Inverse Scattering Investigation

Final Report

1 October 1965 - 30 September 1966

by

V.H. Weston, J.J. Bowman and Ergun Ar

November 1966

Contract AF 19(628) - 4884

Prepared for

Electronic Systems Division, ESSXK
AFSC, USAF
Laurence G. Hanscom Field
Bedford, Massachusetts 01731

ABSTRACT

The problem in question consists of determining the means of solving the inverse scattering problem where the transmitted field is given and the received fields are measured, and this data is used to discover the nature of the target. The concept of equivalent sources is introduced wherein the scattered field may be thought of as arising from a set of equivalent sources located on or within the body. This concept is introduced since the radii of the minimum convex surface which encloses the equivalent sources is related to the convergence of any expansion technique utilized to derive the near scattered field of the target from the observed far field. One particular expansion technique is investigated. It is based upon the expansion of the field in the form of an inverse power series in r, multiplied by the factor exp (ikr), where (r, θ, ϕ) are the coordinates of a spherical polar coordinate system. The approach to the inverse scattering problem based upon the representation of the scattered field in terms of plane waves is investigated. An explicit expression for the scattered field, valid in the half-space which depends upon the coordinate axis, is given in terms of an integral operating on the far scattered field. It is shown that the plane wave representation converges part way inside smooth convex portions of the body, thus establishing the concept that the minimum convex shape enclosing the equivalent sources often may be inside the actual scattering body. For non-magnetic and non-perfectly conducting bodies, the exact total field inside the body could be represented in terms of a plane wave expansion involving the far field quantities. This representation involves an appropriate split-up of the far field data, and a fundamental problem still exists to uniquely determine the split-up from the knowledge of the far field data alone. The monostatic-bistatic theorem is used to determine the material characteristics of the scatterer.

TABLE OF CONTENTS

Table of Contents (cont'd)

LIST OF FIGURES

I
INTRODUCTION

The problem in question consists of determining means of solving the inverse scattering problem where the transmitted field is given and the received fields are measured, and this data is used to discover the nature of the target. In connection with this, a review of the literature is first given in Chapter II.

In Chapter III, the concept of equivalent sources is introduced wherein the scattered field may be thought of as arising from a set of equivalent sources located on or within the body. This concept is introduced since the radii of the minimum convex surface which encloses the equivalent sources is related to the convergence of any expansion technique utilized to derive the near scattered field of the target from the observed far field. Thus in some cases the expansions may be convergent part way inside the body. (This is investigated in more detail in Chapter VIII.) It is shown in the particular case of the sphere that the expansion for the scattered field is convergent down to the center.

In Chapter IV, one particular expansion technique is investigated. It is based upon the representation of the field in the form

$$\underline{E} = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{\underline{E}_n(\theta,\phi)}{r^n}$$

where (r,θ,ϕ) are the coordinates of a spherical polar coordinate system. The leading term given by $n = 0$ is the observed far field, from which the remaining terms can be derived through a set of recurrence relations. This expansion is convergent outside the minimum sphere enclosing the equivalent sources. By changing the origin of the coordinate system one would obtain an expansion outside a different minimum sphere. Thus by repeatedly changing the origin of the coordinate system and obtaining the associated expansions, one can obtain expressions for the near scattered field everywhere outside of a minimum convex shape enclosing the equivalent sources.

In connection with the above expansion, the relationship to the size of the scatterer and the second term $\underline{E}_1$ is investigated. It is shown generally that the ratio of the second term to the first term is in the order of $1/2\,kD^2$, where D is the distance from the farthest point of the body to a line directed from the origin of the coordinate system to the receiver. For specular scattering it is shown that on changing the origin of the coordinate system to lie on the "ray" directed from the specular point to the observation point, and at the effective phase center, the second term $\underline{E}_1$ can be made quite small.

In Chapters V through IX, the approach to the inverse scattering problem is based upon the representation of the scattered field in terms of plane waves. An explicit expression for the scattered field, valid in a half-space which depends upon the coordinate axis, is given in terms of an integral operating on the far scattered field. By rotation of axes the same expression can be used to find the near field everywhere outside the minimum convex shape enclosing the equivalent sources. It is shown that the plane wave representation is ideally suited for high frequency in which case the integral representation may be directly evaluated for many cases by means of stationary phase of other asymptotic techniques. This is demonstrated for specular scattering.

In Chapter VIII it is shown that the plane wave representation converges part way inside smooth convex portions of the body, thus establishing the concept that the minimum convex shape enclosing the equivalent sources often may be inside the acutal scattering body.

For non-magnetic and non-perfectly conducting bodies, it is shown in Chapter IX that the exact total field inside the body could be represented in terms of a plane wave expansion involving the far field quantities. This representation involves an appropriate split up of the far field data, and a fundamental problem still exists to uniquely determine the split up from the knowledge of the far field data alone. This indicates that additional information will be needed; perhaps knowledge of the complete scattering matrix for all frequencies. This is an important problem since its solution will yield both the shape and material of the body. In connection with representations which are valid in the interior of the minimum convex shape enclosing the equivalent sources, a practical representation is given in the second half of Chapter IX. This representation, based on an expansion in terms of spherical harmonics and Bessel functions, can be used to find the scattered field in cavity regions that penetrate the body or the minimum convex shape.

In any numerical approach where one is given the far field and obtains the near scattered field by use of one of the many representation, the problem remains to determine the shape of the body. If it is assumed that the body is a perfect conductor, then one has to find the surface for which the total tangential electric field vanishes. A necessary (but not sufficient) condition is given in Chapter X, for a point to be on the surface of a perfect conductor. This condition requires a simple calculation involving the total electric field at a point.

In Chapter XI the monostatic-bistatic theorem is used to determine the material characteristics of the scatterer. In particular, it is shown that two polarization measurements of cross section at one non-zero bistatic angle and at the zero bistatic angle (backscattering) determine the relative surface impedance of $\eta = u \pm iv$ apart from the sign in the imaginary part. Such surfaces would correspond to poor conductors, or absorber coated conductors. However, the case where the ratio of the bistatic to monostatic cross section is unity for both polarizations produced incomplete results. In this case, it could only be concluded the $u = 0$.

II

REVIEW OF INVERSE SCATTERING TECHNIQUES

There seems to be a variety of problems that are named "the inverse scattering problem" and what follows is certainly not a complete account of the existing literature on the subject.

Connected with the inverse scattering problems is the question of to what extent knowledge of the far-field from sources of finite extension determines the distribution of the same sources. The relationship has been investigated by Müller (1954, 1956) for the scalar and vector case respectively. The results are that the far-field determines the radius of the smallest sphere such that the far-field can be generated by sources all of which are inside the sphere. Furthermore, if the far-field vanishes the total field is identically zero out-side every region, such that it contains all sources. These results also follow from an expansion theorem given by Wilcox (1956). The statements are also true in two dimensions where an expansion theorem is due to Karp (1961). It should be noted that the smallest sphere mentioned above does not necessarily determine the extension of the real sources. For instance, if the sources are distributed over a certain volume in such a manner that the far-field can be expanded in a finite number of surface harmonics, an identical far-field can be obtained from a number of multipoles, i.e. from sources inside an infinitesimal sphere around the origin.

Some acoustic and electromagnetic scattering problems can be formulated in terms of the Schrödinger equation. A group of one-dimensional problems has been treated by Moses and deRidder (1963), and a three-dimensional scalar problem by Kay (1962). The physical problem considered by Kay is to find the variation of electron density in a weakly ionized gas from a knowledge of the scattering amplitude resulting from the incidence of a plane electromagnetic wave. However, his results seem to be applicable to scalar scattering by a plane wave from an arbitrary isotropic body. Instead of the ordinary time-independent Schrödinger equation, Kay considers the integro-differential equation

$$\Delta u + k^2 u - \int v(\underline{x}, \underline{x}') \, u(\underline{x}', \underline{k}) \, d\underline{x}' = 0 \qquad (2.1)$$

which takes the form of the ordinary Schrödinger equation if v is a distribution of the form $v(\underline{x}, \underline{x}') = V(x)\,\delta(\underline{x}-\underline{x}')$. He required knowledge of the scattering amplitude over a hemisphere from a wave incoming from the same half-space in an arbitrary direction for all values of k to determine the function $v(\underline{x}, \underline{x}')$. The question of uniqueness and existence of $v(\underline{x}, \underline{x}')$ under any general condition is not touched upon. Instead, a particular condition on the solution $u(\underline{x}, \underline{k})$ is introduced which leads to a unique $v(\underline{x}, \underline{x}')$.

An extensive bibliography, to the date of publication, of the quantum mechanics inverse scattering problem is given by Faddeyev (1963).

A three-dimensional scalar problem is also treated by Petrina (1960). The scattering body is there assumed to be homogeneous and isotropic so the Helmholtz equation with a wave number k_1 is satisfied inside the body and the same equation with a wave number k_0 is satisfied outside the body. Petrina gives the following relation between the scattering amplitude and the shape of the scattering body.

$$\left. \frac{\partial f(k_0, k_1, \underline{\tau})}{\partial (k_0^2)} \right|_{k_0^2 = k_1^2} = -\frac{1}{4\pi} \int_B e^{i \underline{\tau} \cdot \underline{y}} \, d\underline{y} \ . \qquad (2.2)$$

The integration is to be performed over the volume of the scattering body and

$$\underline{\tau} = \underline{k_0} - \frac{k_0}{|\underline{x}|} \underline{x} ,$$

where $\underline{k}_0$ is the wave vector of the incident plane wave and $\underline{x}$ is in the direction of observation. The integral on the right hand side of (2.2) can be considered as the Fourier transform of a function which takes the value 1 inside the scatterer and vanishes outside. Thus, knowledge of the left hand side for all $\underline{\tau}$ determines the shape of the scatterer. However, this means determination of the behavior of the scattering amplitude when the wave constant of the surrounding medium is changed, which is not measurable in a physical situation.

Some results for two-dimensional acoustically soft or hard obstacles is given by Karp (1961). He forms determinants whose elements are f_{ij} where $f_{ij} = f(\theta_i, \theta_j)$ is the scattering amplitude at an angle of observation θ_i for an angle of incidence θ_j of the plane incoming wave. Necessary and sufficient conditions for a point to be on the surface of the scatterer is thus derived for the special case that $\det(f_{ij})$ vanishes, where $\theta_i = \theta_1, \theta_2, \ldots \theta_n$ are n different angles. Furthermore, it is shown that if $f(\theta, \theta_0)$ only depends on the difference $\theta - \theta_0$, the scatterer must be a circle.

The inverse scattering problem in geometrical optics has been investigated by Keller (1959). If the scattering amplitude and reflection coefficient are known, explicit formulas determining the illuminated part of the surface can be obtained for two-dimensional problems. In the three-dimensional case the bistatic radar cross section is proportional to the reflection coefficient and the product of the principal radii of curvature R_1, R_2 at the point of reflection. The problem of determining a surface when its Gaussian curvature, $G = 1/(R_1 R_2)$, for all directions of the normal to the surface is given, is known as Minkowski's problem It has a unique solution for any sufficiently smooth convex body (c.f. Nirenberg (1953)). If the differential scattering cross section is known for two different incident waves coming from opposite directions and the reflection coefficient is also known, the Gaussian curvature is determined everywhere and the inverse problem has a unique solution. It also follows that the problem of determining the shape of scatterer from knowledge of the backscattering cross section in all directions has a unique solution in the geometrical optics formulation for a smooth convex body.

The geometric optics method no longer applies of the scattering body has any section where one principal radius of curvature is infinite. For a body of revolution where this is the case or, where the radius of the cross section varies slowly along the axis of revolution, an approximate method due to Blasberg is described in Altman et al (1964). Using the physical optics approximation the backscattered far field is shown to be proportional to the Fourier transform of the function $r(x)e^{-ikr(x)}$ at the point $k \sin \theta$, where $r(x)$ is the radius of the cross section as a function of a coordinate x along the axis of revolution. The relation is valid for small values of θ where $\pi/2-\theta$ is the angle between the direction of propagation of the incident plane wave and the axis of revolution of the body. Consequently, if a substantial part of the backscattering is confined to small angles, the inverse Fourier transform of the scattered far field with respect to $d = k \sin \theta$ integrated over $\theta = -\pi/2$ to $\theta=\pi/2$ will be a function which is close to $r(x)e^{-ikr(x)}$ for x values inside the body and close to zero for points outside. According to Brindley (1965) the Blasberg approximation has been successfully used to determine the shape of objects from empiric data.

Another theory of high frequency scattering is employed by Freedman (1963). There the incoming wave consists of a modulated pulse and the scattered field in an arbitrary direction in the lit region is a superposition of pulses of the same form. Each discontinuity in

$$\frac{d^n A(R)}{dR^n} \quad (n = 0, 1, 2, \ldots)$$

where A(R) is the projection towards the transmitter of those parts of the scatterer which are within distance R, is assumed to generate a component towards the scattered signal. The magnitude of each scattering component is proportional to the size of its generating discontinuity. A more sophisticated treatment of the impulse response from a finite object is given by Kennaugh and Moffat (1965).

III
THE CONCEPT OF EQUIVALENT SOURCES

3.1 Equivalent Sources

The scattered field may be thought of as produced by a set of equivalent sources located on or within the surface of the scattering body. These sources are not necessarily the same as the measurable currents induced, for example, on the surface of a conducting object, but are related to the mathematical representation of the scattered field in the space exterior to the body. As we shall see, there are various methods for obtaining field representations valid everywhere outside the scattering object from a knowledge of the complete radiation pattern of the scattered field. In addition, however, these representations may also be analytic in a region inside the scattering surface, in which case the resultant field expression there may be conceived of as being produced by fictional sources in the absence of the actual body. At points outside the geometrical surface of the body, the field produced by these fictional sources is identical to the scattered field of the body; the sources are therefore called equivalent sources for the scattered field.

The equivalent sources are not, in general, unique--although they are confined to some finite region of space. The problem of determining the location and extent of this region is fundamental to the inverse scattering investigation and, of course, is intimately connected with the radius of convergence of the mathematical field representation.

3.2 Scattering by a Sphere

The problem of scattering by a sphere provides an example in which the formal series solution for the scattered field converges absolutely and uniformly inside as well as outside the body. To show this, it is sufficient to prove convergence for the radial components of the field; convergence of the remaining components follows easily. Thus consider the Mie series for the radial electromagnetic field

$$\underline{r} \cdot \underline{H}^{sc} = -\frac{\sin \phi}{ik} \sum_{n=1}^{\infty} i^n (2n+1) a_n h_n(kr) P_n^1(\cos \theta),$$

$$\underline{r} \cdot \underline{E}^{sc} = -\frac{\cos \phi}{ik} \sum_{n=1}^{\infty} i^n (2n+1) b_n h_n(kr) P_n^1(\cos \theta), \qquad (3.1)$$

where, for a perfectly conducting sphere of radius a, the coefficients a_n and b_n are given by

$$a_n = -\frac{j_n(ka)}{h_n(ka)}$$

$$b_n = -\frac{\left[(ka) j_n(ka)\right]'}{\left[(ka) h_n(ka)\right]'} \qquad (3.2)$$

The prime denotes differentiation with respect to (ka). The incident plane wave is assumed to have unit intensity and the free-space constants ϵ_o, μ_o are taken to be unity.

Simplification is obtained by noting the inequality (Weil et al, 1956)

$$\left| \frac{P_n^1(\cos \theta)}{\sin \theta} \right| \leq \frac{n(n+1)}{2} \quad \text{for } \theta \leq \pi \qquad (3.3)$$

which leads to

$$\left| \underline{r} \cdot \underline{H}^{sc} \right| \leq \frac{1}{2k} \sum_{n=1}^{\infty} n(n+1)(2n+1) \left| a_n h_n(kr) \right| ,$$

$$\left| \underline{r} \cdot \underline{E}^{sc} \right| \leq \frac{1}{2k} \sum_{n=1}^{\infty} n(n+1)(2n+1) \left| b_n h_n(kr) \right| . \tag{3.4}$$

Employing the following asymptotic approximations as $n \to \infty$

$$j_n(x) \sim \frac{x^n}{(2n+1)!!} , \quad h_n(x) \sim -i \frac{(2n-1)!!}{x^{n+1}} \tag{3.5}$$

where $(2n+1)!! = (2n+1)(2n-1)\ldots 3.1$, one finds

$$\left| a_n h_n(kr) \right| \sim \left| b_n h_n(kr) \right| \sim \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} . \tag{3.6}$$

It follows that the above series converge uniformly in r for all $r \geq \delta > 0$. Finally, with the aid of the above inequality concerning the Legendre polynomial and the following inequality (Weil et al, 1956)

$$\left| \frac{dP_n^1(\cos\theta)}{d\theta} \right| \leq \frac{n(n+1)}{2} \quad \text{for } \theta \leq \pi , \tag{3.7}$$

the remaining components of the scattered field may similarly be shown to converge absolutely and uniformly for all $r \geq \delta > 0$.

In the case of a homogeneous dielectric sphere with relative material parameters ϵ and μ, we find for large n

$$\left| a_n h_n (kr) \right| \sim \left| \frac{\mu - 1}{\mu + 1} \right| \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} ,$$

$$\left| b_n h_n (kr) \right| \sim \left| \frac{\epsilon - 1}{\epsilon + 1} \right| \frac{a}{r} \left(\frac{ka^2}{r} \right)^n \frac{1}{(2n+1)!!} , \qquad (3.8)$$

and convergence for $r \neq 0$ is again evident. The result may be extended to include the case of concentric spheres of different dielectric materials.

The field scattered by a sphere may thus be conceived of as a superposition of electric and magnetic multipole fields whose sources are located at a single point--the center of the sphere. Upon deletion of this point, the field representation for the scattered field Eq. (3.1) is analytic everywhere in space.

IV
THE WILCOX EXPANSION

4.1 Determination of the Near Field from the Far Field.

A problem of fundamental significance in inverse scattering theory is that of determining the electromagnetic field at all points in space from a knowledge of the field in the far zone. In this regard, the expansion theorem due to Wilcox (1956) may be employed, once the far-field pattern is known, to determine the field everywhere in the region outside the smallest sphere enclosing the sources of the field. These sources may be thought of as equivalent sources for some scattered field or as real sources for some radiation field. In connection with applications to inverse scattering, a problem of major concern is the radius of convergence of the Wilcox expansion. The radius of convergence may yield important information concerning the extent and location of the equivalent sources for the scattered field.

The far-zone scattered field will be represented in the form

$$\underline{E}^{sc}(r,\theta,\phi) \underset{r \to \infty}{\sim} \frac{e^{ikr}}{r} \underline{E}_o(\theta,\phi) \qquad (4.1)$$

where (r,θ,ϕ) is a spherical polar coordinate system with the origin in the vicinity of, or in the interior of, the body. Let r, θ, ϕ denote the unit vectors associated with the coordinate system, then the complete radiation pattern (both phase and amplitude) takes the form

$$\underline{E}_o = E_o^2 \hat{\theta} + E_o^3 \hat{\phi} \quad . \qquad (4.2)$$

From Wilcox (1956) it is seen that the fields can be uniquely determined in the region outside the smallest sphere enclosing a set of equivalent sources. To be more precise, let r = c be the radius of the smallest sphere (with the center at the origin of the coordinate system) enclosing a set of sources. Then $\underline{E}$ can be expressed in the form

$$\underline{E} = e^{ikr}/r \sum_{n=0}^{\infty} \frac{\underline{E}_n(\theta,\phi)}{r^n} \quad , \quad r > c + \epsilon \tag{4.3}$$

where ϵ is an arbitrary small number greater than zero. The higher order coefficients $\underline{E}_n$ are uniquely determined from the following recurrence relations

$$\begin{aligned}
ikE_1^1 &= -r\, \underline{\nabla} \cdot \underline{E}_o \\
2iknE_{n+1}^1 &= n(n-1)E_n^1 + D E_n^1 \ , && n = 1,2,3\ldots \\
2iknE_n^2 &= n(n-1)\, E_{n-1}^2 + D E_{n-1}^2 + D_\theta \underline{E}_{n-1} && n = 1,2,3\ldots \\
2iknE_n^3 &= n(n-1)\, E_{n-1}^3 + D E_{n-1}^3 + D_\phi \underline{E}_{n-1} && n = 1,2,3\ldots
\end{aligned} \tag{4.4}$$

where the operators D, D_θ and D_ϕ are defined by

$$\begin{aligned}
Df &= \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\phi^2} \ , \\
D_\theta \underline{F} &= \frac{2\partial F^1}{\partial\theta} - \frac{1}{\sin^2\theta} F^2 - \frac{2\cos\theta}{\sin^2\theta} \frac{\partial F^3}{\partial\phi} \ , \\
D_\phi \underline{F} &= \frac{2}{\sin\theta} \frac{\partial F^1}{\partial\phi} + \frac{2\cos\theta}{\sin^2\theta} \frac{\partial F^2}{\partial\phi} - \frac{1}{\sin^2\theta} F^3 \ .
\end{aligned} \tag{4.5}$$

Thus, complete knowledge of the far field pattern implies the determination of the field outside the smallest sphere enclosing an equivalent source.

The next problem is to relate this result to the scattering body. Consider a body composed, at the present, of arbitrary material. Let the exterior surface of the body be given by S. A plane wave of harmonic time dependence $\exp(-i\omega t)$ will be assumed to be incident upon the body generating a scattered field. The scattered field at a point $\underline{x}$ outside the body can be represented in terms of the total field generated on the surface as follows,

$$\underline{E}^{sc}(\underline{x}) = -\frac{1}{4\pi}\int_S \left[i\omega\mu_0(\underline{n}\wedge\underline{H})\, g + (\underline{n}\wedge\underline{E})\wedge\nabla' g + (\underline{n}\cdot\underline{E})\nabla' g\right] ds' \qquad (4.6)$$

where $\underline{n}$ is the unit outward normal to the surface and the Green's function g is

$$g = e^{ikR}/R \qquad (4.7)$$

with $R = \left|\underline{x} - \underline{x}'\right|$, the vector $\underline{x}'$ being a point of integration on the surface.

Employing the expansion

$$e^{ikR}/R = ik \sum_{n=0}^{\infty} (2n+1)\, j_n(kr')\, h_n^{(1)}(kr) P_n(\cos\gamma) , \qquad (4.8)$$

one can derive the following expansion for g in terms of r, where r is the distance from the origin to the point $\underline{x}$,

$$g = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{A_n}{r^n} \qquad (4.9)$$

where

$$A_n = \frac{1}{(2k)^n (n)!} \sum_{p=0}^{\infty} (2n+2p+1)(-i)^p \frac{(2n+p+1)}{(p+1)}\, j_{n+p}(kr') P_{n+p}(\cos\gamma) \qquad (4.10)$$

with

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \quad . \tag{4.11}$$

The angle γ is the angle between the two vectors $\underline{x}$ and $\underline{x}'$.

In particular it can be shown that the scalar quantities A_n obey recursion formulas

$$A_0 = \exp\left[-ikr'\cos\gamma\right] \tag{4.12}$$

and

$$2iknA_n = \left[n(n-1) + D\right] A_{n-1}$$

where D is the differential operator defined previously in Eq. (4.5). Investigating the behavior of A_n for $n \to \infty$, one can show that the expansion given by Eq. (4.9), is uniformly convergent for $r > r' + \delta > r'$. From the relationship

$$\underline{\nabla}' g = -\underline{\nabla} g \tag{4.13}$$

where the prime indicates differentation with respect to the source variable $\underline{x}'$, one obtains

$$\nabla' g = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{\underline{B}_n}{r^n} \tag{4.14}$$

where the vector quantity $\underline{B}_n$ is derived from A_n by means of the relation

$$\underline{B}_n = \hat{r}\left[-ik A_n + nA_{n-1}\right] - \hat{\theta}\frac{\partial A_{n-1}}{\partial\theta} - \hat{\phi}\frac{1}{\sin\theta}\frac{\partial A_{n-1}}{\partial^n\phi} \quad . \tag{4.15}$$

Interchanging the order of summation and integration it follows that expression Eq. (4.6) can be represented in the form of a Wilcox expansion

$$\underline{E}(\underline{x}) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{\underline{E}_n(\theta, \phi)}{r^n} \tag{4.16}$$

with $\underline{E}_n(\theta, \phi)$ given explicitly by

$$\underline{E}_n = -\frac{1}{4\pi} \int_S \left[i\omega\mu_o(\underline{n} \wedge \underline{H})A_n + (\underline{n} \wedge \underline{E}) \wedge \underline{B}_n + (\underline{n} \cdot \underline{E})\underline{B}_n \right] ds' \tag{4.17}$$

provided that $r > r'$ for all values of r' associated with the points of integration.

Expansion Eq. (4.16) is uniformly convergent for all points outside the minimum sphere enclosing the body, and represents the scattered field outside this sphere. However, it is possible that the expansion may be uniformly convergent inside the minimum sphere, in which case it will represent there the field produced by an equivalent source. The problem requiring investigation is to obtain an estimate of the radius of convergence of (4.16). In the case of perfectly conducting bodies with induced edge or tip singularities, the expansion will not converge inside the minimum sphere containing these sharp corners or edges. Thus, if the scattered field is singular on sufficient portions of the surface of the body, then the radius of convergence of the expansion in this case will determine the radius of the minimum sphere enclosing the body itself. On the other hand, for smooth convex shapes, convergence inside the minimum sphere enclosing the body is expected. As we have seen, the problem of scattering by a sphere furnishes an example in which the formal series solution for the scattered field converges absolutely and uniformly inside, as well as outside, the body.

It was shown that for a fixed origin, Wilcox's expansion represented the scattered field only outside the minimum sphere (with center at the origin) enclosing the body. By changing the origin of the coordinate system, one will get a new minimum sphere enclosing the body, outside of which Wilcox's expansion will represent the scattered field. By considering a sequence of translations of the origin, a sequence of minimum spheres enclosing the body will be obtained, such that the envelope will be a convex shape enclosing the body. On considering the sequence of minimum spheres, Wilcox's expansion will give the scattered field outside the convex envelope enclosing the body. Thus only for convex scattering shapes, can one obtain by Wilcox's expansion alone, the scattered field everywhere outside the body.

Assuming, however, that the Wilcox expansion converges inside the minimum sphere enclosing the body, it should be possible, by moving the origin of coordinates, to generate a minimum convex shape within the body such that the field is analytic outside this minimum region. The equivalent sources of the field lie within such a region.

For smooth convex scattering shapes, then, it is expected that the Wilcox expansion can be employed to determine a minimum source region inside the body. Knowing the field everywhere outside this region, the problem reduces to seeking a scattering surface for a particular boundary condition. In case of perfect conductors, for example, one would look for the surface on which the sum of the tangential components of the scattered and incident electric field vectors vanishes. The requirement that the surface remain unchanged as the frequency varies is sufficient to determine a unique perfect conductor. A similar technique could be used for bodies whose material properties can be represented in terms of an impedance boundary condition of the form

$$\underline{E} - (\underline{E} \cdot \underline{n})\, \underline{n} = \eta \sqrt{\frac{\mu_0}{\epsilon_0}}\; \underline{n} \wedge \underline{H} \tag{4.18}$$

where $\underline{n}$ is the unit outward normal to the surface of the body, and η is a parameter depending upon material characteristics.

4.2 Specular Scattering from a Convex Shape.

It is of interest to know for what value of r the second term in the Wilcox expansion is the order of the leading term. In this section the second term and its relationship to the size of the scatterer is investigated. The case will be considered where the dominant scattered field in a particular angular sector arises from a single scattering center such as a specular point on a smooth convex body. Assuming perfect conductivity, the first two terms in the expansion

$$\underline{E}(r,\theta,\phi) = \frac{e^{ikr}}{r} \sum \frac{\underline{E}_n}{r^n} \tag{4.19}$$

are given by

$$\underline{E}_o = -\frac{1}{4\pi} \int_S \left[i\omega\mu_o (\underline{n} \wedge \underline{H}) - ik(\underline{n} \cdot \underline{E}) \hat{r} \right] A_o \, ds' \tag{4.20}$$

$$\underline{E}_1 = -\frac{1}{4\pi} \int_S \left[i\omega\mu_o (\underline{n} \wedge \underline{H}) - ik(\underline{n} \cdot \underline{E}) \hat{r} \right] A_1 ds' \tag{4.21}$$

$$- \frac{1}{4\pi} \int_S \left[\hat{r} A_o - \hat{\theta} \frac{\partial A_o}{\partial \theta} - \hat{\phi} \frac{1}{\sin\theta} \frac{\partial A_o}{\partial \phi} \right] \underline{n} \cdot \underline{E} \, ds'$$

where

$$A_o = \exp\left[-ikr' \cos\gamma\right]$$

$$A_1 = A_o \left[\frac{1}{2} ikr'^2 \sin^2\gamma + r' \cos\gamma\right] . \tag{4.22}$$

The integration is over the surface of the body. In the above expressions, r' is the distance from the origin to the point of integration on the surface of the body S, and γ is the angle between the radius vectors directed to the observation point (r,θ,ϕ) and the point of integration (r',θ',ϕ') and is given by the relation

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

Since we are considering specular scattering, the dominant contribution of the above integrals will arise from a small region about the specular point (r'_s,θ'_s,ϕ'_s). The coordinates of the specular point depend upon the position of the observation point (r,θ,ϕ).

Since the dominant contribution of the integrals arises from the vicinity of the specular point (r'_s,θ'_s,ϕ'_s), and can be evaluated by the method of stationary phase, it follows for $kr'_s \sin\alpha_s > 1$, that

$$\underline{E}_1 \sim \frac{ik}{2} (r'_s \sin\gamma_s)^2 \, \underline{E}_o \quad . \tag{4.23}$$

This indicates that the value of r for which $\left|E_1\right| \sim r \left|E_o\right|$ is given approximately by

$$r_o \sim \frac{1}{2} kd^2 \tag{4.24}$$

where d is the distance from the specular point to the line directed from the origin to the observation point (r, θ, ϕ). (A more precise value of r_o is given below.)

4.3 Effect of Changing the Origin of the Coordinate System

For a general body with dimensions much greater than a wavelength, and such that the scattered field is comprised of components arising from many scattering centers, it follows in the same manner as indicated in Section 4.2, that

$$\left| E_1 \right| \sim r \left| E_o \right| \quad \text{when}$$

$$r \sim 0 \; (1/2\, kD^2). \tag{4.25}$$

Here D is the distance from the farthest point of the body to the line directed from the origin to the observation point. The distance D is a function of (θ, ϕ) varying with changing position of the far field observation point.

Changing the origin of the coordinate system will increase or decrease the distance D, thus effectively increasing or decreasing the ratio of the second term to the first term in Wilcox's expansion. The effect of changing the origin of the coordinate system of the far field pattern is to produce an additional phase factor. This can be seen as follows. Let the origin of the coordinate system (r_i, θ_i, ϕ_i) be displaced a distance $\underline{\ell}$, resulting in a new coordinate system (r, θ, ϕ). For a point in the far field $r \sim r_i - \underline{i}_r \cdot \underline{\ell}$, $\theta \sim \theta_i$, and $\phi \sim \phi_i$. Thus if the far field pattern (phase and amplitude) is given by

$$\frac{e^{ikr_i}}{r_i} \; \underline{E}_o(\theta_i, \phi_i) \tag{4.26}$$

in the initial coordinate system, it will be given by

$$\frac{e^{ikr}}{r} \; \underline{E}_o(\theta, \phi) \exp(-ik\, \hat{r} \cdot \underline{\ell}) \tag{4.27}$$

in the final system. If the vector $\underline{\ell}$ has components

$$(-\ell \sin\theta_\ell \cos\phi_\ell \ , \ -\ell \sin\theta_\ell \sin\phi_\ell \ , \ -\ell \cos\theta_\ell) \tag{4.28}$$

with regard to the final coordinate system, then

$$\hat{r} \cdot \underline{\ell} = -\ell \cos\gamma_\ell = -\ell \left[\cos\theta \cos\theta_\ell + \sin\theta \sin\theta_\ell \cos(\phi-\phi_\ell)\right]. \tag{4.29}$$

For further illustration of the effect of changing the origin of the coordinate system, we will return again to the case of specular scattering. Using the notation of Section 4.2, $(r'_s, \theta'_s, \phi'_s)$ will refer to the specular point on the body, giving rise to the scattered field at the ray (θ, ϕ) in the far field. The dominant portion of the integral (4.20) arises from the vicinity of the specular point. It can be shown that

$$\underline{E}_o(\theta,\phi) \sim \underline{\mathcal{E}} \, \text{expik} \left[\underline{k}_i \cdot \underline{r'}_s - \underline{r'}_s \cdot \underline{r}_o\right] \tag{4.30}$$

where $\underline{k}_i$ is the unit vector indicating the direction of incident propagation, and $\underline{r}_o$ is the unit vector directed from origin to the receiver or observation point (r, θ, ϕ). The amplitude factor $\underline{\mathcal{E}}$ is a slowly varying function of θ and ϕ. For simplification the direction of the axis of the coordinate system will be chosen so that $\underline{k}_i = -\underline{i}_z$.

The phase factor

$$g(\theta,\phi) = \underline{r'}_s \cdot \left[\underline{r}_o - \underline{k}_i\right] \tag{4.31}$$

can be shown to have the form

$$g(\theta,\phi) = 2 \cos\frac{\theta}{2} \ \underline{r'}_s \cdot \underline{n}_s \tag{4.32}$$

where $\underline{n}$ is the unit outward normal at the specular point (see Fig. 4-1). Since the position of the specular point is a function of θ and ϕ, it follows that $\underline{r}'_s$ and $\underline{n}_s$ are functions of θ and ϕ also.

$$\frac{\partial g}{\partial \theta} = \underline{r}'_s \cdot \hat{\theta} \tag{4.33}$$

$$\frac{\partial g}{\partial \phi} = \sin\theta \, \underline{r}'_s \cdot \hat{\phi} \tag{4.34}$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\cos\frac{\theta}{2}}{2G\rho_2} - \underline{r}'_s \cdot \hat{r} \tag{4.35}$$

$$\frac{\partial^2 g}{\partial \phi^2} = -\sin\theta \left[\sin\theta \, \hat{r} + \cos\theta \, \hat{\theta}\right] \cdot \underline{r}'_s + \sin\frac{\theta}{2} \sin\theta \frac{1}{\rho_1 G} \tag{4.36}$$

where (r, θ, ϕ) are the unit vectors associated with the spherical polar coordinate system at the far field observation point (r, θ, ϕ). In these equations G is the Gaussian curvature of the surface at the specular point, and ρ_1 and ρ_2 are the radii of curvature of the surface at the specular point in the ($\underline{k}_i$, $\underline{r}_o$) plane and perpendicular to this plane respectively.

Setting

$$\underline{\mathcal{E}} = \mathcal{E}^2 \hat{\theta} + \mathcal{E}^3 \hat{\phi} \tag{4.37}$$

the leading term in Wilcox's expansion has the form

$$\underline{E}_o = \mathcal{E}^2 e^{-ikg} \hat{\theta} + \mathcal{E}^3 e^{-ikg} \hat{\phi} \quad . \tag{4.38}$$

Using relation (4.4) together with Eqs. (4.33) to (4.36), it can be shown that the θ component of the second term in Wilcox's expansion has the form

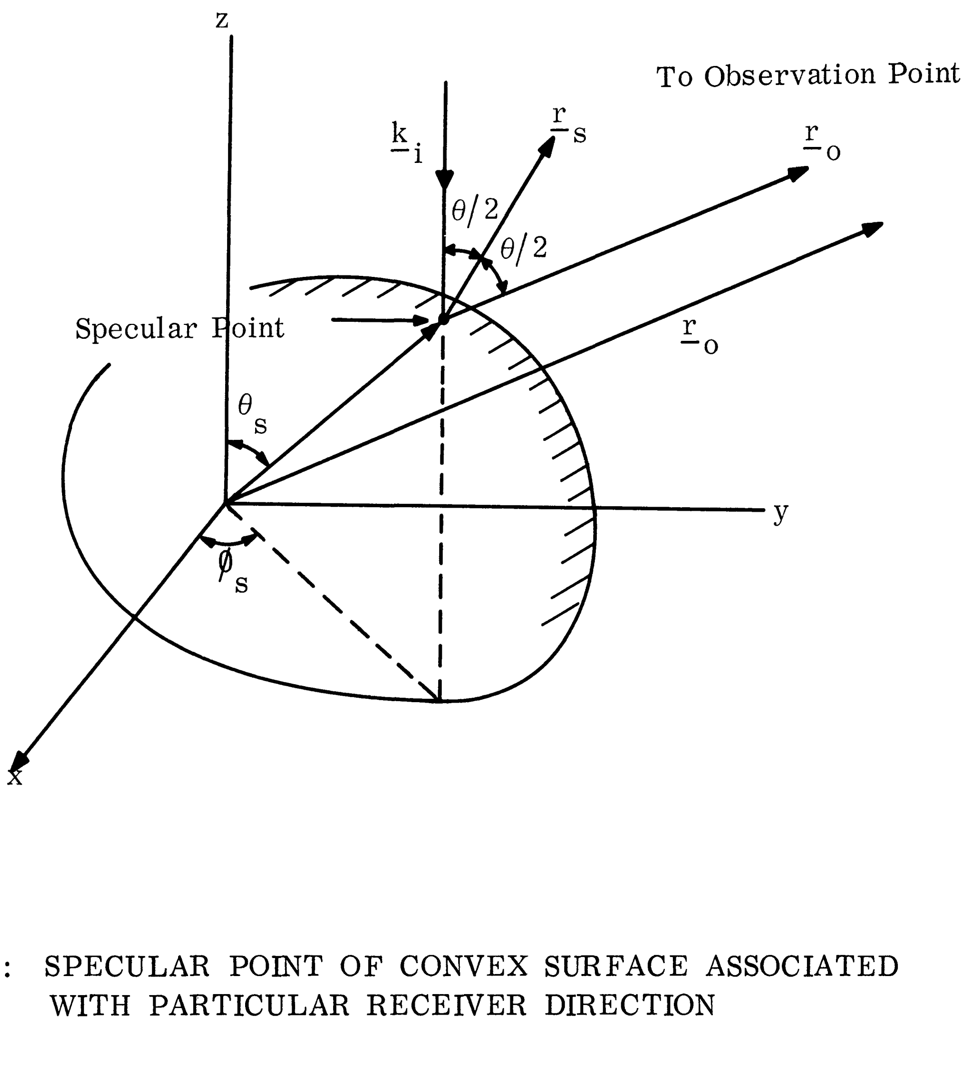

FIG. 4-1: SPECULAR POINT OF CONVEX SURFACE ASSOCIATED WITH PARTICULAR RECEIVER DIRECTION

$$2ikE_1^{\,2} = -E_o^{\,2}\left[(kd)^2 + ik\left(\frac{\cos\theta/2}{2G\rho_2} + \frac{\sin\theta/2}{G\rho_1\sin\theta} - 2\underline{r}'_s\cdot\hat{r}\right)\right]$$

$$+ e^{-ikg}\left[D\mathcal{E}^2 - \frac{\mathcal{E}^2}{\sin^2\theta}\right] - 2ik\left[\frac{\partial g}{\partial\theta}\frac{\partial\mathcal{E}^2}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial g}{\partial\phi}\frac{\partial\mathcal{E}^2}{\partial\phi}\right]e^{-ikg}$$

$$-2\frac{\cos\theta}{\sin^2\theta}\left[\frac{\partial\mathcal{E}^3}{\partial\phi} - ik\frac{\partial g}{\partial\phi}\mathcal{E}^3\right]e^{-ikg}\,, \qquad (4.39)$$

where d is the same parameter given in Section 4.2.

It is seen that on changing the origin of the coordinate system, so that

$$\underline{r}'_s = r_s\hat{r} \qquad (4.40)$$

$$2r_s = \left[\frac{\cos\theta/2}{2G\rho_2} + \frac{\sin\theta/2}{G\rho_1\sin\theta}\right], \qquad (4.41)$$

expression (4.39) reduces to

$$2ikE_1^{\,2} = \left[D\mathcal{E}^2 - \frac{1}{\sin^2\theta}\left(\mathcal{E}^2 + 2\cos\theta\,\frac{\partial\mathcal{E}^2}{\partial\phi}\right]e^{-ikg}\,. \qquad (4.42)$$

The right-hand side of Eq. (4.42) is very small, since $\mathcal{E}^2$ and $\mathcal{E}^1$ are slowly varying functions (both porportional to G^{-1}).

Relations (4.40) and (4.41) indicate that on changing the origin of the coordinate system to lie on the "ray" directed from the specular point to the observation point, and at the effective phase center, the second term in Wilcox's expansion can be made quite small. Changing the origin of the coordinate system to reduce the second term in Wilcox's expansion is equivalent to tracing back the reflected wave front to the specular point.

This points out the importance of the phase information in the scattered far field, for in this case the knowledge of the phase leads to the angular position of the specular point on the body, whereas amplitude knowledge by itself will not indicated this, only yielding the Gaussian curvature at the specular point.

V
THE PLANE-WAVE EXPANSION

As an analytical tool for investigating the inverse scattering problem, the Wilcox expansion has several shortcomings. Most important of these is the fact that one cannot go inside the minimum convex shape enclosing the equivalent sources by means of the Wilcox expansion alone. One may, of course, speak mathematically of continuing the field to the inside; however, it is felt that other representations of the electromagnetic field are available which may yield more concrete results and provide a greater insight into the analytical problems of inverse scattering.

The approach to the inverse scattering problem based upon the representation of the scattered field in terms of plane waves is here investigated. Such a technique has been used for partiuclar direct scattering problems, and modified versions of it appropriate to the high frequency case, have been employed in geometric optics (Kline and Kay, 1965). This technique is shown to have several advantages. If the scattered field is thought of as arising from a set of discrete sources, the field can be obtained everywhere outside and between each individual source, i.e. it is not restricted to the region outside the minimum convex shape enclosing the sources. This could have practical uses for investigating cavities or antennas mounted on the surface of the body. In addition, if the scattered field (phase and amplitude) is known only over some angular bistatic sector, the near field (in the high frequency case) can be still obtained in certain regions. Thus, if it is assumed a priori that the body was a perfect conductor, then those portions of the scattering body giving rise to the observed portions of the scattered field can be found.

5.1 Determination of the Near Field from the Far Field.

A fundamental representation of the electromagnetic field in free space may be obtained as a combination of infinite plane waves whose amplitude factors are given by the far field and whose directions of propagation are, in general, complex. Although the representation discussed in this section is valid only at points outside the sources of the field, the extension to points within a source region is under investigation and will be discussed in Chapter IX.

Consider the electromagnetic field produced by a given volume distribution of electric currents $\underline{j}$ varying harmonically with time ($e^{-i\omega t}$) and located in some finite volume V of free space. The field everywhere in space may be expressed in terms of the vector potential $\underline{A}$ given by

$$\underline{A}(\underline{x}) = \frac{\mu_o}{4\pi} \int_V \underline{j}(\underline{x}') \frac{e^{ikR}}{R} d\underline{x}' \tag{5.1}$$

where $R = \left| \underline{x} - \underline{x}' \right|$, and the far-field distribution has the form

$$\underline{A}(r,\theta,\phi) \underset{r \to \infty}{\sim} \frac{e^{ikr}}{r} \underline{A}_o(\theta,\phi) \tag{5.2}$$

with

$$\underline{A}_o(\theta,\phi) = \frac{\mu_o}{4\pi} \int_V \underline{j}(\underline{x}') e^{-i\underline{k} \cdot \underline{x}'} d\underline{x}' \tag{5.3}$$

$$\underline{k} = k (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta). \tag{5.3}$$

The currents $\underline{j}$ may again be thought of as equivalent sources for some scattered field or as real sources for some radiation field.

For points exterior to V the Green's function e^{ikR}/R can be expanded into plane waves. We shall employ the well-known integral representation due to Weyl (1919) (see also Stratton, 1941, p. 578)

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta\ (z \geq z') \qquad (5.4)$$

where now $\underline{k} = k(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$ is a function of the variables of integration α and β running from 0 to $\pi/2 - i\infty$ and 0 to 2π, respectively. It is seen that in this expansion of the spherical wave e^{ikR}/R all possible plane-wave directions within the limits $0 \leq \beta \leq 2\pi$, $0 \leq \alpha \leq (\pi/2)$ are included; values of α lying in $(\pi/2 \leq \alpha \leq \pi$ correspond to plane waves traveling in from infinity in the half-space $z \geq z'$, and are, therefore, excluded. In addition, however, inhomogeneous plane waves with an exponentially decreasing amplitude in the z - direction (for $z > z'$) are included in order to yield the necessary singularity at $R \rightarrow 0$. These waves correspond to that part of the integration path running from $\alpha = (\pi/2)$ to $\alpha = (\pi/2) - i\infty$. An alternative representation valid in the half-space $z \leq z'$ may be obtained by selecting a different path of integration in the α-plane; thus, for example, we may write

$$\frac{e^{ikR}}{R} = -\frac{ik}{2\pi}\int_0^{2\pi}\int_{\pi}^{\frac{\pi}{2}+i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta\ \ (z \leq z'). \qquad (5.5)$$

When (5.4) is introduced in (5.1) and the orders or integration interchanged, one obtains for the vector potential $\underline{A}$ the following result

$$\underline{A}(\underline{x}) = \frac{ik}{2\pi}\frac{\mu_o}{4\pi}\int_V \underline{j}(\underline{x}')\left(\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta\, d\underline{x}'\right.$$

$$= \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\left\{\frac{\mu_o}{4\pi}\int_V \underline{j}(\underline{x}')e^{-i\underline{k}\cdot\underline{x}'}d\underline{x}'\right\}\sin\alpha\, d\alpha\, d\beta, \qquad (5.6)$$

and upon recognizing in view of (5.3) that the quantity contained in $\{\}$ immediately above is merely $\underline{A}_o(\alpha,\beta)$, one finds

$$\underline{A}(\underline{x}) = \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\underline{A}_o(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta \qquad (5.7)$$

provided $\underline{x}$ lies in the half-space formed by the portion of the x-axis above the source volume V, that is, $z > z'_{max}$. In this upper half-space, then, (5.7) provides a representation of the near field in terms of the far-field data. For $\underline{x}$ lying in the lower half-space below the source region, $z < z'_{min}$ we have by virtue of (5.5)

$$\underline{A}(\underline{x}) = -\frac{ik}{2\pi}\int_0^{2\pi}\int_\pi^{\frac{\pi}{2}+i\infty} e^{i\underline{k}\cdot\underline{x}}\underline{A}_o(\alpha,\beta)\sin d\alpha\, d\beta\ . \qquad (5.8)$$

The integrals (5.7) and (5.8) together give the field everywhere in space except in the region $z'_{min} \le z \le z'_{max}$ which sandwiches the sources. It is clear, however, that other paths of integration in the α-plane depending on the observation angle θ may be selected to yield results even within $z'_{min} \le z \le z'_{max}$, although the source region must still be excluded. Choosing other paths of integration is tantamount to rotating the reference axes and will be discussed shortly.

5.2 Additional Comments.

As we have seen, the integral representation of the near field in terms of the far field requires integration over a surface element $d\,\Omega = \sin\alpha\, d\alpha\, d\beta$ of the complex unit sphere Ω. It is interesting to note that integration over the real portion of the unit sphere yields a result which contains both incoming and outgoing waves. Thus, in view of the representation (Stratton, 1941, p. 410)

$$\frac{\sin kR}{R} = \frac{k}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta\ , \tag{5.9}$$

one finds

$$\frac{k}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{i\underline{k}\cdot\underline{x}}\underline{A}_o(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta = \frac{\mu_o}{4\pi}\int_V \underline{j}(\underline{x}')\frac{\sin kR}{R}\, d\underline{x}'. \tag{5.10}$$

On the other hand, complex values of β as well as α may be included since, by a straightforward modification of the Weyl formula (5.4), we have

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty}\int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta, \quad (z \geq z') \tag{5.11}$$

and thus

$$\underline{A}(\underline{x}) = \frac{ik}{2\pi}\int\int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\underline{A}_o(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta\ . \tag{5.12}$$

Finally, it should be mentioned that the plane-wave expansion of the near field in terms of the far field could have been derived by starting with a surface integral representation of the vector potential. For example, in the case of a perfectly conducting body, the vector potential for the scattered field is given by

$$\underline{A}^{sc}(\underline{x}) = \frac{\mu_o}{4\pi} \int_S (\underline{n} \wedge \underline{H}) \frac{e^{ikR}}{R} \, ds' \qquad (5.13)$$

where S is the exterior surface of the scattering body and $\underline{H}$ is the total magnetic field generated on the surface. Introducing (5.4) into (5.13) and interchanging the orders of integration, one finds

$$\underline{A}^{sc}(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \, \underline{A}_o^{sc}(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta \qquad (5.14)$$

provided $\underline{x}$ lies in the half-space formed by the portion of the z axis above the scattering surface S. The last condition may be too restrictive, however, and the integral in (5.14) may be convergent for points $\underline{x}$ lying inside the scattering surface, where it will represent the field produced by an equivalent source. This very important observation is considered in Chapter VIII where the domain of convergence is extended into smooth, convex sections of a perfectly conducting, scattering surface.

5.3 Rotations of the Reference Axes.

The integral representations (5.7) and (5.8) taken together exclude certain portions of free space, namely the free-space points lying within the region $z'_{min} \leq z \leq z'_{max}$ sandwiching the source volume V. These points may be included by means of a rotation of the reference axes. Consider the integral (5.4)

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha\, d\alpha\, d\beta \quad (z \geq z')$$

where $\underline{k} = k(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$. This integral is invariant to a rotation of the reference axes, thus change to new variables α', β' defined by direction cosine relations,

$$\sin\alpha'\cos\beta' = \sin\alpha\cos\theta_o\cos(\beta-\phi_o) - \cos\alpha\sin\theta_o$$

$$\sin\alpha'\sin\beta' = \sin\alpha\sin(\beta-\phi_o). \tag{5.15}$$

$$\cos\alpha' = \sin\alpha\sin\theta_o\cos(\beta-\phi_o) + \cos\alpha\cos\theta_o,$$

where θ_o, ϕ_o are arbitrary (real) angles. The inverse transformation, defining the old variables (α,β) in terms of the new variables (α',β') is

$$\sin\alpha\cos\beta = \cos\alpha'\sin\theta_o\cos\phi_o + \sin\alpha'\left[\cos\theta_o\cos\beta'\cos\phi_o - \sin\beta'\sin\phi_o\right],$$

$$\sin\alpha\sin\beta = \cos\alpha'\sin\theta_o\sin\phi_o + \sin\alpha'\left[\cos\theta_o\cos\beta'\sin\phi_o + \sin\beta'\cos\phi_o\right],$$

$$\cos\alpha = \cos\alpha'\cos\theta_o - \sin\alpha'\sin\theta_o\cos\beta'. \tag{5.16}$$

Further

$$d\Omega = \sin\alpha\, d\alpha\, d\beta = \sin\alpha'\, d\alpha'\, d\beta' = d\Omega' \tag{5.17}$$

and the limits of integration may remain the same. The integral representation is essentially unchanged in form:

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\sin\alpha'\,d\alpha'\,d\beta' , \qquad (5.18)$$

where the directional cosines of $\underline{k}$ are to be obtained from the inverse transformation relations given in (5.16). The integral, however, now converges at $\alpha' = (\pi/2) - i\infty$ provided

$$(x-x')\sin\theta_o\cos\phi_o + (y-y')\sin\theta_o\sin\phi_o + (z-z')\cos\theta_o > 0 \qquad (5.19)$$

that is, provided

$$\underline{x}\cdot\hat{x}_o > \underline{x}'\cdot\hat{x}_o \qquad (5.20)$$

where $\hat{x}_o$ is the unit vector

$$\hat{x}_0 = (\sin\theta_o\cos\phi_o,\ \sin\theta_o\sin\phi_o,\ \cos\theta_o). \qquad (5.21)$$

We have, therefore,

$$\underline{A}(\underline{x}) = \frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\,\underline{A}_o(\alpha,\beta)\sin\alpha'\,d\alpha'\,d\beta' \qquad (5.22)$$

provided

$$\underline{x}\cdot\hat{x}_o > \max(\underline{x}'\cdot\hat{x}_o). \qquad (5.23)$$

This representation is thus valid for all $\underline{x}$ lying in the half-space formed by the portion of the $\hat{x}_o$ axis not containing the sources. Since $\hat{x}_o$ is an arbitrarily directed vector, it is clear that portions of free space within $z'_{min} \leq z \leq z'_{max}$, which were previously excluded, may now be included. In particular, by rotating the $\hat{x}_o$ vector we may generate the field everywhere in the space outside some minimum convex shape surrounding the sources.

5.4 Relationship to Spherical Harmonics.

Assume the far field is known as an expansion in spherical harmonics

$$\underline{A}_o(\theta,\phi) = \sum_n \sum_m \underline{a}_{nm} P_n^m(\cos\theta)\, e^{im\phi} . \qquad (5.24)$$

Then, because of the integral representation derived by Erdélyi (1937):

$$i^n h_n^{(1)}(kr)\; P_n^m(\cos\,\theta)\; e^{\pm im\phi}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\, P_n^m(\cos\alpha)\, e^{\pm im\beta} \sin\alpha\, d\alpha\, d\beta\ , \qquad (5.25)$$

we have immediately from (5.7)

$$\underline{A}(\underline{x}) = ik \sum_n \sum_m \underline{a}_{nm}\, i^n h_n^{(1)}(kr)\, P_n^m(\cos\,\theta) e^{im\phi}\ , \qquad (5.26)$$

thereby giving the field as an expansion in spherical wave functions.

5.5 Relationship to Aperture Problems.

Assume the source currents $\underline{j}(\underline{x}')$ are confined to the plane $z' = 0$ and denote the directional cosines of $\underline{k}$ by u, v, w, where

$$u = \sin\alpha\cos\beta,$$

$$v = \sin\alpha\sin\beta. \qquad (5.27)$$

$$w = \cos\alpha.$$

The far-field amplitude (5.3) may be written in the form

$$\underline{A}_o(u,v) = \frac{\mu_o}{4\pi}\int \underline{j}(x',y')\, e^{-ik(ux'+vy')}\, dx'\, dy' \qquad (5.28)$$

and the spherical wave (5.4) may be expressed as

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\iint_{-\infty}^{\infty} e^{ik\left[u(x-x')+v(y-y')+w(z-z')\right]}\frac{dudv}{w} \qquad (5.29)$$

since $\sin\alpha\, d\alpha\, d\beta = (1/w)\, dudv$. Thus, remembering $z' = 0$, we have

$$\underline{A}(\underline{x}) = \frac{\mu_o}{4\pi}\int \underline{j}(x',y')\,\frac{e^{ikR}}{R}\, dx'\, dy'$$

$$= \frac{ik}{2\pi}\iint_{-\infty}^{\infty} e^{ik(ux+vy+wz)}\,\underline{A}_o(u,v)\,\frac{dudv}{w} \,. \qquad (5.30)$$

When $z = 0$ this leads to the well-known result that polar diagrams and aperture distributions are related by two-dimensional Fourier transformations (see e.g. Bouwkamp, 1954).

5.6 The Field Quantities.

The electromagnetic field is derived from the vector potential by means of relations

$$\underline{H} = \frac{1}{\mu_o} \text{ curl } \underline{A}$$
(5.31)
$$\underline{E} = \frac{i}{\epsilon_o \mu_o \omega} \text{ curl curl } \underline{A} \, .$$

In the far zone these equations give

$$\underline{H}(r,\theta,\phi) \sim \frac{e^{ikr}}{r} \underline{H}_o(\theta,\phi) = \frac{e^{ikr}}{r} \left(\frac{i}{\mu_o}\right) \underline{k} \wedge \underline{A}_o(\theta,\phi),$$
(5.32)
$$\underline{E}(r,\theta,\phi) \sim \frac{e^{ikr}}{r} \underline{E}_o(\theta,\phi) = \frac{e^{ikr}}{r} \left(\frac{i}{\epsilon_o \mu_o \omega}\right) \underline{k} \wedge \underline{k} \wedge \underline{A}_o(\theta,\phi).$$

When these relations are applied to the integral representation (5.7) of the vector potential, one finds.

$$\underline{H}(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \left(\frac{i}{\mu_o}\right) \underline{k} \wedge \underline{A}_o(\alpha,\beta) \sin \alpha \, d\alpha \, d\beta$$
(5.33)
$$= \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \underline{H}_o(\alpha,\beta) \sin \alpha \, d\alpha \, d\beta,$$

and similarly for the electric field

$$\underline{E}(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \underline{E}_o(\alpha,\beta) \sin \alpha \, d\alpha \, d\beta. \qquad (5.34)$$

The near fields may thus be represented directly in terms of the electromagnetic fields in the far zone.

5.7 Determination of Field in Free-Space Region Between Distinct Sources.

If the original source region V can be separated into a number of disjointed volumes V_i in free space, then it may be possible to determine the field in the space between these distinct source regions. For example, let the currents $\underline{j}$ be located in two finite volumes V_1 and V_2 , and further assume that V_1 lies within the range $z_1 < z < z_2$ while V_2 lies within the range $z_3 < z < z_4$ with $z_2 < z_3$. Also let $\underline{H}_o^{(1)}(\theta, \phi)$ and $\underline{H}_o^{(2)}(\theta, \phi)$ denote the far-field amplitudes due to the sources in V_1 and V_2, respectively. Then the field in the free-space region $z_2 < z < z_3$ between the two volumes V_1, V_2 can be represented in the form

$$\underline{H}(\underline{x}) = \frac{ik}{2\pi} \left\{ \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \underline{H}_o^{(1)}(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta \right.$$
$$\left. - \int_0^{2\pi} \int_0^{\frac{\pi}{2} + i\infty} e^{i\underline{k} \cdot \underline{x}} \underline{H}_o^{(2)}(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta \right\} . \qquad (5.35)$$

The first integral above converges for $z > z_2$ while the second converges for $z < z_3$; hence, this representation is valid in the desired region $z_2 < z < z_3$. This has an immediate application in providing a means of separating out distinct sources of scattered field that may occur, such as an antenna or other protuberance mounted on a smooth surface.

VI

ANALYTIC CONTINUATION

The field $\underline{H}_o(\theta, \phi)$ in the far zone is measurable only for real values of θ, ϕ in the ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. However, in order to obtain the near field by means of the integral representation discussed in the previous section, it is necessary to know $\underline{H}_o(\alpha, \phi)$ where $\alpha = \theta + i\xi$. Therefore, we need an extension into the complex α-plane based upon the measured quantity $\underline{H}_o(\theta, \phi)$.

Now $\underline{H}_o(\theta, \phi)$ is immediately known for the range $-\pi \leq \theta \leq \pi$. This follows from the definition

$$\underline{H}_o(\theta, \phi) = \frac{i}{4\pi} \int_V \underline{k} \wedge \underline{j}(x') \, e^{i\underline{k} \cdot \underline{x}'} \, d\underline{x}' \tag{6.1}$$

and the fact that $\underline{k}$ as a function of θ and ϕ satisfies

$$\underline{k}(-\theta, \phi) = \underline{k}(\theta, \phi \pm \pi); \tag{6.2}$$

hence

$$\underline{H}_o(-\theta, \phi) = \underline{H}_o(\theta, \phi \pm \pi). \tag{6.3}$$

In addition, $\underline{H}_o$ is periodic with period 2π in both θ and ϕ.

To obtain an expression for $\underline{H}_o(\alpha, \phi)$ in the complex α-plane, we observe from (6.1) that $\underline{H}_o(\alpha, \phi)$ is a harmonic function in the variables θ and ξ; that is

$$\left(\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\xi^2}\right) \underline{H}_o(\theta + i\xi, \phi) = 0 . \tag{6.4}$$

As such, $\underline{H}_o(\alpha, \phi)$ may be expressed in the following form

$$\underline{H}_o(\alpha, \phi) = \sum_{n=-\infty}^{\infty} \underline{a}_n \, e^{in(\theta + i\xi)} \tag{6.5}$$

where the coefficients $\underline{a}_n$ are derived by means of the relation

$$\underline{a}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta'} \, \underline{H}_o(\theta', \phi) \, d\theta' \tag{6.6}$$

This provides an extension into the complex α-plane. The series (6.5) may be partially summed and put into closed form as follows:

$$\begin{aligned} \underline{H}_o(\alpha, \phi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(\theta + i\xi)} \, \underline{H}_o(\theta', \phi)}{e^{i(\theta + i\xi)} - e^{i\theta'}} \, d\theta' \, + \\ &+ \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{in(\theta + i\xi)} \int_{-\pi}^{\pi} \underline{H}_o(\theta', \phi) \, e^{in\theta'} \, d\theta' \end{aligned} \tag{6.7}$$

for $\xi < 0$, and

$$\begin{aligned} \underline{H}_o(\alpha, \phi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta'} \, \underline{H}_o(\theta', \phi)}{e^{i\theta'} - e^{i(\theta + i\xi)}} \, d\theta' \, + \\ &+ \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{-in(\theta + i\xi)} \int_{-\pi}^{\pi} \underline{H}_o(\theta', \phi) \, e^{in\theta'} \, d\theta' \end{aligned} \tag{6.8}$$

for $\xi > 0$.

To investigate the convergence of the series (6.5) we examine the behavior of $\underline{a}_n$ and $n \to \infty$. Now

$$\underline{a}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\left\{\frac{i}{4\pi}\int_V \underline{k}\wedge\underline{j}(x')e^{-i\underline{k}\cdot\underline{x}'}d\underline{x}'\right\}d\theta =$$

$$= \frac{-i}{8\pi^2}\int_V \underline{j}(\underline{x}')\wedge\int_{-\pi}^{\pi}\underline{k}e^{-in\theta-i\underline{k}\cdot\underline{x}'}d\theta\, d\underline{x}' \tag{6.9}$$

where

$$\underline{k} = k(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta). \tag{6.10}$$

Write $\underline{k}$ in the form

$$\underline{k} = \frac{k}{2}\left(e^{i\theta}\underline{t} + e^{-i\theta}\underline{t}^*\right) \tag{6.11}$$

with

$$\underline{t} = (-i\cos\phi,\ -i\sin\phi,\ 1), \tag{6.12}$$

then

$$\underline{a}_n = \frac{-ik}{16\pi^2}\int_V \underline{j}(\underline{x}')\wedge\left\{\underline{t}\int_{-\pi}^{\pi}e^{-i(n-1)\theta-i\underline{k}\cdot\underline{x}'}d\theta + \right.$$

$$\left. + \underline{t}^*\int_{-\pi}^{\pi}e^{-i(n+1)\theta-i\underline{k}\cdot\underline{x}'}d\theta\right\}d\underline{x}' . \tag{6.13}$$

But $\underline{k} \cdot \underline{x}'$ may be written in the form

$$\underline{k} \cdot \underline{x}' = kr' \left[\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \right] \tag{6.14}$$
$$= kr'\rho \cos(\theta - \psi),$$

where

$$\rho^2 = \cos^2\theta' + \sin^2\theta' \cos^2(\phi - \phi'), \tag{6.15}$$
$$\tan\psi = \tan\theta' \cos(\phi - \phi'),$$

and, in view of the integral representation

$$e^{-im\psi} J_m(kr'\rho) = \frac{i^n}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta - ikr'\rho\cos(\theta - \psi)} \, d\theta, \tag{6.16}$$

we have the closed form expression

$$\underline{a}_{-n} = \frac{(-i)^n k}{8\pi} \int_V \underline{j}(\underline{x}') \wedge \left\{ \underline{t} e^{i\psi} J_{n-1}(kr'\rho) - \underline{t}^* e^{-i\psi} J_{n+1}(kr'\rho) \right\} e^{-in\nu} \, d\underline{x}'. \tag{6.17}$$

As n tends to infinity the dominant contribution is due to the first term within the braces in the integrand

$$\underline{a}_{-n} \underset{n \to \infty}{\sim} \frac{(-i)^n k}{8\pi \Gamma(n)} \int_V \underline{j}(\underline{x}') \wedge \underline{t} \left(\frac{kr'\rho}{2} e^{-i\psi} \right)^{n-1} d\underline{x}' \tag{6.18}$$

hence

$$\left| \underline{a}_{-n} \right|_{n \to \infty} \leqslant \frac{\text{Constant}}{\Gamma(n)} \left(\frac{kR}{2} \right)^{n-1} \tag{6.19}$$

where R represents the maximum value of r'. A similar result holds for $\left| \underline{a}_{-n} \right|$. The convergence of the series (6.5) is therefore secured for all $\alpha = \theta + i\xi$ because of the gamma function in the denominator of (6.19).

VII

HIGH-FREQUENCY SCATTERING

It will often happen at high frequencies that the far scattered field from a body can be characterized either by a rapidly varying phase function (e.g. specular scattering) or by a rapidly varying amplitude function (e.g. scattering by a flat plate). Under these conditions the contour integral representation of the near field in terms of the far field is particularly convenient because the powerful methods of asymptotic analysis are then at one's disposal. As an application of the integral representation at high-frequencies we shall here consider the geometrical optics field produced by a plane wave incident on a perfectly conducting sphere.

The incident field is taken in the form

$$\underline{E}^i = \hat{x}\, e^{-ikz},$$

$$\underline{H}^i = -\sqrt{\frac{\epsilon_o}{\mu_o}}\; \hat{y}\, e^{-ikz}, \tag{7.1}$$

and the scattered field is given by

$$\underline{E}^s(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k}\cdot\underline{x}}\, \underline{E}_o^s(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta,$$

$$\underline{H}^s(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k}\cdot\underline{x}}\, \underline{H}_o^s(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta. \tag{7.2}$$

The geometrical optics field in the far zone may be written as

$$\underline{E}_o^s(\alpha,\beta) = -\frac{a}{2}\,\hat{e}(\alpha,\beta)\, e^{-2ika\cos(\alpha/2)} ,$$

$$\underline{H}_o^s(\alpha,\beta) = -\sqrt{\frac{\epsilon_o}{\mu_o}}\,\frac{a}{2}\,\hat{h}(\alpha,\beta)\, e^{-2ika\cos(\alpha/2)} , \qquad (7.3)$$

where $\hat{e}(\alpha,\beta)$ and $\hat{h}(\alpha,\beta)$ are the unit vectors

$$\hat{e}(\alpha,\beta) = \hat{x}(\cos\alpha\cos^2\beta + \sin^2\beta) - \hat{y}(1-\cos\alpha)\sin\beta\cos\beta - \hat{z}\sin\alpha\cos\beta,$$

$$\hat{h}(\alpha,\beta) = -\hat{x}(1-\cos\alpha)\sin\beta\cos\beta + \hat{y}(\cos\alpha\sin^2\beta + \cos^2\beta) - \hat{z}\sin\alpha\sin\beta. \qquad (7.4)$$

The exponential behavior of the integrands in (7.2) is thus governed by the factor

$$e^{ikf(\alpha,\beta)} \qquad (7.5)$$

where

$$f(\alpha,\beta) = \hat{k}\cdot\underline{x} - 2a\cos\frac{\alpha}{2}$$

$$= r\left[\sin\theta\sin\alpha\cos(\phi-\beta) + \cos\theta\cos\alpha\right] - 2a\cos\frac{\alpha}{2} . \qquad (7.6)$$

Upon examining the convergence of the integral as $\alpha \to (\pi/2) - i\infty$, one finds that the integrand decays exponentially so long as $r\cos\theta > 0$, that is, $z > 0$. When $z = 0$, however, the integrand grows exponentially as $\alpha \to (\pi/2) - i\infty$, hence the representation (7.2) with (7.3) is valid for all $z > 0$, or what is the same, $0 \le \theta < (\pi/2)$.

As $k \to \infty$ the dominant contribution to the integral arises from the vicinity of the stationary phase point ($\beta = \phi$, $\alpha = \alpha_o$) where α_o satisfies the equation

$$r \sin(\alpha_o - \theta) = a \sin(\alpha_o/2) . \tag{7.7}$$

The physical interpretation of this equation is shown in Fig. 7-1. The quantity $p = a \sin(\alpha_o/2)$ may be interpreted as the impact parameter associated with an incident ray, and this is precisely the incident ray that reaches the observation point P after being reflected at the surface according to the laws of geometrical optics. The angle α_o is twice the angle of incidence.

By means of a first order stationary phase evaluation we obtain immediately

$$\underline{E}^s(\underline{x}) \sim -\frac{a}{2}\sqrt{\frac{\sin\alpha_o}{\left[r\cos(\alpha_o-\theta)-\frac{a}{2}\cos\frac{\alpha_o}{2}\right]r\sin\theta}}\ \hat{e}(\alpha_o,\phi)\,e^{ikf(\alpha_o,\phi)} , \tag{7.8}$$

and similarly for the magnetic field. If we let s denote the distance along the reflected ray from the point of reflection

$$\begin{aligned} s &= r\cos(\alpha_o-\theta) - a\cos(\alpha_o/2), \\ r\sin\theta &= s\sin\alpha_o + a\sin(\alpha_o/2), \end{aligned} \tag{7.9}$$

then the result (7.8) may be written in the form

$$\underline{E}^s(\underline{x}) \sim -\left[\frac{D(0)}{D(s)}\right]^{1/2} \hat{e}(\alpha_o,\phi)\,e^{iks-ika\cos(\alpha_o/2)} , \tag{7.10}$$

where

$$D(s) = \left(\cos\frac{\alpha_o}{2} + \frac{2s}{a}\right)\left(1 + \frac{2s}{a}\cos\frac{\alpha_o}{2}\right) . \tag{7.11}$$

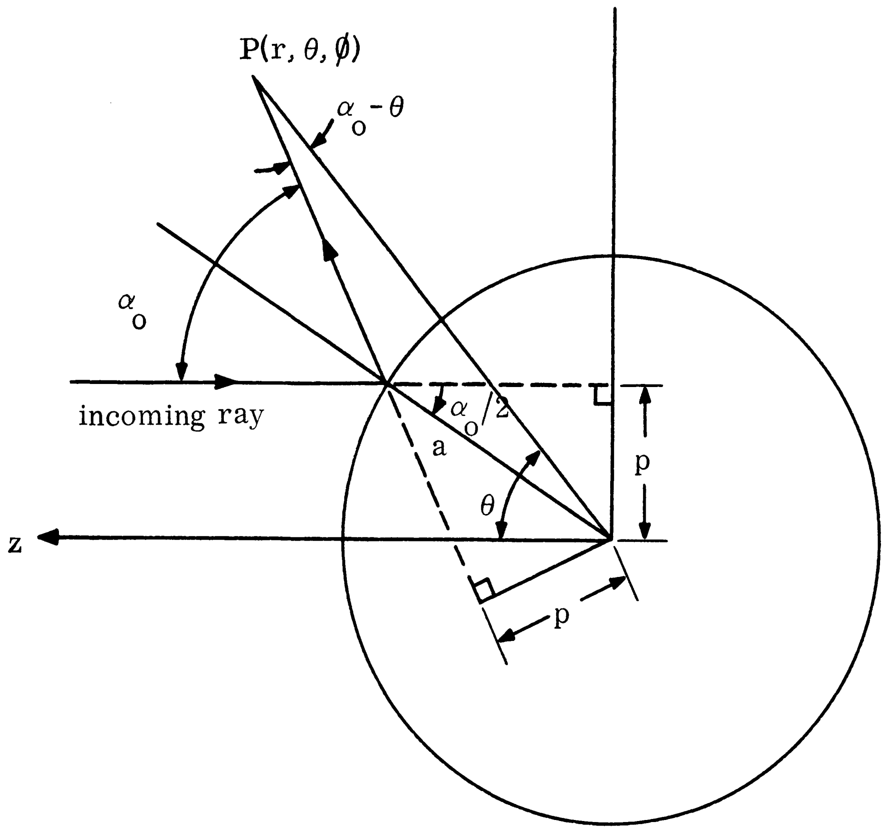

FIG. 7-1: PHYSICAL INTERPRETATION OF STATIONARY PHASE POINT α_0 FOR HIGH-FREQUENCY (GEOMETRICAL OPTICS) SCATTERING BY CONDUCTING SPHERE.

The magnetic field takes the analogous form

$$\underline{H}^s(\underline{x}) \sim -\sqrt{\frac{\epsilon_o}{\mu_o}} \left[\frac{D(0)}{D(s)}\right]^{1/2} \hat{h}(\alpha_o, \phi)\, e^{iks-ika\cos(\alpha_o/2)} \tag{7.12}$$

The amplitude factor $\left[D(0) / D(s)\right]^{1/2}$ in the equations above accounts for the divergence of the rays after reflection at the surface. This factor has been derived on the basis of geometrical optics for reflection from an arbitrary convex body by Fock (1948), and it is easy to verify that the quantity (7.11) agrees with the expression given by Fock in the case of the sphere. The phase of the field is also in agreement with standard geometrical optics considerations.

The ordinary stationary phase evaluation fails in the vicinity of the caustic given by the equation

$$r\cos(\alpha_o - \theta) - \frac{a}{2}\cos\frac{\alpha_o}{2} = 0. \tag{7.13}$$

However, it must be emphasized that the behavior of the field near the caustic may still be examined by applying a modified asymptotic analysis to the integral representation of (7.2). The elegance and simplicity of this representation for application to high-frequency scattering is evident.

VIII

THE VALIDITY OF THE FIELD REPRESENTATION INTERIOR TO THE SMOOTH AND CONVEX PARTS OF THE SCATTERING BODY

We recall that the vector potential $\underline{A}(\underline{x})$ can be represented as

$$\underline{A}(\underline{x}) = \frac{ik}{2\pi}\int_0^{2\pi} d\beta \int_0^{\frac{\pi}{2}-i\infty} d\alpha \sin\alpha \; e^{i\underline{k}\cdot\underline{x}} \; \underline{A}_o(\alpha,\beta) \quad , \tag{8.1}$$

provided the position vector $\underline{x}$ lies in the half-space formed by the portion of the z-axis above the source volume V, that is, $z > z'_{max}$. For $\underline{x}$ lying in the lower half-space below the source region, $z < z'_{min}$, we have

$$\underline{A}(\underline{x}) = -\frac{ik}{2\pi}\int_0^{2\pi} d\beta \int_0^{\frac{\pi}{2}+i\infty} d\alpha \sin\alpha \; e^{i\underline{k}\cdot\underline{x}} \; \underline{A}_o(\alpha,\beta) \tag{8.2}$$

In Eq. (8.1) and (8.2) $\underline{A}_o(\alpha,\beta)$ is the vector function of the complex variable α which is obtained from the far-zone potential (taking $\mu_o = 1$)

$$\underline{A}_o(\theta,\beta) = \frac{1}{4\pi}\int_V \underline{j}(\underline{x}') \; e^{-i\underline{k}\cdot\underline{x}'} \; d\underline{x}' \;, \tag{8.3}$$

by means of the analytic continuation

$$\alpha = \theta + i\xi \; . \tag{8.4}$$

Here

$$0 \leq \theta \leq \pi \;, \quad 0 \leq \beta \leq 2\pi. \tag{8.5}$$

$$\underline{k} = k\,(\sin\theta\cos\beta,\; \sin\theta\sin\beta,\; \cos\theta), \tag{8.6}$$

and the currents $\underline{j}$ may be thought of as equivalent sources for some scattered field or as real sources for some radiation field. We can of course determine the field $(\underline{E}, \underline{H})$ immediately from (8.1) and (8.2) in the usual manner.

For a perfectly conducting surface S the integral (8.3) may be written explicitly in terms of the (physically) induced surface currents; i.e.,

$$\underline{A}_o(\alpha,\beta) = \int_S (\underline{n} \wedge \underline{H})\, e^{-ik \cdot x'} dS, \quad x' \in S , \tag{8.7}$$

where $\underline{H}$ is the total magnetic field. We let $\alpha = \frac{\pi}{2} - it$, and find that

$$e^{i\underline{k} \cdot \underline{x}}\, \underline{A}_o(\alpha,\beta) = \int_S dS\, (\underline{n} \wedge \underline{H}) \exp k \left\{ i \cosh t \left[\cos\beta\, (x-x') + \sin\beta\, (y-y') \right] - \sinh t\, (z-z') \right\} . \tag{8.8}$$

In Eq. (8.8), letting $t \to \infty$, we see that the representation (8.1) is valid at the upper limit $\alpha = \frac{\pi}{2} - i\infty$ for $z > z'$. Here, the important question arises; namely, is (8.1) valid for $z < z'$? If so how far can one go inside the survace S before the representation (8.1) falls? It is the main purpose of this chapter to study this question in detail. However, before we consider more general shapes, it would be enlightening to study this question for the perfectly conducting sphere, with the further assumption that we are in the physical optics region.

8.1 A Special Case: The Sphere

Consider a perfectly conducting sphere of radius a, and let a plane wave propagating in the direction of negative z-axis be incident upon it (Fig. 8-1). Thus

$$\underline{E}^i = \hat{x}\, e^{-ikz}$$
$$\underline{H}^i = -\hat{y}\, e^{-ikz} . \tag{8.9}$$

With the physical optics assumption $\hat{n} \wedge \underline{H} = 2\, \hat{n} \wedge H^i$, $(0 \leq \theta \leq \frac{\pi}{2})$, (8.7) reduces to

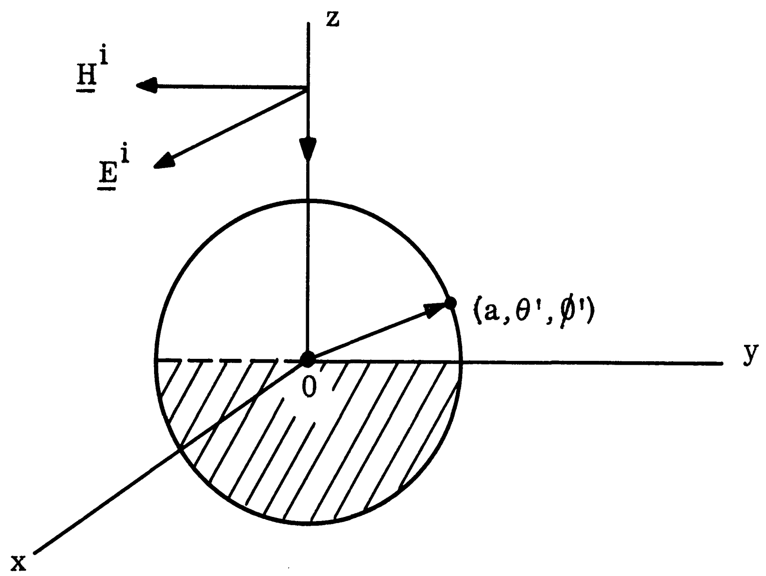

FIG. 8-1: SPHERE GEOMETRY.

$$\underline{A}_o(\alpha,\beta) = \frac{a^2}{2\pi} \int_0^{2\pi} d\phi' \int_0^{\pi/2} \sin\theta'\, d\theta' (x \cos\theta' - z \sin\theta' \cos\phi')\, e^{-ika(\cos\theta' + \cos\gamma)} \tag{8.10}$$

where

$$\cos\gamma = \sin\alpha \sin\theta' \cos(\phi' - \beta) + \cos\alpha \cos\theta' . \qquad (8.11)$$

We now proceed to study the behavior of the x and z components of $\underline{A}_o$ as $\alpha \rightarrow \frac{\pi}{2} - i\infty$. We have (Stratton (1937) p. 367)

$$J_n(\rho) = \frac{i^{-n}}{2\pi} \int e^{i\rho\cos\phi + in\phi}\, d\phi \quad . \qquad (8.12)$$

Here the path of integration is extended along the real axis of ϕ on any segment of length 2π.

Using Eq. (8.12) we carry out first the ϕ' integration and find the x-component of $\underline{A}_o$ to be

$$A_{o_x}(\alpha,\beta) = a^2 \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \cdot$$
$$\cdot\, e^{-ika\cos\theta'(1+\cos\alpha)} J_o(-ka\sin\alpha\sin\theta'). \qquad (8.13)$$

Since about $\pi/2$ cos is odd and sin is even, (8.13) reduces to

$$A_{o_x}(\alpha,\beta) = \frac{a^2 i}{2} \int_0^{\pi} \sin\left[-ka(1+\cos\alpha)\cos\theta'\right] \cdot$$
$$\cdot\, J_o(-ka\sin\alpha\sin\theta')\cos\theta'\sin\theta'\, d\theta' \quad . \qquad (8.14)$$

We have (Watson, 1952, p. 379)

$$\int_0^{\pi} \sin (z \cos \theta \cos \psi) J_{\nu-1/2}(z \sin\theta \sin\psi) C_r^{\nu} (\cos\theta)\sin^{\nu+1/2}\theta \, d\theta =$$

$$= \begin{cases} 0 & (r \quad \text{even}) \\ (-1)^{1/2(r-1)} (\frac{2\pi}{z}) \sin^{\nu-1/2}\psi \, C_r^{\nu}(\cos\psi) J_{\nu+r}(z) & (r \quad \text{odd}), \end{cases} \tag{8.15}$$

and (Magnus and Oberhettinger, 1954, p. 77)

$$C_{n-\ell}^{\ell+1/2} (t) = (-1)^{\ell} \frac{(1-t^2)^{-\ell/2} \ell! \, 2^{\ell}}{(2\ell)!} P_n^{\ell} (t) \tag{8.16}$$

where C_r^{ν} is the Gegenbauer polynomial. By means of (8.15) and (8.16) we obtain

$$A_{o_x}(\alpha,\beta) \sim \frac{a^2 i}{2} (\frac{2\pi}{-ka})^{1/2} C_1^{1/2} (\cos \alpha) J_{3/2} (-ka) =$$

$$= \frac{a^2}{\sqrt{2}} \sqrt{\frac{\pi}{ka}} \cos \alpha \, J_{3/2} (-ka). \tag{8.17}$$

or, since

$$J_{3/2}(-ka) = \sqrt{2} \, i \sqrt{\frac{ka}{\pi}} \, j_1 (-ka) \quad ,$$

$$A_{o_x} (\alpha, \beta) \sim a^2 i \cos \alpha \, j_1 (-ka) \; . \tag{8.18}$$

By means of somewhat similar analysis we can obtain the z-component in the form

$$A_{o_z}(\alpha,\beta) \sim a^2 \sin\alpha \cos\beta \; j_1(-ka) \; . \tag{8.19}$$

With $\alpha = \frac{\pi}{2} - it$, this becomes

$$\underline{A}_o\left(\frac{\pi}{2} - it, \beta\right) \sim a^2 j_1(-ka)\left[-\hat{x}\sinh t + z\cos\beta\cosh t\right] , \tag{8.20}$$

and

$$e^{i\underline{k}\cdot\underline{x}} = \exp k\left\{i\cosh t\,(x\cos\beta + y\sin\beta) - z\sinh t\right\} , \tag{8.21}$$

Therefore $e^{i\underline{k}\cdot\underline{x}}\underline{A}_o \rightarrow 0$ as $t \rightarrow \infty$ ($\alpha \rightarrow \frac{\pi}{2} - i\infty$) for $x \geq 0$, $y \geq 0$, and $z > 0$.

This, in turn, implies that the representation (8.1) is valid all the way down to (and including) the xy-plane which is punctured at the origin.

8.2 The Asymptotic Analysis for the More General Cases

As noted earlier for a perfectly conducting surface S' we have

$$\underline{A}_o(\alpha,\beta) = \int_S (\underline{n} \wedge \underline{H})\, e^{-i\underline{k}\cdot\underline{x}}\, dS \; . \tag{8.22}$$

With $\alpha = \pi/2 - it$, $\underline{k}$ is given by

$$\underline{k} = k(\cosh t\cos\beta,\; \cosh t\sin\beta,\; i\sinh t). \tag{8.23}$$

We shall now study the asymptotic behavior of the integral (8.22) for $t \rightarrow \infty$. To simplify the analysis we choose the coordinate system so that the origin is on the surface S, and that the positive z-axis is normal to S (pointing outwards).

Further, we orient the coordinate system so as to have

$$\underline{k} \cdot \underline{x} = k\,(x \cosh t + i z \sinh t). \tag{8.24}$$

We let a plane $z = z_o (z_o < 0)$ intersect the surface S, and denote the portion of S above this plane by S_o. We require S_o to have the following properties:

(i) S_o has the representation $z = f(x,y)$, $x, y \in A$, (8.25)

(ii) S_o is smooth; i.e., f_x, f_y are continuous functions for $x, y \in A$,

(iii) S_o is convex,

(iv) f is single valued,

(v) f is analytic.

Equation (8.22) reduces to

$$\underline{A}_o(\alpha, \beta) \sim \int_{S_o} e^{-i\underline{k} \cdot \underline{x}} (\underline{n} \wedge \underline{H})\, dS + O\left(e^{k z_o \sinh t}\right), \tag{8.26}$$

as $t \to \infty$. Hence we study the integral

$$\underline{I} = \int_{S_o} (\underline{n} \wedge \underline{H})\, e^{-i\underline{k} \cdot \underline{x}}\, dS \tag{8.27}$$

as $t \to \infty$.

Since

$$\hat{n} = \frac{\left(-\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}, \tag{8.28}$$

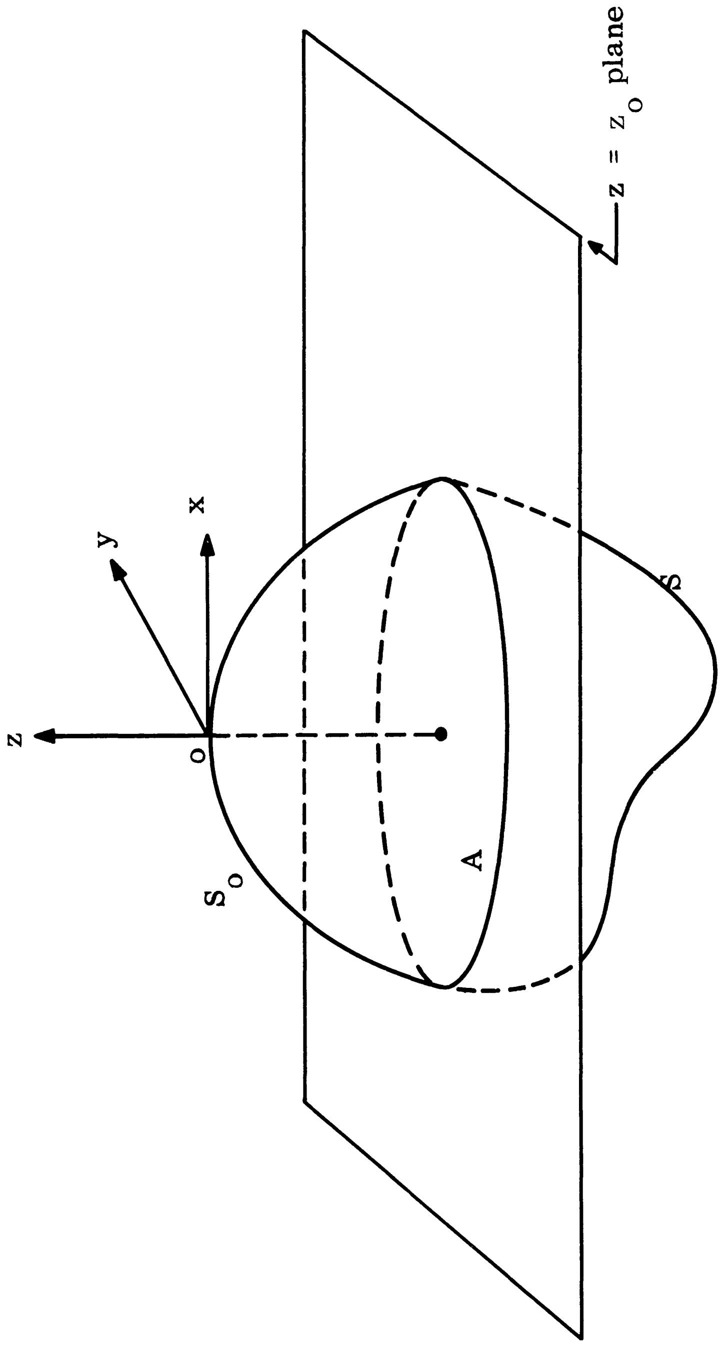

FIG. 8-2: GEOMETRY OF A GENERAL SHAPE.

we can write (8.27) as

$$\underline{I} = \iint_A \exp\left\{-ik(x\cosh t + if\sinh t)\right\} \cdot$$

$$\cdot \left[-\frac{\partial f}{\partial x}\hat{x} - \frac{\partial f}{\partial y}\hat{y} + \hat{z}\right] \wedge \underline{H}\, dx\, dy, \qquad (8.29)$$

where A is the projected area of S_o on the $z = z_o$ plane. We integrate (8.29) with respect to y first. As $t \to \infty$ the dominant behavior comes from the neighborhood of the point y_o, where

$$\left(\frac{\partial f}{\partial y}\right)_{y_o} = 0. \qquad (8.30)$$

With this, (8.23) reduces by saddle point integration to

$$\underline{I} \sim \frac{2\pi}{\sqrt{k\sinh t}} \int_{x_1}^{x_2} \exp\left\{-ik\left[x\cosh t + if(x, y_o)\sinh t\right]\right\} \underline{B}\, dx, \qquad (8.31)$$

where

$$\underline{B} = \left[\left(-\frac{\partial f}{\partial x}\hat{x} - \frac{\partial f}{\partial y}\hat{y} + \hat{z}\right) \wedge \underline{H}\right]_{y_o} \left(-\frac{\partial^2 f}{\partial y^2}\right)_{y_o}^{-1/2} . \qquad (8.32)$$

Note that because of the assumptions (8.25) we have $\left(-\frac{\partial^2 f}{\partial y^2}\right)_{y_o} > 0$ and there is only one such point y_o.

Now let us cut the surface by the plane $x = x'$ $(x_1 < x' < x_2)$.

For $z > z_o$, $x_1 < x' < x_2$, the function $f(x', y)$ is monotonically increasing if $y < y_o$, and it is monotonically decreasing if $y > y_o$. It has a maximum at $y = y_o$. This holds for all the curves determined by planes $x = x'(x_1 < x' < x_2)$. Since the analysis will have to hold for all angles β, these observations must remain valid for all cuts parallel to the z-axis of the surface S_o. Thus the surface S_o, $z = f(x,y)$, has a maximum at the origin and monotonically decreases away from the origin. Thus it appears that the dominant contribution to (8.31) arises from the point $z = 0$ because of the term $\exp\left[kf(x,y_o)\sinh t\right]$. However the remaining term $\exp\left[-ikx\cosh t\right]$ is a rapidly varying function and in the limit when $t \to \infty$ this can negate the contribution of the integral in the neighborhood of $x = 0$. In order to properly evaluate the integral (8.31) we proceed as follows. We define the complex plane $w = x + iv$. Because of our assumptions (8.25) about the surface S_o, we have

$$z = f(x,y) = \sum_{m+n=2} a_{mn} x^m y^n \text{ (m, n positive integers).} \qquad (8.33)^*$$

The function $f(w, y_o)$ is also analytic in some domain D_1 which contains the line segment $x_1 \leq x \leq x_2$. Assume that the vector function $\underline{B}(w, y_o)$ is also analytic in a domain D_2. This domain also must contain the segment $x_1 \leq x \leq x_2$. We see that analyticity of $\underline{B}$ presupposes the analyticity of $\underline{H}$ on S_o. Here, without going into the details, we shall simply remark that we can

---

* Note that assumptions (8.25) (iii), (v) imply that planar parts are excluded from the surface S_o. This fact is inherent in the power series expansion (8.33). Thus we are assuming the surface S_o to be locally parabolic.

always use the high frequency physical optics approximation to $\underline{H}$ when S_o is always completely in the illuminated region, thereby assuring ourselves of the analyticity of $\underline{H}$.

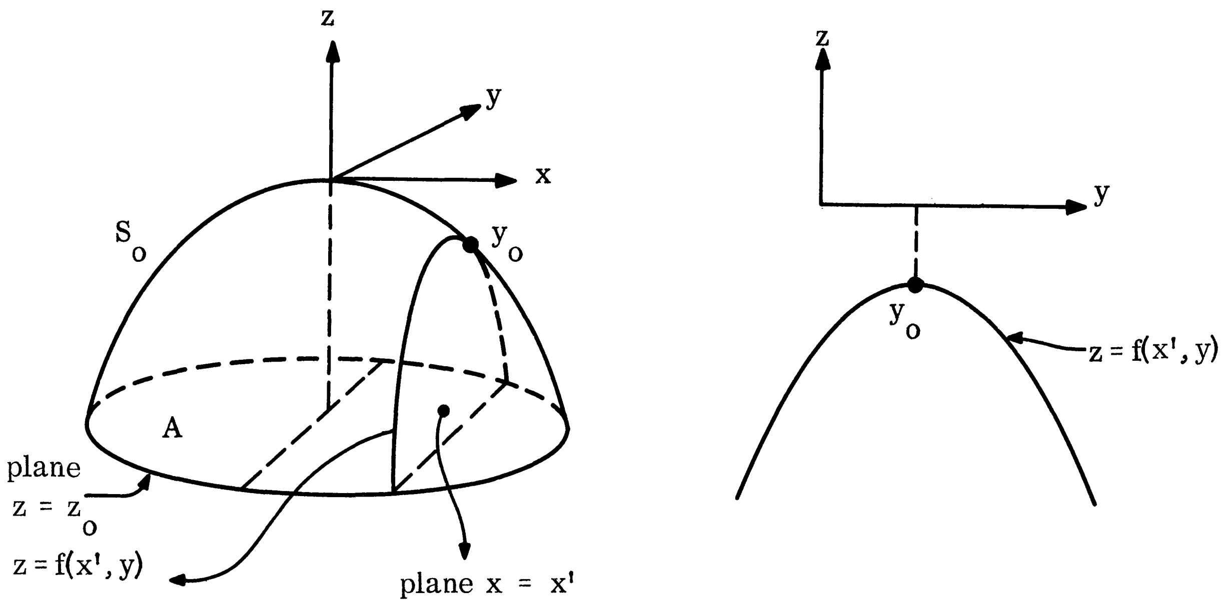

FIG. 8-3: CUT OF SURFACE S BY PLANE x = x'.

Now we write the expression (8.31) in the form

$$\underline{I} \sim \sqrt{\frac{2\pi}{k \sinh t}} \int_C \exp\Big[- i k w \cosh t + + k f(w, y_o) \sinh t \Big] \underline{B}(w, y_o)\, dw, \tag{8.34}$$

where the contour C lies in the domain $D = D_1 \cap D_2$, with end-points $w_1 = x_1$,

$w_2 = x_2$. The contour C will consist of three parts: the line segment $x_1 \leq x \leq \tilde{x}_1$, the contour C_1 with end-points $\tilde{x}_1$ and $\tilde{x}_2$, and the line segment $\tilde{x}_2 \leq x \leq x_2$. Since

$$\mathrm{Re}\left\{ - i\,k\,w \cosh t + k\,f(w, y_o) \sinh t \right\}$$

$$= \mathrm{Re}\left\{ -i\,k\,(x + iv) \cosh t \right\} + \mathrm{Re}\left\{ kf(w, y_o) \sinh t \right\} \qquad (8.35)$$

$$= k \sinh t \left\{ v \coth t + \mathrm{Re}\left[f(w, y_o) \right] \right\} ,$$

we shall require the contour C_1 to have the property that for $w \in C_1$

$$\mathrm{Re}\left\{ f(w, y_o) \right\} + v \coth t = f(\tilde{x}_1, y_o) = f(\tilde{x}_2, y_o) \qquad (8.36)$$

Also, since

$$\mathrm{Im}\left\{ -i\,k\,w \cosh t + k\,f(w, y_o) \sinh t \right\}$$

$$= k \left\{ -x \cosh t + \sinh t \ \mathrm{Im} \left[f(w, y_o) \right] \right\} , \qquad (8.37)$$

we see that the integral (8.34) has the asymptotic form for $t \rightarrow \infty$

$$\underline{I} \sim \sqrt{\frac{2\pi}{k \sinh t}} \cdot \exp\left\{ k\, f(\tilde{x}_1, y_o) \sinh t \right\} \cdot$$

$$\cdot \int_{C_1} \exp\left\{ i\,k \left[-x \cosh t + \mathrm{Im} \ f(w, y_o) \sinh t \right] \right\} \underline{B}(w)\, dw \,. \qquad (8.38)$$

The numbers $\tilde{x}_1$ and $\tilde{x}_2$, not specified at present, are restricted by the requirement that C_1 must lie in domain D. An additional restriction can be placed when the function

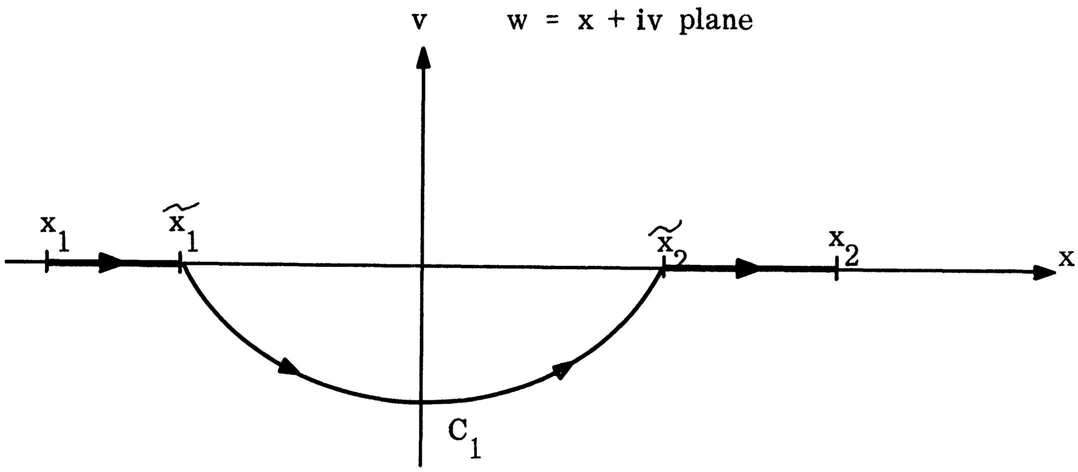

FIG. 8-4: CONTOUR $C = \overline{x_1 \tilde{x}_1} + C_1 + \overline{\tilde{x}_2 x_2}$

$$- i w \coth t + f(w, y_o) \qquad (8.39)$$

has a saddle point at $w_o = x_o + iv_o$. In this case, by the nature of the saddle point, we cannot have a "closed curve" joining the points $\tilde{x}_1$ and $\tilde{x}_2$ when $\tilde{x}_2 > \tilde{x}_2^o$ (see Fig. 8-5), where

$$\mathrm{Re} \left\{ f(w_o, y_o) \right\} + v_o \coth t = f(\tilde{x}_2^o, y_o) \quad . \qquad (8.40)$$

8.3 Application I: Elliptic Paraboloid

Consider the elliptic paraboloid given by

$$z = - a x^2 - b x y - c y^2 = f(x, y) \quad , \qquad (8.41)$$

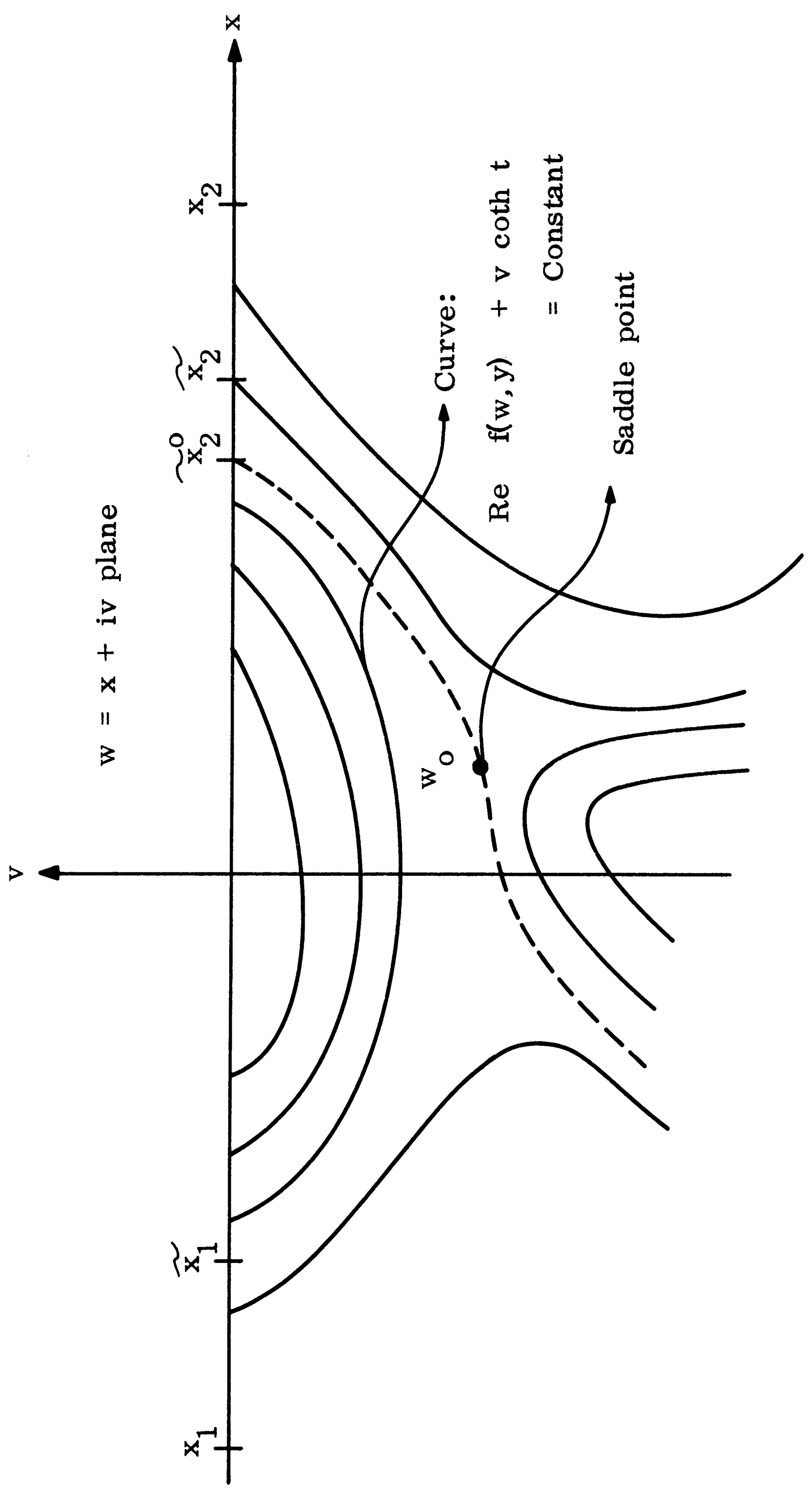

FIG. 8-5: SADDLE POINT.

whose principal radii of curvature R_1 and R_2 are given by

$$2(a + c) = 1/R_1 + 1/R_2$$
$$2\sqrt{(a - c)^2 + b^2} = 1/R_1 - 1/R_2 \quad . \tag{8.42}$$

The stationary point y_o [Eq. (8.30)] is given by

$$- b x - 2 c y_o = 0, \tag{8.43}$$

and for $y = y_o$, we have

$$f(x, y_o) = - \frac{x^2}{4c} \left[4 a c - b^2 \right] . \tag{8.44}$$

For this surface $x_1 = - \infty$ and $x_2 = \infty$. We need to consider the possible stationary points of the function [Eq. (8.39)]

$$- i w \coth t + f(w, y_o) \quad \text{for } t \rightarrow \infty \quad . \tag{8.45}$$

From

$$\lim_{t \rightarrow \infty} \frac{\partial}{\partial w} \left[- i w \coth t + f(w, y_o) \right]$$

$$= \lim_{t \rightarrow \infty} \left[- i \coth t - \frac{w}{2c} (4a c - b^2) \right] = 0,$$

we see that the stationary point is at $w = w_o$, where

$$w_o = \frac{-2 c i}{(4 a c - b^2)} \quad . \tag{8.46}$$

Thus the value of $\tilde{x}_2^o$ to be taken is given by [Eq. (8.40)]

$$f(\tilde{x}_2^o, y_o) = \mathrm{Re}\left\{ f(w_o, y_o) \right\} + v_o, \quad \text{for } t \rightarrow \infty , \tag{8.47}$$

i.e.,

$$-\frac{(\tilde{x}_2^o)^2}{4c}(4ac - b^2) = \frac{c}{4ac - b^2} - \frac{2c}{4ac - b^2} = -\frac{c}{4ac - b^2} ,$$

therefore,

$$(\tilde{x}_2^o)^2 = \frac{4c^2}{(4ac - b^2)^2} \tag{8.48}$$

Therefore, in this case we find that

$$f(\tilde{x}_2^o, y_o) = -\frac{c}{4ac - b^2} . \tag{8.49}$$

From (8.41) and (8.42) we observe that

$$R_1 R_2 = \frac{1}{4ac - b^2} , \tag{8.50}$$

and that the curvature of the curve $z = f(x, y)$, $x = 0$ is

$$\frac{-2c}{(1 + 4c^2 y^2)^{3/2}} . \tag{8.51}$$

If we let

$$-\frac{1}{R_y} = \left. \frac{-2c}{(1 + 4c^2 y^2)^{3/2}} \right|_{y=0} = -2c , \tag{8.52}$$

then we can write (8.49) as

$$f(\tilde{x}_2^o, y_o) = -\frac{R_1 R_2}{2R_y} . \tag{8.53}$$

Now we go back and from (8.38) observe that the integral $\underline{I}$ has the dominant asymptotic behavior

$$\underline{I} \sim O\left[\exp\left(-k \cdot \sinh t \cdot \frac{R_1 R_2}{2R_y}\right)\right] . \tag{8.54}$$

Note that in the original expression [Eq. (8.22), (8.23)] the angle β occurred. The coordinate system was rotated to remove the dependence on β, so that the proper asymptotic behavior for all angles β requires a rotation of the coordinate system. This effectively changes the value of R_y. It goes from R_1 to R_2 where R_1 is the minimum radius of curvature. Therefore,

$$\underline{I} \sim \exp\left(-k \sinh t \cdot \frac{R_{min}}{2}\right) . \tag{8.55}$$

From (8.22), (8.23), (8.24) and (8.55) we see that $e^{i\underline{k} \cdot \underline{x}} \underline{A}_o \rightarrow 0$, exponentially as

$$\alpha \rightarrow \frac{\pi}{2} - i\infty , \tag{8.56}$$

for $z > z^*$ where

$$z^* = \frac{R_{min}}{2} . \tag{8.57}$$

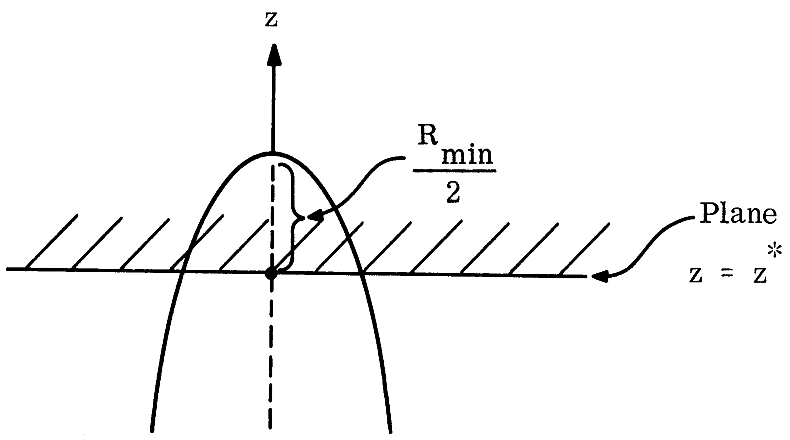

FIG. 8-6: REGION OF CONVERGENCE FOR THE PLANE WAVE EXPANSION FOR AN ELLIPTIC CYLINDER.

8.4 Application II: Spheroid

We now consider a body of revolution, and take the z-axis to be the axis of revolution. In this case the orientation of the x, y axes does not matter.

The surface is given by

$$z = f(x, y) = -b + b\sqrt{1 - \frac{p^2}{a^2}}\ , \qquad (8.58)$$

$$p^2 = x^2 + y^2\ .$$

The stationary point $y = y_o$ for which $\frac{\partial f}{\partial y} = 0$ is $y_o = 0$, and we have

$$f(w, y_o) = -b + b\sqrt{1 - \frac{w^2}{a^2}}\ . \qquad (8.59)$$

This function will be taken to have cuts along the x-axis, $x > a$, and $x < -a$.

Let w_o be the stationary point of

$$-i\,w + f(w, y_o) \qquad (t = \infty) \ . \tag{8.60}$$

This is given by

$$i = \frac{\partial f}{\partial w} = -\frac{b \cdot w/a^2}{\sqrt{1 - \frac{w^2}{a^2}}} \tag{8.61}$$

to be

$$w_o = \frac{-\,i\,a}{\sqrt{\frac{b^2}{a^2} - 1}} \qquad \text{provided that } b > a. \tag{8.62}$$

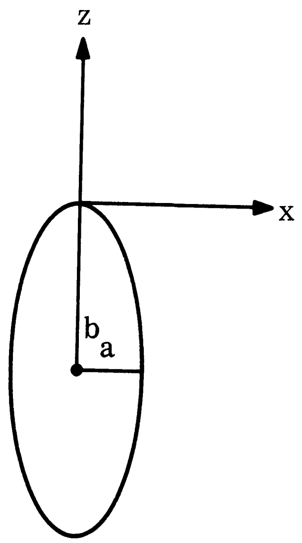

FIG. 8-7: SPHEROID

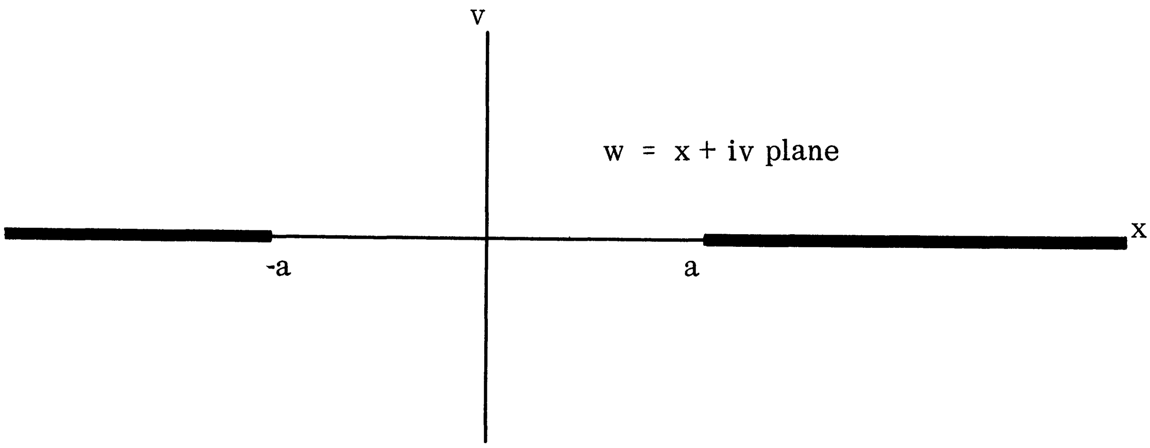

FIG. 8-8: CUTS FOR $f(w, y_o)$.

From Eq. (8.40), for $t = \infty$, we determine $\tilde{x}_2^o$ so that

$$\mathrm{Re}\left\{ f(w_o, y_o) \right\} + v_o = f(\tilde{x}_2^o, y_o) \quad . \tag{8.63}$$

Thus,

$$-b + b\sqrt{1 - \frac{(\tilde{x}_2^o)^2}{a^2}} = -b + b\sqrt{1 + \frac{1}{\frac{b^2}{a^2} - 1}} - \frac{a}{\frac{b^2}{a^2} - 1} \, ,$$

or,

$$b\sqrt{1 - \frac{(x_2^o)^2}{a^2}} = \frac{1}{\sqrt{\frac{b^2}{a^2} - 1}} \left\{ \frac{b^2}{a^2} - a \right\} = a\sqrt{\frac{b^2}{a^2} - 1} \, ,$$

or,

$$1 - \frac{(\tilde{x}_2^o)^2}{a^2} = \frac{a^2}{b^2}\left(\frac{b^2}{a^2} - 1\right).$$

Therefore,

$$\tilde{x}_2^o = \frac{a^2}{b} = R_o \qquad (8.64)$$

where R_o is the radius of curvature at the tip. We find that

$$f(\tilde{x}_2^o, y_o) = -b + b\sqrt{1 - \frac{(\tilde{x}_2^o)^2}{a^2}} = -b + \sqrt{b^2 - a^2} \; . \qquad (8.65)$$

Thus the asymptotic behavior of $\underline{I}$ for $t \to \infty$ is given be

$$\underline{I} \sim \exp\left\{ -k\left[\sinh t \;\; b - \sqrt{b^2 - a^2}\right]\right\} . \qquad (8.66)$$

This means that the region of convergence of the plane wave expansion holds for $z > z^*$ with

$$z^* = -b + \sqrt{b^2 - a^2} \; , \; (b > a) \; . \qquad (8.67)$$

This plane goes through the focus of the spheroid.

Finally, we note that the results obtained earlier for the sphere follow immediately if we put $b = a + \epsilon$ and take $\epsilon > 0$ to be arbitrarily small.

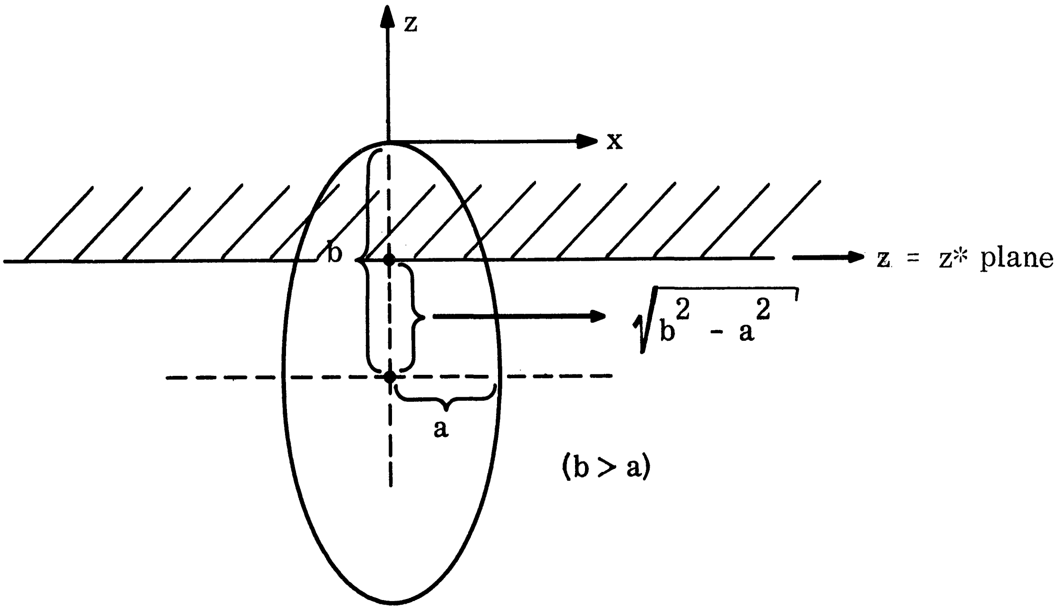

FIG. 8-9: REGION OF CONVERGENCE FOR THE PLANE WAVE EXPANSION FOR THE SPHEROID.

IX

REPRESENTATION OF THE FIELD INSIDE THE SCATTERING BODY AND CAVITY REGIONS

9.1 Continuation of the Field Inside of the Body.

In the previous sections the scattered field was represented in terms of a vector potential involving currents that were physical or otherwise; i.e. the fact that the scattered field arose from induced sources was not prescribed, only that it arose from some current distribution. Outside the source region the scattered field was then expressed in terms of an integral operator acting on the far-zone scattered field components. In this section the possibility of obtaining an expression for the total field inside the scattering body, in terms of the far-zone scattered field is examined.

As a preliminary, the derivation of the total field in terms of a vector potential relating to the actual induced currents (conduction and polarization) is reviewed. It will be assumed that the scattering body is contained in a finite volume V_s. The material of the body will be taken to be non-magnetic (i.e. $\mu = \mu_o$), and characterized by the relative permittivity ϵ' which may be complex allowing for conductivity. For present purposes the conductivity will be taken to be finite (but can be extremely large) thus ruling out the mathematical concept of a perfect conductor. Let the incident field be generated by a current source $\underline{J}_o$ outside the body. The source will first be taken a finite distance from the body, then later allowed to go to infinity, to account for plane wave incidence. Maxwell's equations become

$$\underline{\nabla} \cdot \underline{H} = 0 \qquad (9.1)$$

$$\omega\epsilon_o \underline{\nabla} \cdot \epsilon' \underline{E} = i\, \underline{\nabla} \cdot \underline{J}_o \qquad (9.2)$$

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o \underline{H} \qquad (9.3)$$

$$\underline{\nabla} \wedge \underline{H} = i\omega\epsilon_o \epsilon' \underline{E} + \underline{J}_o \qquad (9.4)$$

The field quantities $\underline{H}$ and $\underline{E}$ will be represented in terms of a vector potential $\underline{A}$ and a scalar potential ϕ as follows

$$\mu_o \underline{H} = \underline{\nabla} \wedge \underline{A} \tag{9.5}$$

$$\underline{E} = i\omega \underline{A} + \underline{\nabla}\, \phi \quad . \tag{9.6}$$

Equations (9.1) and (9.3) are automatically satisfied by the potential representation. Equations (9.2) and (9.5) become

$$i\omega\, \underline{\nabla} \cdot \epsilon' \underline{A} + \underline{\nabla} \cdot \epsilon' \underline{\nabla} \phi = \frac{i\omega\mu_o}{k_o^2}\, \underline{\nabla} \cdot \underline{J}_o \tag{9.7}$$

$$\nabla^2 \underline{A} - \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) + k_o^2\, \epsilon' \underline{A} - i\epsilon' \nabla (\omega\mu_o \epsilon_o \phi) = \mu_o \underline{J}_o \; . \tag{9.8}$$

Since in place of $\underline{A}$ one could have used $\underline{A} + \underline{\nabla}\, \psi$ where ψ is arbitrary, and still automatically satisfy (9.1) and (9.3), one can impose an additional condition of the potentials in terms of a gauge transformation. The particular choice will be taken as follows

$$\omega\mu_o \epsilon_o \phi = i\, \underline{\nabla} \cdot \underline{A} \tag{9.9}$$

Equation (3.8) reduces to

$$\nabla^2 \underline{A} - (1 - \epsilon')\, \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) + k_o^2\, \epsilon' \underline{A} = \mu_o \underline{J}_o \quad . \tag{9.10}$$

Taking the divergence of this equation, one obtains (9.7) automatically.

Thus, it is seen that with condition (9.9), the vector potential $\underline{A}$ must satisfy (9.10). Outside the body $\epsilon' = 1$, and this reduces to the free space Helmholtz equation operating on the components of $\underline{A}$.

The above equation can be placed in a different form useful for deriving an integral expression for $\underline{A}$. Eliminating the term $(1 - \epsilon') \, \underline{\nabla}(\underline{\nabla} \cdot \underline{A})$ from (9.10), with the help of relation (9.6) and (9.9), one obtains

$$\nabla^2 \underline{A} + k_o^2 \underline{A} = \mu_o \underline{J}_o - i\omega\mu_o\epsilon_o (1-\epsilon') \underline{E} . \tag{9.11}$$

It follows that $\underline{A}$ can be expressed in the form

$$\underline{A}(\underline{x}) = \frac{\mu_o}{4\pi} \int_V \left[\underline{J}_o + \underline{J} \right] \frac{e^{ikR}}{R} \, d\underline{x}' \tag{9.12}$$

where $R = \left| \underline{x} - \underline{x}' \right|$ and

$$\underline{J} = i\omega\epsilon_o (1 - \epsilon') \underline{E} . \tag{9.13}$$

This can be represented in the form

$$\underline{A}(\underline{x}) = \underline{A}^i(\underline{x}) + \frac{\mu_o}{4\pi} \int_{V_s} \underline{J}(\underline{x}') \frac{e^{ikR}}{R} \, d\underline{x}'$$

where $\underline{A}^i(\underline{x})$ is the vector potential of the incident field.

The magnetic field is thus given by

$$\underline{H}(\underline{x}) = \underline{H}^i(\underline{x}) + \frac{1}{4\pi} \int_{V_s} \underline{J}(\underline{x}') \wedge \underline{\nabla}' \left(\frac{e^{ikR}}{R} \right) d\underline{x}' \tag{9.14}$$

The source $\underline{J}_o$ giving rise to the incident field $\underline{H}^i$ can now be taken to infinity, in which case $\underline{H}^i$ will represent an incident plane wave. The current $\underline{J}(\underline{x}) = i\omega\epsilon_o(1-\epsilon') \underline{E}(\underline{x})$ is the current induced in the scattering body, being composed of conduction and polarization currents. Both the vector $\underline{H}$ and $\underline{A}$ will be continuous everywhere, due to the assumption that $\mu = \mu_o$ everywhere, and that ϵ' is finite. In the limiting case when the body is a perfect conductor, $\underline{H}$ is then discontinuous. It can be shown that in the limiting case when $\mathrm{Im}\,\epsilon' \to \infty$, (i.e. a perfect conductor), the volume integral in (9.14) reduces to a surface integral, and expression (9.14) reduces to

$$\underline{H}(\underline{x}) = \underline{H}^i(\underline{x}) + \frac{1}{4\pi}\int_S (\underline{n} \wedge \underline{H}) \wedge \underline{\nabla}' \frac{e^{ikR}}{R}\, ds$$

where S is the surface of the conductor and $\underline{n}$ is the unit outward normal.

Having considered the above preliminary work, we are now in a position to discuss the possible representation of the field inside the body in terms of the far scattered field. The notation $\underline{A}^s$ will be used to represent that part of the vector potential which results from the induced currents, i.e.:

$$\underline{A}^s(\underline{x}) = \frac{\mu_o}{4\pi}\int_{V_s} \underline{J}(\underline{x}) \frac{e^{ikR}}{R}\, d\underline{x}' \qquad (9.15)$$

The scattering body designated by the volume V_s will be split up into the following parts $V_+(\zeta)$, $V_-(\zeta)$ and $V_\delta(\zeta)$, where $V_+(\zeta)$ is the intersection of V_s and the half-space $z \leq \zeta$, $V_-(\zeta)$ is the intersection of V_s and the half-space $z \geq \zeta$, and $V_\delta(\zeta)$ the intersection of V_s and the slab $\zeta - \delta \leq z \leq \zeta + \delta$. The decomposition is displayed in Fig. 9-1.

Associated with the above, the following vector potentials will be considered,

$$\underline{A}_+(\underline{x}) = \frac{\mu_o}{4\pi}\int_{V_+(z-\delta)} \underline{J}(\underline{x}') \frac{e^{ikR}}{R}\, d\underline{x}' \qquad (9.16)$$

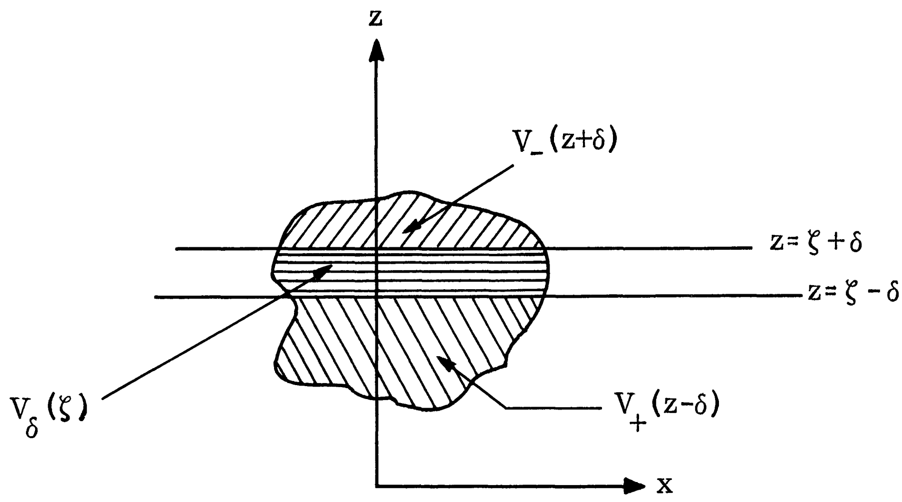

FIG. 9-1: DECOMPOSITION OF THE SCATTERING BODY

$$\underline{A}^s_-(\underline{x}) = \frac{\mu_o}{4\pi} \int_{V_-(z-\delta)} \underline{J}(\underline{x}') \frac{e^{ikR}}{R} d\underline{x}' \tag{9.17}$$

$$\underline{A}^s_\delta(\underline{x}) = \frac{\mu_o}{4\pi} \int_{V_\delta(z)} \underline{J}(\underline{x}') \frac{e^{ikR}}{R} d\underline{x}' \tag{9.18}$$

Using the relations

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot (\underline{x}-\underline{x}')} \sin\alpha \, d\alpha \, d\beta \quad z \geq z'$$

$$= - \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} + i\infty} e^{i\underline{k} \cdot \underline{x}} \sin\alpha \, d\alpha \, d\beta \quad z \leq z'$$

$\underline{A}^s_+(x)$ and $\underline{A}^s_-(x)$ become

$$\underline{A}^s_+(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k}\cdot\underline{x}} \int_{V_+(z-\delta)} \frac{\mu_o}{4\pi} \underline{J}(\underline{x}') \cdot e^{-i\underline{k}\cdot\underline{x}'} d\underline{x}' \sin\alpha \, d\alpha \, d\beta \qquad (9.19)$$

$$\underline{A}^s_-(\underline{x}) = -\frac{ik}{2\pi} \int_0^{2\pi} \int_{\pi}^{\frac{\pi}{2} - i\infty} e^{i\underline{k}\cdot\underline{x}} \int_{V_-(z+\delta)} \frac{\mu_o}{4\pi} \underline{J}(\underline{x}') \cdot e^{-i\underline{k}\cdot\underline{x}'} d\underline{x}' \sin\alpha \, d\alpha \, d\beta \qquad (9.20)$$

provides that $\underline{J}(\underline{x}')$ is absolutely integrable, allowing the order of integration to be interchanged.

If $\underline{J}$ is bounded, it follows that each component of $\underline{A}^s_\delta(\underline{x})$ is bounded

$$\left| A^s_\delta(\underline{x}) \right| \le \frac{\mu_o}{4\pi} M \int_0^{2\pi} \int_0^{a} \int_{-\delta}^{\delta} \frac{\rho d\rho d\phi d\zeta}{\sqrt{\rho^2 + \zeta^2}} = \mu_o M \int_0^{\delta} \sqrt{a^2 + \zeta^2} \, d\zeta$$

where $\rho^2 = (x-x')^2 + (y-y')^2$, and a is the maximum value of ρ such that the cylinder ρ = a encloses V_δ. It is easily seen then, that when $\delta \to 0$

$$\underline{A}^s_\delta(\underline{x}) \to 0. \qquad (9.21)$$

The above condition that $\underline{J}$ be bounded may be weakened, by allowing certain types of integrable singularities. However, these cases will not be considered at the present time.

Letting $\delta \rightarrow 0$, the vector potential $\underline{A}$ can be expressed as follows

$$\underline{A} = \underline{A}^i + \underline{A}_+^s(\underline{x}) + \underline{A}_-^s(\underline{x}) \tag{9.22}$$

where

$$\underline{A}_+^s(\underline{x}) = \frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \, \underline{A}_o^+(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta \tag{9.23}$$

and

$$\underline{A}_-^s(\underline{x}) = -\frac{ik}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} e^{i\underline{k} \cdot \underline{x}} \, \underline{A}_o^-(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta \tag{9.24}$$

with the vector $\underline{k}(\alpha, \beta) = k(\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha)$. The quantities $\underline{A}_o^+(\alpha, \beta)$ and $\underline{A}_o^-(\alpha, \beta)$ are the far field components

$$\underline{A}_o^+(\alpha, \beta) = \frac{\mu_o}{4\pi} \int_{V_+(z-0)} \underline{J}(\underline{x}') \, e^{-i\underline{k} \cdot \underline{x}'} \, d\underline{x}' \tag{9.25}$$

$$\underline{A}_o^-(\alpha, \beta) = \frac{\mu_o}{4\pi} \int_{V_-(z+0)} \underline{J}(\underline{x}') \, e^{-i\underline{k} \cdot \underline{x}'} \, d\underline{x}' \tag{9.26}$$

arising from an appropriate decomposition of the quantity

$$\underline{A}_o(\alpha,\beta) = \frac{\mu_o}{4\pi}\int_{V_s} \underline{J}(\underline{x}')\, e^{-i\underline{k}\cdot\underline{x}'}\, d\underline{x}' \tag{9.27}$$

defined previously, i.e.

$$\underline{A}_o(\alpha,\beta) = \underline{A}_o^+(\alpha,\beta) + \underline{A}_o^-(\alpha,\beta). \tag{9.28}$$

It can be shown that the same results hold for the magnetic field, in which case

$$\underline{H} = \underline{H}^i + \underline{H}_+^s + \underline{H}_-^s \tag{9.29}$$

where

$$\underline{H}_+^s(x) = \frac{-k}{2\pi\mu_o}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{i\underline{k}\cdot\underline{x}}\, \underline{k}\wedge\underline{A}_o^+(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta \tag{9.30}$$

$$\underline{H}_-^s(\underline{x}) = \frac{+k}{2\pi\mu_o}\int_0^{2\pi}\int_0^{\frac{\pi}{2}+i\infty} e^{i\underline{k}\cdot\underline{x}}\, \underline{k}\wedge\underline{A}_o^-(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta\,. \tag{9.31}$$

From the above it is seen that it is possible to obtain the magnetic field inside the body (composed of non-magnetic material with finite conductivity), from a knowledge of the far field data. This follows from the results in Chapter VI, which indicate how $\underline{k}(\alpha,\beta)\wedge\underline{A}_o(\alpha,\beta)$ may be determined for complex values of α where $\alpha = \theta + it$, from the knowledge of the far field quantities $A_\theta^o(\theta,\phi)$ and $A_\phi^o(\theta,\phi)$, measured in the range $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$. However, if the body is inside the slab $z_2 \le z \le z_1$, the appropriate split up of

$\underline{k} \wedge \underline{A}$ must be sought. The key problem remains of determining a method of uniquely performing this decomposition from knowledge of the far field data alone. Additional knowledge will most likely be required, such as, knowledge of the scattered field for all frequencies, or all angles of incidence.

9.2 Continuation into Cavity Regions

A practical technique can be formulated which can be used to find the scattered field in cavity portions of the body or portions of the minimum convex shape enclosing the equivalent sources. To demonstrate the process of continuation, the vector potential will be used, although, in practice the corresponding process would be employed with the field quantities. From (9.15), the vector potential of the scattered field is given by

$$\underline{A}^s(\underline{x}) = \frac{\mu_o}{4\pi} \int_{V_s} \underline{J}(\underline{x}') \frac{e^{ikR}}{R} d\underline{x}' , \tag{9.32}$$

where $\underline{J}(\underline{x}')$ are the physical currents, conduction and polarization. For the special case of a perfectly-conducting body, the volume integral is replaced by a surface integral containing the surface currents $\underline{j} = \underline{n} \wedge \underline{H}$.

Let $\underline{x}_o$ be a point outside the body. Then expression (9.32) can be placed in the form

$$\underline{A}^s(\underline{x}) = \frac{ik\mu_o}{4\pi} \sum_{n=0}^{\infty} j_n(kr) \int_{V_s} \underline{J}(\underline{x}')(2n+1) P_n(\cos\gamma) h_n^{(1)}(kr') d\underline{x}' \tag{9.33}$$

where

$$r' = \left| \underline{x}_o - \underline{x}' \right| , \tag{9.34}$$

$$r = \left| \underline{x} - \underline{x}_o \right| , \tag{9.35}$$

provided that $r < r'_{min}$, where r'_{min} is the minimum distance from $\underline{x}_o$ to the surface of the body. The angle γ is defined by the relation

$$(\underline{x} - \underline{x}_o) \cdot (\underline{x}' - \underline{x}_o) = r\, r' \cos\gamma$$

using a local spherical coordinate system (r, θ, ϕ) centered at $\underline{x}_o$, and the addition theorem

$$P_n(\cos\gamma) = P_n(\cos\theta)\, P_n(\cos\theta') +$$

$$+ 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi') \qquad (9.36)$$

one obtains the following expression

$$\underline{A}^s(\underline{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(kr)\, P_n^m(\cos\theta)\, e^{im\phi}\, \underline{a}_{nm} \qquad (9.37)$$

where the coefficients $\underline{a}_{nm}$ are expressed in terms of a volume integral over the currents, which will not be given here, since their actual form is not important.

Representation (9.37) will be convergent and will represent the scattered field in the domain $r \leq r'_{min}$. However this restriction on the domain was imposed by the derivation, namely by the radius of convergence of the chosen expansion of e^{ikR}/R in terms of the spherical Bessel functions. Thus it is possible that representation (9.37) will be convergent over a larger domain i.e. $0 < r < c$, where $c \geq r_{min}$. Similar results were developed for the expansions involving the far field quantities, given in Chapters IV and VIII. Representation (9.37) provides a practical means of continuing into cavity portions of the minimum convex shape enclosing the body or equivalent sources To illustrate this consider a body containing a cavity, as shown in Fig. 9-2.

Assume that the scattered field has been obtained from far field data for the region $z > z_o$. If the convex portion of the body is smooth and analytic, the plane $z = z_o$ cuts the body (see Chapter VIII). Take a point $\underline{x}_o$ to be in the domain $z > z_o$. From the above results, the vector potential can be represented in the form given by (9.37), and will hold for $r < c$, where c is undetermined but is greater or equal to r_{min} the minimum distance from $\underline{x}_o$ to the surface of the body. The unknown coefficients $\underline{a}_{nm}$ in expression (9.37) can be found since the value of $\underline{A}(\underline{x})$ is known on any sphere lying in the domain $z > z_o$, and centered at $\underline{x}_o$. Thus if r_1 is the radius of such a sphere and (r_1, θ_1, ϕ_1) are the coordinates of a point on this sphere, the coefficients, $\underline{a}_{nm}$ can be calculated from the relation

$$j_n(kr_1)\,\underline{a}_{nm} = \frac{(2n+1)(n-m)!}{4\pi\,(n+m)!}\int_0^{\pi}\int_0^{2\pi} P_n^m(\cos\theta_1)\,\cdot$$

$$\cdot\; e^{im\phi_1}\,\underline{A}(r_1, \theta_1, \phi_1)\,\sin\theta_1\,d\theta_1\,d\phi_1 \quad . \tag{9.38}$$

From this it is seen that the scattered field can be extended into the domain given by the intersection D of the sphere $\left|\underline{x} - \underline{x}_o\right| < c$ and the half-space $z < z_o$. Such an analytic continuation can be extended farther into the cavity region, by using representation (9.37) centered on a new point $\underline{x}_o^*$ located in the extended region D, and repeating this process.

In the above, the analysis has been carried out with the vector potential. A similar analysis will hold for the field quantities themselves.

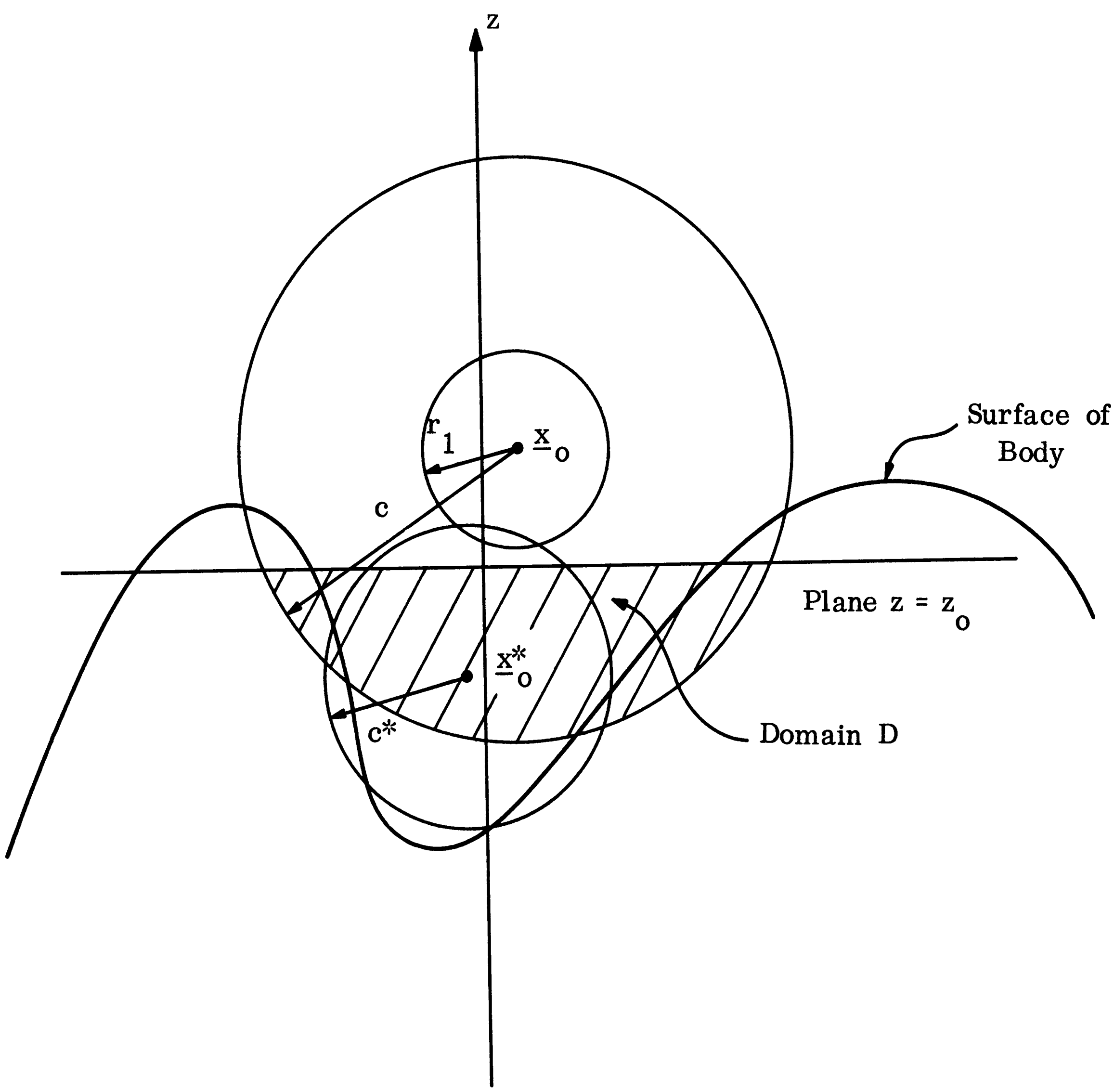

FIG. 9-2: ANALYTIC CONTINUATION IN CAVITY REGIONS.

X

DETERMINATION OF PERFECTLY CONDUCTING SHAPES

Let $\underline{E}$ be the total electric field and assume that

$$\underline{E} = \xi_1 \hat{a} + i\xi_2 \hat{b} \quad . \tag{10.1}$$

Then

$$\underline{E} \wedge \underline{E}^* = -2\, \xi_1 \xi_2\, \hat{a} \wedge \hat{b}. \tag{10.2}$$

On a perfectly conducting surface we have

$$\hat{n} \wedge \underline{E} = 0. \tag{10.3}$$

From this it follows necessarily that

$$\underline{E} \wedge \underline{E}^* = 0; \tag{10.4}$$

because if this were not true we see from (10.2) that $\underline{E}$ would be a vector having a complex direction, and this in turn would imply, since n is a real vector, that $\hat{n} \wedge \underline{E} \neq 0$ on the surface.

By means of the following example we shall see the significance of the observation (10.4). Consider a plane wave incident upon a perfectly conducting plane z = 0. Assume the time dependence $e^{-i\omega t}$ and the polarization so that

$$\underline{E}^i = (\hat{y} \cos\alpha + \hat{z} \sin\alpha)\, e^{ik(y \sin\alpha - z \cos\alpha)} \tag{10.5}$$

$$\underline{H}^i = \hat{x}.$$

Thus, for the reflected field we have

$$\underline{E}^r = (-\hat{y} \cos\alpha + \hat{z} \sin\alpha)\, e^{ik(y \sin\alpha + z \cos\alpha)} \tag{10.6}$$

$$\underline{H}^r = \hat{x}\, .$$

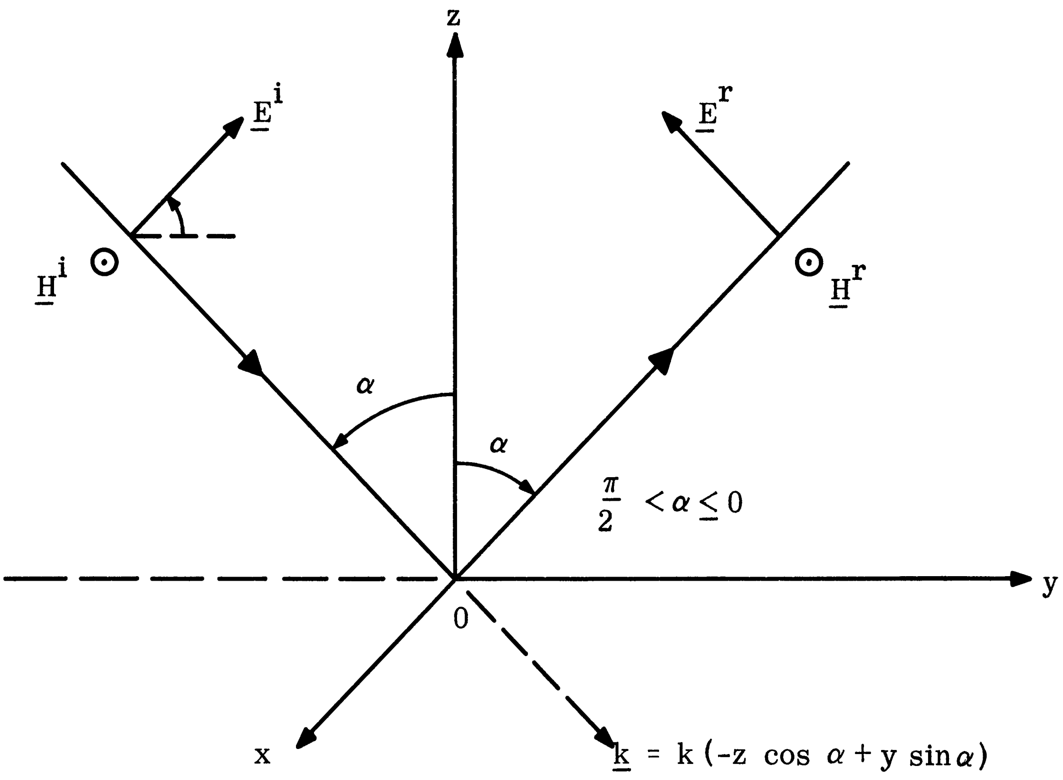

FIG. 10-1: GEOMETRY FOR PLANE WAVE INCIDENCE

We find the total field to be

$$E_y = 2i \cos\alpha \, e^{iky\sin\alpha} \sin(kz\cos\alpha),$$

$$E_z = 2\sin\alpha \, e^{iky\sin\alpha} \cos(kz\cos\alpha). \qquad (10.7)$$

Therefore,

$$\underline{E}^* = 2e^{iky\sin\alpha}\left[\hat{z}\sin\alpha\cos(kz\cos\alpha) - i\hat{y}\cos\alpha\sin(kz\cos\alpha)\right], \qquad (10.8)$$

so that

$$\underline{E} \wedge \underline{E}^* = 2i\hat{x}\, e^{2iky\sin\alpha} \sin(2kz\cos\alpha). \qquad (10.9)$$

We now let

$$\underline{E} \wedge \underline{E}^* = 0 \text{ i.e., } \sin(2kz\cos\alpha) = 0, \qquad (10.10)$$

and find that

$$z = \frac{n\pi}{2k\cos\alpha}, \quad n = 0, \pm 1, \pm 2, \cdots \quad (k \neq 0, \ \alpha \neq \pi/2). \qquad (10.11)$$

We now observe that if we had been given the reflected field (10.6) and had known that the scattering surface was a perfect conductor, then by invoking the necessary surface condition $\underline{E} \wedge \underline{E}^* = 0$ we would have arrived to the conclusion in (10.11). However, since the actual scatterer must be independent of k, we pick out the surface z = 0 out of the family equation (10.11), thereby determining the scatterer uniquely.

However, when the incident polarization is parallel to the generators of two dimensional bodies, then $\underline{E}^* \wedge \underline{E} = 0$ everywhere, as is seen in the above example for the case $\underline{E}^i = \hat{x}$. In this two dimensional case, the condition $\underline{E}^* \wedge \underline{E} = 0$ fails to provide useful information.

The condition $\underline{E} \wedge \underline{E}^* = 0$ will play an important role in any numerical attack on the determination of the surface of a finite three dimensional conducting body from far field measurements.

Next we remark on the uniqueness question for more general case. Consider the scattered field due to a smooth, perfectly conducting, convex surface S and assume that an analytic expression for the field is known everywhere exterior to the equivalent source region which resides inside S. In seeking the surface S by looking for the surface on which the electric field obeys the required boundary condition, it is possible that more than one eligible surface may be found for a particular wave number k.

Let us assume, therefore, that two perfectly conducting surfaces S and S_1 have been found. These surfaces both surround the equivalent source region and are taken to be smooth. In the volume V between the two surfaces, the total electric field satisfies the source-free wave equation

$$(\nabla^2 + k^2)\,\underline{E} = 0 \tag{10.12}$$

together with the equation

$$\mathrm{div}\,\underline{E} = 0 \quad . \tag{10.13}$$

However, solutions of these equations in the simply connected cavity V such that

$$\underline{n} \wedge \underline{E} = 0 \tag{10.14}$$

on the bounding surfaces S, S_1 exist only for a discrete set of eigenfrequencies. Thus, if k varies continuously, the shape of S_1 must change in order to satisfy the boundary condition since by definition the scattering surface S is independent of the wavelength of the incident field. The requirement that S remain unchanged as the frequency is varied continuously therefore allows us to determine the scattering surface uniquely.

XI

USE OF MONOSTATIC-BISTATIC THEOREM TO DETERMINE MATERIAL CHARACTERISTICS

For a class of perfectly-conducting bodies, the monostatic-bistatic cross-section theorem stated as follows, is well known: In the limit of vanishing wavelength, the bistatic cross-section for transmitter direction $\underline{k}$ and receiver direction n_o is equal to the monostatic cross-section for the transmitter-receiver direction $\underline{k} + \underline{n}_o$ with $\underline{k} \neq \underline{n}_o$ for bodies which are sufficiently smooth.

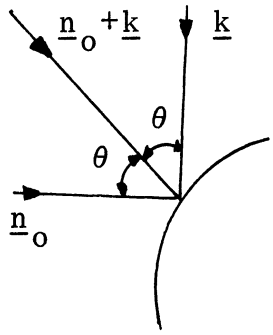

FIG. 11-1: GEOMETRY FOR MONOSTATIC-BISTATIC THEOREM

This theorem may be amended for non-perfectly conducting bodies, thus yielding information on the material characteristics of the body.

Let 2θ be the angle formed by the vectors $\underline{\hat{n}}_o$ and $\underline{k}$ (i.e., the bistatic angle). Let the surface of the body be sufficiently smooth, and let its electrical properties be characterized by a voltage reflection coefficient R, which is a function of the angle of incidence (θ) and polarization; i.e., $R = R_{\parallel}(\theta)$ for polarization in plane of incidence and $R = R_{\perp}(\theta)$ for polarization perpendicular to plane of incidence. Denote $\sigma_{\perp}(\underline{k}, \underline{\hat{n}}_o)$ as the bistatic cross-section where both transmitting and receiving antennas are linearly polarized perpendicular to the plane formed by the vectors $(\underline{k}, \underline{\hat{n}}_o)$. Let $\sigma_{\parallel}(\underline{k}, \underline{\hat{n}}_o)$ denote the bistatic cross-section where both antennas are polarized parallel to the plane $(\underline{k}, \underline{\hat{n}}_o)$.

The bistatic cross-section $\sigma_{\perp}(\underline{k}, \hat{\underline{n}}_o)$ is a product of two factors; a geometrical factor depending upon the radii of curvature of the surface of the body, and a material factor depending upon the reflection coefficient. However, as implied by the monostatic-bistatic theorem given above, the geometrical factor for $\sigma_{\perp}(\underline{k}, \hat{\underline{n}}_o)$ and $\sigma_{\perp}(\underline{k}+\hat{\underline{n}}_o, \underline{k}+\hat{\underline{n}}_o)$ are the same. Thus, it follows that

$$\frac{\sigma_{\perp}(\underline{k}, \hat{\underline{n}}_o)}{\sigma_{\perp}(\underline{k}+\hat{\underline{n}}_o, \underline{k}+\hat{\underline{n}}_o)} = \left| \frac{R_{\perp}(\theta)}{R_{\perp}(0)} \right|^2 = r_{\perp} \; . \tag{11.1}$$

Similarly, it follows that

$$\frac{\sigma_{||}(\underline{k}, \hat{\underline{n}}_o)}{\sigma_{||}(\underline{k}+\hat{\underline{n}}_o, \underline{k}+\hat{\underline{n}}_o)} = \left| \frac{R_{||}(\theta)}{R_{||}(0)} \right|^2 = r_{||} \; . \tag{11.2}$$

The particular case where the material characteristics of the surface can be represented by an impedance boundary condition will be considered to determine the number of measurements needed to prescribe the impedance parameter η. The effect of the surface upon incident energy can be represented in the form

$$\underline{E} - (\underline{E} \cdot \underline{n})\, \underline{n} = \eta \sqrt{\frac{\mu_o}{\epsilon_o}} \; \underline{n} \wedge \underline{H} \tag{11.3}$$

where $\underline{E}$ and $\underline{H}$ are the total fields generated on the surface. Such a condition represents either a poor conductor, or perfect conductors coated with a material of high index of refraction as is encountered in the use of magnetic type absorbers. For a single layer of such material η is given by

$$\eta = -i\sqrt{\frac{\mu}{\epsilon}} \tan(Nk\delta) \tag{11.4}$$

where δ is the thickness of the coating, N is the index of refraction, and μ, ϵ are the relative parameters of the coating.

The voltage reflection coefficients for such a surface can be represented in terms of η and the angle of incidence θ, by the following relations

$$R_{\perp} = \frac{\eta \cos\theta - 1}{\eta \cos\theta + 1} \tag{11.5}$$

$$R_{||} = \frac{\eta / \cos\theta - 1}{\eta / \cos\theta + 1} \tag{11.6}$$

Let the real and imaginary parts of η be given by u and v, that is

$$\eta = u + iv \ . \tag{11.7}$$

It can be shown that

$$\left| R_{\perp}(\theta) \right|^2 = \frac{\left[(u^2 + v^2)\cos^2\theta + 1 - 2u\cos\theta\right]}{\left[(u^2 + v^2)\cos^2\theta + 1 + 2u\cos\theta\right]} \ , \tag{11.8}$$

$$\left| R_{||}(\theta) \right|^2 = \frac{\left[(u^2 + v^2) + \cos^2\theta - 2u\cos\theta\right]}{\left[(u^2 + v^2) + \cos^2\theta + 2u\cos\theta\right]} \ . \tag{11.9}$$

For further simplification, the parameters u and v will be replaced by x and y where

$$x = 2u \tag{11.10}$$

$$y = u^2 + v^2 \ . \tag{11.11}$$

It then follows, provided that $R_{\perp}(0) \neq 0$ and $R_{\parallel}(0) \neq 0$,

$$r_{\perp} = \frac{\left[y \cos^2\theta + 1 - x \cos\theta\right]}{\left[y \cos^2\theta + 1 + x \cos\theta\right]} \frac{\left[y + 1 + x\right]}{\left[y + 1 - x\right]} \tag{11.12}$$

$$r_{\parallel} = \frac{\left[y + \cos^2\theta - x \cos\theta\right]}{\left[y + \cos^2\theta + x \cos\theta\right]} \frac{\left[y + 1 + x\right]}{\left[y + 1 - x\right]} . \tag{11.13}$$

Performing algebraic manipulation, one can rewrite the above equations in the following form

$$\cos\theta\, x^2 + p_1 \cos\theta(1 - \cos\theta)\, xy - \cos^2\theta\, y^2 + p_1(\cos\theta - 1)\, x - (1 + \cos^2\theta)\, y - 1 = 0 \tag{11.14}$$

$$\cos\theta\, x^2 + p_2(\cos\theta - 1)\, xy - y^2 + p_2 \cos\theta(1 - \cos\theta)\, x - (1 + \cos^2\theta)\, y - \cos^2\theta = 0 \tag{11.15}$$

where

$$p_1 = \left[1 + r_{\perp}\right] / \left[1 - r_{\perp}\right] \tag{11.16}$$

$$p_2 = \left[1 + r_{\parallel}\right] / \left[1 - r_{\parallel}\right] . \tag{11.17}$$

The quantities p_1 and p_2 are both real and are greater or equal to unity. The problem reduces to solving the two equations for the unknown quantities x and y, in terms of the parameters p_1 and p_2 which are obtained from the measured quantities $r_{\perp}$ and $r_{\parallel}$. The angle θ is of course known, being one-half the bistatic angle. However, the required solution must lie in the first quadrant of the xy plane. The reason for this is twofold. First from the definition $u^2 + v^2 = y$, and the fact that u and v are real quantities, the required value of y must be greater or equal to zero. Secondly, it can be shown from energy considerations (the surface can only absorb energy), that $u \geq o$, implying that $x \geq o$. Both (11.14) and (11.15) represent conic section, in the xy plane. The solutions are given by the intersections of these two conic sections. However, it is possible that there are no intersections, and if there are, they may lie outside the first quadrant. Thus, the nature of the conic sections will have to be further examined to indicate whether the appropriate solutions exist.

Consider a general conic section in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{11.18}$$

Its center is at the point (k, ℓ) where

$$k = \frac{fh - bg}{ab - h^2} \quad , \quad \ell = \frac{gh - af}{ab - h^2} \quad . \tag{11.19}$$

By transforming the coordinate system (x, y) to a coordinate system (X, Y) centered at (k, ℓ) with the axis centered along the principal axis of the conic section, using the following relations

$$x - k = X \cos \beta - Y \sin \beta \tag{11.20}$$

$$y - \ell = X \sin \beta + Y \cos \beta \tag{11.21}$$

where

$$\tan 2\beta = \frac{2h}{a - b} \tag{11.22}$$

the equation of the conic section in the new coordiante system becomes

$$X^2 A + Y^2 B + \Delta \Big/ \left[ab - h^2\right] = 0 \tag{11.23}$$

where

$$2A = (a + b) + (a - b) \cos 2\beta + 2h \sin 2\beta \tag{11.24}$$

$$2B = (a + b) - (a - b) \cos 2\beta - 2h \sin 2\beta \qquad (11.25)$$

$$D = abc + 2fgh - af^2 - bg^2 - ch^2 \ . \qquad (11.26)$$

In the cases under consideration (a - b) is positive but h may be positive or negative. The angle $|2\beta|$ will be taken to be less than $\pi/2$, in which case

$$\cos 2\beta = (a - b) \Big/ \sqrt{4h^2 + (a - b)^2} \qquad (11.27)$$

$$\sin 2\beta = 2h \Big/ \sqrt{4h^2 + (a - h)^2} \ . \qquad (11.28)$$

Thus 2A and 2B can be given by

$$2A = a + b + \sqrt{4h^2 + (a - b)^2} \qquad (11.29)$$

$$2B = a + b - \sqrt{4h^2 + (a - b)^2} \qquad (11.30)$$

in which case

$$AB = ab - h^2 \ . \qquad (11.31)$$

Define the conic sections given (11.14) and (11.15) by c_1 and c_2 respectively. The various parameters associated with these conic sections are given in Table XI - 1. The parameter H is defined by the relation

$$H = (1 + \cos \theta)^2 \ / \ (1 - \cos \theta)^2 \ . \qquad (11.32)$$

It is seen that the centers of the conic section c_1 and c_2 lie in the right-half and left-half planes respectively. Also the equations of both conic sections can be written in the form

Parameters	c_1	c_2
$ab - h^2$	$-\cos^2\theta\left[\cos\theta + p_1^2(1-\cos\theta)^2/4\right]$	$-\left[\cos\theta + p_2^2(1-\cos\theta)^2/4\right]$
$k(ab - h^2)$	$-p_1\cos\theta\sin^2\theta(1+\cos\theta)/4$	$+p_2\sin^2\theta(1+\cos\theta)/4$
$\ell(ab - h^2)$	$1/2\cos\theta\left[1+\cos^2\theta - 1/2\,p_1^2(1-\cos\theta)^2\right]$	$1/2\cos\theta\left[1+\cos^2\theta - 1/2\,p_2^2(1-\cos\theta)^2\right]$
$\left.\begin{matrix}2A\\ 2B\end{matrix}\right\}$	$\cos\theta(1-\cos\theta)\left[1 \pm \sqrt{p_1^2 + H^2}\right]$	$(1-\cos\theta)\left[-1 \pm \sqrt{p_2^2 + H^2}\right]$
Δ	$1/4\cos\theta\sin^4\theta(p_1^2 - 1)$	$1/4\cos\theta\sin^4\theta(p_2^2 - 1)$
sign β	+	-

TABLE XI - 1.

$$\frac{X^2}{A_1^2} - \frac{Y^2}{B_1^2} = 1.$$

Further information can be found by examining the y-intercepts, the x-intercepts, and the asymptotes. Conic section c_1 has y-intercepts

$$y_1^{\,1} = -1 \text{ and } y_2^{\,1} = -1/\cos^2\theta \tag{11.33}$$

and c_2 has y-intercepts

$$y_1^{\,2} = -1 \text{ and } y_2^{\,2} = -\cos^2\theta\ . \tag{11.34}$$

The x-intercepts of c_1 and the slope of the asymptotes are related. If x_1 and x_2 are the x-intercepts, given by the relations

$$x_1^{\,1} = \left[p_1(1-\cos\theta) + \sqrt{p_1^{\,2}\,(1-\cos\theta)^2 + 4\cos\theta}\,\right] \Big/ (2\cos\theta) \tag{11.35}$$

$$x_2^{\,1} = -\left[x_1\cos\theta\right]^{-1}$$

then the equations for the asymptotes have the form

$$y = x_1^1\, x, \quad y = x_2^1\, x\ . \tag{11.36}$$

There is a similar relationship between of the asymptotes of c_2 and the x-intercepts given by

$$x_1^{\,2} = 1/2\left[-p_2(1-\cos\theta) + \sqrt{p_2^{\,2}(1-\cos\theta)^2 + 4\cos\theta}\,\right] \tag{11.37}$$

$$x_1^{\,2} = -\cos\theta / x_1^{\,2}\ . \tag{11.38}$$

In addition it can be shown that c_1 passes through the points

$$\left(\pm\left[1 + \frac{1}{\cos\theta}\right], \quad \frac{1}{\cos\theta}\right)$$

and c_2 through the points

$$\left(\pm\left[1 + \cos\theta\right], \quad \cos\theta\right) \quad .$$

The conic sections are shown in Fig. 11-2 for a typical case. As indicated there is an intersection in the first quadrant. Except for the case where $p_1 = \infty$ and $p_2 = \infty$ (i.e. $r_\perp = r_{||} = 1$), it can be shown that there will always be one intersection in the first quadrant. This follows from the fact that $1 < x_1^1 \leq \infty$, and $o \leq x_1^2 < 1$. Since x_1^1 and x_1^2 are the slopes of the asymptotes of the branches of the conic section in the first quadrant, these branches intercept, and since x_1^1 and x_1^1 are also the x-intercepts on the positive x-axis, the intersection is in the first quadrant. Thus a solution can be found for which $x \geq o$ and $y \geq o$. However, from relation (11.11) two values of v will be found. This means that the impedance will be determined apart from the sign of the imaginary part i.e; $\eta = u \pm iv$. The determination of the appropriate sign will require measurement of the phases of the scattered field.

As a special case, it should be pointed out, that when $r_\perp = r_{||} = 1$, the solution is not unique, with u = 0 and v undetermined. The most likely possible physical case that would occur in this instance is where v = 0 also, implying the surface is a perfect conductor.

Summing up, it is shown that the two polarization measurements of cross section at one non-zero bistatic angle (backscattering) determines the reactive surface impedance of $\eta = u \pm iv$ apart from the sign in the imaginary part, where such surfaces would correspond to poor conductors, or absorber coated conductors. However, the case where the ratio of the bistatic monostatic cross-section is unity for both polarizations, produced incomplete results. In this case, it could only be concluded that u = 0.

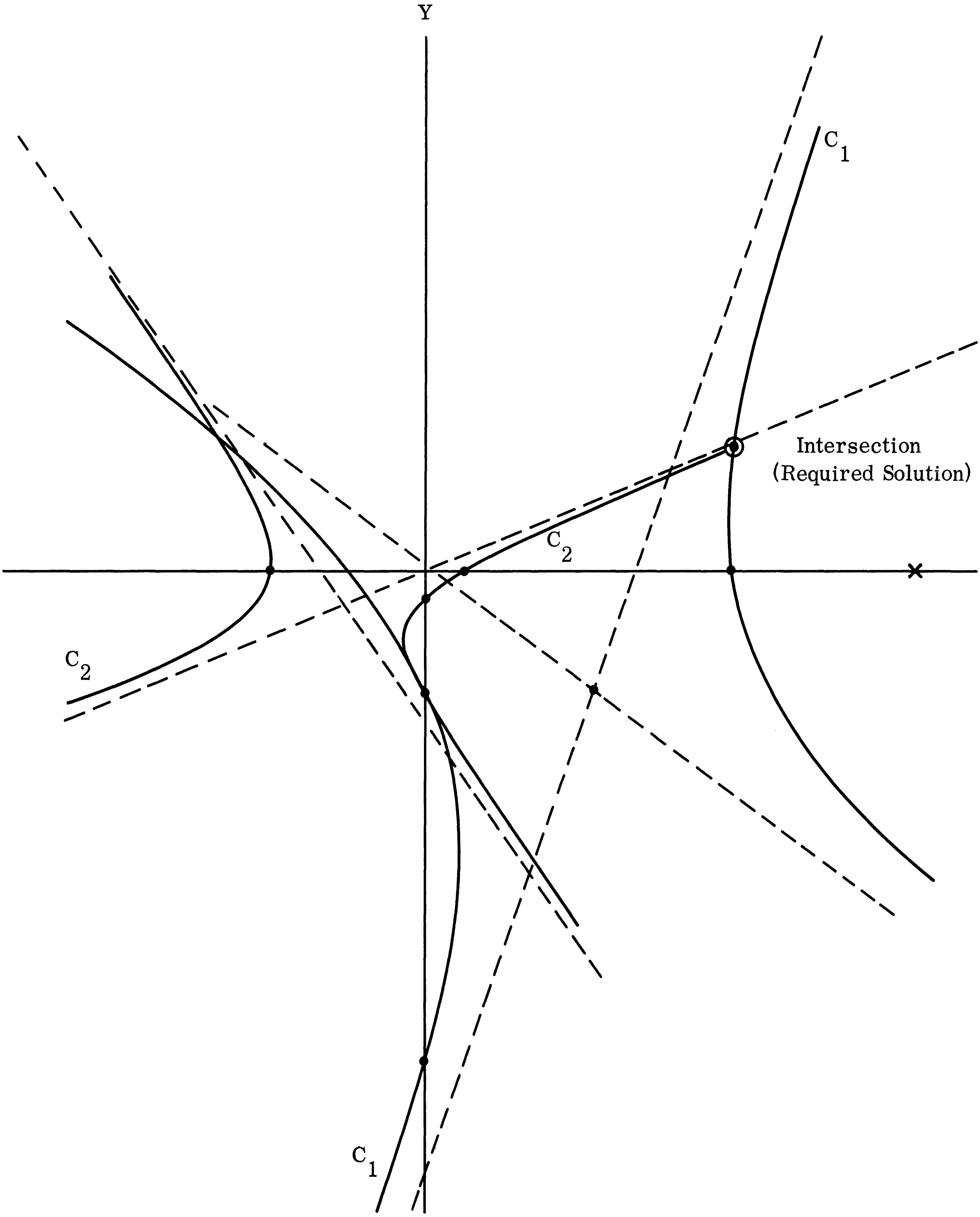

FIG. 11-2: CONIC SECTIONS ASSOCIATED WITH THE DETERMINATION OF SURFACE IMPEDANCE

REFERENCES

Altman, J. L., R.H.T. Bates and E. N. Fowle (1964) "Introductory Notes Relating to Electromagnetic Inverse Scattering," The Mitre Corporation Report No. SR - 121, Bedford, Massachusetts.

Brindley, C. (1965) "Target Recognition," Space/Aeronautics, 43, No. 6, 62-68.

Bouwkamp, C. J. (1954), "Diffraction Theory," Rep. Progr. Phys., 17, 35-100.

Erdélyi, A. (1948), "Zur Theorie der Kugelwellen," Physica, 4, 107-120.

Faddeyev, L.D. (1963) "The Inverse Problem in Quantum Theory of Scattering," J. Math. Phys., 4, 72.

Freedman, A. (1963) "The Portrayal of Body Shape by Sonar or Radar System," J. Brit. Inst. Radio Engrs., 25, 51-64.

Fock, V.A. (1948), "Fresnel Reflection Laws and Diffraction Laws," Uspekhi Fiz. Nauk, 36, 308-319.

Karp, S. N. (1962) "Far Field Amplitudes and Inverse Diffraction Theory," Electromagnetic Waves, (ed.) R. E. Langer, University of Wisconsin Press, Madison.

Karp, S. N. (1961) "A Convergent 'Farfield' Expansion for Two-Dimensional Radiation Functions," Comm. Pure Appl. Math., 14, 427-434.

Kay, I. W. (1962) "The Three-Dimensional Inverse Scattering Problem," New York University Courant Institute of Math. Sciences; Division of Electromagnetic Research, Research Report No. EM 174.

Keller, J. B. (1959) "The Inverse Scattering Problem in Geometrical Optics and the Design of Reflections," IRE TRANS., AP-7, 146-149.

Kennaugh, E. M. and D. L. Moffatt (1965) "Transient and Impulse Response Approximations," Proc. IEEE, 53, No. 8, 893-901.

Kline, M. and I. W. Kay (1965), Electromagnetic Theory and Geometrical Optics, Interscience Publishers, New York.

Magnus, W. and F. Oberhettinger (1954), Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Publishing Company.

Moses, H. E. and C. M. deRidder (1963), "Properties of Dielectrics from Reflection Coefficients in One Dimension," MIT-Lincoln Laboratory Technical Report No. 322.

Müller, C. (1955), "Radiation Patterns and Radiation Fields," J. Rat. Mech. and Anal., 4, 235.

Müller, C. (1956), "Electromagnetic Radiation Patterns and Sources," IRE Trans., AP-4, No. 3, 224-232.

Nirenberg, L. (1953), " The Weyl and Minkowski Problems in Differential Geometry in the Large," Comm. Pure and Appl. Math., 6, 337-394.

Petrina, D. Ia. (1960), "Solution of the Inverse Scattering Problem," Ukraniakii Matem. Zhurnal (USSR) 12, No. 2, 476-479 (Translation: OTS 62-11505; AD 255-678).

Stratton, J.A. (1941), Electromagnetic Theory, McGraw-Hill Book Company, New York.

Watson, G.N. (1952), Theory of Bessel Functions, Cambridge University Press.

Weil, H., M. L. Barasch, and T. A. Kaplan (July 1956), "Studies in Radar Cross Sections X: Scattering of Electromagnetic Waves by Spheres," The University of Michigan Radiation Laboratory Report 2255-20-T, AD 114 234. UNCLASSIFIED.

Weyl, H. (1919) "Ausbreitung elektromagnetischer Wellen über einem ebenen Leiter," Ann. Physik Ser., 4, 60, 481-500.

Wilcox, C. H. (1956), "An Expansion Theorem for Electromagnetic Fields," Comm. Pure Appl. Math., 9, 115-134.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
The University of Michigan Radiation Laboratory Department of Electrical Engineering Ann Arbor, Michigan 48108	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

INVERSE SCATTERING INVESTIGATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Report (1 October 1965 - 30 September 1966)

5. AUTHOR(S) (Last name, first name, initial)

Weston, Vaughan H., Bowman, John J., Ar, Ergun.

6. REPORT DATE	7a TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1966	96	24

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
AF 19(628)-4884	7644-1-F
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Qualified requestors may obtain copies of this report from DDC

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Electronic Systems Division USAF, AFSC L. G. Hanscom Field Bedford, Massachusetts 01731

13. ABSTRACT The problem in question consists of determining the means of solving the inverse scattering problem where the transmitted field is given and the received fields are measured, and this data is used to discover the nature of the target. The concept of equivalent sources is introduced wherein the scattered field may be thought of as arising from a set of equivalent sources located on or within the body. This concept is introduced since the radii of the minimum convex surface which encloses the equivalent sources is related to the convergence of any expansion technique utilized to derive the near scattered field of the target from the observed far field. One particular expansion technique is investigated. It is based upon the expansion of the far field in the form of an inverse power series in r, multiplied by the factor exp(ikr), where (r,θ,ϕ) are the coordinates of a spherical polar coordinate system. The approach to the inverse scattering problem based upon the representation of the scattered field in terms of plane waves is investigated. An explicit expression for the scattered field, valid in the half-space which depends upon the coordinate axis, is given in terms of an integral operating on the far scattered field. It is shown that the plane wave representation converges part way inside smooth convex portions of the body, thus establishing the concept that the minimum convex shape enclosing the equivalent sources often may be inside the actual scattering body. For non-magnetic and non-perfectly conducting bodies, the exact total field inside the body could be represented in terms of a plane wave expansion involving the far field quantities. This representation involves an appropriate split up of the far field data, and a fundamental problem still exists to uniquely determine the split up from the knowledge of the far field data alone. The monostatic-bistatic theorem is used to determine the material characteristics of the scatterer.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Inverse Scattering Electromagnetic Theory Radar Identification of Objects Near Field Representation in Terms of Far Field Data						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.