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# PRESSURE PULSE RECEIVED DUE TO AN EXPLOSION IN THE ATMOSPHERE AT AN ARBITRARY ALTITUDE PART I

STUDIES IN RADAR CROSS SECTIONS XLI

by

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References

## INTRODUCTION

There have been many recent efforts devoted to the study of "free" waves in the atmosphere and the pressure pulse produced by large explosions such as the Krakatoa volcanic eruption of 1883, the great Siberian meteorite of 20 June 1908, and, recently, nuclear explosions. As an example of the latter, the Japanese have recorded such atmospheric waves from nuclear explosions (R. Yamamoto, Ref. 1).

The pressure pulse produced by such explosions can be decomposed into two portions, a low frequency "gravity" wave train, plus high frequency "acoustic" wave trains. Theoretical analysis has been mainly devoted to the "gravity" wave portion of the pressure pulse produced by large explosions on the ground for simple models of the atmosphere. (Pekeris, Refs. 2 and 3, Scorer, Ref. 4.)

Recently Dikii (Ref. 5) has discussed the gravity and acoustical type waves. Penney et al, (Ref. 6) have calculated both the gravity wave and acoustic wave portions of the pressure pulse produced by explosions on the ground.

Here the interest will be mainly concerned with the gravity wave, produced by explosions not only on the ground, but at various heights in the atmosphere. At present the effect of winds is ignored, but will be considered later. They do have an influence upon the pressure pulse, although the high-frequency acoustical portion is affected the most.

It is assumed that the explosion is symmetric about an axis normal to the earth's surface, and that beyond a specific distance away from the center of the explosion the perturbed values of pressure etc., will be small compared to the unperturbed value. Hence beyond this distance (characterized by a surface $\sum$) the hydrodynamical equations may be linearized. The explosion can then be represented in terms of the excess pressure and normal velocity on this surface $\sum$. Unfortunately analytical solutions are not available for large explosions. An analytic solution has been found for an intense spherically symmetric explosion (J. L. Taylor, Ref. 7) but it holds down to only over-pressure of about 20 atmospheres. Hence for good source models, values of the excess pressure and normal velocity must be obtained from observational data.

Hence the hydrodynamical equations are linearized with the viscosity terms neglected (actually these become important at very high altitudes, roughly around 200 km). The time dependence of the equations is removed by taking a Laplace transform of them. A second order differential equation involving the excess pressure transform is obtained. The boundary conditions that are imposed, are, that the vertical velocity must vanish at the earth's surface, and the total kinetic energy in a solid angle subtended at the earth's surface be finite. This latter condition (which is a restriction upon the behavior of the excess pressure at high altitudes) is the same condition Scorer and Penney used.

A ring source Green's function is then defined. It is shown that the excess pressure may be represented in terms of an integral over the surface surrounding the source, the integral containing the Green's function together with the values of the excess pressure and normal velocity on $\sum$.

An eigenfunction expansion is then obtained for the Green's function involving the various modes. Only the contribution due to the modes corresponding to the "free" waves (those waves which propagate around the earth without exponential attenuation) are then retained.

It is shown that for certain temperature models of the atmosphere a good approximation can be made to the "gravity" wave mode. This approximation gives good results for Scorer's model of the atmosphere.

The pressure pulse for the directly received wave (as contrasted to the antipodal wave) is computed for various ranges, for two models of the atmosphere. In doing so, a simple source model is taken, namely a point source in space with a delta function dependence in time. The intensity of the explosion is given in terms of volume of gas introduced. In the calculation of the pressure pulse at various ranges from the source, emphasis is placed upon the main body of the gravity wave portion. The tail of the pulse up to the so-called cut-off point is not considered. In calculating the head of the pulse a new asymptotic technique is introduced which gives very good results, for intermediate and long ranges.

# PRESSURE PULSE RECEIVED DUE TO AN EXPLOSION IN THE ATMOSPHERE AT AN ARBITRARY ALTITUDE PART I

Initial Nomenclature:

(1) Spherical polar coordinates $(r, \theta, \phi)$

(2) Radius of earth $r = a$

(3) $T$ = absolute temperature

$R$ = gas constant

$\gamma$ = ratio of specific heats

$p$ = gas pressure

$\rho$ = gas density

$c_0^2 = \gamma R T_0 = \gamma p_0 / \rho_0$

$\underline{u}$ = velocity = $(u_r, u_\theta, u_\phi)$

(4) Unperturbed state denoted by subscripts zero, i.e. $T_0$, $\rho_0$, etc.

(5) Perturbed state is represented in terms of the unperturbed state plus a perturbation term (which has no subscript) i.e. $p_0 + p$, $T_0 + T$, ...

## 1. Characterization of the Source of the Pressure Pulse

A large explosion in the atmosphere is usually detected at large distances by changes in pressure. Hence the excess pressure $p(\underline{r}, t)$ will be solved for. At a sufficient distance from the source of the explosion it is known that the excess variables, pressure, density and temperature are small in comparison with the corresponding unperturbed variables for the atmosphere.

Let the source of the explosion be at the point $(R_0, 0, 0)$ where $R_0 \geq a$ (the radius of the earth). It will be assumed that the explosive effects are independent of the azimuth angle $\phi$. The source can be enclosed by a surface of revolution $\Sigma_1$ with axis of revolution $\theta = o$ (i.e. the z axis), such that outside this surface, the excess pressure, density, etc. are small in comparison with their respective unperturbed values. The surface $\Sigma_1$ may be a small sphere with centre $(R_0, 0, 0)$. However, its actual form will not be specified.

Let $\mathcal{D}$ represent the region outside $\Sigma_1$ and outside the earth $(r = a)$. Hence for the region $\mathcal{D}$, the assumption that the excess variables are small in comparison with their respective unperturbed values, is valid.+

The source of the pressure pulse will be characterized by boundary conditions on the surface $\Sigma_1$. To specify the problem uniquely $p(r, t)$ and $\underline{n} \cdot \underline{u}$ for $\underline{r}$ on $\Sigma_1$ must be specified. $\underline{n}$ represents the unit outward normal to $\Sigma_1$.

Hence set the initial conditions.

$$\left.\begin{aligned} u(\underline{r}, t) &\equiv 0 \\ p(\underline{r}, t) &\equiv 0 \\ \rho(\underline{r}, t) &\equiv 0 \end{aligned}\right\} \qquad t < 0 \quad \text{and } \underline{r} \in \mathcal{D}$$

+ In addition viscosity and the effects of winds will be neglected for the present.

$$\left.\begin{aligned} p(\underline{r}, t) &\equiv 0 && t < 0 \\ &\equiv f(\underline{r}, t) && t \geq 0 \\ \underline{n} \cdot \underline{u} &\equiv 0 && t < 0 \\ &\equiv g(\underline{r}, t) && t \geq 0 \end{aligned}\right\} \qquad \underline{r} \in \Sigma$$

where $f(\underline{r}, t)$ and $g(\underline{r}, t)$ are given functions such that $f(r, t)$ and $g(\underline{r}, t)$ both vanish when $t \to \infty$.

## 2. The Equations of Motion:

To begin with note that the explosive effects are independent of $\phi$, hence all the variables will be independent of $\phi$.

From Ramsey (Ref. 8), we have the equations of motion

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} + \underline{K} = -\frac{1}{(\rho + \rho_0)} \underline{\nabla}(p + p_0) + (-g, 0, 0) \tag{1}$$

where

$$\underline{K} = \frac{1}{r}\left(-\left[u^2_\theta + u^2_\phi\right], \left[u_r u_\theta - u^2_\phi \cot\theta\right], \left[u_r u_\phi + u_\theta u_\phi \cot\theta\right]\right)$$

the equation of continuity

$$\frac{\partial}{\partial t}(\rho + \rho_0) + (\underline{u} \cdot \underline{\nabla})(\rho + \rho_0) + (\rho + \rho_0)\underline{\nabla} \cdot \underline{u} = 0, \tag{2}$$

the adiabatic energy equation

$$\frac{\partial}{\partial t}(p + p_0) + (\underline{u} \cdot \nabla)(p + p_0) = c^2\left[\frac{\partial}{\partial t}(\rho + \rho_0) + (\underline{u} \cdot \nabla)(\rho + \rho_0)\right], \tag{3}$$

and the equation of state

$$(p + p_o) = (\rho + \rho_o) R (T + T_o)$$

If use is made of the hydrostatic equations for the unperturbed atmosphere and second order pressure and density terms neglected, equation (1) becomes

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} + \underline{K} \simeq -\frac{1}{\rho_o} \nabla p + \left(-\frac{g\rho}{\rho_o}, 0, 0\right). \tag{4}$$

Also equations (2) and (3) become

$$\frac{\partial \rho}{\partial t} + u_r \rho_o' + \rho_o \underline{\nabla} \cdot \underline{u} = 0 \tag{5}$$

$$\frac{\partial p}{\partial t} + u_r p_o' = c^2 \left[\frac{\partial \rho}{\partial t} + u_r \rho_o'\right] \tag{6}$$

where the prime indicates differentiation with respect to r. Now, since all the variables $\underline{u}$, $\rho$, p vanish for $t < 0$ and $\underline{r} \in \mathcal{D}$, take the Laplace transform of the above equations, and set

$$\underline{U}(\underline{r}, s) = \int_0^\infty e^{-st} \underline{u}(\underline{r}, t)\, dt$$

$$P(\underline{r}, s) = \int_0^\infty e^{-st} p(r, t)\, dt \tag{7}$$

$$\mathcal{P}(\underline{r}, s) = \int_0^\infty e^{-st} \rho(\underline{r}, t)\, dt$$

Hence

$$s\underline{U} + \frac{1}{\rho_o}\underline{\nabla} P + \left(\frac{g\mathcal{P}}{\rho_o}, 0, 0\right) = 0 \tag{8}$$

$$s\mathcal{P} + U_r \rho_o' + \rho_o \underline{\nabla} \cdot \underline{U} = 0 \tag{9}$$

$$s P + U_r p_o' = c_o^2 \left[ s\mathcal{P} + U_r \rho_o' \right] \tag{10}$$

[Since $\underline{u}(\underline{r}, t) \rightarrow 0$ or $t \rightarrow \infty$ for all $\underline{r} \in \mathcal{B}$, $|\underline{U}(\underline{r}, t)|$ is bounded above. Hence it can be shown that for sufficiently large s, the transforms of the velocity squared terms are negligible in comparison with $s\underline{U}$ (provided $\underline{U} \not\equiv 0$).]

If the hydrostatic equation

$$p_o' = -g\rho_o$$

is used, equations (10) and (9) become

$$s\left[P - c_o^2 \mathcal{P}\right] = U_r \left[c_o^2 \rho_o' + g\rho_o\right] \tag{11}$$

$$c_o^2 \underline{\nabla} \cdot \underline{U} + \frac{s}{\rho_o} P - gU_r = 0 \tag{12}$$

Using equation (11), eliminate $\mathcal{P}$ from equation (8) to give

$$s\underline{U}\rho_o^{1/2} = -\underline{L}(\rho_o^{-1/2} P) - \hat{\imath}_r P h^{-1} \rho_o^{-1/2} A \tag{13}$$

where

$$h = 1 - \frac{g}{s^2}\left(\frac{\rho_o'}{\rho_o} + \frac{g}{c_o^2}\right) \tag{14}$$

$$A = \left[\frac{g}{c_o^2} + \frac{1}{2}\frac{\rho_o'}{\rho_o}\right] \tag{15}$$

and the operator $\underline{L}$ has components

$$\underline{L} = \left( \frac{1}{h} \frac{\partial}{\partial r}, \; \frac{1}{r} \frac{\partial}{\partial \theta}, \; \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \tag{16}$$

The velocity vector $\underline{U}$ may now be eliminated from (12) and (13) giving

$$\underline{\nabla} \cdot \underline{L} (\rho_o^{-1/2} P) + \frac{q(r)}{r^2} (\rho_o^{-1/2} P) = 0 \tag{17}$$

with

$$q(r) = \left[ - \frac{s^2 r^2}{c_o^2} + \frac{r^2}{h} \left( -A^2 + A' - A \frac{h'}{h} + \frac{2}{r} A \right) \right] \tag{18}$$

3. Boundary Conditions:

For $\underline{r}$ on the surface $\sum$, the following is obtained:

$$\left. \begin{aligned} P(\underline{r}, s) &= \int_0^\infty e^{-st} f(\underline{r}, t)\, dt = E(\underline{r}, s) \\ \underline{n} \cdot \underline{U}(\underline{r}, s) &= \int_0^\infty e^{-st} g(\underline{r}, t)\, dt = F(\underline{r}, s) \end{aligned} \right\} \tag{19}$$

It is also required that $\underline{u} \cdot \underline{n} = 0$ on the earth's surface (i.e. at $r = a$).

This condition becomes

$$U_r = 0 \quad \text{for } r = a.$$

Hence from equation (8) and (10) one obtains

$$\left[ \frac{\partial P}{\partial r} + \frac{g}{c_o^2} P \right] = 0 \qquad r = a \tag{20}$$

or

$$\left[\frac{\partial}{\partial r}(\rho_o^{-1/2} P) + A(\rho_o^{-1/2} P)\right] = 0 \qquad r = a \tag{21}$$

where A is given by (15).

A boundary condition is required for $r \to \infty$. Here the condition that will be placed, is that the total kinetic energy in a solid angle whose vertex is the center of the earth, be finite.

Hence

$$\int_a^\infty r^2 \rho_o(r) |\underline{U}|^2 \, dr < \infty \quad \text{and} \quad \int_a^\infty \rho_o^{-1} |P|^2 \, dr < \infty$$

4. <u>The Unperturbed Variables</u>:

Before proceeding further, several relations are needed involving the unperturbed variables $\rho_o$, $p_o$, and $T_o$. From the hydrostatic equation

$$p_o' = -g \rho_o$$

and equation of state

$$p_o = R \rho_o T_o$$

we obtain on integration the following

$$p_o(r) = p_o(a) \exp\left[-\frac{1}{R}\int_a^r \left(\frac{g}{T_o}\right) dr\right] \tag{22}$$

$$\rho_o(r) = \frac{p_o(a)}{R\, T_o(r)} \exp\left[-\frac{1}{R}\int_a^r \left(\frac{g}{T_o}\right) dr\right] \tag{23}$$

The following relation can be obtained from (23)

$$\frac{\rho_0'}{\rho_0} = -\frac{T_0'}{T_0} - \frac{\gamma g}{c_0^2} \tag{24}$$

5. The Green's Function:

Define a ring source Green's Function $G(\underline{r}, \underline{r}_0)$ to be the solution of

$$\underline{\nabla} \cdot \underline{L}\, G + G \frac{q(r)}{r^2} = \frac{\delta(r - r_0)\, \delta(\theta - \theta_0)}{2\pi \sin\theta\, r^2} \tag{25}$$

which is integrable squared over $(a, \infty)$ and satisfies the boundary condition

$$\left[\frac{\partial G}{\partial r} + A G\right] = 0 \qquad \text{for } r = a \tag{26}$$

Later on an eigenfunction expansion will be obtained for $G(\underline{r}, \underline{r}_0)$. However in the remainder of this section it will be shown how the Laplace transform of $p(\underline{r}, t)$, namely $P(\underline{r}, s)$ satisfying the boundary conditions in Section 3, may be derived from $G(\underline{r}, \underline{r}_0)$. Recall the equation for the excess pressure, namely

$$\underline{\nabla} \cdot \underline{L}(\rho_0^{-1/2} P) + (\rho_0^{-1/2} P) \frac{q(r)}{r^2} = 0 \tag{27}$$

Multiply equations (27) and (25) by $G$ and $\rho_0^{-1/2} P$ respectively, then subtract the two. If in the resulting equation we interchange $\underline{r}$ and $\underline{r}_0$ we have

$$G(\underline{r}_0, \underline{r})\, \underline{\nabla}_0 \cdot \underline{L}_0 \left[\rho_0^{-1/2}(\underline{r}_0)\, P(\underline{r}_0, s)\right] - \left[\rho_0^{-1/2}(r_0)\, P(\underline{r}_0, s)\right] \underline{\nabla}_0 \cdot \underline{L}_0 \left[G(\underline{r}_0, r)\right]$$

$$= -\frac{1}{r_0^2}\left[\rho_0^{-1/2}(r_0)\, P(\underline{r}_0, s)\right] \frac{\delta(r - r_0)\, \delta(\theta - \theta_0)}{2\pi \sin\theta} \tag{28}$$

This may be rewritten in the form

$$\underline{\nabla}_0 \cdot \left\{ G\, \underline{L}_0 (\rho_0^{-1/2} P) - (\rho_0^{-1/2} P)\, \underline{L}_0\, G \right\}$$

$$= -\frac{1}{r_0^2}\left[\rho_0^{-1/2} P\right] \frac{\delta(r - r_0)\, \delta(\theta - \theta_0)}{2\pi \sin\theta_0} \tag{29}$$

Integrate over the domain $\mathcal{D}$ with variables of integration $\underline{r}_0$. We obtain using the divergence theorem,

$$-\iint_S \underline{n} \cdot \left\{ G\, \underline{L}_0 (\rho_0^{-1/2} P) - (\rho_0^{-1/2} P)\, \underline{L}_0\, G \right\} dS_0 = -\rho_0^{-1/2}(\underline{r})\, P(\underline{r}, s) \tag{30}$$

where $\underline{n}$ is the unit normal to S directed inwards in $\mathcal{D}$. The surface S comprises of the surfaces $\Sigma$, $r = a$, and $r = \infty$. Since both G and P are squared integrable on $(a, \infty)$, the surface integral at infinity vanishes.

The integrand for the surface $r = a$ becomes

$$\frac{1}{h}\left[ G \frac{\partial (\rho_0^{-1/2} P)}{\partial r} - (\rho_0^{-1/2} P) \frac{\partial G}{\partial r} \right] \tag{31}$$

which vanishes since G and $(\rho_0^{-1/2} P)$ satisfy the same boundary condition at $r = a$.

Hence (30) becomes

$$P(\underline{r}, s) = \rho_0^{1/2}(\underline{r}) \iint_\Sigma \underline{n} \cdot \left\{ G\, \underline{L}_0 (\rho_0^{-1/2} P) - (\rho_0^{-1/2} P)\, \underline{L}_0\, G \right\} dS_0 \tag{32}$$

To simplify the integral in (28), note from (13) that

$$\underline{L}(\rho_0^{-1/2} P) = -s\, \underline{U}\, \rho_0^{1/2} - \hat{\underline{i}}_r\, P \rho_0^{-1/2} A h^{-1} \tag{33}$$

thus (32) becomes

$$P(\underline{r}, s) = -s\rho_0^{1/2}(r)\iint_{\Sigma} G(\underline{r}, \underline{r}_0)\rho_0^{1/2}(r_0)\underline{n}\cdot\underline{U}(\underline{r}_0, s)\,dS_0$$
$$-\rho_0^{1/2}(r)\iint_{\Sigma}\rho_0^{-1/2}(r_0)P(\underline{r}_0, s)\left\{\left(\frac{A}{h}\right)_0 G\underline{n}\cdot\hat{\underline{i}}_{r_0} + \underline{n}\cdot\underline{L}_0 G\right\}dS_0 \qquad (34)$$

Thus it is seen that if $\underline{n}\cdot\underline{U}_r$ and P are known on the surface $\Sigma$, the transform of the excess pressure $P(\underline{r}, s)$ can be found.

## 6. Eigenfunction Expansion for the Green's Function:

Let $L_r$ and $L_\theta$ represent the differential operators

$$L_r\Psi = \frac{\partial}{\partial r}\left(\frac{r^2}{h}\frac{\partial\Psi}{\partial r}\right) + \Psi q(r) \qquad (a \leq r \leq \infty) \qquad (35)$$

$$L_\theta\Psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) \qquad (0 \leq \theta \leq \pi) \qquad (36)$$

Hence the Green's function is given by

$$(L_r + L_\theta)\,G(\underline{r}, \underline{r}_0) = \frac{\delta(r - r_0)\,\delta(\theta - \theta_0)}{2\pi\sin\theta} \qquad (37)$$

First the operator $L_r$ must be investigated. This can be written in the normal form for the Sturm-Liouville operators

$$L_r\Psi = \frac{\partial}{\partial r}\left[p(r)\frac{\partial\Psi}{\partial r}\right] + \Psi q(r) \qquad \text{with } p(r) = \frac{r^2}{h} \qquad (38)$$

The parameter s may be chosen sufficiently large such that $h(r)$ is positive over the range $(a, \infty)$, hence

$$p(r) > 0 \quad \text{for } (a, \infty).$$

We will consider the case where the parameter $s$ is real and positive. Thus, $p(r)$ and $q(r)$ are real for $(a, \infty)$. The following restriction will also be placed: any model of the atmosphere that is used must have $p(r)$ continuous. The asymptotic values of $p(r)$ and $q(r)$ for large $r$ have to be investigated.

For large $r$, $c^2 = \gamma R T_0$ has the asymptotic value

$$c^2 \sim c^2(\infty) + 0\left(\frac{1}{r}\right) \tag{39}$$

where $c^2(\infty) = \gamma R T_0(\infty)$. Equation (24) obtains $\frac{\rho_0'}{\rho_0} \sim 0\left(\frac{1}{r^2}\right)$ for large $r$. Thus it is seen that

$$\left.\begin{aligned} A &\sim 0\left(\frac{1}{r^2}\right) \\ h &\sim 1 + 0\left(\frac{1}{r^4}\right) \end{aligned}\right\} \tag{40}$$

and

which together with equations (22) and (38) gives the asymptotic values for large $r$

$$\left.\begin{aligned} p(r) &\sim r^2 \\ q(r) &\sim -\frac{r^2 s^2}{c^2(\infty)} \end{aligned}\right\} \tag{41}$$

The equation

$$L_r \Psi - \lambda \Psi = 0$$

has two solutions $w_1(r, \lambda)$ and $w_2(r, \lambda)$ whose asymptotic values are

$$w_1(r, \lambda) \sim \frac{1}{r} e^{rk(\infty)}$$

$$w_2(r, \lambda) \sim \frac{1}{r} e^{-rk(\infty)}$$

where $k(\infty)$ is the positive root of $k^2(\infty) = \dfrac{s^2}{c^2(\infty)}$. The second solution $w_2(r, \lambda)$ is square integrable on $(a, \infty)$.

Let $\Psi(r, \lambda_i)$ represent the set of solutions of

$$L_r \Psi - \lambda_i \Psi = 0 \tag{42}$$

with the property that

$$\int_a^\infty (\Psi)^2 \, dr < \infty \tag{43}$$

and

$$\left[ \frac{d\Psi}{dr} + A\Psi \right] = 0 \qquad \text{for } r = a. \tag{44}$$

From Friedrichs (Refs. 9 ,10 ) it may be shown using (41) that the spectrum of the operator $L_r$ is totally discrete, and if $\mathcal{H}$ represents the manifold of all functions $\Psi(x)$ such that

$$\int_a^\infty |\Psi|^2 \, dr < \infty .$$

then for $\Psi \in \mathcal{H}$ and the $\Psi$ such that the functions $L_r \Psi \in \mathcal{H}$, the following expansion holds

$$\Psi \sim \sum a_i \, \Psi(r, \lambda_i) \tag{45}$$

$$L_r \Psi \sim \sum \lambda_i \, a_i \Psi(r, \lambda_i) \tag{46}$$

Therefore, the Green's functions $G(\underline{r}, \underline{r}_0)$ may be expanded in such a form.

In order to simplify the analysis set $\lambda = -(\mu^2 + \frac{1}{4})$ and let $\phi(r, \mu_i)$ represent the solutions of

$$L_r\phi + (\mu_i^2 + \frac{1}{4})\phi = 0 \tag{47}$$

so that

$$\int_a^\infty \left|\phi\right|^2 dr < \infty$$

and

$$\left[\frac{d\phi(r, \mu_i)}{dr} + A\phi(r, \mu_i)\right] = 0 \qquad \text{at } r = a. \tag{48}$$

For the present let $\phi(r, \mu)$ be the solution of

$$L_r\phi + (\mu^2 + \frac{1}{4})\phi = 0 \tag{49}$$

which belongs to $\mathcal{H}$. Multiply equations (47) and (49) by $\phi(r, \mu)$ and $\phi(r, \mu_i)$ respectively. Then subtract the resulting two equations and integrate from a to $\infty$. Using (48) one obtains

$$\int_a^\infty \phi(r, \mu)\phi(r, \mu_i)\, dr = \frac{+a^2\phi(a, \mu_i)}{(\mu^2 - \mu_i^2)(h)_{r=a}}\left[\frac{d\phi(r, \mu)}{dr} + A\phi(r, \mu)\right]_{r=a} \tag{50}$$

Let $\mu = \mu_j$ where $i \neq j$, then it can be seen immediately using (48) that

$$\int_a^\infty \phi(r, \mu_i)\,\phi(r, \mu_j)\, dr = 0 \qquad i \neq j \tag{51}$$

Let $\mu$ approach $\mu_i$ and take the limit, obtaining

$$\int_a^\infty \phi(r, \mu_i)^2\, dr = \frac{a^2\phi(a, \mu_i)}{2\mu_i(h)_{r=a}}\left\{\frac{\partial}{\partial\mu}\left[A\phi(a, \mu) + \frac{\partial\phi(a, \mu)}{\partial a}\right]\right\}_{\mu=\mu_i} \tag{52}$$

Expand the Green's function $G(\underline{r}, \underline{r}_0)$ in terms of $\phi(r, \mu_i)$ by

$$G(\underline{r}, \underline{r}_0) = \sum_i a_i \, \phi(r, \mu_i) \tag{53}$$

Substitute expansion (53) into equation (37). Using the property given by (46), one obtains

$$\sum_i \phi(r, \mu_i) \left\{ -(\mu_i^2 + \frac{1}{4}) + L_\theta \right\} a_i = \frac{\delta(r - r_0)\, \delta(\theta - \theta_0)}{2\pi \sin\theta} \tag{54}$$

Multiply both sides of (54) by $\phi(r, \mu_j)$ and integrate from $a$ to $\infty$, thus giving

$$\left[ L_\theta - (\mu_j^2 + \frac{1}{4}) \right] a_j = \frac{\delta(\theta - \theta_0)\, \phi(r_0, \mu_j)}{2\pi \sin\theta \int_a^\infty \phi(r, \mu_j)^2 \, dr} \tag{55}$$

Set

$$a_j = \frac{\phi(r_0, \mu_j)}{\int_a^\infty \phi(r, \mu_j)^2 \, dr} X_j(\theta) \tag{56}$$

hence equation (55) becomes

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial X_j}{\partial\theta}) - (\mu_j^2 + \frac{1}{4}) X_j = \frac{\delta(\theta - \theta_0)}{2\pi \sin\theta} \tag{57}$$

Solutions of the homogeneous equation corresponding to (57) are the Legendre functions

$$P_{-1/2 + i\mu_j}(\cos\theta) \qquad \text{and} \qquad P_{-1/2 + i\mu_j}(-\cos\theta) \tag{58}$$

the first of which is finite at $\theta = 0$, and the second finite at $\theta = \pi$.

By the usual techniques it can be shown that the solution of (57) which is finite over the range $(0 \leqslant \theta \leqslant \pi)$ is given by*

$$X_j = -\frac{1}{4\cosh \pi\mu_i} \begin{cases} P_{-1/2+i\mu_j}(\cos\theta)\, P_{-1/2+i\mu_j}(-\cos\theta_0) & 0 \leqslant \theta \leqslant \theta_0 \\ P_{-1/2+i\mu_j}(\cos\theta_0)\, P_{-1/2+i\mu_j}(-\cos\theta) & \theta_0 \leqslant \theta \leqslant \pi \end{cases} \tag{59}$$

The Green's function expansion is now given by

$$G(\underline{r}, \underline{r}_0) = \sum_j \left[\phi(r_0, \mu_j)\, \phi(r, \mu_j)\, X_j(\theta)\right] \Big/ \int_a^\infty \phi(r, \mu_j)^2\, dr \tag{60}$$

However the interest for the particular problem at hand is in evaluating the excess pressure $p(\underline{r}, t)$ on the earth's surface i.e., at $r = a$, at a long distance from the pressure pulse source. Hence the value of $G(\underline{r}, \underline{r}_0)$ for $r = a$, and $\theta$ such that $\theta > \theta_0$ is required. In this case (60) simplifies giving

$$G(a, r_0) = \sum_j \frac{f_j(a, \theta; r_0, \theta_0)}{2\cosh \pi\mu_j} \tag{61}$$

with

$$f_j = -\frac{\mu_j\, \phi(r_0, \mu_j)\, P_{t_j}(\cos\theta_0)\, P_{t_j}(-\cos\theta)}{\left\{\frac{r^2}{h}\frac{\partial}{\partial\mu}\left[A\,\phi(r, \mu) + \frac{\partial}{\partial r}\phi(r, \mu)\right]\right\}_{\begin{pmatrix}\mu = \mu_j \\ r = a\end{pmatrix}}}$$

where for simplification $t_j$ has been set equal to $i\mu_j - 1/2$. However to evaluate the inverse transform, we require the Green's function for s complex. But for s complex, $p(r)$ given by (38) is complex. For this case, one cannot deduce

* See Friedlander (Ref. 14) page 170.

that there exists a totally discrete set of eigenvalues $\{\mu_j\}$, hence the expansion given by (6) is questionable. For $p(r)$ discontinuous we apply the boundary condition that the pressure and vertical component of the velocity must be continuous at any points of discontinuity of $T_0'$, the orthogonality properties of the eigenfunctions are retained. Thus we may write

$$G(a, \theta; r_0, \theta_0) = \sum_{j=1}^{n(s)} \frac{t_j}{2 \cosh \pi\mu_j} + R(\mu) \qquad (63)$$

where the summation is over the "free" wave modes (those modes such that $i\mu_j$ is real for $s$ pure imaginary, or, setting $s = i\omega$, for real frequency $\omega$). The remainder $R(\mu)$ represents the contribution due to the remaining modes. For present purposes it does not matter whether the remaining eigenvalues form a totally discrete set or not.*

7. An Approximate Technique to the Determination of the Low Frequency Free Wave Modes:

We now require the values of the eigenvalues $\mu_j$ and the eigenfunctions corresponding to the free wave modes for real frequencies $\omega$. This requires a solution of the differential equation (47) with $s$ replaced by $i\omega$. However, this equation is difficult to solve for most temperature models of the atmosphere. Simple solutions can be found for an isothermal atmosphere (Pekeris, Refs. 2, 3) or an atmosphere composed of isothermal strata (Yamamoto, Ref. 1; Penney, Ref. 6). However, for the case of Scorer's model, (Ref. 4) (isothermal stratosphere and tropsphere with a constant temperature gradient), numerical methods have to be used.

* This is explained in more detail in Section 9.

For these various models, it is shown that there is a low frequency mode which we will call the gravity wave. This low frequency mode possesses a continuous spectrum in $\omega$ from $\omega = 0$*to some cut-off frequency $\omega_c$. The cut-off frequency depends upon the model of the atmosphere. The low frequency mode is essentially a surface wave, the energy being propagated around the earth. However above cut-off, the character of the wave changes into a wave which propagates energy upward.

The temperature of the actual atmosphere increases quite rapidly above 100 km. For instance, at 140 km the temperature is given as $880^0$ K. For models of the atmosphere possessing this characteristic, namely that the upper atmosphere is the warmest, there exists a set of higher frequency modes with a continuous spectrum. Penney et al have shown these modes or "branches" for models comprised of isothermal layers, with the top layer the warmest.

We will give here a new approach to the problem which will enable one to find good approximate analytical solutions to the gravity waves for some simple atmospheric models. The approach here is based upon an approximate technique for solving the radial equation for the radial component $U_r$ of the velocity. Set

$$\mu^2 + \frac{1}{4} = -\omega^2 \lambda^2 a^2 \qquad (64)$$

$$\mu = i\left[\omega^2 \lambda^2 a^2 + \frac{1}{4}\right]^{1/2} \qquad (65)$$

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* For some models of the atmosphere there is a lower frequency cut-off at a non-zero value of $\omega$.

where $\lambda^{-1}$ is the phase velocity of the wave*. From Appendix B it is shown that

$$U_r(r, \theta, \omega) = P_{i\mu - 1/2}(\cos\theta) \frac{\left[r^2 / c_o^2 - \lambda^2 a^2\right]^{1/2}}{r^2 \rho_o^{1/2}} M(r) \qquad (66)$$

where M is a solution of the equation

$$M'' + M\left\{ -\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{1}{\alpha}\left[A\alpha' + \frac{\alpha''}{2}\right] - A' - A^2 + \frac{h\alpha}{r^2}\right\} = 0 \qquad (67)$$

where
$$\alpha(r) = \omega^2\left[\frac{r^2}{c_o^2} - \lambda^2 a^2\right]$$

The best way to make use of this equation is to break the atmosphere into two+ regions specified by (i) $a \leq r \leq b$ and (ii) $b \leq r$. In the lower region $(a \leq r \leq b)$, the atmosphere will be composed of the troposphere and possibly the lower portion of the stratosphere. In the upper region $(b \leq r)$, the temperature will be assumed to be a very slowly varying function of altitude. We

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* This is seen using the asymptotic relations for large real $\omega\lambda a$, $-i\mu \sim \omega\lambda a$, and $P_{-\omega\lambda a - 1/2}(\cos\theta) \sim \left(\frac{2}{\pi \sin\theta\, \omega\lambda a}\right)^{1/2} \cos\left[\omega\lambda a\theta - \frac{\pi}{4}\right]$ and noting that $a\theta$ is the arc length along the earth's surface. Hence for harmonic time dependence, $\lambda^{-1}$ is the phase velocity of the wave propagating around the earth.

+ Note: At the interface of the two regions we will assume that the temperature is continuous (although the derivatives may or may not be continuous).

will use two different approximate techniques for finding M and hence $U_r(r, \omega)$ in each of these regions. The solutions will be matched across the interface $r = b$, by requiring that the vertical velocity and pressure be continuous functions. The latter condition can be shown to be equivalent to the criteria that the derivative of the vertical velocity with respect to r, be continuous.

Solution for the Upper Atmosphere $(r \geq b)$

Here $T_0$ is a slowly varying function of r, in fact we may take $T_0$ constant. If the region is isothermal we will take $T_0 = T_s$, and if the temperature should vary slightly, we will take the value of $T_0$ at $r = b$ to be $T_s$.

For this region rewrite equation (26) in the form

$$M'' + M\left\{ -\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{1}{2}\left(\frac{\alpha''}{\alpha}\right) + \alpha\left(\frac{\beta}{\alpha}\right)' - \beta^2 + R(r) \right\} = 0 \tag{68}$$

where we have set
$$\beta = \left[A^2 - h\frac{\alpha}{r^2}\right]^{1/2} \tag{69}$$

and
$$R(r) = -\alpha\left[\frac{A + \beta}{\alpha}\right]' \tag{70}$$

Since the temperature is slowly varying or constant we may neglect the remainder term $R(r)$. Solving for M with R set equal to zero we have

$$M = \alpha^{-1/2} \exp\left\{ -\int_b^r \beta(r)\, dr \right\} \tag{71}$$

This is essentially the W. K. B. approximation. Of course for this to be sufficiently accurate, we require that the remainder R be sufficiently small, and this in turn holds provided that $\alpha$ does not vanish in, or close to the region $b \leq r$. In addition, at sufficient distances out there may be a turning point given approximately by the value for r for which $\beta$ vanishes. However, we will assume* that the turning point is sufficiently far away from $r = b$, so that expression (71) is valid at least for the lower part of the upper atmosphere.

We now have

$$U_r(r, \omega) = \frac{C_1(\lambda)}{r^2 \rho_o^{1/2}} \exp\left\{-\int_b^r \beta(r)\, dr\right\} \tag{72}$$

where

$$\beta = \sqrt{A^2 - \frac{h\alpha}{r^2}} \tag{73}$$

which for the isothermal case has the explicit form

$$\beta = \left\{\left(\frac{2-\gamma}{2}\right)^2 \frac{g^2}{c_s^4} - \frac{1}{r^2}\left(\frac{r^2}{c_s^2} - \lambda^2 a^2\right)\left[\omega^2 + (1-\gamma)\frac{g^2}{c_s^2}\right]\right\}^{1/2} \tag{74}$$

The constant $C_1(\lambda)$ is determined from the boundary conditions at $r = b$.

Solution for the Lower Atmosphere $(a \leq r \leq b)$

Because $\lambda^{-1}$ is the phase velocity of the gravitational wave we expect that there is some point in or just outside the interval $(a \leq r \leq b)$ for which

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* This assumption breaks down when we approach the "so-called" cut-off frequency.

$(r^2 / c^2 - \lambda^2 a^2)$ vanishes. Hence the equation (67) has one or more singular points. Performing an order analysis it is seen that the term $h\alpha / r^2$ is small compared to the other terms for low frequencies (periods greater than three minutes). Thus a good approximation is given by the equation

$$M'' + M \left\{ -\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{1}{\alpha}\left(A\alpha' + \frac{\alpha''}{2}\right) - A' - A^2 \right\} = 0 \tag{75}$$

and the general exact solution of (34) is

$$M = \alpha^{-1/2} e^{\phi(r)} \left\{ C_2(\lambda) + \int_a^r \alpha e^{-2\phi(x)} dx \right\} \tag{76}$$

with $\phi(r) = \int_a^r A dx$

$$= \int_a^r \left[ \left(\frac{2-\gamma}{2}\right) \frac{g}{c^2} - \frac{1}{2} \frac{T_o'}{T_o} \right] dr \tag{77}$$

Using the first approximation given by (76), higher order approximations may be obtained by using an iterative process (which will be discussed below). For present purposes we will consider the first approximation only, given by

$$U_r^1(r, \omega) = \frac{1}{r^2 \rho_o^{1/2}} e^{\phi(r)} \left\{ C_2(\lambda) + \int_a^r \alpha e^{-2\phi(x)} dx \right\} \tag{78}$$

Determination of the Eigenvalue $\lambda = \lambda^*$ (first approximation)

First of all we must match the solutions for the two regions at $r = b$, from the boundary conditions that $U_r'$, and $U_r$ are continuous.

Using the continuity condition of $U_r$ at $r = b$, we have from (72) and (78)

$$C_1(\lambda) = e^{\phi(b)} \left\{ C_2(\lambda) + \int_a^b \alpha e^{-2\phi(x)}\, dx \right\} \tag{79}$$

Before we employ the continuity condition of $U_r'$ at $r = b$, we note that the corresponding derivatives have the form

$$U_r' = U_r \left\{ -\frac{2}{r} + \frac{1}{2}\frac{\gamma g}{c_s^2} - \beta(r) \right\} \qquad b \leq r \tag{80}$$

and

$$U_r^{1\prime} = U_r^1 \left[ -\frac{2}{r} + \frac{g}{c^2} \right] + \frac{\alpha e^{-\phi(r)}}{r^2 \rho_o^{1/2}} \qquad a \leq r \leq b \tag{81}$$

where in (80) we have taken the temperature constant $T_o = T_s$, and in (81) we have used the relation

$$\phi(r)' = A = \frac{g}{c^2} + \frac{1}{2}\frac{\rho_o'}{\rho_o}$$

Thus for $r = b$ we may equate the right hand sides of (80) and (81) giving

$$U_r^1(b, \omega) \left[ \beta(b) + \left( \frac{2-\gamma}{2} \right) \frac{g}{c_s^2} \right] = - \left[ \frac{\alpha(r, \lambda) e^{-\phi(r)}}{r^2 \rho_o^{1/2}} \right]_{r = b} \tag{82}$$

Substitute the value for $U_r^1(b, \omega)$ from (78), and rearrange terms to give

$$C_2(\lambda) = - \int_a^b \alpha e^{-2\phi(x)}\, dx - \frac{\alpha(b, \lambda) e^{-2\phi(b)}}{\left[ \beta(b) + \left| \frac{2-\gamma}{2} \right| \frac{g}{c_s^2} \right]} \tag{83}$$

Thus the values $C_1(\lambda)$ and $C_2(\lambda)$ given by (79) and (83) are determined. However we must satisfy the remaining boundary condition namely that $U_r^1$ must vanish at $r = a$. Since the solution (78) in the region $a \leq r \leq b$, is uniquely determined apart from $\lambda$, we require a condition upon $\lambda$ in order to satisfy the boundary condition. From (78) it is easily seen that for $U_r^1$ to vanish at $r = a$ we must have

$$C_2(\lambda) = 0 \tag{84}$$

where we shall denote the real solutions of (84) by $\lambda^*$. The first approximation to $\lambda^*$ is given by equation (84).

<u>Higher Order Approximation for $U_r(r, \omega)$ in $a \leq r \leq b$</u>

The problem is to find higher order approximations to the equation

$$M'' + M\left[f(r) + g(r)\right] = 0 \tag{85}$$

in the range $a \leq r \leq b$, and where for convenience of analysis we set

$$f(r) = -\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{1}{\alpha}\left(\frac{\alpha''}{2} + A\alpha'\right) - A' - A^2 \tag{86}$$

$$g(r) = \frac{\alpha h}{r^2} \tag{87}$$

The two independant solutions $M_1$ and $M_2$ of

$$M'' + Mf(r) = 0 \tag{88}$$

are

$$M_1 = \alpha^{-1/2} e^{\phi(r)} \tag{89}$$

$$M_2 = \alpha^{-1/2} e^{\phi(r)} \int^r \alpha e^{-2\phi(x)} dx \tag{90}$$

As the first approximation to M we take

$$M^1 = \alpha^{-1/2} e^{\not{\phi}(r)} \left[ C_2(\lambda) + \int_a^r \alpha e^{-2\not{\phi}(x)} dx \right] \tag{91}$$

where $C_2(\lambda)$ is given by equation (83).

To find higher order approximations of (85) we write the equation in the form

$$M'' + f(r) M = -g(r) M \tag{92}$$

and treat the right-hand side as the inhomogeneous portion of the differential equation. Equation (92) may be solved by the method of variation of parameters giving

$$M = M^1 + \int_r^b g(t) M(t) \left[ M_1(t) M_2(r) - M_2(t) M_1(r) \right] dt \tag{93}$$

or expressing it in terms of vertical velocity we have

$$U_r(r, \omega) = U_r^1(r, \omega) + \int_r^b U_r(t, \omega) k(r, t) dt \tag{94}$$

where

$$k(r, t) = \frac{h(t)}{r^2} \frac{\rho_o^{1/2}(t)}{\rho_o^{1/2}(r)} e^{\not{\phi}(t) + \not{\phi}(r)} \int_t^r \alpha(x, \lambda) e^{-2\not{\phi}(x)} dx \tag{95}$$

and $U_r^1(r, \omega)$ is the first-order approximation. This is a Volterra integral equation which can be solved explicitly by successive iterations.

We may take the second order approximation

$$U_r^2(r,\omega) = U_r^1(r,\omega) + \int_r^b U_r^1(t,\omega)\, k(r,t)\, dt \tag{96}$$

and obtain similar expressions for the higher order approximations.

The next question that arises concerns the matching of the boundary conditions at $r = b$. However, it can be shown that the higher approximations, in fact, even the exact solution found by the iteration technique, are such that the boundary conditions, namely, that $U_r$, $U_r'$ be continuous at $r = a$, are automatically satisfied where for $r \geqslant b$ we have taken the solution given by equation (72) with $C_1(\lambda)$ specified by (79).

The remaining problem is to consider higher order approximations to $\lambda^*$. Let us consider the second order approximation only. We require that $U_r$ vanish at $r = a$. From (96) and (78), we have

$$\frac{C_2(\lambda)}{a^2 \rho_o^{1/2}(a)} + \int_a^b U_r^1(t,\omega)\, k(r,t)\, dt = 0$$

i.e.,

$$C_2(\lambda) + \int_a^b dt \frac{h(t)}{t^2} e^{2\oint(t)} \left\{ C_2(\lambda) + \int_a^t \alpha\, e^{-2\oint(x)}\, dx \right\} \int_t^a \alpha(x,\lambda).$$

$$e^{-2\oint(x)}\, dx = 0 \tag{97}$$

Solving equation (97) will give the second approximation to $\lambda^*$.

From equations (B.1), (B.2) and (B.3) of Appendix B, the radial component of the excess pressure $P(r)$ is related to the vertical velocity component by

$$\rho_o^{-1/2}(r)\,P(r) = \frac{i\omega}{\alpha}\,e^{\emptyset(r)}\left[e^{-\emptyset(r)}\,r^2\,\rho_o^{1/2}(r)\,U_r\right]' \tag{98}$$

hence the second approximation to $P(r)$ is on using (96)

$$\rho_o^{-1/2}(r)\;P(r) = i\omega e^{-\emptyset(r)}\left[1+\int_r^b \frac{h(t)}{t^2}\,e^{2\emptyset(t)}\left[C_2(\lambda)+\int_a^t \alpha\,e^{-2\emptyset(x)}\,dx\right]dt\right] \tag{99}$$

8. Comments on the Accuracy of the Approximations:

In order to show the accuracy of the approximations, we give two examples. First we compare our first and second approximations with the results obtained by Scorer.

Scorer considered the case where the temperature gradient was constant in the troposphere and the stratosphere was isothermal, i.e.,

$$a \leqslant r \leqslant b \qquad T_o = T_1\left[1-\frac{(r-a)}{\ell}\left(\frac{T_1-T_s}{T_1}\right)\right]$$

$$b \leqslant r \qquad T_o = T_s$$

where $\ell = b - a$ is the height of the troposphere

$T_1 = 286.91^{o}K$ is the ground temperature

$T_s = 229.53^{o}K$ is the stratosphere temperature

The acceleration due to gravity $g$, was considered constant with the value $9.806 \times 10^{-3}$ km sec$^{-2}$. In addition the constants $\ell$, $\gamma$ are given

$$\gamma = 1.403$$

$$\ell = 9.6137 \text{ km}$$

Scorer solved the differential equation for a "modified" pressure function numerically, and hence obtained the eigenvalue $K^*$ which corresponds to our $\lambda^* \omega$. We compare the results between Scorer's theory and our approximate theory with the Table I below where $\lambda^*$ is in units $km^{-1}$ sec.

TABLE I

| $5000\,\omega^2$ | First Approximation to $\lambda^*$ | Second Approximation to $\lambda^*$ | Scorer's Value of $\lambda^*$ |
|---|---|---|---|
| 0 | 3.1899 | 3.18947 | 3.18950 |
| 1 | 3.1913 | 3.19148 | 3.19148 |
| 2 | 3.1929 | 3.19355 | 3.19356 |
| 3 | 3.1945 | 3.19570 | 3.19574 |
| 4 | 3.1962 | 3.19793 | 3.19793 |
| 5 | 3.1980 | 3.20025 | 3.20022 |
| 6 | 3.2000 | 3.20266 | 3.20271 |
| 7 | 3.2021 | 3.20517 | 3.20521 |
| 8 | 3.2045 | 3.20778 | 3.20781 |
| 9 | | 3.21052 | |
| 10 | | 3.21338 | 3.21342 |
| 11 | | 3.21638 | |
| 12 | | 3.21953 | 3.21956 |
| 13 | | 3.22286 | |
| 14 | | 3.22637 | 3.22643 |

We immediately see that our first approximation gives the phase velocities $(1/\lambda^*)$ accurate to three figures, and for the lower frequency modes, the error is one unit in the fourth figure. These errors really are insignificant compared to the errors introduced through the simplification of the atmospheric model, i.e., neglect of winds could produce a change in the phase velocity in the second figure. The second approximation is extremely accurate.

For the second case, we take an atmosphere whose region below 50 geopotential kilometers corresponds to measured values. Above the 50 geopotential kilometers we take the atmosphere as isothermal. We will consider the case where the measured values of temperature are given as follows (where for convenience of later reference we will call this atmosphere II):

| Altitude (geopotential km) | Temperature (degrees centigrade) |
|---|---|
| surface | 29 |
| 1.5 | 18 |
| 3.1 | 10 |
| 5.9 | - 5 |
| 9.7 | -31 |
| 12.4 | -53 |
| 14.2 | -67 |
| 16.6 | -81 |
| 20 | -66 |
| 25 | -55 |
| 30 | -45 |
| 35 | -34 |
| 40 | -20 |
| 45 | - 5 |
| 50 | + 7 |

with this temperature model, we calculate $\lambda^*$ from the second approximation by setting equation (103) equal to zero. In Table II we compare the second approximations to $\lambda^*$ to those calculated exactly (this is discussed below).

TABLE II

| $5000\,\omega^2$ | Second Approximation to $\lambda^*$ | Exact Value of $\lambda^*$ |
|---|---|---|
| 0 | 3.1519 | 3.1528 |
| 1 | 3.1577 | 3.1582 |
| 2 | 3.1624 | 3.1628 |
| 3 | 3.1663 | 3.1670 |
| 4 | 3.1697 | 3.1709 |
| 5 | 3.1727 | 3.1740 |
| 6 | 3.1753 | 3.1784 |
| 7 | 3.1777 | 3.1821 |
| 8 | 3.1799 | 3.1857 |

It is seen that the approximate value of $\lambda^*$ is accurate to four figures for small values of $\omega$, but for higher values the accuracy decreases to three figures. For this model of the atmosphere the accuracy is not as good as in Scorer's model. The main reason for this is that the error is proportional to $(\omega \ell)^4$ for $\omega \neq 0$, where $\ell = b - a$, is the interval of integration (in this case $\ell = 49$ km, in Scorer's model $\ell = 9.6137$ km). The approximate technique is good for low frequency waves and not too large non-isothermal intervals.

However, for models of the atmosphere composed of alternating isothermal and constant temperature gradients sections, an approximate technique of this type should give good results. In the isothermal sections, analytical solutions in the form of exponentials can be found, and in the sections with constant temperature gradient, good approximate analytical solutions can be found.

For large intervals and high frequencies, numerical methods are necessary. However, for very high frequencies asymptotic techniques may be used (see Friedlander, Ref. 14).

To obtain a numerical solution we will return to the equation for the radial component $P(r)$ of the excess pressure. From equation (B.2) and (B.4) of Appendix B we have

$$\frac{d}{dr}\left(\frac{r^2}{h}\frac{d}{dr}\left[\rho_o^{-1/2} P\right]\right) + \left[q(r) - \omega^2 \lambda^2 a^2\right] \rho^{-1/2} P = 0 \tag{100}$$

where for the free waves $q(r)$ is given by (18) with $s = i\omega$.

Set
$$\Psi = \rho_o^{-1/2} P(r) \exp \int_a^r A\,dx \tag{101}$$

with A given by (15), then (100) becomes

$$\Psi'' + \Psi'\left[-2A - \frac{h'}{h} + \frac{2}{r}\right] + h\omega^2\left[\frac{1}{c_o^2} - \frac{\lambda^2 a^2}{r^2}\right]\Psi = 0 \tag{102}$$

Setting $s = i\omega$ in (13) and using (101) we obtain

$$\rho_o^{1/2}\, i\omega\, U_r = -\frac{1}{h}\Psi' \exp\left[-\int_a^r A\,dx\right]$$

thus the boundary condition at $r = a$, namely that $U_r$ vanish requires

$$\Psi' = 0 \qquad \text{at } r = a.$$

and the boundary condition at a discontinuity in $T_o'$ (but not $T_o$) requiring that $U_r$, P be continuous, sets the condition that both $\Psi'/h$ and $\Psi$ be continuous.

The upper atmosphere was assumed isothermal and the upper boundary condition at the junction $r = b$, is

$$\Psi' = \left[-\beta + \left(\frac{2-\gamma}{2}\right)\frac{g}{c_s^2}\right]\frac{h_t}{h_s} \tag{103}$$

where $\beta$ is given by (74) and $h_t$, $h_s$ are the values of h for the lower and upper atmospheres at $r = b$, respectively.

Equation (102) was solved numerically using the method of Runge-Kutta. In addition g was assumed constant, the term $a^2/r^2$ was set to unity and the term $2/r$ neglected.

9. Simple Source Model:

Consider the case where the surface $\sum$ may be taken as a small sphere with centre $(R_o, 0)$ and radius $\epsilon$. In addition we assume that locally the disturbance produced by the source is spherically symmetric about $(R_o, 0)$. If we take the limit as $\sum$ shrinks to a point, it is shown in Appendix A that we can write in place of (34)

$$P(a, \theta, s) = \rho_0^{1/2}(a)\, G(a, \theta; R_0, 0)\, I(s, R_0) \tag{104}$$

where $I(s, R_0)$ is a function of $s$ representing the behavior of the source at $(R_0, 0)$.

10. Method of Evaluation of $p(a, \theta, t)$:

The excess pressure at $(a, \theta)$ may be evaluated from (104) by means of the inverse Laplace transform giving

$$p(a, \theta, t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \rho_0^{1/2}(a)\, G(a, \theta; R_0, 0)\, I(s, R_0)\, ds \tag{105}$$

where the line Real $s = b$ is to the right of all the poles of the integrand of (105).

We shall assume that the position of the source $(R_0, 0)$ and the observer $(a, \theta)$ is such that the contribution of the modes other than the free wave modes is negligible. It can be shown that the other modes which satisfy the required boundary conditions at $r = a$ and $r = \infty$ will attenuate since the phase velocities are complex. We shall place the restriction that the observer should be in the geometrical shadow region i.e.,

$$\theta > \cos^{-1}\left(\frac{a}{R_0}\right)$$

In the illuminated region, the modes other than the "free" wave modes become important. In addition, appealing to known results of acoustic and electromagnetic theory, the Green's function cannot solely be represented by a residue series, but must be represented by a contour integral in the $\mu$ plane. Hence for the geometric illuminated region, other techniques than the modal expansion must be used.

From (63) and (105) we have

$$p(a, \theta, t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{st} \rho_0^{1/2}(a) \, I(s, R_0) \sum_{j=1}^{n(s)} \frac{f_j(a, \theta; R_0, 0)}{2\cosh(\pi\mu j)} \, ds \tag{106}$$

where we have considered the contribution of the "free" wave modes only. Making the following decomposition

$$\frac{1}{2\cosh(\pi\mu j)} = e^{-\pi\mu j} - \frac{e^{-2\pi\mu j}}{\cosh \pi\mu j} \tag{107}$$

equation (106) becomes

$$p(a, \theta, t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{st} \rho_0^{1/2}(a) \, I(s, R_0) \sum_{j=1}^{n(s)} f_j \, e^{-\pi\mu j} \, ds$$

$$- \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{st} \rho_0^{1/2}(a) \, I(s, R_0) \sum_{j=1}^{n(s)} f_j \, \frac{e^{-2\pi\mu j}}{\cosh \pi\mu j} \, ds \tag{108}$$

Using the relation

$$P_{i\mu j - 1/2}(\cos\theta_0) = 1 \quad \text{for } \theta_0 = 0 \tag{109}$$

and the asymptotic expression

$$P_{i\mu j - 1/2}(-\cos\theta) = \sqrt{\frac{2}{\pi \sin\theta}} \cdot \frac{\Gamma(i\mu_j + 1/2)}{\Gamma(i\mu_j + 1)} \cos\left[ i\mu_j(\pi - \theta) - \frac{\pi}{4} \right] \qquad (110)$$

for $0 < \theta < \pi$ together with the fact that for s in the right half-plane

$$\text{Real } \mu_j > 0 \qquad (111)$$

it can be shown that the second integral* of (108) represents the contributions to the excess pressure of the modes that have travelled at least once completely around the earth. Hence if we consider only the contribution to the excess pressure pulse that arrives directly, we may take

$$p(a, \theta, t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{st} \rho_0^{1/2}(a)\, I(s, R_0) \sum_{j=1}^{n(s)} f_j e^{-\pi\mu_j}\, ds \qquad (112)$$

We may now take b = 0 and set $s = i\omega$ in (112), giving

$$p(a, \theta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \rho_0^{1/2}(a)\, I(i\omega, R_0) \sum_{j=1}^{n} f_j e^{-\pi\mu_j}\, d\omega \qquad (113)$$

For certain models of the atmosphere (in the sense of the variation of the atmospheric temperature $T_0$), there exists only one free mode below some critical frequency $\omega_c$. For this case we obtain

$$p(a, \theta, t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{i\omega t - \pi\mu^*} \rho_0^{1/2}(a)\, f^*(a, \theta; R_0, 0)\, I(i\omega, R_0)\, d\omega \qquad (114)$$

---

* See footnote, page 171 of Reference 14.

where we specify the eigenvalue by $\mu^*$. For models of the atmosphere in which there exists a set of higher frequency modes with continuous spectrum in $\omega$ greater than their respective cut-off frequencies, we must of course, include their integrals corresponding to the integral given by (114) for the low frequency mode.

However, for the present, we will consider the excess pressure pulse produced by the low frequency gravity wave.

From (64) we have

$$\mu^* = i\left[(\omega\lambda^* a)^2 + \frac{1}{4}\right]^{1/2} \sim i\omega\lambda^* a \qquad \text{for } \omega\lambda^* a >> 1 \tag{115}$$

From (110) we have

$$P_{i\mu^* - 1/2}(-\cos\theta) \sim \left(\frac{2}{|\omega|\,\lambda^*\, a\,\pi \sin\theta}\right)^{1/2} \cos\left[|\omega|\,\lambda^*\, a\,(\pi-\theta) - \frac{\pi}{4}\right] \tag{116}$$

hence we see that from (116) we may write

$$\rho_0^{1/2}(a)\, f^*(a,\theta;R_0,0)\, I(i\omega,R_0) = K(a,\theta,R_0,\omega)\cos\left[|\omega|\lambda^*\, a\,(\pi-\theta) - \frac{\pi}{4}\right] \tag{117}$$

where from (61), (109) and (116) we have, on using the approximate relation

$\mu \sim i\omega\lambda a$

$$K = I(i\omega, R_0)\left(\frac{2\rho_0(a)\lambda^*}{\pi\, a \sin\theta|\omega|}\right)^{1/2} \frac{\omega^2\,\phi(R_0,\lambda^*)}{\left\{\frac{\partial}{\partial\lambda}\left[\frac{\phi' + A\phi}{h}\right]\right\}_{\begin{pmatrix} r=a \\ \lambda=\lambda^* \end{pmatrix}}} \tag{118}$$

The integral (114) can be written in the form

$$p(a,\theta,t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega t - i\omega\lambda^* a\pi} \cos\left[|\omega|\,\lambda^*\, a\,(\pi-\theta) - \frac{\pi}{4}\right] K\, d\omega \tag{119}$$

Decompose K in the following manner

$$K = K_e + K_o$$

where $K_e$ and $K_o$ are the even and odd functions of $\omega$. Making use of the fact that $\lambda^*$ is an even function of $\omega$ we can rewrite (119) in the form

$$p(a, \theta, t) = \frac{1}{\pi}\int_0^{\omega_c} \cos\left[\omega(t - \lambda^* a\pi)\right] \cos\left[\omega\lambda^* a(\pi - \theta) - \frac{\pi}{4}\right] K_e \, d\omega$$

$$+ \frac{i}{\pi}\int_0^{\omega_c} \sin\left[\omega(t - \lambda^* a\pi)\right] \cos\left[\omega\lambda^* a(\pi - \theta) - \frac{\pi}{4}\right] K_o \, d\omega \tag{120}$$

Decompose $p(a, \theta, t)$ into two parts

$$p(a, \theta, t) = p^1(a, \theta, t) + p^2(a, \theta, t)$$

where

$$p^1(a, \theta, t) = \frac{1}{2\pi}\int_0^{\omega_c} K_e \cos\left[\omega t - \omega\lambda^* a\theta - \frac{\pi}{4}\right] d\omega$$

$$+ \frac{i}{2\pi}\int_0^{\omega_c} K_o \sin\left[\omega t - \omega\lambda^* a\theta - \frac{\pi}{4}\right] d\omega \tag{121}$$

$$p^2(a, \theta, t) = \frac{1}{2\pi}\int_0^{\omega_c} K_e \cos\left[\omega t - \omega\lambda^* a(2\pi - \theta) + \frac{\pi}{4}\right] d\omega$$

$$+ \frac{i}{2\pi}\int_0^{\omega_c} K_o \sin\left[\omega t - \omega\lambda^* a(2\pi - \theta) + \frac{\pi}{4}\right] d\omega \tag{122}$$

The physical significance of expressions $p^1(a, \theta, t)$ and $p^2(a, \theta, t)$ is that $p^1(a, \theta, t)$ represents the pressure wave which has travelled directly to the observer, whereas $p^2(a, \theta, t)$ is the pressure wave which has travelled through the antipode.

For a simple point source at altitude $z_o = (R_o - a)$, the source function (given in Appendix A) is

$$I(i\omega, R_o) = -i\omega \rho_o^{1/2}(R_o) V \tag{123}$$

where V is the volume of gas introduced. In this case $K_e = 0$, and (118) becomes

$$K = K_o = i\omega \left( \frac{2 \rho_o(a) \rho_o(R_o) \lambda^*}{\pi a \sin \theta |\omega|} \right)^{1/2} V Q(\omega) \tag{124}$$

where for convenience we have set

$$Q(\omega) = \frac{-\omega^2 \phi(R_o, \lambda^*)}{\left\{ \frac{\partial}{\partial \lambda} \left[ \frac{\phi' + A\phi}{h} \right] \right\}_{\substack{r = a \\ \lambda = \lambda^*}}} \tag{125}$$

Hence the pressure waves which have travelled both directly to the observer and by way of the antipodes are given respectively by

$$p^1(a, \theta, t) = \left( \frac{\rho_o(a) \rho_o(R_o)}{2\pi a \sin \theta} \right)^{1/2} \frac{V}{\pi} \int_0^{\omega_c} (\omega\lambda^*)^{1/2} Q(\omega) \cos \left[ \omega t - \omega\lambda^* a \theta + \frac{\pi}{4} \right] d\omega \tag{126}$$

$$p^2(a, \theta, t) = \left( \frac{\rho_o(a) \rho_o(R_o)}{2\pi a \sin \theta} \right)^{1/2} \frac{V}{\pi} \int_0^{\omega_c} (\omega\lambda^*)^{1/2} Q(\omega) \cos \left[ \omega t - \omega\lambda^* a (2\pi - \theta) + \frac{3\pi}{4} \right] d\omega \tag{127}$$

The integrals in (126) and (127) may be equated to the real parts of

$$\int_0^{\omega_c} (\omega\lambda^*)^{1/2} Q(\omega) \exp i \left[ \omega t - \omega\lambda^* a \theta + \frac{\pi}{4} \right] d\omega \tag{128}$$

and

$$\int_0^{\omega_c} (\omega\lambda^*)^{1/2}\, Q(\omega)\, \exp i\left[\omega t - \omega\lambda^* a\,(2\pi - \theta) + 3\frac{\pi}{4}\right] d\omega \tag{129}$$

respectively. For large ranges ($a\theta$ and $(2\pi - \theta)\,a$ sufficiently large) these integrals may be evaluated by the method of stationary phase.

If $\omega_0$ is the value of $\omega$ for which

$$t - a\theta\,(\omega\lambda^*)' = 0 \tag{130}$$

where the prime indicates the derivative with respect to $\omega$, then the stationary phase technique gives the asymptotic approximation to (128)

$$\sqrt{\frac{2\pi}{a\theta}}\; Q(\omega_0)\; \sqrt{\frac{\omega_0\lambda^*}{(\omega_0\lambda^*)''}}\; \cos\left[\omega_0 t - \omega_0\lambda^* a\theta\right] \tag{131}$$

Similarly if $\omega_0$ is the value of $\omega$ for which

$$t - a\,(2\pi - \theta)\,(\omega\lambda^*)' = 0 \tag{132}$$

then the stationary phase technique gives the asymptotic approximation to (129)

$$\sqrt{\frac{2\pi}{a\,(2\pi - \theta)}}\; Q(\omega_0)\; \sqrt{\frac{\omega_0\lambda^*}{(\omega_0\lambda^*)''}}\; \cos\left[\omega_0 t - \omega_0\lambda^* a\,(2\pi - \theta) + \frac{\pi}{2}\right] \tag{133}$$

However the stationary phase technique for (128) is not accurate unless $(\omega\lambda)''' / \left[(\omega\lambda)''\right]^{3/2} < \sqrt{a\theta}$ and $(\omega\lambda)^{iv} / \left[(\omega\lambda)''\right]^{2} < a\theta$. It has been shown by Penney (Ref. 6 ), that for most models of the atmosphere under consideration, this critique fails when the range is of the order of 1000 km or less. For these cases it is best at present to perform numerical integration to evaluate (128) in the manner that is suggested in Reference (6).

In addition, even at very large ranges, the stationary phase technique fails for the initial part of the received pressure pulse. We shall give here an asymptotic technique for calculating the initial part or head of the pulse. We will consider integral (128).

Define the following series

$$\lambda^*(\omega) = \sum_{n=0}^{\infty} \lambda_n \omega^{2n} \tag{134}$$

$$(\lambda^*)^{1/2} Q(\omega) = \lambda_o^{1/2} Q(o) \sum_{n=0}^{\infty} d_n \omega^{2n} \tag{135}$$

where $d_o = 1$, and the time interval

$$\tau = t - a\theta\lambda_o . \tag{136}$$

By the stationary phase technique, the condition $\tau = 0$ or $t = a\theta\lambda_o$ would indicate the beginning of the pulse. However for $|\tau|$ small, the stationary phase technique fails and we shall show that $\tau = 0$ does not represent the beginning of the pulse, but a point near the peak of the first compression.

The integral (128) may now be written in the form

$$\lambda_o^{1/2} Q(o) \int_0^{\omega_c} \omega^{1/2} \cdot \sum_{n=0} d_n \omega^{2n} \cdot \left[1 - ia\theta(\omega^5\lambda_2 + \omega^7\lambda_3 + \ldots) + \ldots\right] \cdot$$

$$\exp i\left[\omega\tau - \omega^3\lambda_1 a\theta + \frac{\pi}{4}\right] d\omega \tag{137}$$

Making the substitution

$$\omega = \sigma (\lambda_1 a \theta)^{-1/3} \tag{138}$$

$$q = \tau (\lambda_1 a \theta)^{-1/3} \tag{139}$$

and defining the integral

$$F_n(q) = \int_0^{\infty} \sigma^{-1/2+n} e^{i\left[\sigma q - \sigma^3\right]} d\sigma \tag{140}$$

expression (137) has the asymptotic form

$$\sqrt{\frac{\lambda_o}{\lambda_1 a \theta}}\, Q(o)\, e^{i\pi/4} \left\{ F_1(q) + \frac{d_1}{(\lambda_1 a \theta)^{2/3}} F_3(q) + \frac{d_2}{(\lambda_1 a \theta)^{4/3}} F_5(q) + \ldots \right.$$

$$\left. - \frac{i a \theta}{(\lambda_1 a \theta)^{5/3}} \lambda_2 F_6(q) - \frac{i a \theta}{(\lambda_1 a \theta)^{7/3}} \left[\lambda_3 + d_1 \lambda_2\right] F_8(q) + \ldots \right\} \tag{141}$$

In Appendix C, $F_o(q)$, $F_1(q)$ and $F_2(q)$ are expressed in terms of Airy integrals, and $F_n(q)$ for $n \geqslant 3$ are expressed in terms of $F_m(q)$; $m = 0, 1, 2$. Thus expression (141) may be represented in terms of Airy integrals (which are tabulated, Ref. 13). After much algebraic manipulation, the <u>real</u> part of (141) has the following asymptotic form

$$2\, Q(o) \sqrt{\frac{\lambda_o \pi^3}{3 \lambda_1 a \theta}} \left\{ - A_i A_i' C_1 + A_i^2 C_2 - \left[(A_i')^2 + y (A_i)^2\right] C_3 \right\} \tag{142}$$

where $A_i(y)$ is the Airy integral, with $A_i'$ being its derivative. The argument $y$ is given by

$$y = - \frac{\tau}{(12 \lambda_1 a \theta)^{1/3}} \tag{143}$$

The functions $C_i(\tau)$ have the forms

$$C_1(\tau) = 1 + \frac{\tau}{(12\lambda_1 a\theta)}\left(4d_1 - \frac{20\lambda_2}{3\lambda_1}\right) + \frac{8\tau^2}{(12\lambda_1 a\theta)^2}\left(2d_2 - 7\frac{[\lambda_3 + d_1\lambda_2]}{\lambda_1}\right)$$

$$C_2(\tau) = \frac{1}{(12\lambda_1 a\theta)^{2/3}}\left(d_1 - \frac{7\lambda_2}{6\lambda_1}\right) + \frac{4\tau}{(12\lambda_1 a\theta)^{5/3}}\left(d_2 - 3\frac{[\lambda_3 + d_1\lambda_2]}{\lambda_1}\right)$$

$$C_3(\tau) = \frac{10}{(12\lambda_1 a\theta)^{4/3}}\left(d_2 - \frac{11[\lambda_3 + d_1\lambda_2]}{6\lambda_1}\right) + \frac{\tau^2}{(12\lambda_1 a\theta)^{4/3}}\,\frac{4\lambda_2}{3\lambda_1}$$

$$+ \frac{\tau^3}{(12\lambda_1 a\theta)^{7/3}}\,\frac{16[\lambda_3 + d_1\lambda_2]}{3\lambda_1} \tag{144}$$

For the real part of integral (129) one may derive in a similar manner the asymptotic expression

$$-2Q(o)\sqrt{\frac{\lambda_o\pi^3}{3\lambda_1 a(2\pi - \theta)}}\left[A_i'(y)B_i(y) + \frac{1}{2\pi}\right] \tag{145}$$

where the argument of the Airy integrals in this case is given by

$$y = -\frac{\tau}{[12\lambda_1 a(2\pi - \theta)]^{1/3}} \tag{146}$$

with

$$\tau = t - a(2\pi - \theta)\lambda_o \tag{147}$$

Because the distance around the earth by way of the antipodal route is extremely large, the higher order terms in the asymptotic expansions are unnecessary, except for very large $\tau$ in which case, the stationary phase technique is valid.

The directly received pulse $p^1(a, \theta, t)$ is calculated for Scorer's model of the atmosphere for explosions both on the ground and at a height of 9.6137 km, and the pulse forms are plotted in Fig. 1.

The integral representing the pulse was calculated using the stationary phase technique for $\tau = t - a\theta\lambda_0 > 160$ sec. and using the asymptotic expansion involving the Airy integrals for $\tau \leq 160$ sec. Even though the use of the stationary phase technique for the case when $a\theta = 3600$ km (the distance from the source) is somewhat a delicate matter, there is good matching of the results obtained by the two techniques. The complete pulse forms are not given, the dashed line indicating that the remaining portion of the pulses are not given. The transit time of the first crest (given approximately by $t = a\theta\lambda_0$) is 3 hr. 11 min. and 5 hr. 19 min. for distances 3600 and 6000 km respectively.

In addition, the received pressure pulse $p^1(a, \theta, t)$ is calculated for atmosphere II for a range of 7000 km for explosions on the ground and at a height of 39 km. A table of the intensity function required in the evaluation of the pressure pulse integral, and the $\lambda$'s are given below. The value of the $\lambda$'s for the higher values of $\omega$'s are not calculated.

The stationary phase technique was used for $\tau \geq 260$ secs., the other asymptotic technique for $\tau \leq 260$ secs. The main body of the pressure pulses observed at ground level at range of 7000 km from both an explosion on the ground and an explosion at 39 km are plotted in Fig. 2. The transit time of these pulses is roughly 6 hr. 8 min.

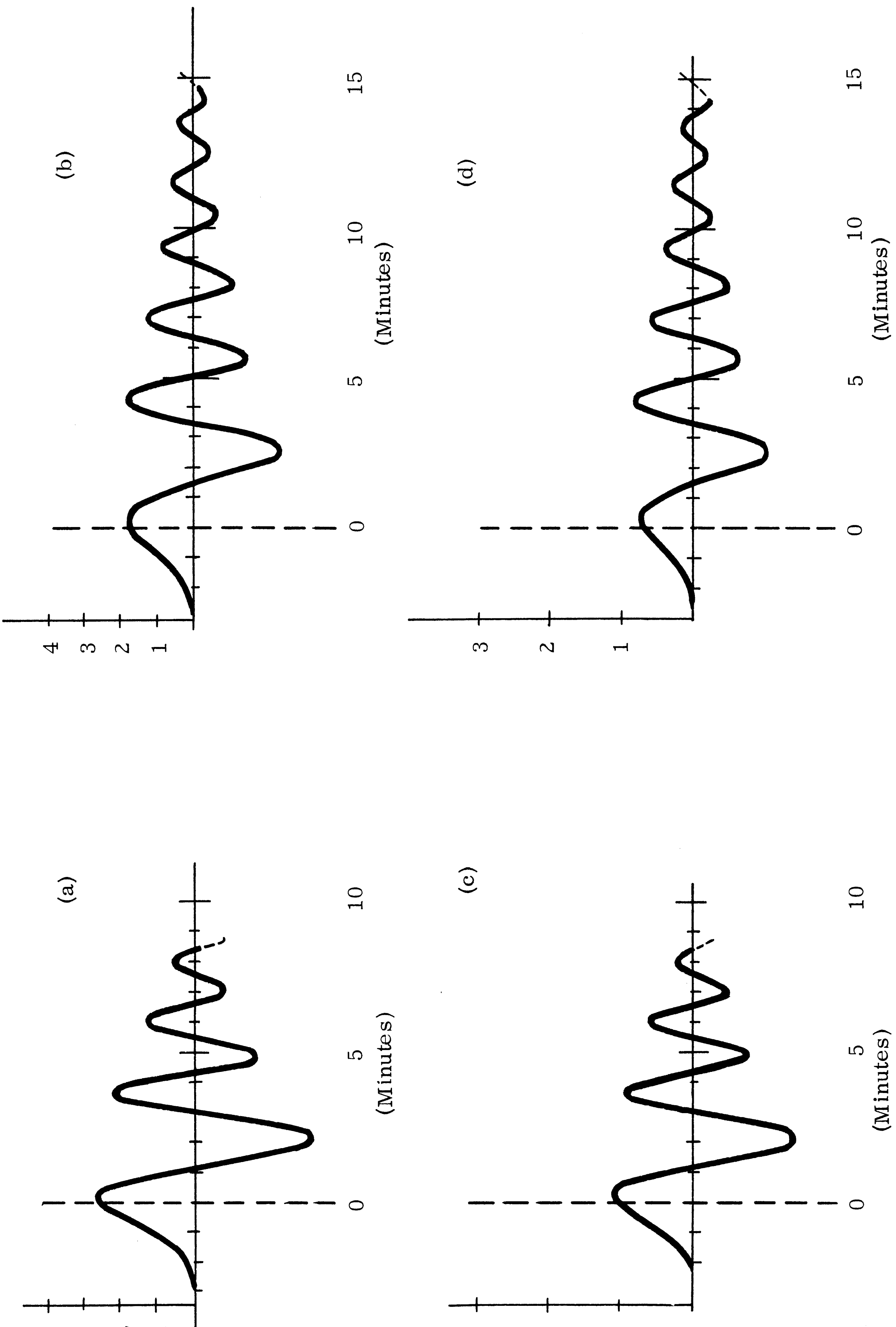


FIG. 1: THE PRESSURE PULSE AT THE GROUND AT A DISTANCE OF 3600 km (a) and (c), and 6000 km (b) and (d) FOR SCORER'S ATMOSPHERE, FOR AN EXPLOSION ON THE GROUND (a) and (b), AND AT A HEIGHT OF 9.6137 km (c) and (d). ONE UNIT OF AMPLITUDE CORRESPONDS TO 0.614 $\mu$ BARS, PER 1 km$^3$ OF GAS RELEASED.

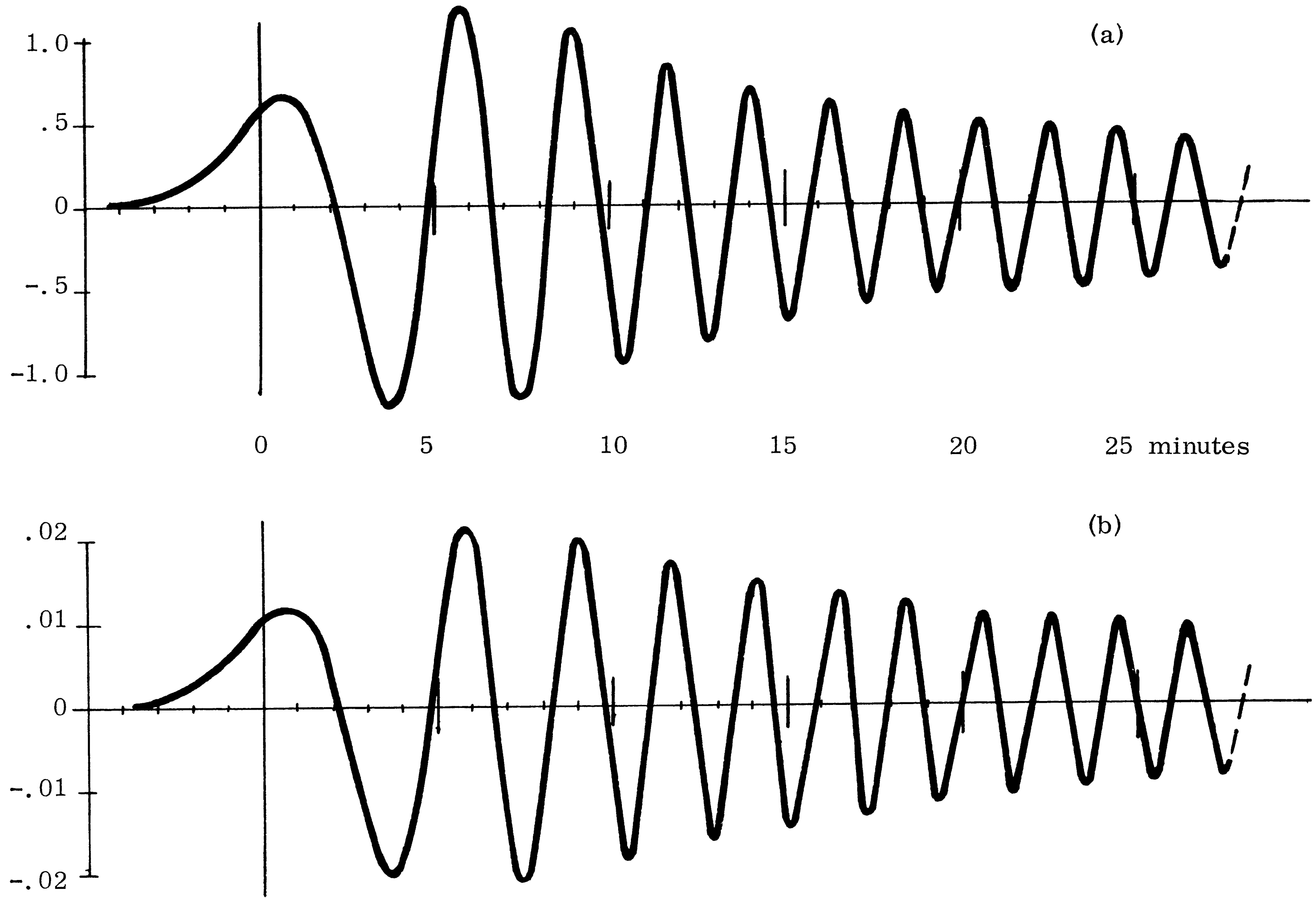


FIG. 2: THE HEAD OF THE PRESSURE PULSE AT GROUND LEVEL AT A DISTANCE OF 7000 km FROM AN EXPLOSION ON THE GROUND (a), AND AT A HEIGHT OF 39 km (b), FOR ATMOSPHERE MODEL II. ONE UNIT OF AMPLITUDE CORRESPONDS TO 1 $\mu$ BAR, PER 1 km$^3$. OF GAS RELEASED. (The tail of the pulse which extends for a very long time interval beyond the dashed lines is not given.)

| $5000\,\omega^2$ | $\lambda^*(\omega)$ | $10^3 \sqrt{\frac{\rho_0(R_0)}{\rho_0(a)}}\; Q(\omega, R_0)$ | |
|---|---|---|---|
| | | Explosions on the ground | Explosions at 39 km |
| 0 | 3.15283 | 10.71 | .1856 |
| 1 | 3.15819 | 10.83 | .1848 |
| 2 | 3.16277 | 10.90 | .1862 |
| 3 | 3.16695 | 10.68 | .1843 |
| 4 | 3.17091 | 10.42 | .1829 |
| 5 | 3.17472 | 10.09 | .1807 |
| 6 | 3.17844 | 9.74 | .1783 |
| 7 | 3.18209 | 9.37 | .1758 |
| 8 | 3.18569 | 8.99 | .1730 |
| 9 | 3.18923 | 8.61 | .1699 |
| 10 | 3.19274 | 8.25 | .1671 |
| 11 | 3.19620 | 7.86 | .1635 |
| 12 | 3.19962 | 7.54 | .1609 |
| 13 | 3.20301 | 7.19 | .1576 |
| 14 | 3.20635 | 6.86 | .1542 |
| 15 | 3.20965 | 6.51 | .1500 |
| 16 | 3.21290 | 6.19 | .1462 |
| 17 | 3.21611 | 5.89 | .1424 |
| 18 | 3.21928 | 5.60 | .1387 |
| 19 | 3.22239 | 5.30 | .1344 |
| 20 | 3.22545 | 5.03 | .1304 |
| 21 | 3.22846 | 4.77 | .1263 |
| 22 | 3.23142 | 4.51 | .1221 |
| 23 | 3.23433 | 4.30 | .1187 |
| 24 | 3.23718 | 4.09 | .1151 |
| 25 | 3.23998 | 3.87 | .1111 |

One important difference between explosions on the ground and at an altitude of 39 km is that the gravity wave portion of the pressure pulse is less attenuated for the latter case.

As expected the amplitude on the ground of the pressure pulses is less for high altitude explosions. For atmosphere II, the ratio of amplitudes of the pressure pulses observed on the ground due to an explosion at 39 km and on the ground is .017, where the same volume of gas is released in both cases. An estimate of the ratio of the amplitudes of the gravity wave portion of the pressure pulses is given by

$$\exp\left\{-\int_{z_1}^{z_2} \frac{g}{c_o^2(z)}\, dz\right\}$$

where $z_2$, $z_1$ are the altitude of the explosions. However this holds for idealistic explosions characterized by the sudden release of a specific amount of gas.

11. Comments:

The pulse forms in the previous section have been obtained from an idealistic source model. The source representation could be more generalized by taking a different time dependence (like the ones mentioned in Ref. 6) rather than the delta function. Another question arises as to the relation between the volume of gas introduced and the total energy released by an actual explosion, for instance a nuclear explosion. Penney et al do give some estimates for explosions on the ground. However, explosions at various altitudes will require different constants of proportionality.

A problem arises with regard to the upper boundary condition, namely that the total kinematic energy be finite. For an isothermal stratosphere or upper atmosphere, the excess pressure has the following functional dependence on altitude z

$$\exp\left[\frac{\gamma g}{2\, c_s^2} - \beta\right] z$$

where the flat earth approximation is taken for simplication by equating r to a, and taking g constant in the expression for $\beta$ given by (74). For Scorer's model $\gamma g/(2c_s^2)$ is greater than $\beta$ for the gravity wave. Hence the excess pressure increases exponentially with altitude. At high altitudes the hydrodynamics equations can no longer be linearized, as the excess variables approach and even become greater than the magnitude of the unperturbed variables. For atmosphere II, $\gamma g/(2c_s^2)$ is greater than $\beta$ for $5000\,\omega^2$ less than 13. In this case the excess pressure of the gravity wave increases exponentially only for the lower frequencies. This suggests that when taking an isothermal model for the upper atmosphere, it probably is best to start the isothermal level above 100 or 110 km, in which case the temperature may be taken very large, say $350^{\circ}$ K or higher. For this case one would expect that the excess pressure of the gravity wave attenuates with increasing altitude except for the extreme lower frequencies. However, at high altitudes an additional factor becomes important, namely viscosity. This should be included in the model of a very warm isothermal layer at high altitudes.

Winds have an effect upon the gravity wave although not so much as on the high frequency acoustical waves. Apart from jet streams, there are good strong, steady winds at high altitudes. For instance, in the winter there are winds of magnitudes of 50 m/sec and 100 m/sec at heights 40 and 60 km respectively around latitudes $30^{\circ}$ to $60^{\circ}$. (See Ref. 15.) These winds can cause an appreciable change in the phase and group velocities of the gravity wave.

In Part II, the effect of winds and better models of the upper atmosphere will be investigated.

Acknowledgement

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# APPENDIX A

## Simple Source Representation:

Given the inverse Laplace transform of the excess pressure in terms of an integral over a surface $\sum$, namely

$$P(a, \theta, s) = - s\rho_o^{1/2}(a) \iint_{\sum} G(a, \theta; r_o, \theta_o) \, \underline{n} \cdot \underline{U}(\underline{r}_o) \, \rho_o^{1/2}(r_o) \, dS_o$$

$$- \rho_o^{1/2}(a) \iint_{\sum} \rho_o^{-1/2}(r_o) P(\underline{r}_o) \left\{ \underline{n} \cdot \hat{\underline{i}}_{r_o} \left( \frac{A}{h} \right)_o + \underline{n} \cdot \underline{L}_o \right\} G \, dS_o$$

where the variables of integration are $(r_o, \theta_o)$, we will consider the case where $\sum$ is a small sphere with center $(R_o, 0)$ and radius $\epsilon$. We will assume in addition that the disturbance (characterized by the transform of the normal velocity to the surface and the pressure) is spherically symmetric about $(R_o, 6)$, in which case $\underline{n} \cdot \underline{U}(\underline{r}_o)$ and $P(\underline{r}_o)$ are functions of $s$ and $\epsilon$ only.

We shall assume that, the velocity and the pressure satisfy the acoustic equation at the surface $\sum$, in which case $\underline{U}$ and $P$ are proportional to $\epsilon^{-2}$ and $\epsilon^{-1}$ respectively. Retaining only the dominant terms for small $\epsilon$ we have

$$P(a, \theta, s) \sim - s\rho_o^{1/2}(a) \, \rho_o^{1/2}(R_o) \, G(a, \theta; R_o, 0) \iint_{\sum} \underline{n} \cdot \underline{U} \, dS_o$$

$$\sim \rho_o^{1/2}(a)\, G(a, \theta; R_o, 0)\, I(s, R_o)$$

For an instantaneous velocity source at $\underline{r} = (\underline{R}_o, 0, 0)$ at time $t_o$ we obtain

$$\underline{n} \cdot \underline{U}(\underline{r}_o, s) = \underline{n} \cdot \underline{U}(r_o)\, e^{st_o}$$

and $\iint_{\Sigma} \underline{n} \cdot \underline{U}(\underline{r}_o)\, dS_o = V$ is the volume of gas introduced

Thus for a simple point source at time $t_o$

$$I(s, R_o) = - s\rho_o^{1/2}(R_o)\, V\, e^{st_o}$$

and for $t_o = 0_+$, this becomes

$$I(s, R_o) = - s\rho_o^{1/2}(R_o)\, V.$$

In the special case where the explosion is on the ground, the surface $\Sigma$ is a hemisphere, and V will be the volume of gas passing through it.

APPENDIX B

Radial Equation for the Vertical Velocity:

For the convenience of analysis set

$$i\omega\rho_o^{1/2}\ U_r(r,\ \theta,\ \omega) = \chi(r)\, P_{i\mu - 1/2}(\cos\theta) \tag{B. 1}$$

then using the relationship

$$\rho_o^{1/2}\ \phi(r)\ P_{i\mu - 1/2}(\cos\theta) = P(r,\ \theta,\ \omega) \tag{B. 2}$$

together with radial component of equation (13), we obtain

$$\frac{\phi' + A\phi}{h} = -\chi \tag{B. 3}$$

But we have from Eqns. (47) and (64)

$$\frac{d}{dr}\left(\frac{r^2}{h}\ \frac{d\phi}{dr}\right) + \left[q(r) - \omega^2\lambda^2 a^2\right]\phi = 0 \tag{B. 4}$$

Hence from (B. 3) substitute $-\left[\chi + A\,\phi\, h^{-1}\right]$ in place of $\phi'\, h^{-1}$ in (B. 4) and repeating the procedure after differentiation we obtain

$$\frac{dr^2\ \chi}{dr} - Ar^2\chi - \phi\left[-\omega^2\lambda^2 a^2 + \frac{\omega^2 r^2}{c^2}\right] = 0 \tag{B. 5}$$

Setting $r^2 \chi = \psi$ (B. 6)

$$\alpha = \omega^2 \left[ \frac{r^2}{c^2} - \lambda^2 a^2 \right] \tag{B. 7}$$

This reduces to

$$\frac{\psi' - A\psi}{\alpha} = \phi \tag{B. 8}$$

Eliminate $\phi$ from (B. 8) and (B. 3) to obtain

$$\psi'' - \frac{\alpha'}{\alpha} \psi' + \psi \left[ -A^2 - A' + A \frac{\alpha'}{\alpha} \right] = -\psi \frac{h\alpha}{r^2} \tag{B. 9}$$

Setting $\psi = \alpha^{1/2} M$ we obtain (B. 10)

$$M'' + M \left\{ -\frac{3}{4} \left( \frac{\alpha'}{\alpha} \right)^2 + \frac{\alpha''}{2\alpha} + \frac{A\alpha'}{\alpha} - A' - A^2 \right.$$

$$\left. + \frac{h}{r^2} \alpha \right\} = 0 \tag{B. 11}$$

where now from (B. 1), (B. 6), and (B. 10) we have

$$U_r = \sqrt{\frac{r^2/c^2 - \lambda^2 a^2}{\rho_0(r)}} \frac{M(r)}{r^2} P_{i\mu - 1/2} (\cos \theta) \tag{B. 12}$$

## APPENDIX C

Evaluation of the Integrals: $F_n(q) = \int_0^\infty \omega^{-1/2+n} \exp i \left[\omega q - \omega^3\right] d\omega$

From Franz (Ref. 12) we have the relation

$$\int_0^\infty \sigma^{-1/2} \; e^{\alpha\sigma - \sigma^3} \, d\sigma = \sqrt{\frac{3}{\pi}} \; 2^{5/3} \; A\left(\frac{e^{i\pi/3}\alpha}{4^{1/3}}\right) \; A\left(\frac{e^{-i\pi/3}\alpha}{4^{1/3}}\right) \qquad (C.1)$$

where

$$A(x) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \; e^{i\left[x\tau - \tau^3\right]} \qquad (C.2)$$

However we may express $A(x)$ in terms of the usual Airy integral $A_i(x)$, by the relation

$$A(x) = - 3^{-1/3} \pi \; A_i(-3^{-1/3} x) \qquad (C.3)$$

where $$A_i(x) = \frac{1}{\pi} \int_0^\infty \cos\left[xt + t^3/3\right] dt \qquad (C.4)$$

Hence the integral (C.1) may be given in the better known notation

$$\int_0^\infty \sigma^{-1/2} \; e^{\alpha\sigma - \sigma^3} \, d\sigma = \pi^{3/2} \, 2^{5/3} \, 3^{-1/6} \; A_i\left(e^{2i\pi/3} y\right) A_i(y) \qquad (C.5)$$

where
$$y = -e^{-i\pi/3}\ \alpha(12)^{-1/3} \tag{C.6}$$

Performing the transformation

$$\begin{aligned} \alpha &= q\,e^{i\pi/3} \\ \sigma &= \omega\,e^{i\pi/6} \end{aligned} \tag{C.7}$$

Equation (C.5) may be written in the form

$$F_o(q) = \pi^{3/2}\, 2^{5/3}\, 3^{-1/6}\, e^{-i\pi/12}\, A_i(y)\, A_i\left(e^{2i\pi/3}y\right) \tag{C.8}$$

with
$$y = -(12)^{-1/3}\, q \tag{C.9}$$

From Miller (Ref. 13) we have

$$A_i\left(e^{2i\pi/3}y\right) = -\frac{1}{2}\, e^{i4\pi/3}\left[A_i(y) - iB_i(y)\right] \tag{C.10}$$

with

$$B_i(y) = \frac{1}{\pi}\int_0^\infty \left\{\exp\left[-\frac{1}{3}t^3 + yt\right] + \sin\left[\frac{1}{3}t^3 + yt\right]\right\} dt \tag{C.11}$$

hence (C.8) becomes

$$F_o(q) = \pi^{3/2}\, 2^{2/3}\, 3^{-1/6}\, e^{i\pi/4}\, A_i(y)\left[A_i(y) - iB_i(y)\right] \tag{C.12}$$

The other integrals $F_n(q)$ may be evaluated from $F_o(q)$ by differentiation with respect to $q$, i e!

$$F_n(q) = i^{-n} \frac{d^n F_o(q)}{dq^n} \tag{C.13}$$

$$= (12)^{-n/3} i^n \frac{d^n F_o(q)}{dy^n}$$

In particular we obtain

$$F_1(q) = \pi^{3/2} 3^{-1/2} e^{i3\pi/4} \Big[ 2A_i(y) A_i'(y) - i \Big\{ A_i'(y) B_i(y) + A_i(y) B_i'(y) \Big\} \Big]$$

which on using the Wronskian relation

$$A_i B_i' - A_i' B_i = \frac{1}{\pi} \tag{C.14}$$

reduces to

$$F_1(q) = \pi^{3/2} 2\,3^{-1/2} e^{i3\pi/4} \left[ A_i(y) A_i(y)' - i \left\{ A_i'(y) B_i(y) + \frac{1}{2\pi} \right\} \right] \tag{C.15}$$

In addition we obtain

$$F_2(q) = 2(\pi^3/3)^{1/2}(12)^{-1/3} e^{i5\pi/4} \Big[ (A_i')^2 + y(A_i)^2$$

$$- i\,(y A_i B_i + A_i' B_i') \Big] \tag{C.16}$$

The remaining $F_n(q)$ may be expressed in terms of $F_o$, $F_1$, $F_2$ by the relation

$$F_n(q) = \alpha_n (12)^{-n/3}\, i^n\, F_o(q) + \beta_n (12)^{-(n-1)/3}\, i^{(n-1)}\, F_1(q)$$

$$+ \gamma_n (12)^{-(n-2)/3}\, i^{(n-2)}\, F_2(q)$$

where the functions $\alpha_n(q)$, $\beta_n(q)$ and $\gamma_n(q)$ are given in the table below

| n | $\alpha_n$ | $\beta_n$ | $\gamma_n$ |
|---|---|---|---|
| 3 | 2 | 4y | 0 |
| 4 | 0 | 6 | 4y |
| 5 | 8y | $16y^2$ | 10 |
| 6 | 28 | 80y | $16y^2$ |
| 7 | $32y^2$ | $108 + 64y^3$ | 112y |
| 8 | 288y | $672y^2$ | $(220 + 64y^3)$ |

REFERENCES

1. R. Yamamoto, "A Dynamical Theory of the Microbarographic Oscillations Produced by the Explosions of Hydrogen Bombs," Journal of the Meteorological Society of Japan, Series II, 35:5, (1957).

2. C. L. Pekeris, "The Propagation of a Pulse in the Atmosphere," Proceedings of the Royal Society, A 171, 434, (1939).

3. C. L. Pekeris, "The Propagation of a Pulse in the Atmosphere, Part II," Physical Review, 73:2, 145, (1948).

4. R. C. Scorer, "The Dispersion of a Pressure Pulse in the Atmosphere," Proceedings of the Royal Society, A 201 137, (1950).

5. L. A. Dikii, "Acoustical and Gravity Vibrations in the Atmosphere," Izvestiia A.N. SSSR; ser. geofiz, (8): 1186-1194, (1959).

6. J. N. Hunt, R. Palmer, and Sir William Penney, "Atmospheric Waves Caused by Large Explosions," Phil. Trans. Roy. Soc. London, 18 February 1960.

7. J. L. Taylor, "An Exact Solution of the Spherical Blast Problem," Phil. Mag., Vol. 46, 317, (1954).

8. A. S. Ramsey, A Treatise on Hydromechanics, Part II, C. Bell and Son, (1949).

9. K. O. Friedrichs, "Criteria for Discrete Spectra," Symposium on Electromagnetic Theory, p. 85, Interscience, (1950).

10. K. O. Friedrichs, Courant Anniversary Volume, Interscience, New York, pp. 145-160, (1948).

11. E. C. Titchmarsh, Eigenfunction Expansion, Part I, Oxford, (1946).

12. W. Franz, "On the Green's Function of the Cylinder and Sphere," Z. für Naturforschung, 9a, 705-714, (1954).

13. J. C. P. Miller, "The Airy Integral," British Assoc. for the Advancement of Science, Math. Tables Part - Vol. B, Univ. Press, Cambridge, (1946).

14. F. G. Friedlander, Sound Pulses, Cambridge Univ. Press, (1958).

15. R. J. Murgatroyd, "Winds and Temperatures Between 20 km and 40 km -- a Review," Quart. Journ. of Royal Met. Soc., 83, No. 358, 417-458, October, 1957.