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On The Formulation and Solution of the E-Field Integral Equation

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1 Introduction

In this report the discretization and solution of the E-field integral equation is considered for scattering by open and closed resistive boundaries. Two specific forms of this integral equation are studied and their solutions are compared . One form of the integral equation involves the usual unknown surface current density and the other is in terms of the charge distribution.

Provided the charge and current expansions are chosen to satisfy the continuity equation, it is shown that the two integral equations result in identical systems. This is analytically verified and to do so it was necessary to employ linear weighting functions for both integral equations. The conclusion of this study is that by resorting to higher order basis for expansion and weighting, one can readily reduce the singularity of the kernel without a need to reformulate the integral equation.

2 Derivation of the Integral Equations

Let us consider the two dimensional surface shown in figure 1. This surface may be open and it is therefore a curved strip whose unit tangent and normal shall be denoted by $\hat{s}$ and $\hat{n}$, respectively. The strip has a normalized resistivity R and is illuminated by an H-polarized (TE) plane wave

$$\mathbf{H}^i = \hat{z}e^{jk_o(x\cos\phi_o+y\sin\phi_o)}$$

$$\mathbf{E}^i = -\frac{jZ_o}{k_o}\left(\hat{x}\frac{\partial H_z^i}{\partial y}-\hat{y}\frac{\partial H_z^i}{\partial x}\right) = Z_o\left(\hat{x}\sin\phi_o-\hat{y}\cos\phi_o\right)e^{jk_o(x\cos\phi_o+y\sin\phi_o)} \tag{1}$$

where k_o denotes the free space wavenumber and Z_o is the free space intrinsic impedance. To derive an integral equation for the current supported by the resistive strip, we recall the boundary condition

$$E_s = Z_o R J_s \tag{2}$$

where $E_s = \hat{s}\cdot\mathbf{E}$ is the s component of the total electric field at the surface of the resistive strip, J_s denotes the surface current density along the surface at (x, y) and R is the normalized resistivity of the strip also at (x, y). The total electric field is given by the sum of the incident and scattered field, i.e.

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \tag{3}$$

and since $\mathbf{E}^s$ is caused by the strip current we have

$$\begin{aligned}\mathbf{E}^s = & - \frac{k_o Z_o}{4} \int_C \hat{s}' J_{s'}(\rho') H_o^{(2)}(k_o r)\, ds' \\ & - \frac{Z_o}{4k_o} \int_C \hat{s}' J_{s'}(\rho') \cdot \nabla\nabla H_o^{(2)}(k_o r)\, ds' \end{aligned} \tag{4}$$

In this $H_o^{(2)}(\cdot)$ is the zeroth order Hankel function of the second kind and

$$r = |\rho - \rho'| = \sqrt{(x-x')^2 + (y-y')^2}$$

is the distance between the observation and integration points. Also, by refering to figure 2 we observe that $\hat{s}'$ is the unit tangent vector to the strip at the integration point (x', y').

From (4) we can now extract the s component of $\mathbf{E}^s$ to be used in (2). We have

$$\begin{aligned}E_s^s = \hat{s}\cdot\mathbf{E}^s = & - \frac{k_o Z_o}{4} \int_C J_{s'}(\rho') \left\{\hat{s}\cdot\hat{s}' H_o^{(2)}(k_o r)\right\} ds' \\ & - \frac{Z_o}{4k_o} \int_C J_{s'}(\rho') \left\{(\hat{s}'\cdot\nabla)(\hat{s}\cdot\nabla) H_o^{(2)}(k_o r)\right\} ds' \end{aligned} \tag{5}$$

and upon noting that $\hat{s}'\cdot\nabla = -\hat{s}'\cdot\nabla' = -\frac{\partial}{\partial s'}$, we can rewrite (5) as

$$\begin{aligned}E_s^s = -\frac{k_o Z_o}{4} & \left\{ \int_c J_{s'}(\rho') \left[\hat{s}\cdot\hat{s}' H_o^{(2)}(k_o r)\right] ds' \right. \\ & \left. - \frac{1}{k_o^2} \int_C J_s'(\rho') \frac{\partial}{\partial s'}\frac{\partial}{\partial s} H_o^{(2)}(k_o r) \right\} ds' \end{aligned} \tag{6}$$

Substituting this into (3) and then into (2) we obtain the integral equation

$$Y_o E_s^i(s) = R J_s(s) + \frac{k_o}{4}\int_C J_{s'}(s')\left\{(\hat{s}\cdot\hat{s}')\, H_o^{(2)}(k_o r)\right\} ds'$$
$$-\frac{1}{4k_o}\int_C J_{s'}(s')\left\{\frac{\partial}{\partial s'}\frac{\partial}{\partial s} H_o^{(2)}(k_o r)\right\} ds' \qquad (7)$$

where $Y_o = 1/Z_o$ and for convenience we have replaced $J_{s'}(\rho')$ by $J_{s'}(s')$in which s' denotes the distance along the strip. This is the standard E-field integral equation [1,2].

By applying integration by parts we can obtain other integral equations which are equivalent to (7) and more suitable for numerical implementation. Noting that $J_s(s)$ vanishes at the strip ends upon applying integration by parts to the second integral of (7) we obtain

$$Y_o E_s^i(s) = R J_s(s) \;+\; \frac{k_o}{4}\int_C J_{s'}(s')\left\{(\hat{s}\cdot\hat{s}')\, H_o^{(2)}(k_o r)\right\} ds'$$
$$+\; \frac{1}{4k_o}\int_C \frac{dJ_{s'}(s')}{ds'}\frac{\partial}{\partial s} H_o^{(2)}(k_o r)\, ds' \qquad (8)$$

To do the same for the first integral we first define the function

$$G_I(k_o r) = \int_a^{s'} (\hat{s}\cdot\hat{s}'')\, H_o^{(2)}(k_o r')\, ds'' \qquad (9)$$

where a is an arbitrary constant and

$$r' = \sqrt{(x - x'')^2 + (y - y'')^2}$$

with the integration being with respect to the double primed variables. Then $G_I(k_o r)$ satisfies the identity

$$\frac{dG_I(k_o r)}{ds'} = (\hat{s}\cdot\hat{s}')\, H_o^{(2)}(k_o r') \qquad (10)$$

permitting us to rewrite (8) as

$$Y_o E_s^i(s) = RJ_s(s) - \frac{k_o}{4}\int_C \phi(s')G_I(k_o r)ds'$$

$$+\frac{1}{4k_o}\int_C \phi(s')\frac{\partial}{\partial s}H_o^{(2)}(k_o r')\,ds' \tag{11}$$

in which

$$\phi(s) = \frac{dJ_s(s)}{ds} \tag{12}$$

is a quantity proportional to the electric charge on the strip. It can also be shown (see Appendix) that

$$J_s(s) = -\frac{1}{2}\int_C \left(\frac{L-2\delta}{L}\right)[\phi(s+\delta) - \phi(s-\delta)]\,d\delta \tag{13a}$$

where L denotes the length of the curved strip and

$$\phi(L-s) = \phi(s) \tag{13b}$$

should be used in evaluating the integral. Substituting this into (11) yields the integral equation

$$\begin{aligned} Y_o E_s^i(s) = & - \frac{1}{2}R(s)\int_C ds'\left(\frac{L-2s'}{L}\right)[\phi(s+s') - \phi(s-s')] \\ & - \frac{k_o}{4}\int_C \phi(s')G_I(k_o r)ds' \\ & + \frac{1}{4k_o}\int_C \phi(s')\frac{\partial}{\partial s}H_o^{(2)}(k_o r)ds' \end{aligned} \tag{14}$$

Clearly, (14) is solely in terms of the charge distribution on the strip whereas (8) is the tranditional E-field integral equation involves only the current density. Consequently we shall refer to (8) as the (E-field) current integral equation whereas (14) shall be referred to as the charge integral equation.

A standard procedure for solving (8) or (14) is the method of weighted residuals. In applying this technique the integral equation is multiplied by a testing functions $w_i(s)$ and then integrated over the domain of the weighting function. Doing so, (8) and (14) become,

$$\begin{aligned} Y_o \int_{C_i} E_s^i(s) w_i(s) ds &= \int_{C_i} R(s) J_s(s) w_i(s) ds \\ &+ \frac{k_o}{4} \int_{C_i} w_i(s) \int_C J_{s'}(s') \left\{ (\hat{s} \cdot \hat{s}') H_o^{(2)}(k_o r) \right\} ds' ds \\ &- \frac{1}{4k_o} \int_{C_i} \frac{dw_i(s)}{ds} \int_C \frac{dJ_{s'}(s')}{ds'} H_o^{(2)}(k_o r) ds' ds \qquad (15) \end{aligned}$$

$$\begin{aligned} Y_o \int_{C_i} E_s^i(s) w_i(s) ds = &- \frac{1}{2} \int_{C_i} R(s) w_i(s) \int_C \left(\frac{L - 2s'}{L} \right) [\phi(s+s') - \phi(s-s')] \, ds' ds \\ &- \frac{k_o}{4} \int_{C_i} w_i(s) \int_C \phi(s') G_I(k_o r) ds' ds \\ &- \frac{1}{4k_o} \int_{C_i} \frac{dw_i(s)}{ds} \int_C \phi(s') H_o^{(2)}(k_o r) ds' \qquad (16) \end{aligned}$$

for $i = 1, 2, ..., N$ such that $\sum_{i=1}^{N} C_i = C$. That is, C_i represents the i th segment of the discretized strip. Also, in obtaining the last integral in (15) and (16) we employed integration by parts. Thus, in their present form both integral equations are associated with kernels which have an integrable logarithmic singularity. In contrast the kernel for the original integral equation (6) has a non-integrable singularity which precludes its implementation in any rigorous mathematical fashion. Nevertheless, it has been implemented [1] using pulse basis expansion function with reasonably good success.

The implementation of (15) and (16) requires that we first expand the current/charge as

$$J_s(s) = \sum_{j=1}^{N} J_j L_j(s) \qquad (17)$$

$$\phi(s) = \sum_{j=1}^{N} \phi_j \psi_j(s) \qquad (18)$$

where $L_i(s)$ and $\psi_j(s)$ are subdomain basis functions to be chosen. That is they are non-zero over the segment C_j and vanish elsewhere on the surface of the strip. Substituting these expansions into (15) and (16) yields

$$\begin{aligned} Y_o \int_{C_i} E_s^i(s) w_i(s) ds &= \sum_{j=1}^{N} J_j \int_{C_i} R(s) L_j(s) w_i(s) ds \\ &+ \frac{k_o}{4} \sum_{j=1}^{N} J_j \int_{C_i} w_i(s) \int_{C_j} L_j(s') \left\{ (\hat{s} \cdot \hat{s}') H_o^{(2)}(k_o r) \right\} ds' ds \\ &- \frac{1}{4k_o} \sum_{j=1}^{N} J_j \int_{C_i} \frac{dw_i(s)}{ds} \int_{C_j} \frac{dL_j(s')}{ds'} H_o^{(2)}(k_o r) ds' ds \quad (19) \end{aligned}$$

$$\begin{aligned} Y_o \int_{C_i} E_s^i(s) w_i(s) ds &= -\frac{1}{2} \sum_{j=1}^{N} \int_{C_i} R(s) w_i(s) \int_{C_j} \frac{(L - 2s')}{L} \left[\phi(s + s') - \phi(s - s') \right] ds' ds \\ &- \frac{k_o}{4} \sum_{j=1}^{N} \phi_j \int_{C_i} w_i(s) \int_{C_j} \psi_j(s') G_I(k_o r) ds' ds \\ &- \frac{1}{4k_o} \sum_{j=1}^{N} \phi_j \int_{C_i} \frac{dw_i(s)}{ds} \int_{C_j} \psi_j(s') H_o^{(2)}(k_o r) ds' ds \quad (20) \end{aligned}$$

and it remains to specify the testing and expansion basis functions for a solution of the current/charge. Since $\phi(s)$ is equal to the derivative of the current $J_s(s)$, it is logical to choose $\psi_j(s)$ so that

$$\psi_j(s) = \frac{dL_j(s)}{ds} \quad (21)$$

That is, if $L_j(s)$ is chosen to be the triangle function $\Lambda_j(s)$ as shown in figure 3, then in accordance with (21), $\psi_j(s)$ becomes the doublet function $D_j(s)$ also shown in figure 3. With this choice of $\psi_j(s)$, it is seen that J_j is equal to ϕ_j and if $w_i(s)$ are chosen to be the same for both integral equations, it follows that the last integral of (19) is identical to the last integral of (20). It can also be shown that the second integral of (19) can be readily written (via integration by parts) in a form that matches the corresponding integral in (20). This simply proves that by using appropriate choices for the expansion and weighting functions, the resulting systems from (19) and (20) are identical. Of course, in principle, one could choose the expansion and weighting functions for (19) and (20) to be different. However, in view of the charge conservation requirements, the above choice for $\psi_j(s)$ is the most appropriate. Expansion basis other than the triangle function $\Lambda_j(s)$ were also considered. However, we have found that if pulse basis are chosen for expanding the charge along with point-matching, the resulting system is unstable. After going through various implementations of (20) with different combinations of testing and expansion functions, it was found that the best lowest order choice for the weighting function is $\Lambda_j(s)$ and the same holds for the expansion functions $L_j(s)$. In accordance with (21) this also implies that an appropriate choice for $\psi_j(s)$ is the doublet function $D_j(s)$.

Next we consider the evaluation of the matrix elements associated with current and charge integral equations. First we compute the matrix elements for a flat perfectly conducting strip.

3 Solution of the current integral equation for the flat strip

Consider the perfectly conducting flat strip shown in figure 4. Choosing the weighting and expansion functions as

$$w_j(x) = L_j(x) = \Lambda_j(x) = \begin{cases} \frac{x-(j-1)\Delta x}{\Delta x} & (j-1)\Delta x < x < j\Delta x \\ \frac{(j+1)\Delta x - x}{\Delta x} & j\Delta x < x < (j+1)\Delta x \end{cases}$$

from (19) we obtain the system

$$[A_{ij}]\{J_j\} = \{b_i\} \tag{22}$$

In this, $[A_{ij}]$ is a square $N \times N$ matrix whereas $\{b_i\}$ and $\{J_j\}$ denote column matrices for the excitation and unknown current density. Directly from (19), the elements of the excitation column are

$$b_i = Y_o \int_{(i-1)\Delta x}^{(i+1)\Delta x} \Lambda_i(x) E_x^i(x) ds \tag{23}$$

$$\approx Y_o E_x^i(i\Delta x)\Delta x$$

and those for the impedance matrix $[A_{ij}]$ can be written as

$$A_{ij} = \frac{k_o}{4}\left(a_{ij} - \frac{1}{k_o^2}a'_{ij}\right) \tag{24}$$

in which

$$a_{ij} = \int_{(i-1)\Delta x}^{(i+1)\Delta x} \Lambda_i(x) \int_{(j-1)\Delta x}^{(j+1)\Delta x} \Lambda_j(x) H_o^{(2)}\left(k_o\left|x - x'\right|\right) dx' dx \tag{25}$$

$$a'_{ij} = \int_{(i-1)\Delta x}^{(i+1)\Delta x} \frac{d\Lambda_i(x)}{dx} \int_{(j-1)\Delta x}^{(j+1)\Delta x} \frac{d\Lambda_j(x')}{dx'} H_o^{(2)}\left(k_o\left|x - x'\right|\right) dx' dx \tag{26}$$

To evaluate a_{ij} we may employ the 5-point integration formula. We have

$$a_{ij} = \sum_{n=-2}^{2} \alpha_n \sum_{m=-2}^{2} \alpha_m I_{ij}^{nm} \tag{27}$$

where

$$I_{ij}^{nm} = \int_{\left(i+\frac{2n-1}{5}\right)\Delta x}^{\left(i+\frac{2n+1}{5}\right)\Delta x} \int_{\left(j+\frac{2m-1}{5}\right)\Delta x}^{\left(j+\frac{2m+1}{5}\right)\Delta x} H_o^{(2)}\left(k_o\left|x - x'\right|\right) dx' dx \tag{28}$$

and

$$\alpha_n = \begin{cases} \frac{1}{5} & n = -2, 2 \\ \frac{3}{5} & n = -1, 1 \\ 1 & n = 0 \end{cases}$$

The integral I_{ij}^{nm} can be readily evaluated numerically using the mid-point integration formula provided $i \neq j$ or $n \neq m$. When $i = j$ and $n = m$ the integrand has a logarithmic singularity and we must then evaluate I_{ii}^{nn} analytically. In this case $k_o\left|x - x'\right| < 2k_o\frac{\Delta x}{3} << 1$ and we can thus employ the small argument approximation for the Hankel function,

$$H_o^{(2)}(z) = 1 - j\frac{2}{\pi} ln \frac{\gamma z}{2} \tag{29}$$

Substituting this into (29) we obtain

$$
\begin{aligned}
I_{ii}^{nn}(\delta) &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left(1 - j\frac{2}{\pi}ln\left|\frac{\gamma k_o(x-x')}{2}\right|\right)dx'dx \\
&= \delta^2 - j\frac{2}{\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} ln\left|\frac{\gamma k_o(x-x')}{2}\right|dx'dx \\
&= \delta^2 - j\frac{2}{\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left\{\int_{-\frac{\delta}{2}}^{x} ln\frac{\gamma k_o(x-x')}{2}dx'\right. \\
&\quad \left. + \int_{x}^{\frac{\delta}{2}} ln\frac{\gamma k_o(x'-x)}{2}dx'\right\}dx \\
&= \delta^2 - j\frac{2}{\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left\{-\left[(x-x')ln\frac{\gamma k_o(x-x')}{2e}\right]_{-\frac{\delta}{2}}^{x}\right. \\
&\quad \left. + \left[(x-x')ln\frac{\gamma k_o(x-x')}{2e}\right]_{x}^{\frac{\delta}{2}}\right\}dx \\
&= \delta^2 - j\frac{2}{\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left\{(x+\frac{\delta}{2})ln\frac{\gamma k_o\left(x+\frac{\delta}{2}\right)}{2e}\right. \\
&\quad \left. +(\frac{\delta}{2}-x)ln\frac{\gamma k_o\left(\frac{\delta}{2}-x\right)}{2e}\right\}dx \\
&= \delta^2 - j\frac{2}{\pi}\left\{\int_{o}^{\delta} yln\frac{\gamma k_o y}{2e}dy - \int_{\delta}^{o} yln\frac{\gamma k_o y}{2e}dy\right\} \\
&= \delta^2 - j\frac{4}{\pi}\int_{o}^{\delta} yln\frac{\gamma k_o y}{2e}dy \qquad (30)
\end{aligned}
$$

where for this case $\delta = \frac{2\Delta x}{5}$. To evaluate the remaining integral we let $z = \frac{\gamma k_o y}{2e}$ giving

$$
I_{ii}^{nn}(\delta) = \delta^2 - j\frac{4}{\pi}\left(\frac{2e}{\gamma k_o}\right)^2\int_{o}^{\frac{\gamma k_o \delta}{2e}} zlnzdz
$$

$$= \delta^2 - j\frac{4}{\pi}\left(\frac{2e}{\gamma k_o}\right)^2 z^2 \left[\frac{1}{2}lnz - \frac{1}{4}\right]_o^{\frac{\gamma k_o \delta}{2e}}$$

$$= \delta^2 - j\frac{2}{\pi}\left(\frac{2e}{\gamma k_o}\right)^2 \left(\frac{\gamma k_o \delta}{2e}\right)^2 ln\frac{\gamma k_o \delta}{2e}$$

$$= \left[1 - j\frac{2}{\pi}ln\left(\frac{\gamma k_o \delta}{2e}\right)\right]\delta^2 \qquad (31)$$

We now turn our attention to the evaluation of a'_{ij}. Upon substituting the expressions for $\Lambda_{ij}(x)$ and differentiating we obtain

$$a'_{ij} = \left\{\left[\int_{(i-1)\Delta x}^{i\Delta x} \frac{1}{\Delta x} - \int_{i\Delta x}^{(i+1)\Delta x} \frac{1}{\Delta x}\right]\left[\int_{(j-1)\Delta x}^{j\Delta x} \frac{1}{\Delta x} - \int_{j\Delta x}^{(j+1)\Delta x} \frac{1}{\Delta x}\right] \cdot H_o^{(2)}\left(k_o\left|x - x'\right|\right) dx'dx\right\}$$

$$= \frac{1}{(\Delta x)^2}\left\{\left[\int_{(i-1)\Delta x}^{i\Delta x} - \int_{i\Delta x}^{(i+1)\Delta x}\right]\left[\int_{(j-1)\Delta x}^{j\Delta x} - \int_{j\Delta x}^{(j+1)\Delta x}\right] \cdot H_o^{(2)}\left(k_o\left|x - x'\right|\right) dx'dx\right\}$$

$$= \frac{1}{(\Delta x)^2}\left\{\int_o^{\Delta x}\int_o^{\Delta x}\left[2H_o^{(2)}\left(k_o\left|(i-j)\Delta x + x - x'\right|\right) - H_o^{(2)}\left(k_o\left|(i-j-1)\Delta x + x - x'\right|\right) - H_o^{(2)}\left(k_o\left|(i-j+1)\Delta x + x - x'\right|\right)\right] dxdx'\right.$$

$$(33)$$

which can be evaluated numerically provided the integrands are not singular there. This gives

$$a'_{ij} = 2H_o^{(2)}\left(k_o\left|i-j\right|\Delta x\right) - H_o^{(2)}\left(k_o\left|i-j-1\right|\Delta x\right) - H_o^{(2)}\left(k_o\left|i-j+1\right|\Delta x\right) \qquad (34)$$

When the integrands are singular, which occurs for the first term when $i = j$, for the second term when $i = j + 1$, and for the third term when $i = j - 1$, we may use the result of (31). In particular, we have

$$\int_o^{\Delta x} \int_o^{\Delta x} H_o^{(2)} \left(k_o \left|x - x'\right|\right) dx dx' = \left[1 - j\frac{2}{\pi} ln \left(\frac{\gamma k_o \Delta x}{2e}\right)\right] (\Delta x)^2 \qquad (35)$$

4 Solution of Charge Integral Equation

For the solution of the charge integral equation we again choose the same weighting functions defined in (23) but in view of the requirement for charge conservation, the expansion functions are chosen to be

$$\psi_j(x) = D_j(x) = \begin{cases} 1 & (j-1)\Delta x < x < j\Delta x \\ -1 & j\Delta x < x < (j+1)\Delta x \end{cases} \qquad (36)$$

Substituting these into (20) with $R = 0$ we obtain the system

$$\left[\tilde{A}_{ij}\right] \{\phi_i\} = \{b_i\} \qquad (37)$$

where b_i is again given by (24). The elements of the matrix $\left[\tilde{A}_{ij}\right]$ are defined by

$$\tilde{A}_{ij} = \frac{k_o}{4}\left(\tilde{a}_{ij} - \frac{1}{k_o^2}\tilde{a}'_{ij}\right) \qquad (38)$$

with

$$\begin{aligned} \tilde{a}'_{ij} &= -\int_{(i-1)\Delta x}^{(i+1)\Delta x} \Lambda_i(x) \int_{(j-1)\Delta x}^{(j+1)\Delta x} D_j(x') H_o^{(2)} \left(k_o \left|x - x''\right|\right) dx'' dx' dx \\ &= -\int_{(i-1)\Delta x}^{(i+1)\Delta x} \Lambda_i(x) \int_{(j-1)\Delta x}^{(j+1)\Delta x} \frac{d\Lambda_j(x')}{dx'} G_I \left(k_o \left|x - x'\right|\right) dx' dx \\ &= \int_{(i-1)\Delta x}^{(i+1)\Delta x} \Lambda_i(x) \int_{(j-1)\Delta x}^{(j+1)\Delta x} \Lambda_j(x) H_o^{(2)} \left(k_o \left|x - x'\right|\right) dx' dx \\ &= a_{ij} \end{aligned} \qquad (39)$$

and

$$\begin{aligned}
\tilde{a}'_{ij} &= \int_{(i-1)\Delta x}^{(i+1)\Delta x} \frac{d\Lambda_i(x)}{dx} \int_{(j-1)\Delta x}^{(j+1)\Delta x} D_j(x') H_o^{(2)}\left(k_o \left|x - x'\right|\right) dx' dx \\
&= \int_{(i-1)\Delta x}^{(i+1)\Delta x} \frac{d\Lambda_i(x)}{dx} \int_{(j-1)\Delta x}^{(j+1)\Delta x} \frac{d\Lambda_j(x')}{dx'} H_o^{(2)}\left(k_o \left|x - x'\right|\right) dx' dx \\
&= a'_{ij} \qquad (40)
\end{aligned}$$

That is, the elements of the matrix $\left[\tilde{A}_{ij}\right]$ are identical to those of the matrix $[A_{ij}]$ associated with the current integral equation. This was, of course, to be expected since as noted in the previous section $\phi_j = J_j$.

5 Appendix: The strip current in terms of charges

Mitzner [3] derived that for a closed two-dimensional surface whose circumferential length is L (see figure A1), the surface current density satisfies the relation

$$K_s(s) = \tilde{K}_s - \frac{1}{2}\int_o^{\frac{L}{2}} \left(\frac{L-2\delta}{L}\right)[g(s+\delta) - g(s-\delta)]\, d\delta \qquad (A1)$$

where

$$\tilde{K}_s = \int_o^L K_s(s)ds = \text{average current on C} \qquad (A2)$$

$$g(s) = \frac{dK_s}{ds} \qquad (A3)$$

and

$$\int_o^L g(s)ds = K_s(L) - K_s(0) = 0 \qquad (A4)$$

since $K_s(s) = K_s(s+L)$, i.e. K_s is a periodic function having a period L.

We like to derive an expression similar to (A1) for the current on a curved strip. Equation (A1) is still applicable for the strip except that C is now made-up of the top and bottom surfaces of the strip (see figure A1). The net current on the strip is given by

$$J_s = K_{s+} - K_{s-} = K_s(s) - K_s(L-s) \qquad (A5)$$

and from (A1) on letting $s \to L-s$ we get

$$K_s(L-s) = \tilde{K}_s - \frac{1}{2}\int_o^{\frac{L}{2}} \left(\frac{L-2\delta}{L}\right)[g(L-s+\delta) - g(L-s-\delta)]\, d\delta \qquad (A6)$$

Combining (A1), (A5) and (A6) yields

$$J_s = K_s(s) - K_s(L-s)$$
$$= -\frac{1}{2}\int_o^{\frac{L}{2}} \left(\frac{L-2\delta}{L}\right)[g(s+\delta) - g(s-\delta) - g(L-s+\delta) + g(L-s-\delta)]\, d\delta \qquad (A7)$$

However, since

$$\frac{dJ_s}{ds} = \frac{d\left[K_s(s) - K_s(L-s)\right]}{ds} = \frac{dK_s(s)}{ds} - \frac{dK_s(L-s)}{ds}$$
$$= \frac{dK_s(s)}{ds} + \frac{dK_s(L-s)}{d(L-s)}$$

it follows that

$$\frac{dJ_s}{ds} = g(s) + g(L-s) = \phi(s) \tag{A8}$$

where $\phi(s)$ is proportional to the net charge on the surface of the strip. Using now (A8) into (A7) gives

$$J_s = -\frac{1}{2}\int_o^{\frac{L}{2}} \left(\frac{L-2\delta}{L}\right)\left[\phi(s+\delta) - \phi(s-\delta)\right] d\delta \tag{A9}$$

and it should be noted that

$$\phi(s) = g(s) + g(L-s) = g(L-s) + g(L-L+s) = \phi(L-s) \tag{A10}$$

Expression (A9) was implemented and found to hold when $J_s = 0$ at the ends of the strip. An example calculation of the relation between the current and charge on a 3λ flat perfectly conducting strip is given in figure A2. Also figure A3 shows the same quantities for a 1λ flat strip. It is clear from both of these figures that the charge distribution has a much larger dynamic range than the current density. This probably implies that the computation of the charge density is more difficult.

To test the validity of (A9) we can substitute the charge density shown in figures A2 or A3 and attempt to recover the original current density distribution. The results of our first attempt is shown in figure A4 and as seen the re-calculated current, albeit close, is not precisely in agreement with the original current density (see routine in figure A6 used to implement (A9)). As it turned out, this is because the original current density was not exactly zero at the ends of the strip (because the MoM code does not sample at the strip ends) which violated the conditions for which (A9) holds. Since the re-calculated current density does vanish at the ends of the strip, this can then be used to validate (A9). Differentiating the current given in figure A4 and substituting the result in (A9) it was found that the original "modified" current density is recovered exactly as demonstrated in figure A5.

References

[1] E.F. Knott and T.B.A. Senior, "Non-Specular Radar Cross Section Study," University of Michigan Radiation Laboratory Report 11764-1-T (also U.S. Air Force Report AFAL-TR-73-422), Jan.1974.

[2] M.A. Ricoy and J.L. Volakis, "Integral Equations with Reduced Unknowns for the Simulation of Two Dimensional Composite Structures," University of Michigan technical Report 389492-2-T, Nov. 1987.

[3] K. Mitzner, personal communication.

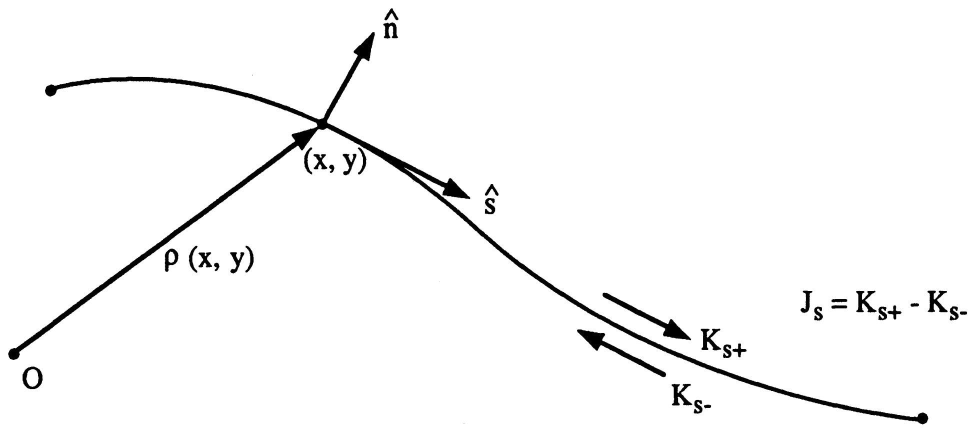


Figure 1. Geometry of the 2D curved surface (strip).

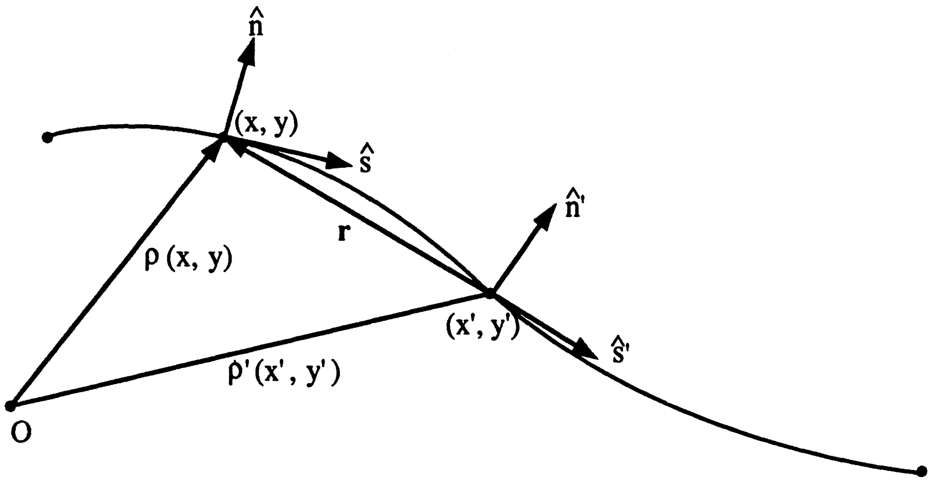


Figure 2. Illustration of the observation and integration point parameters.

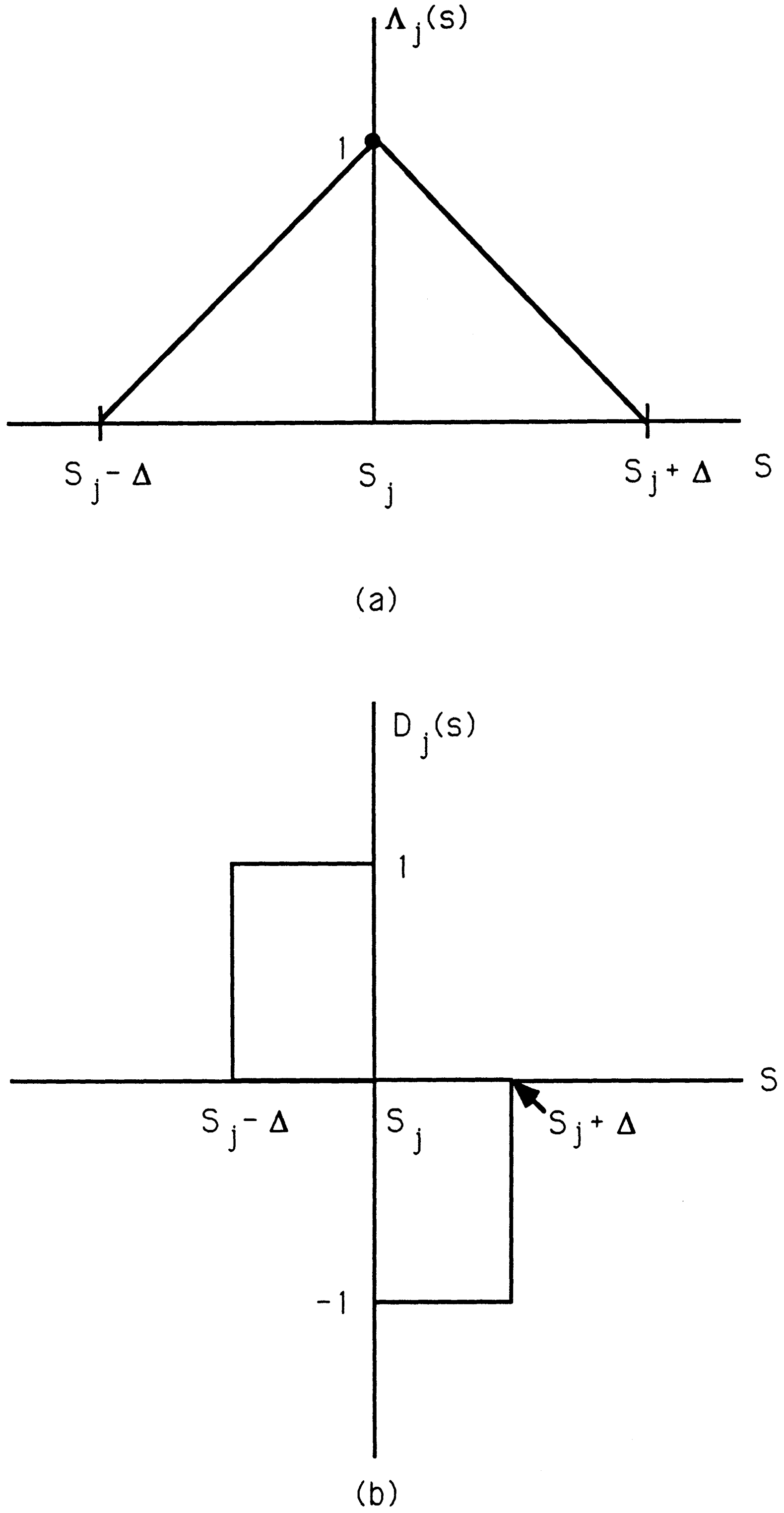


Figure 3. Expansion functions. (a) Linear expansion functions for the current (b) corresponding expansion functions for the charge.

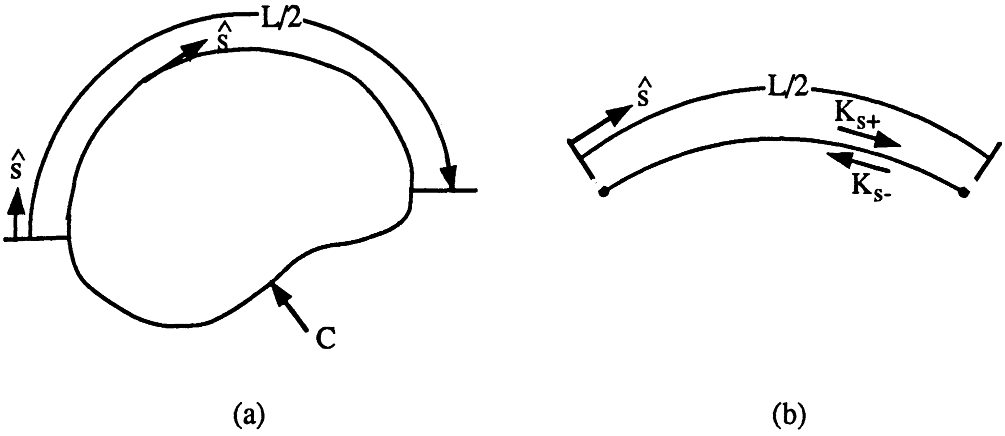


Figure A1. Geometry of a closed surface. (a) Original geometry of circumference L (b) Currents on a closed surface collapsed to a strip.

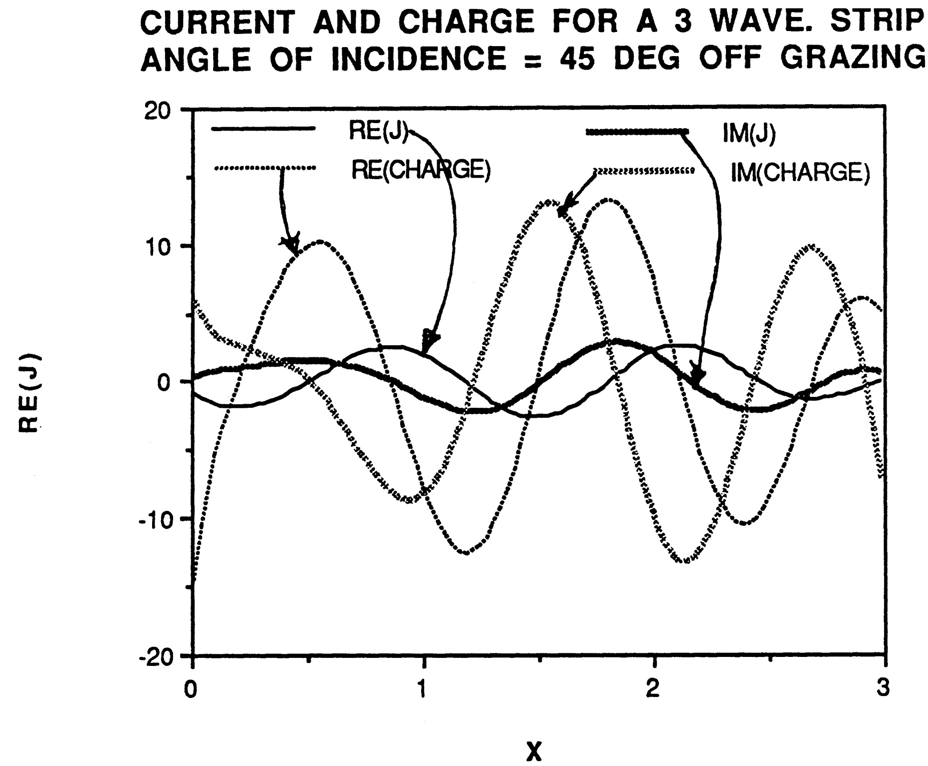


Figure A2. Current and charge densities on a 3λ flat strip.

Routine to compute charge (CHARG) from current (PHI)

```
      DO 66 I=1,M
      IF(I.EQ.1) THEN
       CHARG(I)=(4.*PHI(I+1)-3.*PHI(I)-PHI(I+2))/(2.*DSQQ)
      ELSE IF(I.EQ.M) THEN
       CHARG(I)=(3.*PHI(I)-4.*PHI(I-1)+PHI(I-2))/(2.*DSQQ)
      ELSE IF((I.NE.1).AND.(I.NE.M)) THEN
       CHARG(I)=(PHI(I+1)-PHI(I-1))/(2.*DSQQ)
      ENDIF
66     CONTINUE
```

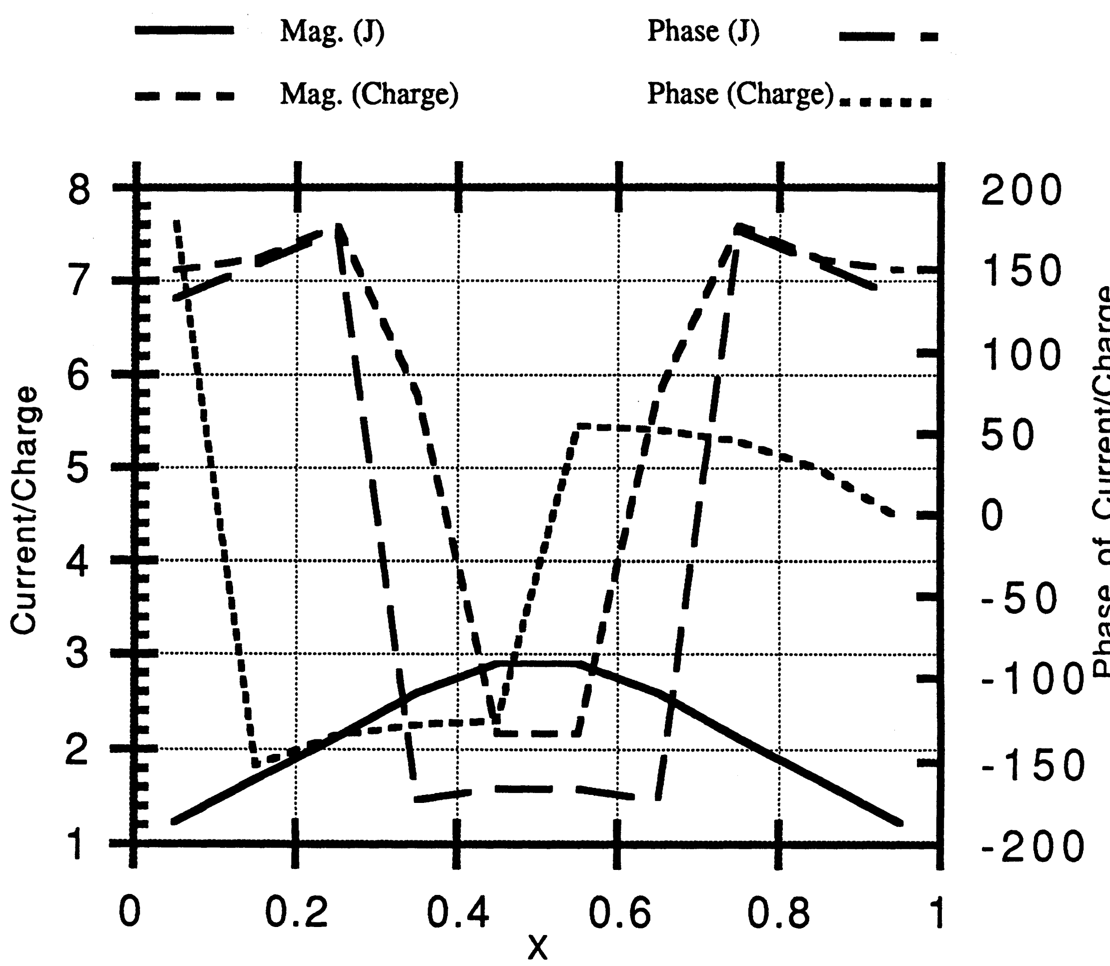


Figure A3. Current and charge densities on a 1λ flat strip.

Solid lines : Current Density (original from MoM code)

Dashed : Recalculated using (A9)

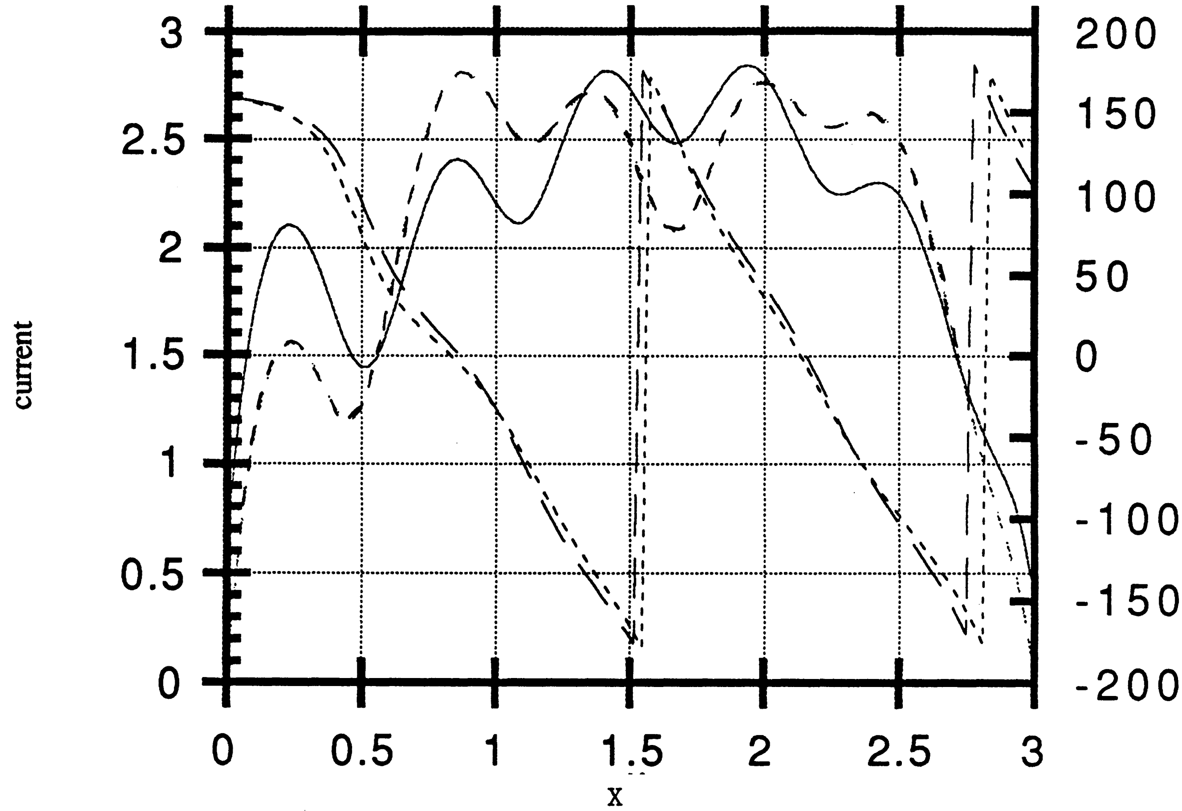


Figure A4. Recalculation of J_s from ϕ using the formula (A9). As seen, J_s is not precisely zero at the ends.

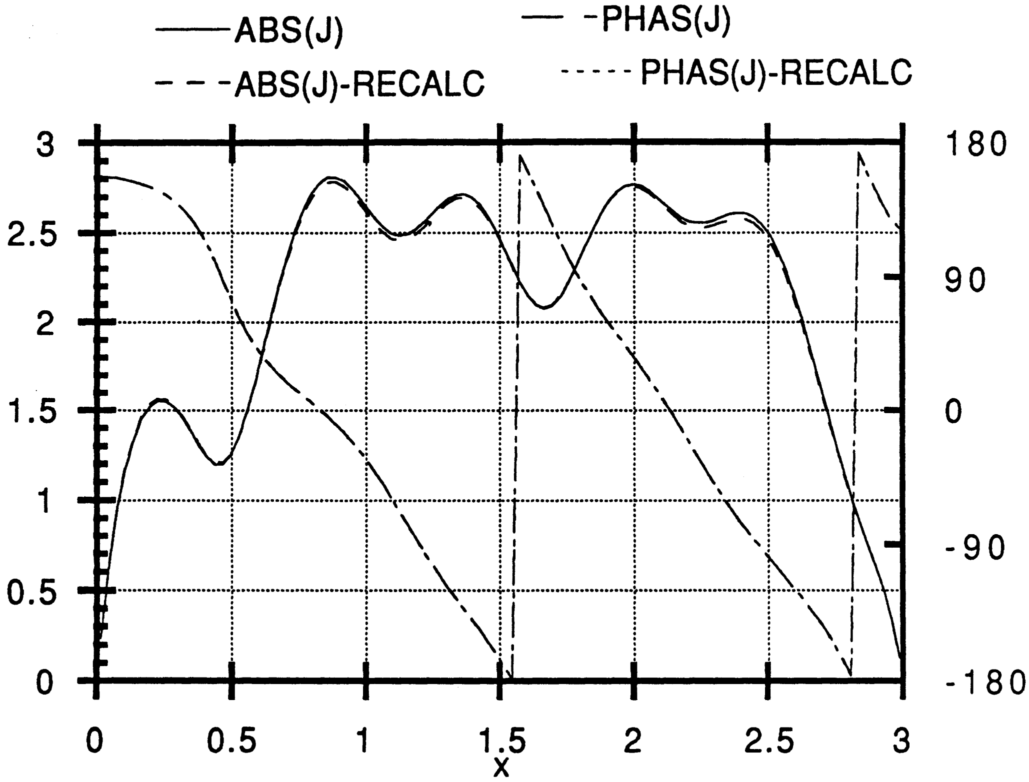
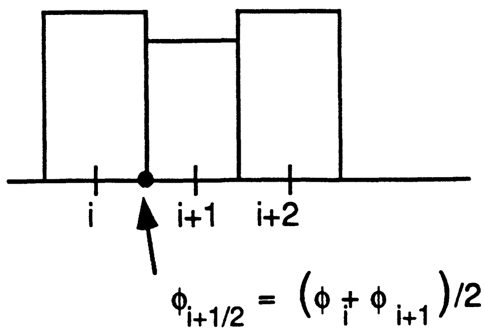


Figure A5. Recalculation of J_S from ϕ using equation (A9). In this case the original J_S is given by the dashed curve in figure A4.

Routine which Implements Equation (A9)

$$J_s = -\frac{1}{2}\int_0^{L/2} d\delta \left(\frac{L-2\delta}{L}\right)\left[\phi(s+\delta) - \phi(s-\delta)\right]$$

L/2 = M * DSQQ
PHI = J_s
CHARG = ϕ




```
C REcover current density
        DO 68 J=1,M
        PHJ(J)=(0.,0.)
         DO 67 I=1,M
         RII=I-1.+.5
         RJJ=J-1.+.5
         RI1=RII+RJJ
         RI2=RJJ-RII
         I1=I+J-1
         I2=I-J
         IF(I1.LT.0)I1=2*M+I1
         IF(I2.LT.0)I2=2*M+I2
         IF(I1.GT.M)I1=2*M-I1
         IF(I2.GT.M)I2=2*M-I2
         IF(I1.EQ.0) THEN
           CHP=CHARG(1)-0.25*(4.*CHARG(2)-3.*CHARG(1)-CHARG(3))
         ELSE IF(I1.EQ.M) THEN
           CHP=CHARG(M)+0.25*(3.*CHARG(M)-4.*CHARG(M-1)+CHARG(M-2))
         ELSE
          CHP=(CHARG(I1)+CHARG(I1+1))/2.
         ENDIF
         IF(I2.EQ.0) THEN
           CHM=CHARG(1)-0.25*(4.*CHARG(2)-3.*CHARG(1)-CHARG(3))
         ELSE IF(I2.EQ.M) THEN
           CHM=CHARG(M)+0.25*(3.*CHARG(M)-4.*CHARG(M-1)+CHARG(M-2))
         ELSE
          CHM=(CHARG(I2)+CHARG(I2+1))/2.
         ENDIF
         PHJ(J)=((2.*M-2.*RII)*(CHP-CHM)/(2.*M))+PHJ(J)
67       CONTINUE
        PHJ(J)=-.5*DSQQ*PHJ(J)
        PHIA=CABS(PHI(J))
        PHASI=dig*atan3(aimag(PHI(J)),real(PHI(J)))
        PHJA=CABS(charg(J))
        PHASJ=dig*atan3(aimag(charg(J)),real(charg(J)))
        WRITE(8,950) PHJ(J)
        WRITE(9,952) (J-0.5)*DSQQ,TAB,PHIA,TAB,PHASI,TAB,PHJA,TAB
    &,PHASJ
68      CONTINUE
```

Figure A6. Routine to recover the current from charge.