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FINAL REPORT

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FOREWORD

This report (8579-1-F) was prepared by the Radiation Laboratory of the Department of Electrical Engineering of The University of Michigan. The work was performed under Contract F19628-67-C-0190, "Inverse Scattering Investigation," and covers the period 3 March 1967 to 3 March 1968. Dr. Vaughan H. Weston is the Principal Investigator and the contract is under the direction of Professor Ralph E. Hiatt, Head of the Radiation Laboratory. The contract is administered under the direction of the Electronic Systems Division, Air Force Systems Command, United States Air Force, Laurence G. Hanscom Field, Bedford, Massachusetts, 01730, by Lt. L. E. Nyman, ESSXS. This report was submitted by the authors on 30 April 1968.

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ABSTRACT

A brief review of the salient features of the theoretical investigation of the c.w. bistatic inverse scattering problem is presented. The effect of changing the origin of the coordinate system upon the convergent properties etc., of the spherical vector wave function representation of the near scattered field and the surface loci $\underline{E} \times \underline{E}^* = 0$, is discussed. It is pointed out that a great deal of analysis remains to be done in this area. The determination of the surface of the scattering body from knowledge of the local total electric field is given. Emphasis is placed upon the generalization of the condition $\underline{E} \times \underline{E}^* = 0$ as applied to perfectly-conducting bodies, to scattering surfaces characterized by the impedance boundary condition. Properties of the Matrix inversion associated with the determination of the expansion coefficients from far field data are discussed. Some numerical results are presented, and restrictions upon the choice of aspect angles are deduced.

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II

REVIEW OF INVERSE SCATTERING THEORY DEVELOPED FOR C.W. BISTATIC SYSTEM

2.1 THE PLANE WAVE EXPANSION

It is shown here that the total field produced by a plane wave incident upon a scattering body, can be expressed as the sum of two terms, the incident field and the Fourier transform of a quantity which is related to the scattering matrix. The resulting expression is valid for all points in space including the interior of the scattering body.

To begin, the analysis will be restricted to non-magnetic bodies (although it could be easily generalized to include such cases) and the geometry of the scattering body will be limited by placing certain restrictive analytical properties on the surface S which encloses the volume V of the scattering body. The classes of surfaces chosen, will belong to class C^3 defined by Barrar and Dolph as follows:

A surface S is said to belong to class C^3 if there exists a finite number m of images, $x^\nu = x^\nu(u, v)$, $y^\nu = y^\nu(u, v)$, $z^\nu = z^\nu(u, v)$, $\nu = 1.2.3. . . m$ of the disk $u^2 + v^2 \leq 1$ that cover the surface S, and such that the third derivatives of x^ν, y^ν and z^ν with respect to u and v exist and are continuous.

Harmonic time dependence exp(-iωt) will be taken in which case Maxwell's equations become

$$\underline{\nabla} \wedge \underline{E} = ik \sqrt{\mu_o / \epsilon_o} \; \underline{H} \quad ,$$

$$\sqrt{\mu_o / \epsilon_o} \; \underline{\nabla} \wedge \underline{H} = -i (\underline{E}/k) \begin{Bmatrix} k_1^2 & \text{inside the body} \\ k^2 & \text{outside the body} \end{Bmatrix} \quad ,$$

where k_1^2, the square of the propagation constant in the body, is given by

$$k_1^2 = \omega^2 \epsilon_1 \mu_o + i\omega\mu_o \sigma_1 \; .$$

The incident electric intensity will be expressed in the form

$$\underline{E}^{i} = (2\pi)^{-3/2} \, \underline{a} \; e^{i\underline{k} \cdot \underline{x}} \tag{2.1}$$

where $\underline{k}$ is the direction of the incident wave, and $\underline{a}$, the unit vector denoting polarization.

The homogeneous body will be treated first. In this case, it follows from Barrar and Dolph, that the total electric intensity $\underline{E}$ induced by the plane wave Eq. (2.1), incident upon the body, satisfies the following integral equation (which has a unique solution)

$$\underline{E}(\underline{x}) = \underline{E}^{i}(\underline{x}) + \frac{\left(k_1^2 - k^2\right)}{4\pi} \left\{ \int_V \phi \, \underline{E} \, dv - \frac{1}{k^2} \underline{\nabla} \cdot \int_S \phi (\underline{E}^{-} \cdot \underline{n}) \, ds \right\} \tag{2.2}$$

where

$$\phi = \frac{e^{ikR}}{R} \, , \qquad R = (\underline{x} - \underline{x}')$$

and $\underline{E}^{-}$ is the value of $\underline{E}$ obtained by approaching the surface from the interior of the body. If $\underline{E}^{+}$ is the value obtained by approaching the surface from the exterior of the body, it follows from the continuity relations that

$$k_1^2 \, \underline{E}^{-} \cdot \underline{n} = k^2 \, \underline{E}^{+} \cdot \underline{n} \quad . \tag{2.3}$$

The appropriate expression for the magnetic field is given by

$$\underline{H}(\underline{x}) = \underline{H}^{i}(\underline{x}) + \frac{\left(k_1^2 - k^2\right)}{4\pi \, i\omega\mu_o} \, \underline{\nabla} \wedge \int_V \underline{E} \, \phi \, dv \tag{2.4}$$

Before deriving a plane wave representation for $\underline{E}(\underline{x})$, the following vector will be introduced.

$$\underline{T}(\underline{k}', \underline{k}) = \frac{\left(k_1^2 - k^2\right)}{(2\pi)^{3/2}} \left\{ \int_V e^{-i\underline{k}'\cdot\underline{x}'} \, \underline{E}(\underline{x}', \underline{k})\, dv' - \frac{i\underline{k}'}{k^2} \int_S e^{-i\underline{k}'\cdot\underline{x}'} \, \underline{E}^- \cdot \underline{n}\, ds' \right\} . \tag{2.5}$$

From Eq. (2.2), it is seen that when $|x| \to \infty$, in the direction given by the vector $\underline{k}'$, then the total field becomes

$$\underline{E}(\underline{x}) \underset{|\underline{x}|\to\infty}{\sim} \underline{E}^i(\underline{x}) + \sqrt{\frac{\pi}{2}} \, \frac{e^{ik|\underline{x}|}}{|\underline{x}|} \, \underline{T}(\underline{k}', \underline{k}) \tag{2.6}$$

indicating that the vector $\underline{T}(\underline{k}', \underline{k})$ is related to the scattering matrix. When $|k'| = |k|$, $\underline{T}(\underline{k}', \underline{k})$ is a measurable function being proportioned to the far scattered field in direction $\underline{k}'$, which is produced by a plane wave of frequency ck, incident upon the body with direction of incidence given by $\hat{\underline{k}}$. The following theorem may now be proven.

<u>Theorem:</u>

If $$\underline{\mathcal{E}}(\underline{k}', \underline{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\underline{k}'\cdot\underline{x}} \, \underline{E}(\underline{x}, \underline{k})\, d\underline{x} \, , \tag{2.7}$$

then $$\underline{\mathcal{E}}(\underline{k}'\cdot\underline{k}) = \delta(\underline{k} - \underline{k}')\, \underline{a} - \lim_{\epsilon\to 0} \frac{\underline{T}(\underline{k}', \underline{k})}{k^2 - k'^2 + i\epsilon} \quad . \tag{2.8}$$

<u>Proof:</u>

It follows from Eq. (2.2) that

$$\underline{\mathcal{E}}(\underline{k}', \underline{k}) = \delta(\underline{k} - \underline{k}')\, \underline{a} + \frac{\left(k_1^2 - k^2\right)}{4\pi} \left\{ \frac{1}{(2\pi)^{3/2}} \int_V \underline{E}(\underline{x}', \underline{k}) \int e^{-i\underline{k}'\cdot\underline{x}} \, \phi \, d\underline{x}\, dv \right.$$

$$\left. - \frac{1}{k^2} \int \frac{e^{-i\underline{k}'\cdot\underline{x}}}{(2\pi)^{3/2}} \, \underline{\nabla} \cdot \int_S \phi \, (\underline{E}^- \cdot \underline{n})\, ds \right\} .$$

On setting

$$\psi = \int_S \phi (\underline{E}^- \cdot \underline{n})\, ds$$

one can show that

$$\int e^{-i\underline{k}'\cdot\underline{x}}\, \underline{\nabla}\psi\, d\underline{x} = \lim_{R_\infty \to \infty} \int_{S_\infty} e^{-i\underline{k}'\cdot\underline{x}}\, \psi\, ds + i\underline{k}' \int e^{-i\underline{k}'\cdot\underline{x}}\, \psi\, d\underline{x}$$

when S_∞ is the surface of a sphere of radius R_∞. Letting k have a small imaginary part, it is seen that the surface integral will vanish when $R_\infty \to \infty$. The resulting integrals can be reduced as follows

$$\frac{1}{(2\pi)^{3/2}} \int e^{-i\underline{k}'\cdot\underline{x}}\, \phi\, d\underline{x} = \sqrt{\frac{2}{\pi}}\, \frac{1}{(2\pi)^3} \iint d\underline{p}\, d\underline{x}\, \frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}') - i\underline{k}'\cdot\underline{x}}}{p^2 - k^2 - i\epsilon}$$

$$= \sqrt{\frac{2}{\pi}} \int d\underline{p}\, \frac{e^{-i\underline{p}\cdot\underline{x}'}}{p^2 - k^2 - i\epsilon}\, \delta(\underline{p} - \underline{k}')$$

$$= -\sqrt{\frac{2}{\pi}}\, \frac{e^{-i\underline{k}'\cdot\underline{x}}}{k^2 - k'^2 + i\epsilon} \quad .$$

Combining the above expressions, one obtains the result

$$\underline{\xi}(\underline{k}'\cdot \underline{k}) = \delta(\underline{k} - \underline{k}')\, \underline{a} - \lim_{\epsilon \to 0} \frac{\underline{T}(\underline{k}', \underline{k})}{k^2 - k'^2 + i\epsilon} \quad .$$

As an immediate consequence of the above theorem, the total field everywhere in space including the interior of the body, can be expressed in terms of the incident field and the quantity $\underline{T}(\underline{k}', \underline{k})$, as follows

$$\underline{E}(\underline{x},\underline{k}) = \underline{E}^i(\underline{x},\underline{k}) + \frac{1}{(2\pi)^{3/2}} \int \frac{e^{i\underline{p}\cdot\underline{x}}}{p^2 - k^2} \, \underline{T}(\underline{p},\underline{k}) \, d\underline{p} \quad . \tag{2.9}$$

If the integration space $\underline{p}$ is expressed in spherical polar coordinates (p, θ_p, ϕ_p) where the range of the variables are $-\infty < p < \infty$, $0 \leq \theta_p \leq \pi/2$, and $0 \leq \phi_p \leq 2\pi$, then in the above integral, the contour of the variable p bends above the pole at $p = -k$, and below the pole at $p = k$.

It can be shown in a similar manner that the magnetic field can be expressed in a similar form

$$\underline{H}(\underline{x},\underline{k}) = \underline{H}^i(\underline{x},\underline{k}) + \frac{1}{\omega \mu_o (2\pi)^{3/2}} \int \frac{e^{i\underline{p}\cdot\underline{x}}}{p^2 - k^2} \, \underline{p} \wedge \underline{T}(\underline{p},\underline{k}) \, d\underline{p} \quad . \tag{2.10}$$

The above results were derived for a homogeneous body. The results may be extended to include inhomogeneous non-conducting bodies, i.e. where $k_1^2 = \omega^2 \epsilon \mu$ (a real function) and k_1^2 varies continuously in the medium. In this case, the appropriate integral equation for the total electric field is (Barrar and Dolph)

$$\underline{E} = \underline{E}^i + \frac{1}{4\pi} \int_V \phi \left(k_1^2 - k^2\right) \underline{E} \, dv - \frac{\nabla}{4\pi k^2} \int_S \phi \left(k_1^2 - k^2\right) (\underline{E}^- \cdot \underline{n}) \, ds$$

$$+ \frac{\nabla}{4\pi} \int_V \left(\underline{\nabla} k_1^2 \cdot \underline{E}\right) \frac{\phi}{k_1^2} \, dv \quad .$$

Unfortunately from the standpoint of rigor, only the uniqueness of the solution of this integral equation has been proven. It's existance has not yet been shown.

Additional relationships involving $\underline{T}(\underline{k}', \underline{k})$ have been shown in 8579-4-Q (1968), among which is an integral equation involving $\underline{T}(\underline{k}', \underline{k})$

and the Fourier transform of the body.

2.2 ALTERNATIVE REPRESENTATION FOR POINTS EXTERIOR TO THE BODY

For a fixed direction of incidence and frequency, the quantity $\underline{T}(\underline{k}', \underline{k})$ is required for all values of $\underline{k}'$, in order to obtain expressions for the field everywhere in space. Unfortunately, for fixed $\underline{k}$, $\underline{T}(\underline{k}', \underline{k})$ is a measurable function (far scattered field), only for values of $\underline{k}'$ such that $|\underline{k}'| = |\underline{k}|$. However it will be shown that restricting the requirements on the knowledge of $\underline{T}(\underline{k}', \underline{k})$ to the values of $\underline{k}'$ which lie on the sphere $|\underline{k}'| = |\underline{k}|$, the total field can be obtained everywhere outside the minimum convex surface enclosing the body. To show this, let the plane $z = z_o$ be the tangent plane of the body, such that the body lies in the half-space $z \leq z_o$. For points in the half-space $z > z_o$, the following representation of the near scattered field

$$\underline{E}^s(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{i\underline{p}\cdot\underline{x}}}{p^2 - k^2} \underline{T}(\underline{p}, \underline{k})\, d\underline{p} \tag{2.11}$$

can be reduced as follows:

Set
$$p_x = k \sin\alpha \cos\beta$$
$$p_y = k \sin\alpha \sin\beta$$
$$p_z = kq$$

when the domains of integration are

$$0 \leq \beta \leq 2\pi, \quad 0 \leq \alpha \leq \pi/2 - i\infty, \quad \text{and} \quad -\infty \leq q \leq \infty .$$

Expression (2.11) becomes

$$\underline{E}^s = \frac{k}{(2\pi)^{3/2}} \int_0^{\pi/2 - i\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\underline{T}(\underline{p}, \underline{k})}{q^2 - \cos^2\alpha - i\epsilon}\, g(\alpha, \beta, q) \cos\alpha \sin\alpha \, d\alpha\, d\beta\, dq,$$

$$g = \exp\left\{ik \sin\alpha\,(x\cos\beta + y\sin\beta) + iqzk\right\} . \tag{2.12}$$

From expression (2.5), it can be seen that for $z > z_o$, the contour in the following integral

$$\int_{-\infty}^{\infty} \frac{e^{ikzq}\, \underline{T}(\underline{p},\underline{k})\, dq}{q^2 - \cos^2\alpha - i\epsilon}$$

may be deformed, to yield the following

$$\pi\, i\, e^{ik\cos\alpha\, z}\ \underline{T}(k\sin\alpha\cos\beta,\ k\sin\alpha\sin\beta,\ k\cos\alpha;\ \underline{k}) \quad .$$

Hence expression (2.12) may be placed in the form

$$\underline{E}^s = \frac{ik}{2\pi}\int_0^{\pi/2 - i\infty}\int_0^{2\pi} e^{i\underline{k}'\cdot\underline{x}}\ \underline{E}_o(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta \qquad (2.13)$$

where

$$\underline{k}' = k(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$$

and $\underline{E}_o$ is the amplitude and phase of the far scattered field in direction given by (θ, ϕ), i.e.

$$\underline{E}^s \sim \frac{e^{ikR}}{R}\ \underline{E}_o(\theta,\phi) \ . \qquad (2.14)$$

Thus it can be seen that for c.w. bistatic scattering, where $\underline{E}_o(\theta,\phi)$ is a measurable function, the scattered field for all points in the domain $z > z_o$, can be obtained from expression (2.13) provided that the analytic continuation of $\underline{E}_o(\theta,\phi)$ can be computed for complex values of θ, from knowledge of real values of θ. It has been shown (Weston, Bowman, Ar, 1966) that the analytic continuation to complex values of θ is possible provided that $\underline{E}_o(\theta,\phi)$ is known over the complete domain $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, and a procedure for doing so is presented.

By rotating the coordinate system, equivalent expressions to (2.13) may be derived which yield the scattered field everywhere outside a different half-space excluding the body. Since the union of all the half-spaces exterior to the body is bounded by the minimum convex surface enclosing the body, it can be deduced that the scattered field at any point outside this convex surface can be obtained from the knowledge of the far field bistatic data (measured over all angles).

At this point, it should be mentioned that expression (2.13) is not the only expression that can be used to evaluate the near scattered field from far field data. It has been shown that expression (2.13) can be re-expressed in terms of an expansion of the spherical vector wave functions. For a fixed region of the coordinate system, the expansion will be convergent down to at least the minimum sphere enclosing the body. Alternatively it was shown (8579-2-Q) that expressions (2.13) could be placed in the form

$$\underline{E}^s = \frac{e^{ikR}}{R} \sum_{n=0} R^{-n} \underline{E}_n(\theta, \phi)$$

where $\underline{E}_n$ are derived from the $\underline{E}_{n-1}$ by a recurrance relation (Wilcox expansion).

2.3 EXTENSION OF THE DOMAIN OF CONVERGENCE AND THE CONCEPT OF EQUIVALENT SOURCES

It was shown above that for a fixed direction of incidence (fixed transmitter position), one could derive the near field everywhere outside the minimum convex surface enclosing the body, from measurements of the phase, amplitude and polarization of the far scattered field made over all directions. This result has been extended. Considering representation (2.13) it was shown that it was convergent down to the tangent plane $z = z_o$, and represented the scattered field everywhere on the half-space $z > z_o$. However for perfectly-conducting smooth convex bodies, it has been shown (Weston, Bowman, Ar, 1966) that this expression was convergent down to the plane $z = z^*$ where $z^* > z_o$, where the value of z^* depends upon the global geometry of the body. Since the true total field displays a discontinuity across the physical surface of the body, expression (2.13) combined with the incident field, cannot represent the true field in the intersection of the half-space $z > z^*$ and the volume of the body. Therefore in this region, expression (2.13) is said to represent the field arising from a set of equivalent sources. In particular it was shown that equivalent sources for a prolate spheroid lie on a line joining the focal point, and for a sphere, the point at the center. Thus for smooth convex bodies, the domains of convergence are not limited by the minimum convex surface enclosing the body, but by a much smaller volume, that is contained by the minimum convex surface enclosing the equivalent sources. As an example, for the prolate spheroid with its center lying at the origin of the coordinate system and oriented so that its major axis lies along the z-axis, representation (2.13) is convergent in the half space $z > c$, where c is the focal length, and both Wilcox's representation and the representation involving the vector spherical wave functions, will be convergent down to the sphere of radius c.

The extension of the domain of convergence will hold for piecewise smooth conducting bodies. However the minimum convex surface enclosing the equivalent sources must enclose the curves or points of surface discontinuties such as edges.

For cavity regions that penetrate the minimum convex surface that encloses the equivalent sources, alternative representation for the scattered or near field must be employed. Such representations will not be of the exterior type, the plane wave representation (2.13), the expansion in vector spherical wave functions, or Wilcox's expansion, which are based upon knowledge of the far scattered field as a starting point, but must be of interior type, i.e. based upon knowledge of the total field in a domain adjacent to the body. Such representations are only convergent in a finite volume. One particular type of such a representation is given in 7466-1-F (Weston Bowman, Ar, 1966).

2.4 DETERMINATION OF THE SURFACE OF PERFECTLY-CONDUCTING BODIES.

Given expressions for the total field (obtained from far field data) which can be computed in the vicinity of the surface of the body, the next step is to employ techniques which will locate the surface of the body. For perfectly conducting bodies, the tangential component of $\underline{E}$ (the total field) must vanish. This implies that at a point on the surface $\underline{E} = \underline{n}\,\mathcal{E}$ where $\underline{n}$ in the unit normal to the surface and $\mathcal{E}$ is a complex quantity. For a general point in space the total field can be decomposed into the real vectors $\underline{\mathcal{E}}_I$ and $\underline{\mathcal{E}}_R$ as follows

$$\underline{E} = \underline{\mathcal{E}}_R + i\underline{\mathcal{E}}_I$$

For a point on the surface $\underline{\mathcal{E}}_I \wedge \underline{n} = 0$ and $\underline{\mathcal{E}}_R \wedge \underline{n} = 0$, which implies that $\underline{\mathcal{E}}_I \wedge \underline{\mathcal{E}}_R = 0$. Thus a necessary condition for a point to be on the surface of a perfectly conducting body is that, at that point, the total field must satisfy the condition

$$\underline{E} \wedge \underline{E}^* = 0\ ,$$

when $\underline{E}^*$ is the complex conjugate of $\underline{E}$. The advantage of this condition is that it is a local condition, requiring only the calculation of the total field at the point in question. Since the above condition is not sufficient, additional requirements would be required, such that $\underline{E} \times \underline{E}^*$ must vanish at adjacent points $\underline{x}$ and $\underline{x} + \Delta\underline{x}$, and that

$$\lim_{\Delta \underline{x} \to 0} \frac{\underline{E}(\underline{x}) \cdot \Delta \underline{x}}{|\Delta \underline{x}|} = 0$$

which implies that $\underline{E}$ is normal to surface. In addition, the resulting surface formed by the set of points must be closed. However, even if a surface S_o

was found such that it satisfied these conditions, it would not necessarily be the correct surface. Since for the enclosed volume formed by two closed surface, on which $\underline{n} \times \underline{E}$ vanishes, there is a discrete spectrum (in frequency), this implies that at a particular frequency, there may exist additional surfaces S_n for which $\underline{n} \times \underline{E}$ vanishes. However the geometry of the additional surfaces depend upon frequency, and these can be separated out from the proper surface by employing at least two different frequencies.

For the illuminated region of the body, an auxillary condition was developed which considerably helped discriminate the proper surface from the loci $\underline{E} \times \underline{E}^* = 0$. This condition given as $|\underline{E}^i| - |\underline{E}^s| = 0$, yields an approximate surface, which approaches smooth convex portions of the correct surface in the limit of high frequency scattering.

From the numerical standpoint where there are errors due to input information, etc., it was demonstrated both theoretically and numerically that the condition $\underline{E} \times \underline{E}^* = 0$, should be replaced by finding the minima of $|\underline{E} \times \underline{E}^*|$.

2.5 FAR FIELD INFORMATION LIMITED TO A SOLID ANGLE.

For most practical solutions, the far scattered field (phase, amplitude and polarization) is measurable only over a region of limited bistatic angles. In this case it is important to know what information can be determined about the body, where measurements of the scattered far field are confined to a limited domain of aspect angles. This in turn depends upon the accuracy to which the near scattered field can be computed using the limited far field data. A general discussion of this latter point is presented in 8579-1-Q. However, the employment of high frequency asymptotic results yields a far more illuminating picture (8579-2-Q). These results will be briefly presented here. When the scattered far field is measured over a limited aspect region (example: the cone $0 \leq \theta \leq \theta_0$), the analytic continuation of quantity $\underline{E}_0(\alpha, \beta)$ given in expression (2.13), cannot be found for the complete complex α plane. In this case the following approximation to expression (2.13) will be employed

$$\underline{E}^s = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\theta_o} e^{i\underline{k}'\cdot\underline{x}} \, \underline{E}_o(\alpha,\beta) \sin\alpha \, d\alpha \, d\beta \ . \qquad (2.15)$$

For the case of the perfectly conducting sphere of radius a , illuminated by the plane wave

$$\underline{E}^i = \hat{\underline{x}} \, e^{-ikz}$$

the geometric optics scattered far field has the form

$$\underline{E}_o(\alpha,\beta) = -\frac{a}{2} \, \hat{\underline{e}}(\alpha,\beta) \exp\left[-2ika \cos(\alpha/2)\right] \ ,$$

where

$$\hat{\underline{e}}(\alpha,\beta) = \hat{\underline{x}} \left[\cos\alpha \cos^2\beta + \sin^2\beta\right] - \hat{\underline{y}} \left[1 - \cos\alpha\right] \sin\beta \cos\beta - $$
$$- \hat{\underline{z}} \sin\alpha \cos\phi \ .$$

From relation (2.15) the near scattered field is given approximately by

$$\tilde{\underline{E}}^s(\underline{x}) = -\frac{ika}{4\pi} \int_0^{2\pi} \int_0^{\theta_o} e^{ikf(\alpha,\beta)} \, \hat{\underline{e}}(\alpha,\beta) \sin\alpha \, d\alpha \, d\beta \ , \qquad (2.16)$$

where

$$f(\alpha,\beta) = r\left[\sin\theta \sin\alpha \cos(\phi - \beta) + \cos\theta \cos\alpha\right] - 2a \cos\frac{\alpha}{2} \ .$$

As $k \to \infty$, the dominant contribution to the integral arises from the vicinity of the stationary phase point $(\beta = \phi, \ \alpha = \alpha_o)$ where α_o satisfies the equation

$$r \sin(\alpha_o - \theta) = a \sin\left(\frac{\alpha_o}{2}\right)$$

provided that $0 \le \alpha_o \le \theta_o$. By means of first order stationary phase evaluation we obtain immediately

$$\underset{\sim}{\tilde{E}}^s(\underline{x}) = - \left[\frac{D(o)}{D(s)}\right]^{1/2} \hat{\underline{e}}\,(\alpha_o, \phi)\, e^{iks - ika \cos\left(\frac{\alpha_o}{2}\right)} ,$$

where

$$D(s) = \left(\cos \frac{\alpha_o}{2} + \frac{2s}{a}\right)\left(1 + \frac{2s}{a} \cos \frac{\alpha_o}{2}\right) .$$

The distance s given by the relation

$$s = r \cos (\alpha_o - \theta) - a \cos\left(\frac{\alpha_o}{2}\right)$$

is the distance along the reflected ray from the point of reflection. The resulting expression for $\hat{\underline{E}}^s(\underline{x})$ is the geometric optics near field expression. When the far field information is limited to the cone $0 \le \theta < \theta_o$, the near field expression will be limited to a volume of space such that

$$r \sin (\alpha_o - \theta) = a \sin\left(\frac{\alpha_o}{2}\right) \qquad 0 \le \alpha_o \le \theta_o , \qquad (2.17)$$

where (r, θ, ϕ) are the coordinates of a point in this volume. In order to find the portion of the scattering surface which can be determined, set $r = a$, in which case it is seen that

$$\theta = \frac{\alpha_o}{2}, \qquad 0 \le \alpha_o \le \theta_o .$$

This then implies that the near scattered field given by expression (2.15) can be found on the portion of the sphere given by the cone $0 \le \theta \le \theta_o/2$. For points outside the volume of space given by Eq. (2.17), stationary phase techniques cannot be employed, in which case the asymptotic approximation to Eq. (2.16) is obtained by alternative means.

The above example illustrates the fact that in the case of high frequency, asymptotic evaluation of expression (2.15) is equivalent to tracing back the rays to the portion of the scattering surface from which they arose. The result is not confined to just perfectly smooth convex shapes but can be applied to piece-wise smooth convex shapes, the far scattered field is decomposable into components of the form

$$\underline{E}_o(\theta,\phi) = \sum_n \underline{A}_n(\theta,\phi)\, e^{ik\psi_n(\theta,\phi)}$$

and stationary phase evaluation of the expression (2.15) for each individual component, effectively traces backthe various rays to their originating portions of the scattering object. The result is applicable to flat portions, in which case the far field component approaches a delta function of the angular variables as $k \to \infty$. The example of the flat plate given in 8579-2-Q illustrates this.

III

EFFECTS OF A DISPLACED ORIGIN ON CONVERGENCE PROPERTIES

In employing the Wilcox theorem or the vector spherical wave function expansion, it can be shown that the near field representation, derived from the far scattered radiation pattern, is uniformly convergent for all points outside the minimum sphere enclosing the equivalent sources of a smooth convex shaped body. From the knowledge of the near field everywhere outside this minimum sphere, it is sought to find the associated scattering surface by proper application of suitable boundary conditions. These Inverse Scattering Boundary conditions, in general, ought not involve any constituent parameters (surface normal $\underline{n}$, surface locus S(x, y, z), relative impedance η) of the unknown scatterer and must be given solely in terms of the known nearfield representations of the incident and the scattered field with respect to a fixed origin which lies within the scatterer. For a perfectly conducting surface it was found that the following two boundary conditions can be successfully applied:

a)

$$\left\{ |\underline{E}^i| - |\underline{E}^s| \right\} = 0 \tag{3.1}$$

where $\underline{E}^i$ denotes the incident field and $\underline{E}^s$ the nearfield representation of the scattered field. This boundary condition represents the geometrical optics limiting approximation and in general, is applicable only within a narrow cone about the specular point (8579-3-Q, 8579-4-Q).

b)

$$\left\{ \underline{E}^T \times \underline{E}^{T*} \right\} = 0, \text{ where } \underline{E}^T = \underline{E}^i + \underline{E}^s \tag{3.2}$$

This boundary condition was derived by V. H. Weston (Final Report 7644-1-F, Weston, Bowman, Ar). Although it is only a necessary but not sufficient condition, namely producing a family of concentric surface loci in addition to the proper one, its application together with $\left\{ |\underline{E}^i| - |\underline{E}^s| \right\} = 0$ proved to be indispensable since that portion of the proper locus within the conical section exterior to the Wilcox minimum sphere can be determined with great accuracy for a minimum number of expansion terms, (8579-3-Q, 8579-4-Q).

To be more precise, the combined boundary conditions will yield the proper surface locus only on those portions of the surface for which the chosen expansion for the scattered field is convergent, for a prolate spheroid, the boundary conditions applied to the complete vector spherical wave function expansion associated with the origin located at the center of the body, will yield portions of the surface outside the sphere $r = c$ where c is the semi-focal length. In order to obtain the side portions one has to displace the origin in the plane passing through the center of the body and perpendicular to the axis of the body. Thus in using far field data one needs to know the domain of convergence of the expansion. Generally this can be only obtained if the complete expansion were known, which in practice is not achievable since the number of receiver locations will be a finite number. If the domain of convergence is not known, then the application of the boundary conditions to the expansion may yield some correct portions of the body, but what the correct portions are cannot be prescribed unless some additional condition is employed. However if it was known apriori that the origin of the coordinate system was located at the center of the body, and the portions of the body farthest removed from the origin where smooth and convex, then the application of the boundary condition will yield the correct surface locus (to within some numberical error) for these portions. Thus if the approximate location of the origin was not known, some additional criteria should be employed, such as the one described below.

A possible additional condition would be to displace the origin of the coordinate system, and on employing the boundary condition to the scattered field expansion derived in the displaced coordinate system to obtain a new surface locus, one would then determine the shift in the surface locus due to the displacement. If the shift is negligible for portions of the surface, then those portions may be the proper surface. For example, if the origin of the coordinate system located at the center of a prolate spheroid, was shifted slightly down the axis, the loci for the top and bottom of the prolate spheroid derived from the boundary conditions applied to the near field expansions, should be the same as before the shift. These remarks are strictly qualitative. A quantitative study should be made and is outlined below.

Let $\underline{x}$ represent the points in a cartesian frame of reference the origin of which is located in the vicinity of the scattering body (its precise location was not established) Let a plane wave be incident upon the body producing a scattered field, the far field

$$\underline{E}^s \sim \frac{e^{ikR}}{R} \underline{E}_o(\theta, \phi)$$

being measured at a finite number (N) locations $\left\{\theta_n, \phi_n\right\}$. Let $\underline{\tilde{E}}^s(\underline{x}, N)$ be a finite expansion of the near field derived from the far field data (the expansion may in vector spherical harmonics, or a Wilcox type expansion)

Suppose that the origin is displaced a distance $\underline{d}$, such that if $\underline{x}'$ represents a point in the displaced coordinate system then

$$\underline{x}' = \underline{x} + \underline{d} \quad .$$

In the displaced coordinate system, the far field directions $\left\{\theta_n, \phi_n\right\}$ remain the same, (i.e. for $R' \rightarrow \infty$, $\theta' = \theta$, $\phi' = \phi$), however the far field quantity $\underline{E}'_o(\theta', \phi')$ is related to $\underline{E}_o(\theta, \phi)$ by the relation

$$\underline{E}'_o(\underline{\theta}', \underline{\phi}') = \underline{E}_o(\theta, \phi) \exp(-i\underline{k}\ \underline{d})$$

where $\underline{k}$ is a vector in the radial direction. Let $\underline{\tilde{E}}'^s(\underline{x}', N)$ be a finite expansion of the near field derived from far field data (N receivers) in the displaced coordinate system.

If $\underline{x}$ is a point in space where the complete ($N = \infty$) expansions are convergent the difference

$$\underline{\tilde{E}}^s(\underline{x}, \infty) - \underline{\tilde{E}}'(\underline{x} + \underline{d}, \infty)$$

would be zero. However when a finite number of terms are used in the expansions (finite number of receivers) the difference will not be zero, except of course at the receiver positions in the far field. Thus it would be fruitful to determine quantitatively a number $\delta(\underline{d})$ such that if

$$\lim_{d \to o} \left| \frac{\underline{E}^s(\underline{x}) - \underline{E}'^s(\underline{x} + \underline{d})}{d} \right| \leq \delta(\hat{\underline{d}}) \left| \underline{E}^i \right| ,$$

where $\underline{E}^i$ is the incident field, the application of the boundary conditions on the total field at points $\underline{x}$ for which the above inequality holds will yield the correct surface to a specified degree of accuracy.

Such a condition given above may have to be obtained through numerical examples. It will depend upon what expansion for the near field is chosen, and how the expansion is matched to far field data (i.e., by matrix inversion, least squares, or otherwise).

To obtain the portions of the surface outside the domain of convergence (if the complete expansion were used or outside the domain given by the above inequality for a finite expansion, the origin of the coordinate system would have to be significantly changed. For this new location of the origin, the domain validity of a finite expansion would have to be chosen by displacing slightly the origin and employing the above inequality.

Some initial investigations of the effects of displacing the origin for a finite expansion in vector spherical wave functions, upon the surfaces derived through the boundary condition $\underline{E} \times \underline{E}^* = 0$, is discussed in internal memo (W. Boerner). However, this memo is just an initial attempt to understand the problem and will not be reported here.

IV

ASPECTS ON CONTINUOUS WAVE INVERSE SCATTERING BOUNDARY CONDITIONS FOR THE DETERMINATION OF THE SHAPE AND THE MATERIAL CONSTITUENTS OF A CONDUCTING BODY

4.1 INTRODUCTION

Although methods have been outlined on how to determine the material characteristics of a conducting body, employing the monostatic-bistatic theorem (7644-1-F, Chapter XI), no suitable method has been found so far which will simultaneously determine the shape and the associated material constituents of an unknown body, since the inverse scattering boundary condition $\left\{ \underline{E}^T \times \underline{E}^{T*} = 0, \text{ where } \underline{E}^T = \underline{E}^i + \underline{E}^s \right\}$, derived by Weston (7644-1-F, Chapter X) holds only for the case of a perfectly conducting body. This condition $\left\{ \underline{E}^T \times \underline{E}^T = 0 \right\}$, though necessary but not sufficient, may however be considered as a first step of formulating a more general c.w. Inverse Scattering Boundary Condition.

An attempt of generalizing this condition will be prescribed in this section. Before any C. W. I. S. boundary condition can be applied, a suitable near field representation of the scattered field from the measured far field radar data must be sought. This can be achieved by either employing a series expansion into proper vector wave functions and if so required their associated plane wave integral representation (7644-1-F, Chapter V; 8579-1-Q, 8579-2-Q) or an expansion method derived by Wilcox (1956) and Müller (1956) as discussed in 7466-1-F (Chapter IV). The associated expansion coefficients may be obtained from a matrix inversion technique (8579-2-Q and 8579-1-F). Assuming that such a sufficiently accurate nearfield representation of both the electric and the magnetic field vector is found, the question of how to derive suitable I. S. boundary conditions arises which as well can be applied to the determination of the characteristic parameters of conducting bodies and their shapes.

In contrast to problems of direct scattering and diffraction for which the shape and the material constituents of the scatterer are assumed to be known a priori together with the prespecified incident field vector $\underline{E}^i$ as regards the computational coordinate system and thus may be incorporated into the boundary conditions, in problems of inverse scattering such boundary conditions must be sought which in particular do not depend upon either the shape or the material properties of the scattering body and its enclosing surface, but allow to specify those characteristic parameters uniquely. And this solely from the near field representation of the electric and the magnetic field vectors, derived from the measured far field data as described above. Now the question arises as to how many and which characteristic parameters must be defined to uniquely determine the shape and the material constituents of the unknown scatterer. In the scalar case it is sufficient to employ the following parameters:

i) the local surface normal $\hat{n}(R, \theta, \phi)$ of the proper surface,

ii) a relative surface impedance $\eta(R, \theta, \phi)$ which is a scalar quantity, or the interior propagation constant $k_{int.}(R, \theta, \phi)$.

Thus at least three independent characteristic equations, solely expressed in terms of the near field quantities $\underline{E}^T = \underline{E}^i + \underline{E}^s$ and $\underline{H}^T = \underline{H}^i + \underline{H}^s$, must be found to determine the surface locus $S(R, \theta, \phi)$ and the surface impedance $\eta = |\eta| \exp i\psi$.

Whereas in the vector case the relative surface impedance must be formulated in dyadic or tensor formulation, involving a further unknown parameter which may be described by a local polarization angle $\epsilon(R, \theta, \phi; \hat{k}_i, \hat{e}_i)$ or a local depolarization angle $\delta(R, \theta, \phi; \hat{k}_i, \hat{e}_i)$. These angles can be expressed in terms of the surface normal $\hat{n}(R, \theta, \phi)$ as well as the unit direction vector of the incident wave $\hat{k}_i$ and its associated polarization vector $\hat{e}_i$. Thus the impedance dyadic $\overrightarrow{\overline{\eta}}$ will be a function of both the material properties of the scattering surface, the surface locus $S(R, \theta, \phi)$ or its associated local normal $\hat{n}(R, \theta, \phi)$, and the prespecified properties $\hat{k}_i$, $\hat{e}_i$ of the incident wave. This additional complication may ask for one or two more independent characteristic equations for the elimination of ϵ or δ.

If such a set of independent scalar and vector equations exist which can be employed to uniquely determine the surface locus $S(R, \theta, \phi)$ and the material characteristic η, then one may argue that the I.S. boundary condition $\underline{E}^T \times \underline{E}^{T*} = 0$ may constitute the remaining part of such a set of independent equations for the degenerate case of $\eta = 0$. With this goal in mind more detailed properties of $\left\{\underline{E}^T \times \underline{E}^{T*} = 0,\ \underline{E}^T \times \underline{E}^i + \underline{E}^s\right\}$ will be discussed next, as well as properties of the Leontovich boundary condition and its complementary formulations. Then it will be shown that for the scalar case a set of independent scalar and vector equations can be derived, based upon the concept of describing the material surface properties by a Leontovich boundary condition, which will in the limit as $\eta = 0$, degenerate into the condition $\underline{E}^T \times \underline{E}^{T*} = 0$ as derived by Weston in 7644-1-F (Chapter X).

4.2 THE C.W.I.S. BOUNDARY CONDITION $\underline{E}^T \times \underline{E}^{T*} = 0$ FOR THE DEGENERATE CASE OF A PERFECTLY CONDUCTING SCATTERER, i.e. $\eta = 0$.

In chapter X of 7644-1-F, the necessary but not sufficient condition $\underline{E}^T \times \underline{E}^{T*} = 0$ was derived intuitively based upon the physical properties that the total field $\underline{E}^T = \underline{E}^i + \underline{E}^s$ must satisfy the boundary condition $\hat{n} \times \underline{E}^T = 0$ on the surface of a perfectly conducting body, and since the surface normal $\hat{n}$ is a real vector also $\underline{E}^T$ cannot be of complex direction, thus $\underline{E}^T \times \underline{E}^{T*} = 0$. The application of this boundary condition is verified for spherical and prolate spheroidal test cases in 8579-3-Q and 8579-4-Q, together with the condition $\left\{|\underline{E}^i| - |\underline{E}^s|\right\} = 0$ as derived from the geometrical optics approximation. Both conditions yield the most accurate results if the incident polarization is parallel to the generators of the scattering body, since then $\underline{E}^i + \underline{E}^s \equiv 0$ identically. Although $\underline{E}^T \times \underline{E}^{T*} = 0$ is only a necessary condition but not sufficient in that an infinite set of exterior concentrical (hyperbolic) surface loci, and for scatterers of larger electrical measure ka a limited set of interior pseudo loci (emanating from the associated interior caustic) in addition to the proper one are generated, this proper surface locus can be determined with least error. In fact, in 8579-3-Q and 8579-4-Q, it is verified that with $\underline{E}^T \times \underline{E}^{T*} = 0$ the proper surface locus can be identified more accurately as with the condition $|\underline{E}^i| - |\underline{E}^s| = 0$ for which only one locus is obtained, however with pronounced deviations from the proper one (8579-4-Q). The family of resulting surface loci of $\underline{E}^T \times \underline{E}^{T*} = 0$ are equidistantly spaced, concentrical hyperboloids with $\Delta X = k \Delta R = \pi/2$ spacing along the axis of symmetry for rotationally symmetric bodies with end-on incidence. The resulting plots for one and the same scatterer versus the electric length are identical for different frequencies. If however, the resulting loci are plotted versus the geometrical length $R = X/k$, then it can be shown (8579-4-Q) that the proper surface locus is stationary whereas all the other interior as well as exterior additional loci will shift, since the proper surface locus is independent of the wavelength and all further loci correspond to a discrete set of eigenfrequencies (7644-1-F, chapter X). Thus the application of the boundary condition $\underline{E}^T \times \underline{E}^{T*} = 0$ at only two different frequencies may be sufficient to uniquely determine the shape of a smooth convex-shaped, perfectly conducting body in those regions which are exterior to the Wilcox minimum sphere. Those portions of the scattering surface within the Wilcox minimum sphere can be determined with $\underline{E}^T \times \underline{E}^{T*} = 0$ as well, but not with the condition $\{|\underline{E}^i| - |\underline{E}^s|\} = 0$, if properties of displacing the computational origin from the geometrical center are employed at two different frequencies by matching the geometrical length of the displacement vector so that a stationary surface locus within the angular domain along the direction of the displacement vector is found, (8579-1-F).

Thus the boundary condition $\underline{E}^T \times \underline{E}^{T*} = 0$ must be considered indispensable in computational problems of C.W. Inverse Scattering as applied to the identification of smooth, convex-shaped, perfectly conducting bodies.

Finally the question arises whether the boundary condition $\underline{E}^T \times \underline{E}^{T*} = 0$ may not bear more physical information and whether its generalization to conducting bodies may reveal a relationship with Maxwell's Stress-Energy Tensor at the bounding surface of a scatterer. It was therefore attempted to employ the Leontovich boundary condition of the scalar case to study this question in more detail.

4.3 THE LEONTOVICH BOUNDARY CONDITION AND ITS VARIOUS COMPLEMENTARY FORMULATIONS.

In problems of C.W. inverse scattering, the concept of an impedance boundary condition may be employed to its best, if such an impedance can be defined so that it describes the averaged local electromagnetic properties of the unknown scattering surface. The boundary condition (4.3.1), known as Leontovich condition (Leontovich, 1948) suggests to offer the desired formulation for the scalar case:

$$\underline{E}^T - (\hat{n} \cdot \underline{E}^T)\,\hat{n} = \eta\, Z_o\, \hat{n} \times \underline{H}^T \tag{4.3.1}$$

where

$\underline{E}^T = \underline{E}^i + \underline{E}^s$, the total electric field

$\underline{H}^T = \underline{H}^i + \underline{H}^s$, the total magnetic field in the region surrounding the body,

$Z_o = \sqrt{\dfrac{\mu_o}{\epsilon_o}} = \dfrac{1}{Y_o} = 120\,\pi\,\Omega$, the intrinsic impedance of free space,

$\hat{n} = \hat{n}^*$, the unit outward normal as regards the surface which is a <u>purely real</u> vector.

η, the relative impedance, a complex scalar quantity, designated as Leontovich impedance.

The degenerate cases for which (4.3.1) is not applicable are treated in detail in Senior (1959, 1962) and Weston (CAA-0020-10-TR) and will be excluded from further discussion. The direct application of (4.3.1) would imply the a priori knowledge of $n = n(R, \theta, \phi)$ and $\eta = \eta(R, \theta, \phi)$. Thus other formulations of

(4.3.1) are required which, however, must contain identical information such that the surface locus $S(R, \theta, \phi)$ and $\eta(R, \theta, \phi)$ can be found solely from $\underline{E}^T$ and $\underline{H}^T$. The first additional equation is found by applying a vector product operation of $\hat{n}$ onto (4.3.1), yielding

$$\underline{E}^T \times \hat{n} = \eta Z_o \left[\underline{H}^T - (\hat{n} \cdot \underline{H}^T)\hat{n}\right] \tag{4.3.2}$$

which merely represents an alternative statement of (4.3.1) and may be rewritten with $\xi = 1/\eta$ into

$$\underline{H}^T - (\hat{n} \cdot \underline{H}^T)\hat{n} = -\xi Y_o (\hat{n} \times \underline{E}^T) \tag{4.3.2a}$$

and thus corresponds to (4.3.1) under the transformation $\underline{E}^T \to \underline{H}^T$, $Z_o \underline{H}^T \to -Y_o \underline{E}^T$) and $\eta \to \xi$, where ξ denotes the relative admittance. Senior (1962a) has shown the affinities of this transformation with Babinet's principle and has proven the invariance of this transformation, attributing η to the material constituents of the body.

In addition to (4.3.1) and (4.3.2) its conjugated formulations will be introduced as:

$$\underline{E}^{T*} - (\hat{n} \cdot \underline{E}^{T*})\hat{n} = \eta^* Z_o \hat{n} \times \underline{H}^{T*} \tag{4.3.3}$$

and

$$\underline{E}^{T*} \times \hat{n} = \eta^* Z_o \left[\underline{H}^{T*} - (\hat{n} \cdot \underline{H}^{T*})\hat{n}\right] \tag{4.3.4}$$

or

$$\left[\underline{H}^* - (\hat{n} \cdot \underline{H}^{T*})\hat{n}\right] = -\xi^* Y_o (\hat{n} \times \underline{E}^{T*}) \quad . \tag{4.3.4a}$$

The validity of statements (4.3.3) and (4.3.4) must strictly assume that all implied field quantities $\underline{E}^T$ and $\underline{H}^T$, as well as the relative impedance

$\eta = \eta(R, \theta, \phi)$ are analytical functions, and $\hat{n} = \hat{n}(R, \theta, \phi)$ is piecewise continuous, satisfying the set of linear equations (4.3.1) and (4.3.2) which in turn satisfy Maxwell's equations. The particular constraints on η and n for which this is not the case are discussed by Weston (CAA-0020-10-TR), and scatterers of these particular features will be excluded. The conjugation of (4.3.1) and (4.3.2) solely implies the reversal of the reactive character of all implied electromagnetic quantities, thus the pair (4.3.3) - (4.3.4a) bears similar affinities as does the pair (4.3.1) - (4.3.2a), now however, for a surface of reversed reactive character.

Assuming that the electromagnetic behavior in the vicinity of a scatterer satisfies the Leontovich boundary condition (4.3.1), the question must be answered whether a set of scalar and vector equations can be derived from the complementary set (4.3.1) to (4.3.4) of the Leontovich boundary condition which may be employed to determine the surface locus $S(R, \theta, \phi)$, the surface normal $\hat{n}(R, \theta, \phi)$, the relative surface impedance $\eta(R, \theta, \phi)$ solely from the knowledge of the near field representations of $\underline{E}^T$ and $\underline{H}^T$. This may be achieved by proper scalar and vector product operations of (4.3.1) to (4.3.4) onto one another which must result in the following set of independent equations:

i) A vector or scalar equation which ought to be independent of η and may thus determine the surface locus $S(R, \theta, \phi)$ of the unknown target. This equation may possess a discrete set of additional solutions, resulting in a family of concentric surface loci.

ii) A set of independent scalar equations necessary to define both the amplitude $|\eta|$ and the phase $\eta/|\eta| = \exp i\psi$ of the relative surface impedance. These equations should result in a pair of quadratic equations such that a fourfold complementary solution is obtained, corresponding to the complementary character of equations (4.3.1) to (4.3.4).

iii) A closed form expression for the surface normal n, which must yield a purely real vector quantity and ought not depend upon η, otherwise the correct solution of η, satisfying (4.3.1) cannot be found.

To this end the scalar formulation of the Leontovich boundary condition was considered only which in general cannot be employed for problems of inverse scattering as applied to the determination of the shape and material constituents of closed convex-shaped bodies. The vector wave nature of the inverse scattering problem requires the definition of a dyadic formulation of the relative impedance, where the following equations must be satisfied on the surface of a non-perfectly conducting body

$$\underline{E}^T - (\hat{n} \cdot \underline{E}^T)\,\hat{n} = Z_o \overleftrightarrow{\eta} \cdot (\hat{n} \times \underline{H}^T) \tag{4.3.5a}$$

$$\underline{E}^T \times \hat{n} = Z_o \overleftrightarrow{\eta} \cdot (\underline{H}^T - (\hat{n} \cdot \underline{H}^T)\,\hat{n}) \quad . \tag{4.3.5b}$$

The impedance dyadic is symmetric and will contain all terms except for the case of plane wave incidence onto a planar surface with invariance along one axis normal to the plane of incidence for which the diagonal terms are involved only. In all other cases a local angle of polarization as well as a local angle of depolarization together with the radii of curvature will be involved. These properties however may not allow the formulation of the required independent equations since the impedance dyadic will become a function of the radiant dependence along the outward normal (Morse and Feshbach, 1953). Before any further assumptions can be made, the scalar case will be investigated in detail.

4.4 FORMULATION OF A SET OF INVERSE SCATTERING BOUNDARY CONDITIONS FOR A SCATTERING SURFACE, SATISFYING THE SCALAR LEONTOVICH BOUNDARY CONDITION $(\underline{E}^T - (\hat{n} \cdot \underline{E}^T)\,\hat{n}) = \eta Z_o \hat{n} \times \underline{H}^T$.

Applying scalar and vector product operations of equations (4.3.1) to (4.3.4) onto one another a complex set of interdependent scalar and vector equations results which are given in the appendix. Inspecting these equations it was found practical to employ the following three pairs of vector expressions as fundamental vectors for further analysis (where $\underline{E} = \underline{E}^T$, and $\underline{H} = \underline{H}^T$) :

$$\underline{A}_{I,II} = \underline{A}_1 \mp \underline{A}_2 = \left[(\underline{E} \times \underline{E}^*) \mp \eta\,\eta^*(\underline{H} \times \underline{H}^*)\right] \quad (4.4.1)$$

$$\underline{A}_I^* = -\underline{A}_I \quad (4.4.1a)$$

$$\underline{A}_{II}^* = -\underline{A}_{II} \quad (4.4.1b)$$

Thus $\underline{A}_I$ and $\underline{A}_{II}$ are purely imaginary veotor quantities

$$\underline{B}_{I,II} = \underline{B}_1 \mp \underline{B}_2 = \left[\eta\,(\underline{E}^* \times \underline{H}) \mp \eta^*(\underline{E} \times \underline{H}^*)\right] \quad (4.4.2)$$

$$\underline{B}_I^* = -\underline{B}_I \quad (4.4.2a)$$

$$\underline{B}_{II}^* = \underline{B}_{II} \quad (4.4.2b)$$

$$\underline{C}_{I,II} = \underline{C}_1 \pm \underline{C}_2 = \left[\eta\,(\underline{E} \times \underline{H}) \mp \eta^*(\underline{E}^* \times \underline{H}^*)\right] \quad (4.4.3)$$

$$\underline{C}_I^* = -\underline{C}_I \quad (4.4.3a)$$

$$\underline{C}_{II}^* = \underline{C}_{II} \quad . \quad (4.4.3b)$$

In addition to these three pairs of fundamental vectors, it was found useful to introduce the following notations:

$$\underline{D}_I = \underline{A}_I \times \underline{B}_I = \underline{D}_I^* \quad (4.4.4)$$

$$\underline{D}_{II} = \underline{A}_{II} \times \underline{B}_{II} = -\underline{D}_{II}^* \quad (4.4.5)$$

$$\underline{E}_I = \underline{A}_I \times \underline{A}_{II} = \underline{E}_I^* = 2\,\underline{A}_1 \times \underline{A}_2 = 2\,\eta\,\eta^*(\underline{E} \times \underline{E}^*) \times (\underline{H} \times \underline{H}^*) \quad (4.4.6)$$

$$\underline{E}_{II} = \underline{B}_I \times \underline{B}_{II} = -\underline{E}_{II}^* = 2\,\underline{B}_1 \times \underline{B}_2 = 2\,\eta\,\eta^*(\underline{E}^* \times \underline{H}^*) \times (\underline{E} \times \underline{H}^*) \quad (4.4.7)$$

$$\underline{E}_{III} = \underline{C}_I \times \underline{C}_{II} = -\underline{E}^*_{III} = 2\,\underline{C}_1 \times \underline{C}_2 = 2\,\eta\,\eta^*(\underline{E} \times \underline{H}) \times (\underline{E}^* \times \underline{H}^*) \qquad (4.4.8)$$

$$\underline{F}_I = -\underline{F}^*_I = \underline{A}_I \times \underline{B}_{II} \qquad (4.4.9)$$

$$\underline{F}_{II} = -\underline{F}^*_{II} = \underline{B}_I \times \underline{A}_{II} \quad . \qquad (4.4.10)$$

Employing properties of the derived set of scalar and vector equations as given in the appendix, three orthogonal vectors out of the thirteen vectors defined must be found to which all the remaining vectors can be simply associated. Inspecting equations(4.6.6a, 9b, 13) of the appendix, two of the required vectors may be found in $\underline{A}_I$ and $\underline{B}_I$, since

$$\hat{n} \cdot \underline{A}_I = 0 \qquad (4.4.11a)$$

$$\hat{n} \cdot \underline{B}_I = 0 \qquad (4.4.11b)$$

$$\underline{B}_I = \hat{n} \times \underline{A}_I \qquad (4.4.11c)$$

$$\underline{A}_I = -\hat{n} \times \underline{B}_I \quad . \qquad (4.4.11d)$$

Thus the purely imaginary (4.4.1a, 4.4.2a) vector quantities $\underline{A}_I$ and $\underline{B}_I$ are perpendicular to each other (4.4.11c, 4.4.11d) and in fact tangent to the local scattering surface (4.4.11a, 4.4.11b) . Furthermore, it can be shown that

$$\underline{A}_I \cdot \underline{A}_I = (\underline{B}_I \times \hat{n}) \cdot \underline{A}_I = (\underline{B}_I \times \hat{n}) \cdot (\underline{B}_I \times \hat{n}) =$$

$$= \underline{B}_I \cdot \underline{B}_I - (\hat{n} \cdot \underline{B}_I)^2 = \underline{B}_I \cdot \underline{B}_I \qquad (4.4.12a)$$

and

$$\underline{A}_I \cdot \underline{B}_I = 0 \quad . \qquad (4.4.12b)$$

Since $\underline{A}_I$ and $\underline{B}_I$ are tangent to the local scattering surface and perpendicular to one another, its cross product $\underline{D}_I$ (4.4.4), apurely real vector quantity, must be directed along the local outward normal $\hat{n}$ of the scattering surface, where

$$\underline{D}_I = \underline{A}_I \times \underline{B}_I = -(\underline{A}_I \cdot \underline{A}_I)\hat{n} = -(\underline{B}_I \cdot \underline{B}_I)\hat{n} = \left[\hat{n} \cdot (\underline{A}_I \times \underline{B}_I)\right]\hat{n} \quad (4.4.13)$$

and with equation (. .), where it was shown that:

$$\left[\hat{n} \cdot (\underline{A}_I \times \underline{B}_I)\right] = -2\,\underline{C}_1 \cdot \underline{C}_2 = -2\,(\underline{A}_1 \cdot \underline{A}_2) + \underline{B}_1 \cdot \underline{B}_2\,) \quad (4.4.14a)$$

$$\underline{D}_I = -2\,\hat{n}\,\underline{C}_1 \cdot \underline{C}_2 = -2\,\hat{n}\,(\underline{A}_1 \cdot \underline{A}_2 + \underline{B}_1 \cdot \underline{B}_2) \quad . \quad (4.4.14b)$$

Since no further triplet of orthogonal vectors was found and the remaining vector quantities as defined in (4.4.1) to (4.4.10) can be uniquely decomposed along the directions of $\underline{A}_I$, $\underline{B}_I$, and $\underline{D}_I$, the following theorem can be formulated:

Theorem: If the electromagnetic behavior in the vicinity of a scatterer satisfies the Leontovich Boundary condition

$$\underline{E}^T - (\hat{n} \cdot \underline{E}^T)\,\hat{n} = \eta\, Z_o\, \hat{n} \times \underline{H}^T$$

then the following two purely imaginary vectors

$$\underline{A}_I = (\underline{E}^T \times \underline{E}^{T*}) - \eta\,\eta^*(\underline{H}^T \times \underline{H}^{T*})$$

and

$$\underline{B}_I = \eta\,(\underline{E}^{T*} \times \underline{H}^T) - \eta^*\,(\underline{E}^T \times \underline{H}^{T*})$$

are orthogonal vector quantities which lie in the local tangent plane of the scatterer and its cross product, a purely real vector quantity,

$$\underline{D}_I = \underline{A}_I \times \underline{B}_I = \hat{n}\,\left[\hat{n} \cdot (\underline{A}_I \times \underline{B}_I)\right]$$

is directed along the outward local normal of the scatterer. The absolute values of these three vector quantities are identical, where

$$\underline{A}_I \cdot \underline{A}_I = \underline{B}_I \cdot \underline{B}_I = -(\hat{n} \cdot \underline{D}_I)$$

and $\underline{A}_I$, $\underline{B}_I$, $\underline{D}_I$ constitute a right-handed orthogonal vector triplet.

These peculiar properties result in the following characteristic equations from which a set of inverse scattering boundary conditions may possibly derived. Since with (4.4.13)

$$\underline{A}_I \cdot \underline{A}_I = \underline{B}_I \cdot \underline{B}_I = -(\hat{\underline{n}} \cdot \underline{D}_I) = +2\,\underline{C}_1 \cdot \underline{C}_2$$

$$= 2\,(\underline{A}_1 \cdot \underline{A}_2 + \underline{B}_1 \cdot \underline{B}_2)$$

two characteristic equations for the determination of the amplitude and the phase of the relative impedance η are found:

$$(\underline{E} \times \underline{E}^*) \cdot (\underline{E} \times \underline{E}^*) + (\eta\,\eta^*)^2\,(\underline{H} \times \underline{H}^*) \cdot (\underline{H} \times \underline{H}^*) = -2\,\eta\,\eta^*(\underline{E}^* \times \underline{H}) \cdot (\underline{E} \times \underline{H}^*) \qquad (4.4.15a)$$

and

$$\eta^2\,(\underline{E}^* \times \underline{H}) \cdot (\underline{E}^* \times \underline{H}) + \eta^{*2}\,(\underline{E} \times \underline{H}^*) \cdot (\underline{E} \times \underline{H}^*) = -2\eta\,\eta^*(\underline{E} \times \underline{E}^*) \cdot (\underline{H} \times \underline{H}^*). \qquad (4.4.15b)$$

For the derivation of (4.4.15a) and (4.4.15b), equation (4.4.12b) was not used, thus the orthogonality condition of $\underline{A}_I$ and $\underline{B}_I$ may be employed as the characteristic equation for the determination of the surface locus, i.e.

$$\underline{A}_I \cdot \underline{B}_I = 0 \quad \text{or} \qquad (4.4.16a)$$

$$\left[(\underline{E} \times \underline{E}^*) - \eta\,\eta^*(\underline{H} \times \underline{H}^*)\right] \cdot \left[\eta\,(\underline{E}^* \times \underline{H}) - \eta^*(\underline{E} \times \underline{H}^*)\right] = 0$$

which can be rewritten into

$$\left[(\underline{E} \cdot \underline{E}^*) - \eta\,\eta^*(\underline{H} \cdot \underline{H}^*)\right]\left[\eta \cdot (\underline{E}^* \cdot \underline{H}) + \eta^*(E \cdot H^*)\right] =$$

$$= \eta\,(\underline{E} \cdot \underline{H})\left[(\underline{E}^* \cdot \underline{E}^*) - \eta^{*2}\,(\underline{H}^* \cdot \underline{H}^*)\right] + \eta^*(\underline{E}^* \cdot \underline{H}^*)\left[(\underline{E} \cdot E) - \eta^2\,(\underline{H} \cdot \underline{H})\right]. \qquad (4.4.16b)$$

Although (4.4.16) has not been employed in the derivation of (4.4.15a, b), these three equations may be linearly dependent which still must be investigated.

The fourth characteristic equation for the determination of the surface normal, then is given by

$$\hat{n} = \frac{\underline{D}_I}{\hat{n} \cdot \underline{D}_I} = \frac{\underline{A}_I \times \underline{B}_I}{\hat{n} \cdot [\underline{A}_I \times \underline{B}_I]} = - \frac{\underline{A}_I \times \underline{B}_I}{2\, \underline{C}_1 \cdot \underline{C}_2} \tag{4.4.17b}$$

$$= \frac{- \left\{ \left[(\underline{E} \times \underline{E}^*) - \eta \eta^* (\underline{H} \times \underline{H}^*) \right] \times \left[\eta (\underline{E}^* \times \underline{H}) - \eta^* (\underline{E} \times \underline{H}^*) \right] \right\}}{2 \eta \eta^* (\underline{E} \times \underline{H}) \cdot (\underline{E}^* \times \underline{H}^*)}$$

which is dependent upon η and so far could not be formulated solely in terms of the field quantities.

Inspecting these characteristic equations it is found that a fourfold solution for the relative impedance η results as was predicted, furthermore these η represent a set of four complementary solutions. For the degenerate case $\eta \rightarrow 0$, equation (4.4.15a) degenerates into the inverse scattering boundary condition $\underline{E}^T \times \underline{E}^{T*} = 0$ which was derived by Weston in 7644-1-F (chapter X). To determine the proper amplitude and phase of η, the formulation (4.3.1) of the Leontovich equation must be satisfied which would require a characteristic equation for the surface normal $\hat{n}$ being independent from η, otherwise a unique solution may not be found from the orthogonality condition (4.4.16a). Namely were η a function of frequency then also $\hat{n}$ may become frequency dependent. However it may be possible to express $\underline{D}_I$ in terms of cross procuts of $\underline{A}_{1,2}$, $\underline{B}_{1,2}$ or $\underline{C}_{1,2}$ such that the normal becomes a closed form expression in terms of the field quantities only.

Although the characteristic equations (4.4.15, 16 and 17) cannot be applied in the general problem of inverse scattering for the determination of the constituent parameters $S(R, \theta, \phi)$, $\hat{n}(R, \theta, \phi)$ and $\eta(R, \theta, \phi)$ due to the inherent vector nature, the presented derivation suggests that such a formulation of local inverse scattering boundary conditions may exist. A more sophisticated analysis, employing the vector form of the Leontovich boundary condition, may yield the desired characteristic equations.

4.5 SUMMARY

It was shown that it is possible to derive a set of scalar and vector equations from the complementary set of the four formulations of the Leontovich boundary condition which may be employed as local boundary conditions to the problem of continuous wave inverse scattering. Although the derivations are based upon the formulation of a scalar Leontovich impedance, this first approach reveals a rather important result. Namely inspecting the characteristic equation (4.4.15a) it can be shown that the associated scattering surface locus can be found if solely the absolute value of the impedance is known. Thus if the material character of the surface is known, the following local inverse scattering boundary conditions may be applied to determine the surface locus $S(R, \theta, \phi)$, i.e. the shape of the scatterer:

α) $\eta = 0$ (perfectly conducting scatterer):

$$(\underline{E}^T \times \underline{E}^{T*}) \cdot (\underline{E}^T \times \underline{E}^{T*}) = 0 \qquad (4.5.1)$$

β) $\eta = \infty$ (perfect magnetic conductor or scatterer with plasma coating):

$$(\underline{H}^T \times \underline{H}^{T*}) \cdot (\underline{H}^T \times H^{T*}) = 0 \qquad (4.5.2)$$

γ) $0 < \eta < \infty$, $|\eta| \neq 1$ (conducting scatterer for which only the absolute value of the relative Leontovich impedance has to be known):

$$(\underline{E}^T \times \underline{E}^{T*}) \cdot (\underline{E}^T \times \underline{E}^{T*}) + (\eta\eta^*)^2 (\underline{H}^T \times \underline{H}^{T*}) \cdot (\underline{H}^T \times H^{T*}) =$$

$$= -2\,\eta\eta^* (\underline{E}^{T*} \times \underline{H}^T) \cdot (\underline{E}^T \times \underline{H}^{T*}) \quad . \qquad (4.5.3)$$

These boundary conditions are necessary but not sufficient,however they are local conditions which permit a point by point determination of the surface locus and its application of two or more different operational frequencies may result in the unique specification of the proper locus out of the infinite set of the obtained loci as was discussed in detail in section (4.2).

Furthermore the properties of the formulated theorem of section (4.4) suggest the formulation of a more general set of local inverse scattering boundary conditions for which the a priori knowledge of the relative Leontovich impedance is not required. Such a formulation requires first of all the proof that the characteristic equations (4.4.15a), (4.4.15b) and (4.4.16a) are linearly independent. Then a formulation of the surface normal must be sought which is independent of η so that the impedance corresponding to the proper plane can be selected.

So far no explicite physical interpretation on the nature of these boundary conditions was presented, preliminary investigations however suggest that a relationship between the electromagnetic stress tensor (J.A. Stratton, 1941) and the properties of the above state theorem exists. It is therefore proposed to investigate such existing relationships in all detail.

4.6 APPENDIX

For convenience, the total magnetic field vector $\underline{H}^T = \underline{H}$ is normalized with respect to the total electric field vector $\underline{E}^T = \underline{E}$ by the intrinsic impedance of free space $Z_o = 120\pi\Omega$, where Eqs. (4.3.1) to (4.3.4) will assume the following form

$$\left\{\left[\underline{E} - (\hat{n} \cdot \underline{E})\,\hat{n}\right] - \eta\,(\hat{n} \times \underline{H})\right\} = 0 \qquad \text{I}$$

$$\left\{(\underline{E} \times \hat{n}) - \eta\left[\underline{H} - (\hat{n} \cdot \underline{H})\,\hat{n}\right]\right\} = 0 \qquad \text{II}$$

$$\left\{\left[\underline{E}^* - (\hat{n} \cdot E^*)\,\hat{n}\right] - \eta^*\,(\hat{n} \times \underline{H}^*)\right\} = 0 \qquad \text{III}$$

$$\left\{(\underline{E}^* \times \hat{n}) - \eta^*\left[\underline{H}^* - (\hat{n} \cdot H^*)\,\hat{n}\right]\right\} = 0 \; . \qquad \text{IV}$$

The scalar and vector product operations of these equations onto another result into the following relationships:

4.6.1 SCALAR PRODUCTS

I · I or II · II

$$(\underline{E} \cdot \underline{E}) - \eta^2\,(\underline{H} \cdot \underline{H}) = (\hat{n} \cdot \underline{E})^2 - \eta^2\,(\hat{n} \cdot \underline{H})^2 \qquad (4.6.1a)$$

$$\eta\left[\underline{E}\cdot(\hat{n}\times\underline{H})\right] = (\underline{E}\cdot\underline{E}) - (\hat{n}\cdot\underline{E})^2 = \eta\left[\underline{H}\cdot(\underline{E}\times\hat{n})\right] =$$

$$\eta^2(\underline{H}\cdot\underline{H}) - \eta^2(\hat{n}\cdot\underline{H})^2 \qquad (4.6.1b)$$

or

$$2\eta\left[\hat{n}\cdot(\underline{H}\times\underline{E})\right] = \left[(\underline{E}\cdot\underline{E}) + \eta^2(\underline{H}\cdot\underline{H})\right] - \left[(\hat{n}\cdot\underline{E})^2 + \eta^2(\hat{n}\cdot\underline{H})^2\right] \qquad (4.6.1c)$$

<u>I* · I* or II* · II*</u>

$$(\underline{E}^*\cdot\underline{E}^*) - \eta^{*2}(\underline{H}^*\cdot\underline{H}^*) = (\hat{n}\cdot\underline{E}^*)^2 - \eta^{*2}(\hat{n}\cdot\underline{H}^*)^2 \qquad (4.6.2a)$$

$$\eta^*\left[\hat{n}\cdot(\underline{H}^*\times\underline{E}^*)\right] = \eta^*\left[\underline{E}^*\cdot(\hat{n}\times\underline{H}^*)\right] = \eta^*\left[\underline{H}^*\cdot(\underline{E}^*\times\hat{n})\right] =$$

$$= (\underline{E}^*\cdot\underline{E}^*) - (\hat{n}\cdot\underline{E}^*)^2 = \eta^{*2}(\underline{H}^*\cdot\underline{H}^*) - \eta^{*2}(\hat{n}\cdot\underline{H}^*)^2 \qquad (4.6.2b)$$

or

$$2\eta^*\left[\hat{n}\cdot(\underline{H}^*\times\underline{E}^*)\right] = \left[(\underline{E}^*\cdot\underline{E}^*) + \eta^{*2}(\underline{H}^*\cdot\underline{H}^*)\right] - \left[(\hat{n}\cdot\underline{E}^*)^2 + \eta^{*2}(\hat{n}\cdot\underline{H}^*)^2\right]$$

(4.6.2c)

<u>I · II and I* · II*</u>

$$\eta(\underline{E}\cdot\underline{H}) = \eta(\hat{n}\cdot\underline{E})(\hat{n}\cdot\underline{H}) \qquad (4.6.3a)$$

$$\eta^*(\underline{E}^*\cdot\underline{H}^*) = \eta^*(\hat{n}\cdot\underline{E}^*)(\hat{n}\cdot\underline{H}^*) \qquad (4.6.3b)$$

<u>I · I*</u>

$$(\hat{n}\times\underline{E})\cdot(\hat{n}\times\underline{E}^*) - \eta\eta^*(\hat{n}\times\underline{H})\cdot(\hat{n}\times\underline{H}^*) = 0 \qquad (4.6.4a)$$

or

$$(\underline{E}\cdot\underline{E}^*) - \eta\eta^*(\underline{H}\cdot\underline{H}^*) = (\hat{n}\cdot\underline{E})(\hat{n}\cdot\underline{E}^*) - \eta\eta^*(\hat{n}\cdot\underline{H})(\hat{n}\cdot\underline{H}^*) \qquad (4.6.4b)$$

<u>II · II*</u>

$$\hat{n}\cdot\left\{\eta(\underline{E}^*\times\underline{H}) - \eta^*(\underline{E}\times\underline{H}^*)\right\} = 0 \qquad (4.6.5)$$

I · II*

$$\hat{n} \cdot \left\{ (\underline{E} \times \underline{E}^*) - \eta\eta^*(\underline{H} \times \underline{H}^*) \right\} = 0 \qquad (4.6.6a)$$

$$\eta\,(\hat{n} \times \underline{E}^*) \cdot (\hat{n} \times \underline{H}) + \eta^*(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{H}^*) = 0 \qquad (4.6.6b)$$

or

$$\eta\,(\underline{E}^* \cdot H) + \eta^*(\underline{E} \cdot \underline{H}^*) = \eta\,(\hat{n} \cdot \underline{E}^*) + \eta^*(\hat{n} \cdot \underline{E})\,(\hat{n} \cdot \underline{H}^*) \qquad (4.6.6c)$$

4.6.2 VECTOR PRODUCTS

I x I and II x II

$$(\underline{E} \times \underline{E}) \equiv \eta^2\,(\underline{H} \times \underline{H}) \equiv 0 \qquad (4.6.7a)$$

I* x I* and II* x II*

$$(\underline{E}^* \times \underline{E}^*) = \eta^{*2}\,(\underline{H}^* \times \underline{H}^*) = 0 \qquad (4.6.7b)$$

I x II and I* x II*

$$\eta\,(\underline{E} \times \underline{H}) = \left\{ \left[\eta\,\hat{n} \cdot (\underline{H} \times \underline{E})\right] - \left[(\underline{E} \cdot \underline{E}) + \eta^2\,(\underline{H} \cdot \underline{H})\right] \right\} \hat{n} + \left[(\hat{n} \cdot \underline{E})\,\underline{E} + \eta^2\,(\hat{n} \cdot \underline{H})\,\underline{H}\right]$$

$$= \left[\eta^2\,(\underline{H} \cdot \underline{H}) - (\hat{n} \cdot \underline{E})^2\right] \hat{n} + \left[(\hat{n} \cdot \underline{E})\,\underline{E} + \eta^2\,(\hat{n} \cdot \underline{H})\,\underline{H}\right] =$$

$$= \left[(\underline{E} \cdot \underline{E}) - \eta^2\,(\hat{n} \cdot \underline{H})^2\right] \hat{n} + \left[(\hat{n} \cdot \underline{E})\,\underline{E} + \eta^2\,(\hat{n} \cdot \underline{H})\,\underline{H}\right] \qquad (4.6.8a)$$

$$\eta^*(\underline{E}^* \times \underline{H}^*) = \left\{ \left[\eta^*\hat{n} \cdot (\underline{H}^* \times \underline{E}^*)\right] - \left[(\underline{E}^* \cdot \underline{E}^*) + \eta^{*2}\,(\underline{H}^* \cdot \underline{H}^*)\right] \right\} \hat{n} +$$

$$+ \left[(\hat{n} \cdot \underline{E}^*)\,\underline{E}^* + \eta^{*2}\,(\hat{n} \cdot \underline{H}^*)\,\underline{H}^*\right] =$$

$$= \left[\eta^*(\underline{H}^* \cdot \underline{H}^*) - (\hat{n} \cdot \underline{E}^*)^2\right] \hat{n} + \left[(\hat{n} \cdot \underline{E}^*)\,\underline{E}^* + \eta^*(\hat{n} \cdot \underline{H}^*)\,\underline{H}^*\right] =$$

$$= \left[(\underline{E}^* \cdot \underline{E}^*) - \eta^{*2}\,(\hat{n} \cdot \underline{H}^*)^2\right] \hat{n} + \left[(\hat{n} \cdot \underline{E}^*)\,\underline{E}^* + \eta^{*2}\,(\hat{n} \cdot \underline{H}^*)\,\underline{H}^*\right]$$

(4.6.8b)

$\underline{I \times I^* \text{ and } II \times II^*}$

$$\hat{n} \cdot \left[(\underline{E} \times \underline{E}^*) + \eta\eta^*(\underline{H} \times \underline{H}^*)\right] = \left[\eta^*(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{H}^*) - \eta(\hat{n} \times \underline{E}^*) \cdot (\hat{n} \times \underline{H})\right] \quad (4.6.9a)$$

since

$$\eta(\hat{n} \times \underline{E}^*) \cdot (\hat{n} \times \underline{H}) = -\hat{n} \cdot (\underline{E} \times \underline{E}^*) = -\eta\eta^*\hat{n} \cdot (\underline{H} \times \underline{H}^*) \quad (4.6.9b)$$

or

$$\eta^*(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{H}) = \hat{n} \cdot (\underline{E} \times \underline{E}^*) = \eta\eta^*\hat{n} \cdot (\underline{H} \times \underline{H}^*) \quad (4.6.9c)$$

thus

$$\hat{n} \cdot \left[(\underline{E} \times \underline{E}^*) - \eta\eta^*(\underline{H} \times \underline{H}^*)\right] = 0 \quad (4.6.9d)$$

and

$$\hat{n}\left[\eta(\hat{n} \times \underline{E}^*) \cdot (\hat{n} \times \underline{H}) + \eta^*(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{H}^*)\right] = 0 \quad (4.6.9e)$$

$\underline{I \times II^*}$

$$\hat{n}\left[(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{E}^*) - \eta\eta^*(\hat{n} \times \underline{H}) \cdot (\hat{n} \times \underline{H}^*)\right] = 0 \quad (4.6.10a)$$

$$\hat{n}\left\{(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{E}^*) + \eta\eta^*(\hat{n} \times \underline{H}) \cdot (\hat{n} \times \underline{H}^*)\right\} =$$

$$= \eta^*\hat{n} \times \left[(\underline{E} \times \underline{H}^*) \times \hat{n}\right] + \eta(\hat{n} \times \underline{H}) \times (\hat{n} \times \underline{E}^*) - \eta^*(\underline{E} \times \underline{H}^*) \quad (4.6.10b)$$

$\underline{II \times I^*}$

$$\hat{n}\left\{(\hat{n} \times \underline{E}) \cdot (\hat{n} \times \underline{E}^*) + \eta\eta^*(\hat{n} \times \underline{H}) \cdot (\hat{n} \times \underline{H}^*)\right\} =$$

$$= \eta^*(\hat{n} \times \underline{H}^*) \times (\hat{n} \times \underline{E}) - \eta(\underline{E}^* \times \underline{H}) + \eta\hat{n} \times \left[(\underline{E}^* \times \underline{H}) \times \hat{n}\right] \quad (4.6.10c)$$

$\underline{(I \times II^*) - (II \times I^*)}$

$$\left[\eta(\underline{E}^* \times \underline{H}) - \eta^*(\underline{E} \times \underline{H}^*)\right] = \eta^*(\hat{n} \times \underline{H}^*) \times (\hat{n} \times \underline{E}) - \eta(\hat{n} \times \underline{H}) \times (\hat{n} \times \underline{E}^*)$$

$$+ \eta\hat{n} \times \left[(\underline{E}^* \times \underline{H}) \times \hat{n}\right] - \eta^*\hat{n} \times \left[(\underline{E} \times \underline{H}^*) \times \hat{n}\right] =$$

$$= \hat{n} \times \left[(\underline{E} \times \underline{E}^*) - \eta\eta^*(\underline{H} \times \underline{H}^*)\right] \quad (4.6.11)$$

Equation (4.6.11) constitutes an important result namely that a set of orthogonal vectors is found which are tangent to the local scattering surface since by (4.6.5) and (4.6.6a)

$$\hat{n} \cdot \left[\eta (\underline{E}^* x \underline{H}) - \eta^* (\underline{E} x \underline{H}^*) \right] = 0$$

and

$$\hat{n} \cdot \left[(\underline{E} x \underline{E}^*) - \eta \eta^* (\underline{H} x \underline{H}^*) \right] = 0 \quad .$$

Verification of (4.6.11)

$$\left[\eta (\underline{E}^* x \underline{H}) - \eta^* (\underline{E} x \underline{H}^*) \right] = \left[\underline{E}^* x (\underline{E} x \hat{n}) - \underline{E} x (\underline{E}^* x \hat{n}) \right]$$

$$+ \eta \eta^* \left\{ (\hat{n} \cdot \underline{H}) \left[(\hat{n} x \underline{H}^*) x \hat{n} \right] - (\hat{n} \cdot \underline{H}^*) \left[(\hat{n} x \underline{H}) x \hat{n} \right] \right\} =$$

$$= \left[(\underline{E}^* \cdot \hat{n}) \underline{E} - \hat{n} (\underline{E} \cdot \underline{E}^*) - (\underline{E} \cdot \hat{n}) \underline{E}^* + \hat{n} (\underline{E} \cdot \underline{E}^*) \right]$$

$$+ \eta \eta^* \left[(\underline{H} \cdot \hat{n}) \underline{H}^* - (\hat{n} \cdot \underline{H}) (\hat{n} \cdot \underline{H}^*) \hat{n} - (\hat{n} \cdot \underline{H}^*) + (\hat{n} \cdot \underline{H}^*) (\hat{n} \cdot \underline{H}) \hat{n} \right] =$$

$$= \left[(\underline{E}^* x \underline{E}) x \hat{n} \right] + \eta \eta^* \left[(\underline{H} x \underline{H}^*) x \hat{n} \right] -$$

$$= \hat{n} x \left[(\underline{E} x \underline{E}^*) - \eta \eta^* (\underline{H} x \underline{H}^*) \right] \quad .$$

There exists another formulation of (4.6.11) given by

$$\left[(\underline{E} x \underline{E}^*) - \eta \eta^* (\underline{H} x \underline{H}^*) \right] = -\hat{n} x \left[\eta (\underline{E}^* x \underline{H}) - \hat{n}^* (\underline{E} x \underline{H}^*) \right] \qquad (4.6.12)$$

which may be verified as follows:

$$\left[(\underline{E} x \underline{E}^*) - \eta \eta^* (\underline{H} x \underline{H}^*) \right] = \left\{ \hat{\eta}^* \left[\underline{E} x (\hat{n} x \underline{H}^*) \right] - \eta \left[\underline{H} x (\underline{E}^* x \hat{n}) \right] \right\}$$

$$+ \left\{ (\hat{n} \cdot \underline{E}^*) \left[\underline{E} x \hat{n} \right] - \eta \eta^* (\hat{n} \cdot \underline{H}^*) \left[\underline{H} x \hat{n} \right] \right\} =$$

$$= \hat{n} \left\{ \underbrace{\left[\eta^* (\underline{E} \cdot \underline{H}^*) + \eta (\underline{E}^* \cdot \underline{H}) \right] - \left[\eta^* (\hat{n} \cdot \underline{E})(n \cdot \underline{H}^*) + \eta (\hat{n} \cdot \underline{E}^*) (\hat{n} \cdot \underline{H}) \right]}_{= \ 0 \text{, according to (4.6.6c)}} \right\}$$

$$+ \left\{ \left[\eta (\hat{n} \cdot \underline{E}^*) \underline{H} - \eta (\hat{n} \cdot \underline{H}) \underline{E}^* \right] + \left[\eta^* (\hat{n} \cdot \underline{H}^*) \underline{E} - \eta^* (\underline{E} \cdot \hat{n}) \underline{H}^* \right] \right\} =$$

$$= \eta \left[\hat{n} x (\underline{H} x \underline{E}^*) \right] + \eta^* \left[\hat{n} x (\underline{E} x \underline{H}^*) \right] =$$

$$= - \hat{n} x \left[\eta (\underline{E}^* x \underline{H}) - \eta^* (\underline{E} x \underline{H}^*) \right] \quad \text{q.e.d.}$$

This concludes the derivation of scalar and vector product operations which resulted into the following identities:

$$\left\{\left[(\underline{E} \times \underline{E}^*) - \eta\eta^*(\underline{H} \times \underline{H}^*)\right] + \hat{n} \times \left[\eta\,(\underline{E}^* \times \underline{H}) - \eta^*(\underline{E} \times \underline{H}^*)\right]\right\} = 0 \qquad (4.6.13a)$$

or

$$\left\{\left[(\underline{E} \times \underline{E}^*) - \eta\eta^*(\underline{H} \times \underline{H}^*)\right] \times \hat{n} + \left[\eta\,(\underline{E}^* \times \underline{H}) - \eta^*(\underline{E} \times \underline{H}^*)\right]\right\} = 0 \quad . \qquad (4.6.13b)$$

4.6.3 DERIVATION OF ADDITIONAL VECTOR IDENTITIES

It can be shown that $\underline{D}_I = \hat{n}\,(\underline{D}_I \cdot \hat{n})$ can be reformulated as:

$$\begin{aligned}
\hat{n} \cdot \underline{D}_1 &= \hat{n} \cdot (\underline{A}_I \times \underline{B}_I) = -2\left\{\left[\eta\eta^*(\hat{n} \cdot \underline{E})(\hat{n} \cdot \underline{E}^*)(\underline{H} \cdot \underline{H}^*)\right.\right. \\
&\quad \left. + \eta\eta^*(\hat{n} \cdot \underline{H})(\hat{n} \cdot \underline{H}^*)(\underline{E} \cdot \underline{E}^*)\right] - \left[\eta\eta^*(\hat{n} \cdot \underline{H})(\hat{n} \cdot \underline{E}^*)(\underline{E} \cdot \underline{H}^*)\right. \\
&\quad \left.\left. + \eta\eta^*(\hat{n} \cdot \underline{H}^*)(\underline{E} \cdot \hat{n})(\underline{E}^* \cdot \underline{H})\right]\right\} = \\
&= -2\,\eta\eta^*\left[(\underline{E} \times \underline{H}) \cdot (\underline{E}^* \times \underline{H}^*)\right] = -2\,\eta\eta^*\left[(\underline{E} \times \underline{E}^*) \cdot (\underline{H} \times \underline{H}^*)\right. \\
&\quad \left. + (\underline{E}^* \times \underline{H}) \cdot (\underline{E} \times \underline{H}^*)\right] = -2\,\eta\eta^*\left[(\underline{E} \cdot \underline{E}^*)(\underline{H} \cdot \underline{H}^*) - \eta\eta^*(\underline{E}^* \cdot \underline{H})(\underline{E} \cdot \underline{H}^*)\right]
\end{aligned}$$

(4.6.14)

V

MATRIX INVERSION FOR THE CASE OF END-ON INCIDENCE ONTO A ROTATIONALLY SYMMETRIC SCATTERER

5.1 FORMULATION OF THE MATRIX

In problems of C.W. inverse scattering it is assumed that for a given incident field the measured far scattered field can be obtained for a sufficiently large number of bistatic angles. If the direction of the incident wave is chosen along and in direction of the negative $\hat{z}$ - axis of a spherical coordinate system and the polarization vector $\hat{e}_t$ of the transmitted wave is parallel and in direction of the positive $\hat{x}$ - axis, then the scattered field may be represented by a series expansion into vector spherical wave functions:

$$\underline{E}_c^s(R_c, \theta_c, \phi_c) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[(i)^{n+1} a_{\substack{e\\o}mn} \underline{M}_{\substack{e\\o}mn} + (i)^n b_{\substack{e\\o}mn} \underline{N}_{\substack{e\\o}mn} \right] \tag{5.1.1}$$

where the vector spherical wave functions are defined by:

$$\begin{aligned} \underline{M}_{\substack{e\\o}mn}(R, \theta, \phi) = & \mp \left\{ h_n^{(1)}(kR)\, S_n^m(\theta) \begin{matrix}\sin\\ \cos\end{matrix} (m\phi) \right\} \hat{\theta} \\ & - \left\{ h_n^{(1)}(kR)\, R_n^m(\theta) \begin{matrix}\cos\\ \sin\end{matrix} (m\phi) \right\} \hat{\phi} \end{aligned} \tag{5.1.2}$$

$$\begin{aligned} \underline{N}_{\substack{e\\o}mn}(R, \theta, \phi) = & \left\{ \frac{n(n+1)}{kR}\, h_n^{(1)}(kR)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} (m\phi) \right\} \hat{R} \\ & + \left\{ \frac{1}{kR} \frac{d}{dR} \left[R\, h_n^{(1)}(kR) \right] R_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} (m\phi) \right\} \hat{\theta} \\ & \mp \left\{ \frac{1}{kR} \frac{d}{dR}\, R\, h_n^{(1)}(kR)\; S_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} (m\phi) \right\} \hat{\phi} \end{aligned} \tag{5.1.3}$$

with

$$\begin{aligned} S_n^m(\theta) = \frac{m}{\sin\theta} P_n^m(\cos\theta) = & \frac{1}{2}\cos\theta \left[(n-m+1)(n+m)\, P_n^{m-1}(\cos\theta) + \right. \\ & \left. + P_n^{m+1}(\cos\theta) \right] + m \sin\theta\, P_n^m(\cos\theta) \end{aligned} \tag{5.1.4}$$

$$R_n^m(\theta) = \frac{\partial P_n^m(\cos\theta)}{\partial\theta} = \frac{1}{2}\left[(n-m+1)(n+m)\ P_n^{m-1}(\cos\theta) - P_n^{m+1}(\cos\theta)\right] \quad .(5.1.5)$$

(5.1.5)

For a perfectly conducting sphere the expansion coefficients $a_{\substack{e\\o}mn}$; $b_{\substack{e\\o}mn}$ bear the following relationship with the expansion coefficients given by Stratton (1941), where $m = 1$ for nose-on incidence onto a rotationally symmetric body:

$$a_{oln} = i(-1)^{n+1}\frac{(2n+1)}{(n+1)n}\ a_n \tag{5.1.6a}$$

$$b_{eln} = i(-1)^{n}\frac{(2n+1)}{(n+1)\,n}\ b_n \quad . \tag{5.1.6b}$$

Employing the asymptotic approximation of the Hankel functions

$$\lim_{x\to\infty}\left\{h_n^{(1)}(x)\right\} = (-i)^{n+1}\ \frac{\exp ix}{x} \tag{5.1.7a}$$

$$\lim_{x\to\infty}\left\{\frac{\frac{d}{dx}\left[x\ h_n^{(1)}(x)\right]}{x}\right\} = (-i)^{n}\ \frac{\exp ix}{x} \tag{5.1.7b}$$

and extracting the factor $\left(\frac{\exp ix}{x}\right)$ from the measured field components, the far scattered field is given by

$$E^s_{\theta_c}(\theta_c, \phi_c) = \sum_{n=1}^{N} \cos\phi_c\ \left[a_{oln}\ S_n^1(\cos\theta_c) + b_{eln}\ R_n^1(\cos\theta_c)\right] \tag{5.1.8a}$$

$$E^s_{\phi_c}(\theta_c, \phi_c) = -\sum_{n=1}^{N} \sin\phi_c\ \left[a_{oln}\ R_n^1(\cos\theta_c) + b_{eln}\ S_n^1(\cos\theta_c)\right] \tag{5.1.8b}$$

where N denotes the total number of receiver aspect angles, and for the c^{th} receiver the field components are given by

$$E^s_{\theta_c}(\theta_c, \phi_c), \; E^s_{\phi_c}(\theta_c, \phi_c) \quad .$$

For the formulation of a suitable near field representation, the unknown expansion coefficients a_{oln}, b_{eln} must be computed, To do so, the field components $E^s_{\theta_c}$, $E^s_{\phi_c}$, and the inherent terms $S^1_n(\theta)$, $R^1_n(\theta)$, $\cos \phi_c$ must be arranged in a matrix formulation so that the most suitable matrix representation results which satisfies stability criterions, encountered in the inversion of such a matrix.

Inspecting Eqs. (5.1.8a,b) which represent the basis of further analysis, these equations may be rewritten into the formulation

$$E = A X \tag{5.1.9}$$

where the transpose E^T of E is given by:

$$E^T = \Big[E_{\theta_1}(\theta_1, \phi_1), \; E_{\theta_2}(\theta_2, \phi_2), \; E_{\theta_3}(\theta_3, \phi_3), \; \ldots$$

$$\ldots \; E_{\theta_N}(\theta_N, \phi_N); \; E_{\phi_1}(\theta_1, \phi_1) \; E_{\phi_2}(\theta_2, \phi_2), \; \ldots$$

$$\ldots \; E_{\phi_N}(\theta_N, \phi_N) \Big] \tag{5.1.10a}$$

which consists of 2N complex elements, and so does the transpose X^T of X which represents the unknown coefficients:

$$X^T = (a_{o11}, b_{e11}, a_{o12}, b_{e12}, \ldots a_{oln}, b_{eln}) \quad . \tag{5.1.10b}$$

With this arrangement the matrix elements of A are defined as well, where the r^{th} row for $1 \leq r \leq N$ is given by

$$\Big[S^1_1(\cos \theta_r) \cos \phi_r, \; R^1_1(\cos \theta_r) \cos \phi_r, \; S^1_2(\cos \theta_r) \cos \phi_r, \; R^1_2(\cos \theta_r) \cos \phi_r,$$

$$\ldots \; S^1_N(\cos \theta_r) \cos \phi_r, \; R^1_N(\cos \theta_r) \cos \phi_r \Big] \tag{5.1.11a}$$

and the $(N + r)^{th}$ row for $1 \leq r \leq N$ is given by

$$\Big[- R_1^1(\cos \theta_r) \sin \phi_{r_1} - S_1^1(\cos \theta_r) \sin \phi_r, \; - R_2^1(\cos \theta_r) \sin \phi_r ,$$

$$- S_2^1 (\cos \theta_r) \sin \phi_r, \; \ldots , \; - R_N^1(\cos \theta_r) \sin \phi_r , \; - S_N^1(\cos \theta_r) \sin \phi_r \Big] \quad . \tag{5.1.11b}$$

The resulting matrix A can however be decomposed into the product of two square matrices:

$$A (\theta_c, \phi_c, n) = B(\phi_c) \, C(\theta_c, n) \quad . \tag{5.1.12}$$

The properties of this matrix must be investigated in detail to gain more information about stability criterions.

5.2 PROPERTIES OF MATRIX $A (\theta_c, \phi_c, n) = B(\phi_c) \, C (\theta_c, n)$

Matrix A is a square matrix with 2N x 2N purely real elements for real angles ϕ_c, and θ_c. Matrix $B(\phi_c)$ is diagonal, solely incorporating the ϕ - dependence and its elements are given by:

$$b_{rr} = \cos \phi_c \qquad \text{for } 1 \leq r \leq N \tag{5.2.1a}$$

$$b_{N+r, N+r} = - \sin \phi_c \quad \text{for } N < N+r \leq 2N \quad . \tag{5.2.1b}$$

The determinant of B is always smaller than unity, where

$$\text{Det} \left\{B\right\} = \frac{(-1)^N}{2^N} \prod_{c = 1}^{N} \sin 2 \phi_c \quad . \tag{5.2.2}$$

This matrix will become singular for

$$\phi_c = 0 + m \;\; \pi/2 \quad , \quad m = 0, 1, 2 \; . \; . \; . \tag{5.2.3a}$$

and is maximum for

$$\phi_c = \pi/4 + m \; \pi/2 \tag{5.2.3b}$$

and it can be seen that otherwise no restrictions are imposed, e.g. the choice of $\phi_c = \phi_1 = \phi_2 = \ldots = \phi_N$ is permissible. However, it may be advisable to extract the ϕ - dependence entirely from the field components before inverting the matrix, since the ϕ - dependence solely reduces the magnitude of determinant Det $\{A\}$.

Matrix $C(\theta_c)$, solely incorporating the θ - dependence in terms of Legendre functions, needs further thorough investigation. Inspecting Eqs. (5.1.8a, b) it is recognized that always two of the 2N x 2N elements are identical, and for even N matrix C can be decomposed into the sum of two matrices each containing non-identical coefficients, where for N = 2 we find:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} = C_1 + C_2 = \begin{bmatrix} c_{11} & 0 & c_{13} & 0 \\ 0 & c_{22} & 0 & c_{24} \\ c_{31} & 0 & c_{33} & 0 \\ 0 & c_{42} & 0 & c_{44} \end{bmatrix} + \begin{bmatrix} 0 & c_{12} & 0 & c_{14} \\ c_{21} & 0 & c_{23} & 0 \\ 0 & c_{32} & 0 & c_{34} \\ c_{41} & 0 & c_{43} & 0 \end{bmatrix} \tag{5.2.4}$$

where

$$\begin{aligned} c_{11} &= c_{32} = S_1^1(\theta_r) = 1 & c_{13} &= c_{34} = S_2^1(\theta_1) = 3\cos\theta_1 \\ c_{21} &= c_{42} = S_1^1(\theta_2) = 1 & c_{23} &= c_{44} = S_2^1(\theta_2) = 3\cos\theta_2 \\ c_{12} &= c_{31} = R_1^1(\theta_1) = \cos\theta_1 & c_{14} &= c_{33} = R_2^1(\theta_1) = 3\cos 2\theta_1 \\ c_{22} &= c_{41} = R_1^1(\theta_2) = \cos\theta_2 & c_{24} &= c_{43} = R_2^1(\theta_2) = 3\cos 2\theta_2 \end{aligned} \tag{5.2.5}$$

Furthermore, it can be shown that

$$\mathrm{Det}\left\{C_1\right\} = \mathrm{Det}\left\{C_2\right\} \tag{5.2.6a}$$

since

$$C_2 = T\, C_1\, R \tag{5.2.6b}$$

where

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \tag{5.2.6c}$$

This property may be employed to derive the determinant of C more easily, where

$$C = C_1 + C_2 = C_1\,(I + C_1^{-1}\, C_2) = (C_1\, C_2^{-1} + I)\; C_2 \tag{5.2.7a}$$

where

$$\mathrm{Det}\left\{C\right\} = \mathrm{Det}\left\{C_1\right\} \mathrm{Det}\left\{I + C_1^{-1}\, C_2\right\} . \tag{5.2.7b}$$

For the chosen case $N = 2$, the determinants result into:

$$\mathrm{Det}\left\{C_1\right\} = -\,(c_{12}\, c_{13} - c_{11}\, c_{14})\,(c_{22}\, c_{23} - c_{21}\, c_{24}) =$$

$$= -\,(3 \sin^2\theta_1)\,(3 \sin^2\theta_2) \tag{5.2.8a}$$

$$\mathrm{Det}\left\{I + C_1^{-1}\, C_2\right\} = \left[2\,(\cos\theta_1 - \cos\theta_2)\right]^2 \tag{5.2. 8b}$$

thus

$$\mathrm{Det}\left\{C\right\}_{N=2} = -\left[3 \sin^2\theta_1\right]\left[3 \sin^2\theta_2\right]\left[2\,(\cos\theta_1 - \cos\theta_2)\right]^2 . \tag{5.2.8c}$$

Employing this scheme, the determinants for the cases N = 4 and N = 6 were evaluated. There it was found convenient to formulate the determinants into a continued fraction expansion for increasing N, where

$$\text{Det}\left\{C\right\}_N = \qquad\qquad N$$

$$= C_1 \sin^2 \theta_1 \qquad\qquad 1$$

$$\frac{1}{C_2 \sin^2 \theta_2 \left[\cos\theta_2 - \cos\theta_1\right]^2} \qquad\qquad 2$$

$$\frac{1}{C_3 \sin^2 \theta_3 \left[\cos\theta_3 - \cos\theta_2\right]^2 \left[\cos\theta_3 - \cos\theta_1\right]^2} \qquad\qquad 3$$

$$\frac{1}{C_4 \sin^2 \theta_4 \left[\cos\theta_4 - \cos\theta_3\right]^2 \left[\cos\theta_4 - \cos\theta_2\right]^2 \left[\cos\theta_4 - \cos\theta_1\right]^2} \qquad\qquad 4$$

$$\frac{1}{C_5 \sin^2 \theta_5 \left[\cos\theta_5 - \cos\theta_4\right]^2 \left[\cos\theta_5 - \cos\theta_3\right]^2 \left[\cos\theta_5 - \cos\theta_2\right]^2 \left[\cos\theta_5 - \cos\theta_1\right]^2} \qquad\qquad 5$$

$$\frac{1}{C_6 \sin^2 \theta_6 \left[\cos\theta_6 - \cos\theta_5\right]^2 \cdots\cdots\cdots \left[\cos\theta_6 - \cos\theta_1\right]^2} \qquad\qquad 6$$

etc.

$$\vdots$$

$$\frac{1}{C_N \sin^2 \theta_N \left[\cos\theta_N - \cos\theta_{N-1}\right]^2 \cdots\cdots\cdots \left[\cos\theta_N - \cos\theta_1\right]^2} \qquad\qquad N$$

(5.2.9a)

The constant multipliers were found to be:

$$C_1 = 1 = (-1)^{1+1} \frac{1^2}{1^2} \cdot 1^2$$

$$C_2 = -2^2 \cdot 3^2 = (-1)^{2+1} \frac{1^2 \cdot 3^2}{1^2} \; 2^2$$

$$C_3 = \frac{3^4 \cdot 5^2}{2^2} = (-1)^{3+1} \frac{1^2 \cdot 3^2 \cdot 5^2}{1^2 \cdot 2^2} \cdot 3^2$$

$$C_4 = -2^2 \cdot 5^2 \cdot 7^2 = (-1)^{4+1} \cdot \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{1^2 \cdot 2^2 \cdot 3^2} \cdot 4^2$$

$$C_5 = \frac{3^4 \cdot 5^4 \cdot 7^2}{2^6} = (-1)^{5+1} \cdot \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{1^2 \cdot 2^2 \cdot 3^7 \cdot 4^2} \cdot 5^2$$

$$C_6 = \frac{-3^6 \cdot 7^2 \cdot 11^2}{2^4} = (-1)^{6+1} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2} \cdot 6^2$$

$$\vdots$$

$$C_c = (-1)^{c+1} \frac{\left[(2c-1)!!\right]^2}{\left[(c-1)!\right]^2} \; c^2 \tag{5.2.9b}$$

$$(2c-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (2c-1)$$

where computational results are given in Table V-1. Inspecting (5.2.9) it can be seen that $\text{Det}\left\{C(\theta_c, 2N \times 2N)\right\}$ can be decomposed into the product of two $N \times N$ determinants:

$$C(\theta_c, 2N \times 2N) = D(\sin^2\theta_c, N \times N) \cdot E^2(\cos\theta_c, N \times N) \tag{5.2.10a}$$

D is a diagonal matrix with diagonal element of the form

$$d_{cc} = (-1)^{c+1} \frac{\left[(2c-1)!!\right]^2}{\left[(c-1)!\right]^2} \; c^2 \sin^2\theta_c \tag{5.2.10b}$$

and E is of the Vandermonde type, where Vandermonde's determinant is here given by:

$$V=E=\begin{bmatrix} 1\cos\theta_1\cos^2\theta_1\cdots\cos^{N-1}\theta_1 \\ 1\cos\theta_2\cos^2\theta_2 \quad \cos^{N-1}\theta_2 \\ 1\cos\theta_3\cos^2\theta_3 \quad \cos^{N-1}\theta_3 \\ \vdots \\ 1\cos\theta_N\cdots\cdots\cos^{N-1}\theta_N \end{bmatrix} = \left\{ \begin{array}{l} (\cos\theta_2-\cos\theta_1)(\cos\theta_3-\cos\theta_1)\cdots(\cos\theta_N-\cos\theta_1) \\ (\cos\theta_3-\cos\theta_2)(\cos\theta_4-\cos\theta_2)\cdots(\cos\theta_N-\cos\theta_2) \\ \cdot \\ \cdot \\ \cdot \\ (\cos\theta_N - \cos\theta_{N-1}) \end{array} \right\} =$$

$$= \prod_{N\geq r>s\geq 1}^{N} (\cos\theta_r - \cos\theta_s) = \prod_{N\geq r>s\geq 1}^{N} 2\sin\left(\frac{\theta_s+\theta_r}{2}\right)\sin\left(\frac{\theta_s-\theta_r}{2}\right) . \quad (5.2.10c)$$

Thus instead of the continuous fraction expansion a closed-form solution for $\mathrm{Det}\left\{C(\theta_c, N)\right\}$ can be given, where

$$\mathrm{Det}\left\{C(\theta_c, N)\right\} = \prod_{c=1}^{N} (-1)^{c+1}\left[\frac{c(2c-1)!!}{(c-1)!} \sin\theta_c\right]^2 \prod_{N\geq r>s\geq 1}^{N} (\cos\theta_r-\cos\theta_s)^2 =$$

$$= \prod_{c=1}^{N}(-1)^{c+1}\left[\epsilon\sin\theta_c\right]^2 \prod_{N\geq r>s\geq 1}^{N}\left[\frac{2(2s+1)}{s}\sin\left(\frac{\theta_s+\theta_r}{2}\right)\sin\left(\frac{\theta_s-\theta_r}{2}\right)\right]^2 . \quad (5.2.11)$$

Inspecting (5.2.11) it is found that for a set of N aspect angles $R = \sum_{c=1}^{N} c$ roots will be encountered, where N of these roots are the $\sin^2\theta_c$ terms and the remainder is the product of the squares of the differences of the argument $\cos\theta_c$ of the Legendre functions, for all mutations of the N polar angles θ_c. Thus the following restrictions must be imposed onto the θ_c-dependence:

i) $\theta_c \neq 0$ or π

ii) $\theta_c \neq \theta_{c-1}$, $c = 2, 3, \ldots . N$

iii) The distribution of the polar angles θ_c must be chosen so that the determinant neither becomes too much nor too large but of the order of unity where for each different number N an individual optimization procedure must be employed.

5.3 SUMMARY

The obtained results may be summarized in the following theorem:

Theorem: V-1

If an expansion of the scattered field into spherical vector wave functions is employed for the first dipole case (m = 1), i.e., end-on incidence onto a rotationally symmetric and perfectly conducting scatterer, then the determinant

$$\mathrm{Det}\left\{A(\theta_c, \phi_c; N)\right\} = \mathrm{Det}\left\{B(\phi_c)\right\} \cdot \mathrm{Det}\left\{C(\theta_c, N)\right\} =$$

$$= (-1)^N \prod_{c=1}^{N} (-1)^{c+1} \left[\frac{\sin 2\phi_c}{2}\right] \left[\frac{c \cdot (2c-1)!!}{(c-1)!} \sin\theta_c\right]^2 \prod_{N \geq r > s \geq 1}^{N} (\cos\theta_r - \cos\theta_s)^2$$

of the associated scattering matrix $A(\theta_c, \phi_c; N)$ will become singular for

i) $\phi_c = p\,\frac{\pi}{2}$, $p = 0 \quad \pm 1, \pm 2, \pm 3$

ii) $\theta_c = q\pi$, $q = 0, 1$

iii) $\theta_c = \theta_{c+1}$, $c = 1, 2, \ldots (N-1)$.

In addition, pseudo-singular behavior may be encountered even though (i), (ii) and (iii) are not satisfied if the computational aspect angles exclusively lie within narrow cones about the z-axis of the computational coordinate system, thus reducing the value of the determinant to almost zero.

These properties are illustrated in Figs. 5-1 and 5-2 for transmitter-receiver configurations which may occur in practice most frequently. For both cases the receiver aspect angles are assumed to be well distributed within a narrow cone whose invariant axis a is oriented perpendicular to back scatter direction. For simplicity it is assumed that $\phi_c = \pi/4$ or $\phi_c = \pi/4 + \pi$, and main attention will be attributed to the dependence of the polar angle θ_c.

The determinant associated with the configuration of Fig. 5-1 will become pseudo-singular, whereas the second configuration constitutes the optimum choice as regards the orientation of the conical section relative to the computational z-axis. This properly is demonstrated by the added $\sin^2\theta$ and $\cos\theta$ - curves in which the respective $\Omega_c = \sin^2\theta_c$ and $\Delta_{\mu r} = (\cos\theta_\mu - \cos\theta_r)$ are plotted.

In general, the optimum distribution of the aspect angles (θ_c, ϕ_c) depends upon the given number N of receiver locations, and must be determined for each individual N separately. Yet from the general structure of determinant $C(\theta_c, N)$ it may be concluded that such an optimum distribution will be obtained if the aspect angles are placed in the now identical maxima of the Legendre functions $P_N^1(\cos\theta)$ for respective N.

5.4 COMPUTATIONAL RESULTS

With the evaluation of the first four determinants as described in section 3.2, the properties of matrix $C(\theta_c, N)$ were established, where

$$\text{Det}\left\{C(\theta_c, N)\right\} = Q_N \cdot P_N(\theta_c) \tag{5.4.1a}$$

$$Q_N = \prod_{c=1}^{N} C_c \tag{5.4.1b}$$

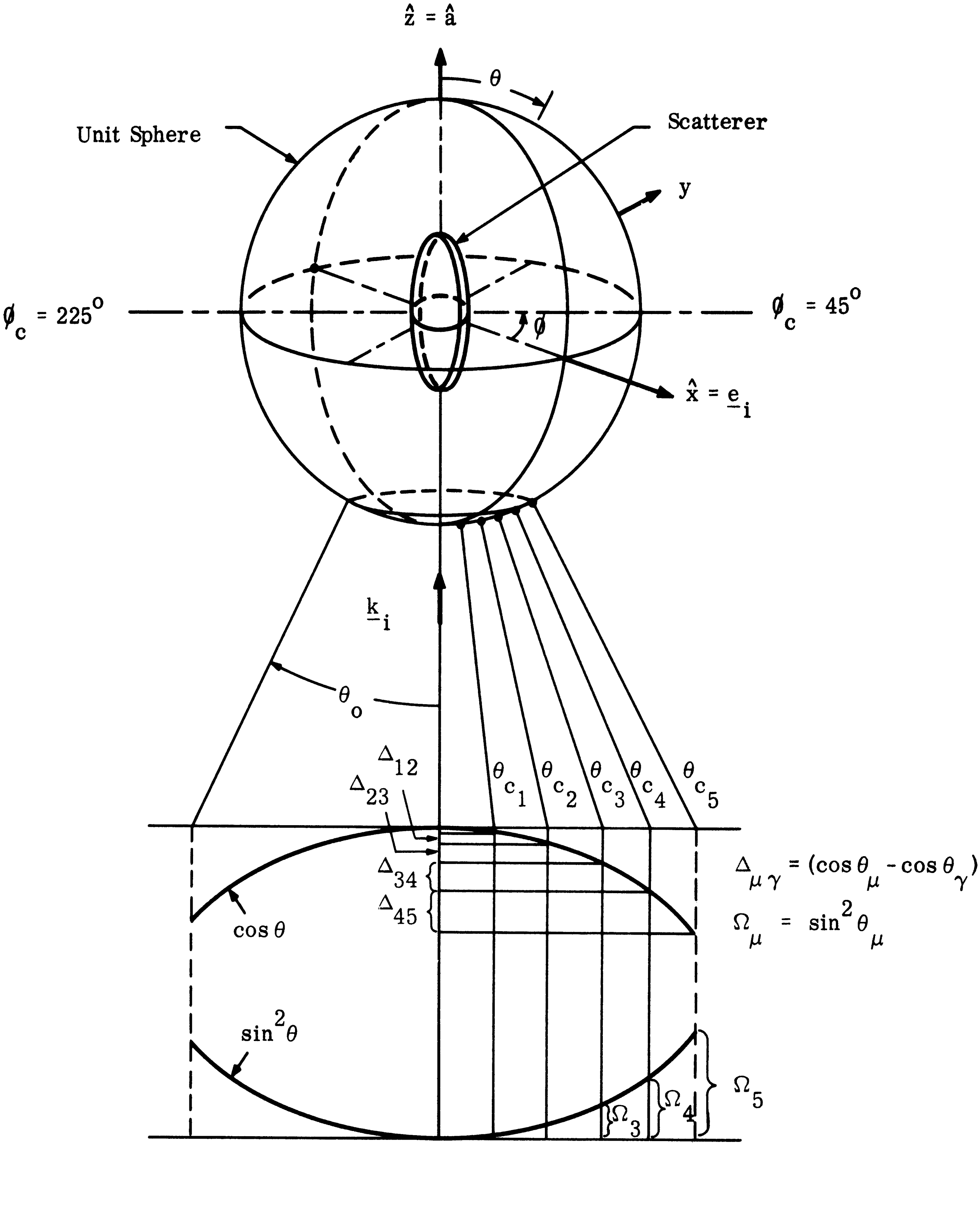

FIG.5-1:"PSEUDO-SINGULAR" DISTRIBUTION OF ASPECT ANGLES

(Invariant axis $\hat{a}$ of conical section (θ_o) identical with the computational z - axis)

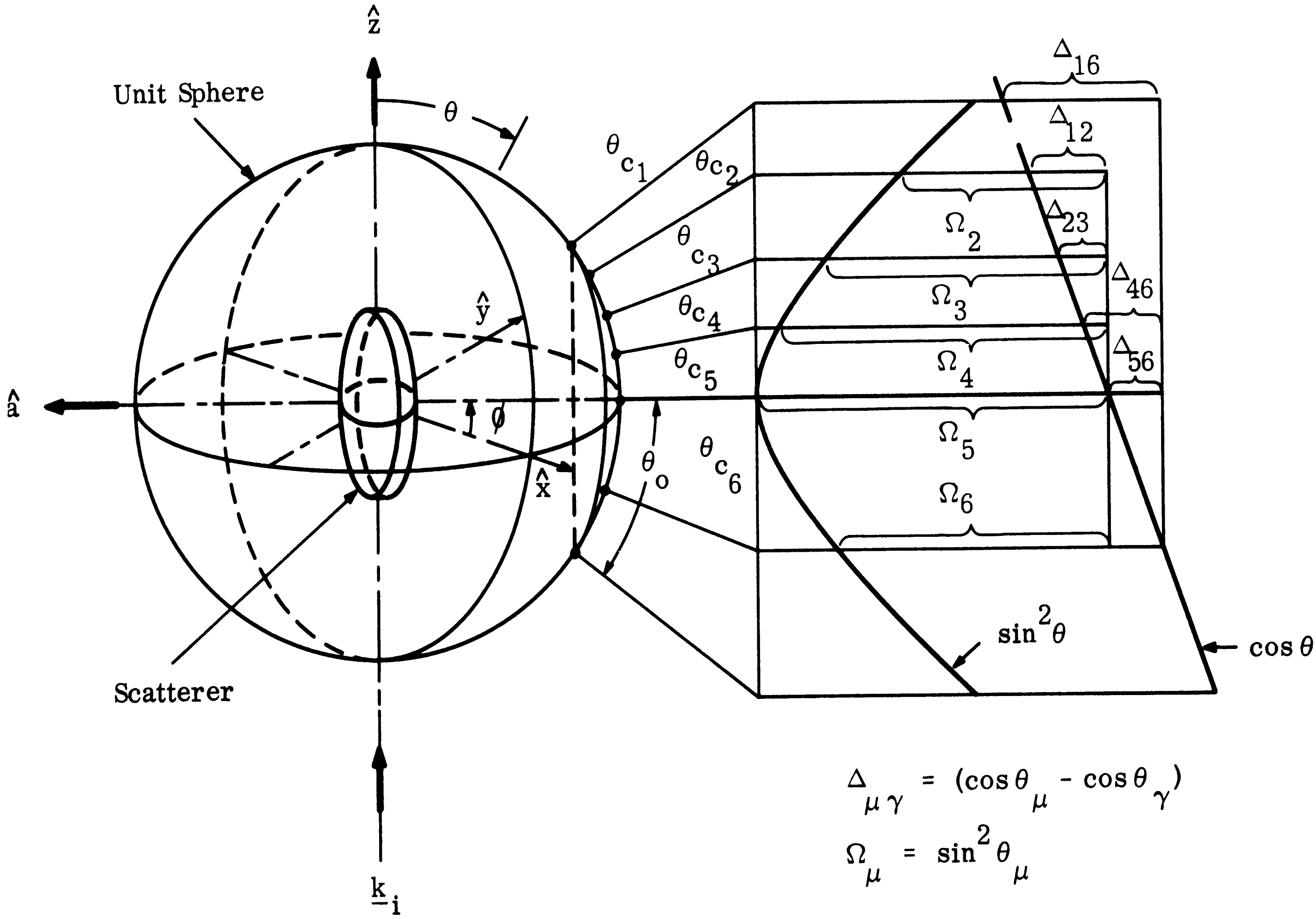

FIG.5-2: STABLE DISTRIBUTION OF ASPECT ANGLES.

(Invariant axis a of conical section (θ_o) oriented perpendicularlly to the backscatter direction).

$$P_N(\theta_c) = \prod_{c=1}^{N} \sin^2\theta_c \prod_{N \geq r > s \geq 1} (\cos\theta_r - \cos\theta_s)^2 \qquad (5.4.1c)$$

where C_c denotes the c^{th} constant multiplier of Eq. (5.2.9a). The derivation of a general algorithm for the constant multipliers C_c required additional computation of the next higher order determinants Det $C(\theta_c, N)$ together with the corresponding $P_N(\theta_c)$. In table V-1 numerical results are presented, where Det $C(\theta_c, N)$ was computed directly from the matrix formulation as derived from (5.1.8a) and (5.1.8b). Apart from negligible round-off errors the computational results verify the assumed expression of C_c to its best, where

$$C_c = \frac{[(2c-1)!!]^2}{[(c-1)!]^2} c^2 \qquad (5.4.2a)$$

$$Q_N = \prod_{c=1}^{N} C_c = \prod_{c=1}^{N} \frac{c^2 [(2c-1)!!]^2}{[(c-1)!]^2} . \qquad (5.4.2b)$$

For the spherical test case of electric measure $ka = 2$ (8579-3-Q) the matrix inversion was tested for a varying number of N of aspect angles and for different choices of the sets of aspect angles (θ_c, ϕ_c, $C = 1, 2, \ldots, N$). In the following tables V-2 and V-3, test results for $N = 4$ and $N = 6$ are presented to verify properties of stability as they can be predicted from the determinant given by (5.2.9). To do so the far field components for the chosen sets of aspect angles are computed from known expansion coefficients as given by Table III- of 8579-3-Q with (5.1.6), where the coefficient $\left(\frac{\exp ikR}{kR}\right)$ is extracted.

Employing the matrix inversion technique, as described above, the expansion coefficients are computed. For a computational check the far-field components were recomputed with these coefficients. It was observed that the minimum deviation occurs for those chosen sets of aspect angles (θ_c, $\phi_c = 45^\circ$, $C = 1, 2, \ldots, N$) for which the magnitude of the determinant is of the order of unity.

TABLE V-1

COMPUTATIONAL DERIVATION OF THE CONSTANT MULTIPLIERS C_c

C/N	1	2	3	4	5	6	7	8	9
θ_c	90^o	75^o	105^o	60^o	120^o	45^o	135^o	30^o	150^o
$\sin\theta_c$	1.000000	.965926	.866025	.866025	.866025	.707107	.707107	.5000000	.500000
$\cos\theta_c$	.000000	.258819	-.258819	.50000	-.500000	.707107	-.707107	.866025	-.866025
$\log\left\{\mathrm{Det}\left[C(\theta_c, N)\right]\right\}$	.000000	.352170	1.28048	2.76864	5.15498	8.05340	12.05830	16.20901	21.59388
$\log\left\{P_N(\theta_c)\right\}$	.000000	.79586-2	.01976-3	.81779-6	.61572-8	.08254-10	.86038-13	.98609-16	.59889-18
$\log\left\{Q_N\right\}$ Computed	.000000	1.55631	4.26072	7.95090	12.53926	17.97086	24.20789	31.22292	38.99499
$\log\left\{C_c\right\}$ Computed	.000000	1.55631	2.70441	3.69018	4.58836	5.43160	6.23704	7.01503	7.77207
$\left\{C_c\right\}$ Computed	1	$2^2\cdot 3^2$	$2^{-2}\cdot 3^4\cdot 5^2$	$2^2\cdot 5^2\cdot 7^2$	$\frac{3^4\cdot 5^4\cdot 7^2}{2^6}$	$\frac{3^6\cdot 7^2\cdot 11^2}{2^4}$	$\frac{3^2\cdot 7^4\cdot 11^2\cdot 13^2}{2^8}$	$\frac{3^4\cdot 5^2\cdot 11^2\cdot 13^2}{2^2}$	$\frac{3^8\cdot 5^2\cdot 11^2\cdot 13^2\cdot 17^2}{2\cdot 4}$
$\left\{\frac{[(2c-1)!!]^2 c^2}{[(c-1)!]^2}\right\} = C_c$	$\frac{1^2}{1^2}\cdot 1^2 = 1$	$\frac{1^2\cdot 3^2}{1^2}\cdot 2^2 = 2^2\cdot 3^2$	$(\frac{1\cdot 3\cdot 5}{1\cdot 2}\cdot 3)^2 = \frac{3^4\cdot 5^2}{2^2}$	$(\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\cdot 4)^2$	$(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot 5)^2$	$(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 6)^2$	$(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot}\cdot 7)^2$	$(\frac{9\cdot 11\cdot 13\cdot 15}{2\cdot 4\cdot 6}\cdot 8)^2$	$(\frac{9\cdot 11\cdot 13\cdot 15\cdot 17}{2\cdot 4\cdot 6\cdot 8}\cdot 9)^2$
$\log\left\{\frac{[(2c-1)!!]^2 c^2}{[(c-1)!]^2}\right\}$	.000000	1.55630	2.70439	3.69020	4.58839	5.43159	6.23707	7.015037	7.772061

TABLE V-2-1a: COMPUTATIONAL RESULTS FOR N = 4.

T	R	THETA	PHI
1	1	0.7854000	0.7854000
1	2	1.047199	0.7854000
1	3	1.308999	0.7854000
1	4	1.570800	0.7854000

NR = Using parametric number of aspect angle

DF = 0 NR = 4 NT = 1 Computation of Matrix Elements

T = 1 R = 1 THETA(T,R) = 0.7854000 PHI(T,R) = 0.7854000

1	0.7071055	0.7071081	0.0
2	0.2499967	1.499999	1.500005
3	-0.1767801	1.590981	5.303308
4	-0.4062512	0.6249819	9.374977

T = 1 R = 2 THETA(T,R) = 1.047199 PHI(T,R) = 0.7854000

1	0.4999985	0.8660263	0.0
2	-0.1250023	1.299035	2.250004
3	-0.4375003	0.3247495	5.624991
4	-0.2890596	-1.353173	4.218595

T = 1 R = 3 THETA(T,R) = 1.308999 PHI(T,R) = 0.7854000

1	0.2588170	0.9659265	0.0
2	-0.3995207	0.7499944	2.799041
3	-0.3448823	-0.9636111	3.622195
4	0.1434330	-1.581923	-3.716404

T = 1 R = 4 THETA(T,R) = 1.570800 PHI(T,R) = 0.7854000

1	-0.3500762E-05	1.000000	0.0
2	-0.5000000	-0.1050229E-04	3.000000
3	0.5251140E-05	-1.500000	-0.5251140E-04
4	0.3750000	0.2625570E-04	-7.500000

A MATRIX (A is defined by equation (5.1.12))

1	0.7071053	0.4999979	1.499992	-0.8766528D-05
	1.590974	-2.625011	0.6249781	-6.187173
2	0.7071055	0.3535517	1.060654	-1.060664
	0.2651563	-3.844884	-1.104856	-3.535487
3	0.7071055	0.1830109	0.5490327	-1.837117
	-0.7054105	-2.743846	-1.158044	2.328167
4	0.7071055	-0.2475408D-05	-0.7426225D-05	-2.121316
	-1.060658	0.4084420D-04	0.1856554D-04	5.303291
5	-0.4999998	-0.7071078	0.8766560D-05	-1.499998
	2.625021	-1.590980	6.187195	-0.6249803
6	-0.3535529	-0.7071080	1.060668	-1.060658
	3.844897	-0.2651573	3.535501	1.104860
7	-0.1830115	-0.7071081	1.837124	-0.5490347
	2.743855	0.7054130	-2.328176	1.158048
8	0.2475417D-05	-0.7071081	2.121324	0.7426252D-05
	-0.4084434D-04	1.060661	-5.303310	-0.1856561D-04

A1 MATRIX = $B(\phi_c) \cdot C_1(\theta_c)$ as defined by (5.2.1) and (5.2.4)

1	0.7071053	0.0	1.499992	0.0
	1.590974	0.0	0.6249781	0.0
2	0.0	0.3535517	0.0	-1.060664
	0.0	-3.844884	0.0	-3.535487
3	0.7071055	0.0	0.5490327	0.0
	-0.7054105	0.0	-1.158044	0.0
4	0.0	-0.2475408D-05	0.0	-2.121316
	0.0	0.4084420D-04	0.0	5.303291

TABLE V-2-1b: COMPUTATIONAL RESULTS FOR N=4.

5	-0.4999998	0.0	0.8766560D-05	0.0
	2.625021	0.0	6.187195	0.0
6	0.0	-0.7071080	0.0	-1.060658
	0.0	-0.2651573	0.0	1.104860
7	-0.1830115	0.0	1.837124	0.0
	2.743855	0.0	-2.328176	0.0
8	0.0	-0.7071081	0.0	0.7426252D-05
	0.0	1.060661	0.0	-0.1856561D-04

A2 MATRIX $= B(\phi_c)\cdot C_2(\theta_c)$ (see 5.2.6)

1	0.0	0.4999979	0.0	-0.8766528D-05
	0.0	-2.625011	0.0	-6.187173
2	0.7071055	0.0	1.060654	0.0
	0.2651563	0.0	-1.104856	0.0
3	0.0	0.1830109	0.0	-1.837117
	0.0	-2.743846	0.0	2.328167
4	0.7071055	0.0	-0.7426225D-05	0.0
	-1.060658	0.0	0.1856554D-04	0.0
5	0.0	-0.7071078	0.0	-1.499998
	0.0	-1.590980	0.0	-0.6249803
6	-0.3535529	0.0	1.060668	0.0
	3.844897	0.0	3.535501	0.0
7	0.0	-0.7071081	0.0	-0.5490347
	0.0	0.7054130	0.0	1.158048
8	0.2475417D-05	0.0	2.121324	0.0
	-0.4084434D-04	0.0	-5.303310	0.0

DETA = 8.199224 = Det$\{B\}\cdot$ Det$\{C\}$ (defined by 5.1.12)

A INVERSE

1	-24.18691	43.94923	-32.83723	10.47621
	-24.95881	44.14295	-32.59357	9.782425
2	24.95891	-44.14315	32.59373	-9.782481
	24.18681	-43.94904	32.83707	-10.47616
3	8.626573	-15.34219	11.52027	-3.555206
	7.916068	-13.74816	9.759778	-2.517213
4	-7.916098	13.74822	-9.759824	2.517228
	-8.626534	15.34212	-11.52021	3.555185
5	-2.468427	4.499455	-3.373234	1.006895
	-2.463475	4.137175	-2.557613	0.5151431
6	2.463485	-4.137193	2.557626	-0.5151468
	2.468416	-4.499433	3.373217	-1.006889
7	0.6154678	-1.078557	0.7737976	-0.2207796
	0.4351979	-0.5392716	0.2002681	0.1675255D-05
8	-0.4351990	0.5392729	-0.2002686	-0.1600814D-05
	-0.6154646	1.078551	-0.7737928	0.2207780

DETA1 = 8.557744 = Det$\{B\}\cdot$ Det$\{C_1\}$

A1 INVERSE

1	8.147164	0.0	-10.54787	0.0
	-4.114108	0.0	-3.499777	0.0
2	0.0	0.7542295	0.0	1.131351
	0.0	-3.016924	0.0	1.979821
3	-7.372744	0.0	11.07329	0.0
	4.042033	0.0	3.254776	0.0
4	0.0	-1.077481	0.0	-1.279510

TABLE V-2-1c: COMPUTATIONAL RESULTS FOR N = 4.

	0.0	2.693675	0.0	-3.232408
5	4.440036	0.0	-6.500642	0.0
	-2.298307	0.0	-1.682486	0.0
6	0.0	0.5028201	0.0	0.7542380
	0.0	-2.011284	0.0	2.262690
7	-1.225360	0.0	1.905596	0.0
	0.8042444	0.0	0.4309940	0.0
8	0.0	-0.4309958	0.0	-0.3232471
	0.0	1.077484	0.0	-1.292979

DETA2 = 8.557728 = Det B · Det C_2

A2 INVERSE

1	0.0	3.016933	0.0	-1.979826
	0.0	-0.7542263	0.0	-1.131347
2	4.114134	0.0	3.499800	0.0
	-8.147157	0.0	10.54786	0.0
3	0.0	-2.693683	0.0	3.232417
	0.0	1.077476	0.0	1.279505
4	-4.042058	0.0	-3.254797	0.0
	7.372738	0.0	-11.07328	0.0
5	0.0	2.011290	0.0	-2.262696
	0.0	-0.5028179	0.0	-0.7542349
6	2.298321	0.0	1.682497	0.0
	-4.440031	0.0	6.500636	0.0
7	0.0	-1.077487	0.0	1.292983
	0.0	0.4309940	0.0	0.3232456
8	-0.8042488	0.0	-0.4309969	0.0
	1.225359	0.0	-1.905594	0.0

DET = 0.9581058 = Det $\left\{I + C_1^{-1} C_2\right\}$

A INVERSE

1	-24.18691	43.94923	-32.83723	10.47621
	-24.95881	44.14295	-32.59357	9.782425
2	24.95891	-44.14315	32.59373	-9.782481
	24.18681	-43.94904	32.83707	-10.47616
3	8.626573	-15.34219	11.52027	-3.555206
	7.916068	-13.74816	9.759778	-2.517213
4	-7.916098	13.74822	-9.759824	2.517228
	-8.626534	15.34212	-11.52021	3.555185
5	-2.468427	4.499455	-3.373234	1.006895
	-2.463475	4.137175	-2.557613	0.5151431
6	2.463485	-4.137193	2.557626	-0.5151468
	2.468416	-4.499433	3.373217	-1.006889
7	0.6154678	-1.078557	0.7737976	-0.2207796
	0.4351979	-0.5392716	0.2002681	0.1675255D-05
8	-0.4351990	0.5392729	-0.2002686	-0.1600814D-05
	-0.6154646	1.078551	-0.7737928	0.2207780

R	T	ER (Real Part)	EI (Imaginary Part)	
1	1	0.4782311	-0.3603460	E_{θ_μ}
2	1	0.7822469	-0.2040950	
3	1	1.075998	-0.1207969	
4	1	1.272995	-0.1538157	
5	1	-0.2931354	0.5000026	E_{ϕ_μ}
6	1	-0.4874914	0.3285223	

TABLE V-2-1d: COMPUTATIONAL RESULTS FOR N = 4 .

7	1	-0.6937162	0.1035299	E_{ϕ_μ}
8	1	-0.8679340	-0.1695977	

R	T	XR (Real Part)	XI (Imaginary Part)	
1	1	0.7327698	-0.9099416	a_{oln}
2	1	0.6708078	0.4146107	
3	1	-0.2099670	0.5676624E-01	
4	1	-0.3726765	-0.2303392	
5	1	0.2382536E-01	-0.9745567E-03	b_{eln}
6	1	0.4174312E-01	0.3002857E-02	
7	1	-0.1420681E-02	0.4492320E-05	
8	1	-0.1969776E-02	-0.8617624E-05	

R	T	FRC (Real Part)	FIC (Imaginary Part)	
1	1	0.4782311	-0.3603460	E_{θ_μ}
2	1	0.7822468	-0.2040950	
3	1	1.075997	-0.1207969	
4	1	1.272994	-0.1538158	
5	1	-0.2931353	0.5000024	E_{ϕ_μ}
6	1	-0.4874913	0.3285222	
7	1	-0.6937160	0.1035298	
8	1	-0.8679339	-0.1695977	

TABLE V-2-2: COMPUTATIONAL RESULTS FOR N = 4.

T	R	THETA	PHI
1	1	1.047195	0.7853979
1	2	2.094394	0.7853979
1	3	1.308995	0.7853979
1	4	1.570794	0.7853979

DETA =	410.7537	= Det B · Det C	(see 5.1.12)
DETA1 =	1.075514	= Det B · Det C_1	(see 5.2.1 and 5.2.4)
DETA2 =	1.075526	= Det B · Det C_2	(see 5.2.1 and 5.2.6)
DET =	381.9136	= Det $\left\{I + C_I^{-1} \cdot C_2\right\}$	(see 5.2.7b)

R	T	FR (Real Part)	FI (Imaginary Part)	Group
1	1	0.7822435	-0.2040975	$E_{\theta\mu}$
2	1	1.071161	-0.5945868	$E_{\theta\mu}$
3	1	1.075997	-0.1207976	$E_{\theta\mu}$
4	1	1.272995	-0.1538140	$E_{\theta\mu}$
5	1	-0.4874874	0.3285240	$E_{\phi\mu}$
6	1	-0.9387797	-0.7960107	$E_{\phi\mu}$
7	1	-0.6937119	0.1035333	$E_{\phi\mu}$
8	1	-0.8679292	-0.1695909	$E_{\phi\mu}$

R	T	XR (Real Part)	XI (Imaginary Part)	Group
1	1	0.7327468	-0.9099419	a_{oln}
2	1	0.6708306	0.4146110	a_{oln}
3	1	-0.2099591	0.5676632E-01	a_{oln}
4	1	-0.3726835	-0.2303393	a_{oln}
5	1	0.2382302E-01	-0.9745664E-03	b_{eln}
6	1	0.4174504E-01	0.3002902E-02	b_{eln}
7	1	-0.1420165E-02	0.4491291E-05	b_{eln}
8	1	-0.1970025E-02	-0.8636775E-05	b_{eln}

R	T	FRC (Real Part)	FIC (Imaginary Part)	Group
1	1	0.7822435	-0.2040975	$E_{\theta\mu}$
2	1	1.071160	-0.5945868	$E_{\theta\mu}$
3	1	1.075996	-0.1207976	$E_{\theta\mu}$
4	1	1.272994	-0.1538140	$E_{\theta\mu}$
5	1	-0.4874873	0.3285239	$E_{\phi\mu}$
6	1	-0.9387795	-0.7960106	$E_{\phi\mu}$
7	1	-0.6937118	0.1035333	$E_{\phi\mu}$
8	1	-0.8679290	-0.1695908	$E_{\phi\mu}$

TABLE V-2-3: COMPUTATIONAL RESULTS FOR N = 4.

T	R	THETA	PHI
1	1	1.178097	0.7853979
1	2	1.308995	0.7853979
1	3	1.439896	0.7853979
1	4	1.570796	0.7853979

DETA = 0.1172265E-01 = Det B · Det C (see 5.1.12)

DETA1 = 0.1375974 = Det B · Det C_1 (see 5.2.1 and 5.2.4)

DETA2 = 0.1375988 = Det B · Det C_2 (see 5.2.1 and 5.2.6)

DET = 0.8519518E-01 = $\text{Det}\left\{I + C_I^{-1} \cdot C_2\right\}$ (see 5.2.7b)

R	T	ER (Real Part)	EI (Imaginary Part)	
1	1	0.9354352	-0.1500959	E_{θ_μ}
2	1	1.075997	-0.1207976	
3	1	1.192410	-0.1210306	
4	1	1.272995	-0.1538140	
5	1	-0.5915630	0.2225537	E_{ϕ_μ}
6	1	-0.6937119	0.1035333	
7	1	-0.7879701	-0.2765677E-01	
8	1	-0.8679292	-0.1695909	

R	T	XR (Real Part)	XI (Imaginary Part)	
1	1	0.7327408	-0.9099416	a_{oln}
2	1	0.6708390	0.4146115	
3	1	-0.2099559	0.5676665E-01	
4	1	-0.3726856	-0.2303391	
5	1	0.2382234E-01	-0.9745671E-03	b_{eln}
6	1	0.4174596E-01	0.3002943E-02	
7	1	-0.1419870E-02	0.4560921E-05	
8	1	-0.1970182E-02	-0.8590585E-05	

R	T	ERC (Real Part)	EIC (Imaginary Part)	
1	1	0.9354351	-0.1500959	E_{θ_μ}
2	1	1.075996	-0.1207976	
3	1	1.192410	-0.1210306	
4	1	1.272994	-0.1538140	
5	1	-0.5915629	0.2225536	E_{ϕ_μ}
6	1	-0.6937119	0.1035333	
7	1	-0.7879701	-0.2765677E-01	
8	1	-0.8679291	-0.1695908	

TABLE V-2-4: COMPUTATIONAL RESULTS FOR N = 4.

T	R	THETA	PHI
1	1	0.5235986	0.7853979
1	2	0.7853979	0.7853979
1	3	1.047195	0.7853979
1	4	1.308995	0.7853979

DETA = 0.3053473

DETA1 = 0.6143691

DETA2 = 0.6143543

DET = 0.4970094

R	T	ER	EI
1	1	0.2273244	-0.5316700
2	1	0.4782298	-0.3603482
3	1	0.7822435	-0.2040975
4	1	1.075997	-0.1207976
5	1	-0.1399181	0.6185058
6	1	-0.2931336	0.5000023
7	1	-0.4874874	0.3285240
8	1	-0.6937119	0.1035333

R	T	XR	XI
1	1	0.7327479	-0.9099605
2	1	0.6708288	0.4146293
3	1	-0.2099591	0.5677189E-01
4	1	-0.3726825	-0.2303446
5	1	0.2382298E-01	-0.9759669E-03
6	1	0.4174458E-01	0.3004137E-02
7	1	-0.1420120E-02	0.4689246E-05
8	1	-0.1969865E-02	-0.8783497E-05

R	T	FRC	FIC
1	1	0.2273245	-0.5316700
2	1	0.4782298	-0.3603482
3	1	0.7822435	-0.2040975
4	1	1.075996	-0.1207977
5	1	-0.1399182	0.6185057
6	1	-0.2931336	0.5000023
7	1	-0.4874873	0.3285239
8	1	-0.6937118	0.1035333

TABLE V-3-1a: COMPUTATIONAL RESULTS FOR N = 6.

T	R	THETA	PHI
1	1	0.7854000	0.7854000
1	2	1.047199	0.7854000
1	3	1.308999	0.7854000
1	4	1.570800	0.7854000
1	5	0.3927000	0.7854000
1	6	0.5236000	0.7854000

NR

DF = 0 NR = 6 NT = 1 Computation of Matric Elements

T = 1 R = 1 THETA(T,R) = 0.7854000 PHI(T,R) = 0.7854000

1	0.7071055	0.7071081	0.0
2	0.2499967	1.499999	1.500005
3	-0.1767801	1.590981	5.303308
4	-0.4062512	0.6249819	9.374977
5	-0.3756481	-0.9943888	9.280675
6	-0.1484324	-2.296881	1.640434

T = 1 R = 2 THETA(T,R) = 1.047199 PHI(T,R) = 0.7854000

1	0.4999985	0.8660263	0.0
2	-0.1250023	1.299035	2.250004
3	-0.4375003	0.3247495	5.624991
4	-0.2890596	-1.353173	4.218695
5	0.8984733E-01	-1.928251	-4.921961
6	0.3232428	-0.4972618	-14.15039

T = 1 R = 3 THETA(T,R) = 1.308999 PHI(T,R) = 0.7854000

1	0.2588170	0.9659265	0.0
2	-0.3995207	0.7499944	2.799041
3	-0.3448823	-0.9636111	3.622195
4	0.1434330	-1.581923	-3.716404
5	0.3427269	0.2833002	-10.12999
6	0.4309556E-01	2.059617	-0.7062798

T = 1 R = 4 THETA(T,R) = 1.570800 PHI(T,R) = 0.7854000

1	-0.3500762E-05	1.000000	0.0
2	-0.5000000	-0.1050229E-04	3.000000
3	0.5251140E-05	-1.500000	-0.5251140E-04
4	0.3750000	0.2625570E-04	-7.500000
5	-0.6563923E-05	1.875000	0.1837899E-03
6	-0.3125000	-0.4594748E-04	13.12500

T = 1 R = 5 THETA(T,R) = 0.3927000 PHI(T,R) = 0.7854000

1	0.9238793	0.3826842	0.0
2	0.7803288	1.060661	0.4393414
3	0.5856298	1.875781	2.029491
4	0.3615928	2.629439	5.464164
5	0.1328186	3.121162	11.08569
6	-0.7636195E-01	3.188541	18.60277

T = 1 R = 6 THETA(T,R) = 0.5236000 PHI(T,R) = 0.7854000

1	0.8660249	0.5000010	0.0
2	0.6249981	1.299039	0.7500029
3	0.3247566	2.062498	3.247604
4	0.2343416E-01	2.435688	7.968758
5	-0.2232749	2.167948	14.20821
6	-0.3740247	1.207664	19.89249

A MATRIX (see equation 5.1.12)

1	0.7071053	0.4999979	1.499992	-0.8766528D-05
	1.590974	-2.625011	0.6249781	-6.137173
	-0.9943840	-7.265548	-2.296866	-2.784084

TABLE V-3-1b: COMPUTATIONAL RESULTS FOR N = 6.

2	0.7071055	0.3535517	1.060654	-1.060664
	0.2651563	-3.844884	-1.104856	-3.535487
	-1.574405	2.693146	-0.4060120	9.802808
3	0.7071055	0.1830109	0.5490327	-1.837117
	-0.7054105	-2.743846	-1.158044	2.328167
	0.2073893	7.216644	1.507739	0.8896422
4	0.7071055	-0.2475408D-05	-0.7426225D-05	-2.121316
	-1.060658	0.4084420D-04	0.1856554D-04	5.303291
	1.325823	-0.1346002D-03	-0.3248971D-04	-9.280760
5	0.7071054	0.6532800	1.959838	1.499993
	3.465977	1.767078	4.858549	0.6249722
	5.767127	-2.510626	5.891620	-7.710974
6	0.7071052	0.6123707	1.837111	1.060653
	2.916801	0.2296237	3.444567	-2.651671
	3.065926	-7.391532	1.707885	-12.58701
7	-0.4999998	-0.7071078	0.8766560D-05	-1.499998
	2.625021	-1.590980	6.187195	-0.6249803
	7.265574	0.9943876	2.784094	2.296875
8	-0.3535529	-0.7071080	1.060668	-1.060658
	3.844897	-0.2651573	3.535501	1.104860
	-2.693155	1.574410	-9.802843	0.4060135
9	-0.1830115	-0.7071081	1.837124	-0.5490347
	2.743855	0.7054130	-2.328176	1.158048
	-7.216670	-0.2073901	-0.8896454	-1.507745
10	0.2475417D-05	-0.7071081	2.121324	0.7426252D-05
	-0.4084434D-04	1.060661	-5.303310	-0.1856561D-04
	0.1346006D-03	-1.325828	9.280793	0.3248983D-04
11	-0.6532824	-0.7071080	-1.499998	-1.959846
	-1.767085	-3.465989	-0.6249745	-4.858566
	2.510634	-5.767148	7.711001	-5.891642
12	-0.6123729	-0.7071078	-1.060657	-1.837117
	-0.2296245	-2.916812	2.651681	-3.444579
	7.391559	-3.065937	12.58705	-1.707891

A1 MATRIX = $B(\phi)\, C_1(\theta)$

1	0.7071053	0.0	1.499992	0.0
	1.590974	0.0	0.6249781	0.0
	-0.9943840	0.0	-2.296866	0.0
2	0.0	0.3535517	0.0	-1.060664
	0.0	-3.844884	0.0	-3.535487
	0.0	2.693146	0.0	9.802808
3	0.7071055	0.0	0.5490327	0.0
	-0.7054105	0.0	-1.158044	0.0
	0.2073893	0.0	1.507739	0.0
4	0.0	-0.2475408D-05	0.0	-2.121316
	0.0	0.4084420D-04	0.0	5.303291
	0.0	-0.1346002D-03	0.0	-9.280760
5	0.7071054	0.0	1.959838	0.0
	3.465977	0.0	4.858549	0.0
	5.767127	0.0	5.891620	0.0
6	0.0	0.6123707	0.0	1.060653
	0.0	0.2296237	0.0	-2.651671
	0.0	-7.391532	0.0	-12.58701
7	-0.4999998	0.0	0.8766560D-05	0.0
	2.625021	0.0	6.187195	0.0
	7.265574	0.0	2.784094	0.0
8	0.0	-0.7071080	0.0	-1.060658
	0.0	-0.2651573	0.0	1.104860
	0.0	1.574410	0.0	0.4060135

TABLE V-3-1c: COMPUTATIONAL RESULTS FOR N = 6.

9	-0.1830115	0.0	1.837124	0.0
	2.743855	0.0	-2.328176	0.0
	-7.216670	0.0	-0.8896454	0.0
10	0.0	-0.7071081	0.0	0.7426252D-05
	0.0	1.060661	0.0	-0.1856561D-04
	0.0	-1.325828	0.0	0.3248983D-04
11	-0.6532824	0.0	-1.499998	0.0
	-1.767085	0.0	-0.6249745	0.0
	2.510634	0.0	7.711001	0.0
12	0.0	-0.7071078	0.0	-1.837117
	0.0	-2.916812	0.0	-3.444579
	0.0	-3.065937	0.0	-1.707891

A2 MATRIX = $B(\emptyset_c)\,C_2(\theta_c)$

1	0.0	0.4999979	0.0	-0.8756578D-05
	0.0	-2.625011	0.0	-6.187173
	0.0	-7.265548	0.0	-2.784084
2	0.7071055	0.0	1.060654	0.0
	0.2651563	0.0	-1.104856	0.0
	-1.574405	0.0	-0.4060120	0.0
3	0.0	0.1830109	0.0	-1.837117
	0.0	-2.743846	0.0	2.328167
	0.0	7.216644	0.0	0.8896422
4	0.7071055	0.0	-0.7426225D-05	0.0
	-1.060658	0.0	0.1856554D-04	0.0
	1.325823	0.0	-0.3248971D-04	0.0
5	0.0	0.6532800	0.0	1.499993
	0.0	1.767078	0.0	0.6249722
	0.0	-2.510626	0.0	-7.710974
6	0.7071052	0.0	1.837111	0.0
	2.916801	0.0	3.444567	0.0
	3.065926	0.0	1.707885	0.0
7	0.0	-0.7071078	0.0	-1.499998
	0.0	-1.590980	0.0	-0.6249803
	0.0	0.9943876	0.0	2.296875
8	-0.3535529	0.0	1.060668	0.0
	3.844897	0.0	3.535501	0.0
	-2.693155	0.0	-9.802843	0.0
9	0.0	-0.7071081	0.0	-0.5490347
	0.0	0.7054130	0.0	1.158048
	0.0	-0.2073901	0.0	-1.507745
10	0.2475417D-05	0.0	2.121324	0.0
	-0.4084434D-04	0.0	-5.303310	0.0
	0.1346006D-03	0.0	9.280793	0.0
11	0.0	-0.7071080	0.0	-1.959846
	0.0	-3.465989	0.0	-4.858566
	0.0	-5.767148	0.0	-5.891642
12	-0.6123729	0.0	-1.060657	0.0
	-0.2296245	0.0	2.651681	0.0
	7.391559	0.0	12.58705	0.0

DETA = -7.864776 = Det$\{B\}\cdot$ Det$\{C\}$

A INVERSE

1	-2076.057	824.8329	-235.4175	36.47884
	-4829.994	5242.467	-2061.957	813.2379
	-229.4479	34.48322	-4822.696	5227.653
2	2061.938	-813.2238	229.4407	-34.48125

TABLE V-3-1d: COMPUTATIONAL RESULTS FOR N = 6.

	4822.681	-5227.630	2076.055	-824.8388
	235.4223	-36.48045	4829.960	-5242.438
3	773.0667	-306.8711	87.52279	-13.40418
	1798.773	-1952.241	756.1236	-293.4619
	80.44323	-11.23036	1789.795	-1934.655
4	-756.1076	293.4531	-80.43952	11.22946
	-1789.769	1934.624	-773.0571	306.8698
	-87.52362	13.40464	-1798.739	1952.208
5	-287.7432	114.1203	-32.43706	4.892121
	-669.8868	727.0248	-274.0968	103.4287
	-26.79751	3.337342	-662.7319	712.7876
6	274.0853	-103.4232	26.79549	-3.336910
	662.7093	-712.7620	287.7342	-114.1178
	32.43678	-4.892206	669.8516	-726.9986
7	89.14685	-35.27639	9.953992	-1.476195
	207.7711	-225.4182	81.18503	-29.08981
	6.845014	-0.7152721	203.4543	-216.9684
8	-81.17951	29.08693	-6.844049	0.7150863
	-203.4405	216.9531	-89.14116	35.27445
	-9.953588	1.476175	-207.7565	225.4027
9	-19.88525	7.822536	-2.183500	0.3186573
	-46.46694	50.39694	-16.69391	5.446818
	-1.091031	0.8101490D-01	-44.75387	47.01133
10	16.69138	-5.445745	1.090693	-0.8095460D-01
	44.74829	-47.00520	19.88293	-7.821692
	2.183295	-0.3186361	46.46121	-50.39080
11	2.400715	-0.9341004	0.2574637	-0.3707457D-01
	5.675387	-6.136266	1.695998	-0.4660073
	0.6620728D-01	0.8364774D-04	5.242401	-5.312328
12	-1.695390	0.4657577	-0.6613227D-01	-0.9609588D-04
	-5.241025	5.310824	-2.400137	0.9338818
	-0.2574065	0.3706728D-01	-5.673994	6.134767

DETAI = -3.975948 = Det $\{B\}$ Det $\{C_1\}$

AT INVERSE

1	-553.6156	0.0	-107.8239	0.0
	262.1271	0.0	-204.3067	0.0
	-17.79314	0.0	-272.3880	0.0
2	0.0	14.99275	0.0	3.550754
	0.0	11.44198	0.0	14.00711
	0.0	16.22107	0.0	-14.23709
3	604.6294	0.0	115.9178	0.0
	-284.6938	0.0	221.2133	0.0
	18.62790	0.0	297.2351	0.0
4	0.0	-17.46460	0.0	-4.013707
	0.0	-13.44339	0.0	-17.46238
	0.0	-19.40593	0.0	16.49383
5	-435.3287	0.0	-80.60011	0.0
	203.6143	0.0	-157.3388	0.0
	-12.51176	0.0	-214.1189	0.0
6	0.0	13.98429	0.0	3.021337
	0.0	10.96294	0.0	14.37258
	0.0	15.64476	0.0	-13.53111
7	215.4509	0.0	37.30367	0.0
	-99.46158	0.0	76.28499	0.0
	5.571632	0.0	105.9758	0.0
8	0.0	-8.123169	0.0	-1.569300
	0.0	-6.514678	0.0	-8.802463

TABLE V-3-1e: COMPUTATIONAL RESULTS FOR N = 6.

	0.0	-8.723768	0.0	7.822831
9	-68.38198	0.0	-10.61173	0.0
	30.95625	0.0	-23.36369	0.0
	-1.524127	0.0	-33.68645	0.0
10	0.0	3.191295	0.0	0.5233298
	0.0	2.667959	0.0	4.027597
	0.0	3.110316	0.0	-3.231770
11	10.67905	0.0	1.422088	0.0
	-4.652337	0.0	3.456405	0.0
	0.1967558	0.0	5.361962	0.0
12	0.0	-0.6498930	0.0	-0.8706933D-01
	0.0	-0.6498923	0.0	-1.038577
	0.0	-0.5493498	0.0	0.7001620

DETA2 = -3.975176 = Det $\{B\}$ Det $\{C_2\}$

A2 INVERSE

1	0.0	-14.00701	0.0	-16.22097
	0.0	14.23700	0.0	-14.99255
	0.0	-3.550709	0.0	-11.44183
2	204.3484	0.0	17.79669	0.0
	272.4440	0.0	553.7254	0.0
	107.8451	0.0	-262.1788	0.0
3	0.0	17.46228	0.0	19.40581
	0.0	-16.49373	0.0	17.46436
	0.0	4.013655	0.0	13.44321
4	-221.2587	0.0	-18.63176	0.0
	-297.2960	0.0	-604.7488	0.0
	-115.9408	0.0	284.7501	0.0
5	0.0	-14.37250	0.0	-15.64467
	0.0	13.53103	0.0	-13.98411
	0.0	-3.021296	0.0	-10.96280
6	157.3714	0.0	12.51452	0.0
	214.1626	0.0	435.4145	0.0
	80.61663	0.0	-203.6547	0.0
7	0.0	8.802417	0.0	8.723711
	0.0	-7.822785	0.0	8.123062
	0.0	1.569277	0.0	6.514594
8	-76.30100	0.0	-5.572988	0.0
	-105.9974	0.0	-215.4931	0.0
	-37.31180	0.0	99.48148	0.0
9	0.0	-4.027582	0.0	-3.110293
	0.0	3.231753	0.0	-3.191253
	0.0	-0.5233213	0.0	-2.667926
10	23.36874	0.0	1.524554	0.0
	33.69325	0.0	68.39530	0.0
	10.61430	0.0	-30.96253	0.0
11	0.0	1.038575	0.0	0.5493452
	0.0	-0.7001590	0.0	0.6498849
	0.0	0.8706772D-01	0.0	0.6498855
12	-3.457200	0.0	-0.1968226	0.0
	-5.363034	0.0	-10.68115	0.0
	-1.422493	0.0	4.653328	0.0

DET = 1.978087 = Det $\left\{I + C_1^{-1} C_2\right\}$

A INVERSE

1	-2076.057	824.8329	-235.4175	36.47884

TABLE V-3-1f: COMPUTATIONAL RESULTS FOR N = 6.

	-4829.994	5242.467	-2061.957	813.2379
	-229.4479	34.48322	-4822.696	5227.653
2	2061.938	-813.2238	229.4407	-34.48125
	4822.681	-5227.630	2076.055	-824.8388
	235.4223	-36.48045	4829.960	-5242.438
3	773.0667	-306.8711	87.52279	-13.40418
	1798.773	-1952.241	756.1236	-293.4619
	80.44323	-11.23036	1789.795	-1934.655
4	-756.1076	293.4531	-80.43952	11.22946
	-1789.769	1934.624	-773.0571	306.8699
	-87.52362	13.40464	-1798.739	1952.208
5	-287.7432	114.1203	-32.43706	4.892171
	-669.8868	727.0248	-274.0968	103.4287
	-26.79751	3.337342	-662.7319	712.7876
6	274.0853	-103.4232	26.79549	-3.336910
	662.7093	-712.7620	287.7342	-114.1178
	32.43678	-4.892206	669.8616	-726.9986
7	89.14685	-35.27639	9.953992	-1.476195
	207.7711	-225.4182	81.18603	-29.08981
	6.845014	-0.7152722	203.4543	-216.9684
8	-81.17951	29.08693	-6.844050	0.7150863
	-203.4405	216.9531	-89.14116	35.27445
	-9.953588	1.476175	-207.7565	225.4027
9	-19.88525	7.822536	-2.183500	0.3186573
	-46.46694	50.39694	-16.69391	5.446818
	-1.091031	0.8101490D-01	-44.75387	47.01133
10	16.69138	-5.445745	1.090693	-0.8095460D-01
	44.74829	-47.00520	19.88293	-7.821692
	2.183295	-0.3186361	46.46121	-50.39080
11	2.400715	-0.9341004	0.2574637	-0.3707457D-01
	5.675387	-6.136266	1.695998	-0.4660073
	0.6620729D-01	0.8364764D-04	5.242401	-5.312328
12	-1.695390	0.4657577	-0.6613227D-01	-0.9609580D-04
	-5.241025	5.310824	-2.400137	0.9338818
	-0.2574065	0.3706728D-01	-5.673994	6.134767

R	T	FR	FI	
1	1	0.4777118	-0.3603461	E_{θ_μ}
2	1	0.7823263	-0.2040949	
3	1	1.076478	-0.1207968	
4	1	1.273079	-0.1538157	
5	1	0.1342068	-0.6048777	
6	1	0.2270187	-0.5316691	
7	1	-0.2927000	0.5000026	E_{ϕ_μ}
8	1	-0.4875162	0.3285223	
9	1	-0.6941013	0.1035299	
10	1	-0.8680329	-0.1695977	
11	1	-0.8427382E-01	0.6589078	
12	1	-0.1397491	0.6185065	

R	T	XR	XI	
1	1	0.7329465	-0.9100457	a_{oln}
2	1	0.6706334	0.4147143	
3	1	-0.2100334	0.5680502E-01	
4	1	-0.3726118	-0.2303776	
5	1	0.2385065E-01	-0.9890206E-03	
6	1	0.4171939E-01	0.3016903E-02	b_{eln}
7	1	-0.1428704E-02	0.8983790E-05	

TABLE V-3-1g: COMPUTATIONAL RESULTS FOR N = 6.

8	1	-0.1962515E-02	-0.1287782E-04	b_{eln}
9	1	0.5388186E-04	-0.1014100E-05	
10	1	0.6386307E-04	0.9279858E-06	
11	1	-0.1541888E-05	0.1228147E-06	
12	1	-0.1411965E-05	-0.1046130E-06	

R	T	FRC	FIC	
1	1	0.4777117	-0.3603460	E_{θ_μ}
2	1	0.7823262	-0.2040949	
3	1	1.076477	-0.1207967	
4	1	1.273078	-0.1538157	
5	1	0.1342067	-0.6048777	
6	1	0.2270186	-0.5316680	
7	1	-0.2926999	0.5000025	E_{ϕ_μ}
8	1	-0.4875162	0.3285223	
9	1	-0.6941013	0.1035299	
10	1	-0.8680328	-0.1695977	
11	1	-0.8427376E-01	0.6589078	
12	1	-0.1397491	0.6185064	

TABLE V-3-2: COMPUTATIONAL RESULTS FOR N = 6.

T	R	THETA	PHI
1	1	1.570794	0.7853979
1	2	1.439896	0.7853979
1	3	1.308995	0.7853979
1	4	1.178097	0.7853979
1	5	1.047195	0.7853979
1	6	0.9162976	0.7853979

DETA = -0.2481404E-02 = Det B Det C

DETA1 = -0.5461611E-02 = Det B Det C_1

DETA2 = -0.5461816E-02 = Det B Det C_2

DET = 0.4543355 = $\text{Det}\left\{I + C^{-1} C_2\right\}$

R	T	FR (Real Part)	FI (Imaginary Part)	Component
1	1	1.273079	-0.1538140	E_{θ_μ}
2	1	1.192777	-0.1210305	
3	1	1.076477	-0.1207975	
4	1	0.9358051	-0.1500959	
5	1	0.7823229	-0.2040975	
6	1	0.6266248	-0.2765667	
7	1	-0.8680280	-0.1695910	E_{ϕ_μ}
8	1	-0.7882850	-0.2765675E-01	
9	1	-0.6940970	0.1035333	
10	1	-0.5918370	0.2225537	
11	1	-0.4875122	0.3285241	
12	1	-0.3863768	0.4210207	

R	T	XR (Real Part)	XI (Imaginary Part)	Component
1	1	0.7327083	-0.9100252	a_{oln}
2	1	0.6708680	0.4146953	
3	1	-0.2099424	0.5679531E-01	
4	1	-0.3726938	-0.2303743	
5	1	0.2381674E-01	-0.9854243E-03	
6	1	0.4174785E-01	0.3016957E-02	
7	1	-0.1417519E-02	0.7195473E-05	b_{eln}
8	1	-0.1969696E-02	-0.1422371E-04	
9	1	0.5133791E-04	-0.5793227E-06	
10	1	0.6510974E-04	0.1509818E-05	
11	1	-0.1117200E-05	-0.5812608E-07	
12	1	-0.1382133E-05	-0.3721185E-06	

R	T	FRC (Real Part)	FIC (Imaginary Part)	Component
1	1	1.273078	-0.1538140	a_{oln}
2	1	1.192776	-0.1210306	
3	1	1.076476	-0.1207975	
4	1	0.9358050	-0.1500959	
5	1	0.7823229	-0.2040975	
6	1	0.6266248	-0.2765667	
7	1	-0.8680279	-0.1695910	b_{eln}
8	1	-0.7882848	-0.2765676E-01	
9	1	-0.6940969	0.1035333	
10	1	-0.5918369	0.2225537	

TABLE V-3-3: COMPUTATIONAL RESULTS FOR N = 6.

T	R	THETA	PHI
1	1	0.1308995	0.7853979
1	2	0.2617993	0.7853979
1	3	0.3926989	0.7853979
1	4	0.5235986	0.7853979
1	5	0.6544982	0.7853979
1	6	0.7853979	0.7853979

DETA = -0.1069765E-16

DETA1 = -0.2249567E-15

DETA2 = -0.6205844E-15

DET = 0.4778413E-01

R	T	FR	FI
1	1	0.2564795E-01	-0.6975904
2	1	0.6663364E-01	-0.6616657
3	1	0.1342062	-0.6048795
4	1	0.2270181	-0.5316702
5	1	0.3427395	-0.4478709
6	1	0.4777105	-0.3603483
7	1	-0.2000465E-01	0.7042730
8	1	-0.4419711E-01	0.6873667
9	1	-0.8427298E-01	0.6589066
10	1	-0.1397482	0.6185058
11	1	-0.2097238	0.5656968
12	1	-0.2926982	0.5000023

R	T	XR	XI
1	1	0.7536438	-0.7004206
2	1	0.6528825	0.2076693
3	1	-0.2177883	-0.1088313E-01
4	1	-0.3684531	-0.1630270
5	1	0.2660180E-01	0.1988587E-01
6	1	0.4105565E-01	-0.1826363E-01
7	1	-0.2179982E-02	-0.5125280E-02
8	1	-0.2060676E-02	0.5146421E-02
9	1	0.1989400E-03	0.8107808E-03
10	1	0.1203925E-03	-0.8427415E-03
11	1	-0.1488529E-04	-0.6590229E-04
12	1	-0.9949403E-05	0.6816187E-04

R	T	FRC	FIC
1	1	0.2480265E-01	-0.6982132
2	1	0.6589258E-01	-0.6619473
3	1	0.1336178	-0.6046321
4	1	0.2266036	-0.5307584
5	1	0.3424911	-0.4462211
6	1	0.4775945	-0.3579487
7	1	-0.1911997E-01	0.7049143
8	1	-0.4330888E-01	0.6877364
9	1	-0.8338946E-01	0.6589058
10	1	-0.1388899	0.6181347

TABLE V-3-4: COMPUTATIONAL RESULTS FOR N = 6.

T	R	THETA	PHI
1	1	1.570794	0.7853979
1	2	1.308995	0.7853979
1	3	1.832595	0.7853979
1	4	1.047195	0.7853979
1	5	2.094394	0.7853979
1	6	0.7853979	0.7853979

DETA = -1766895.

DETA1 = -0.1647988E-02

DETA2 = -0.1648334E-02

DET = 0.1072153E 10

R	T	ER	EI
1	1	1.273079	-0.1538140
2	1	1.076477	-0.1207975
3	1	1.286346	-0.3183991
4	1	0.7823229	-0.2040975
5	1	1.070888	-0.5945868
6	1	0.4777105	-0.3603483
7	1	-0.8680280	-0.1695910
8	1	-0.6940970	0.1025333
9	1	-0.9605555	-0.4770909
10	1	-0.4875122	0.3285241
11	1	-0.9385232	-0.7960107
12	1	-0.2926982	0.5000023

R	T	XR	XI
1	1	0.7327417	-0.9099416
2	1	0.6708375	0.4146109
3	1	-0.2099560	0.5676627E-01
4	1	-0.3726858	-0.2303391
5	1	0.2382186E-01	-0.9744931E-03
6	1	0.4174646E-01	0.3002873E-02
7	1	-0.1419418E-02	0.4501432E-05
8	1	-0.1970374E-02	-0.8580729E-05
9	1	0.5178574E-04	0.4580560E-08
10	1	0.6558617E-04	0.1169489E-07
11	1	-0.1207627E-05	0.6668106E-08
12	1	-0.1568012E-05	0.1027658E-07

R	T	ERC	EIC
1	1	1.273078	-0.1538140
2	1	1.076476	-0.1207976
3	1	1.286345	-0.3183992
4	1	0.7823229	-0.2040976
5	1	1.070887	-0.5945868
6	1	0.4777105	-0.3603483
7	1	-0.8680279	-0.1695909
8	1	-0.6940970	0.1025333
9	1	-0.9605554	-0.4770908
10	1	-0.4875121	0.3285240

TABLE V-3-5: COMPUTATIONAL RESULTS FOR N = 6.

T	R	THETA	PHI
1	1	1.178097	0.7853979
1	2	1.047195	0.7853979
1	3	0.9162976	0.7853979
1	4	0.7853979	0.7853979
1	5	0.6544982	0.7853979
1	6	0.5235986	0.7853979

DETA = -0.1030830E-06

DETA1 = -0.9475437E-06

DETA2 = -0.9465356E-06

DET = 0.1087894

R	T	FR	FI
1	1	0.9358051	-0.1500959
2	1	0.7823227	-0.2040975
3	1	0.6266248	-0.2765667
4	1	0.4777105	-0.3603483
5	1	0.3427395	-0.4478709
6	1	0.2270181	-0.5316702
7	1	-0.5918370	0.2225537
8	1	-0.4875122	0.3285241
9	1	-0.3863768	0.4210207
10	1	-0.2926982	0.5000023
11	1	-0.2097238	0.5655968
12	1	-0.1397482	0.6185058

R	T	XR	XI
1	1	0.7315806	-0.9122691
2	1	0.6720412	0.4170080
3	1	-0.2095725	0.5754462E-01
4	1	-0.3731167	-0.2311934
5	1	0.2370379E-01	-0.1219536E-02
6	1	0.4189685E-01	0.3300967E-02
7	1	-0.1392134E-02	0.6379363E-04
8	1	-0.2013426E-02	-0.9356308E-04
9	1	0.4783488E-04	-0.9268817E-05
10	1	0.7416171E-04	0.1718685E-04
11	1	-0.1070560E-05	0.6109376E-06
12	1	-0.2523401E-05	-0.1886111E-05

R	T	FRC	FIC
1	1	0.9358056	-0.1500967
2	1	0.7823236	-0.2040983
3	1	0.6266256	-0.2765676
4	1	0.4777114	-0.3603492
5	1	0.3427404	-0.4478717
6	1	0.2270190	-0.5316709
7	1	-0.5918363	0.2225530
8	1	-0.4875116	0.3285233
9	1	-0.3863765	0.4210200
10	1	-0.2926981	0.5000018

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13. ABSTRACT

A brief review of the salient features of the theoretical investigation of the c.w. bistatic inverse scattering problem is presented. The effect of changing the origin of the coordinate system upon the convergent properties etc., of the spherical vector wave function representation of the near scattered field and the surface loci $\underline{E} \times \underline{E}^* = 0$, is discussed. It is pointed out that a great deal of analysis remains to be done in this area. The determination of the surface of the scattering body from knowledge of the local total electric field is given. Emphasis is placed upon the generalization of the condition $\underline{E} \times \underline{E}^* = 0$ as applied to perfectly-conducting bodies, to scattering surfaces characterized by the impedance boundary condition. Properties of the matrix inversion associated with the determination of the expansion coefficients from far field data are discussed. Some numerical results are presented, and restrictions upon the choice of aspect angles are deduced.

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