ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

THE HEAT TRANSPORT BETWEEN TWO PARALLEL PLATES AS FUNCTIONS OF THE KNUDSEN NUMBER

By

C. S. WANG CHANG

G. E. UHLENBECK

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I. INTRODUCTION

The method of the previous report[1] for treating transport phenomena by starting immediately from the Boltzmann equation (instead of using the equations of motions of the fluid) has been applied to the problem of heat conduction between two parallel plates. We assume, as in the previous report, that the "Mach number", which in this case is measured by the ratio of the temperature difference, $2\Delta T$, between the plates and the average temperature T is small, so that only terms of the first order in $\Delta T/T$ are kept. The Knudsen number, d/λ, where d is the distance between the plates and λ is some sort of mean free path is, however, arbitrary. In this way the transition from the Clausius to the Knudsen regime can be described more completely than in an earlier report.[2]

Formal expressions for the heat flux and the temperature distribution between the plates can be derived for arbitrary values of d/λ. The limiting cases for the Knudsen gas ($d/\lambda \ll 1$) and for the Clausius gas ($d/\lambda \gg 1$) can be deduced from the general expressions. All observable quantities like the heat flux and the temperature distribution are functions of the Knudsen number $K = d/\lambda$. It is to be noted at the outset that only for small values of K can a power-series development in K be obtained. It is not possible to find a series expansion in inverse powers of K, since $K = \infty$ is an essential singularity. The approach to the Clausius regime is therefore more complicated than previously assumed, due to the development of successive types of boundary layers, as will be explained in Sec. VI. This insight is the main result of this report.

[1] C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in Monatomic Gases", Univ. of Mich., Eng. Res. Inst., Proj. M999, Oct. 1952.
[2] C. S. Wang Chang and G. E. Uhlenbeck, "Transport Phenomena in Very Dilute Gases", CM 579, UMH-3-F, Univ. of Mich., Eng. Res. Inst., Proj. M604-6, Nov. 15, 1949.

II. FORMULATION OF THE PROBLEM

As in Ref. 2, we take the (y-z) plane halfway between the plates. The upper plate, $x = d/2$, has a temperature $T - \Delta T$, while the lower plate at $x = -d/2$ is kept at $T + \Delta T$. We make the assumptions that

1) $\Delta T \ll T$, and

2) the accommodation coefficient is α; i.e., α is the fraction of the molecules that is re-emitted by the wall with the temperature of the wall.

We will use the notation of Ref. 1; the distribution function is

$$f = f_0 (1 + h(\vec{c}, x)),$$

where $\vec{c}$ is the dimensionless velocity (unit $(m/2kT)^{1/2}$) and f_0 is the complete equilibrium distribution. Because of the first assumption the Boltzmann equation becomes a linear integral differential equation for the disturbance h, of the form

$$c_x \frac{\partial h}{\partial x} = n J(h), \tag{1}$$

where n is the number density and J is the collision operator

$$J(h) = \frac{1}{\pi^{3/2}} \int d\vec{c}_1 \, e^{-c_1^2} \int d\Omega \, g I(g, \theta) (h' + h_1' - h - h_1), \tag{2}$$

which has the dimension of an area and has the order of magnitude of a collision cross section.

The boundary conditions are formulated as follows: In the complete distribution function f, we distinguish between the molecules going up and those going down. Calling these distribution functions f^+ and f^- respectively, where the plus and minus signs are the signs of the x-component of the velocity, we write:

$$f = f^+ + f^- .$$

The boundary conditions are then:

$$f^+(-\tfrac{d}{2}) = \alpha f_0 \left[1 + \frac{\Delta T}{T}(c^2 + B^+)\right] + (1-\alpha) f^-(-c_x, -\tfrac{d}{2}) \qquad c_x > 0 \tag{3a}$$

$$f^-(+\tfrac{d}{2}) = \alpha f_0 \left[1 - \frac{\Delta T}{T}(c^2 + B^-)\right] + (1-\alpha) f^+(-c_x, +\tfrac{d}{2}) \qquad c_x < 0 \tag{3b}$$

where

$$f_0 = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-c^2} .$$

Eq (3a) expresses the fact that the distribution function of the molecules going up at the lower plate consists of two parts: a fraction $(1 - \alpha)$ of the total molecules specularly reflected by the plate so that this portion has the distribution function of the molecules going down at $d/2$ with the velocity component c_x reversed, and a fraction α of the total molecules re-emitted by the plate with the temperature $T + \Delta T$. Eq (3b) is a similar statement for the molecules leaving the upper plate. B^+ and B^- are two constants which take care of the different densities at the two plates. They are to be determined by the following two conditions:

a) the total number of molecules per unit area between the plates is nd, and

b) there is no streaming velocity in the x-direction.

In terms of the disturbance h, Eqs (3a) and (3b) are:

$$h^+(-\tfrac{d}{2}) = \frac{\alpha \Delta T}{T}(c^2 + B^+) + (1-\alpha) h^-(-c_x, -\tfrac{d}{2}) \qquad c_x > 0 \tag{4a}$$

$$h^-(+\tfrac{d}{2}) = -\frac{\alpha \Delta T}{T}(c^2 + B^-) + (1-\alpha) h^+(-c_x, +\tfrac{d}{2}) \qquad c_x < 0 \tag{4b}$$

and the conditions for the determination of B^+ and B^- are:

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} dx \int d\vec{c}\, e^{-c^2} h = 0 \tag{5a}$$

and

$$\int d\vec{c}\, e^{-c^2} c_x h = 0 . \tag{5b}$$

There is, furthermore, one symmetry property of the functions $h^{\pm}$, namely:

$$h^{\pm}(c_x, x) = -h^{\mp}(-c_x, -x) \, . \tag{6}$$

This symmetry property is a consequence of the first assumption and can easily be seen by reversing the x-axis while keeping the temperatures unchanged. From Eq (6), it follows that

$$h(c_x, x) = -h(-c_x, -x) \, . \tag{7}$$

Thus Eqs (4a) and (4b) are equivalent to each other provided

$$B^+ = B^- ,$$

and Eq (5a) is then automatically satisfied.

To summarize, our problem is to solve the linear integral equation, Eq (1) with the boundary condition Eq (4a), namely:

$$h^+(c_x, -\tfrac{d}{2}) = \frac{\alpha \Delta T}{T}(c^2 + B) + (1-\alpha)\, h^-(-c_x, -\tfrac{d}{2}) \qquad c_x > 0 ,$$

where $h^{\pm}$ have the symmetry property Eq (6) and where B is to be determined by Eq (5b). The heat flux and the temperature distribution are then readily calculated.

$$q = \frac{nkT}{\pi^{3/2}}\sqrt{\frac{2kT}{m}} \int d\vec{c}\, e^{-c^2} c^2 c_x h$$
$$T(x) = T\left[1 - \frac{2}{3\pi^{3/2}} \int d\vec{c}\, e^{-c^2}\left(\frac{3}{2} - c^2\right) h\right] . \tag{8}$$

It follows from Eq (1) and the conservation of energy that the heat flux q is a constant.

III. DERIVATION OF THE LIMITING RESULTS: THE KNUDSEN LIMIT

Before turning to the formal solution of the problem stated above, it is instructive first to derive the limiting results, valid for small and large values of the Knudsen number K.

When the Knudsen number is very small, in the zeroth approximation the collision term on the right-hand side of Eq (1) can be neglected. Writing:

$$h = \sum_{i=0}^{\infty} n^i h_i \tag{9a}$$

and

$$B = \sum_{i=0}^{\infty} n^i B_i \quad , \tag{9b}$$

we have:

1) Zeroth approximation:

$$\frac{\partial h_o}{\partial x} = 0$$

$$h_o = K_o(c^2, c_x).$$

The boundary condition now becomes:

$$h_o^+ (c^2, c_x) = \frac{\alpha \Delta T}{T} (c^2 + B_o) \frac{1 + \operatorname{sign} c_x}{2} + (1-\alpha) h_o^- (c^2, -c_x)$$

$$= \frac{\alpha \Delta T}{T} (c^2 + B_o) \frac{1 + \operatorname{sign} c_x}{2} - (1-\alpha) h_o^+ (c^2, c_x)$$

by use of the symmetry property. Thus

$$(2-\alpha) h_o^+ (c^2, c_x) = \frac{\alpha \Delta T}{T} (c^2 + B_o) \frac{1 + \operatorname{sign} c_x}{2}$$

and

$$(2-\alpha) h_o^- (c^2, c_x) = - \frac{\alpha \Delta T}{T} (c^2 + B_o) \frac{1 - \operatorname{sign} c_x}{2} \quad ,$$

or together:

$$h_o(c^2, c_x) = K_o(c^2, c_x) = \frac{\alpha}{2-\alpha} \frac{\Delta T}{T} (c^2 + B_o) \operatorname{sign} c_x . \tag{10a}$$

B_o as determined from Eq (5b) has the value -2. The complete distribution function to the Knudsen approximation is therefore:

$$f^{(0)} = n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-c^2} \left\{ 1 + \frac{\alpha}{2-\alpha} \frac{\Delta T}{T} (c^2 - 2) \operatorname{sign} c_x \right\} , \tag{10b}$$

from which it follows that the temperature between the plates is constant and is equal to T and the heat flux $q^{(0)}$ is

$$q^{(0)} = \frac{\alpha}{2-\alpha} \cdot 2nkT\sqrt{\frac{2kT}{\pi m}}\,\frac{\Delta T}{T_0} \tag{11}$$

independent of the molecular model as is to be expected. For perfect accommodation $\alpha = 1$, Eq (11) is the well-known Knudsen expression. For $\alpha = 0$, $q^{(0)} = 0$; in fact it will be found that for $\alpha = 0$, $q = 0$ to all approximations as is to be expected.

2) First approximation: In this approximation

$$\frac{\partial h_1^{\pm}}{\partial x} = \frac{1}{c_x} J(h_0)\,\frac{1 \pm \operatorname{sign} c_x}{2},$$

the solutions of which are:

$$h_1^{\pm} = \frac{x}{c_x} J(h_0)\,\frac{1 \pm \operatorname{sign} c_x}{2} + K_1^{\pm}(c^2, c_x), \tag{12}$$

where K_1^+ and K_1^- are integration constants like K_0. Because of the symmetry property, Eq (6), K_1^+ and K_1^- are related by the equation

$$K_1^{\pm}(c_x) = -K_1^{\mp}(-c_x).$$

Substituting Eq (12) into the boundary condition Eq (4a), one finds:

$$K_1^{\pm}(c_x) = \pm\frac{\alpha}{2-\alpha}\left(\frac{\Delta T}{T} B_1 + \frac{d}{2c_x} J(h_0)\right)\frac{1 \pm \operatorname{sign} c_x}{2}$$

and therefore:

$$h_1(c_x, x) = \frac{\alpha}{2-\alpha}\,\frac{\Delta T}{T} B_1 \operatorname{sign} c_x + \frac{1}{c_x} J(h_0)\left(x + \frac{\alpha}{2-\alpha}\,\frac{d}{2}\operatorname{sign} c_x\right).$$

B_1 as determined by Eq (5b) is found to be:

$$\begin{aligned} B_1 &= -\frac{T}{\Delta T}\,\frac{d}{2\pi}\int d\vec{c}\, e^{-c^2} \operatorname{sign} c_x\, J(h_0) \\ &= -\frac{\alpha}{2-\alpha}\,\frac{d}{2\pi}\left[\operatorname{sign} c_x,\ (c^2-2)\operatorname{sign} c_x\right], \end{aligned}$$

using the bracket symbol:

$$[A, B] = [B, A] = \int d\vec{c}\, e^{-c^2} A\, J(B).$$

Hence the first-order disturbance function h_1 is:

$$h_1(c_x, x) = -\left(\frac{\alpha}{2-\alpha}\right)^2 \frac{\Delta T}{T} \frac{d}{4\pi} \operatorname{sign} c_x \left[\operatorname{sign} c_x,\ (c^2-2)\operatorname{sign} c_x\right] + \frac{1}{c_x} J(h_0)\left(x + \frac{\alpha}{2-\alpha}\frac{d}{2}\operatorname{sign} c_x\right) \tag{13}$$

The first-order correction to the heat flux, $q^{(1)}$, is

$$\begin{aligned} q^{(1)} &= \frac{nkT}{\pi^{3/2}}\sqrt{\frac{2kT}{m}} \int d\vec{c}\, e^{-c^2} c^2 c_x \cdot n h_1 \\ &= q^{(0)} \cdot \frac{\alpha}{2-\alpha}\frac{nd}{4\pi}\left[(c^2-2)\operatorname{sign} c_x,\ (c^2-2)\operatorname{sign} c_x\right]. \end{aligned} \tag{14}$$

Since the square bracket [A, A] is always negative, the ratio $q^{(1)}/q^{(0)}$ is always negative. For $\alpha = 1$. this reduces to the expression given in Ref. 2.* For elastic spheres and Maxwell molecules, the square bracket has been evaluated in Ref. 2. The results are:

For elastic spheres (diameter σ):

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{\sqrt{2}\, nd\pi\sigma^2}{32}\frac{\alpha}{2-\alpha}\left(16\sqrt{2} - 5\right) \tag{15}$$

For Maxwell molecules (force law κ/r^5):**

$$\begin{aligned} \frac{q^{(1)}}{q^{(0)}} = -\frac{nd}{8}\frac{\alpha}{2-\alpha}\sqrt{\frac{\kappa}{kT}} \int_0^\pi d\theta \sin\theta\, F(\theta) \cdot \\ \cdot \Bigg\{ 7\pi - 2(4-3\cos\theta)\sin^{-1}\frac{1-\cos\theta}{2} - 2(4+3\cos\theta)\sin^{-1}\frac{1+\cos\theta}{2} + \\ + \frac{2(2-\cos\theta)(1-\cos\theta)\sqrt{1+\cos\theta}}{(3-\cos\theta)^{3/2}} + \frac{2(2+\cos\theta)(1+\cos\theta)\sqrt{1-\cos\theta}}{(3+\cos\theta)^{3/2}} \Bigg\} \end{aligned} \tag{16}$$

For Maxwell molecules, one can also make use (as in Ref. 1) of the eigenvalues and eigenfunctions of the collision operator J. The h's are expanded in terms of the eigenfunctions $\psi_{r\ell}$ with coefficients $a_{r\ell}$. In this way one finds:

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*The results given on p. 35 of Ref. 2 for $\alpha \neq 1$ are not correct.

**The function $F(\theta, \kappa, 5)$ used in Ref. 2 is $(2\kappa/m)^{1/2} F(\theta)$, where $F(\theta)$ is the dimensionless function used in Ref. 1.

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{\alpha}{2-\alpha}\frac{nd}{4\pi}\sqrt{\frac{x}{kT}}\sum_{r,\ell}\lambda_{r\,2\ell+1}\frac{2\ell+\frac{3}{2}}{(2\ell+r+\frac{3}{2})!}\frac{(\ell^2-\frac{\ell}{2}-2r)^2\left[(\ell+r-\frac{3}{2})!\right]^2}{r!}, \tag{17}$$

where

$$\lambda_{r\ell} = 2\pi\int_0^\pi d\theta\sin\theta F(\theta)\left[\cos^{2r+\ell}\frac{\theta}{2}P_\ell(\cos\frac{\theta}{2})+\sin^{2r+\ell}\frac{\theta}{2}P_\ell(\sin\frac{\theta}{2})-(1+\delta_{r0}\delta_{\ell 0})\right]$$

are the eigenvalues of J. The sums in Eq (17) can be carried out and this leads again to Eq (16) for $q^{(1)}/q^{(0)}$. This serves as a check of the result given in Ref. 2.

3) Second approximation. The calculation goes as before. The function h_2 is found to be:

$$\begin{aligned} h_2 = {} & \frac{1}{2c_x}\left(x^2-\frac{d^2}{4}\right)J\left(\frac{1}{c_x}J(h_0)\right) + \\ & +\frac{\alpha}{2-\alpha}\frac{dx}{2c_x}\left\{J\left(\frac{\operatorname{sign}c_x}{c_x}J(h_0)\right)-\frac{1}{\pi}J(\operatorname{sign}c_x)\left[\operatorname{sign}c_x,h_0\right]\right\} + \\ & +\left(\frac{\alpha}{2-\alpha}\right)^2\frac{d^2}{4c_x}\operatorname{sign}c_x\left\{J\left(\frac{\operatorname{sign}c_x}{c_x}J(h_0)\right)-\frac{1}{\pi}J(\operatorname{sign}c_x)\left[\operatorname{sign}c_x,h_0\right]- \right. \\ & \left. -\frac{c_x}{\pi}\left[\operatorname{sign}c_x,\frac{\operatorname{sign}c_x}{c_x}J(h_0)\right]+\frac{c_x}{\pi^2}\left[\operatorname{sign}c_x,\operatorname{sign}c_x\right]\left[\operatorname{sign}c_x,h_0\right]\right\}. \end{aligned} \tag{18}$$

The second-order correction to the heat flux is again found to be independent of x, as it should be. It is given by the following expression:

$$\begin{aligned} \frac{q^{(2)}}{q^{(0)}} = {} & \left(\frac{\alpha}{2-\alpha}\right)^2\frac{n^2d^2}{8}\left\{\pi\left[(c^2-2)\operatorname{sign}c_x,\frac{\operatorname{sign}c_x}{c_x}J\left((c^2-2)\operatorname{sign}c_x\right)\right]- \right. \\ & \left. -\left[(c^2-2)\operatorname{sign}c_x,\operatorname{sign}c_x\right]^2\right\}. \end{aligned} \tag{19}$$

The ratio $q^{(2)}/q^{(0)}$ is positive, which can be seen as follows: Replacing π in the curly brackets by the integral

$$\pi = \int d\vec{c}\, e^{-c^2} c_x \operatorname{sign} c_x,$$

then

$$\{ \ \} = \left(\int d\vec{c}\,' e^{-c'^2} c_x' \operatorname{sign} c_x'\right) \int d\vec{c}\, e^{-c^2} \frac{\operatorname{sign} c_x}{c_x} \left[J\left((c^2-2)\operatorname{sign} c_x\right)\right]^2 -$$
$$- \left[\int d\vec{c}\, e^{-c^2} \operatorname{sign} c_x\, J\left((c^2-2)\operatorname{sign} c_x\right)\right]^2,$$

which is seen to be positive by use of the Schwarz inequality.

IV. DERIVATION OF THE LIMITING RESULTS: THE CLAUSIUS GAS LIMIT

Even though the approach to the Clausius gas regime is more complicated, so that a development in inverse powers of the Knudsen number K (or of the density n) is not possible, we will, in this section, attempt to obtain a solution for the Clausius gas limit directly by making such a series development; i.e., we will write

$$h = \sum_{i=0}^{\infty} n^{-i} h_i.$$

The procedure adopted will be slightly different from that used in the previous section. We first solve the integral equations for h_i, making use of the symmetry condition Eqs (6) and (7) and of the fact that $\bar{c}_x$ must be zero, but without taking into account the boundary conditions Eq (4a). The solution will contain a number of arbitrary constants, and at the end of the section we will try to determine these constants from the boundary conditions. It will be seen that the integral equations for the successive approximations can be solved without any difficulty. The fact that these solutions cannot be the true solutions of the problem will be reflected by the fact that the boundary conditions cannot be satisfied exactly. Up to which approximation the calculation can be trusted will then also become clear.

Substituting the series expansion for h into the Boltzmann equation and equating terms of equal powers of n, we have:

1) Zeroth approximation:

$$J(h_0) = 0,$$

the general solution of which is:

$$h_0(x, c_x) = a_1^{(0)} + a_2^{(0)} c_x + a_3^{(0)} (c^2 - \tfrac{3}{2}), \tag{20}$$

where the a's are independent of c^2 and c_x but are in general still functions of x. The symmetry property, Eq (7), requires that $a_2^{(0)}$ be even in x and that $a_1^{(0)}$ and $a_3^{(0)}$ be odd in x. The fact that $\overline{c}_x = 0$ requires that $a_2^{(0)}$ be zero, while $a_1^{(0)}$ and $a_3^{(0)}$ may be arbitrary.

2) First approximation:

$$c_x \frac{\partial h_0}{\partial x} = J(h_1). \tag{21}$$

This inhomogeneous integral equation will have a solution only if the left-hand side of the equation is orthogonal to the solutions of the homogeneous equation. Multiplying Eq (21) by e^{-c^2}, $c_x e^{-c^2}$, and $(c^2 - 3/2)e^{-c^2}$ respectively, one sees that Eq (21) will have a solution if

$$\frac{da_1^{(0)}}{dx} + \frac{da_3^{(0)}}{dx} = 0 . \tag{22}$$

Equation (21) becomes therefore:

$$\frac{da_3^{(0)}}{dx}\left(c^2 - \frac{5}{2}\right) c_x = J(h_1), \tag{23}$$

the complete solution of which is:

$$h_1 = a_1^{(1)} + a_2^{(1)} c_x + a_3^{(1)}\left(c^2 - \frac{3}{2}\right) + a_4^{(1)} g(c^2)\, c_x . \tag{24}$$

The form of the particular solution, i.e., the fourth term, follows from the isotropy property of the linear operator J. By symmetry we have again that $a_1^{(1)}$ and $a_3^{(1)}$ must be odd in x, and $a_2^{(1)}$ and $a_4^{(1)}$ must be even in x. In order to make the particular solution completely definite, we may and will require that this term be orthogonal to the solution of the homogeneous equation. This imposes one condition on $g(c^2)$, namely:

$$\int d\vec{c}\, e^{-c^2} c^2 g(c^2) = 0 . \tag{25}$$

Then $\overline{c}_x = 0$ again requires that $a_2^{(1)} = 0$. Putting

$$a_4^{(1)} = \frac{da_3^{(0)}}{dx} \tag{26}$$

the equation for $g(c^2)$ becomes:

$$J\left(g(c^2)\, c_x\right) = \left(c^2 - \frac{5}{2}\right) c_x . \tag{27}$$

Together with the condition Eq (25), this equation is identical (except for notation; $g(c^2)$ has the dimension of a reciprocal area) with the equation solved in the book of Chapman and Cowling. The method is to develop $g(c^2)$ in Sonine polynomials of degree r and order 3/2:

$$g(c^2) = \sum_{r=1}^{\infty} \alpha_r S_{3/2}^{(r)}(c^2)$$

Omitting the term with r = 0 takes care of the condition Eq (25). For the development coefficients α_r, one gets from Eq (27) an infinite set of linear equations, which can be solved by convergent infinite determinants. For further details we refer to Chap. VII of the book of Chapman and Cowling, and we will note only that in our notation the heat conductivity coefficient ν of the gas is given by:

$$[g(c^2)c_x, g(c^2)c_x] = -\frac{\pi^{3/2}}{k}\sqrt{\frac{m}{2kT}}\,\nu \tag{29}$$

and that in terms of ν:

$$\alpha_1 = -\frac{4}{5k}\sqrt{\frac{m}{2kT}}\,\nu . \tag{30}$$

From Eq (8) for the heat flux and Eq (24) it follows that for the heat flux in the first approximation:

$$q^{(1)} = -a_4^{(1)} T\nu = -\nu T \frac{da_3^{(0)}}{dx} . \tag{31}$$

The constancy of $da_3^{(0)}/dx$ follows from the second approximation.

3) Second approximation:

$$c_x \frac{\partial}{\partial x}\left[a_1^{(1)} + a_3^{(1)}\left(c^2 - \frac{3}{2}\right) + a_4^{(1)} g(c^2) c_x\right] = J(h_2)$$

The solubility conditions lead to:

$$\frac{da_1^{(1)}}{dx} + \frac{da_3^{(1)}}{dx} = 0$$

and

$$\frac{da_4^{(1)}}{dx}\int d\vec{c}\, e^{-c^2} c_x^2 \left(c^2 - \frac{3}{2}\right) g(c^2) = 0,$$

from which it follows that $a_4^{(1)}$ must be a constant so that $a_1^{(0)}$ and $a_3^{(0)}$ are proportional to x. The complete second-order solution is:

$$h_2 = a_1^{(2)} + a_2^{(2)} c_x + a_3^{(2)} \left(c^2 - \frac{3}{2}\right) + a_4^{(2)} g(c^2) c_x .$$

By the same argument, $a_2^{(2)}$ must be zero. The function $g(c^2)$ is the same function as in the first approximation. It satisfies the same integral equation and the same auxiliary condition if one puts again:

$$a_4^{(2)} = \frac{d a_3^{(1)}}{dx} .$$

The coefficient $a_4^{(2)}$ will again have to be a constant if one goes to the next approximation. Thus we see that the complete solution of the disturbance h will be:

$$h = a_1 + a_3 \left(c^2 - \frac{3}{2}\right) + a_4 g(c^2) c_x ,$$

where

$$a_1 = \sum_{i=0}^{\infty} n^{-i} a_1^{(i)} , \quad a_3 = \sum_{i=0}^{\infty} n^{-i} a_3^{(i)} , \quad a_4 = \sum_{i=1}^{\infty} n^{-i} a_4^{(i)}$$

and

$$\frac{da_1^{(i)}}{dx} + \frac{da_3^{(i)}}{dx} = 0 , \quad a_4^{(i)} = \frac{da_3^{(i-1)}}{dx} .$$

Therefore $a_3 = -a_1 = n a_4 x$ so that:

$$h = n a_4 x \left(c^2 - \frac{5}{2}\right) + a_4 c_x g(c^2) . \tag{32}$$

The $a_4^{(i)}$ are the constants to be determined from the boundary conditions.

Boundary Condition: The complete solution of the problem depends on the boundary condition Eq (4a), namely:

$$h^+\left(c_x, -\frac{d}{2}\right) = \frac{\alpha \Delta T}{T}(c^2 + B)\frac{1 + \operatorname{sign} c_x}{2} + (1-\alpha)\, h^-\left(-c_x, -\frac{d}{2}\right) , \tag{33}$$

where the $h^{\pm}$ (c_x, x) must fulfill the symmetry conditions Eq (6),

$$h^{\pm}(c_x, x) = -h^{\mp}(-c_x, -x) ,$$

and the Boltzmann equation, which we write in the form

$$\frac{\partial h^{\pm}}{\partial x} = \frac{n}{c_x} J(h) \frac{1 \pm \operatorname{sign} c_x}{2} \tag{34}$$

Since we know that $J(h) = a_4 (c^2 - 5/2) c_x$, we can conclude from Eq (34):

$$h^{\pm}(c_x, x) = n a_4 x (c^2 - \tfrac{5}{2}) \frac{1 \pm \operatorname{sign} c_x}{2} + g^{\pm}(c^2, c_x).$$

From the symmetry condition it follows that $g^{\pm}(c^2, c_x) = -g^{\mp}(c^2, -c_x)$, while from Eq (32) it follows that:

$$g^{+}(c^2, c_x) + g^{-}(c^2, c_x) = a_4 c_x g(c^2).$$

Developing $g^{\pm}(c^2, c_x)$ in Legendre polynomials in $\cos\theta = c_x/c$, one can conclude that $g^{\pm}(c^2, c_x)$ must be of the form:

$$g^{\pm}(c^2, c_x) = \pm b_0(c^2) + \tfrac{1}{2} a_4 c_x g(c^2) \pm \sum_{n=1}^{\infty} b_{2n}(c^2) P_{2n}(\cos\theta), \tag{37}$$

where $b_0(c^2)$ and $b_{2n}(c^2)$ are still undetermined functions. Turning to the boundary conditions Eq (33), it follows from the fact that in the development of $(1/2)(1 + \operatorname{sign} c_x)$ in Legendre polynomials the polynomials $P_{2n}(\cos\theta)$, $n \neq 0$ do not appear, that in Eq (37) all the coefficients $b_{2n}(c^2)$ must be zero. Eq (33) can therefore be written as:

$$\frac{\alpha \Delta T}{T}(c^2 + B) \frac{1 + \operatorname{sign} c_x}{2} = -\frac{\alpha n a_4 d}{2}(c^2 - \tfrac{5}{2}) \frac{1 + \operatorname{sign} c_x}{2} - (2-\alpha)\left[b_0 + \tfrac{1}{2} a_4 c_x g(c^2)\right]. \tag{38}$$

It is clear that this equation cannot be fulfilled identically in c_x and c^2, so that some compromise must be made. Multiplying* Eq (38) by $c_x e^{-c^2}$ and $(c^2 - 5/2) c_x e^{-c^2}$ and integrating, one obtains:

$$\begin{aligned} \frac{\Delta T}{T}\left(1 + \frac{B}{2}\right) &= \frac{n a_4 d}{8} \\ \frac{\Delta T}{T}\left(1 - \frac{B}{2}\right) &= -\frac{9 n a_4 d}{8} + \frac{2-\alpha}{\alpha} a_4 \alpha_1 \frac{5\sqrt{\pi}}{4}. \end{aligned} \tag{39}$$

Solved for B and a_4, these equations give:

---

*It is to be noted that 1 and $c^2 - 5/2$ are the zero- and first-order Sonine polynomial $S^{(r)}_{3/2}(c^2)$ used in the development of $g(c^2)$.

$$a_4 = \frac{2\Delta T}{T} \Big/ \left(-nd + \frac{2-\alpha}{\alpha}\frac{5\sqrt{\pi}}{4}\alpha_1\right)$$

$$B = -\frac{5}{2}\left(1 - \frac{2-\alpha}{\alpha}\frac{\sqrt{\pi}\,\alpha_1}{nd}\right) \Big/ \left(1 - \frac{2-\alpha}{\alpha}\frac{5\sqrt{\pi}\,\alpha_1}{4nd}\right). \tag{40}$$

With these values of a_4 and B, one then obtains from Eq (38) by integrating over all angles of $\vec{c}$

$$b_6 = 2\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}\frac{\Delta T}{T} \Big/ \left(1 + \frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}\right).$$

Using Eq (30) for α_1, one obtains for the heat flux:

$$q = -n\nu a_4 = \frac{\nu}{1 + \frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}} \cdot \frac{2\Delta T}{d}$$

and for the temperature distribution

$$T(x) = T(1 + a_3) = T\left\{1 - \frac{x}{1 + \frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}} \cdot \frac{2\Delta T}{Td}\right.$$

so that the temperature slip is approximately:

$$T\left(\frac{d}{2}\right) - (T - \Delta T) = \frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}\,\Delta T,$$

which agrees with Maxwell's results.[3]

It should be emphasized that these results cannot be the exact solution of the problem, since only two of the moments of the boundary condition Eq (38) have been fulfilled. It is still of interest to remark that Eq (40) can be developed in inverse powers of n, according to:

$$a_4 = \sum_{i=0}^{\infty} n^{-i} a_4^{(i)} \quad , \quad B = \sum_{i=0}^{\infty} n^{-i} B^{(i)}$$

and one gets:

[3]Maxwell collected papers.

$$B^{(0)} = -\frac{5}{2} \qquad a_4^{(1)} = -\frac{2}{d}\frac{\Delta T}{T}$$

and for i > 0:

$$B^{(i)} = -\frac{1}{2}\left(-\frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}\right)^i$$

$$a_4^{(i+1)} = -\frac{2}{d}\frac{\Delta T}{T}\left(-\frac{2-\alpha}{\alpha}\frac{\nu}{nkd}\sqrt{\frac{\pi m}{2kT}}\right)^i$$

Since $b_o(c^2)$ developed in powers of $1/n$ begins with a term $\sim 1/n$, one sees that with the Eqs (40) values for a_4 and B, the boundary condition Eq (38) is in the zeroth approximation identically fulfilled in c_x and c^2, so that up to this order at least the development in powers of λ/d is consistent.

V. GENERAL SOLUTION

In this section the general solution of the Boltzmann equation, Eq (1), subjected to the boundary condition Eq (4a) and the condition $\bar{c}_x = 0$, will be obtained. For this purpose we expand the function h in the set of normalized eigenfunctions $\psi_{r\ell}$ belonging to the collision operator J for the Maxwell molecules:

$$h = \sum_{r,\ell} a_{r\ell}(x)\psi_{r\ell}(c^2, c_x) \tag{43}$$

where

$$\psi_{r\ell} = N_{r\ell}\, c^\ell P_\ell\left(\frac{c_x}{c}\right) S_{\ell+\frac{1}{2}}^{(r)}(c^2).$$

$N_{r\ell}$ is the normalization constant, P_ℓ is the Legendre polynomial of order ℓ, and $S_{\ell+1/2}^{(r)}$ is the Sonine polynomial of degree r and order $\ell + 1/2$. The Boltzmann equation becomes then a doubly infinite set of linear differential equations for the expansion coefficients $a_{r\ell}$:

$$\frac{da_{r\ell}}{dx} = n\sum a_{r\ell'}\left[\frac{1}{c_x}\psi_{r\ell}, \psi_{r\ell'}\right]. \tag{44}$$

As a consequence of the symmetry property of h and the evenness and oddness property of $\psi_{r\ell}$ for even and odd ℓ respectively, and $a_{r\ell}$'s have the symmetry property:

$$a_{r\ell}(x) = \mp a_{r\ell}(-x) \text{ according to } \ell \begin{cases}\text{even}\\ \text{odd}\end{cases} \tag{45}$$

The requirement that $\bar{c}_x = 0$ means that

$$a_{01} = 0$$

and the boundary condition Eq (4a) becomes the following equations;

$$a^{+}_{r2l+1}\left(-\frac{d}{2}\right) = \frac{\alpha\delta T}{T}\int d\vec{c}\, e^{-c^2}(c^2+B)\psi_{r2l+1}\frac{1+\mathrm{sign}\, c_x}{2} - (1-\alpha)\, a^{-}_{r2l+1}\left(-\frac{d}{2}\right). \qquad (46)$$

The choice of the functions $\psi_{r\ell}$ has the additional advantage that the physical quantities we are interested in are then all expressible in terms of the first few development coefficients $a_{r\ell}$.

For instance:

$$n(x) = n\left[1 + \frac{a_{00}(x)}{\pi^{3/2} N_{00}}\right],$$

$$T(x) = T\left[1 - \frac{a_{10}(x)}{\pi^{3/2} N_{10}}\right], \qquad (47)$$

$$p_{xx} = nkT\left[1 + \frac{1}{\pi^{3/2}}\left(\frac{a_{02}}{N_{02}} - \frac{a_{10}}{N_{10}} - \frac{3}{2}\frac{a_{00}}{N_{00}}\right)\right],$$

and

$$q_x = -\frac{nkT}{\pi^{3/2}}\sqrt{\frac{2kT}{m}}\,\frac{a_{11}}{N_{11}} \quad .$$

The conservation theorems lead to simple results for some of the first $a_{r\ell}$'s.

One obtains:

1) From the conservation of number

$$\frac{da_{01}}{dx} = 0\ ;$$

so a_{01} = constant, which can be taken to be zero:

$$a_{01} = 0 \qquad (48)$$

2) From the conservation of linear momentum:

$$\frac{d}{dx}\left(\frac{a_{02}}{N_{02}} - \frac{a_{10}}{N_{10}} - \frac{3}{2}\frac{a_{00}}{N_{00}}\right) = 0\ ; \qquad ?$$

so $p_{xx} = nkT\ (1 + \text{const.})$, and the constant must be zero since the $a_{r\ell}$ with even ℓ are odd in x.

3) From the conservation of energy:

$$\frac{da_{11}}{dx} = 0 \quad ;$$

so

$$a_{11} = \text{const} = b_{11} \; . \tag{49}$$

The set of equations, Eqs (44), has constant coefficients and a formal solution can therefore be easily obtained. However, for the actual solution, we will simplify the problem slightly by separating the equations for even and odd ℓ's. Because of the property that:

$$\left[\frac{\psi_{r\ell}}{c_x}, \psi_{r''\ell''}\right] \neq 0 \quad \text{only when } \ell - \ell'' = \text{odd}$$

Eqs (44) can be written as two sets, for odd and even ℓ respectively. With the following matrix notations:

$$\mathcal{A}_e = (a_{r2\ell})$$

$$\mathcal{A}_o = (a_{r2\ell+1})$$

$$\mathcal{R} = \left(\left[\frac{1}{c_x}\psi_{r2\ell}, \psi_{r'2\ell'+1}\right]\right)$$

$$\tilde{\mathcal{R}} = \left(\left[\frac{1}{c_x}\psi_{r2\ell+1}, \psi_{r'2\ell'}\right]\right),$$

these are:

$$\begin{aligned} \frac{d\mathcal{A}_e}{dx} &= n\mathcal{R}\mathcal{A}_o \\ \frac{d\mathcal{A}_o}{dx} &= n\tilde{\mathcal{R}}\mathcal{A}_e \; . \end{aligned} \tag{50}$$

On eliminating $\mathcal{A}_e$, one obtains

$$\frac{d^2\mathcal{A}_o}{dx^2} = n^2\tilde{\mathcal{R}}\mathcal{R}\mathcal{A}_o \; . \tag{51}$$

Eq (51) gives an infinite set of linear homogeneous equations for all the $a_{r,2\ell+1}$'s. The even $a_{r2\ell}$'s are then obtained from $a_{r2\ell+1}$ by Eq (50a).

Making the Ansatz:

$$\mathcal{A}_o = \mathcal{B}_o e^{pnx}$$

where $\mathcal{B}_o = (b_{r2\ell+1})$ is a column matrix with elements which are constants, and substituting into Eq (51), one finds:

$$p^2 \mathcal{B}_o = \tilde{\mathcal{R}} \mathcal{R} \mathcal{B}_o \ .$$

This infinite set of linear homogeneous equations will have a solution if the determinant

$$\Delta = \left| p^2 I - \tilde{\mathcal{R}} \mathcal{R} \right| = 0, \tag{52}$$

where I is the unit matrix. We see that the roots p_i always appear in pairs with opposite signs. It can also be shown that the p_i's are all real.* From the conservative theorems and the property of the square bracket, it follows that

$$\tilde{R}_{01,r\ell} = \tilde{R}_{11,r\ell} = 0$$

so that among the roots of Eq (52) there are four having the value zero. Hence the general solution of Eq (51) is

$$\mathcal{A}_o = \mathcal{B}_1 + \mathcal{B}_1' x + \mathcal{B}_1'' \frac{x^2}{2} + \mathcal{B}_1''' \frac{x^3}{6} + \frac{1}{2} \sum \mathcal{B}_1^{(i)} e^{p_i n x} , \tag{53}$$

where the sum goes over all the nonzero roots p_i.

The symmetry condition Eq (45) requires that:

$$\mathcal{B}_1' = 0 \ ,$$

$$\mathcal{B}_1''' = 0 \ ,$$

and

$$\mathcal{B}_1^{(+i)} = \mathcal{B}_1^{(-i)} \ ,$$

where the $\pm i$ refer to the two roots $\pm p_i$.

Of the elements $b^{(i)}_{r,2\ell+1}$ of the matrix $B^{(\pm i)}_1$, one set for fixed r and ℓ can be taken as arbitrary. We will choose for this set the $b^{(i)}_{03}$. Then:

---

*For proof, see Appendix I.

$$b^{(i)}_{r\,2\ell+1} = \frac{\Delta^{(i)}_{r\,2\ell+1}}{\Delta^{(i)}_{03}}\, b^{(i)}_{03}$$

where $\Delta^{(i)}_{r\ell}$ designates the first minor of the $(r\ell, r\ell)$ element of the determinant Δ in which p is replaced by the ith root p_i. It is easily seen that:

$$b^{(i)}_{01} = b^{(i)}_{11} = 0 \qquad \text{for all i}$$

The constant matrices $\mathcal{B}_1$ and $\mathcal{B}_1''$ satisfy the following equations:

$$0 = n^2 \tilde{\mathcal{R}}\, \mathcal{R}\, \mathcal{B}_1'' \tag{54}$$

$$\mathcal{B}_1'' = n^2 \tilde{\mathcal{R}}\, \mathcal{R}\, \mathcal{B}_1 \, . \tag{55}$$

Since $\tilde{R}_{01,r\ell} = \tilde{R}_{11,r\ell} = 0$, the rank of the determinant $|\tilde{R}R|$ is less than the number of variables by two. Therefore, so far as the set of linear homogeneous equations Eq (54) is concerned, two of the elements of $\mathcal{B}_1''$ are arbitrary. To satisfy Eqs (48) and (49) we must have:

$$(\mathcal{B}_1'')_{01} = (\mathcal{B}_1'')_{11} = 0 \; ;$$

hence $\mathcal{B}_1''$ must be identically zero. Eqs (55) now become linear homogeneous equations like Eqs (54). By the same reasoning, two of the elements of $\mathcal{B}_1$ are arbitrary. These must be assigned the values

$$(\mathcal{B}_1)_{01} = 0, \qquad (\mathcal{B}_1)_{11} = b_{11}$$

in order that Eqs (54) and (55) be satisfied. The rest of the $\mathcal{B}_1$'s must then all be proportional to b_{11}. The solution of the set of equations, Eq (50) can thus be written as:

$$\mathcal{A}_0 = \mathcal{B}\, b_{11} + \sum_i \mathcal{B}^{(i)} b^{(i)}_{03} \cosh p_i n x \, . \tag{56}$$

The matrices $\mathcal{B}$ and $\mathcal{B}^{(i)}$ are known constants independent of b_{11} and $b^{(i)}_{03}$, the determination of which depends on the boundary conditions. The coefficients $a_{r2\ell}$ are given by:

$$\mathcal{A}_e = n b_{11} \mathcal{R} \mathcal{B} x + \mathcal{R} \sum_i \frac{\mathcal{B}^{(i)} b^{(i)}_{03}}{p_i} \sinh p_i n x \, . \tag{57}$$

Before making use of the boundary conditions Eq (46), we first have to find expressions for $a^{\pm}_{r2\ell+1}$. The differential equation for these quantities is

$$\frac{da^{\pm}_{r2\ell+1}}{dx} = n \sum_{r',\ell'} a_{r'\ell'} \int d\vec{c}\, e^{-c^2} \frac{\psi_{r2\ell+1}}{c_x} J(\psi_{r'\ell'}) \frac{1 \pm \text{sign}\, c_x}{2}$$

$$= n \left\{ \sum a_{r'2\ell'} \int d\vec{c}\, e^{-c^2} \frac{\psi_{r2\ell+1}}{c_x} J(\psi_{r'2\ell'}) \frac{1 \pm \text{sign}\, c_x}{2} + \right.$$

$$\left. + \sum a_{r'2\ell'+1} \int d\vec{c}\, e^{-c^2} \frac{\psi_{r2\ell+1}}{c_x} J(\psi_{r'2\ell'+1}) \frac{1 \pm \text{sign}\, c_x}{2} \right\}$$

$$= \frac{1}{2} \frac{d}{dx} a_{r2\ell+1} + n \sum a_{r'2\ell'+1} T^{\pm}_{r2\ell+1,\, r'2\ell'+1} ,$$

where

$$T^{\pm}_{r2\ell+1,\, r'2\ell'+1} = \left[\frac{\psi_{r2\ell+1}}{c_x} \frac{1 \pm \text{sign}\, c_x}{2} ,\ \psi_{r'2\ell'+1} \right] .$$

In matrix notation we have:

$$\frac{d\mathcal{A}_0^{\pm}}{dx} = \frac{1}{2} \frac{d\mathcal{A}_0}{dx} + n \mathcal{T}^{\pm} \mathcal{A}_0$$

with

$$\mathcal{T}^{\pm} = \left(T^{\pm}_{r2\ell+1,\, r'2\ell'+1} \right) .$$

Substituting Eq (56) into the above equation and integrating, one obtains:

$$\mathcal{A}_0^{\pm} = \left(\frac{1}{2} + n\mathcal{T}^{\pm} x\right) \mathcal{B}\, b_{11} + \sum_i \left(\frac{1}{2} \cosh p_i n x + \frac{\mathcal{T}^{\pm}}{p_i} \sinh p_i n x \right) \mathcal{B}^{(i)} b_{03}^{(i)} \qquad (58)$$

There is no integration constant. The symmetry property requires that any integration constant $c^{+}_{r2\ell+1}$ be equal to $c^{-}_{r2\ell+1}$. On the other hand, since

$$a^{+}_{r2\ell+1} + a^{-}_{r2\ell+1} = a_{r2\ell+1} ,$$

$c^{+}_{r2\ell+1} + c^{-}_{r2\ell+1}$ must be zero. Hence both $c^{+}_{r2\ell+1}$ and $c^{-}_{r2\ell+1}$ must be zero.

Putting Eq (58) into Eq (46) and calling

$$\mathcal{P} = \left(\int d\vec{c}\, e^{-c^2} c^2 \psi_{r2\ell+1} \frac{1 + \text{sign}\, c_x}{2} \right)$$

$$Q = \left(\int d\vec{c}\, e^{-c^2} \psi_{r2l+1} \frac{1+\operatorname{sign} c_x}{2} \right),$$

one is led to the following equation:

$$-\frac{\Delta T}{T} P = \frac{\Delta T}{T} Q \mathcal{B} + \left[-\frac{2-\alpha}{2\alpha} + \frac{nd}{2} J^{+} \right] \mathcal{B} b_{11} + \sum_i \left[-\frac{2-\alpha}{2\alpha} \cosh p_i n \frac{d}{2} + \frac{I^{+}}{p_i} \sinh p_i n \frac{d}{2} \right] \mathcal{B}^{(i)} b_{03}^{(i)} \tag{59}$$

Eq (59) is an infinite set of inhomogeneous equations for B, b_{11}, and $b_{03}^{(i)}$. It can easily be seen that this set of equations determines the unknown constants in the following sense. Suppose the basic infinite determinant Δ is broken off at some value of r and ℓ; then with this finite set of values for r and ℓ, the number of equations, Eq (59) will be just equal to the number of unknowns.

We have not been able to discuss the convergence of this procedure. It is possible to deduce the limiting results for the Knudsen and Clausius gas from the general expression. The zeroth order for the Knudsen gas is especially simple, since in this case all the hyperbolic sine and cosine functions can be replaced by zero and one respectively and since n can be neglected compared to unity, Eq (59) gives two equations for B and b_{11}, namely:

$$-\frac{\Delta T}{T} P_{01} = \frac{\Delta T}{T} Q_{01} B$$

$$-\frac{\Delta T}{T} P_{11} = \frac{\Delta T}{T} Q_{11} B - \frac{2-\alpha}{2\alpha} b_{11} .$$

By solving for b_{11} one obtains the Knudsen limit, Eq (11), for the heat flux.

We will discuss the Clausius gas limit in detail in the next section for the case of Maxwell molecules. The complex nature of this limit is clearly due to the fact that the density (or the Knudsen number) occurs in the argument of the hyperbolic functions.

VI. MAXWELL MOLECULES

For Maxwell molecules:

$$J(\psi_{r\ell}) = \lambda_{r\ell} \psi_{r\ell} ,$$

where according to the definition of J used here $\lambda_{r\ell}$ has the dimension of an area; it is $\sqrt{\kappa/kT}$ times the $\lambda_{r\ell}$ in Ref. 1, where κ is the force constant. Thus

$$R_{r\ell,r'\ell'} = \lambda_{r'\ell'} L_{r\ell,r'\ell'}$$

where we define:

$$L_{r\ell,r'\ell'} = \int d\vec{c}\; e^{-c^2} \frac{1}{c_x} \psi_{r\ell} \psi_{r'\ell'}$$

which is symmetric in the two pairs of indices. Furthermore, for Maxwell molecules $R_{r,2\ell;11}$ are zero except $R_{10,11}$ and $R_{00,11}$. From the conservation theorems, $R_{r,2\ell+1,00}$ and $R_{r2\ell+1,11}$ are zero, so that all the coefficients of b_{11} in the system of equations, Eq (55), are zero. As a result, all the elements of the matrix B except b_{11} must be zero. The solutions of Eq (44) are slightly simplified, and can be written in the expanded form:

$$\begin{aligned}
a_{01} &= 0 \\
a_{11} &= b_{11} \\
a_{r2\ell+1} &= \sum_i \frac{\Delta^{(i)}_{r2\ell+1}}{\Delta^{(i)}_{03}} b^{(i)}_{03} \cosh np_i x \\
a_{r2\ell} &= \sum_{r'\ell'} \lambda_{r'2\ell'+1} L_{r2\ell,r'2\ell'+1} \sum_i \frac{1}{p_i} \frac{\Delta^{(i)}_{r'2\ell'+1}}{\Delta^{(i)}_{03}} b^{(i)}_{03} \sinh np_i x \\
a_{00} &= n b_{11} \lambda_{11} L_{00,11} x + \sum_{r'\ell'} \lambda_{r'2\ell'+1} L_{00,r'2\ell'+1} \sum_i \frac{1}{p_i} \frac{\Delta^{(i)}_{r'2\ell'+1}}{\Delta^{(i)}_{03}} b^{(i)}_{03} \sinh np_i x \\
a_{10} &= n b_{11} \lambda_{11} L_{10,11} x + \sum_{r'\ell'} \lambda_{r'2\ell'+1} L_{10,r'2\ell'+1} \sum_i \frac{1}{p_i} \frac{\Delta^{(i)}_{r'2\ell'+1}}{\Delta^{(i)}_{03}} b^{(i)}_{03} \sinh np_i x
\end{aligned} \qquad (60)$$

The p's are the positive nonzero roots of the basic infinite determinant

$$\Delta = \left| p^2 \delta_{rr''} \delta_{\ell\ell''} - \lambda_{r''2\ell''+1} \sum_{r'\ell'} \lambda_{r'2\ell'} L_{r'2\ell',r''2\ell''+1} L_{r'2\ell',r2\ell+1} \right| = 0 .$$

The set of infinite inhomogeneous equations for the determination of B, b_{11}, and $b^{(i)}_{03}$ becomes

$$\begin{aligned}
-\frac{\Delta T}{T} P_{r2\ell+1} &= \frac{\Delta T}{T} Q_{r2\ell+1} B + \left(\frac{nd}{2} \lambda_{11} L^{+}_{r2\ell+1,11} - \frac{2-\alpha}{2\alpha} \delta_{r1} \delta_{\ell 0}\right) b_{11} + \\
&+ \sum_i \left\{ \frac{1}{p_i} \frac{1}{\Delta^{(i)}_{03}} \sinh \frac{np_i d}{2} \sum \lambda_{r'2\ell'+1} L^{+}_{r2\ell+1,r'2\ell'+1} \Delta^{(i)}_{r'2\ell'+1} - \frac{2-\alpha}{2\alpha} \frac{\Delta^{(i)}_{r2\ell+1}}{\Delta^{(i)}_{03}} \cosh \frac{np_i d}{2} \right\} b^{(i)}_{03} .
\end{aligned} \qquad (61)$$

where

$$L^{+}_{r2\ell+1,\, r'2\ell'+1} = \int d\vec{c}\, e^{-c^2} \frac{1}{c_x} \psi_{r2\ell+1} \psi_{r'2\ell'+1} \frac{1+\operatorname{sign} c_x}{2} .$$

We will use the same successive approximation method as in Ref. 1. In the zeroth approximation ("ideal fluid"), only the eigenfunctions ψ_{00}, ψ_{10}, and ψ_{01} are used. In the first approximation ("Stokes Navier"), ψ_{11} and ψ_{02} are added; in the second approximation ("Burnett") three more functions (ψ_{03}, ψ_{20}, ψ_{12}) are used; and so on.

It is easily seen that in the zeroth approximation one does not get a heat flux, and so we pass to the higher approximations.

1) First approximation: For this approximation we take ψ_{00}, ψ_{10}, ψ_{01}, ψ_{11} and ψ_{02}. The solutions for the development coefficients are:

$$a_{01} = 0$$

$$a_{11} = b_{11}$$

$$a_{02} = 0$$

$$a_{00} = n b_{11} \lambda_{11} L_{00,11}\, x$$

$$a_{10} = n b_{11} \lambda_{11} L_{10,11}\, x ,$$

where b_{11} and the constant B are to be determined by Eqs (61) which reduce to the two linear equations,

$$-\frac{\Delta T}{T} P_{01} = \frac{\Delta T}{T} Q_{01} B + \frac{nd}{2} \lambda_{11} L^{+}_{01,11} b_{11}$$

$$-\frac{\Delta T}{T} P_{11} = \frac{\Delta T}{T} Q_{11} B + \left(\frac{nd}{2} \lambda_{11} L^{+}_{11,11} - \frac{2-\alpha}{2\alpha}\right) b_{11} .$$

All the coefficients can easily be evaluated.* Solving for b_{11}, we find:

$$b_{11} = -\frac{4\Delta T}{T} \frac{\lambda}{d} \frac{1}{N_{11}} \cdot \frac{1}{1 + \frac{2-\alpha}{2\alpha} \frac{5\sqrt{\pi}\lambda}{d}}$$

*Tables for some of the elements of the matrices P, Q, L, and L^{+} will be found in Appendix II.

where λ is a measure of the free path defined by:

$$\lambda \equiv \left(nA_2\sqrt{\varkappa/kT}\right)^{-1}.$$

Substituting into Eqs (47d) and (47b), the heat flux is found to be

$$q_x = 5nk\Delta T\sqrt{\frac{2kT}{m}}\,\frac{\lambda}{d}\left(1+\frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\right)^{-1}, \tag{62}$$

and the temperature distribution is given by

$$T(x) = T\left\{1-\frac{2\Delta T}{T}\frac{x}{d}\left(1+\frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\right)^{-1}\right\}. \tag{63}$$

Thus there is a temperature slip at the wall equal to:

$$T_s \equiv T\left(\frac{d}{2}\right)-(T-\Delta T) = \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\Delta T\left(1+\frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\right)^{-1}. \tag{64}$$

These results are the same as those obtained in Sect. IV if these are specialized to Maxwell molecules.

2) Second approximation: For this approximation, one stops with eight ψ's: ψ_{00}, ψ_{01}, ψ_{10}, ψ_{11}, ψ_{02}, ψ_{03}, ψ_{12}, and ψ_{20}. It is in this approximation that the hyperbolic sine and cosine functions begin to enter. The determinant Δ is simply:

$$\Delta = p^2-\lambda_{03}\sum_{r'\ell'}\lambda_{r'2\ell'}\left(L_{r'2\ell',03}\right)^2 = 0.$$

Making use of the values of $\lambda_{r\ell}$, and $L_{r\ell,r'\ell'}$ one finds

$$p = \pm\frac{\sqrt{15}}{4}A_2',$$

where

$$A_2' \equiv A_2\sqrt{\varkappa/kT}\ .$$

The linear equations for the determination of the constants are:

$$-\frac{\Delta T}{T}P_{01} = \frac{\Delta T}{T}Q_{01}B+\frac{nd}{2}\lambda_{11}\overset{+}{L}_{01,11}b_{11}+\frac{b_{03}\lambda_{03}}{p}\overset{+}{L}_{01,03}\sinh\frac{npd}{2}$$

$$-\frac{\Delta T}{T}P_{11} = \frac{\Delta T}{T}Q_{11}B+\left(\frac{nd}{2}\lambda_{11}\overset{+}{L}_{11,11}-\frac{2-\alpha}{2\alpha}\right)b_{11}+\frac{b_{03}\lambda_{03}}{p}\overset{+}{L}_{11,03}\sinh\frac{npd}{2}$$

$$-\frac{\Delta T}{T}P_{03} = \frac{\Delta T}{T}Q_{03}B+\frac{nd}{2}\lambda_{11}\overset{+}{L}_{03,11}b_{11}+\left(\frac{\lambda_{03}\overset{+}{L}_{03,03}}{p}\sinh\frac{npd}{2}-\frac{2-\alpha}{2\alpha}\cosh\frac{npd}{2}\right)b_{03},$$

so that

$$b_{11} = -\frac{\Delta T}{T}\frac{\lambda}{d}\frac{4}{N_{11}}\left\{1 + \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\frac{\frac{39}{10\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c}{\frac{15}{4\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c}\right\}^{-1},$$

where we have written s and c for sinh (ndp/2) and cosh (ndp/2) respectively. The heat flux is therefore given by

$$q_x = 5nk\Delta T\sqrt{\frac{2kT}{\pi m}}\frac{\lambda}{d}\left\{1 + \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\frac{\frac{39}{10\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c}{\frac{15}{4\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c}\right\}^{-1}.$$

The coefficient a_{10} is in this approximation:

$$a_{10} = -\sqrt{\frac{3}{10}}\frac{x}{\lambda}b_{11} + \frac{b_{03}\lambda_{03}}{p}L_{10,03}\sinh npx$$

$$= -\sqrt{\frac{3}{10}}\frac{x}{\lambda}b_{11} - \frac{2}{5\sqrt{3}}b_{03}\sinh npx,$$

which gives for the temperature distribution

$$T(x) = T\left\{1 - \frac{2\Delta T}{T}\left[\frac{x}{d}\left(\frac{15}{4\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c\right) + \frac{2-\alpha}{2\alpha}\frac{\lambda}{\sqrt{15}d}\sinh\frac{\sqrt{15}}{4}\frac{x}{\lambda}\right]\cdot\right.$$

$$\left.\cdot\left[\frac{15}{4\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c + \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\left(\frac{39}{10\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c\right)\right]^{-1}\right\}.$$

Eq (66) leads to the following expression for the temperature slip:

$$T_s = T\left(\frac{d}{2}\right) - (T-\Delta T) = \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\Delta T\,\frac{\frac{7}{2\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c}{\frac{15}{4\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c + \frac{2-\alpha}{2\alpha}\frac{5\sqrt{\pi}\lambda}{d}\left(\frac{39}{10\sqrt{15\pi}}s + \frac{2-\alpha}{2\alpha}c\right)}.$$

3) Third approximation: In this approximation three additional functions are taken. They are ψ_{04}, ψ_{21}, and ψ_{13}. The basic determinant Δ becomes:

$$\begin{vmatrix} p^2 - \frac{15}{16}A_2'^2 & 0 & -\frac{5\times 1.176\sqrt{2}}{18}A_2'^2 \\ 0 & p^2 - \frac{9}{28}A_2'^2 & -\frac{4\times 1.176\sqrt{21}}{63}A_2'^2 \\ -\frac{5\sqrt{2}}{16}A_2'^2 & -\frac{\sqrt{21}}{21}A_2'^2 & p^2 - \frac{41\times 1.176}{36}A_2'^2 \end{vmatrix} = 0$$

Table I gives the roots p and the coefficients $b^{(i)}_{r2\ell+1}$ in terms of $b^{(i)}_{03}$

Table I

i	1	2	3
p_i	$0.4798\ A_2^{\frac{1}{2}}$	$0.8344\ A_2^{\frac{1}{2}}$	$1.293\ A_2^{\frac{1}{2}}$
$b_{03}^{(i)}/b_{03}^{(i)}$	1	1	1
$b_{21}^{(i)}/b_{03}^{(i)}$	5.700	-0.4787	0.4036
$b_{13}^{(i)}/b_{03}^{(i)}$	-1.519	-0.5244	1.593

The set of linear inhomogeneous equations, Eq (61), now consists of five members, so that the coefficients b_{11} and $b_{03}^{(i)}$ are ratios of two 5 x 5 determinants. By straightforward calculation, using Table I and the tables in Appendix II, one finds that the heat flux can be written as:

$$q = 5nkT\sqrt{\frac{2kT}{\pi m}}\,\frac{\Delta T}{T}\,\frac{1}{\dfrac{d}{\lambda} + \dfrac{5\sqrt{\pi}}{2}\dfrac{D_1}{D_2}},$$

where

$$D_1 = \begin{vmatrix} -0.5000 - 1.0937\,t_1 & -0.5000 - 0.5722\,t_2 & -0.5000 - 0.5749\,t_3 \\ 2.850 - 3.828\,t_1 & 0.2394 + 0.2018\,t_2 & -0.2018 - 0.2430\,t_3 \\ 0.7595 + 0.8938\,t_1 & 0.2622 + 0.2540\,t_2 & -0.7965 - 0.7953\,t_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} -0.5000 - 0.6898\,t_1 & -0.5000 - 0.5520\,t_2 & -0.5000 - 0.5696\,t_3 \\ 2.850 - 3.076\,t_1 & 0.2394 + 0.2393\,t_2 & -0.2018 - 0.2335\,t_3 \\ 0.7595 + 0.6083\,t_1 & 0.2622 + 0.2397\,t_2 & -0.7965 - 0.7991\,t_3 \end{vmatrix}$$

and we have written t_i for tanh $(np_i\ d/2)$. The temperature distribution turns out to be:

$$T(x) = T\Big\{1 - \frac{2\Delta T}{T}\frac{x}{d}\Big(1 + \frac{5\sqrt{\pi}}{2}\frac{\lambda}{d}\frac{D_1}{D_2}\Big)^{-1} +$$

$$+ \frac{\Delta T}{T}\frac{\lambda}{d}\Big(1 + \frac{5\sqrt{\pi}}{2}\frac{\lambda}{d}\frac{D_1}{D_2}\Big)^{-1}\sum_i \frac{A_2'}{p_i}\frac{E_i}{D_2 c_i}\,b_{03}^{(i)}\sinh\frac{p_i x}{\lambda}\Big(\frac{1}{4} + \frac{3}{2\sqrt{42}}\frac{b_{21}^{(i)}}{b_{03}^{(i)}} - \frac{\sqrt{2}\times 1.176}{27}\frac{b_{13}^{(i)}}{b_{03}^{(i)}}\Big)\Big\}$$

with $c_i = \cosh(np_i\, d/2)$ and

$$E_1 = -(1.274 + 1.342\, t_2 + 1.325\, t_3 + 1.392\, t_2\, t_3)$$

$$E_2 = -(1.917 + 1.986\, t_1 + 1.976\, t_3 + 2.068\, t_1\, t_3)$$

$$E_3 = -(0.2973 - 0.0245\, t_1 + 0.2225\, t_2 - 0.1196\, t_1\, t_2).$$

The corresponding temperature slip is

$$T_s = \frac{5\sqrt{\pi}}{2}\Delta T \frac{D_1}{D_2}\left(\frac{d}{\lambda} + \frac{5\sqrt{\pi}}{2}\frac{D_1}{D_2}\right)^{-1} +$$

$$+ \frac{\Delta T}{D_2}\left(\frac{d}{\lambda} + \frac{5\sqrt{\pi}}{2}\frac{D_1}{D_2}\right)^{-1}(3.465\, t_1 E_1 + 0.2055\, t_2 E_2 + 0.1897\, t_3 E_3).$$

In this way one can go on to higher approximations. However, since, contrary to the case of the sound propagation, the successive-approximation method changes, at each stage, the results of the previous stage, and the calculations grow more and more involved, we have not gone any further.

For the discussion of the results obtained so far, we have made some numerical computations and plotted three sets of curves. To simplify matters we have assumed that $\alpha = 1$. Figure I is a family of curves for q/q_K against the Knudsen number $K = d/\lambda$. The straight line gives the initial slope for such a curve expected from an exact theory. The value is taken from Ref. 2, with adjustment for the change of the definition of λ. The initial slopes for the successive approximation results can also be evaluated easily. For comparison they are listed below:

	Initial Slope
Exact theory:	-0.907
First approximation:	-0.226
Second approximation:	-0.236
Third approximation:	-0.306

The convergence is therefore quite slow, as is to be expected for small K. The differences between the successive approximations are quite small, so that Fig. I may already give a good idea of the dependence of the heat flux on the Knudsen number.

Figure II is a family of curves for the temperature slip measured in ΔT as a function of d/λ. They have the same general feature as the curves in Fig. I, except that the difference between successive approximations is larger. The initial slopes of these curves have also been computed. They are:

	Initial Slope
First approximation:	-0.226
Second approximation:	-0.282
Third approximation:	-0.390

We have not computed the initial slope for the exact result.

Figure III presents plots of the temperature distribution. On account of the small differences between successive approximations, we have plotted instead the difference $T_1(x) - T_i(x)$ measured in ΔT against $2x/d$, where $T_i(x)$ is the temperature distribution for the ith approximation and $T_1(x)$ is the linear temperature distribution, as follows from the first approximation. The rapid rise of the curves near $2x/d = 1$ can be taken as an indication of the so-called boundary-layer phenomena. This rise is a consequence of the appearance of the hyperbolic sine and cosine functions, which is also responsible for making development in λ/d impossible.

APPENDIX I. PROOF THAT ALL THE ROOTS p_i OF THE BASIC DETERMINANT ARE REAL

For the proof it is easier to start from Eq (44), namely

$$\frac{da_{r\ell}}{dx} = n \sum_{r'\ell'} a_{r'\ell'} \left[\frac{1}{c_x} \psi_{r\ell}, \psi_{r'\ell'} \right]$$

Expanding the function $1/c_x\ \psi_{r\ell}$ in terms of the eigenfunctions,

$$\frac{1}{c_x} \psi_{r\ell} = \sum_{r''\ell''} \psi_{r''\ell''} \int d\vec{c}\,' e^{-c'^2} \frac{1}{c_x'} \psi_{r\ell} \psi_{r''\ell''} ,$$

so that

$$\frac{da_{r\ell}}{dx} = \sum_{r'\ell'} \sum_{r''\ell''} a_{r'\ell'} \left[\psi_{r''\ell''}, \psi_{r'\ell'} \right] \int d\vec{c}\, e^{-c^2} \frac{1}{c_x} \psi_{r\ell} \psi_{r''\ell''}$$

In matrix notations, letting

$$\mathcal{A} = (a_{r\ell})$$

$$\mathcal{R}_0 = \left(\left[\psi_{r\ell}, \psi_{r'\ell'} \right] \right)$$

$$\mathcal{L} = \left(\int d\vec{c}\, e^{-c^2} \frac{1}{c_x} \psi_{r\ell} \psi_{r'\ell'} \right),$$

where both $\mathcal{R}_0$ and $\mathcal{L}$ are symmetric, the above equation becomes

$$\frac{d\mathcal{A}}{dx} = n \mathcal{R}_0 \mathcal{L} \mathcal{A} .$$

The Ansatz $\mathcal{A} = \mathcal{B} e^{-npx}$ leads to

$$p \mathcal{B} = \mathcal{R}_0 \mathcal{L} \mathcal{B} .$$

Since $\mathcal{L}$ is not singular, one can define $\mathcal{B} = \mathcal{L}^{-1} \mathcal{B}'$ so that we have

$$p \mathcal{L}^{-1} \mathcal{B}' = \mathcal{R}_0 \mathcal{B}' .$$

Both $\mathfrak{L}^{-1}$ and $\mathbb{R}_o$ being symmetric, the determinant for p is symmetric and thus all the roots of the secular determinant must be real.

APPENDIX II. COLLECTION OF TABLES USED FOR NUMERICAL CALCULATION

1) Table of eigenvalues $\lambda_{r\ell}$ ($A_2' = A_2 \sqrt{\kappa/kT}$, $A_4' = 0.636\ A_2'$)

$r \backslash \ell$	0	1	2	3	4
0	0	0	$-3A_2'/4$	$-9A_2'/8$	$-\frac{7A_2'}{4} + \frac{35A_4'}{64}$
1	0	$-A_2'/2$	$-7A_2'/8$	$-\frac{11A_2'}{8} + \frac{5A_4'}{16} = -1.176A_2'$	
2	$-A_2'/2$	$-3A_2'/4$			

2) Tables for $P_{r\ell}$ and $Q_{r\ell}$

	01	11	03	21	13
$P_{r\ell}$	$\sqrt{2}\ \pi^{1/4}$	$-\frac{1}{\sqrt{5}}\pi^{1/4}$	$-\frac{3}{\sqrt{30}}\pi^{1/4}$	$-\frac{1}{2\sqrt{35}}\pi^{1/4}$	$-\frac{1}{2\sqrt{15}}\pi^{1/4}$
$Q_{r\ell}$	$\frac{1}{\sqrt{2}}\pi^{1/4}$	$\frac{1}{2\sqrt{5}}\pi^{1/4}$	$-\frac{1}{\sqrt{30}}\pi^{1/4}$	$\frac{3}{4\sqrt{35}}\pi^{1/4}$	$-\frac{1}{2\sqrt{15}}\pi^{1/4}$

3) Table for $L_{r\ell,\ r'\ell'}$

$r'\ell' \backslash r,\ell$	00	10	02	12	20	04
11	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{6}{5}}$	0	0	0	0
03	$-2\sqrt{\frac{2}{15}}$	$\frac{4}{3\sqrt{5}}$	$\frac{\sqrt{10}}{3}$	0	0	0
21	$\frac{4}{\sqrt{35}}$	$2\sqrt{\frac{6}{35}}$	0	0	$\sqrt{\frac{6}{7}}$	0
13	$-\frac{8}{3\sqrt{15}}$	$-\frac{2}{9}\sqrt{\frac{2}{5}}$	$\frac{2}{9}\sqrt{5}$	$\frac{1}{9}\sqrt{70}$	$\frac{4\sqrt{2}}{9}$	0

4) Table for $L^{+}_{r\ell,r'\ell'}$

$r'\ell'$ \ r,ℓ	01	11	03	21	13
01	$\frac{1}{\sqrt{\pi}}$	$\frac{1}{\sqrt{10\pi}}$	$-\frac{1}{\sqrt{15\pi}}$	$\frac{3}{2\sqrt{70\pi}}$	$-\frac{1}{\sqrt{30\pi}}$
11	$\frac{1}{\sqrt{10\pi}}$	$\frac{9}{10\sqrt{\pi}}$	$\frac{1}{5\sqrt{6\pi}}$	$\frac{19}{20\sqrt{7\pi}}$	$-\frac{3}{10\sqrt{3\pi}}$
03	$-\frac{1}{\sqrt{15\pi}}$	$\frac{1}{5\sqrt{6\pi}}$	$\frac{14}{15\sqrt{\pi}}$	$\frac{1}{10\sqrt{42\pi}}$	$\frac{14}{45\sqrt{2\pi}}$
21	$\frac{3}{2\sqrt{70\pi}}$	$\frac{19}{20\sqrt{7\pi}}$	$\frac{1}{10\sqrt{42\pi}}$	$\frac{233}{280\sqrt{\pi}}$	$\frac{9}{20\sqrt{21\pi}}$
13	$-\frac{1}{\sqrt{30\pi}}$	$-\frac{3}{10\sqrt{3\pi}}$	$\frac{14}{45\sqrt{2\pi}}$	$\frac{9}{20\sqrt{21\pi}}$	$\frac{119}{135\sqrt{\pi}}$

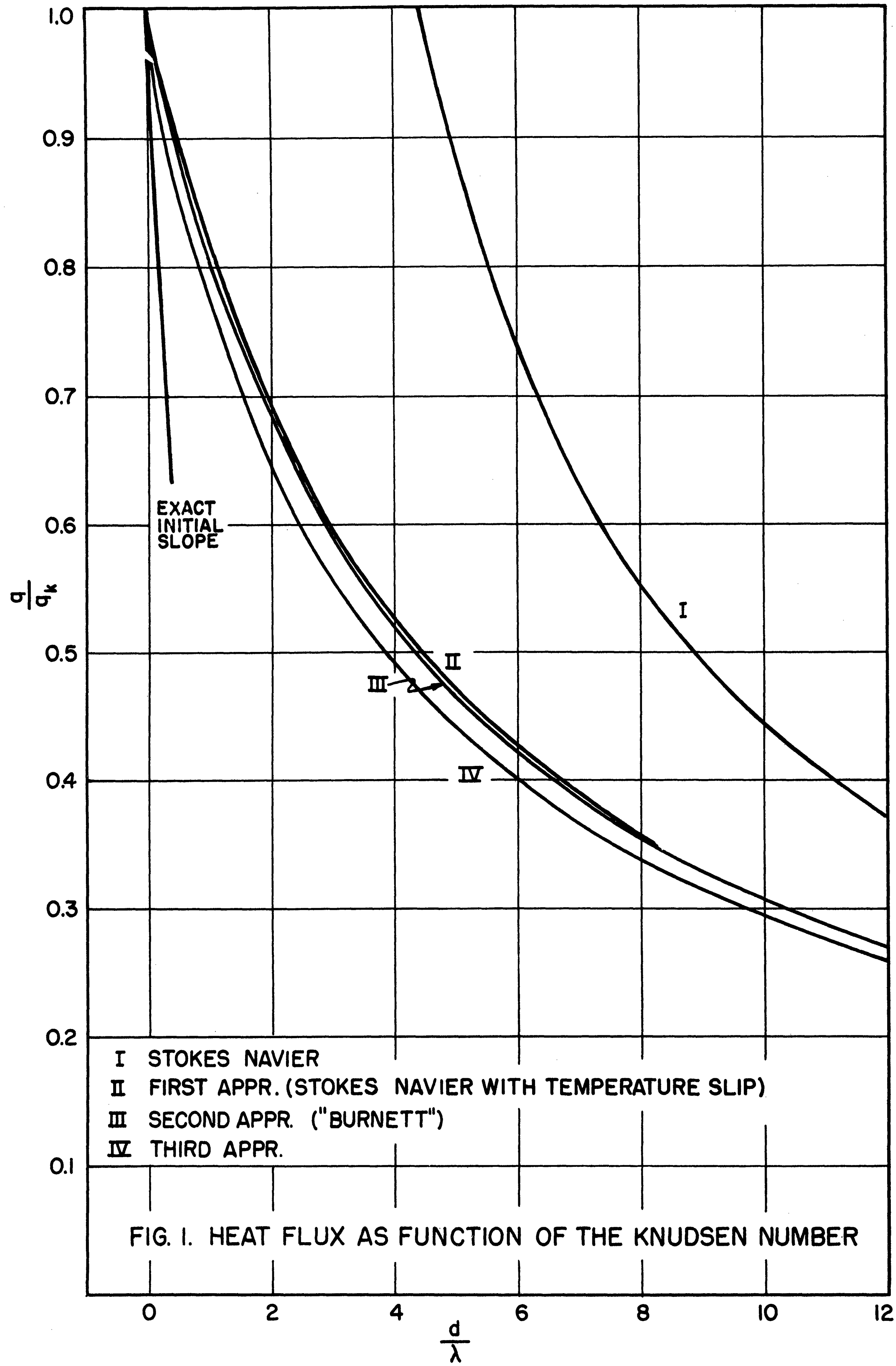

FIG. 1. HEAT FLUX AS FUNCTION OF THE KNUDSEN NUMBER

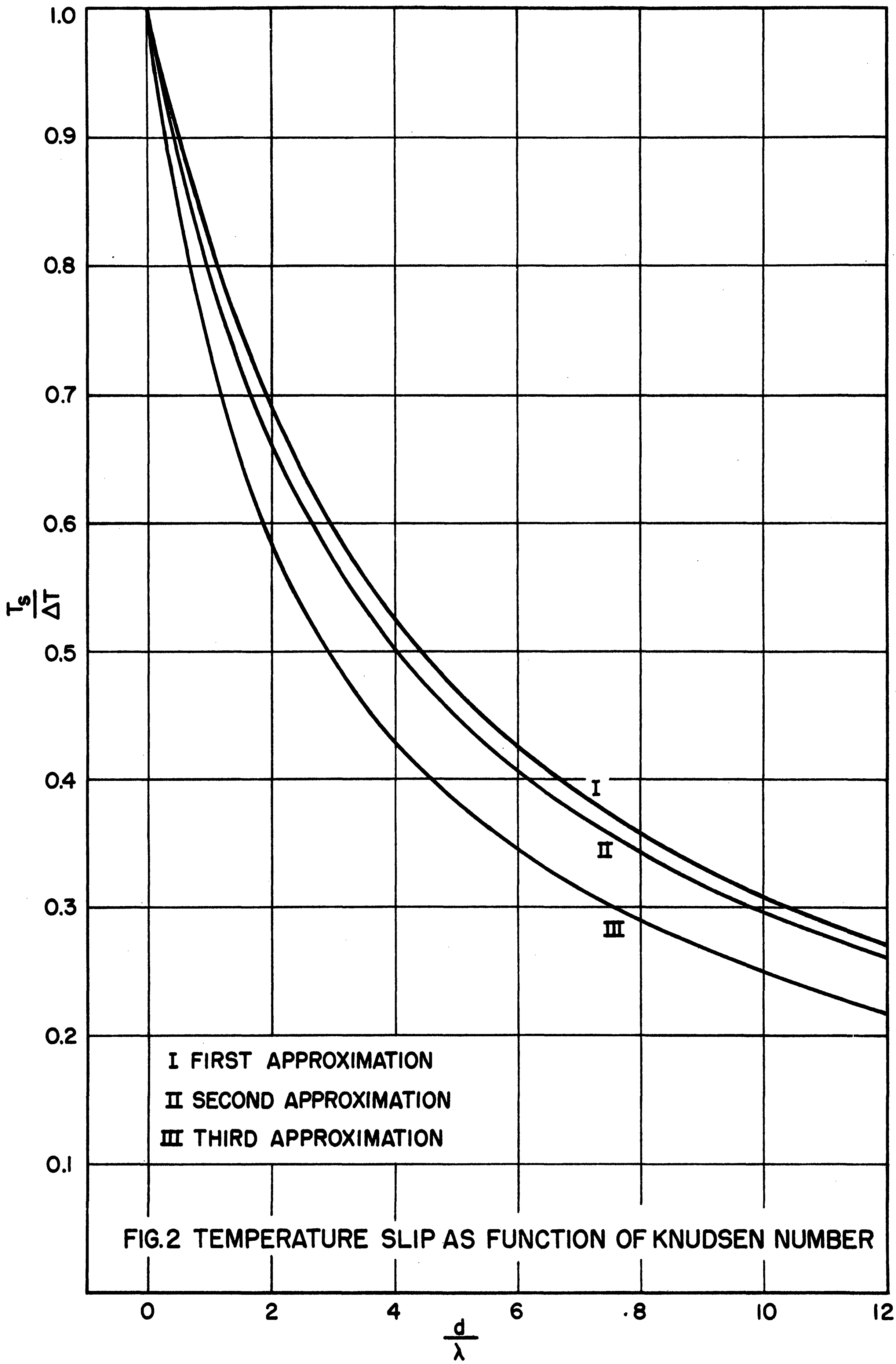

FIG. 2 TEMPERATURE SLIP AS FUNCTION OF KNUDSEN NUMBER

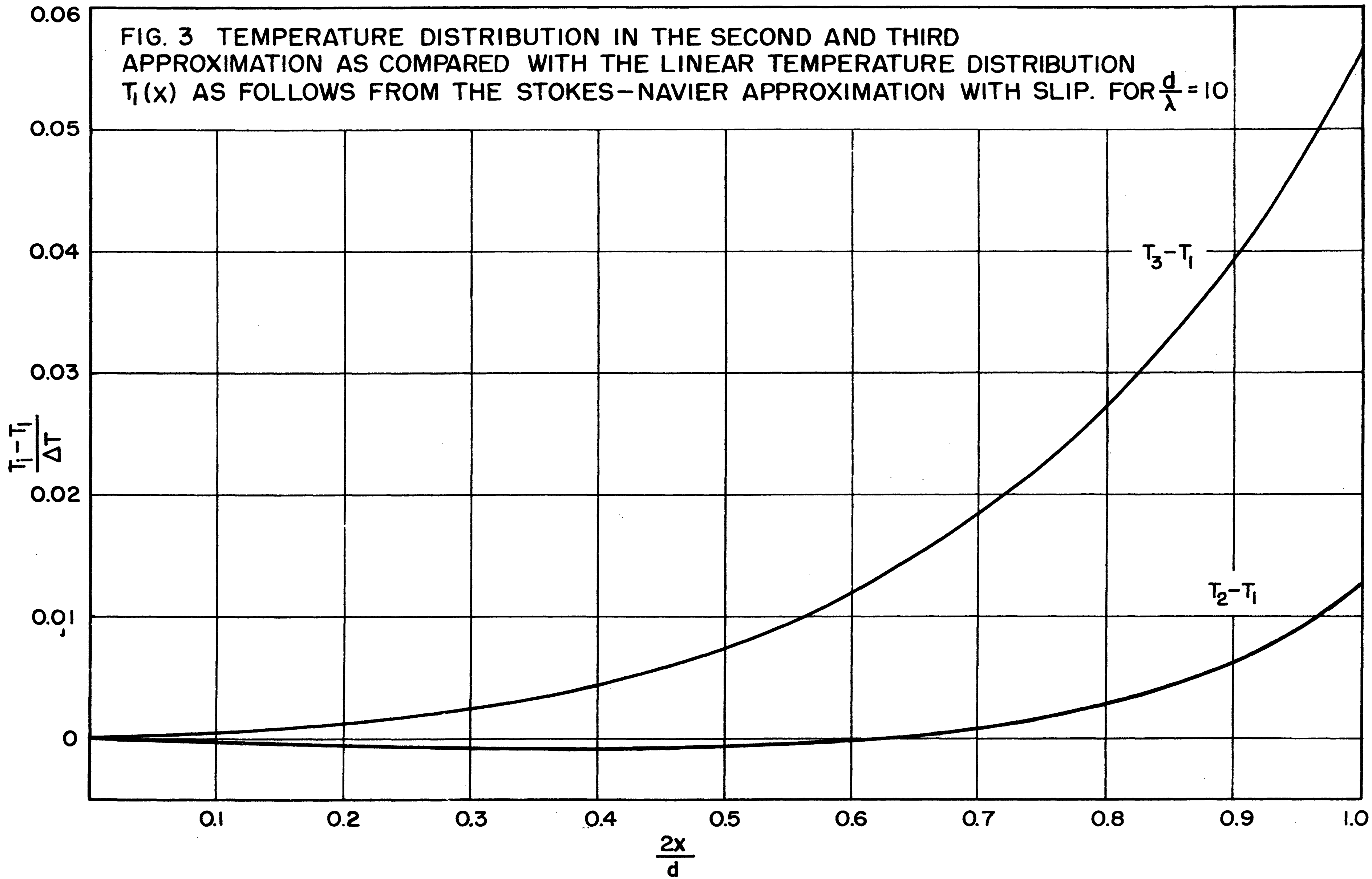

FIG. 3 TEMPERATURE DISTRIBUTION IN THE SECOND AND THIRD APPROXIMATION AS COMPARED WITH THE LINEAR TEMPERATURE DISTRIBUTION $T_1(x)$ AS FOLLOWS FROM THE STOKES–NAVIER APPROXIMATION WITH SLIP. FOR $\frac{d}{\lambda} = 10$