ABSTRACT

The use of Wilcox's recurrence relation method (described in the First Technical Report, 7644-1-T) of determining the body shape from the scattered far field data, is investigated in more detail. The problem of concern is the radius of convergence of the series. When the body is composed of sharp edges connected by smooth segments, with at least one of the principal curvature zero, the radius of convergence in this case is given by the radius of the smallest sphere enclosing the body. However, when the scattering body is a sphere whose center is at the origin of the coordinate system, the radius of convergence is vanishingly small.

The second term of Wilcox's expansion and its relationship to the size of the scatterer is investigated. Two special cases are considered; with the body a perfectly conducting flat plate, and a smooth convex shape giving rise to specular scattering. Generally, it is shown that the ratio of the second term to the first term in the expansion is the order of $1/2kD^2$ where D is the distance from the farthest point of the body to a line directed from the origin of the coordinate system to the receiver. For specular scattering, D is the distance from the specular point to the aforementioned line. Further investigation yielded the importance of phase, at least insofar as specular scattering was concerned. It is shown that the knowledge of phase information determined the location of the phase center of the specular point.

Further consideration of employing the monostatic-bistatic theorem to determine the material characteristics of the scatterer is undertaken. In particular, it is shown that the two polarization measurements of cross section at one non-zero bistatic angle and at the zero bistatic angle (backscattering) determines the reactive surface impedance of $\eta = u \pm iv$ apart from the sign in the imaginary part. Such surfaces would correspond to poor conductors, or absorber coated conductors. However, the case where the ratio of the bistatic to monostatic cross section is unity for both polarizations, produced incomplete results. In this case, it could only be concluded that $u = 0$.

TABLE OF CONTENTS

I
EQUIVALENT SOURCES

1.1 Radius of Convergence of the Wilcox Expansion

The scattered field at a point $\underline{x}$ outside an arbitrary body can be represented in terms of the total fields generated on the surface as follows

$$\underline{E}^{sc}(\underline{x}) = -\frac{1}{4\pi}\int_S \left[i\omega\mu_o(\hat{n}\wedge\underline{H})\phi + (\hat{n}\wedge\underline{E})\wedge\nabla'\phi + (\hat{n}\cdot\underline{E})\nabla'\phi\right] dS' \quad (1.1)$$

where $\hat{n}$ is the unit outward normal to the surface and $\phi = e^{ik|\underline{x}-\underline{x}'|}\Big/|\underline{x}-\underline{x}'|$. For points exterior to a sphere centered about the origin of coordinates and completely enclosing the body, Expression (1.1) can be represented in the form of a Wilcox expansion

$$\underline{E}^{sc}(\underline{x}) = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{\underline{E}_n(\theta,\phi)}{r^n} \quad (1.2)$$

with $\underline{E}_n(\theta,\phi)$ given explicitly by

$$\underline{E}_n = -\frac{1}{4\pi}\int_S \left[i\omega\mu_o(\hat{n}\wedge\underline{H})A_n + (\hat{n}\wedge\underline{E})\underline{B}_n + (\hat{n}\cdot\underline{E})\underline{B}_n\right] dS' \quad (1.3)$$

The scalar quantities A_n are determined by recursion formulas

$$A_o = \exp\left[-ikr'\cos\gamma\right], \qquad 2iknA_n = \left[n(n-1)+D\right]A_{n-1}. \quad (1.4)$$

where D is the differential operator

$$D = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad (1.5)$$

and γ is the angle between the two vectors $\underline{x}$ and $\underline{x}'$,

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{1.6}$$

The vector quantity $\underline{B}_n$ is derived from A_n by means of the relation

$$\underline{B}_n = \underline{i}_r \left[-ikA_n + nA_{n-1} \right] - \underline{i}_\theta \frac{\partial A_{n-1}}{\partial\theta} - \underline{i}_\phi \frac{1}{\sin\theta} \frac{\partial A_{n-1}}{\partial\phi} \tag{1.7}$$

where $\underline{i}_r, \underline{i}_\theta, \underline{i}_\phi$ denote the unit vectors of the spherical polar coordinate system associated with the observation point $\underline{x}$.

Expansion (1.2) is uniformly convergent for all points outside the minimum sphere enclosing the body, and represents the scattered field outside this sphere. However, it is possible that the expansion may be uniformly convergent inside the minimum sphere, in which case it will represent there the field produced by an equivalent source. The problem requiring investigation is to obtain an estimate of the radius of convergence of Expansion (1.2). In the case of perfectly conducting bodies with induced edge or tip singularities, the expansion will not converge inside the minimum sphere containing these sharp corners or edges. On the other hand, for smooth convex shapes, convergence inside the minimum sphere enclosing the body is expected. Work is proceeding on this matter with expectations that the surface must be locally analytic in order to carry the expansion inside.

Assuming the Wilcox expansion to converge inside the minimum sphere enclosing the body, it should be possible by moving the origin of coordinates to generate a minimum convex shape within the body such that the field is analytic outside this minimum region. The equivalent sources of the field lie within such a region.

For smooth convex scattering shapes, then, it is expected that the Wilcox expansion can be employed to determine a minimum source region inside the body. Knowing the field everywhere outside this region, the problem reduces to seeking a scattering surface for a particular boundary condition. In the case of perfect conductors, for example, one would look for the surface on which vanishes the sum of the tangential components of the scattered and incident electric field vectors. The requirement that the surface remain unchanged as the frequency varies is sufficient to determine a unique perfect conductor.

1.2 Scattering by a Sphere

The problem of scattering by a sphere provides an example in which the formal series solution for the scattered field converges absolutely and uniformly inside as well as outside the body. To show this, it is sufficient to prove convergence for the radial components of the field; convergence of the remaining components follows easily. Thus consider the Mie series for the radial electromagnetic field

$$\underline{r} \cdot \underline{H}^{sc} = -\frac{\sin\phi}{ik} \sum_{n=1}^{\infty} i^n (2n+1) b_n h_n(kr) P_n^1(\cos\theta),$$

$$\underline{r} \cdot \underline{E}^{sc} = -\frac{\cos\phi}{ik} \sum_{n=1}^{\infty} i^n (2n+1) b_n h_n(kr) P_n^1(\cos\theta) , \tag{1.8}$$

where, for a perfectly conducting sphere of radius a, the coefficients a_n and b_n are given by

$$a_n = -\frac{j_n(ka)}{h_n(ka)}$$

$$b_n = -\frac{\left[(ka)j_n(ka)\right]'}{\left[(ka)h_n(ka)\right]'} \tag{1.9}$$

The prime denotes differentiation with respect to (ka). The incident plane wave is assumed to have unit intensity and the free-space constants ϵ_o, μ_o are taken to be unity.

Simplification is obtained by noting the inequality (Weil et al, 1956)

$$\left| \frac{P_n^1(\cos\theta)}{\sin\theta} \right| \leqslant \frac{n(n+1)}{2} \quad \text{for } \theta \leqslant \pi \tag{1.10}$$

which leads to

$$\left| \underline{r} \cdot \underline{H}^{sc} \right| \leqslant \frac{1}{2k} \sum_{n=1}^{\infty} n(n+1)(2n+1) \left| a_n h_n(kr) \right| ,$$

$$\left| \underline{r} \cdot \underline{E}^{sc} \right| \leqslant \frac{1}{2k} \sum_{n=1}^{\infty} n(n+1)(2n+1) \left| b_n h_n(kr) \right| . \tag{1.11}$$

Employing the following asymptotic approximations as $n \longrightarrow \infty$

$$j_n(x) \sim \frac{x^n}{(2n+1)!!} , \quad h_n(x) \sim -i \frac{(2n-1)!!}{x^{n+1}} \tag{1.12}$$

where $(2n+1)!! = (2n+1)(2n-1) \ldots 3 \cdot 1$, one finds

$$\left| a_n h_n(kr) \right| \sim \left| b_n h_n(kr) \right| \sim \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} \tag{1.13}$$

It follows by an application of the ratio test that the above series converge uniformly for all r different from zero. Finally, with the aid of the above inequality concerning the Legendre polynomial and the following inequality (Weil et al, 1956)

$$\left| \frac{dP_n^{\,1}(\cos\theta)}{d\theta} \right| \leqslant \frac{n(n+1)}{2} \quad \text{for } \theta \leqslant \pi , \tag{1.14}$$

the remaining components of the scattered field may similarly be shown to converge absolutely and uniformly for all $r \neq 0$.

In the case of a homogeneous dielectric sphere with relative material parameters ϵ and μ, we find for large n

$$\left| a_n h_n(kr) \right| \sim \left| \frac{\mu - 1}{\mu + 1} \right| \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} \ ,$$

$$\left| b_n h_n(kr) \right| \sim \left| \frac{\epsilon - 1}{\epsilon + 1} \right| \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} \ , \qquad (1.15)$$

and convergence for $r \neq 0$ is again evident. The result may be extended to include the case of concentric spheres of different dielectric materials.

The field scattered by a sphere may thus be conceived of as a superposition of electric and magnetic multipole fields whose sources are located at a single point--the center of the sphere.

1.3 The Question of Uniqueness

Consider the scattered field due to a smooth, perfectly conducting, convex surface S and assume that an analytic expression for the field is known everywhere exterior to the equivalent source region which resides inside S. In seeking the surface S by looking for the surface on which the electric field obeys the required boundary condition, it is possible that more than one eligible surface may be found for a particular wave number k. The question of uniqueness naturally arises.

Let us assume, therefore, that two perfectly conducting surfaces S and S_1 have been found. These surfaces both surround the equivalent source region and are taken to be smooth. In the simply connected volume V between the two surfaces, the total electric field satisfies the source-free wave equation

$$(\nabla^2 + k^2)\underline{E} = 0 \qquad (1.16)$$

together with the equation

$$\text{div } \underline{E} = 0. \qquad (1.17)$$

However, solutions of these equations in the simply connected cavity V such that

$$\underline{n} \wedge \underline{E} = 0 \qquad (1.18)$$

on the bounding surfaces S, S_1 exist only for a discrete set of eigenfrequencies. Thus, if k varies continuously, the shape of S_1 must change in order to satisfy the boundary condition since by definition the scattering surface S is independent of the wavelength of the incident field. The requirement that S remain unchanged as the frequency is varied continuously therefore allows us to determine the scattering surface uniquely.

II

RELATIONSHIPS BETWEEN THE SIZE OF THE SCATTERER AND THE SECOND TERM IN WILCOX'S EXPANSION

In employing Wilcox's expansion, it is of interest to know for what value of r the second term in the expansion is the order of the leading term. To obtain estimates of the value of r two types of scattering bodies are considered first.

2.1 The Flat Plate

As a representative of the class of bodies which produce a narrow main lobe in the scattered field, a rectangular flat plate will be considered. It's dimensions will be taken to be much greater than a wavelength. For simplicity, the coordinate system will be chosen so that the z-axis lies normal to the flat plate, the origin is in the center of the plate, and the x and y axes are parallel to the sides. The plane of incidence will be the xz plane and the angle of incidence denoted by α (Fig. 2-1).

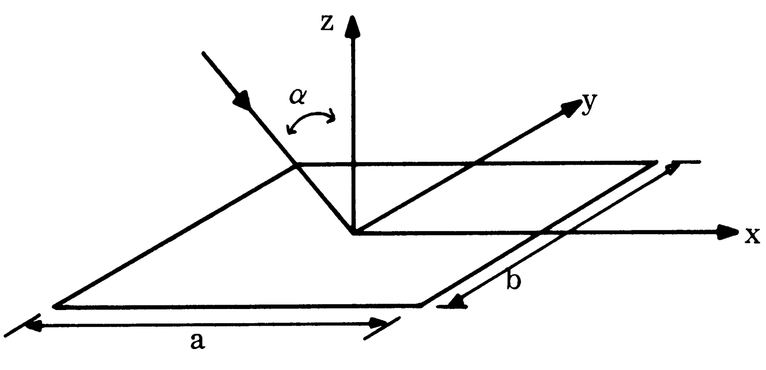

FIG. 2-1: FLAT PLATE GEOMETRY

The incident wave will be assumed to be polarized in the plane of incidence. Taking the plate to be a perfect conductor, the scattered field at a point $\underline{x}$, is given by

$$\underline{H}^s(\underline{x}) = \frac{1}{4\pi} \int (\underline{n} \times \underline{H}) \times \underline{\nabla}' \phi \, dx'dy' \qquad (2.1)$$

where $\underline{x}'$ is a point on the surface of the plate, and

$$\underline{\nabla}' \phi = (ik - \frac{1}{R}) \frac{e^{ikR}}{R} (\frac{\underline{x}' - \underline{x}}{R}) \qquad (2.2)$$

with

$$R = \left| \underline{x} - \underline{x}' \right|$$

If the observation point is taken to be in the far field with coordinate (r, θ, ϕ), then

$$\underline{H}^s = \frac{ike^{ikr}}{4\pi r} \iint_A (\underline{n} \times \underline{H}) \times \underline{a} \, e^{-ikf} dx'dy' \qquad (2.3)$$

where

$$\underline{a} = \underline{i}_y \sin\theta\sin\phi + \underline{i}_z \cos\theta \qquad (2.4)$$

$$f = x' \sin\theta\cos\phi + y' \sin\theta\sin\phi \qquad (2.5)$$

Using the physical optics approximation, where $\underline{n} \times \underline{H} = 2\underline{n} \times \underline{H}^i$, with $\underline{H}^i$ taken to be the incident magnetic intensity,

$$\underline{H}^i(\underline{x}) = \underline{i}_y \, \text{expik} \, (-z\cos\alpha + x\sin\alpha) \qquad (2.6)$$

The above expression can be approximated as follows

$$\underline{H}^s(r, \theta, \phi) = \frac{ike^{ikr}}{2\pi r} \iint_A (\underline{i}_x \times \underline{a}) \exp(-ikg) dx'dy' \qquad (2.7)$$

where

$$g(x',y') = f(x', y') \quad x'\sin\alpha \tag{2.8}$$

This can be evaluated, yielding

$$\underline{H}^s(r,\theta,\phi) = \frac{-ikab}{2\pi r} e^{ikr} \frac{\sin u}{u} \frac{\sin v}{v} \left[\underline{i}_\theta \sin\phi + \underline{i}_\phi \cos\theta\cos\phi\right] \tag{2.9}$$

where

$$u = \frac{ka}{2} (\sin\theta\cos\phi - \sin\alpha) \tag{2.10}$$

$$v = \frac{kb}{2} \sin\theta\sin\phi \tag{2.11}$$

Using the relation

$$\underline{E}^s = -\sqrt{\frac{\mu_o}{\epsilon_o}} \left[-\underline{i}_\theta H^s_\phi + \underline{i}_\phi H^s_\theta\right]$$

which holds for the far field, the electric intensity in the far field, has the form

$$\underline{E}^s = \frac{e^{ikr}}{r} \underline{E}_o(\theta,\phi) \tag{2.12}$$

where

$$\underline{E}_o = ik \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{ab}{2\pi} \frac{\sin u}{u} \frac{\sin v}{v} \left[-\underline{i}_\theta \cos\theta\cos\phi + \underline{i}_\phi \sin\phi\right] \tag{2.13}$$

The direction of the main scattering lobe is given by $\phi = 0$ and $\theta = \alpha$. In this direction $\underline{E}_o$ reduces to

$$\underline{E}_o = -ik \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{ab}{2\pi} \cos\alpha\, \underline{i}_\theta \tag{2.14}$$

Using Wilcox's notation for the vector components given as follows,

$$\underline{E} = E^1 \underline{i}_r + E^2 \underline{i}_\theta + E^3 \underline{i}_\phi \qquad , \tag{2.15}$$

the θ component of the second term in the expansion

$$\frac{e^{ikr}}{r} \sum_{n=0} \frac{\underline{E}_n}{r^n} \tag{2.16}$$

is given by the recurrence relations

$$2ikE_1^{\ 2} = DE_o^{\ 2} - \frac{1}{\sin^2\theta} E_o^{\ 2} - \frac{2\cos\theta}{\sin^2\theta} \frac{\partial E_o^{\ 3}}{\partial \phi} \tag{2.17}$$

where the operator D is

$$D = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \tag{2.18}$$

Using the expressions for $E_o^{\ 2}$ and $E_o^{\ 3}$ given by Eq. (2.13), it can be shown that for $\theta = \alpha$, $\phi = 0$,

$$2ikE_1^{\ 2} = \frac{ik}{12} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{ab}{2\pi} \left[(ka)^2 \cos^2 \alpha + (kb)^2 + O(1)\right] \quad . \tag{2.19}$$

Thus it follows that

$$\left|E_1^{\ 2}\right| \sim \left|E_o^{\ 2}\right| \frac{(ka)^2 \cos^2\alpha + (kb)^2 + O(1)}{24k} \tag{2.20}$$

The second term in Expansion (2.16) is approximately equal to the first term in the direction of the main lobe ($\theta = \alpha$, $\phi = 0$) when $r = r_o$ where

$$kr_o = \frac{1}{24} \left[(ka)^2 \cos^2\alpha + (kb)^2 \right] \tag{2.21}$$

For a square plate, this becomes

$$r_o = \frac{\pi A}{\lambda 12} \left[1 + \cos^2\alpha \right] \tag{2.22}$$

where A is the area of the plate. It follows that the second term in Wilcox's expansion becomes the same order of magnitude as the first term, for r such that kr is the area of the plate, in square wavelengths.

The analysis developed above holds for the origin at the center of the plate. Different results will occur when the origin is elsewhere. Some comments on these are given below.

2.2 Specular Scattering From a Convex Shape

The case will be considered where the dominant scattered field in a particular angular sector, arises from a single scattering center such as a specular point. Assuming perfect conductivity, the first two terms in the expansion

$$\underline{E}\,(r,\theta,\phi) = \frac{e^{ikr}}{r} \sum \frac{\underline{E}_n}{r^n} \tag{2.23}$$

are given by

$$\underline{E}_o = -\frac{1}{4\pi} \int_S \left[i\omega\mu_o (\underline{n} \times \underline{H}) - ik(\underline{n} \cdot \underline{E})\, \underline{i}_r \right] A_o\, ds' \tag{2.24}$$

$$\underline{E}_1 = -\frac{1}{4\pi} \int_S \left[i\omega\mu_o (\underline{n} \times \underline{H}) - ik(\underline{n} \cdot \underline{E})\underline{i}_r \right] A_1 ds' \tag{2.25}$$

$$-\frac{1}{4\pi} \int_S \left[\hat{\underline{i}}_r A_o - \hat{\underline{i}}_\theta \frac{\partial A_o}{\partial \theta} - \hat{\underline{i}}_\phi \frac{1}{\sin\theta} \frac{\partial A_o}{\partial \phi} \right] \underline{n} \cdot \underline{E} ds'$$

where

$$A_o = \exp\left[-ikr'\cos\gamma\right] \tag{2.26}$$

$$A_1 = A_o \left[\frac{1}{2} ikr'^2 \sin^2\gamma + r'\cos\gamma\right] \tag{2.27}$$

The integration is over the surface of the body. In the above expressions, r' is the distance from the origin to the point of integration on the surface of the body S, and γ is the angle between the radius vectors directed to the observation point (r, θ, ϕ) and the point of integration (r', θ', ϕ') and is given by the relation

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{2.28}$$

Since we are considering specular scattering, the dominant contribution of the above integrals will arise from a small region about the specular point $(r'_s, \theta'_s, \phi'_s)$. The coordinates of the specular point depend upon the position of the observation point (r, θ, ϕ).

Since the dominant contribution of the integrals arises from the vicinity of the specular point $(r'_s, \theta'_s \phi'_s)$, and can be evaluated by the method of stationary phase, it follows that for $kr'_s \sin\alpha_s > 1$, that

$$\underline{E}_1 \sim \frac{ik}{2} (r'_s \sin\gamma_s)^2 \underline{E}_o \tag{2.29}$$

This indicates that the value of r for which $\left| E_1 \right| \sim r \left| E_o \right|$ is given approximately by

$$r_o \sim \frac{1}{2} kd^2 \tag{2.30}$$

where d is the distance from the specular point to the line directed from the origin to the observation point (r, θ, ϕ). (A more precise value of r_o is given below).

2.3 Effect of Changing the Origin of the Coordinate System

For a general body with dimensions much greater than a wavelength, and such that the scattered field is comprised of components arising from many scattering centers, it follows in the same manner as indicated in Section (2.3), that

$$\left|E_1\right| \sim r \left|E_o\right| \quad \text{when}$$

$$r \sim 0\left(\frac{1}{2} kD^2\right). \tag{2.31}$$

Here D is the distance from the farthest point of the body to the line directed from the origin to the observation point. The distance D is a function of (θ, ϕ) varying with changing position of the far field observation point.

Changing the origin of the coordinate system will increase or decrease the distance D, thus effectively increasing or decreasing the ratio of the second term to the first term in Wilcox's expansion. The effect of changing the origin of the coordinate system of the far field pattern is to produce an additional phase factor. This can be seen as follows. Let the origin of the coordinate system (r_i, θ_i, ϕ_i) be displaced a distance $\underline{\ell}$, resulting in a new coordinate system (r, θ, ϕ). For a point in the far field $r \sim r_i - \underline{i}_r \cdot \underline{\ell}$, $\theta \sim \theta_i$, and $\phi \sim \phi_i$. Thus if the far field pattern (phase and amplitude) is given by

$$\frac{e^{ikr_i}}{r_i} \underline{E}_o(\theta_i, \phi_i) \tag{2.32}$$

in the initial coordinate system, it will be given by

$$\frac{e^{ikr}}{r} \underline{E}_o(\theta, \phi) \exp(-ik \underline{i}_r \cdot \underline{\ell}) \tag{2.33}$$

in the final system. If the vector $\underline{\ell}$ has components

$$(-\ell\sin\theta_\ell\cos\phi_\ell\ ,\ -\ell\sin\theta_\ell\sin\phi_\ell\ ,\ -\ell\cos\theta_\ell) \tag{2.34}$$

with regard to the final coordinate system, then

$$\underline{i}_r \cdot \underline{\ell} = -\ell\cos\gamma_\ell = -\ell\left[\cos\theta\,\cos\theta_\ell + \sin\theta\sin\theta_\ell\cos(\phi - \phi_\ell)\right] \tag{2.35}$$

For further illustration of the effect of changing the origin of the coordinate system, we will return again to the case of specular scattering. Using the notation of Section (2.2), $(r'_s, \theta'_s, \phi'_s)$ will refer to the specular point on the body, giving rise to the scattered field at the ray (θ, ϕ) in the far field. The dominant portion of the integral (2.24) arises from the vicinity of the specular point. It can be shown that

$$\underline{E}_o\,(\theta,\phi) \sim \underline{\mathcal{E}}\,\text{expik}\left[\underline{k}_i \cdot \underline{r}'_s - \underline{r}'_s \cdot \underline{r}_o\right] \tag{2.36}$$

where $\underline{k}_i$ is the unit vector indicating the direction of incident propagation, and $\underline{r}_o$ is the unit vector directed from origin to the receiver or observation point (r, θ, ϕ). The amplitude factor $\underline{\mathcal{E}}$ is a slowly varying function of θ and ϕ. For simplification the direction of the axis of the coordinate system will be chosen so that $\underline{k}_i = -\underline{i}_z$.

The phase factor

$$g(\theta,\phi) = \underline{r}'_s \cdot \left[\underline{r}_o - \underline{k}_i\right] \tag{2.37}$$

can be shown to have the form

$$g(\theta,\phi) = 2\cos\frac{\theta}{2}\,\underline{r}'_s \cdot \underline{n}_s \tag{2.38}$$

where $\underline{n}$ is the unit outward normal at the specular point (see Fig. 2-2). Since the position of the specular point is a function of θ and ϕ, it follows that $\underline{r}'_s$ and $\underline{n}_s$ are functions of θ and ϕ also.

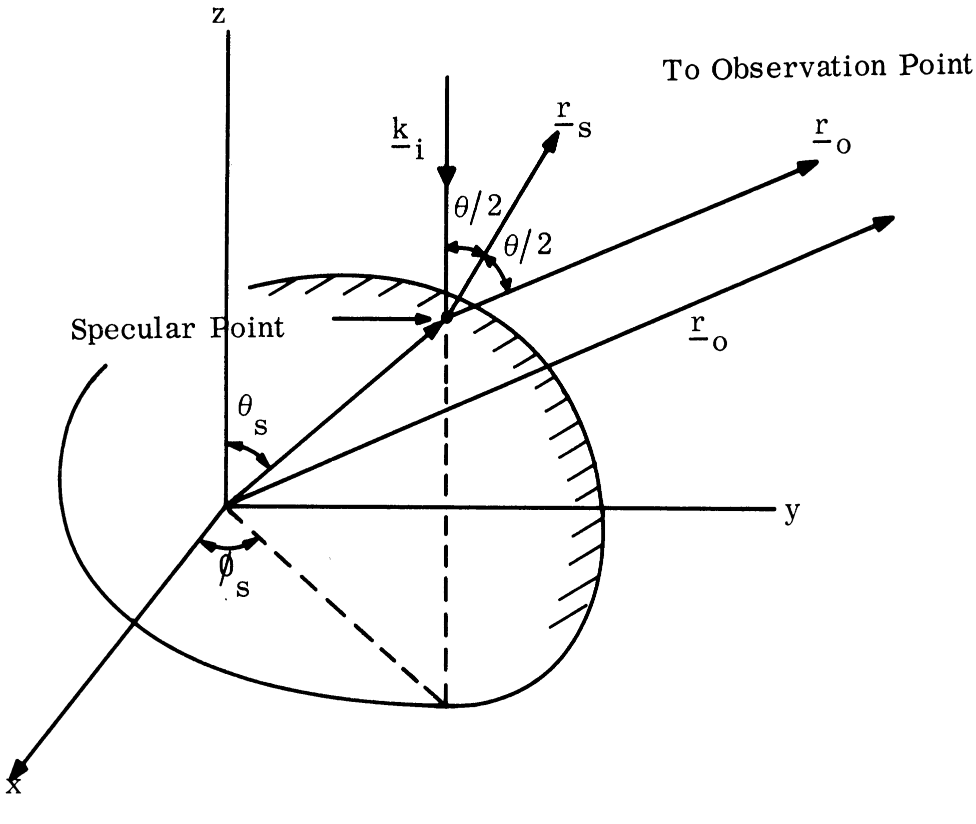

FIG. 2-2: SPECULAR POINT OF CONVEX SURFACE ASSOCIATED WITH PARTICULAR RECEIVER DIRECTION

It can be shown that the derivatives of the phase factor $g(\theta, \phi)$ satisfies the following relations

$$\frac{\partial g}{\partial \theta} = \underline{r}'_s \cdot \underline{i}_\theta \tag{2.39}$$

$$\frac{\partial g}{\partial \phi} = \sin\theta \, \underline{r}'_s \cdot \underline{i}_\phi \tag{2.40}$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\cos\frac{\theta}{2}}{2G\rho_2} \; -\underline{r}'_s \cdot \underline{i}_r \tag{2.41}$$

$$\frac{\partial^2 g}{\partial \phi^2} = -\sin\theta \left[\sin\theta \, \underline{i}_r + \cos\theta \, \underline{i}_\theta\right] \cdot \underline{r}'_s + \sin\frac{\theta}{2}\sin\theta \, \frac{1}{\rho_1 G} \tag{2.42}$$

where $(\underline{i}_r, \underline{i}_\theta, \underline{i}_\phi)$ on the unit vectors associated with the spherical polar coordinate system at the far field observation point (r, θ, ϕ).

G is the Gaussion curvature of the surface at the specular point, and ρ_1 and ρ_2 are the radii of curvature of the surface at the specular point in the $(\underline{k}_i, \underline{r}_o)$ plane and perpendicular to this plane respectively.

Setting

$$\underline{\mathcal{E}} = \mathcal{E}^2 \underline{i}_\theta + \mathcal{E}^3 \hat{\underline{i}}_\phi \tag{2.43}$$

the leading term in Wilcox's expansion has the form

$$\underline{E}_o = \mathcal{E}^2 e^{-ikg} \underline{i}_\theta + \mathcal{E}^3 e^{-ikg} \underline{i}_\phi \tag{2.44}$$

Using relation (2.17) together with Eqs. (2.39) to (2.42), it can be shown that the θ component of the second term in Wilcox's expansion has the form

$$2ikE_1^{\,2} = -E_o^{\,2}\left[(kd)^2 + ik\left(\frac{\cos\theta/2}{2G\rho_2} + \frac{\sin\theta/2}{G\rho_1\sin\theta} - 2\underline{r}'_s\cdot\underline{i}_r\right)\right]$$

$$+e^{-ikg}\left[D\mathcal{E}^2 - \frac{\mathcal{E}^2}{\sin^2\theta}\right] - 2ik\left[\frac{\partial g}{\partial\theta}\frac{\partial\mathcal{E}^2}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial g}{\partial\phi}\frac{\partial\mathcal{E}^2}{\partial\phi}\right]e^{-ikg}$$

$$-2\frac{\cos\theta}{\sin^2\theta}\left[\frac{\partial\mathcal{E}^3}{\partial\phi} - ik\frac{\partial g}{\partial\phi}\mathcal{E}^3\right]e^{-ikg} \qquad (2.45)$$

where d is the same parameter given in Section 2.2.

It is seen that on changing the origin of the coordinate system, so that

$$\underline{r}'_s = r_s\underline{i}_r \qquad (2.46)$$

$$2r_s = \left[\frac{\cos\theta/2}{2G\rho_2} + \frac{\sin\theta/2}{G\rho_1\sin\theta}\right] \qquad (2.47)$$

expression (2.46) reduces to

$$2ikE_1^{\,2} = \left[D\mathcal{E}^2 - \frac{1}{\sin^2\theta}\left(\mathcal{E}^2 + 2\cos\theta\frac{\partial\mathcal{E}^2}{\partial\phi}\right)\right]e^{-ikg} \qquad (2.48)$$

The right-hand side of Eq. (2.48) is very small, since $\mathcal{E}^2$ and $\mathcal{E}^1$ are slowly varying functions (both porportional to G^{-1}).

Relations (2.46) and (2.47) indicate that on changing the origin of the coordinate system to lie on the "ray" directed from the specular point to the observation point, and at the effective phase center, the second term in Wilcox's expansion can be made quite small. Changing the origin of the coordinate system to reduce the second term in Wilcox's expansion is equivalent to tracing back the reflected wave front to the specular point.

This points out the importance of the phase information in the scattered far field, for in this case the knowledge of the phase leads to the angular position of the specular point on the body, whereas amplitude knowledge by itself will not indicate this, only yielding the Gaussion curvature at the specular point.

III

USE OF MONOSTATIC-BISTATIC THEOREM TO DETERMINE MATERIAL CHARACTERISTICS

It was pointed out in the last report that in the limit of vanishing wavelength, the monostatic-bistatic theorem could be used to determine the reflection coefficient of the body. The ratios of the bistatic cross section to the monostatic cross-section gives the ratios

$$\left| \frac{R_{\perp}(\theta)}{R_{\perp}(0)} \right|^2 = r_{\perp} \tag{3.1}$$

and

$$\left| \frac{R_{\parallel}(\theta)}{R_{\parallel}(0)} \right|^2 = r_{\parallel} \tag{3.2}$$

where $R_{\perp}(\theta)$, and $R_{\parallel}(\theta)$ are the voltage reflection coefficients for polarization perpendicular and parallel to the plane of incidence. The angle θ is the angle of incidence to the surface measured from the normal, and is such that 2θ is the bistatic angle.

The particular case where the material characteristics of the surface can be represented by an impedance boundary condition will be considered to determine the number of measurements needed to prescribe the impedance parameter η. The effect of the surface upon incident energy can be represented in the form

$$\underline{E} - (\underline{E} \cdot \underline{n})\underline{n} = \eta \sqrt{\frac{\mu_o}{\epsilon_o}}\, \underline{n} \times \underline{H} \tag{3.3}$$

where $\underline{E}$ and $\underline{H}$ are the total fields generated on the surface. Such a condition represents either a poor conductor, or perfect conductors coated with a material of high index of refraction as is encountered in the use of magnetic type absorbers. For a single layer of such material η is given by

$$\eta = -i \sqrt{\frac{\mu}{\epsilon}} \tan(Nk\delta) \qquad (3.4)$$

where δ is the thickness of the coating, N is the index of refraction, and μ, ϵ are the relative parameters of the coating.

The voltage reflection coefficients for such a surface can be represented in terms of η and the angle of incidence θ, by the following relations

$$R_{\perp} = \frac{\eta \cos\theta - 1}{\eta \cos\theta + 1} \qquad (3.5)$$

$$R_{\parallel} = \frac{\eta / \cos\theta - 1}{\eta / \cos\theta + 1} \qquad (3.6)$$

Let the real and imaginary parts of η be given by u and v, that is

$$\eta = u + iv$$

It can be shown that

$$\left| R_{\perp}(\theta) \right|^2 = \left[(u^2 + v^2)\cos^2\theta + 1 - 2u\cos\theta \right] \Big/ \left[(u^2 + v^2)\cos^2\theta + 1 + 2u\cos\theta \right] \qquad (3.8)$$

$$\left| R_{\parallel}(\theta) \right|^2 = \left[(u^2 + v^2) + \cos^2\theta - 2u\cos\theta \right] \Big/ \left[(u^2 + v^2) + \cos^2\theta + 2u\cos\theta \right] \qquad (3.9)$$

For further simplification, the parameters u and v will be replaced by x and y where

$$x = 2u \qquad (3.10)$$

$$y = u^2 + v^2 \qquad (3.11)$$

It then follows, provided that $R_\perp(0) \neq 0$ and $R_\parallel(0) \neq 0$,

$$r_\perp = \frac{\left[y\cos^2\theta + 1 - x\cos\theta\right]}{\left[y\cos^2\theta + 1 + x\cos\theta\right]} \; \frac{\left[y + 1 + x\right]}{\left[y + 1 - x\right]} \tag{3.12}$$

$$r_\parallel = \frac{\left[y + \cos^2\theta - x\cos\theta\right]}{\left[y + \cos^2\theta + x\cos\theta\right]} \; \frac{\left[y + 1 + x\right]}{\left[y + 1 - x\right]} \tag{3.13}$$

Performing algebraic manipulation, one can rewrite the above equations in the following form

$$\cos\theta x^2 + p_1\cos\theta(1 - \cos\theta)xy - \cos^2\theta y^2 + p_1(\cos\theta - 1)x - (1 + \cos^2\theta)y - 1 = 0 \tag{3.14}$$

$$\cos\theta x^2 + p_2(\cos\theta - 1)xy - y^2 + p_2\cos\theta(1 - \cos\theta)x - (1 + \cos^2\theta)y - \cos^2\theta = 0 \tag{3.15}$$

where

$$p_1 = \left[1 + r_\perp\right] \Big/ \left[1 - r_\perp\right] \tag{3.16}$$

$$p_2 = \left[1 + r_\parallel\right] \Big/ \left[1 - r_\parallel\right] \tag{3.17}$$

The quantities p_1 and p_2 are both real and are greater or equal to unity. The problem reduces to solving the two equations for the unknown quantities x and y, in terms of the parameters p_1 and p_2 which are obtained from the measured quantities $r_\perp$ and $r_\parallel$. The angle θ is of course known, being one-half the bistatic angle. However, the required solutions must lie in the first quadrant of the xy plane. The reason for this is twofold. First from the definition $u^2 + v^2 = y$, and the fact that u and v are real quantities, the required value of y must be greater or equal to zero. Secondly, it can be shown from energy considerations (the surface can only absorb energy), that $u \geqslant o$, implying that $x \geqslant o$. Both Eq. (3.14) and (3.15) represent conic sections, in the xy plane. The solutions are given by the intersections of these two conic sections. However, it is possible that there are no intersections, and if there are, they may lie outside the first quadrant. Thus, the nature of the conic sections will have to be further examined to indicate whether the appropriate solutions exist.

Consider a general conic section in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{3.18}$$

Its center is at the point (k, ℓ) where

$$k = \frac{fh - bg}{ab - h^2} \, , \quad \ell = \frac{gh - af}{ab - h^2} \tag{3.19}$$

By transforming the coordinate system (x, y) to a coordinate system (X, Y) centered at (k, ℓ) with the axis centered along the principal axis of the conic section, using the following relations

$$x - k = X\cos\beta - Y\sin\beta \tag{3.20}$$

$$y - \ell = X\sin\beta + Y\cos\beta \tag{3.21}$$

where

$$\tan 2\beta = \frac{2h}{a - b} \tag{3.22}$$

the equation of the conic section in the new coordinate system becomes

$$X^2A + Y^2B + \Delta / \left[ab - h^2\right] = 0 \tag{3.23}$$

where

$$2A = (a + b) + (a - b)\cos 2\beta + 2h\sin 2\beta \qquad (3.24)$$

$$2B = (a + b) - (a - b)\cos 2\beta - 2h\sin 2\beta \qquad (3.25)$$

$$D = abc + 2fgh - af^2 - bg^2 - ch^2 \qquad (3.26)$$

In the cases under consideration (a - b) is positive but h may be positive or negative. The angle $|2\beta|$ will be taken to be less than $\pi/2$, in which case

$$\cos 2\beta = (a - b) \Big/ \sqrt{4h^2 + (a - b)^2} \qquad (3.27)$$

$$\sin 2\beta = 2h \Big/ \sqrt{4h^2 + (a - h)^2} \qquad (3.28)$$

Thus 2A and 2B can be given by

$$2A = a + b + \sqrt{4h^2 + (a-b)^2} \qquad (3.29)$$

$$2B = a + b - \sqrt{4h^2 + (a - b)^2} \qquad (3.30)$$

in which case

$$AB = ab - h^2 \qquad (3.31)$$

Define the conic sections given (3.14) adn (3.15) by c_1 and c_2 respectively.

The various parameters associated with these conic sections are given in Table III-I below.

TABLE III-1

Parameters	c_1	c_2
$ab - h^2$	$-\cos^2\theta\left[\cos\theta + p_1^2(1-\cos\theta)^2/4\right]$	$-\left[\cos\theta + p_2^2(1-\cos\theta)^2/4\right]$
$k(ab - h^2)$	$-p_1\cos\theta\sin^2\theta(1+\cos\theta)/4$	$+p_2\sin^2\theta(1+\cos\theta)/4$
$\ell(ab - h^2)$	$1/2\cos\theta\left[1+\cos^2\theta - 1/2\, p_1^2(1-\cos\theta)^2\right]$	$1/2\cos\theta\left[1+\cos^2\theta - 1/2\, p_2^2(1-\cos\theta)^2\right]$
$\left.\begin{matrix}2A\\2B\end{matrix}\right\}$	$\cos\theta(1-\cos\theta)\left[1 \pm \sqrt{p_1^2 + H^2}\right]$	$(1-\cos\theta)\left[-1 \pm \sqrt{p_2^2 + H^2}\right]$
Δ	$1/4\cos\theta\sin^4\theta(p_1^2 - 1)$	$1/4\cos\theta\sin^4\theta(p_2^2 - 1)$
sign β	+	-

The parameter H is defined by the relation

$$H = (1 + \cos\theta)^2 \,/\, (1 - \cos\theta)^2 \tag{3.32}$$

It is seen that the centers of the conic section c_1 and c_2 lie in the right-half and left-half planes respectively. Also the equations of both conic sections can be written in the form

$$\frac{X^2}{A_1^2} - \frac{Y^2}{B_1^2} = 1$$

Further information can be found by examining the y-intercepts, the x-intercepts, and the asymptotes.

Conic section c_1 has y-intercepts

$$y_1^1 = -1 \text{ and } y_2^1 = -1/\cos^2\theta \tag{3.33}$$

and c_2 has y-intercepts

$$y_1^2 = -1 \text{ and } y_2^2 = -\cos^2\theta \tag{3.34}$$

The x-intercepts of c_1 and the slope of the asymptotes are related. If x_1 and x_2 are the x-intercepts, given by the relations

$$x_1^1 = \left[p_1(1 - \cos\theta) + \sqrt{p_1^2(1 - \cos\theta)^2 + 4\cos\theta}\right] \Big/ (2\cos\theta) \tag{3.35}$$

$$x_2^1 = -\left[x_1\cos\theta\right]^{-1}$$

then the equations for the asymptotes have the form

$$y = x_1^{\,1} x\,, \quad y = x_2^{\,1} x \tag{3.36}$$

There is a similar relationship between of the asymptotes of c_2 and the x-intercepts given by

$$x_1^{\,2} = 1/2\left[-p_2(1-\cos\theta) + \sqrt{p_2^{\,2}(1-\cos\theta)^2 + 4\cos\theta}\,\right] \tag{3.37}$$

$$x_1^{\,2} = -\cos\theta / x_1^{\,2} \tag{3.38}$$

In addition it can be shown that c_1 passes through the points

$$\left(\pm\left[1 + \frac{1}{\cos\theta}\right], \quad \frac{1}{\cos\theta}\right)$$

and c_2 through the points

$$\left(\pm\left[1 + \cos\theta\right], \quad \cos\theta\right)$$

The conic sections are shown in Fig. 3-1 for a typical case. As indicated there is an intersection in the first quadrant. Except for the case where $p_1 = \infty$ and $p_2 = \infty$ (ie. $r_\perp = r_\parallel = 1$), it can be shown that there will always be one intersection in the first quandrant. This follows from the fact that $1 < x_1^{\,1} \leqslant \infty$, and $o \leqslant x_1^{\,2} < 1$. Since $x_1^{\,1}$ and $x_1^{\,2}$ are the slopes of the asymptotes of the branches of the conic section in the first quadrant, these branches intercept, and since $x_1^{\,1}$ and $x_1^{\,2}$ are also the x-intercepts on the positive x-axis, the intersection is in the first quadrant. Thus a solution can be found for which $x \geqslant o$ and $y \geqslant o$. However, from relation (3.11) two values of v will be found. This means that the impedance will be determined apart from the sign of the imaginary part ie;

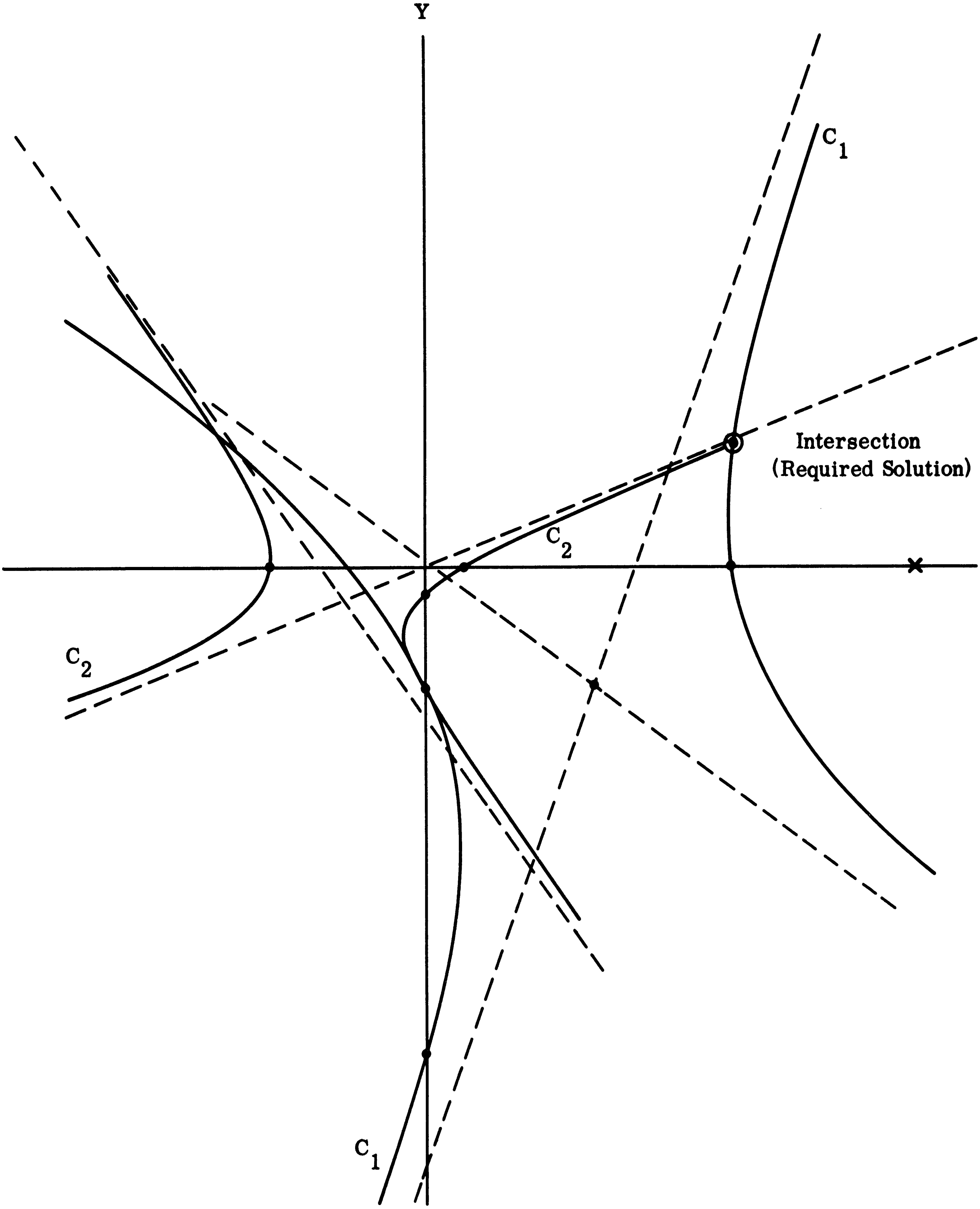

FIG. 3-1: CONIC SECTIONS ASSOCIATED WITH THE DETERMINATION OF SURFACE IMPEDANCE

$$\eta = u \pm iv$$

The determination of the appropriate sign will require measurement of the phases of the scattered field.

As a special case, it should be pointed out, that when $r_{\perp} = r_{\parallel} = 1$, the solution is not unique, with u = o and v undetermined. The most likely possible physical case that would occur in this instance is where v = o also, implying the surface is a perfect conductor.

REFERENCE

Weil, H., M.L. Barasch, and T.A. Kaplan, (1956) "Scattering of Electromagnetic Waves by Spheres," Studies in Radar Cross Sections X, University of Michigan Radiation Laboratory, Report 2255-20-T (July).

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13. ABSTRACT The use of Wilcox's recurrence relation method (described in the First Technical Report, 7644-1-T) of determining the body shape from the scattered far field data, is investigated in more detail. The problem of concern is the radius of convergence of the series. When the body is composed of sharp edges connected by smooth segments, with at least one of the principal curvature zero, the radius of convergence in this case is given by the radius of the smallest sphere enclosing the body. However, when the scattering body is a sphere whose center is at the origin of the coordinate system, the radius of convergence is vanishingly small.

The second term of Wilcox's expansion and its relationship to the size of the scatterer is investigated. Two special cases are considered; with the body a perfectly conducting flat plate, and a smooth convex shape giving rise to specular scattering. Generally, it is shown that the ratio of the second term to the first term in the expansion is the order of $1/2\ kD^2$ where D is the distance from the farthest point of the body to a line directed from the origin of the coordinate system to the receiver. For specular scattering, D is the distance from the specular point to the aforementioned line. Further investigation yielded the importance of phase, at least insofar as specular scattering was concerned. It is shown that the knowledge of phase information determined the location of the phase center of the specular point.

Further consideration of employing the monostatic-bistatic theorem to determine the material characteristics of the scatterer is undertaken. In particular, it is shown that the two polarization measurements of cross section at one non-zero bistatic angle and at the zero bistatic angle (backscattering) determines the reactive surface impedance of $\eta = u \pm iv$ apart from the sign in the imaginary part. Such surfaces would correspond to poor conductors, or absorber coated conductors. However, the case where the ratio of the bistatic to monostatic cross section is unity for both polarizations, produced incomplete results. In this case, it could only be concluded that u = 0.

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