9)}{n(n-1)^5}$
K_{222222}	0	$\frac{720n(n+1)(n^2+15n-4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}$	$\frac{1728n(n+1)(n^2-5n+2)}{(n-1)^2(n-2)^2(n-3)^2}$	$\frac{108n^2(n^2+27n-70)}{(n-1)^3(n-2)^3}$	$\frac{(n+1)(n+3)(n+5)(n+7)(n+9)}{(n-1)^5}$	$\frac{108n^2(n^2+27n-70)}{(n-1)^3(n-2)^3}$	$\frac{(n+1)(n+3)(n+5)(n+7)(n+9)}{(n-1)^5}$
0	1	k_6^2	k_4^3	k_3^4	k_2^6	k_3^4	k_2^6
$K_{10}K_2$	$[k_{10}k_2]$				$-\frac{6}{n^4}$		$-\frac{6}{n^4}$
K_9K_3	$[k_9k_3]$			$-\frac{4}{n^2}$		$-\frac{4}{n^2}$	
K_8K_4	$[k_8k_4]$		$-\frac{3}{n}$				
K_6^2	$[k_6^2]$	-1					
$K_{82}K_2$	$[k_{82}k_2]$				$-\frac{30(n+7)}{n^3(n-1)}$		$-\frac{30(n+7)}{n^3(n-1)}$
K_8K_{22}	$[k_8k_{22}]$						$-\frac{15(n+1)}{n^3(n-1)}$
$K_{73}K_2$	$[k_{73}k_2]$				$-\frac{480(n-2)}{n^3(n-1)^2}$		$-\frac{480(n-2)}{n^3(n-1)^2}$
$K_{72}K_3$	$[k_{72}k_3]$			$-\frac{108}{n(n-1)}$		$-\frac{108}{n(n-1)}$	
$K_{64}K_2$	$[k_{64}k_2]$				$-\frac{60(n^3+17n^2-45n+35)}{n^3(n-1)^3}$		$-\frac{60(n^3+17n^2-45n+35)}{n^3(n-1)^3}$
$K_{62}K_4$	$[k_{62}k_4]$		$-\frac{48}{n-1}$				$-\frac{60(n+5)}{n^3(n-1)}$
K_6K_{42}	$[k_6k_{42}]$					$-\frac{54}{n(n-1)}$	$-\frac{60(n+3)}{n^3(n-1)}$
$K_{63}K_3$	$[k_{63}k_3]$			$-\frac{12(n^2+25n-35)}{n(n-1)^2}$		$-\frac{12(n^2+25n-35)}{n(n-1)^2}$	

TABLE 8.8 (CONT.)

Exp.	Est.	$M(6^2)$	$M(4^3)$	$M(3^4)$	$M(2^6)$	$K(3^4)$	$K(2^6)$
$K_6 K_{33}$	$[k_6 k_{33}]$					$\frac{-6(n+3)}{n(n-1)}$	$\frac{-80(n-2)}{n^3(n-1)^2}$
$K_{55} K_2$	$[k_{55} k_2]$				$-\frac{96(n-2)(6n^2-12n+7)}{n^3(n-1)^4}$		$-\frac{96(n-2)(6n^2-12n+7)}{n^3(n-1)^4}$
$K_{54} K_3$	$[k_{54} k_3]$			$-\frac{108(4n-7)}{n(n-1)^2}$		$-\frac{108(4n-7)}{n(n-1)^2}$	
$K_{53} K_4$	$[k_{53} k_4]$		$-\frac{144}{n-1}$				$-\frac{480(n-2)}{n^3(n-1)^2}$
$K_{44} K_4$	$[k_{44} k_4]$		$-\frac{3(n+33)}{n-1}$				$-\frac{15(3n^3+23n^2-63n+45)}{n^3(n-1)^3}$
$K_{622} K_2$	$[k_{622} k_2]$				$-\frac{60(n+5)(n+7)}{n^2(n-1)^2}$		$-\frac{60(n+5)(n+7)}{n^2(n-1)^2}$
$K_{62} K_{22}$	$[k_{62} k_{22}]$						$-\frac{60(n+1)(n+5)}{n^2(n-1)^2}$
$K_6 K_{222}$	$[k_6 k_{222}]$					$-\frac{36}{(n-1)(n-2)}$	$-\frac{20(n+1)(n+3)}{n^2(n-1)^2}$
$K_{532} K_2$	$[k_{532} k_2]$				$-\frac{960(n-2)(n+7)}{n^2(n-1)^3}$		$-\frac{960(n-2)(n+7)}{n^2(n-1)^3}$
$K_{53} K_{22}$	$[k_{53} k_{22}]$						$-\frac{480(n-2)(n+1)}{n^2(n-1)^3}$
$K_{522} K_3$	$[k_{522} k_3]$			$-\frac{216(4n-7)}{(n-1)^2(n-2)}$		$-\frac{216(4n-7)}{(n-1)^2(n-2)}$	
$K_{442} K_2$	$[k_{442} k_2]$				$-\frac{30(n+7)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$		$-\frac{30(n+7)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$
$K_{44} K_{22}$	$[k_{44} k_{22}]$						$-\frac{15(n+1)(3n^3+23n^2-63n+45)}{n^2(n-1)^4}$
K_{42}^2	$[k_{42}^2]$					$-\frac{243}{(n-1)^2}$	$-\frac{90(n+3)^2}{n^2(n-1)^2}$
$K_{422} K_4$	$[k_{422} k_4]$		$-\frac{216n}{(n-1)(n-2)}$				$-\frac{90(n+3)(n+5)}{n^2(n-1)^2}$
$K_{433} K_2$	$[k_{433} k_2]$				$-\frac{240(n^3+20n^2-79n+70)}{n^2(n-1)^4}$		$-\frac{240(n^3+20n^2-79n+70)}{n^2(n-1)^4}$
$K_{42} K_{33}$	$[k_{42} k_{33}]$					$-\frac{54(n+8)}{(n-1)^2}$	$-\frac{240(n-2)(n+3)}{n^2(n-1)^3}$
$K_{432} K_3$	$[k_{432} k_3]$			$-\frac{108(n^2+27n-70)}{(n-1)^2(n-2)}$		$-\frac{108(n^2+27n-70)}{(n-1)^2(n-2)}$	
$K_{332} K_4$	$[k_{332} k_4]$		$-\frac{432n}{(n-1)(n-2)}$				$-\frac{240(n-2)(n+5)}{n^2(n-1)^3}$
$K_{333} K_3$	$[k_{333} k_3]$			$-\frac{4(n^4+21n^3+131n^2-850n+1120)}{(n-1)^2(n-2)^2}$		$-\frac{4(n^4+21n^3+131n^2-850n+1120)}{(n-1)^2(n-2)^2}$	
K_{33}^2	$[k_{33}^2]$					$-\frac{3(n+8)^2}{(n-1)^2}$	$-\frac{160(n-2)^2}{n^2(n-1)^4}$
$K_{4222} K_2$	$[k_{4222} k_2]$				$-\frac{60(n+3)(n+5)(n+7)}{n(n-1)^3}$		$-\frac{60(n+3)(n+5)(n+7)}{n(n-1)^3}$
$K_{422} K_{22}$	$[k_{422} k_{22}]$						$-\frac{90(n+1)(n+3)(n+5)}{n(n-1)^3}$
$K_{42} K_{222}$	$[k_{42} k_{222}]$					$-\frac{324n}{(n-1)^2(n-2)}$	$-\frac{60(n+1)(n+3)^2}{n(n-1)^3}$
$K_{2222} K_4$	$[k_{2222} k_4]$		$-\frac{72n(n+1)}{(n-1)(n-2)(n-3)}$				$-\frac{15(n+1)(n+3)(n+5)}{n(n-1)^3}$
$K_{3322} K_2$	$[k_{3322} k_2]$				$-\frac{240(n-2)(n+5)(n+7)}{n(n-1)^4}$		$-\frac{240(n-2)(n+5)(n+7)}{n(n-1)^4}$
$K_{332} K_{22}$	$[k_{332} k_{22}]$						$-\frac{240(n-2)(n+1)(n+5)}{n(n-1)^4}$
$K_{33} K_{222}$	$[k_{33} k_{222}]$					$-\frac{36n(n+8)}{(n-1)^2(n-2)}$	$-\frac{80(n-2)(n+1)(n+3)}{n(n-1)^4}$
$K_{3222} K_3$	$[k_{3222} k_3]$			$-\frac{72n(n^2+27n-70)}{(n-1)^2(n-2)^2}$		$-\frac{72n(n^2+27n-70)}{(n-1)^2(n-2)^2}$	
$K_{22222} K_2$	$[k_{22222} k_2]$				$-\frac{6(n+1)(n+3)(n+5)(n+7)}{(n-1)^4}$		$-\frac{6(n+1)(n+3)(n+5)(n+7)}{(n-1)^4}$
$K_{2222} K_{22}$	$[k_{2222} k_{22}]$						$-\frac{15(n+1)^2(n+3)(n+5)}{(n-1)^4}$
K_{222}^2	$[k_{222}^2]$						$-\frac{10(n+1)^2(n+3)^2}{(n-1)^4}$
$K_8 K_2^2$	$[k_8 k_2^2]$				$\frac{15}{n^3}$		$\frac{30}{n^3}$
$K_6 K_4 K_2$	$[k_6 k_4 k_2]$						$\frac{120}{n^3}$
$K_6 K_3^2$	$[k_6 k_3^2]$			$\frac{6}{n}$		$\frac{12}{n}$	
K_4^3	$[k_4^3]$		2				$\frac{30}{n^3}$
$K_{62} K_2^2$	$[k_{62} k_2^2]$				$\frac{60(n+5)}{n^2(n-1)}$		$\frac{120(n+5)}{n^2(n-1)}$

TABLE 8.8 (CONCLD.)

Exp.	Est.	M(6²)	M(4³)	M(3⁴)	M(2⁶)	K(3⁴)	K(2⁶)
$K_6K_2K_{22}$	$[k_6k_2k_{22}]$						$\frac{120(n+1)}{n^2(n-1)}$
$K_{53}K_2^2$	$[k_{53}k_2^2]$				$\frac{480(n-2)}{n^2(n-1)^2}$		$\frac{960(n-2)}{n^2(n-1)^2}$
$K_{44}K_2^2$	$[k_{44}k_2^2]$				$\frac{15(3n^3+23n^2-63n+45)}{n^2(n-1)^3}$		$\frac{30(3n^3+23n^2-63n+45)}{n^2(n-1)^3}$
$K_{42}K_4K_2$	$[k_{42}k_4k_2]$						$\frac{360(n+3)}{n^2(n-1)}$
$K_{42}K_3^2$	$[k_{42}k_3^2]$			$\frac{54}{n-1}$		$\frac{108}{n-1}$	
$K_{33}K_4K_2$	$[k_{33}k_4k_2]$						$\frac{480(n-2)}{n^2(n-1)^2}$
$K_{33}K_3^2$	$[k_{33}k_3^2]$			$\frac{6(n+8)}{n-1}$		$\frac{12(n+8)}{n-1}$	
$K_{422}K_2^2$	$[k_{422}k_2^2]$				$\frac{90(n+3)(n+5)}{n(n-1)^2}$		$\frac{180(n+3)(n+5)}{n(n-1)^2}$
$K_{42}K_{22}K_2$	$[k_{42}k_{22}k_2]$						$\frac{360(n+1)(n+3)}{n(n-1)^2}$
$K_4K_{22}^2$	$[k_4k_{22}^2]$						$\frac{90(n+1)^2}{n(n-1)^2}$
$K_{222}K_4K_2$	$[k_{222}k_4k_2]$						$\frac{120(n+1)(n+3)}{n(n-1)^2}$
$K_{332}K_2^2$	$[k_{332}k_2^2]$				$\frac{240(n-2)(n+5)}{n(n-1)^3}$		$\frac{480(n-2)(n+5)}{n(n-1)^3}$
$K_{33}K_{22}K_2$	$[k_{33}k_{22}k_2]$						$\frac{480(n-2)(n+1)}{n(n-1)^3}$
$K_{222}K_3^2$	$[k_{222}k_3^2]$			$\frac{36n}{(n-1)(n-2)}$		$\frac{72n}{(n-1)(n-2)}$	
$K_{222}K_{22}K_2$	$[k_{222}k_{22}k_2]$						$\frac{120(n+1)^2(n+3)}{(n-1)^3}$
K_{22}^3	$[k_{22}^3]$						$\frac{30(n+1)^3}{(n-1)^3}$
$K_{2222}K_2^2$	$[k_{2222}k_2^2]$				$\frac{15(n+1)(n+3)(n+5)}{(n-1)^3}$		$\frac{30(n+1)(n+3)(n+5)}{(n-1)^3}$
$K_6K_2^3$	$[k_6k_2^3]$				$-\frac{20}{n^2}$		$-\frac{120}{n^2}$
$K_4^2K_2^2$	$[k_4^2k_2^2]$						$-\frac{270}{n^2}$
K_3^4	$[k_3^4]$			-3		-6	
$K_{42}K_2^3$	$[k_{42}k_2^3]$				$-\frac{60(n+3)}{n(n-1)}$		$-\frac{360(n+3)}{n(n-1)}$
$K_{22}K_4K_2^2$	$[k_{22}k_4k_2^2]$						$-\frac{540(n+1)}{n(n-1)}$
$K_{33}K_2^3$	$[k_{33}k_2^3]$				$-\frac{80(n-2)}{n(n-1)^2}$		$-\frac{480(n-2)}{n(n-1)^2}$
$K_{222}K_2^3$	$[k_{222}k_2^3]$				$-\frac{20(n+1)(n+3)}{(n-1)^2}$		$-\frac{120(n+1)(n+3)}{(n-1)^2}$
$K_{22}^2K_2^2$	$[k_{22}^2k_2^2]$						$-\frac{270(n+1)^2}{(n-1)^2}$
$K_4K_2^4$	$[k_4k_2^4]$				$\frac{15}{n}$		$\frac{360}{n}$
$K_{22}K_2^4$	$[k_{22}k_2^4]$				$\frac{15(n+1)}{n-1}$		$\frac{360(n+1)}{n-1}$
K_2^6	$[k_2^6]$				-5		-120

Also, since

$$(5.6) \qquad k_p k_q = \sum_{\rho=0}^{q} \rho(n) \sum (p_1 \cdots p_\rho)(q_1 \cdots q_\rho)\, k_{p_1 \cdots p_\rho + q_1 \cdots q_\rho} \ ,$$

where we assume $p \geq q$ and $\rho(n)$ is Fisher's (1928) function $\sum_{i=1}^{\rho} \frac{(i-1)!}{i} \frac{\Delta^i(0^p)}{n^{(i)}}$ associated with a two-column pattern having ρ rows ($\Delta^i(0^p)$ being the leading i^{th} advancing difference of the series $0^\rho, 1^\rho, 2^\rho, \ldots$) and the second summation is over all pairs of ρ-part partitions of p and q, we have

$$(5.7) \qquad M(pq) = \sum_{\rho=0}^{q} \rho(n) \quad (p_1 \cdots p_\rho)(q_1 \cdots q_\rho)\, K_{p_1 \cdots p_\rho + q_1 \cdots q_\rho} - K_p K_q \ .$$

A special case is

$$(5.8) \qquad M(p2) = K_{p2} + \frac{1}{n} K_{p\ 2} + \frac{2}{n-1} \sum (p_1 p_2)\, K_{p_1+1,\ p_2+1} - K_p K_2$$

In generalization of (5.7),

$$\begin{aligned} M(p_1 \cdots p_u) &= E_N \prod_{i=1}^{u} (k_{p_i} - K_{p_i}) \\ &= E_N \sum_{I \subset P} (-1)^{\bar{I}} \prod_{p_i \in I} K_{p_i} \prod_{p_{i'} \in P-I} k_{p_{i'}} \\ (5.9) \qquad &= \sum_{I \subset P} (-1)^{\bar{I}} \prod_{p_i \in I} K_{p_i} \left[\prod_{p_{i'} \in P-I} K_{p_{i'}} \right]_n \ , \end{aligned}$$

where I is a subset of $P = \{ p_1, \ldots, p_u \}$ and $\bar{I}$ is the cardinality of I. We can also write

$$(5.10) \qquad M(p_1 \ldots p_u) = \sum (-1)^h K_{p_{i_1}} \cdots K_{p_{i_h}} \left[K_{p_1} \ldots) K_{p_{i_1}} (\ldots) K_{p_{i_h}} (\ldots K_{p_u} \right]_n ,$$

where the sum is over all subsets $I = \{ p_{i_1}, \ldots, p_{i_h} \}$ of $P = \{ p_1, \ldots, p_u \}$ and $p_i \neq p_{i_1}, \ldots, p_{i_h}$ for the product in the square brackets.

As an example,

(5.11) $$M(p_1 p_2 p_3) = \left[K_{p_1} K_{p_2} K_{p_3}\right]_n - K_{p_1}\left[K_{p_2} K_{p_3}\right]_n - K_{p_2}\left[K_{p_1} K_{p_3}\right]_n - K_{p_3}\left[K_{p_1} K_{p_2}\right]_n + 2K_{p_1} K_{p_2} K_{p_3} \ .$$

At times, however, semi-general formulae of Table 6 can be used to write a more general moment formula. For example, from (5.11),

$$M(p22) = \left[K_p K_2^2\right]_n - K_p\left[K_2^2\right]_n - 2K_2\left[K_p K_2\right]_n + 2K_p K_2^2$$

$$= K_{p22} + \frac{1}{n} K_{p+2,2} + \frac{1}{n} K_{p4} - \frac{2}{n(n-1)} K_{p+1,3} + \frac{2}{n-1} \sum(p_1 p_2)\, K_{p_1+1,p_2+1,2}$$

$$+ \frac{2}{n-1} K_{p22} + \frac{1}{n} K_{p+2,2} + \frac{1}{n^2} K_{p+4} + \frac{2}{n(n-1)} \sum \left((p+2)_1 , (p+2)_2\right) K_{(p+2)_1+1,(p+2)_2+1}$$

$$+ \frac{2}{n-1} \sum(p_1 p_2)\, K_{p_1+1,p_2+1,2} + \frac{2}{n(n-1)} \sum(p_1 p_2)\, K_{p_1+3,p_2+1}$$

$$+ \frac{2}{n(n-1)} \sum(p_1 p_2)\, K_{p_1+1,p_2+3}$$

$$- \frac{4}{n(n-1)^2} \sum(p_1 p_2)\, K_{p_1+2,p_2+2} + \frac{4}{(n-1)^2} \sum(p_1 p_2)\left((p_1+1)_1,(p_1+1)_2\right) K_{(p_1+1)_1+1,(p_1+1)_2+1,p_2+1}$$

$$+ \frac{4}{(n-1)^2} \sum(p_1 p_2)\left((p_2+1)_1 , (p_2+1)_2\right) K_{p_1+1,(p_2+1)_1+1,(p_2+1)_2+1}$$

(5.12) $$- \frac{1}{n} K_p K_4 - \frac{n+1}{n-1} K_p K_{22} - 2K_2 K_{p2} - \frac{2}{n} K_{p+2} K_2 - \frac{4}{n-1}\sum(p_1 p_2)\, K_{p_1+1,p_2+1} K_2 + 2K_p K_2^2 \ ,$$

where $\left((p_1+1)_1 , (p_1+1)_2\right)$ denotes the partition coefficient of (p_1+1), etc. This formula can be used to write $M(2^3)$, $M(32^2)$,...,$M(62^2)$.

Formulae involving the Mean

Some general results are now presented for determining expressions for product moments involving the sample mean k_1. For Fisher's (1928) infinite case, a very simple rule was found, which gave for example $\chi(r1^s) = \frac{\chi_{r+s}}{n^s}$. The relationship is not so simple for the finite case. Formulae of the type $M(p1^r)$, $M(pq1^r)$, $M(p2^2 1^r)$ are sufficient for writing all the finite formulae through weight 8. These are now obtained.

Dwyer (1962) has shown, using deviates from the mean, that

$M(1^r) = E_N(k_1^r)$, assuming $K_1 = 0$

$= E_N\left[\sum \frac{(r_1, \ldots, r_\rho)}{n^{r-\rho}} k_{r_1, \ldots, r_\rho}\right]$, where the summation is over all ρ-part partitions $r_1, \ldots, r_\rho$ ($\rho = 1, \ldots, r$) of r

(5.13) $$= \sum (r_1 \ldots, r_\tau) \alpha_{r_1 - 1} \cdots \alpha_{r_\tau - 1} K_{r_1, \ldots, r_\tau},$$

where the summation is over all non-unitary partitions $r_1, \ldots, r_\tau$ of r (i.e. partitions not involving unit parts) and

$$\alpha_{r-1} = \sum_{j=0}^{r-2} (-1)^j \binom{r}{j} \left(\frac{1}{n}\right)^{r-j-1} \left(\frac{1}{N}\right)^j + \left(-\frac{1}{N}\right)^{r-1} (r-1) .$$

Barton and David (1961) have also shown that

(5.14) $$M(1^r) = \sum \frac{r!}{(p_1!)^{\pi_1} \cdots (p_s!)^{\pi_s} \, \pi_1! \cdots \pi_s!} A_{p_1}^{\pi_1} \cdots A_{p_s}^{\pi_s} K_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \text{*}$$

where the summation is over all partitions $p_1^{\pi_1} \ldots p_s^{\pi_s}$ of r excluding those with unit parts and

$$A_r = \sum_{i=1}^{r-1} \alpha^i \left(-\frac{1}{n}\right)^{r-i-1}, \quad r \geq 2 \qquad \left(\alpha = \frac{1}{n} - \frac{1}{n}\right)$$

* Their result erroneously contains a $(\pi - 1)!$ in the numerator.

as defined by Abdel-Aty (1954). It may be noted that $A_r = \alpha_{r-1}$ in Dwyer's (1962) notation.

Wishart's (1952) formulae for $M(pl^2)$, $M(pl^3)$, $M(pl^4)$ have also been generalized by Dwyer (1962) for $M(pl^r)$. He shows, using (4.3), that

$$\begin{aligned} M(pl^r) &= E_N(k_p k_1^r) - K_p E_N(k_1^r) \\ &= \sum_{I=1}^{r} \binom{r}{I} \alpha^I \sum (r_1', \ldots, r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{p+I, r_1' \ldots r_{\tau'}'} \\ &\quad - \sum (r_1 \ldots r_\tau) \alpha_{r_1-1} \cdots \alpha_{r_\tau-1} \left[K_p K_{r_1 \ldots r_\tau} - K_{pr_1 \ldots r_\tau} \right], \end{aligned} \tag{5.15}$$

where the second summation in the first term is over all non-unitary partitions $r_1', \ldots, r_{\tau'}'$ of $r' = r-I$ and the summation in the second term is over all non-unitary partitions $r_1, \ldots, r_\tau$ of r. Although (5.15) appears formidable, it is easy to apply for small r. For example,

$$\begin{aligned} M(pl^4) &= \alpha^4 K_{p+4} + 6\alpha^3 K_{p+2,2} + 4\alpha\alpha_2 K_{p+1,3} - \alpha_3 \left[K_p K_4 - K_{p4} \right] \\ &\quad - 3\alpha^2 \left[K_p K_{22} - K_{p22} \right] \end{aligned}$$

which agrees with Wishart (1952, p.9).

Also, with the help of (5.6), we find

$$\begin{aligned} M(pql^r) &= \sum_I \sum_{nup\,r'} \sum \sum_{\rho=0}^{q} \binom{r}{I} \alpha^I \rho(n)(p_1 \ldots p_\rho)(q_1 \ldots q_\rho)(r_1' \ldots, r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{p_1 \ldots p_\rho + q_1 \ldots q_\rho + I, r_1' \ldots r_{\tau'}'} \\ &\quad - K_p \sum_I \binom{r}{I} \alpha^I \sum_{nup\,r'} (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{q+I, r_1' \ldots r_{\tau'}'} \\ &\quad - K_q \sum_I \binom{r}{I} \alpha^I \sum_{nup\,r'} (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{p+I, r_1' \ldots r_{\tau'}'} \\ &\quad + K_p K_q \sum_I (r_1 \ldots r_\tau) \alpha_{r_1-1} \cdots \alpha_{r_\tau-1} K_{r_1 \ldots r_\tau} \end{aligned} \tag{5.16}$$

where the summations $\sum_{nup\,r'}$ are over the non-unitary partitions of r', the third summation in the first term is over pairs of ρ-part partitions of p and q and $p_1 \ldots p_\rho + q_1 \ldots q_\rho + I$ means that the p_i are added to the q_j in all possible ways and the units of I are then added in all possible ways.

As a special case of (5.16), we obtain

$$(5.17) \quad M(pq1) = \alpha\Big[\Sigma \sum_{\rho=0}^{q} \rho\,(n)(p_1 \ldots p_\rho)(q_1 \ldots q_\rho)K_{p_1 \ldots p_\rho + q_1 \ldots q_\rho + 1} - K_p K_{q+1} - K_q K_{p+1}\Big]$$

where the first summation is over pairs of ρ-part partitions of p and q. For q = 2,

$$M(p21) = \alpha\Big[\rho(0)K_{p2+1} + \rho(1)K_{p+3} + \rho(2)\Sigma(p_1p_2)K_{p_1p_2+11+1} - K_pK_3 - K_2K_{p+1}\Big]$$

$$(5.18) \qquad = \alpha\Big[K_{p+1,2} + K_{p3} + \frac{1}{n}K_{p+3} + \frac{2}{n-1}\Sigma(p_1p_2)K_{p_1p_2+21} - K_pK_3 - K_{p+1}K_2\Big].$$

We note that (5.18) checks with Wishart's (1952) result when the latter is appropriately expanded.

Also, by putting r = 2 in (5.16),

$$M(pq1^2) = \alpha^2 \Sigma \sum_{\rho=0}^{q} \rho\,(n)(p_1 \ldots p_\rho)(q_1 \ldots q_\rho)K_{p_1 \ldots p_\rho + q_1 \ldots q_\rho + 2} \overset{*}{+} \alpha \Sigma \sum_{\rho=0}^{q} \rho(n)(p_1 \ldots p_\rho)(q_1 \ldots q_\rho)$$

$$(5.19) \qquad K_{p_1 \ldots p_\rho + q_1 \ldots q_\rho, 2} - K_p\{\alpha^2 K_{q+2} + \alpha K_{q2}\} - K_q\{\alpha^2 K_{p+2} + \alpha K_{p2}\} + \alpha K_pK_qK_2,$$

where the first summations in the first two terms are over pairs of ρ-part partitions of p and q. For q = 2 in (5.19),

$$M(p21^2) = \alpha^2\Big\{K_{p4} - K_pK_4 + K_{p+2,2} - K_{p+2}K_2 + 2K_{p+1,3} + \frac{1}{n}K_{p+4}$$

$$+ \frac{2}{n-1}\Sigma(p_1p_2)K_{p_1p_2+31} + \frac{4}{n-1}\Sigma(p_1p_2)K_{p_1+2,p_2+2}\Big\}$$

$$(5.20) \qquad + \alpha\Big\{K_{p22} - K_pK_{22} - K_{p2}K_2 + K_pK_2^2 + \frac{1}{n}K_{p+2,2} + \frac{2}{n-1}\Sigma(p_1p_2)K_{p_1+1,p_2+1,2}\Big\}.$$

---

* Includes $K_{p_1 \ldots p_\rho + q_1 \ldots q_\rho + 11}$ since $\{\ \} + 2 = \{\{\ \} + 1\} + 1 = \{2\} + \{11\}$.

It is useful to have on record as a special case of (5.16), in order to obtain moment formulae through weight 8,

$$M(p21^r) = \sum_I \sum_{nup r'} \binom{r}{I} \alpha^I (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} \Big\{ K_{p2+I, r_1' \ldots r_{\tau'}'} + \frac{1}{n} K_{p+2+I, r_1' \ldots r_{\tau'}'}$$

$$+ \frac{1}{n-1} \sum (p_1 p_2) K_{p_1 p_2+11+I, r_1' \ldots r_{\tau'}'} \Big\} - K_p \sum_I \binom{r}{I} \alpha^I \sum_{nup r'} (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{2+I, r_1' \ldots r_{\tau'}'}$$

$$- K_2 \sum_I \binom{r}{I} \alpha^I \sum_{nup r'} (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{p+I, r_1' \ldots r_{\tau'}'}$$

$$(5.21) \qquad + K_p K_2 \sum_I (r_1 \ldots r_\tau) \alpha_{r_1-1} \cdots \alpha_{r_\tau-1} K_{r_1 \ldots r_\tau} .$$

Thus,

$$M(221^r) = \sum_I \sum_{nup r'} \binom{r}{I} \alpha^I (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} \Big\{ K_{22+I, r_1' \ldots r_{\tau'}'} + \frac{1}{n} K_{4+I, r_1' \ldots r_{\tau'}'} +$$

$$\frac{1}{n-1} K_{11+11+I, r_1' \ldots r_{\tau'}'} \Big\} - 2 K_2 \sum_I \binom{r}{I} \alpha^I \sum_{nup r'} (r_1' \ldots r_{\tau'}') \alpha_{r_1'-1} \cdots \alpha_{r_{\tau'}'-1} K_{2+I, r_1' \ldots r_{\tau'}'}$$

$$(5.22) \qquad + K_2^2 \sum_I (r_1 \ldots r_\tau) \alpha_{r_1-1} \cdots \alpha_{r_\tau-1} K_{r_1 \ldots r_\tau} .$$

This gives

$$M(221^2) = \alpha^2 \Big\{ 2K_{42} + 2K_{33} + \frac{1}{n} K_6 + \frac{4}{n-1} K_{42} + \frac{4}{n-1} K_{33} \Big\} + \alpha \Big\{ K_{222} + \frac{1}{n} K_{42}$$

$$(5.23) \qquad + \frac{2}{n-1} K_{222} \Big\} - 2K_2 \Big\{ \alpha^2 K_4 + \alpha K_{22} \Big\} + \alpha K_2^3$$

which checks with Wishart's (1952, p.9) result.

Most moment formulae through weight 8 can be written with the help of general results obtained above. For example, for weight 7, (5.22) gives

$$M(2^2 1^3) = \alpha^3 \Big\{ 2K_{52} + 6K_{43} + \frac{1}{n} K_7 + \frac{4}{n-1} K_{52} + \frac{12}{n-1} K_{43} \Big\} + 3\alpha^2 \Big\{ 2K_{322} + \frac{1}{n} K_{52}$$

$$+ \frac{4}{n-1} K_{322} \Big\} + \alpha_2 \Big\{ K_{322} + \frac{1}{n} K_{43} + \frac{2}{n-1} K_{322} \Big\} - 2K_2 \Big\{ \alpha^3 K_5 + 3\alpha^2 K_{32}$$

$$(5.24) \qquad + \alpha_2 K_{32} \Big\} + \alpha K_2^3 + \alpha_2 K_3 K_2^2 .$$

Another formula needed for writing all M(...) through weight 8 is

$$M(p221^r) = \sum_I \binom{r}{I} \alpha^I \sum_{nu\,p\,r'} (r'_1 \cdots r'_{\tau'}) \alpha_{r'_1-1} \cdots \alpha_{r'_{\tau'}-1} \Big\{ K_{p22+I, r'_1 \cdots r'_{\tau'}} + \frac{1}{n} K_{p+2, 2+I, r'_1 \cdots r'_{\tau'}} + \cdots \Big\}$$

$$- \sum_I \binom{r}{I} \alpha^I \sum_{nu\,p\,r'} (r'_1 \cdots r'_{\tau'}) \alpha_{r'_1-1} \cdots \alpha_{r'_{\tau'}-1} K_p \Big\{ \frac{1}{n} K_{4+I} + \frac{n+1}{n-1} K_{22+I} \Big\}$$

$$-2 \sum_I \binom{r}{I} \alpha^I \sum_{nu\,p\,r'} (r'_1 \cdots r'_{\tau'}) \alpha_{r'_1-1} \cdots \alpha_{r'_{\tau'}-1} K_2 \Big\{ K_{p2+I} + \frac{1}{n} K_{p+2+I}$$

(5.25) $$+ \frac{1}{n-1} \sum (p_1 p_2) K_{p_1 p_2 + 11 + I} \Big\} .$$

This gives

$$M(p221) = \alpha \Big[K_{p+1,2,2} + 2K_{p32} + \frac{1}{n} K_{p+3,2} + \frac{1}{n} K_{p+2,3} + \frac{1}{n} K_{p+1,4} + \frac{1}{n} K_{p5}$$

$$- \frac{2}{n(n-1)} K_{p+2,3} - \frac{2}{n(n-1)} K_{p+1,4} + \frac{2}{n-1} \sum (p_1 p_2) \Big\{ K_{p_1+2, p_2+1, 2} +$$

$$K_{p_1+1, p_2+2, 2} + K_{p_1+1, p_2+1, 3} \Big\} + \frac{2}{n-1} K_{p+1,2,2} + \frac{4}{n-1} K_{p32} + \frac{1}{n} K_{p+3,2} + \frac{1}{n} K_{p+2,3}$$

$$+ \frac{1}{n^2} K_{p+5} + \frac{2}{n(n-1)} \sum \Big((p+2)_1 , (p+2)_2 \Big) \Big\{ K_{(p+2)_1+2, (p+2)_2+1} +$$

$$K_{(p+2)_1+1, (p+2)_2+2} \Big\} + \frac{2}{n-1} \sum (p_1 p_2) \Big\{ K_{p_1+2, p_2+1, 2} + K_{p_1+1, p_2+2, 2} +$$

$$K_{p_1+1, p_2+1, 3} \Big\} + \frac{2}{n(n-1)} \sum (p_1 p_2) \Big\{ K_{p_1+4, p_2+1} + K_{p_1+3, p_2+2} \Big\} + \frac{2}{n(n-1)}$$

$$\sum (p_1 p_2) \Big\{ K_{p_1+2, p_2+3} + K_{p_1+1, p_2+4} \Big\} - \frac{4}{n(n-1)^2} \sum (p_1 p_2) \Big\{ K_{p_1+3, p_2+2} +$$

$$K_{p_1+2, p_2+3} \Big\} + \frac{4}{(n-1)^2} \sum (p_1 p_2) \Big((p_1+1)_1 , (p_1+1)_2 \Big) \Big\{ K_{(p_1+1)_1+2, (p_1+1)_2+1, p_2+1} +$$

$$K_{(p_1+1)_1+1, (p_1+1)_2+2, p_2+1} + K_{(p_1+1)_1+1, (p_1+1)_2+1, p_2+2} \Big\} + \frac{4}{(n-1)^2} \sum (p_1 p_2)$$

$$\left((p_2+1)_1,(p_2+1)_2\right)\Big\{ K_{p_1+2,(p_2+1)_1+1,(p_2+1)_2+1} + K_{p_1+1,(p_2+1)_1+2,(p_2+1)_2+1} +$$

$$K_{p_1+1,(p_2+1)_1+1,(p_2+1)_2+2}\Big\} - \frac{1}{n} K_pK_5 - \frac{2(n+1)}{n-1} K_pK_{32} - 2K_2K_{p+1,2} - 2K_2K_{p3}$$

$$- \frac{2}{n} K_2K_{p+3} - \frac{4}{n-1} K_2 \sum(p_1p_2) \Big\{ K_{p_1+2,p_2+1} + K_{p_1+1,p_2+2}\Big\}\Big] . \tag{5.26}$$

Hence,

$$M(3221) = \alpha\Big\{(1 + \frac{14}{n-1} + \frac{48}{(n-1)^2})(K_{422} + 2K_{332}) + (\frac{2}{n} + \frac{16}{n(n-1)})(K_{62} + K_{53}) +$$

$$(\frac{1}{n} + \frac{24}{n(n-1)} - \frac{12}{n(n-1)^2})(K_{53} + K_{44}) + \frac{1}{n^2} K_8 - (\frac{1}{n}K_5 + \frac{2(n+1)}{n-1}K_{32})K_3$$

$$- (2 + \frac{12}{n-1})(K_{42} + K_{33})K_2 - \frac{2}{n} K_6K_2 \Big\} . \tag{5.27}$$

<u>Recursive Approach to Formulae involving the Mean using Substitution Products</u>

As observed by Irwin and Kendall (1944),

$$M(r1) = \alpha K_{r+1} .$$

In terms of substitution products, however,

$$M(r1) = \left[K_rK_1\right]_n - K_rK_1 . \tag{5.28}$$

Proceeding further, we find

$$M(r1^2) = \left[K_rK_1^2\right]_n - 2K_1\left[K_rK_1\right]_n - K_r\left[K_1^2\right]_n + 2K_rK_1^2 , \tag{5.29}$$

and in general,

$$M(r1^s) = E_N\,(k_r - K_r)\Big(\sum_{t=0}^{s} (-1)^t \binom{s}{t} k_1^{s-t} K_1^t\Big)$$

$$= \sum_{t=0}^{s-1} (-1)^t \binom{s}{t} K_1^t \left[K_r K_1^{s-t} \right]_n - \sum_{t=0}^{s-2} (-1)^t \binom{s}{t} K_r K_1^t \left[K_1^{s-t} \right]_n$$

(5.30) $$+ (-1)^s s \, K_r K_1^s .$$

Also, from (5.11),

(5.31) $$M(pq1) = \left[K_p K_q K_1\right]_n - K_p \left[K_q K_1\right]_n - K_q \left[K_p K_1\right]_n - K_1 \left[K_p K_q\right]_n + 2K_p K_q K_1 .$$

And, in general,

$$M(pq1^s) = E_N (k_p - K_p)(k_q - K_q)\left(\sum_{t=0}^{s} (-1)^t \binom{s}{t} k_1^{s-t} K_1^{t}\right)$$

$$= \sum_{t=0}^{s} (-1)^t \binom{s}{t} K_1^t \left[K_p K_q K_1^{s-t}\right]_n - \sum_{t=0}^{s-1} (-1)^t \binom{s}{t} K_p K_1^{t} \left[K_q K_1^{s-t}\right]_n$$

$$- \sum_{t=0}^{s-1} (-1)^t \binom{s}{t} K_q K_1^t \left[K_p K_1^{s-t}\right]_n + \sum_{t=0}^{s-2} (-1)^t \binom{s}{t} K_p K_q K_1^{t} \left[K_1^{s-t}\right]_n$$

(5.32) $$+ (-1)^{s+1} (s+1) K_p K_q K_1^{s} .$$

From (5.9),

(5.33) $$M(p_1 p_2 \ldots p_u p) = \sum_{I \subset P} (-1)^{\bar{I}} \prod_{p_i \in I} K_{p_i} \left[K_p \prod_{p_{i'} \in P-I} K_{p_{i'}} \right]_n - K_p M(p_1 p_2 \ldots p_u) .$$

Hence a recursion formula is

(5.34) $$M(p_1 p_2 \ldots p_u 1) = \sum_{I \subset P} (-1)^{\bar{I}} \prod_{p_i \in I} K_{p_i} \left[K_1 \prod_{p_{i'} \in P-I} K_{p_{i'}} \right]_n - K_1 \, M(p_1 p_2 \ldots p_u) .$$

This can be successively applied to get expression (5.28) to (5.32).

As an example, we consider

$$M(32) = E_N (k_3 - K_3)(k_2 - K_2)$$

$$= \left[K_3 K_2\right]_n - K_3 \left[K_2\right]_n - K_2 \left[K_3\right]_n + K_3 K_2 ^*$$

$$= \frac{1}{n} K_5 + \frac{n+5}{n-1} K_{32} - K_3 K_2 - K_2 K_3 + K_3 K_2$$

---

* For recursion, we use M(32) in this extended form, obtained in derivation, and not as $\left[K_3 K_2\right]_n - K_3 K_2$.

Hence,

$$M(321) = \frac{1}{n}\left\{\frac{K_6}{n}+K_{51}\right\} + \frac{n+5}{n-1}\left\{\frac{K_{42}}{n}+\frac{K_{33}}{n}+K_{321}\right\} - K_3\left\{\frac{K_3}{n}+K_{21}\right\}$$

$$-K_2\left\{\frac{K_4}{n}+K_{31}\right\} + K_3K_2K_1 - K_1M(32)$$

$$= \frac{K_6}{n^2} + \frac{K_{51}}{n} + \frac{n+5}{n(n-1)}K_{42} + \frac{n+5}{n(n-1)}K_{33} + \frac{n+5}{n-1}K_{321} - \frac{K_3^2}{n} - \frac{K_2K_4}{n}$$

$$-\frac{K_1K_5}{n} - K_3K_{21} - K_2K_{31} - \frac{n+5}{n-1}K_1K_{32} + 2K_3K_2K_1$$

and

$$M(3211) = \frac{1}{n^2}\left\{\frac{K_7}{n}+K_{61}\right\} + \frac{1}{n}\left\{\frac{K_{61}}{n}+\frac{K_{52}}{n}+K_{511}\right\} + \frac{n+5}{n(n-1)}\left\{\frac{K_{52}}{n}+\frac{K_{43}}{n}+K_{421}\right\}$$

$$+\frac{n+5}{n(n-1)}\left\{\frac{2K_{43}}{n}+K_{331}\right\} + \frac{n+5}{n-1}\left\{\frac{K_{421}}{n}+\frac{K_{331}}{n}+\frac{K_{322}}{n}+K_{3211}\right\}$$

$$-\frac{K_3}{n}\left\{\frac{K_4}{n}+K_{31}\right\} - K_3\left\{\frac{K_{31}}{n}+\frac{K_{22}}{n}+K_{211}\right\} - \frac{K_2}{n}\left\{\frac{K_5}{n}+K_{41}\right\}$$

$$-K_2\left\{\frac{K_{41}}{n}+\frac{K_{32}}{n}+K_{311}\right\} - \frac{K_1}{n}\left\{\frac{K_6}{n}+K_{51}\right\} - \frac{n+5}{n-1}K_1\left\{\frac{K_{42}}{n}+\frac{K_{33}}{n}+K_{321}\right\}$$

$$+K_1K_2\left\{\frac{K_4}{n}+K_{31}\right\} + K_2K_3\left\{\frac{K_2}{n}+K_{11}\right\} + K_3K_1\left\{\frac{K_3}{n}+K_{21}\right\}$$

$$-K_3K_2K_1^2 - K_1\,M(321).$$

<u>Estimators</u> <u>of</u> <u>Moments</u>

Once the moment formulae are expressed as linear functions of the K..., their estimators can be readily obtained by replacing K... by k..., since k... is an unbiased estimator of K... . For example, from (5.1)

$$(5.35) \qquad \hat{M}(2^2) = \left(\frac{1}{n} - \frac{1}{N}\right)k_4 + \left(\frac{2}{n-1} - \frac{2}{N-1}\right)k_{22}.$$

Symbolically, since

$$(5.2)\qquad M(2^2) = [K_2^2]_n - [K_2^2]_N = [K_2^2]_n - K_2^2 = \left\{\frac{K_4}{n} + \frac{n+1}{n-1}\ K_{22}\right\} - K_2^2\ ,$$

we have

$$(5.36)\qquad \hat{M}(2^2) = [k_2^2]_n - [k_2^2]_N = k_2^2 - [k_2^2]_N = k_2^2 - \left\{\frac{k_4}{N} + \frac{N+1}{N-1}\ k_{22}\right\}$$

The transformation from (5.2) to (5.36) can be described as changing the sign of the expression for $M(2^2)$ and replacing K's by k's and n by N to get $\hat{M}(2^2)$. By a similar operation on the variance-covariance formulae (5.3), (5.4), we get

$$(5.37)\qquad \hat{M}(r^2) = k_r^2 - [k_r^2]_N$$

$$(5.38)\qquad \hat{M}(rs) = k_{rs} - [k_r k_s]_N\ .$$

The use of substitution products allows us to combine the computation and estimation properties in the same expression. Thus, either of (5.2), (5.36) can be used to represent both formulae if we note the transformation described above. To take a few more examples, we can write from (5.5),

$$(5.39)\qquad \hat{M}(r^a) = \sum_{s=0}^{a-2} (-1)^s \binom{a}{s} \left[[k_r^{\ a-s}]_n\ k_r^{\ s}\right]_N + (-1)^{a-1}(a-1)\left[k_r^{\ a}\right]_N .$$

Also, from (5.9), (5.10),

$$(5.40)\qquad \hat{M}(p_1 \dots p_u) = \sum_{I \subset P} (-1)^{\bar{I}} \left[\prod_{p_i \in I} k_{p_i} \left[\prod_{p_{i'} \in P-I} k_{p_{i'}}\right]_n\right]_N$$

$$(5.41)\qquad = \sum (-1)^h \left[k_{p_{i_1}} \dots k_{p_{i_h}} \left[k_{p_1} \dots) k_{p_{i_1}} (\dots) k_{p_{i_h}} (\dots k_{p_u}\right]_n\right]_N$$

in the earlier notation. It should be noted that though the N could be dropped in (5.9), (5.10), since $[K_r K_s \ldots]_N = K_r K_s \ldots$, we can not drop it in (5.40), (5.41). Thus, whereas the finite moment formulae can be written compactly in terms of simple substitution products $[K_{\ldots} K_{\ldots} \ldots]_N$ which are really products $K_{\ldots} K_{\ldots} \ldots$ giving the results of Table 8, the compact form of the estimation formulae is more complicated as it involves complex substitution products like $\left[[k_r k_s \ldots]_n k_p^a\right]_N$ which, of course, can be transformed into simple substitution products by using formulae for $k_r k_s \ldots$.

Estimators of moments and cumulants not involving the sample mean are given in Table 8. (Results for $\hat{M}(\ldots)$ for weight ≤ 8 have been given by Schaeffer and Dwyer (1963)). We pass over the first column and the beginning rows which have a zero under the "estimator" column. The formulae for the estimators of moment functions when we read $\hat{M}(\ldots)$, $\hat{K}(\ldots)$ for $M(\ldots)$, $K(\ldots)$ at the head of the columns are available in terms of substitution products $[k_{\ldots} k_{\ldots} \ldots]_N$. The suffix N has been dropped in Table 8. The relationship between the expectation and estimation formulae is illustrated by the following example. For M(522), we have

$$M(522) = [K_5 K_2^2]_n - \frac{2}{n} K_7 K_2 - \frac{2(n+9)}{n-1} K_{52} K_2 - \frac{40}{n-1} K_{43} K_2$$

$$(5.42) \qquad - \frac{1}{n} K_4 K_5 - \frac{n+1}{n-1} K_{22} K_5 + 2 K_5 K_2^2$$

(in Table 8.6, the expansion of $K_5 K_2^2$ is written using Table 7.1). Now for the estimator,

$$\hat{M}(522) = k_5 k_2^2 - \frac{2}{n} [k_7 k_2]_N - \frac{2(n+9)}{n-1} [k_{52} k_2]_N - \frac{40}{n-1} [k_{43} k_2]_N$$

$$(5.43) \qquad - \frac{1}{n} [k_4 k_5]_N - \frac{n+1}{n-1} [k_{22} k_5]_N + 2 [k_5 k_2^2]_N .$$

Here, the first term $k_5 k_2^2$ need not be expanded and appears with a multiplier 1 in the "estimator" column. Thus the expectation formulae, with the use of substitution products, give estimation formulae. For weight ≤ 8, Schaeffer and Dwyer (1963) have used this fact in constructing a computer program which can be used either for expectation or estimation.

Estimators of Moment Functions involving the Mean

For estimators of moments and cumulants involving the mean k_1, a similar tranformation of substitution products is made. For example, from (5.13),

$$\hat{M}(1^r) = \sum (r_1 \ldots r_\tau) \alpha_{r_1 - 1} \ldots \alpha_{r_\tau - 1} \; k_{r_1 \ldots r_\tau} \,, \tag{5.44}$$

from (5.29),

$$\begin{aligned} \hat{M}(r1^2) &= k_r k_1^2 - 2\left[k_1 [k_r k_1]_n \right]_N - \left[k_r \left[k_1^2 \right]_n \right]_N + 2\left[k_r k_1^2 \right]_N \\ &= k_r k_1^2 - \frac{2}{n}\left[\frac{k_{r+2}}{N} + k_{r+1,1} \right] - 2\left[\frac{k_{r+1,1}}{N} + \frac{k_{r2}}{N} + k_{r11} \right] \\ &\quad - \frac{1}{n}\left[k_r k_2 \right]_N - \left[k_r k_{11} \right]_N + 2\left[k_r k_1^2 \right]_N \,, \end{aligned} \tag{5.45}$$

and from (5.32),

$$\begin{aligned} \hat{M}(pq1^s) &= \sum_{t=0}^{s} (-1)^t \binom{s}{t} \left[k_1^{\,t} \left[k_p k_q k_1^{\,s-t} \right]_n \right]_N - \sum_{t=0}^{s-1} (-1)^t \binom{s}{t} \left[k_p k_1^{\,t} \left[k_q k_1^{\,s-t} \right]_n \right]_N \\ &\quad - \sum_{t=0}^{s-1} (-1)^t \binom{s}{t} \left[k_q k_1^{\,t} \left[k_p k_1^{\,s-t} \right]_n \right]_N + \sum_{t=0}^{s-2} (-1)^t \binom{s}{t} \left[k_p k_q k_1^{\,t} \left[k_1^{\,s-t} \right]_n \right]_N \\ &\quad + (-1)^{s+1} (s+1) \left[k_p k_q k_1^{\,s} \right]_N \,. \end{aligned} \tag{5.46}$$

Infinite Populations

For the infinite case, we denote the moments by $\mu(\ldots)$ and their estimators by $\hat{\mu}(\ldots)$ and similarly for cumulants use $\kappa(\ldots)$. From the variance-covariance formulae (5.3),(5.4), we get

$$(5.47)\qquad \mu(r^2) = [\kappa_r{}^2]_n - \kappa_r{}^2$$

$$(5.48)\qquad \mu(rs) = [\kappa_r\,\kappa_s]_n - \kappa_r\kappa_s$$

where $[\kappa_r\kappa_s]_n = \lim_{N\to\infty} [K_rK_s]_n$ (Schaeffer and Dwyer, 1963).

Taking estimates and noting that $\lim_{N\to\infty} [k_rk_s]_N = k_{rs}$, we obtain the formulae for estimators,

$$(5.49)\qquad \hat{\mu}(r^2) = k_r{}^2 - k_{rr}$$

$$(5.50)\qquad \hat{\mu}(rs) = k_rk_s - k_{rs}$$

as given by Glasser (1962) and Schaeffer and Dwyer (1963).

The M(...) and K(...) formulae of Table 8 give Fisher's (1928) formulae as $N\to\infty$. For example, from Table 8.6,

$$M(522) = K(522) = \frac{1}{n^2} K_9 + \frac{2(n\ 11)}{n(n-1)} K_{72} + \frac{20(3n-4)}{n(n-1)^2} K_{63} + \frac{n^2+98n-139}{n(n-1)^2} K_{54} +$$

$$\frac{(n+9)(n+11)}{(n-1)^2} K_{522} + \frac{40(n+11)}{(n-1)^2} K_{432} + \frac{120}{(n-1)^2} K_{333} - \frac{2}{n} K_7K_2$$

$$- \frac{1}{n} K_5K_4 - \frac{2(n+9)}{n-1} K_{52}K_2 - \frac{40}{n-1} K_{43}K_2 - \frac{n+1}{n-1} K_5K_{22} +$$

$$(5.51)\qquad 2\,K_5K_2^2 .$$

Hence, since $\lim_{N\to\infty} K_{rs} = \kappa_r\kappa_s$,

$$\mu(522) = \kappa(522) = \frac{1}{n^2}\,\kappa_9 + \left\{\frac{2(n+11)}{n(n-1)} - \frac{2}{n}\right\}\kappa_7\kappa_2 + \frac{20(3n-4)}{n(n-1)^2}\,\kappa_6\kappa_3 +$$

$$\left\{\frac{n^2+98n-139}{n(n-1)^2} - \frac{1}{n}\right\} \kappa_5\kappa_4 + \left\{\frac{(n+9)(n+11)}{(n-1)^2} - \frac{2(n+9)}{n-1} - \frac{n+1}{n-1} + 2\right\}\kappa_5\kappa_2^2 +$$

$$\left\{\frac{40(n+11)}{(n-1)^2} - \frac{40}{n-1}\right\}\kappa_4\kappa_3\kappa_2 + \frac{120}{(n-1)^2} \kappa_3^3$$

$$= \frac{1}{n^2} \kappa_9 + \frac{24}{n(n-1)} \kappa_7\kappa_2 + \frac{20(3n-4)}{n(n-1)^2} \kappa_6\kappa_3 + \frac{20(5n-7)}{n(n-1)^2} \kappa_5\kappa_4 +$$

(5.52) $$\frac{120}{(n-1)^2} \kappa_5\kappa_2^2 + \frac{480}{(n-1)^2} \kappa_4\kappa_3\kappa_2 + \frac{120}{(n-1)^2} \kappa_3^3 ,$$

which is formula (18) given by Fisher (1928, pp. 210-211) and provides a useful check.

Also from the formula for $\hat{M}(522)$ in Table 8.6, we can obtain an estimator of $\mu(522)$, using $\lim_{N\to\infty} [k_r k_s]_N = k_{rs}$. Thus from (5.43), we have

$$\hat{\mu}(522) = k_5k_2^2 - \frac{2}{n} k_{72} - \frac{2(n+9)}{n-1} k_{522} - \frac{40}{n-1} k_{432} - \frac{1}{n} k_{54} - \frac{n+1}{n-1} k_{522} + 2k_{522}$$

$$= \frac{1}{n^2} k_9 + \frac{24}{n(n-1)} k_{72} + \frac{20(3n-4)}{n(n-1)^2} k_{63} + \frac{20(5n-7)}{n(n-1)^2} k_{54} +$$

(5.53) $$\frac{120}{(n-1)^2} k_{522} + \frac{480}{(n-1)^2} k_{432} + \frac{120}{(n-1)^2} k_{333} ,$$

using the expansion of $k_5k_2^2$ from Table 7.1. The relationship between (5.52) and (5.53) is immediately visible, since $E(k_{rs\ldots}) = \kappa_r\kappa_s\ldots$.

Expressions for $\mu(\ldots)$, $\hat{\mu}(\ldots)$ can similarly be obtained from all formulae above for $M(\ldots)$, $\hat{M}(\ldots)$. Thus, from (5.10), (5.41),

(5.54) $$\mu(p_1 \ldots p_u) = (-1)^h \kappa_{p_{i_1}} \ldots \kappa_{p_{i_h}} \left[\kappa_{p_1} \ldots)\kappa_{p_{i_1}} (\ldots)\kappa_{p_{i_h}} (\ldots \kappa_{p_u}\right]_n$$

(5.55) $$\hat{\mu}(p_1 \ldots p_u) = (-1)^h \left[k_{p_1} \ldots)k_{p_{i_1}} (\ldots)k_{p_{i_h}} (\ldots k_{p_u} \cdot p_{i_1} \ldots p_{i_h}\right]_n ,$$

where $\left[k_{p_1} \ldots k_{p_u} \cdot p_{i_1} \ldots p_{i_h}\right]_n$ is the expansion of $\left[k_{p_1} \ldots k_{p_u}\right]_n$ with subscript $p_{i_1} \ldots p_{i_h}$ added to each term, extending the notation of

Schaeffer and Dwyer (1963). Thus, from (5.55),

$$\hat{\mu}(rst) = k_r k_s k_t - \left[k_r k_s \cdot t\right]_n - \left[k_s k_t \cdot r\right]_n - \left[k_r k_t \cdot s\right]_n$$

$$+ \left[k_r \cdot st\right]_n + \left[k_s \cdot rt\right]_n + \left[k_t \cdot rs\right]_n - k_{rst}$$

(5.56) $$= k_r k_s k_t - \left[k_r k_s \cdot t\right]_n - \left[k_s k_t \cdot r\right]_n - \left[k_r k_t \cdot s\right]_n + 2k_{rst}$$

and so

$$\hat{\mu}(332) = k_3^2 k_2 - \left[k_3^2 \cdot 2\right]_n - 2\left[k_3 k_2 \cdot 3\right]_n + 2\, k_{332}$$

$$= k_3^2 k_2 - \left\{\left(\frac{1}{n} k_6 + \frac{9}{n-1} k_{42} + \frac{n+8}{n-1} k_{33} + \frac{6n}{(n-1)(n-2)} k_{222}\right) \cdot 2\right\}$$

$$- 2\left\{\left(\frac{1}{n} k_5 + \frac{n+5}{n-1} k_{32}\right) \cdot 3\right\} + 2\, k_{332}$$

$$= k_3^2 k_2 - \left\{\frac{1}{n} k_{62} + \frac{9}{n-1} k_{422} + \frac{n+8}{n-1} k_{332} + \frac{6n}{(n-1)(n-2)} k_{2222}\right\}$$

(5.57) $$-2\left\{\frac{1}{n} k_{53} + \frac{n+5}{n-1} k_{332}\right\} + 2\, k_{332} .$$

Thus the finite moment formulae and their estimators are directly applicable to infinite populations, though a number of terms need collection to provide the infinite formulae.

SUMMARY

The aim of the work was to generalize Fisher's combinatorial technique to write products of generalized k-statistics and to use these to obtain moments of moments when sampling from a finite population.

The basic material was first reviewed, studying the sample symmetric functions and, in particular, the generalized k-statistics were defined in terms of partition coefficients and symmetric means. It was seen how their property of being unbiased estimators of products of parent cumulants for all distributions implies uniqueness and their seminvariance, when no subscript is unity, was established. Fisher's (1928) combinatorial approach to write products of single subscript k's and to obtain their cumulants was described. Tukey's (1956) algebraic method for products of two generalized k-statistics and its modification to a combinatorial method by Dwyer and Tracy (1962) were discussed with the use of an example. The steps needed to write semi-general formulae for products $k_{\{\}}\ k_{...}$ with the use of array types and distinct units were mentioned.

The rules of Dwyer and Tracy (1962) for products of two generalized k-statistics were shown to hold for multiple products. Four additional rules were stated and proved which are useful in determining the coefficients of array types involved in the use of combinatorial method. These coefficients were obtained for some general patterns and tabulated for many commonly occuring patterns, generalizing those given by Fisher (1928), who had a simpler situation in that all rows could be added for any pattern when dealing with single subscript k's.

A generalization of the combinatorial method was next used to write semi-general formulae for multiplication of $k_{\{\}}$ by products of $k_{...}$'s up

to weight 4. The rule of proper parts was very helpful in reducing the number of array types to be considered as only those contributing non-vanishing coefficients had to be retained. These formulae were then applied to write products of seminvariant generalized k-statistics of weights 9 and 10 and selected ones of weight 12. References for product formulae of lesser weight were given. Checks for formulae of both types were indicated.

Lastly, the product formulae were used in writing moment functions in terms of K..., adapted for computation. Formulae not involving the mean k_1 for M(...) and K(...), where it differed from M(...), were given for finite sampling through weight 10 and for special cases of weight 12 in tabular form. Methods for obtaining moment formulae involving k_1 from those not involving it were given and illustrated. Ways to express estimators of all these moments in terms of substitution products (Schaeffer and Dwyer, 1963) were given and estimators of all moment functions tabulated through weight 12 were also incorporated in the tables. Use in checking was made of the fact that as $N \to \infty$, the finite K(...) formulae give Fisher's (1928) cumulant formulae for the infinite case.

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