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# DESIGN OF THICK-WALLED PRESSURE VESSELS FOR ELEVATED-TEMPERATURE SERVICE

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FOR ELEVATED-TEMPERATURE SERVICE

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## ABSTRACT

Typical formulas which have been used to design pressure vessels are reviewed in terms of rupture life at elevated temperatures. It is suggested that creep strength of an alloy is a key variable when the wall thickness is a large fraction of the radius, so that initial stress are nonuniform in distribution and this stress pattern changes with time.

Limiting conditions would be instantaneous leveling of stress gradients as one extreme, and as the other no relaxation of stress concentrations during service. In the first case, analysis indicates that failure should result from excessive deformation of the vessel after the effective stress becomes uniform. Design should properly be based on limiting the total deformation under this uniformed stress to a maximum of around one or two per cent.

Alloys with very high creep strength relative to their rupture strength cannot reduce initial stress gradients appreciably. The appropriate design criterion under constant temperature and pressure should then be rupture strength applied to the initial effective stress at the bore.

## INTRODUCTION

Engineers may not agree completely on methods for handling extreme pressures alone or for elevated temperatures alone, with the other absent, but

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at least the existing codes (1, 2) appear to have provided a reasonable basis for numerous successful designs. No corresponding simple treatment appears possible for all materials when extreme pressure and high temperature act together. Steep stress gradients, characteristic of thick vessel walls, alter under the influence of creep at elevated temperature. The life until fracture depends on this variable stress-time history at critical portions of the structure.

An attempt will be made in this paper to analyze the changes in stress distribution in the walls of a thick cylinder under internal pressure at elevated temperatures and to develop a procedure for calculating the time of rupture from creep and stress-rupture data normally available to a designer.

## DESIGN FORMULAS AND THEIR LIMITATIONS AT ELEVATED TEMPERATURES

Most discussions on design of pressure equipment center around formulas for wall thickness. Their rather universal familiarity to engineering personnel makes such formulas useful as standards against which to measure behavior under new conditions.

The basic relationship for distribution of elastic stress in a uniformly-thick cylinder under internal pressure is that of Lamé, published a century ago (3). Lamé's analysis is for elastic stresses near the center of a closed cylinder long enough that the end closures are too remote to cause any nonuniformity in the axial stress. Under these conditions values of the tangential and radial principal stresses at any radius $r$ are, respectively:

$$S_T = p\left(\frac{a^2}{b^2 - a^2}\right)\left(1 + \frac{b^2}{r^2}\right) \qquad (1)$$

and

$$S_R = p\left(\frac{a^2}{b^2 - a^2}\right)\left(1 - \frac{b^2}{r^2}\right) \quad . \qquad (2)$$

In these expressions p is the internal pressure, while b and a are the radial distances to the outside and inside surfaces.

The uniform axial stress depends directly on the ratio of the area acted on by the pressure to that of the cross section resisting this end thrust:

$$S_Z = p\left(\frac{a^2}{b^2 - a^2}\right) . \qquad (3)$$

Inspection of equations (1) and (2) shows that both the tangential and radial stresses have their maximum magnitude (of opposite signs) at the inside surface. Moreover, though radial stresses are limited in value to the applied pressure, the tangential stress at the bore is always numerically greater than the internal pressure even for wall thickness approaching infinity.

It might also be noted that the Lamé equations do not involve the wall thickness as such, but contain only ratios of radii. Many writers prefer to express the principal-stress distribution in terms of the ratio k = b/a of the outer and inner radii. When "thick" walls are referred to, what is usually meant is a value of k in excess of some minimum value, say 1.5.

Basically, the Lamé equations are not design equations...they simply state the stresses which exist at any distance from the axis for a given vessel geometry and applied internal pressure. They say nothing about how thick the walls must be to withstand a particular pressure and temperature condition until one first defines a criterion of failure for the vessel and until this criterion can be expressed in terms of some limiting combination of the several stresses acting.

Some engineering structures retain their usefulness up to the very point of fracture. In other circumstances "failure" can be said to have occurred the moment any part exceeds the elastic limit and permanent deformation begins. The ultimate tensile strength and the yield point (or the elastic limit) determined in simple tension serve as measures of the relative ability of different materials to resist failure by the different criteria. Either criterion is completely specified by stating a limiting stress level.

At elevated temperature, the amount of plastic deformation and the occurance of rupture depend on time as well as the stress level, so that a series of tests is needed to obtain required data for each material and temperature. However, once the desired operating life and the temperature have been set, a single stress will again characterize fracture and another definite stress level corresponds to the allowable deformation before the part ceases to function adequately.

In the paragraphs to follow the stress in pure tension corresponding to the appropriate type of failure will be designated $\underline{S_F}$. Failure under the actual triaxial stresses of a pressure vessel will then occur when some suitable combination of the three principal stresses reaches a value equivalent to $S_F$.

The largest principal normal stress was early suggested as the only stress affecting failure, independent of stresses in other directions. For a vessel subjected to internal pressure alone the critical normal stress would be the hoop stress at the bore. From equation (1), the maximum stress theory of failure gives the result:

$$S_F = p\left(\frac{a^2}{b^2 - a^2}\right)\left(1 + \frac{b^2}{a^2}\right) = p\left(\frac{b^2 + a^2}{b^2 - a^2}\right) . \qquad (4)$$

This equation itself or some equivalent form is often referred to, perhaps improperly, as the Lamé formula. It may also be found written in terms of the outside diameter $\underline{D} = 2b$ and the wall thickness $\underline{t} = b - a$:

$$S_F = p\left(\frac{0.5\left(\frac{D}{t}\right)^2 - \left(\frac{D}{t}\right) + 1}{(D/t) - 1}\right) . \qquad (4a)$$

When the wall thickness is small compared to the outside diameter, $(D/t) >> 1$ and this last expression can be simplified to give the "common" or bore formula:

$$S_F = p\left(\frac{D}{2t} - 1\right) = p\,\frac{d}{2t}\ , \tag{5}$$

where $d = 2a$ is the inside diameter. This same result follows immediately from static equilibrium between the internal pressure load and the hoop tension when the circumferential stress is assumed to be uniform across the wall thickness. Based on the realization that the circumferential stress is actually not uniform, several modifications to the bore formula have been applied to make it more conservative. The "average" formula assumes the internal pressure to act on the average of the outside and inside diameters:

$$S_F = \frac{p\left(\frac{D + d}{2}\right)}{2t} = p\left(\frac{D}{2t} - \frac{1}{2}\right), \tag{6}$$

and the well-known Barlow formula employs the outside diameter:

$$S_F = pD/2t\ . \tag{7}$$

No valid justification for either of these latter two formulas is apparent. Any agreement with test results at room temperature is probably fortuitous and is not proof of general applicability for other cylinders with different $k$ values.

Numerous studies into the behavior of ductile metals under complex stressing suggest either the maximum shear stress or the shear-stress invariant as a reasonable general criterion of failure. In terms of the principal stresses $S_1 > S_2 > S_3$ at any point, the maximum shear stress is $(S_1 - S_3)/2$. In pure tension the maximum shearing stress is half the tensile stress, wherefore $S_F = S_1 - S_3$. At the bore of a thick cylinder:

$$S_F = (S_T - S_R)_{r=a} = 2p\left(\frac{b_2}{b^2 - a^2}\right) = \frac{1}{2}\,p\left(\frac{(D/t)^2}{(D/t) - 1}\right). \tag{8}$$

The shear stress invariant $\overline{S}$ is related to the individual principal stresses by the expression

$$\overline{S}^2 = 1/2\ [(S_1 - S_2)^2 + (S_1 - S_3)^2 + (S_2 - S_3)^2]\ . \tag{9}$$

For the Lamé distribution, the axial stress is exactly the average of the tangential and radial components at any point. Under these conditions, $\overline{S} = (S_T - S_Z)\sqrt{3}$. Setting $S_F$ equal to the shear-stress invariant at the inner surface, one obtains

$$\left.\begin{aligned} S_F &= \sqrt{3}\, p \left(\frac{b^2}{b^2 - a^2}\right) \\ &= \frac{\sqrt{3}}{4}\, p \left[\frac{(D/t)^2}{\frac{D}{t} - 1}\right] \end{aligned}\right\} \quad . \qquad (10)$$

In the majority of applications for pressure equipment, satisfactory operation would be little affected by a small change in vessel dimensions during service life. Under these conditions the limiting factor in design at elevated temperature is the need to prevent rupture during the desired period of use.

As will be considered in more detail later, time until rupture depends upon the entire history of stress and temperature at critical locations in the vessel wall. Stress conditions depend not only on the acting pressure, but also on loads imposed by vessel supports and by thermal stresses (4, 5, 6). Residual stresses from fabrication and heat treating steps (7) and stress concentrations at imperfections or changes in section exert additional effects. Whatever the combination of such stresses acting on the vessel, the initial stress pattern can presumably be determined. Even at high temperatures the usual methods of elasticity and plasticity should still be usable provided necessary stress-strain properties are evaluated at the temperature involved and provided loading occurs in a brief enough period that creep during this short time may be neglected or corrected for.

For the present, discussion will be limited to conditions of constant service temperature and pressure. Cyclic or variable operations can be considered equivalent to the sum of a number of such fixed-pressure, fixed-temperature periods.

At conditions where stress rupture determines vessel life, initial stress gradients tend to redistribute under the action of the accompanying creep. The degree of such redistribution varies with the material and the vessel geometry, and depends on relative creep strength and rupture strength at service temperature.

At one extreme, negligible relaxation of initial stress concentrations could be assumed. In the absence of stresses from sources other than internal pressure, the initial Lamé stresses at the bore would control and design formula 4, 8 or 9 would apply, according to whether rupture under creep conditions is determined by the maximum normal stress, the maximum shear stress or the shear-stress invariant.

The other extreme possibility would be complete and instantaneous leveling of initial stress gradients. Under conditions where the maximum principal stress criterion of failure applied, the bore formula for thin walls would then be exact, regardless of the wall thickness present. Should the maximum shear stress or the shear-stress invariant theory of failure be found to apply more closely to rupture after extended creep, $S_F$ would equal $(S_T - S_R)$ or $\overline{S}$ calculated from stress components present after equilibrium is reached.

The creep responsible for stress redistribution is most rapid in the direction of the largest gradient, so that the tangential stress changes the fastest. If this stress alone is assumed to even out at very early times the resulting tangential stress is the bore-formula value, pd/2t. For this same condition, the maximum shear theory gives Barlow's formula at the bore:

$$S_F = (S_T - S_R)_{equilibrium} = [(pd/2t) - (-p)] = p\,(d + 2t)/2t = pD/2t.$$

Considering each of the principal stresses to level independently of the others, the equilibrium value in the radial direction would be -p/2. Inserting this average radial stress into the maximum shear stress expression,

$$S_F = [(pd/2t) - (-p/2)] = p\,(D/2t - 1/2)\ ,$$

and the "average" formula (equation 6) reappears.

It thus seems that for alloys with low creep strength the simple "average" and Barlow formulas could give accurate results at high temperatures, even for rather thick walls.

Substitution of the "averaged" stress components into the relationship for the shear stress invariant leads to none of the usual formulas. Comparison of equations (8) and (10) shows that for the Lamé stresses from the same pressure the shear stress invariant would give an effective stress lower than that found for maximum shear in the ratio $\sqrt{3}/2$.

The difference in rupture stress with and without leveling of initial stress gradients is illustrated in Table 1 for five ratios of outside to inside diameter. The comparison made is for the shear-stress invariant controlling fracture.

For pipes and tubing with relatively thin walls (k = 1.2 or less) initial stress gradients are so small that stress redistribution by creep has but limited effect. In the absence of bulging and extraneous stresses from sources other than internal pressure, the simple Barlow formula should never be in error by more than about ten to twenty per cent in stating the rupture stress. This should be true regardless of what failure theory applies or the extent of stress redistribution by creep during service.

Confirmation of this expectation is found in recent results (8) for the rupture of carbon steel tubes under internal pressure at 850°F and 900°F. Data for twenty-four tubes with ratio of outside to inside diameter of just under 1.2 were reported covering rupture lives from near one hour to more than 10,000 hours. Correlation of stress versus rupture time was fairly good when the stress level in the tubes was based on the bore formula. Equally good, or even better, agreement between tube tests and tests with bars in tension results when the "average" formula is applied or when the averaged stress components are introduced into equation (9) and comparisons made on the basis of the shear stress invariant.

Table 1

Effect of Stress Redistribution on Rupture Stress
Assuming the Shear-Stress Invariant Theory of Failure

| k = OD/ID | Equivalent stress in pure tension with same rupture characteristics as vessel under internal pressure p, expressed as a multiple of the pressure. | |
|---|---|---|
| | No Stress Redistribution (Initial Bore Value) | *Complete equalization of Shear-Stress Invariant across Vessel Wall |
| 1.2 | 5.67 p | 4.68 p |
| 1.5 | 3.14 p | 2.02 p |
| 2.0 | 2.31 p | 1.07 p |
| 3.0 | 1.95 p | 0.54 p |
| 4.0 | 1.85 p | 0.34 p |

*Calculated by integration over the wall of the initial $\overline{S}$ values:

$$\overline{S}_{ave} = \frac{\int_{r=a}^{r=b} (2\pi r)(\overline{S})\, dr}{\pi (b^2 - a^2)} \quad .$$

But

$$\overline{S} = (S_T - S_Z)\sqrt{3} = \sqrt{3}p \left(\frac{a^2}{b^2 - a^2}\right)\left(\frac{b^2}{r^2}\right)$$

$$\therefore \quad \overline{S}_{ave} = \frac{2\sqrt{3}p \left(\frac{a^2}{b^2 - a^2}\right)(b^2)}{(b^2 - a^2)} \int_a^b \frac{dr}{r}$$

$$= 2\sqrt{3}p \left(\frac{ab}{b^2 - a^2}\right)^2 \ln\left(\frac{b}{a}\right) \quad . \qquad (11)$$

Returning to Table 1, one should take note of the significant variations in rupture stress possible for k values of 2.0 or more, depending on the relative effectiveness of creep in reducing initial high stresses near the bore. In this range of wall thickness which covers the bulk of anticipated extreme pressure-temperature conditions, either improper choice of the failure theory or omission of an allowance for stress relaxation could result in an error of several fold in $S_F$ compared with the actual rupture stress. No one of the usual formulas can now be expected to hold for all materials and temperatures.

Consequences of erroneous stress evaluations may perhaps be better appreciated if one recalls that ten per cent change in applied stress corresponds roughly to a two-fold change in rupture life for most alloys. Complications arising from extraneous stresses might cause even further departure from the truth. Any satisfactory general method of design for extreme pressure at elevated temperature must be able to handle any pattern of changing stress and any degree of triaxiality of this variable stress. A necessary first step appears to be careful consideration of available data on rupture life under variable simple-tension stress and for steady complex stresses.

## ELEVATED-TEMPERATURE BEHAVIOR UNDER VARIABLE AND COMPLEX STRESSES

The foregoing analysis indicates that redistribution of initial stress gradients should be an important factor in determining rupture life of a thick vessel operating at elevated temperatures. Before this basic premise can be treated quantitatively, one must be able to predict the creep behavior under the complex stress pattern in the vessel wall and to predict the pattern of stress changes which this creep produces. When the history of changing stress pattern is known, it is still necessary to evaluate rupture life of the material under the changing stress level and changing triaxiality during service.

Presumably separate investigations can be used to study the separate effects and the results then applied simultaneously to fibers in the vessel wall where variations in these two factors are concurrent.

<u>Addibility of Rupture Life under Variable Stresses</u>

To estimate the life expectancy of a fiber in a structure with changing stress level, one must know what portion of the total life is expended by a given sojourn at each stress level. In 1952 Robinson (9) proposed without supporting data that the fraction of total life used up at any stress should equal the ratio:

$$\frac{\text{actual time at the given stress level}}{\text{rupture life at that stress in a conventional constant-load test}} .$$

Two of the present authors have conducted an extensive investigation into factors influencing notch sensitivity of heat-resistant alloys at elevated temperatures. This work is being performed at the University of Michigan under the sponsorship of the Materials Laboratory, Wright Air Development Center. Experiments included rupture tests in which one steady stress was maintained for a portion of the test, and then another steady stress was applied for another portion, all at constant temperature. In some cases the stress was raised after an initial period; in others it was lowered. Three of the tests involved an increase followed by a decrease in the applied load. Eighteen tests on three super-strength alloys disclosed a maximum discrepancy of 27% for any single test (10). This is within the normal scatter expected for the usual single-stress rupture test. If the results for all 18 tests are averaged, addition of fractions of rupture life checks experiment within one per cent.

Such addibility of rupture-life fractions cannot reasonably be expected to hold true in the presence of structural alterations during testing. Guarnieri (11) found that an aluminum alloy and a magnesium alloy experienced marked reduction in life from that predicted by the life-fraction rule at temperatures where annealing of strain hardening occurred or where over-aging resulted.

The present paper is an outgrowth of a doctoral thesis research by the first author, still in progress at the University of Michigan. Rupture life at elevated temperature is being determined experimentally for pressure vessels of several diameter-thickness ratios. Materials include an annealed carbon steel at 900°F and at 1050°F, a creep-resisting ferritic steel [Timken "17-22A" (S)] to be tested at 1100°F in the normalized plus tempered condition, and Inconel "X" tested at 1350°F. In the course of this investigation creep rupture tests on tension specimens were conducted on the first two alloys, both with a single steady load and with one or two stress changes during the test.

To check on the initial isotropy of the test materials, many of the specimens were miniature bars sampled in the radial, tangential and longitudinal directions. Constant-load data for all three conditions studied, plotted in Figure 1, show no large or consistent variation with sampling direction and no effect of sample size. Results of the thirteen multiple-stress tests on these two steels are listed in Table 2, along with computations of life fractions based on the smoothed curves of Figure 1. Once more the life fractions add substantially to unity, even though the test temperatures were above those at which these alloys are normally considered to be metallurgically stable.

In view of the consistent results for this entire mass of data, it appears safe to conclude that the observed addibility of life fractions under variable tension stresses should hold to a close degree for all alloys under conditions where they have reasonable structural stability.

## Creep and Stress - Rupture Life under Complex Stresses

The von Mises formulation for the shear-stress invariant appears to give quite satisfactory engineering correlations for both the start of yielding (12) and the occurrence fracture (13, 14) of ductile alloys at room temperature. Considerable research effort has been expended to learn whether elevated-temperature behavior follows a similar pattern.

Table 2

Results of Multiple-Stress Rupture Tests

| Stress Level, psi | [a]Direction Specimen was Sampled | Actual Time at Stress / Rupture Life at Stress |
|---|---|---|
| | Carbon Steel at 900°F | |
| 20,000 | L | 5.5/35 = 0.16 |
| 15,000 | | 469.4/650 = 0.72 |
| | | 0.88 |
| 15,000 | L | 241.3/650 = 0.37 |
| 20,000 | | 17.5/35 = 0.50 |
| | | 0.87 |
| | Carbon Steel at 1050°F | |
| 7,500 | T | 114.5/220 = 0.52 |
| 6,000 | | 304/700 = 0.43 |
| | | 0.95 |
| 6,000 | R | 135.3/700 = 0.19 |
| 10,000 | | 23.7/47 = 0.51 |
| 6,000 | | 174.2/700 = 0.25 |
| | | 0.95 |
| 6,000 | T | 296.5/700 = 0.42 |
| 10,000 | | 21.2/47 = 0.45 |
| | | 0.87 |
| 12,000 | T | 6.5/15 = 0.43 |
| 10,000 | | 25.3/47 = 0.54 |
| | | 0.97 |
| 10,000 | T | 22.4/47 = 0.48 |
| 6,000 | | 291.9/700 = 0.42 |
| | | 0.90 |
| | "17-22A"(S) at 1100°F | |
| 60,000 | L | 0.05/1 = 0.05 |
| 50,000 | | 1/25 = 0.04 |
| 40,000 | | 79/82 = 0.96 |
| | | 1.05 |
| 50,000 | T | 2.4/25 = 0.10 |
| 40,000 | | 20.3/82 = 0.25 |
| 30,000 | | 125.5/320 = 0.40 |
| | | 0.75 |

[a]L = longitudinal, R = radial, T = tangential.

Table 2 (cont.)

| Stress Level, psi | [a]Direction Specimen was Sampled | Actual Time at Stress / Rupture Life at Stress |
|---|---|---|
| 46,000 | T | 11.3/45 = 0.25 |
| 36,000 | | 26/135 = 0.19 |
| 26,000 | | 274.4/520= 0.53 |
| | | 0.97 |
| 45,000 | L | 17/50 = 0.34 |
| 35,000 | | 77/154 = 0.50 |
| | | 0.84 |
| 25,000 | L | 111.1/550 = 0.20 |
| 35,000 | | 109.7/154 = 0.71 |
| | | 0.91 |
| Relaxation from 45,200 to 36,200 psi in 12 min | L | |
| 36,200 | | 0.01 |
| | | 133.1/130 = 1.02 |
| | | 1.03 |

[a]L = longitudinal, R = radial, T = tangential

A rather complete study of elevated-temperature behavior under complex stress has been in progress in England for several years, under the direction of A.E. Johnson. His group has determined that the elastic limit of a low-carbon steel and of an aluminum alloy for different types of loading was certainly not a direct function of the maximum principal stress. Correlation was almost equally close by the shear stress invariant theory or by the maximum shear stress, with slight favor for the former (15).

Creep in thin cylinders of four alloys, each tested at two or three temperatures under different combinations of tension and torsion, obeyed the shear-stress invariant theory in all cases (16).

Short-time plastic strain tests with an aluminum alloy produced fracture at strains of the order of two per cent, so that tubular specimens were virtually undistorted up to fracture. For a variety of stress patterns it was

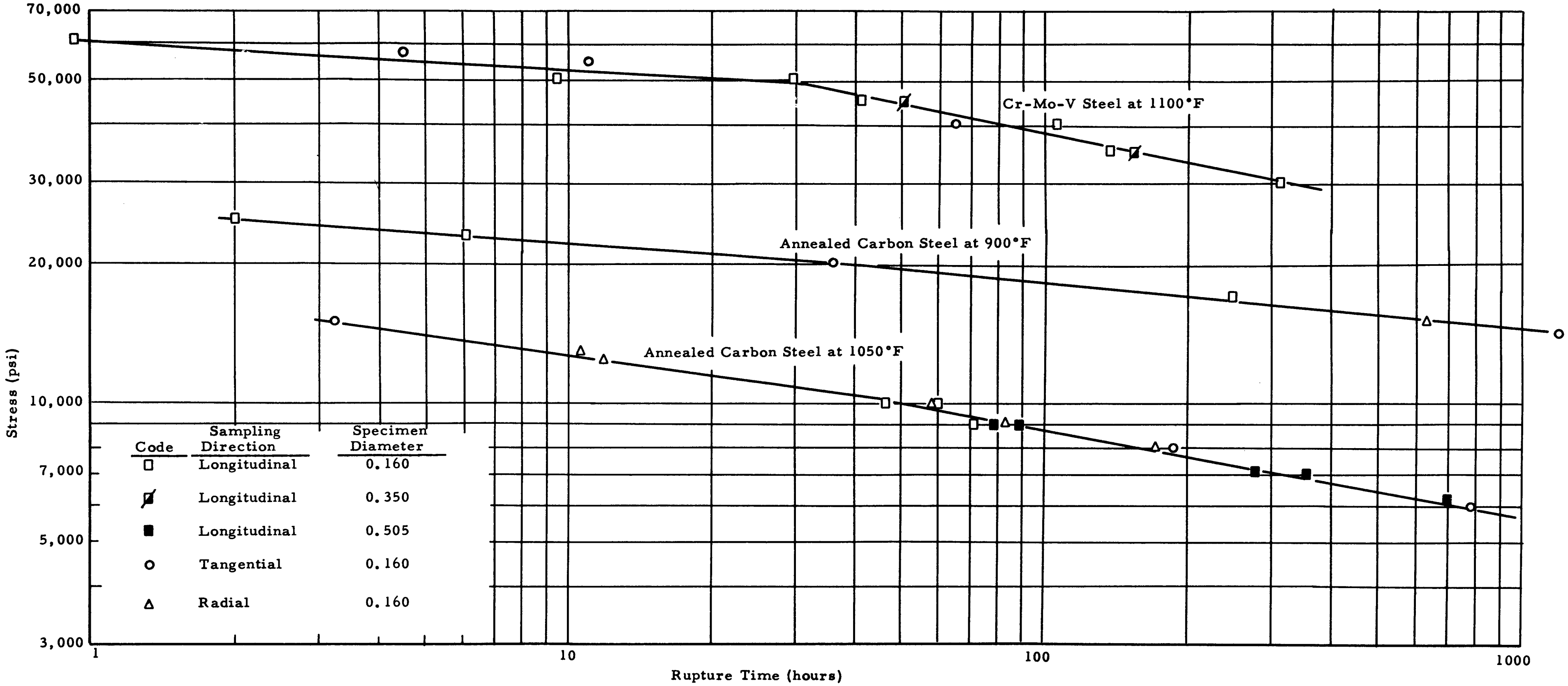



Figure 1. - Stress versus Rupture Time for Three Conditions.

concluded (16) that the criterion of fracture appeared to be between the shear-stress invariant and the maximum shear theories, and that fracture for these conditions was again certainly not a direct function of the largest normal stress.

Rupture after creep under combined stress is difficult to interpret because of the variable reduction of the cross section during third stage of creep for different stress patterns. Published findings to date are limited to preliminary tests reported by Johnson and Frost (17). These particular tests indicated that the criterion for creep-rupture may be that of maximum principal stress. As the above authors remarked, this finding was quite unexpected after all other types of elevated-temperature behavior seemed to follow the von Mises criterion.

The doctoral research referred to in a previous section was formulated before these last data of Johnson and Frost had appeared. With no data in evidence to the contrary, the shear stress invariant was assumed to control rupture life in the original analysis of the problem. When a tentative testing program was set up, no specific plans were made to delve into questions of failure theories. However, some of the data obtained since seem to support the von Mises criterion for stress-rupture under complex stress.

A number of carbon steel tubes had been prepared with 1-inch inside diameter and a wall thickness of 1/8 inch. Such a tube, with $k = 1.25$ can hardly be considered a truly "thin" wall, but variations in stress across the wall are considerably more limited than in other specimens included in this program.

Results of three tests to rupture using these specimens under internal pressure are included in Figure 2, which shows pressure versus rupture life for completed tests on annealed carbon steel tubes at 1050°F. Comparison with Figure 1 shows that for this tube geometry the stress in tension for a given rupture life is about 4.8 to 5.0 times the pressure to give the same life.

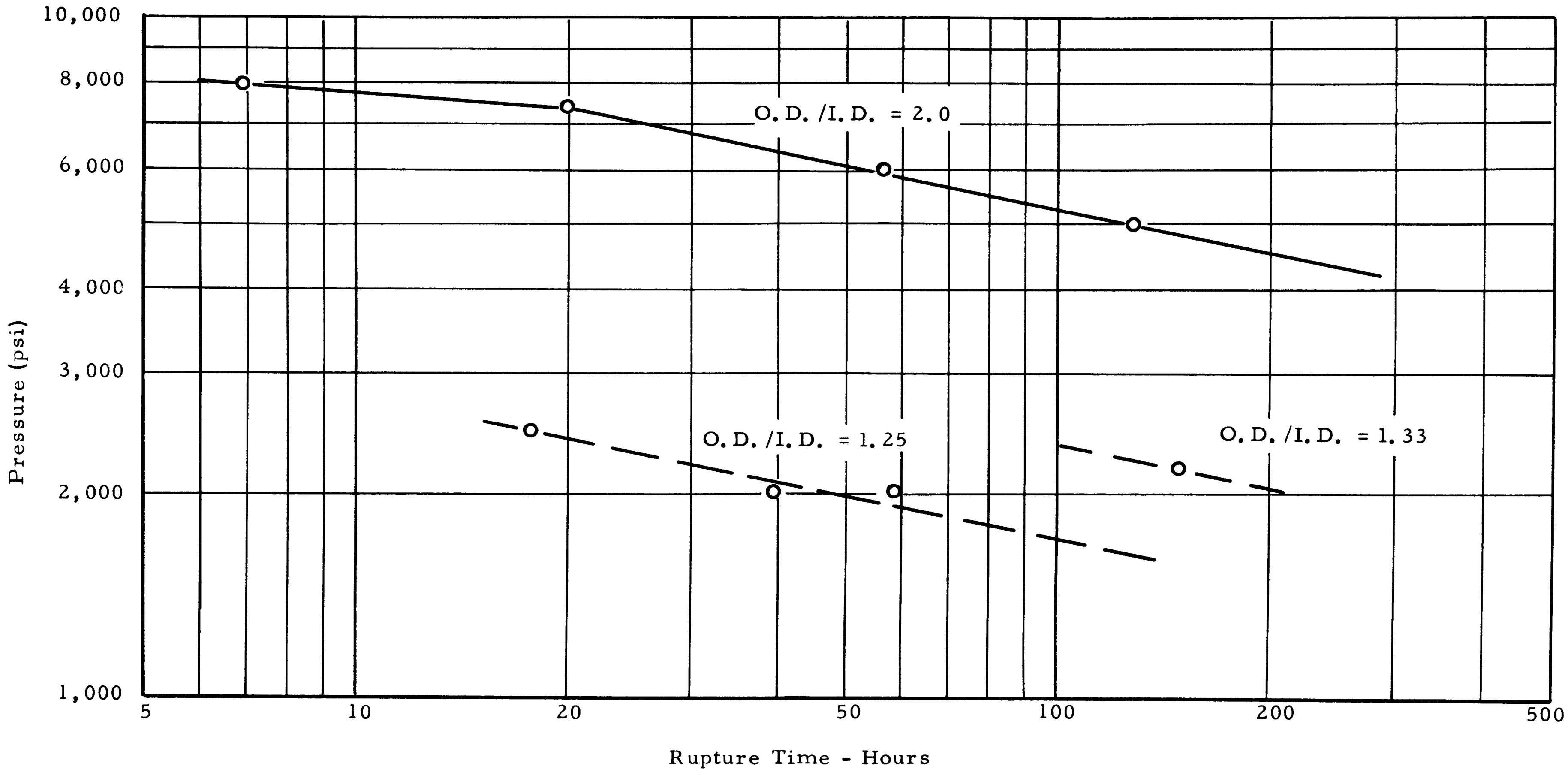



Figure 2. - Internal Pressure versus Rupture Life at 1050°F for Annealed Carbon Steel Vessels.

For the typical dimensions of these specimens (D = 1.270, d = 1.020, t = 0.125 inch) the initial Lamé stresses on loading follow:

| Location | Tangential Component | Axial Component | Radial Component | Shear-Stress Invariant |
|---|---|---|---|---|
| Inner surface | 4.64p | 1.82p | -1.0p | 4.88p |
| Outer surface | 3.64p | 1.82p | 0 | 3.15p |

The maximum (tangential) principal stress is initially about 5% lower than the shear-stress invariant at the bore and about 15% above it at the outer surface. The rather moderate stress gradient across the wall of a thin cylinder should limit the amount of stress redistribution, but any such stress shifts as do occur tend to bring the tangential stress and the shear-stress invariant closer together. Therefore, regardless of which criterion controls rupture life, the tension/pressure ratio found for pressure alone should give a good approximation to the hoop stress. For combined pressure-tension experiments with tubes of these same dimensions and test conditions, the hoop stress may be estimated as 4.9 times the magnitude of the pressure.

Results are available for three tubes under combined pressure and tension. Loadings employed, initial stress patterns at the bore and the rupture lives according to three criteria of failure are listed in Table 3, together with the experimentally-determined rupture lives.

The two failure criteria of greatest interest predict rupture lives too close together for certain differentiation between them. However, the better agreement does seem to lie with the shear-stress invariant. The first two tests do seem to rule out need for further consideration of maximum shearing stress insofar as stress rupture is concerned. Any stress redistribution should tend to extend life for either criterion of failure. With this consideration, the third test is probably a better support for the von Mises theory than a cursory examination of the data indicates.

Table 3

Tests at 1050°F with Annealed Carbon Steel in Combined Internal Pressure and Axial Tension
(D = 1.27, d = 1.02, t = 0.125 inch)

| Internal Pressure (psi) | Additional Axial Load (psi) | Initial Stress Pattern (psi) | | | | Rupture Life for Different Failure Criteria (Hours) | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Tangential | Axial | Radial | $\overline{S}$ | Max. Normal Stress | Max. Shear Stress | Shear-Stress Invariant | Experi-mental |
| 1,480 | 7,340 | 7,250 | 10,030 | -1,480 | 10,400 | 48 | 21 | 40 | 43.3 ($\pm$0.7) |
| 2,050 | 2,830 | 10,050 | 6,560 | -2,050 | 10,790 | 47 | 14 | 34 | 32.1 ($\pm$4.6) |
| 1,700 | 5,210 | 8,300 | 8,300 | -1,700 | 10,000 | 140 | 50 | 50 | 90 ($\pm$2) |

In a written discussion of Reference 8, C.L. Clark mentioned that nearly identical rupture lives had been found when specimens were sampled from the longitudinal and transverse directions of tubes removed after extended pressure service at elevated temperature. This finding appears to refute the role of maximum principal stress in determining rupture life. A number of further tests of this sort have now been run with specimens sampled from thick-walled carbon-steel tubes carried to rupture under pressure in tests at the University of Michigan. Results were somewhat variable but no positive effect of sample orientation on rupture behavior could be found.

Until some clear-cut demonstration is made that the largest normal stress plays a key role in creep-rupture, continued use of the shear-stress invariant to correlate all elevated-temperature behavior under complex stresses appears to be reasonable and justified.

## STEP-WISE CALCULATION OF RUPTURE LIFE OF THICK PRESSURE VESSELS

The portion of life used up by a period of time at any particular stress at constant temperature was shown earlier to be simply the length of time at that stress divided by the rupture life in a test with that stress held fixed. When a thick vessel is held under constant pressure at elevated temperature, complex changes in stress and strain throughout the walls may be expected, with gradual leveling of initial stress gradients. In the actual vessel the stress levels will vary smoothly from point to point without discontinuities, but to facilitate calculations the cross section will be divided into a sufficient number of concentric rings or shells such that conditions at the centroid of any given shell are quite representative of that entire shell. Further, the actual continuous change in stress pattern will be replaced by an equivalent series of time intervals over each of which the creep rate and stress in a given fiber may be considered nearly constant.

The fraction of rupture life expended during each interval is to be calculated for each shell. When the cumulative fraction for any one shell reaches unity, rupture at that location should occur and failure of the entire vessel is imminent.

Stress Redistribution by the Creep-Relaxation Process

Onset of yielding was mentioned previously to depend on an effective stress $\overline{S}$, calculated by equation (9). A similar criterion can be written to define an effective strain $\overline{e}$ in terms of the principal strains:

$$9/2\ (\overline{e})^2 = (e_1 - e_2)^2 + (e_1 - e_3)^2 + (e_2 - e_3)^2 \ . \qquad (12)$$

The effective strain for pure tension equals the axial strain.

Immediately on loading, a fiber in the vessel wall has a unique effective creep rate determined by the initial effective stress.

The component of this plastic strain in any direction may result in elongation (creep) of the body, but it could also replace initial elastic strain, with resultant drop in the stress level of the fiber (relaxation). How the total plastic deformation splits between creep and relaxation depends on the extent of stress gradients in the structure.

In a conventional tensile creep bar, where all fibers are subjected to the same stress until necking occurs, the body can creep as a unit with no reaction of one fiber on another. Such is not the case, however, where the stresses vary continuously from one fiber to the next.

Consider a flat bar with three parallel bands having axial stresses $S_1 > S_2 > S_3$ at their respective centroids. Corresponding axial creep rates, if each were separate from its neighbors, would be $C_1 > C_2 > C_3$. For continuity to be maintained between filaments, the same total deformation must exist on the two sides of the common interface. This does not say that the deformations at the two edges of a particular band will be the same. The creep rate at different points across any such band will deviate slightly from the rate at its centroid, but this latter value should be quite representative if the band chosen is not too wide.

When band 1 has a total creep in excess of band 2, the difference in plastic strain must be made up by elastic strains in the two bands so long as the fibers of 2 do not become stressed above their yield point. This elastic interaction gives a stress reduction (or relaxation) in band 1 and a stress rise in 2. The absolute values of these two elastic changes will be distributed inversely as the areas of the two bands concerned.

In relaxation, plastic strains replace equal but opposite elastic strains initially present so that relaxing fibers exhibit behavior characteristic of both plastic and elastic deformation. In the elastic region the generalized relationship between components of stress and strain may be expressed:

$$S_{ij} = \lambda e_{ii} \delta_{ij} + 2Ge_{ij} , \qquad (13)$$

where $\lambda$ is a proportionality constant and *G* is the shear modulus.

By the assumption of plastic incompressibility inherent to the criterion of yielding and of creep already adopted in this analysis, $e_{ii}$ can be set equal to zero. Therefore changes in principal stresses are equal to *2G* times the corresponding principal strains. The same relationship holds for elastic changes involving effective stress and effective strain. The factor *2G* is related to the elastic modulus (E) and Poisson's ratio ($\nu$) by:

$$2G = E/(1 + \nu) \quad . \qquad (14)$$

Available data indicate that $\nu$ has a value of about 0.32 within the elastic range for many alloys and temperatures of interest.

Simultaneously with the interaction between bands 1 and 2, the elastic stress in band 2 is relaxing and that in 3 increasing due to the like interaction at the 2-3 interface. The net stress change for band 2 is the difference between the gain from 1 and the loss to 3.

This procedure can be applied in turn to strain rates and stresses for each pair of concentric rings in a cylindrical vessel. Calculations for a thick pressure vessel will start with the two innermost rings and procede outward. Stress exchanges at each interface are found for the same short time interval. When all changes are known, the new stress levels in each ring are calculated and the process repeated.

## DETAILS OF PROPOSED CALCULATION METHOD

The initial stress distribution in a thick pressure vessel can normally be obtained from the Lamé equations. For large ratios of outside to inside diameter initial stress gradients in the tangential and radial directions are steep near the bore and quite flat at larger radii. Under such a pattern, any imaginary shell near the bore with essentially-uniform stress level must be extremely thin, while at larger radii a much thicker layer could be considered without undue spread of conditions over its thickness. If the vessel is imagined to consist of concentric shells each with twice the circular cross section of the adjacent shell at smaller radius, a minimum of six such shells appears to give a satisfactory coverage of the total range of stresses across the wall. Table 4 locates the centroids for six such shells in a vessel with b/a = 2. Initial component stresses for Lamé's elastic loading are also listed for each shell. The gradation in stress from shell to shell with this distribution of area is quite uniform during the critical initial period when creep is rapid and creep rates are very unequal at different radii. Moreover, the innermost and outermost shells approach the surfaces in stress level for the shell thicknesses adopted.

Examples will be cited for two different materials at temperatures where extremes of behavior under a stress gradient are shown. First a thick cylinder of the annealed carbon steel of Figure 2 will be considered at 1050°F and 5500 psi pressure, with a wall thickness ratio of 2.0. Experimental creep curves were obtained for this alloy at stresses in tension between 5000 and 13,000 psi.

Table 4

Position and Stress Pattern at Centroids of Shells Chosen for Analysis of Stress Redistribution in a Vessel with OD/ID = 2

| Shell No. | Fraction of Total Cross Section | (r/b) at Centroid | $S_T/p$ | $S_Z/p$ | $S_R/p$ | $\overline{S}/p$ |
|---|---|---|---|---|---|---|
| Bore | | | 1.67 | 0.33 | -1.0 | 2.30 |
| 6 | 1/63 | $\sqrt{43/168}$ | 1.64 | 0.33 | -0.97 | 2.26 |
| 5 | 2/63 | $\sqrt{23/84}$ | 1.55 | 0.33 | -0.88 | 2.10 |
| 4 | 4/63 | $\sqrt{13/42}$ | 1.41 | 0.33 | -0.74 | 1.86 |
| 3 | 8/63 | $\sqrt{8/21}$ | 1.21 | 0.33 | -0.54 | 1.52 |
| 2 | 16/63 | $\sqrt{11/21}$ | 0.97 | 0.33 | -0.30 | 1.10 |
| 1 | 32/73 | $\sqrt{17/21}$ | 0.74 | 0.33 | -0.08 | 0.71 |
| Outer Surface | | | 0.67 | 0.33 | 0 | 0.58 |

Minimum creep rate for each individual test was obtained, along with the average rate of creep for the first per cent of rupture life and the value at 80% of expired life. These results are plotted in Figure 3 as stress versus creep rate, with the fraction of life expired as a parameter. Rates of creep at times before and after the minimum-rate period may thus be estimated. Since the shear-stress invariant in simple tension is the tensile stress itself, the effective (invariant) creep rate $\overline{\dot{e}^p}$ for each shell can be read directly from Figure 3.

From Table 4, the initial pattern of effective stress and effective strain rate may be evaluated and listed:

Initial Conditions for Annealed Carbon Steel Vessel
at 5500 psi (OD/ID = 2)

| Shell No. | $\overline{S}$ (psi) | $\overline{\dot{e}^p}$ (in./in./hr) |
|---|---|---|
| 6 | 12,430 | 0.040 |
| 5 | 11,570 | 0.025 |
| 4 | 10,240 | 0.011 |
| 3 | 8,340 | 0.0027 |
| 2 | 6,050 | 0.0005 |
| 1 | 3,910 | 0.00008 |

Effective creep rates for each pair of shells are tabulated and subtractions performed to determine the differential strain rate at each interface between the hypothetical shells under consideration. For the innermost three shells in this example, values would be as follows:

| Shell No. | Effective Creep Rates (in./in./hr) |
|---|---|
| 6 | 0.040 |
| 5 | 0.025 |
| 6 - 5 | 0.015 |
| 5 | 0.025 |
| 4 | 0.011 |
| 5 - 4 | 0.014 . |

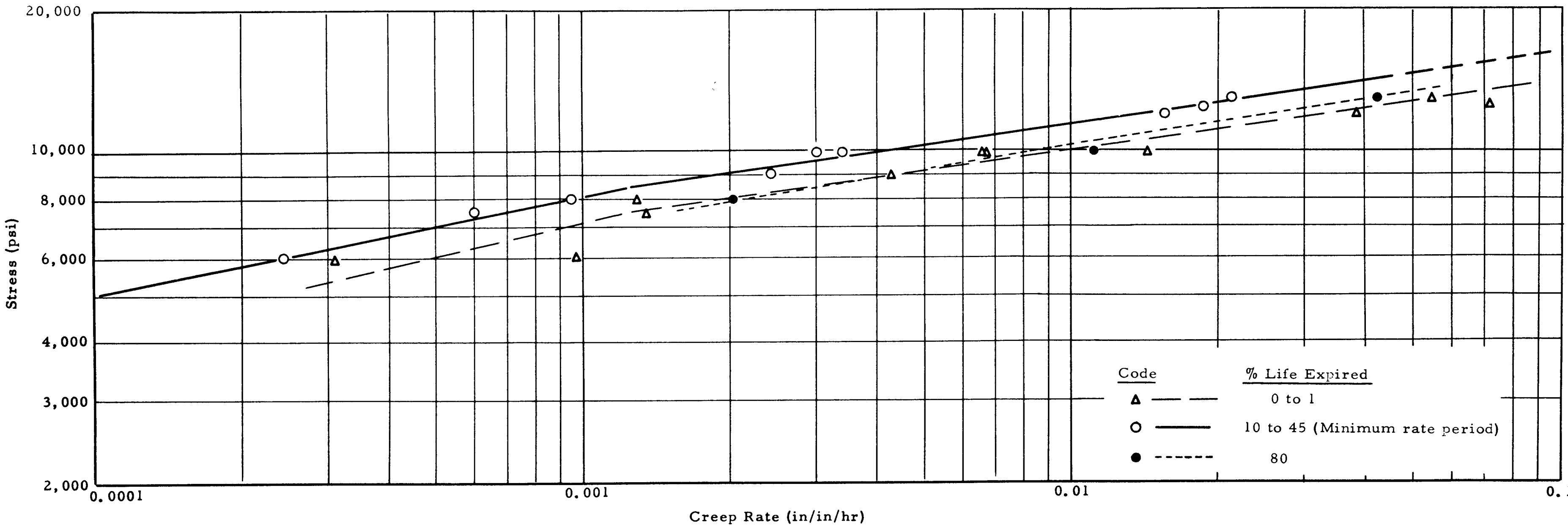



Figure 3.- Stress versus Creep Rate at 1050°F for Annealed Carbon Steel.

The creep-rate difference between shells may be converted to a stress change for a time interval short enough that the assumption of constant creep rates over the interval is not too far in error. As indicated by equation (13), this stress interaction equals the difference in creep rates, multiplied by the time interval and by the shear modulus. For an interval of 0.002 hours, the total stress change for the two pairs of shells tabulated above would be:

(0.015 in./in./hr)(0.002 hr)(24,000,000 psi/in./in.)/1.32 = 546 psi

and

(0.014 in./in./hr)(0.002 hr)(24,000,000 psi/in./in.)/1.32 = 509 psi.

The total stress reaction between adjacent shells is distributed inversely as the respective areas, so that 2/3 of the total affects the one nearer the bore for each pair. The resulting pattern of changes follow for the shells illustrated. (The signs of the stress changes follow from the very nature of relaxation to reduce stress differences.)

| Shell No. | Total Stress Interaction (psi) | $\Delta\bar{S}$ |
|---|---|---|
| 6 | | -(2/3)(546) = - 364 |
| 5 | | +(1/3)(546) = + 182 |
| 6 - 5 | 546 | |
| 5 | | -(2/3)(509) = - 339 |
| 4 | | +(1/3)(509) = + 170 |
| 5 - 4 | 509 | |

After all changes are determined, the net change for each shell may be found. Thus, for shell 6 after the initial interval of 0.002 hour the effective stress would be 12,430 - 364 = 12,066 psi. Shell 5 alters by the net change at the 6-5 and the 5-4 interfaces. Its new effective stress is 11,570 + 182 - 339 = 11,413 psi.

Figure 1 permits one to determine the rupture life for the effective stress at each end of the time interval. In the present instance, for shell 6 the rupture life increased from 12 to 13 hours in the 0.002-hour interval. The fraction of life consumed during that period was only

$$\frac{0.002}{[1/2\ (12 + 13)]}\ (100\%) \ = \ 0.016\%\ .$$

At the new level of stress for each shell the creep rate may once more be found from Figure 3 and the process repeated. For the steel and temperature of this illustration the high rate of stress leveling continues with negligible use of total life. Repeated calculation cycles indicate that by the end of about two hours the effective stress across the entire wall is essentially uniform at about 5800 psi. The maximum expenditure of life (for shell 6) is only about one per cent for this complete process. The stress in shell 6 now corresponds to a rupture life of some 1000 hours contrasted with the initial 12-hour rupture life.

Usually creep rates are not known for other than the minimum-rate period. However the error caused by using only the minimum rate should not be serious when the difference in creep rates across the vessel wall spans several cycles on a logarithmitic scale.

When the effective creep was added together for all the time intervals considered, the total creep at two hours was found to be about 0.42%. In most instances, less than 1% of creep strain should be plenty to allow for substantial uniforming of stress gradients.

One must not stop with the result that rapid creep quickly equalizes stress gradients. At the equalized stress the creep rate for this particular steel has been reduced by a factor of 100 or so but is still appreciable. Continued creep is no longer beneficial, but instead causes progressive and accelerating increase in the uniform effective stress present. The rate of this stress rise should be in proportion to the rate of change in the ratio of diameter to wall thickness. During a period of steady creep, in the absence of localized bulging, the stress at the end of the period should equal the initial stress times the ratio

$$\frac{(1 + \text{creep strain during the interval})}{(1 - \text{creep strain during the interval})} .$$

Completed calculations for the example cited above show that the major shifting of stresses ceased by the end of 1.5 hours. At that point shell 6 had an effective stress of about 6000 psi and an effective creep rate of 0.00048 in./in./hr. A period of 8.5 hours at these conditions would result in (8.5 hours) x (0.00048 in./in./hr) = 0.0041 in./in. of creep. Therefore at the end of a total elapsed time of ten hours the uniform effective stress should be approximately

$$(6000 \text{ psi})(1.0041/0.9959) = 6045 \text{ psi} .$$

This moderate rise in stress corresponds to a significant drop in rupture life in shell 6 from 840 to 780 hours.

Approximately 0.8% of rupture life was consumed during the first 1.5 hours. The next 8.5 hours just considered "uses up" another 8.5/(1/2)(840 + 780) = 1.5%. The stress gain tends to produce a higher creep rate, but this is largely offset at early times by gradual approach to the minimum creep as life expenditure nears 10%.

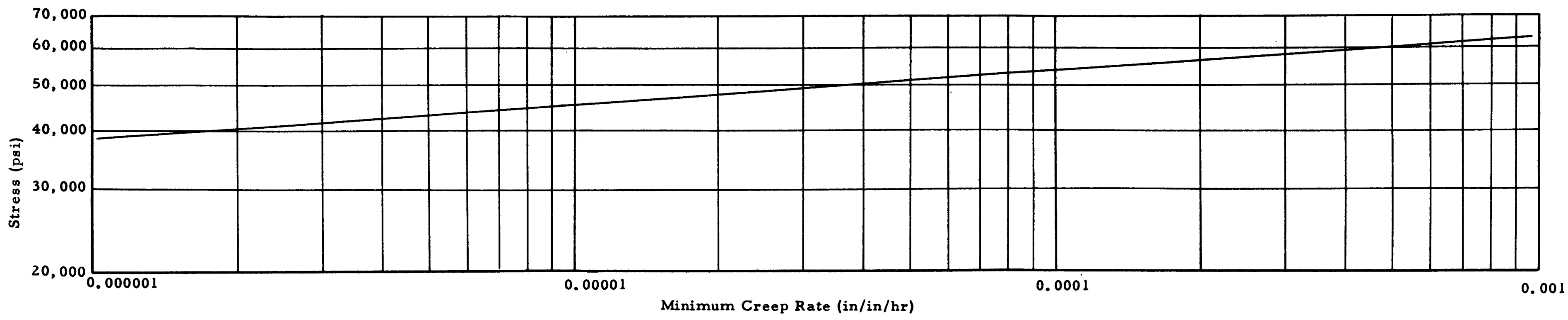
If the calculations are continued, the maximum possible rupture life under ideal conditions would be near 300 hours. To reach such a life would require absolute freedom from out-of-roundness, eccentricity or extraneous stresses. Once the stress at the outside wall reaches the leveled value, a situation corresponding to a completely-plastic vessel is reached. With the rapid creep now present there, the slightest source of localized stress at the outer surface leads to an immediate instability comparable to that in a column bending under an axial compression. The higher stress increases the creep rate which in turn accelerates the local stress rise. The resulting instability can produce bulging and rupture at but a fraction of the time predicted from calculations assuming complete uniformity in wall thickness. It might, for instance, be noted that the experimental result for the above condition is only about 55 hours. (See Figure 2.)

The creep required to cause an instability is surprisingly limited. Even for the idealized conditions considered in the calculations just mentioned, a uniform creep strain of 10% finds the vessel in the last stages of failure.

In the light of these crude calculations one may perhaps appreciate the reasons behind the frequent American practice of designing on the basis of a stress which will restrict creep to 1% in 100,000 hours or some similar value. <u>Thick pressure vessels of alloys with low creep strength will always be limited by the amount of creep they can withstand before instabilities set in, and not by the initial concentrated stress or rupture life at the equalized stress level.</u>

A contrasting example is a vessel of the same proportions but made of Inconel X alloy and operated at 1350°F. Typical curves of rupture life and minimum creep rate as a function of stress are shown in Figure 4 for this alloy in its usual condition of heat treatment for high creep strength. To make this set of calculations comparable to the first, the rupture life of the inside shell (number 6) at the initial stress was set at 12 hours in both instances. For the Inconel X vessel, this requires an effective stress $\overline{S}$ = 60,000 psi, or an internal pressure of 26,100 psi. At the 60,000 psi stress the minimum creep rate is only about 0.0005 in./in./hr. Even if actual rates in early first stage of creep were double or even five times this value, relaxation is nonetheless a lower-order effect than it was for annealed carbon steel.

Calculations based on the available minimum-creep rate data show the shear-stress invariant at the bore to drop only to 58,000 psi in the first 1.5 hours, and while this 2000 psi decline takes place more than ten per cent of the total available life expires. Even at the 30 to 35 hours total time until



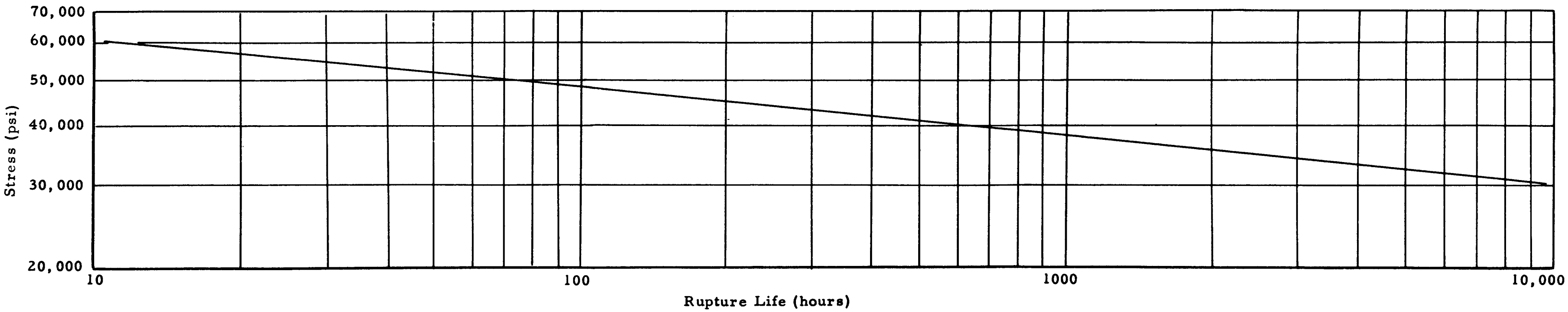




Figure 4. - Typical Creep and Rupture Data for Inconel X at 1350°F.

fracture predicted by the calculations the effective stress level has only fallen to the 50-hour rupture stress.

Although several laboratories are understood to have research well under way for experimental determination of stress-rupture life in heavy-walled tubing of heat-resistant alloys, to date no results seem to have been published. Until the analysis put forth in this paper is supported or refuted by actual tests with quite-refractory alloys, extended discussion of the significance of the calculated results is largely speculative. However a brief examination of some present practices does appear to be in order.

When the thickness of a cylinderical shell made under the ASME Code for Unfired Pressure Vessels exceeds half the inside radius, the design pressure permitted for a seamless shell equals S (Z-1/Z+1), where S is the maximum allowable stress for the alloy and Z is the same ratio of OD/ID previously designated k. Inspection of equation (1) shows the Code formula to give the tangential Lamé stress as the working stress.

Returning to the second example, the calculated rupture life of 30-35 hours corresponds to a rupture stress of approximately 55,000 psi. For the same pressure the initial tangential stress is only 43,500 psi. The actual apparent fracture stress thus exceeds the design level by 11,500/43,500 = 26.5%.

Consider now the same vessel but at half the pressure. Following the von Mises theory, rupture should now occur at 7000 hours, corresponding to the initial effective stress of 30,000 psi at the bore if no redistribution at all of stress takes place. This is almost precisely the situation since at 30,000 psi stress the creep rate is a mere 0.01% or so per 1000 hours.

A sixty-fold increase in rupture life when the stress was halved is accompanied by a drop in creep rate of some 5000 times. An initial calculation interval of 1000 hours under these conditions predicts 13% loss of the total life while reducing the effective bore stress only about 400 psi.

Detailed calculations for this low stress would have questionable accuracy since they must be based completely on extrapolations of available creep-rate data. It can be stated with confidence, however, that the stress redistribution would be proportionally less than that for the same vessel at higher pressure. In extended service at temperatures where alloys exhibit negligible creep the stress promoting rupture should agree closely with the initial shear-stress invariant at the inner surface. Under conditions where no stress leveling took place the actual fracture stress for $k = 2$ would be almost 40% above the tangential stress commonly employed as a design standard in American practice. For higher ratios of OD/ID the discrepancy between actual and design stress is even higher. Under these extreme conditions, even the normally-conservative use of an allowable stress equal to 60% of the 100,000 hour rupture strength could be positively dangerous for an installation designed to operate for many years of service.

The solution does not appear to go to still lower allowable stresses since this would prevent efficient utilization of high-strength alloys. A more rational answer is to adjust design to the method of failure imposed by the vessel geometry and type of material used in its construction. As has been indicated by the opposing results of the examples cited, early failure is promoted by either very rapid creep or by very slow creep, but the mode of approach to eventual rupture in the two cases is quite different. Any attempt to codify design for all elevated-temperature pressure vessels by one simple formula seems doomed from the start.

Until sufficient data are in the hands of the designer to guide him, analysis of each case by a method such as that outlined above is suggested. When variable operations must be handled, calculations for the first set of conditions are carried out as outlined to establish the stress pattern at the end of that period. A subsequent change in pressure can be allowed for by superimposing the elastic pattern for a stress equal in magnitude to the change taking place. Gradual temperature changes require that only the properties used in each step apply to the material at the existing temperature. Rapid changes in temperature can be allowed for by methods outlined in references 4 and 5.

The closer such calculations can evaluate the changing stress pattern, the more one can take full advantage of the inherent strength of alloys employed. But in most instances a high degree of accuracy appears unnecessary, provided it can be determined whether failure will result by stress rupture from concentrated stresses with little change in overall geometry, or whether the final rupture is promoted by extreme general creep and resulting instabilities.

When the calculations indicate rapid stress leveling, followed by progressive thinning of the wall, the appropriate design data are creep rates or, preferably, data on stress levels for specific amounts of total deformation in a given time period. The appropriate design stress is the average toward which the effective stress tends. By a simple integration indicated in Table 1 this average shear-stress invariant is found to be:

$$(\overline{S})_{ave} = 2\sqrt{3}\, p \left(\frac{ab}{b^2 - a^2}\right)^2 \ln (b/a) \,. \qquad (15)$$

A suggested design criterion for alloys with high creep rates at service temperature is 0.8 of the stress for 2% creep in the anticipated life, with this stress applied for $(\overline{S})_{ave}$ as found by equation (15) above. This design method should be valid for the bulk of more-ductile elevated-temperature alloys which exhibit a marked degree of strengthening in the notched-bar rupture test.

Design on such a basis of limited deformation is difficult to justify for alloys treated to produce maximum creep strength. Such materials, generally showing borderline behavior or weakening in the presence of a notch, fail as a consequence of retained high initial stress concentrations. In this case, corresponding to a partially-plastic cylinder at lower temperatures, the total strains at the outer surface may be very small when the first rupture begins at regions of high initial stress concentration. Maximum effective stress expected during normal service appears to be the proper design stress for such a vessel. In most cases, this critical stress is the initial value at the bore. The allowable level of design stress should now be based on stress-rupture data. Limitation to 0.8 of the stress for rupture in the anticipated service life appears to allow ample margin of safety when operating conditions can be reasonably fixed in advance and when the designer has some familiarity with the particular type of alloys employed. For alloys at temperatures where creep is very slow, the suggested design criterion is 0.8 of the stress for the rupture in the anticipated vessel life. This stress is to be applied to the maximum effective stress. This will normally be the initial effective stress at the bore.

It might be well to repeat that the suggestions in this paper are all based on an approximate analysis of probable stress redistribution by a creep-relaxation process. The same general type of analysis appears to have been quite successful in explaining contrasting notch behaviors of heat-resistant alloys with different relaxation strengths. Experimental measurements of rupture lives in thick vessels of carbon steel appear to be in rough agreement with predictions based on the calculations such as outlined in this paper.

Still, until a considerable mass of data becomes available for a variety of alloys, vessel geometry, and test conditions the methods suggested here should be used only as guides for gradual modifications from current good practice where present design methods meet severe limitations.

## A SUGGESTED IMPROVED DESIGN FOR EXTREME CONDITIONS

Since either excessively-rapid or unusually-slow rates of creep can lead to shortened life of a heavy-wall vessel, the range of temperature and pressure is severely limited over which a vessel of any one particular alloy gives superior service. This is especially true when heat is added from the outside.

To extend the useful operating range of the vessel, a high creep rate is desired near the bore to quickly relax initial peak stresses present there, but a slow rate is wanted at the outer surface to restrain overall expansion of the vessel.

A simple duplex design employing two different alloys appears to offer an easy but satisfactory solution. The inner shell would be made of an alloy with only moderate creep strength and capable of extended creep before rupture. Material for the outer shell could be a much more refractory alloy, with quite limited ability to redistribute stresses by creep and need have only a few per cent elongation before fracture. Materials for the two layers can be chosen to accomodate different corrosion and temperature conditions at inner and outer locations.

As yet no duplex vessel designed with this viewpoint is known, but it appears that a substantial degree of leveling of the most severe bore stresses could be attained without need for shrink fits or other prestressing.

No reason is immediately apparent why the two shells could not be assembled with ordinary sliding fits. Creep of the inner shell on the first loading at temperature would bring the two layers into firm contact which would be maintained from then on.

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