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STUDIES IN RADAR CROSS SECTIONS XXXIII - EXACT NEAR FIELD AND FAR FIELD SOLUTION FOR THE BACK-SCATTERING OF A PULSE FROM A PERFECTLY CONDUCTING SPHERE

by

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Preface

This is the thirty-third in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers, and (d) low and high density ionization phenomena.

K. M. Siegel

Abstract

If a short plane-wave pulse comprising the single harmonic frequency ω is incident on a perfectly conducting sphere, there is a significant tail to the back-scattered pulse for the frequency in the resonance region of the sphere. For very large spheres, there is negligible tail to the return pulse.

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Introduction

Scattering of pulses by perfectly conducting bodies has been considered for the electromagnetic case by Keller(1) and other authors, and for the acoustic case by Friedlander(2). However, the investigation centered around the propagation of the diffracted and reflected wave fronts and the zone immediately behind these fronts.

In this report, we are interested in a different aspect of the problem. An incident plane wave pulse of time-length τ, comprised of a single harmonic frequency ω, is incident on a perfectly conducting sphere of radius a. We place an observer in the back-scattering direction at a fixed distance from the center of the sphere. The observer then measures the variation of the back-scattered pulse with respect to time. Besides measuring the initial parts of the received reflected and diffracted wave fronts, the observer measures the field far behind these wave fronts.

There are several questions to be answered. For a given fixed wavelength of the incident pulse, what sizes of spheres will produce a return pulse with a significant tail, i.e. pulse length of back-scattered pulse much longer than that of the incident pulse? Is this tail significantly affected by varying the distance of the observation point from the sphere?

1. Received Pulse:

In considering the back-scattered or received pulse, we will just consider the electric field component.

Since the back-scattered field for the c.w. case is known exactly, we will use the relationship that the pulse solution can be derived from the c.w. solution (where both satisfy the same boundary conditions on the diffracting body), by an inverse Laplace transform.

Consider an incident pulse of pulse length τ and frequency ω, propagating in the direction of the positive z-axis. The electric field is then given by

$$\underline{E}^{i} = \hat{\underline{i}}_x \begin{cases} 0 & t < z/c \\ e^{ikz-i\omega t} & z/c \leq t < z/c + \tau \\ 0 & z/c + \tau \leq t \end{cases} \tag{1.1}$$

This can be expressed in terms of an inverse Laplace transform

$$\underline{E}^{i} = \hat{\underline{i}}_x \frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} \frac{e^{ts}\left[1 - e^{-(s+i\omega)\tau}\right]}{(s + i\omega)} E^{o}\left(\frac{is}{c}\right) ds \tag{1.2}$$

where $E^{o}(k) = e^{ikz}$.

The scattered field in the back-scattering direction is given by

$$\underline{E}^{r} = \hat{\underline{i}}_x \frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} \frac{e^{ts}\left[1 - e^{-(s+i\omega)\tau}\right]}{(s+i\omega)} E^{s}\left(\frac{is}{c}\right) ds \tag{1.3}$$

where $E^s(k)$ is the back-scattered field for the c.w. case, i.e.

$$E^s(k) = -\sum_{n=1}^{\infty} (-i)^n (n+1/2) \left[\frac{h_n^{(1)}(kR)\ j_n\ (ka)}{h_n^{(1)}(ka)} + \frac{i\left[kR\ h_n^{(1)}(kR)\right]' \left[ka\ j_n\ (ka)\right]'}{kR\left[ka\ h_n^{(1)}(ka)\right]'} \right] . \tag{1.4}$$

For $t < (R-2a)/c$, the integrand of (1.3) vanishes exponentially as $|s|$ approaches ∞ for $\mathcal{R}(s) > 0$. Hence we may formally extend the line integral by an integral along an infinite semi-circle to the right of the line $\mathcal{R}(s) > b$, and equate the resulting contour integral to the sum of residues of the poles in the enclosed region. But there are no poles enclosed, since by definition of the inverse Laplace transform, the line $\mathcal{R}(s) = b$ is taken to the right of all the poles of the integrand.

Thus we see that the line integral (1.3) is zero; hence

$$\underline{E}^r = 0 \qquad\qquad t < (R-2a)/c.$$

We wish to examine the time variation of the return pulse, the time being measured from the initial part of the return pulse at a particular fixed observation point R.

Let T be the time measured from the initial part of the return pulse:

$$t = (R - 2a)/c + T \tag{1.5}$$

Define:

$$F(T) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\exp\left[T + (R - 2a)/c\right] s}{(s + i\omega)} \; E^s\left(\frac{is}{c}\right) ds. \tag{1.6}$$

This expression vanishes for $T < 0$. Hence we have

$$\underline{E}^r = \hat{\underline{i}}_x \begin{cases} F(T) - e^{-i\omega\tau} F(T - \tau) & \text{for } T \geqslant \tau \\ F(T) & T < \tau . \end{cases} \tag{1.7}$$

The remaining problem is to compute F(T).

The discussion below will distinguish between two cases: (i) $0 \leqslant T \leqslant 2a/c$ and (ii) $2a/c \leqslant T$. For case (i) we will use an approximate method, and for case (ii) an exact method.

2. Calculation of F(T) for $0 \leqslant T \leqslant 2a/c$

The physical significance of this time zone is that the contribution to the return pulse at the observation point, R, in the back-scattered direction is due purely to the reflected waves alone, without contributions from the geometric shadow region.

Hence we can represent the return pulse in this time zone by an approximation representing the reflected waves. This can be done by obtaining, and solving, a set of "transport equations" for electromagnetic waves, as is done in Friedlander(2) for sound waves; or using the Kline-Luneberg expansion, and the Tauberian theorem which states that, if

$$f(t) = \frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} e^{ts} g(s)\, ds$$

and we replace g(s) by the asymptotic expression for large s, then the resulting transform is an asymptotic expression for f(t) valid for small t.

Now, if we use the Kline-Luneberg expansion for the back scattered field for a harmonic time-dependent field, we have (see Appendix A)

$$E^s(k) \simeq -\frac{a}{(2R-a)} e^{ik(R-2a)} \left[1 + \frac{a_1}{ka} + \frac{a_2}{(ka)^2} + \frac{a_3}{(ka)^3} + \dots\right] \qquad (2.1)$$

with

$$a_1 = -\frac{i\, 2(R-a)^2}{(2R-a)^2}$$

$$a_2 = \frac{a\,(R-a)\,(2R^2 - 4Ra + 3a^2)}{(2R-a)^4}$$

$$a_3 = i\,\frac{139}{24 \cdot 168} + 0\left(\frac{1}{R}\right) .$$

We will assume that we can analytically continue this asymptotic expansion for E^s in the s plane when we replace ka by $\frac{is}{c}$. Hence we have for small T,

$$F(T) \underset{T\to 0}{\simeq} \frac{1}{2\pi i}\left(\frac{-a}{2R-a}\right)\int_{b-i\infty}^{b+i\infty} \frac{e^{Ts}}{(s+i\omega)}\left[1+a_1\left(\frac{c}{isa}\right)+a_2\left(\frac{c}{isa}\right)^2+\right] ds. \tag{2.2}$$

Hence we have

$$F(T) \sim -\left(\frac{a}{2R-a}\right)\left\{e^{-i\omega T}+\frac{a_1}{ka}\left[e^{-i\omega T}-1\right]+\frac{a_2}{(ka)^2}\left[e^{-i\omega T}+i\omega T-1\right]\right.$$

$$\left.+\frac{a_3}{(ka)^3}\left[e^{-i\omega T}-1+i\omega T-\frac{(i\omega T)^2}{\underline{|2}}\right]+\cdots\right. \tag{2.3}$$

3. Calculation of F(T) for T > 2a/c

For T > 2a/c, we may calculate F(T) exactly. To facilitate analysis we define the following

$$\left\{X_n^p\right\} \quad p = 1, 2, \ldots, n, \quad \text{the zeros of } h_n^{(1)}(x) = 0 \tag{3.1}$$

$$\left\{Y_n^q\right\} \quad q = 1, 2, \ldots, n+1, \text{ the zeros of } \left[Y\, h_n^{(1)}(Y)\right]' = 0 \tag{3.2}$$

which are discussed in Appendices B and C. Also set

$$A_n(X_n^p) = \frac{h_n^{(1)}(RX_n^p/a)\; j_n(X_n^p)}{\left[h_n^{(1)}\,(X_n^p)\right]'} \tag{3.3}$$

$$B_n(Y_n^q) = \frac{\left[(RY_n^q/a)\; h_n^{(1)}(RY_n^q/a)\right]'\,\left[Y_n^q\; j_n\;(Y_n^q)\right]'}{(RY_n^q/a)\;\left[Y_n^q\; h_n^{(1)}(Y_n^q)\right]''} \tag{3.4}$$

For $T > 2a/c$ the integrand of (1.6) vanishes exponentially as $|s| \rightarrow \infty$, for real part of $s < 0$. Hence, in (1.6) we may enclose the line integral by a contour to the left extending to infinity, and set the resulting closed contour integral equal to $2\pi i$ times the sum of the residues enclosed. We have then

$$F(T) = \text{Sum of Residues of} \left\{ \frac{e^{\left[T + (R-2a)/c\right]}\;\; E^s(\frac{is}{c})}{(s + i\omega)} \right\} . \tag{3.5}$$

The poles are at

$$s = -\, i\omega$$

$$s = -\,\frac{ic}{a}\; X_n^p, \quad n = 1,\, 2 \ldots: \quad p = 1,\, \ldots,\, n$$

$$s = -\,\frac{ic}{a}\; Y_n^q, \quad n = 1,\, 2 \ldots: \quad q = 1,\, \ldots,\, n + 1.$$

Hence we have

$$F(T) = e^{-i\omega\left[T + (R-2a)/c\right]} E^s(k) - G(T, ka)$$

where

$$G(T, ka) = \sum_{n=1} (-i)^n (n+1/2) \left\{ \sum_{p=1}^{n} \frac{e^{-(T + (R-2a)/c)\frac{icX_n^p}{a}}}{(X_n^p - ka)} A_n(X_n^p) \right.$$

$$\left. + i \sum_{q=1}^{n+1} \frac{e^{-\frac{ic}{a} Y_n^q (T + (R-2a)/c)}}{(Y_n^q - ka)} B_n(Y_n^q) \right\} \qquad (3.6)$$

4. Pulse Return

We will consider separately the two cases $\tau < 2a/c$ and $\tau > 2a/c$. In each case, our return pulse will be divided into four time zones. The time zones and appropriate expression for $\underline{E}^r$ are given as follows:

Case (a) $\tau < 2a/c$

	Time-Zone	$\underline{E}^r$	Appropriate Expressions for F(T)	F(T- τ)	
(i)	$0 \leq T < \tau$	$\hat{\underline{i}}_x F(T)$	(2.3)	-	
(ii)	$\tau \leq T \leq 2a/c$	$\hat{\underline{i}}_x \left\{F(T)-e^{-i\omega\tau} F(T-\tau)\right\}$	(2.3)	(2.3)	
(iii)	$2a/c < T \leq 2a/c+\tau$	$\hat{\underline{i}}_x \left\{F(T)-e^{-i\omega\tau} F(T-\tau)\right\}$	(3.6)	(2.3)	(4.1)
(iv)	$2a/c + \tau < T$	$\hat{\underline{i}}_x \left\{F(T)-e^{-i\omega\tau} F(T-\tau)\right\}$	(3.6)	(3.6)	

Case (b) $2a/c < \tau$

	Time Zone	$\underline{E}^r$	Appropriate Expressions for F(T)	F(T- τ)	
(i)	$0 \leq T < 2a/c$	$\hat{\underline{i}}_x F(T)$	(2.3)	-	
(ii)	$2a/c \leq T < \tau$	$\hat{\underline{i}}_x F(T)$	(3.6)	-	
(iii)	$\tau \leq T \leq 2a/c+\tau$	$\hat{\underline{i}}_x \left\{F(T)-e^{-i\omega\tau} F(T-\tau)\right\}$	(3.6)	(2.3)	(4.2)
(iv)	$2a/c + \tau < T$	$\hat{\underline{i}}_x \left\{F(T)-e^{-i\omega\tau} F(T-\tau)\right\}$	(3.6)	(3.6)	

We will define as "head" of the return pulse that portion corresponding to the time interval $0 \leqslant T < \tau$. The remainder will be defined as the "tail".

The question arises as to how the pulse return varies with respect to change in size of the sphere. Consider case (a). Neglecting the higher order terms for the present, we see that the head of the pulse is given by

$$\underline{E}^r = -\hat{\underline{i}}_x \left(\frac{a}{2R-a}\right) \left[e^{-i\omega T} + \frac{a_1}{ka} \left(e^{-i\omega T} - 1\right) + \cdot\cdot \right] \qquad (4.3)$$

and the initial part of the tail (i.e., for $\tau \leqslant T < 2a/c$) is given by

$$\underline{E}^r = -\hat{\underline{i}}_x \left(\frac{a}{2R-a}\right) \left[\frac{a_1}{ka} \left(e^{-i\omega\tau} - 1\right) + \cdot\cdot \right] \qquad (4.4)$$

Hence for $ka > 1$ we see that the initial part of the tail is of the order of a_1/ka of the head. The coefficient a_1 slowly increases as the observation point given by the coordinate R moves away from the sphere, and in the far field a_1 approaches $-i/2$. This indicates that the magnitude of the initial part of the tail section in comparison to the magnitude of the head is largest in the far field.

On the other hand, in the special case when the initial pulse length τ is such that $\omega\tau$ is a multiple of 2π, the first term in (4.4) vanishes and we have for $\tau \leq T < 2a/c$

$$\underline{E}^r = -\hat{\underline{i}}_x \left(\frac{a}{2R-a}\right) \frac{i\omega\tau}{(ka)^2} \left[a_2 + \frac{a_3}{(ka)} (1 - i\omega T + i\omega\tau/2) + \dots\right] . \quad (4.5)$$

In this case, whether we are in the far or very near field is extremely crucial when $ka > 1$. Since $a_2 \approx a/8R$ as R approaches ∞, a_2 is a near-field term, and does not contribute to the far field. Hence in the far field we have

$$\underline{E}^r \approx -\hat{\underline{i}}_x \frac{a}{2R} \frac{i\omega\tau}{(ka)^3} \left[(1 - i\omega T + i\omega\tau/2) a_3 + \dots\right] . \quad (4.6)$$

Since $\tau < 2a/c$, we have $\omega\tau < ka$. The initial part of the tail ($\tau \leq T < 2a/c$) is the order of $\omega\tau/(ka)^3$ of the magnitude of the head of the received pulse.

The second section of the tail, i.e. $2a/c < T < 2a/c + \tau$, is given by

$$\underline{E}^r = \hat{\underline{i}}_x \left\{ e^{-i\omega\left[T + (R-2a)/c\right]} E^s(k) - G(T, ka) \right.$$

$$+ \left(\frac{a}{2R-a}\right)\left[e^{-i\omega T} + \frac{a_1}{ka}\left(e^{-i\omega T} - e^{-i\omega\tau}\right) + \frac{a_2}{(ka)^2}\left(e^{-i\omega T} - e^{-i\omega\tau}(1 + i\omega\tau - i\omega T)\right)\right.$$

$$\left.\left. + \ldots \right]\right\}. \tag{4.7}$$

which can be written in the form for ka > 1

$$\underline{E}^r = \hat{\underline{i}}_x \left\{ e^{-i\omega T}\left[e^{-ik(R-2a)} E^s(k) + \left(\frac{a}{2R-a}\right)\left(1 + \frac{a_1}{ka} + \frac{a_2}{(ka)^2} + \ldots\right)\right]\right.$$

$$\left. -G(T, ka) - \left(\frac{a}{2R-a}\right) e^{-i\omega\tau}\left[\frac{a_1}{ka} + \frac{a_2}{(ka)^2}(1 + i\omega\tau - i\omega T) + \ldots\right]\right\}. \tag{4.8}$$

The first term in Equation (4.8) (apart from the phase factor) is

$$E^s(k) + \left(\frac{a}{2R-a}\right) e^{ik(R-2a)}\left[1 + \frac{a_1}{ka} + \frac{a_2}{(ka)^2} + \ldots\right] \tag{4.9}$$

This is the back scattered field for the c.w. case with the reflected field component subtracted off. Hence it represents purely the back-scattered diffracted field, i.e., the creeping wave terms.

In Appendix E, it is shown that G(T, ka) may be split up into two parts. One part equals the c.w. creeping wave term, and the other transients of the order of $(a/R)\,(ka)^{-1}$. It can be shown that the c.w. creeping wave terms of G(T, ka) cancel out those of Equation (4.9). Hence since the remaining terms in Equation (4.8) are of the order of $(a/R)\,(ka)^{-1}$, we see that the section of the tail for the received pulse for $2a/c < T < 2a/c + \tau$ is of the order of $(ka)^{-1}$ of the magnitude of the "head" of the pulse.

The remainder of the tail i.e., $T > 2a/c + \tau$ is given by

$$\underline{E}^r = - \hat{\underline{i}}_x \left[G(T,\, ka) - e^{-i\omega\tau} G(T-\tau,\, ka) \right] \qquad (4.10)$$

This is composed of the transients terms. However it can be shown using the decomposition of G(T, ka) in Appendix E that for the far field, we obtain c.w. creeping waves plus additional terms for the period $2a + \pi a < cT < 2a + \pi a + c\,\tau$. A similar analysis holds for the near field.

Now consider case (b). For simplification assume that the pulse length τ is much greater than 2a/c. The head of the pulse is given by

(i) $0 \leqslant T \leqslant 2a/c$

$$\underline{E}^r(T) = -\hat{\underline{i}}_x \left(\frac{a}{2R-a}\right)\left[e^{-i\omega T} + \frac{a_1}{ka}\left(e^{-i\omega T}-1\right) + \dots\right] \qquad (4.11)$$

(ii) $2a/c < T \leqslant \tau$

$$\underline{E}^r(T) = \hat{\underline{i}}_x \left[e^{-i\omega\left[T+(R-2a)/c\right]} E^s(k) - G(T, ka)\right]. \qquad (4.12)$$

We see that initially ($T \sim 0$) the head of the pulse return is of the order of (a/R), this contribution being due to the specular point. The pulse return builds up with contribution from the geometrically reflected waves. Then we obtain direct contribution from the shadow region and finally the contribution due to the c.w. creeping waves. When T approaches τ, the pulse return approaches the c.w. return (for τ, sufficiently large, such that the transients are negligible). Hence if ka is sufficiently greater than 1, the contribution from the specular point is dominant in the head of the pulse return.

The tail of the pulse is given by

(i) $\tau < T \leqslant \tau + 2a/c$

$$\underline{E}^r(T) = \hat{\underline{i}}_x \left\{ e^{-i\omega T} \left[e^{-ik(R-2a)} E^s(k) + \frac{a}{2R-a} \left(1 + \frac{a_1}{ka} + \frac{a_2}{(ka)^2} + \dots\right)\right] - G(T,ka) - \left(\frac{a}{2R-a}\right) e^{-i\omega\tau} \left[\frac{a_1}{ka} + \frac{a_2}{(ka)^2}(1 + i\omega\tau - i\omega T) + \dots\right]\right\} \tag{4.13}$$

(ii) $\tau + 2a/c < T$

$$\underline{E}^r(T) = -\hat{\underline{i}}_x \left[G(T, ka) - e^{-i\omega\tau} G(T-\tau, ka)\right] . \tag{4.14}$$

It can be shown that the initial part of the tail of the pulse given by Equation (4.13), using the decomposition of G(T, ka) given in Appendix E, behaves like the c.w. creeping wave terms (terms the order of $(a/R)\,(ka)^{1/3} \exp\left[-\,(ka)^{1/3}\ \text{constant}\right]$ in the far field), plus terms of the order of $(ka)^{-1}\,(a/R)$. Hence for large ka this tail section is of the order of (1/ka) of the head of the pulse.

The remaining section of the tail Equation (4.14) will give us c.w. creeping waves for a short period of time plus decaying transients which initially are the order of (1/ka) of the head of the pulse.

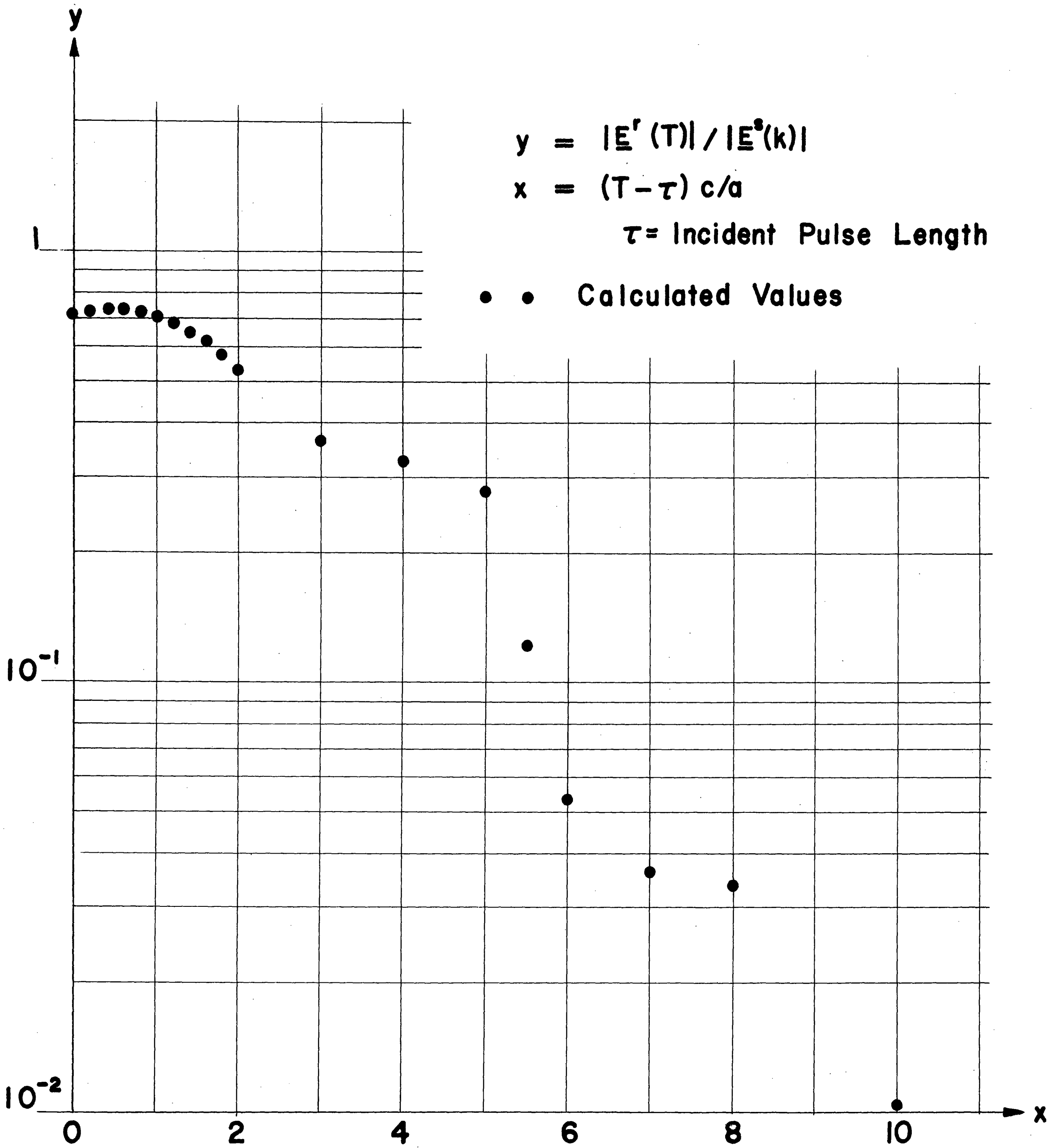

Fig. 1. Tail of the Pulse Return $\underline{E}^r(T)$

The tail of the pulse return is calculated for case (b). The frequency ω is taken such that $ka = 1$, and the distance of the observer R is given by $R/a = 200$. The initial pulse length τ is assumed to be sufficiently large such that the head of the return pulse has approached the c. w. return when $T = \tau - 0$. i.e.,

$$\left|\underline{E}^r(T)\right| = \left|\underline{E}^s(k)\right| \quad \text{for } T = \tau - 0.$$

For this case $ka = 1$, τ should be greater or equal to about 10 a/c.

In Figure 1, the absolute value of the electric field for the tail of the pulse return, normalized through division by the absolute value of the c. w. return, i.e.,

$$y = \left|\underline{E}^r(T)\right| \;/\; \left|\underline{E}^s(k)\right| ,$$

is plotted versus $x = (T - \tau)\,\frac{c}{a}$, where T has been defined as the time measured from the initial part of the pulse return.

The tail may be split up into several regions. For $0 < x < 2$, the tail is composed of contributions from the reflected waves other than that reflected from the specular point, plus creeping wave terms. For the region $2 < x < 2 + \pi$, the contribution is due to all the creeping wave terms plus some transients.

In the neighborhood of the point $x = 2 + \pi$ i.e.,

$$T = \tau + (2a + \pi)/c,$$

there is a drop in the electric intensity. The physical reason for this, is that the observer is no longer receiving the creeping wave contribution which goes half-way around the sphere. This certainly substantiates the physical significance of creeping wave theory.

The distance R of the observer from the center of the sphere will affect the tail. The main effect will be in the creeping wave contribution which will tend to decrease as the observer moves toward the sphere. (see Appendix F), and this becomes significant for distance R such that $ka^2/2R \gtrsim 1$. However, for very large ka, the creeping wave terms become negligible, so condition $ka^2/2R \gg 1$ can be removed for practical purposes. For ka in the resonance region this condition numerically is the same as, saying $R > a$.

The other terms do not significantly change as R decreases provided that $R > a$.

Conclusion

For a fixed frequency ω, the size of the sphere will affect the magnitude of the return pulse. At resonant frequencies there is a significant tail to the return pulse. At high frequencies, the tail is negligible.

For $ka > 1$, the initial part of the tail(i.e., in the time interval $\tau < T < \tau + 2a/c$) will be of the order of $1/ka$ of the head of the return pulse. The remainder of the tail i.e., $T > \tau + 2a/c$, decays quite rapidly.

Appendix A

Kline-Luneberg Expansion for the Back-Scattered Field

When a plane harmonic wave of frequency $\omega = ck$ is incident on a perfectly conducting body, the portion of the scattered field due to reflection only can be represented in the form

$$\underline{E}^r = e^{iks} \left[\underline{E}_o + \underline{E}_1 \lambda + \underline{E}_2 \lambda^2 + \dots \right] \tag{A.1}$$

If the incident radiation is being propagated in the direction of the z-axis and is polarized in the direction of the x-axis, and the scattering body is a body of revolution with the z-axis as symmetry axis, then by Schensted (Ref. 3) we have for the above field components

$$\underline{E}_n = \sqrt{\frac{\rho}{h_1 h_2}} \left[\underline{E}_n^o + \frac{i}{4\pi} \int_f^s \sqrt{\frac{h_1 h_2}{\rho}} \, \nabla^2 \, \underline{E}_{n-1} \, ds \right] \tag{A.2}$$

$$\underline{E}_n^o = \frac{i}{2\pi} \left(\underline{\nabla} \cdot \underline{E}_{n-1} \right) \left(-f' \, \hat{\underline{\rho}} + \underline{s} \right) \quad n \neq 0 \tag{A.3}$$

where

$$z = f(\rho)$$

defines the surface of the body and $\rho = \sqrt{x^2 + y^2}$.

The curvilinear coordinate system (ρ, s, ϕ) is described as follows: Through each point in space (exterior to the body) there passes a ray which has been reflected from the body:

ρ is the distance from the z-axis at which the ray hits the body

s is the sum of f(ρ) and the distance along the ray from the body to the point

ϕ is the angle between the xz-plane and the plane formed by the ray and the z-axis.

The metric coefficients are defined as follows

$$d\ell^2 = h_1^2 \, d\rho^2 + ds^2 + h_2^2 \, d\phi^2$$

and

$$h_1 = 1 + \frac{2f''(s - f)}{(f')^2 + 1} \tag{A.4}$$

$$h_2 = \rho + \frac{2f' \, (s - f)}{(f')^2 + 1} \, .$$

In addition, Schensted (Ref. 3) showed that

$$\underline{E}_o = \sqrt{\frac{\rho}{h_1 \, h_2}} \left[- \cos\phi \, \hat{\underline{\rho}} + \sin\phi \, \hat{\underline{\phi}} \right] \tag{A.5}$$

The problem at present is to evaluate the coefficients $\underline{E}_n$ in the back-scattered direction, for the case when the body of revolution is the sphere.

For a sphere of radius a, the equation of the illuminated face is

$$f(\rho) = a - \sqrt{a^2 - \rho^2} \qquad (A.6)$$

and the back-scattered direction represented in the coordinate system (ρ, s, ϕ) is given by $\rho = 0$. If we use the fact that the back-scattered field will be polarized in the same direction as the incident field, we have that

$$(\underline{E}_n \cdot \hat{\underline{i}}_y)_{(\rho = 0)} = 0$$

$$(\underline{E}_n \cdot \hat{\underline{i}}_z)_{(\rho = 0)} = 0. \qquad (A.7)$$

Thus we just have to evaluate

$$(\underline{E}_n \cdot \hat{\underline{i}}_x)_{(\rho = 0)} \qquad (A.8)$$

$\underline{E}_o \cdot \hat{\underline{i}}_x$ can be expanded in the form

$$\underline{E}_o \cdot \hat{\underline{i}}_x = A_o + \rho^2 \left[B_o \cos 2\phi + C_o \right] + \rho^4 \left[D_o \cos 2\phi + E_o \right] + \ldots \qquad (A.9)$$

and it can be shown that

$$\underline{E}_n \cdot \hat{\underline{i}}_x = A_n + \rho^2 \left[B_n \cos 2\phi + C_n \right] + \rho^4 \left[D_n \cos 2\phi + E_n \right] + \cdots \qquad (A.10)$$

Hence for back-scattering we will require

$$A_o,\ A_1\ \ldots,\ A_n,$$

but for present purposes we will just compute A_o, A_1, and A_2.

Now it can be shown that

$$\underline{E}_2^o \cdot \hat{\underline{i}}_x \sim 0\ (\rho^2) \qquad (A.11)$$

$$\underline{E}_1^o \cdot \hat{\underline{i}}_x \sim 0\ (\rho^4) \qquad (A.12)$$

Hence we have

$$\underline{E}_2 \cdot \hat{\underline{i}}_x = \frac{i}{4\pi} \sqrt{\frac{\rho}{h_1 h_2}} \int_o^s \sqrt{\frac{h_1 h_2}{\rho}}\, \nabla^2 \left(\underline{E}_1 \cdot \hat{\underline{i}}_x \right) ds + 0\ (\rho^2) \qquad (A.13)$$

$$\underline{E}_1 \cdot \hat{\underline{i}}_x = \frac{i}{4\pi} \sqrt{\frac{\rho}{h_1 h_2}} \int_f^s \sqrt{\frac{h_1 h_2}{\rho}}\, \nabla^2 \left(\underline{E}_o \cdot \hat{\underline{i}}_x \right) ds + 0\ (\rho^4). \qquad (A.14)$$

Now in order to compute $\left(\underline{E}_2 \cdot \hat{\underline{i}}_x\right)_{\rho=0}$ we require $\left[\nabla^2 \left(\underline{E}_1 \cdot \hat{\underline{i}}_x\right)\right]_{\rho=0}$.

It can be shown that if

$$u = \left[A_n + \rho^2 C_n + \rho^4 E_n + \dots\right] + \cos 2\phi \left[B_n \rho^2 + D_n \rho^4 + \dots\right] \quad \text{(A.15)}$$

where A_n, C_n, B_n, ... are functions of s only, then

$$\nabla^2 u = \left[A_n^* + \rho^2 C_n^* + \rho^4 E_n^* + \dots\right] + \cos 2\phi \left[B_n^* \rho^2 + D_n^* \rho^4 + \dots\right] \quad \text{(A.16)}$$

where

$$A_n^* = \frac{\partial^2 A_n}{\partial s^2} + \left(\frac{4}{2s + a}\right) \frac{\partial A_n}{\partial s} + 4 C_n \left(\frac{a}{2s + a}\right)^2, \quad \text{(A.17)}$$

and C_n^* depends upon A_n, C_n, E_n only. All the starred coefficients are functions of s only.

Thus in order to compute $\left[\nabla^2 (\underline{E}_1 \cdot \hat{\underline{i}}_x)\right]_{\rho = 0}$ we just require the coefficients A_1 and C_1.

Similarly it can be shown that in order to compute A_1 and C_1, we require A_o, C_o, and E_o.

These in turn are found from the expression

$$\underline{E}_o \cdot \hat{\underline{i}}_x = \sqrt{\frac{\rho}{h_1 h_2}} \left[- \cos \phi \hat{\underline{\rho}} + \sin\phi \hat{\underline{\phi}}\right] \cdot \hat{\underline{i}}_x$$

$$= - \frac{1}{(f')^2 + 1} \sqrt{\frac{\rho}{h_1 h_2}} + \cos 2\phi \frac{(f')^2}{(f')^2 + 1} \sqrt{\frac{\rho}{h_1 h_2}} .$$

$$= \left[A_o + C_o \rho^2 + E_o \rho^4 + \dots\right] + \cos 2\phi \left[B_o \rho^2 + \dots\right] \quad \text{(A.18)}$$

Computing the coefficients, we have

$$\left[\underline{E}_o \cdot \hat{\underline{i}}_x\right]_{\rho=0} = -\frac{a}{(2s+a)} \tag{A.19}$$

$$\left[\underline{E}_1 \cdot \hat{\underline{i}}_x\right]_{\rho=0} = \frac{i}{4\pi}\left[\frac{1}{(2s+a)} - \frac{2a}{(2s+a)^2} + \frac{a^2}{(2s+a)^3}\right] \tag{A.20}$$

$$\left[\underline{E}_2 \cdot \hat{\underline{i}}_x\right]_{\rho=0} = \frac{-1}{16\pi^2}\left[\frac{1}{(2s+a)^2} - \frac{3a}{(2s+a)^3} + \frac{5a^2}{(2s+a)^4} - \frac{3\,a^3}{(2s+a)^5}\right] \tag{A.21}$$

Now for back-scattering $s = R - a$, where R is the distance of the observation point from the center of the sphere. Hence we have

$$\underline{E}^r = -\hat{\underline{i}}_x \frac{a}{(2R-a)} e^{ik(R-2a)}\left[1 + \frac{a_1}{ka} + \frac{a_2}{(ka)^2} + \frac{a_3}{(ka)^3} + .\right] \tag{A.22}$$

and

$$a_1 = -\frac{i\,2(R-a)^2}{(2R-a)^2} \tag{A.23}$$

$$a_2 = \frac{a(R-a)\,(2R^2 - 4Ra + 3a^2)}{(2R-a)^4} . \tag{A.24}$$

The far field limit of the above coefficients i.e.

$$a_1 \sim -\frac{i}{2} + 0\left(\frac{1}{R}\right)$$

$$a_2 \sim 0\left(\frac{1}{R}\right)$$

was checked using an alternative method of computation; this method is based upon a modified Watson transform of the Mie series [Appendix F]. In addition the far field limit of the term a_3 was calculated, yielding

$$a_3 = i\,\frac{139}{24 \cdot 168} + 0\left(\frac{1}{R}\right). \qquad (A.25)$$

Table I

n	Zeros of $h_n^{(1)}(x)$			
1	-1.0i			
2	± .8660254 - 1.5i			
3	±1.754381 - 1.838907i	-2.322185i		
4	±2.65742 - 2.10379i	± .867181 - 2.8962i		
5	±3.571022 - 2.324674i	±1.74266 - 3.35196i	-3.646738i	
6	±4.492673 - 2.51593i	±2.626274 - 3.735705i	± .86750965-4.24836i	
7	±5.420692 - 2.625677i	±3.5171 - 4.0703i	±1.739 -4.758i	-4.971786i

Table II

n	Zeros of $\left[Y\, h_n^{(1)}(y)\right]'$			
1	± .8660254 - .5i			
2	±1.807339 - .7019642i	-1.596072i		
3	± 2.757856 - .8428622i	± .8705692 - 2.157138i		
4	±3.714784 - .9542299i	±1.752303 - 2.5714i	-2.948742i	
5	±4.676410 - 1.047674i	±2.644316 - 2.908062i	± .8689259 -3.554265i	
6	±5.641635 - 1.128905i	±3.54488 - 3.19524i	±1.74305 -4.03354i	-4.284595i
7	±6.609716 - 1.201203i	±4.45256 - 3.4476i	±2.6233 -4.454i	± .86840 -4.89719i

Appendix B

Zeros of $h_n^{(1)}(x)$ (n integer)

From Erdelyi (Ref. 4), Watson (Ref. 5) the number and position of the zeros of $h_n^{(1)}(x)$ are given as follows:

(i) $h_n^{(1)}(x)$ has n zeros;

(ii) these zeros lie in the lower half of the x plane and lie symmetric with respect to the imaginary axis.

The zeros of $h_n^{(1)}(x)$ for n small have been computed, and are given in Table I.

The large zeros of $h_n^{(1)}(x)$ can be estimated as follows:

We will consider just those zeros which lie on the imaginary axis and those in the right half-plane, i.e. for real part $x \geqslant 0$. The remaining zeros are obtained by symmetry.

From Watson (Ref. 5, pgs. 262-267) it can be deduced to the first approximation (in a similar manner as is shown in Ref. 6, pgs. 218-219) that the zeros satisfy the following equations:

$$\frac{n + 1/2}{x} = \cosh \gamma \tag{B.1}$$

$$i(n + 1/2)\left[\text{Tanh}\, \gamma - \gamma\right] + \frac{\pi}{4} = \ell \pi \tag{B.2}$$

where ℓ is an integer lying in the range

$$0 \leq \ell - \frac{1}{4} \leq \frac{n + 1/2}{2}$$

and

$$-\frac{\pi}{2} < \arg(-i \sinh \gamma) < \frac{\pi}{2} .$$

As a special case consider n an odd integer; we can then take $\ell = \frac{n+1}{2}$. Set $\gamma = \alpha + \frac{i\pi}{2}$. Equation (B.2) becomes

$$\left[\coth \alpha - \alpha\right] - \frac{i\pi}{2} = - i \left(\frac{\ell - 1/4}{n + 1/2}\right)\pi = - \frac{i\pi}{2}$$

i.e.

$$\alpha = \coth \alpha .$$

Thus we have $X = - i \frac{n + 1/2}{\sinh \alpha}$. This is the root of $h_n^{(1)}(x)$ which is purely imaginary. However, we are mainly interested in the roots which have the smallest imaginary part. These are given by $\ell = 1, 2, 3,$ etc. For the case ℓ small (B.2) becomes approximately

$$\operatorname{Tanh} \gamma \simeq \gamma$$

i.e.

$$\gamma \sim 0.$$

Thus we have $X \sim (n + 1/2)$ and the roots with the smallest imaginary part lie in the vicinity of $\left| X - (n + 1/2) \right| \sim 0\, (n + 1/2)^{1/3}$. In this region we can obtain better approximations to the zeros by using the Airy integral approximations. [Franz (Ref. 7)].

Let $\bar{q}_\ell$ be the zeros of the Airy integral A(q) where

$$A(q) = \int_0^\infty \cos\,(t^3 - qt)\, dt \,. \qquad \text{(B.3)}$$

Then, if X_n^ℓ represent the roots of $H^{(1)}_{n+1/2}(X) = \sqrt{\frac{2X}{\pi}}\; h_n^{(1)}(X)$, we have the relationship

$$(n + 1/2) = X_n^\ell + \left(\frac{X_n^\ell}{6}\right)^{1/3} e^{i\pi/3}\, \bar{q}_\ell - \left(\frac{6}{X_n^\ell}\right)^{1/3} e^{-i\pi/3}\, \frac{\bar{q}_\ell^2}{180} + \cdot \qquad \text{(B.4)}$$

where $\bar{q}_\ell$ have the values

ℓ	$\bar{q}_\ell$
1	3.372134
2	5.895843
3	7.962025
4	9.788127
5	11.457423

and for large ℓ, $\bar{q}_\ell \sim 3 \left[\frac{\pi}{2} \left(\ell - \frac{1}{4} \right) \right]^{2/3}$.

Thus for given $(n + 1/2)$, the roots in the right half plane with the smallest imaginary part are given by

$$x_n^{\ell} \sim (n + 1/2) - \left(\frac{n + 1/2}{6}\right)^{1/3} e^{i\pi/3}\, \bar{q}_{\ell} - \frac{1}{20}\left(\frac{n + 1/2}{6}\right)^{-1/3} e^{-i\pi/3}\, \bar{q}_{\ell}^{2} \tag{B.5}$$

Appendix C

Zeros of $\left[Y\, h_n^{(1)}(Y)\right]'$

The zeros of $\left[Y\, h_n^{(1)}(Y)\right]'$ lie in the lower half plane and are symmetrical with respect to the imaginary axis. $\left[Y\, h_n^{(1)}(Y)\right]'$ possesses $(n+1)$ zeros. For n small the zeros are computed and tabulated in Table II.

The large zeros may be handled in the same manner as those of $h_n^{(1)}(X)$. We will just consider here the zeros with the smallest imaginary part. These again are found from the Airy integral approximation [Franz (Ref. 7)]. Considering only the zeros in the right half plane we have

$$(n+1/2) = Y_n^{\ell} + \left(\frac{Y_n^{\ell}}{6}\right)^{1/3} e^{i\pi/3}\, q_{\ell} + \left(\frac{6}{Y_n^{\ell}}\right)^{1/3} e^{-i\pi/3} \left(\frac{3}{20 q_{\ell}} - \frac{q_{\ell}^2}{180}\right)$$

where the q_{ℓ} are the zeros of the derivative of the Airy integral $A(q)'$ and have the values

ℓ	q_{ℓ}
1	1.469354
2	4.684712
3	6.951786
4	8.889027
5	10.632519

For large ℓ, $q_\ell \sim 3\left[\frac{\pi}{2}(\ell - \frac{3}{4})\right]^{2/3}$. Thus, for a particular n, the zeros in the right half-plane with the smallest imaginary part are

$$Y_n^\ell = (n+1/2) - \left(\frac{n+1/2}{6}\right)^{1/3} e^{i\pi/3} q_\ell - \left(\frac{6}{n+1/2}\right)^{1/3} e^{-i\pi/3}\left[\frac{3}{20q_\ell} + \frac{q_\ell^2}{20}\right]. \tag{C.1}$$

Appendix D

Estimation of $A_n(X_n^p)$, $B_n(Y_n^q)$ for large n.

Now from (3.3) we have

$$A_n(X_n^p) = \frac{h_n^{(1)}(R/a\, X_n^p)\, h_n^{(2)}(X_n^p)}{2\left[h_n^{(1)}(X_n^p)\right]'} \tag{D.1}$$

and using the Wronskian relations this becomes

$$A_n(X_n^p) = \frac{i \quad h_n^{(1)}(R/a\, X_n^p)}{(X_n^p)^2 \left\{\left[h_n^{(1)}(X_n^p)\right]'\right\}^2} \tag{D.2}$$

We will consider the zeros X_n^p to lie in the right half-plane. We have from Franz (Ref. 7)

$$\frac{h_n^{(2)}(X_n^p)}{\left[h_n^{(1)}(X_n^p)\right]'} \sim - i\pi \left(\frac{X_n^p}{6}\right)^{1/3} \frac{1}{6} \frac{e^{i\pi/3}}{\left[A'(\bar{q}_p)\right]^2} . \tag{D.3}$$

In order to evaluate

$$B_n(Y_n^q) = \frac{\left[(R/a\, Y_n^q)\, h_n^{(1)}(R/a\, Y_n^q)\right]' \left[Y_n^q\, h_n^{(2)}\, (Y_n^q)\right]'}{(\frac{R}{a}\, Y_n^q)\, 2\left[Y_n^q\, h_n^{(1)}(Y_n^q)\right]''} \tag{D.4}$$

we make use of the second order differential equation for the spherical Hankel functions and the Wronskian relation to obtain

$$\frac{\left[Y_n^q\, h_n^{(2)}(Y_n^q)\right]'}{\left[Y_n^q\, h_n^{(1)}(Y_n^q)\right]''} = \frac{-2i}{\left[n(n+1) - (Y_n^q)^2\right]\left[h_n^{(1)}(Y_n^q)\right]^2} \tag{D.5}$$

which is approximately [Franz (Ref. 7)] for zeros in the right half plane

$$\approx i(\pi/2)\left(\frac{Y_n^q}{6}\right)^{2/3} e^{-i\pi/3}\, \frac{(Y_n^q)}{\left[A(q_q)\right]^2 \left[n(n+1) - (Y_n^q)^2\right]} \tag{D.6}$$

Appendix E

Asymptotic Evaluation of G(T,ka) for Large ka

The problem is to determine the behavior of the following series for large ka

$$G(T,ka) = \sum_{n=1}^{\infty} (-i)^n (n+\tfrac{1}{2}) \left\{ \sum_{p=1}^{n} \frac{A_n(X_n^p)}{(X_n^p - ka)} e^{-(T+[R-2a]/c)\, icX_n^p/a} \right. \tag{E.1}$$

$$\left. + i \sum_{q=1}^{n+1} \frac{B_n(Y_n^q)}{(Y_n^q - ka)} e^{-(T+[R-2a]/c)\, icY_n^q/a} \right\}.$$

Although the terms contain factors of the form $(X_n^p - ka)^{-1}$, and $(Y_n^q - ka)^{-1}$, which certainly decrease as ka increases for most X_n^p and Y_n^q, there exists a large number of X_n^p and Y_n^q such that $(X_n^p - ka)$ and $(Y_n^q - ka)$ are the order of $(ka)^{1/3}$. The zeros such that this is true, are those zeros in the right half plane with the smallest imaginary parts, and are given by $n \sim ka$. It can be shown that the terms in Equation (E.1) above corresponding to these zeros, apart from the decaying exponential factors, individually would be the order of a/R. However, because they contain a decaying exponential of the form $\exp\left\{-(T-2a/c)\,(ka)^{1/3}\ \text{constant}\right\}$ these terms become negligible in a very short period of time.

Thus the problem reduces to an evaluation of G(T,ka) for large ka for (T - 2a/c) small.

For simplification of analysis, we will consider the far field. We will take the range of variable T to be

$$2a/c \leq T < 2a/c + \pi a/c.$$

We have for the far field

$$A_n(X_n^p) = \frac{ia(-i)^{n+1}}{R(X_n^p)^3} \frac{e^{iRX_n^p/a}}{\left\{\left[h_n^{(1)}(X_n^p)\right]'\right\}^2} \tag{E.2}$$

$$B_n(Y_n^q) = \frac{a(-i)^{n+1} e^{iRY_n^q/a}}{RY_n^q\left[n(n+1) - (Y_n^q)^2\right]\left[h_n^{(1)}(Y_n^q)\right]^2} . \tag{E.3}$$

At present we will just consider those terms of expression (E.1) corresponding to the zeros of $h_n^{(1)}(X) = 0$. i.e.: the expression

$$\frac{2a}{\pi R} \sum_{n=1}^{\infty} (-1)^n (n+\tfrac{1}{2}) \sum_{p=1}^{n} \frac{e^{-(T-2a/c)icX_n^p/a}}{(X_n^p)^2\left[X_n^p - ka\right]\left\{\left[H_{n+\frac{1}{2}}^{(1)}(X_n^p)\right]'\right\}^2} \tag{E.4}$$

We have defined $\left\{ X_n^p \right\}$ to be the set of n zeros of $h_n^{(1)}(X) = 0$, but we have not identified the number p with any particular zero as yet. The zeros $\left\{ X_n^p \right\}$ lie on the curve $\mathcal{C}$ in Figure 1.

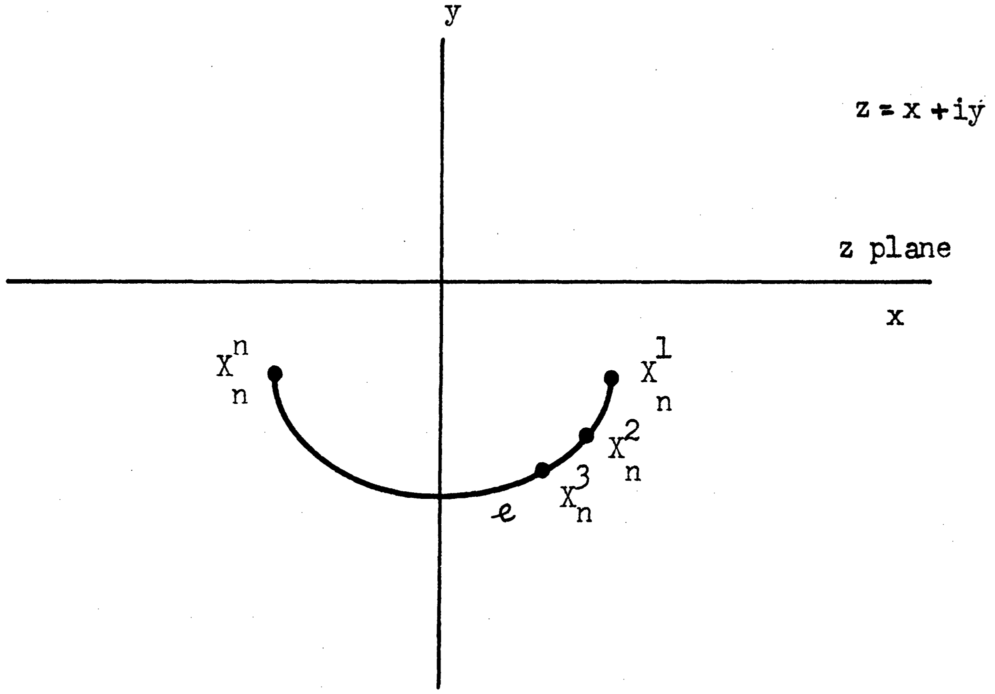

FIGURE E-1

We shall number the zeros in succession p = 1, 2, ..., such that

$$(2\pi - \arg X_n^1) < (2\pi - \arg X_n^2) < (2\pi - \arg X_n^3) < \dots$$

where $\quad 0 < \arg X_n^p < 2\pi$.

Now because of symmetry

$$X_n^n = -\bar{X}_n^{1}$$

$$X_n^{(n-p)} = -\bar{X}_n^{p} \qquad \text{(E.5)}$$

where the bar denotes the complex conjugate. If n is an odd integer, there is a zero $X_n^{(n+1)/2}$ which lies on the negative imaginary axis. Define $\alpha_n^p(ka)$ by the following equation

$$\alpha_n^p(ka) = \frac{e^{-(T-2a/c)icX_n^p/a}}{(X_n^p)^2(X_n^p - ka)\left\{\left[H^{(1)}_{n+1/2}(X_n^p)\right]'\right\}^2} . \qquad \text{(E.6)}$$

Then it follows from (E.5) that

$$\alpha_n^{n-p}(ka) = \frac{-e^{(T-2a/c)ic\bar{X}_n^p/a}}{(\bar{X}_n^p)^2(\bar{X}_n^p + ka)\left\{\left[H^{(1)}_{n+1/2}(-\bar{X}_n^p)\right]'\right\}^2}$$

$$= \frac{e^{(T-2a/c)ic\,\bar{X}_n^p/a}}{(\bar{X}_n^p)^2\,\overline{(X_n^p + ka)}\left\{\overline{\left[H^{(1)}_{n+1/2}(X_n^p)\right]'}\right\}^2}$$

$$= \overline{\alpha_n^p(-ka)} \qquad \text{(E.7)}$$

Hence expression (E.4) becomes

$$\frac{2a}{\pi R} \sum_{n=1,2,\dots}^{\infty} (-1)^n (n+\tfrac{1}{2}) \sum_{p=1}^{\leq n/2} \left[\alpha_n^p (ka) + \overline{\alpha_n^p (-ka)}\right]$$

$$+ \frac{2a}{\pi R} \sum_{n=1,3,\dots} (-1)^n (n+\tfrac{1}{2}) \alpha_n^{(n+1)/2} (ka) \; . \tag{E.8}$$

We shall split the expression (E.8) up into several terms

$$\frac{2a}{\pi R} \sum_{n=1}^{N_o} (-1)^n (n+1/2) \sum_{p=1}^{\leq n/2} \left[\alpha_n^p (ka) + \overline{\alpha_n^p (-ka)}\right]$$

$$+ \frac{2a}{\pi R} \sum_{n=1\cdot 3\cdot 5}^{\infty} (-1)^n (n+1/2) \alpha_n^{(n+1)/2} (ka)$$

$$+ \frac{2a}{\pi R} \sum_{p=1} \sum_{n=N(p)}^{\infty} (-1)^n (n+\tfrac{1}{2}) \left[\alpha_n^p (ka) + \overline{\alpha_n^p (-ka)}\right]$$

$$+ \frac{2a}{\pi R} \sum_{n=N_o+1}^{\infty} (-1)^n (n+\tfrac{1}{2}) \sum_{p=P(n)}^{\leq n/2} \left[\alpha_n^p (ka) + \overline{\alpha_n^p (-ka)}\right] \tag{E.9}$$

where N_o is an integer, and N(p) and P(n) are defined below.

The first series in (E.9) corresponds to all but the pure imaginary zeros of $h_n^{(1)}(x)$ for $n = 1, 2, \ldots, N_o$. The second series corresponds to all the pure imaginary zeros of $h_n^{(1)}(x)$, for $n = 1, 3, 5, 7, \ldots$.

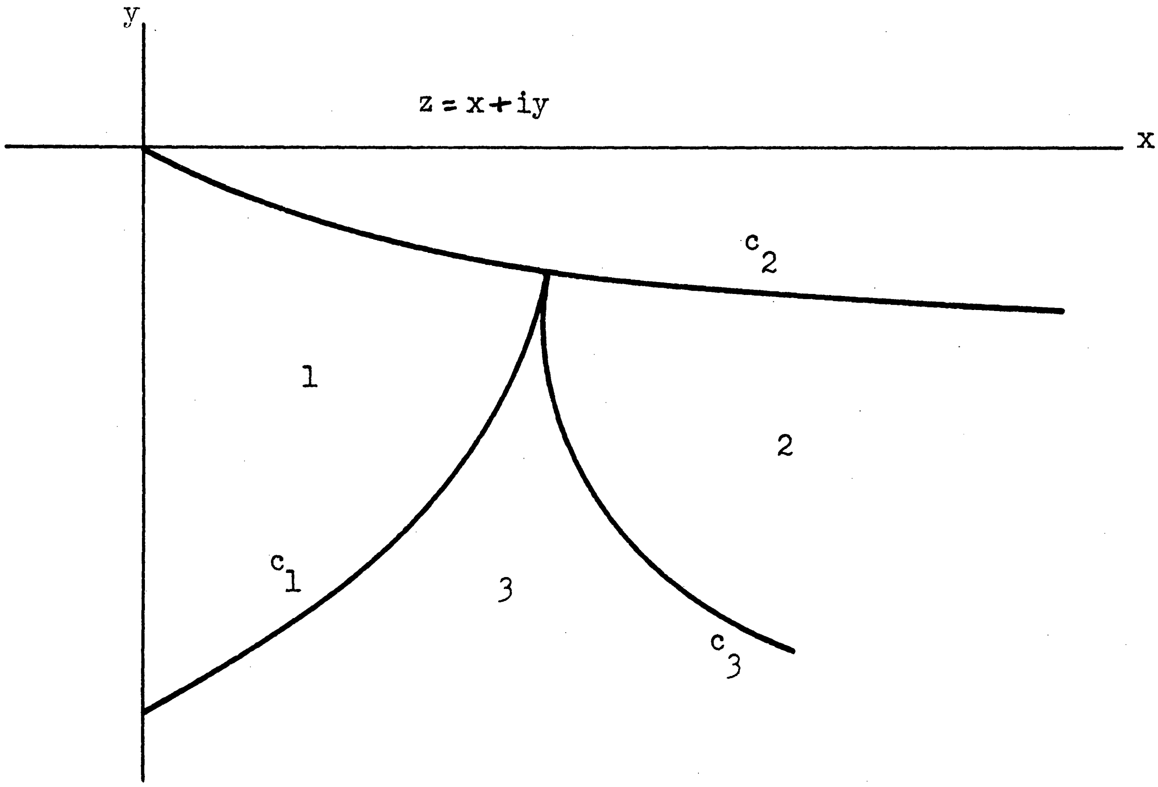

FIGURE E-2

To see what corresponds to the remaining two series we must consider Figure E-2. Region 1 represents the region containing the zeros of $h_n^{(1)}(x)$ for $n = 1, 2, \ldots, N_o$ but not those for $n > N_o$. The third and fourth series in expression (E.9) correspond to the zeros in Region 2 and 3 respectively.

For the particular p, let the zeros X_n^p which are contained in Region 2 be given by

$$n \geqslant N(p).$$

N(p) must have the following properties:

(i) N(p) is a large integer $> N_o$

(ii) $N(1) = N_0 + 1$

(iii) for a given p, N(p) must be such that $h^{(1)}_{N(p)}(x)$ has at least 2p or 2p+1 zeros, for N(p) an even or odd integer respectively.

P(n) is defined such that the fourth series of Equation (E.9) contains the terms corresponding to the remaining zeros, not included in the other three series of (E.9).

In expression (E.9) the problem is to evaluate those terms due to the zeros belonging to Region 2.

For these zeros, the series is slowly converging. Let us consider then

$$\frac{2a}{\pi R} \sum_{p=1} \sum_{n=N(p)}^{\infty} (-1)^n (n+\tfrac{1}{2}) \left[\alpha_n^p(ka) + \overline{\alpha_n^p(-ka)}\right] . \qquad \text{(E.10)}$$

Define the following

$$I_p(ka) = \sum_{n=N(p)} (-1)^n (n+\tfrac{1}{2}) \, \alpha_n^p(ka) . \qquad \text{(E.11)}$$

Then expression (10) becomes

$$\frac{2a}{\pi R} \sum_{p=1} \left\{ I_p (ka) + \overline{I_p(-ka)} \right\} . \qquad \text{(E.12)}$$

Now

$$I_p(ka) = \sum_{n=N(p)}^{\infty} \frac{(-1)^n (n+\frac{1}{2})\, e^{-\left[Tc/a-2\right] i X_n^p}}{(X_n^p)^2 (X_n^p - ka) \left\{ \left[H^{(1)}_{n+\frac{1}{2}} (X_n^p) \right]' \right\}^2} . \qquad \text{(E.13)}$$

For the present set $Tc/a-2 = \tilde{T}$. We are interested in $\tilde{T}$ in the range $0 \leq \tilde{T} < \pi$.

Define $\left\{ Z_p(v) \right\}$ as the zeros of $H_v^{(1)} \left[Z_p(v) \right] = 0$ where

$$\left| v - Z_p(v) \right| < \left| v - Z_{(p+1)}(v) \right| < \quad \text{etc.}$$

$Z_1(v)$, $Z_2(v)$, ... are related to the zeros of the Airy integral $\bar{q}_1$, $\bar{q}_2$, ... for v large. We see that for $v = n + \frac{1}{2}$, and $p \leq n$

$$Z_p(n+\tfrac{1}{2}) = X_n^p .$$

We shall assume that $Z_p(v)$ is an analytic function of v and is uniquely defined. This can be shown to be true for $\left|Z_p(v) - v\right| \sim O(v^{1/3})$, for which we may use the Airy integral representation which holds for $-\pi/2 < \arg v < 3\pi/2$ [Ref. 8]. However we must place a cut along the real axis of v about the origin. Since the number of zeros of $h_v^{(1)}(x)$ for real v, is "the even integer nearest to v, unless v is an integer, in which case the number is v" [Ref. 9], $Z_p(v)$ is not defined on the cut. N(p) has been chosen so that the necessary cut in the v plane for the function $Z_p(v)$ is to the left of N(p). Equation (E.13) may now be written in the contour integral form

$$I_p(ka) = \frac{i}{2} \int_c \frac{v\, e^{-i\tilde{T} Z_p(v)}\, dv}{\cos \pi v \left[Z_p(v)\right]^2 \left[Z_p(v) - ka\right] \left\{\left[H_v^{(1)}(Z_p(v))\right]'\right\}^2} \tag{E.14}$$

Let $F_p(v,ka)$ be the integrand of (E.14).

$$I_p(ka) = \int_c F_p(v, ka)\, dv .$$

Since the integrand vanishes exponentially as $|v| \to \infty$ for $0 \leq \tilde{T} < \pi a/c$, we may set

$$- \int_{c+c'} F_p(v,ka)\, dv = 2\pi i \left\{ \text{Sum of Residues} \right\}$$

where c' is a straight line passing through the real v axis at a point in the interval given by (N(p)-1/2, N(p) + 1/2). The integrand possesses a pole $v = \nu$, where $Z_p(\nu) - ka = 0$. If $v = \nu$ is to the right of c', then

$$\int_c F_p(v, ka) dv = - \int_{c'} F_p(v, ka)\, dv - 2\pi i \text{ (Residue at } v = \nu \text{)}$$

and

$$\int_c F_p(v, ka)\, dv = - \int_{c'} F_p(v, ka)\, dv$$

if $v = \nu$ is to left of c'. Hence we have for $ka > N(p)$

$$I_p(ka) = - \int_{c'} F_p(v, ka)\, dv + \frac{\pi\nu\, e^{-i\tilde{T}ka}}{(ka)^2 \left\{ \left[H_\nu^{(1)}(ka) \right]' \right\}^2 \cos \pi\nu \frac{d}{d\nu} Z_p(\nu)} \,. \tag{E.15}$$

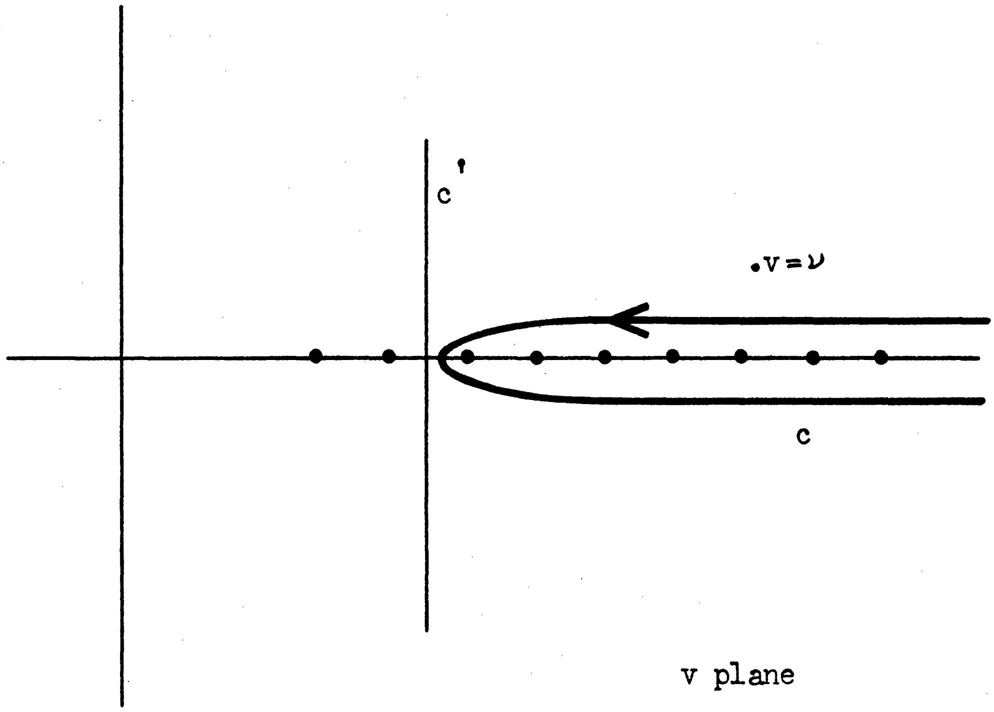

FIGURE E-3

rly it may be shown

$$\overline{I_p(-ka)} = - \left\{ \overline{\int_{c'} F_p(v,-ka)\, dv} \right\} . \qquad (E.16)$$

For p = 1,2,3,... we have

$$\nu = ka + (ka/6)^{1/3} e^{i\pi/3} \bar{q}_p + 0(6/ka)^{1/3} \qquad (E.17)$$

where $\left\{\bar{q}_p\right\}$ are the zeros of the Airy integral. What does the residue term in (E.15) represent? From (E.4), (E.9) and (E.15) the decomposition of G(T, ka) gives us terms of the form

$$+\frac{2a}{R}\;\frac{\nu\; e^{-i\tilde{T}ka}}{(ka)^2\left\{\left[H_\nu^{(1)}(ka)\right]'\right\}^2 \cos\pi\nu\, \frac{dZ_p(\nu)}{d\nu}}\;. \qquad \text{(E.18)}$$

Assuming that $(Z_p(v) - ka)$ possesses a simple zero at $v = \nu$ we have

$$\left[\frac{dH_v^{(1)}(ka)}{d\,ka}\right]_{v=\nu}\left[\frac{dZ_p(v)}{dv}\right]_{v=\nu} = -\left[\frac{dH_v^{(1)}(ka)}{dv}\right]_{v=\nu}.$$

Hence (E.18) becomes

$$-\frac{2a}{R}\;\frac{\nu\; e^{-i\tilde{T}ka}}{(ka)^2\left[H_\nu^{(1)}(ka)\right]' \frac{d}{d\nu}\left[H_\nu^{(1)}(ka)\right]\cos\pi\nu}\;. \qquad \text{(E.19)}$$

But this is the creeping wave term for the c.w. back-scattered field in the far zone (see Appendix F). Hence the residues are c.w. creeping waves.

It can be shown that the integrals over the path c' behave like 1/ka for large ka.

The remaining series in expression (E.9) are rapidly convergent and decrease as 1/ka for large ka.

A similar discussion holds for the terms in expression (E.1) corresponding to the zeros Y_n^q of $\left[Y\, h_n^{(1)}(Y)\right]' = 0$. Thus we see that for the far field, G(T, ka) can be decomposed into two parts for $2a < cT < 2a + \pi a$. One part equals the c.w. creeping wave term. The other represents rapidly decaying transients of order 1/ka.

Appendix F

c.w. Near-Zone Back-Scattering from Large Spheres

If a plane harmonic wave of the form

$$\underline{E}^i = \hat{\underline{i}}_x \; e^{ikz - i\omega t} \tag{F.1}$$

is incident on a large perfectly conducting sphere of radius a, the back scattered field is given by [Ref. 10, p. 564]

$$\underline{E}^s = - \hat{\underline{i}}_x \; e^{-i\omega t} \sum_{n=1}^{\infty} (-i)^n \; (n+1/2) \left[a_n + i \, b_n \right] \tag{F.2}$$

where

$$a_n = \frac{h_n^{(1)}(kR) j_n(ka)}{h_n^{(1)}(ka)} \tag{F.3}$$

$$b_n = \frac{\left[kR \, h_n^{(1)}(kR) \right]' \left[ka \, j_n(ka) \right]'}{kR \left[ka \, h_n^{(1)}(ka) \right]'} \, . \tag{F.4}$$

Henceforth we will drop the time factor $e^{-i\omega t}$. Using the series expansion

$$e^{-ikR} = 2 \sum_{n=0}^{\infty} (-i)^n \; (n+1/2) \; j_n(kR) \tag{F.5}$$

we obtain

$$E^s(k) = (I_1 + iI_2) + \frac{i}{kR} e^{ikR - 2ika} - \frac{i}{2kR} e^{-ikR} - e^{-ikR} \qquad (F.6)$$

where

$$I_1 = \sum_{n=0}^{\infty} (-i)^n (n+1/2) \left[j_n(kR) - a_n(k) \right] \qquad (F.7)$$

$$I_2 = \sum_{n=0}^{\infty} (-i)^n (n+1/2) \left\{ \frac{\left[kR\, j_n(kR)\right]'}{kR} - b_n(k) \right\} . \qquad (F.8)$$

Now we have

$$I_1 = \sqrt{\frac{\pi}{2kR}} \frac{e^{3/4\pi i}}{2i} \int_C \frac{e^{iv\pi} v}{\cos v\pi} \frac{\left[H_v^{(2)}(kR)\, H_v^{(1)}(ka) - H_v^{(1)}(kR)\, H_v^{(2)}(ka)\right]}{2\, e^{iv\pi/2}\, H_v^{(1)}(ka)} dv$$

$$I_2 = \sqrt{\frac{\pi}{2}} \frac{e^{3/4\pi i}}{2i\,(kR)} \int_C \frac{e^{iv\pi} v}{\cos v\pi} \times \qquad (F.9)$$

$$\times \left\{ \frac{\left[\sqrt{kR}\, H_v^{(2)}(kR)\right]' \left[\sqrt{ka}\, H_v^{(1)}(ka)\right]' - \left[\sqrt{kR}\, H_v^{(1)}(kR)\right]' \left[\sqrt{ka}\, H_v^{(2)}(ka)\right]'}{2\, e^{iv\pi/2} \left[\sqrt{ka}\, H_v^{(1)}(ka)\right]'} \right\} dv$$

where C is the contour surrounding the poles $v = 1/2, 3/2, 5/2, \dots$ as shown in Figure F-4.

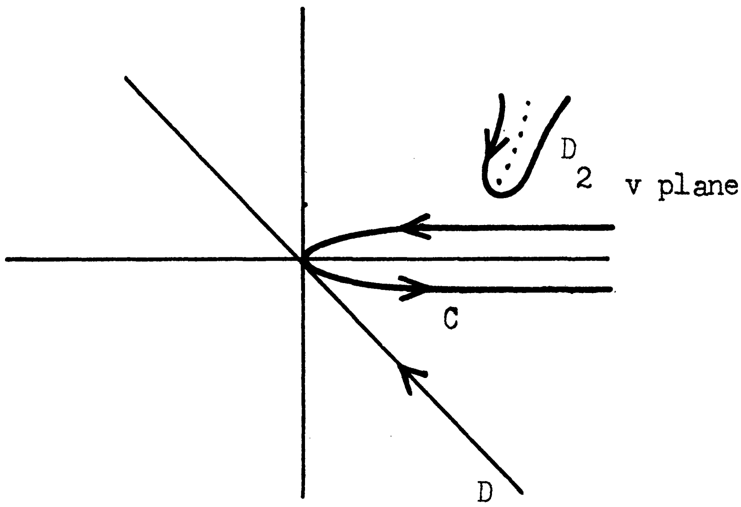

FIGURE F-4

It can be shown that

$$\int_C = -\int_{D_2} - \int_{D_1} \tag{F.10}$$

where D_2 is the contour surrounding the poles of the integrand (i.e., for I_1 the zeros of $H_v^{(1)}(ka)$), and D_1 is a line integral extending through the origin of the v plane. The line integral can be replaced by the integral containing the even part with respect to v of the integrand, and this in turn can be replaced by twice the integral extending from the origin to infinity.

Hence we have

$$I_1 = +\sqrt{\frac{\pi}{2kR}}\, e^{3/4\pi i} \int_0^{\infty e^{-i\beta}} v \frac{\sin v\pi}{\cos v\pi} \frac{\left[H_v^{(2)}(kR)\, H_v^{(1)}(ka) - H_v^{(1)}(kR) H_v^{(2)}(ka)\right]}{2\, e^{iv\pi/2}\, H_v^{(1)}(ka)}\, dv$$

$$+\sqrt{\frac{\pi}{2kR}}\, \frac{e^{3/4\pi i}\, 2\pi i}{2i} \sum_{\ell=1,2} \frac{\bar{v}_\ell\, e^{i\,3/2\,\bar{v}_\ell \pi}}{1+e^{i2\bar{v}_\ell \pi}} \frac{H_{\bar{v}_\ell}^{(1)}(kR)\, H_{\bar{v}_\ell}^{(2)}(ka)}{\frac{d}{d\bar{v}_\ell} H_{\bar{v}_\ell}^{(1)}(ka)} \qquad \text{(F.11)}$$

where $\bar{v}_\ell$ are the zeros of $H_v^{(1)}(ka) = 0$, and $\bar{v}_\ell = ka + \left(\frac{ka}{6}\right)^{1/3} e^{i\pi/3}\, \bar{q}_\ell + 0\left(\frac{1}{(ka)}1/3\right)$;

and $\bar{q}_\ell$ are the zeros of the Airy integral and $0 < \beta < \pi/2$.

Let us first consider the residue portion of Equation (F.11). The dominant part is the term

$$e^{i\,\bar{v}_\ell\,\pi\,3/2}\, H_{\bar{v}_\ell}(kR)\,. \qquad \text{(F.12)}$$

Since $\bar{v}_\ell$ lie above the real axis, $e^{i\bar{v}_\ell \pi 3/2}$ is a decaying exponential. The decaying exponent is of the order of $\bar{q}_\ell\, (ka)^{1/3}$. Now in the far field expression (F.12) behaves like

$$e^{i\bar{v}_\ell \pi 3/2} \sqrt{\frac{2}{\pi kR}}\; e^{ikR - i\bar{v}_\ell \pi/2 - i\pi/4}$$

and in the region where $kR \gg 1$ and $ka^2/R \sim O(1)$, expression (F.12) is

$$\sqrt{\frac{2}{\pi kR}} \exp\left\{ i\bar{v}_\ell \pi - \frac{i\pi}{4} + i\,(kR + ka^2/2R) + i(a/R)(ka/6)^{1/3}\, e^{i\pi/3} \bar{q}_\ell \right\}$$

and in the very near field as $R \rightarrow a$, it becomes the order of

$$e^{i\,\bar{v}_\ell\,\pi\,3/2}$$

Essentially for very large spheres, the residue terms are more important in the far field than near field in the direction of back-scattering. Physically this is what we expect. To reach a point in the near zone the diffracted waves must creep a greater distance around the sphere, and since their attenuation depends upon distance travelled as a surface wave, the diffracted waves represented by the residue terms must be attenuated more in the near zone.

We will now consider the integral portion of (F.11). It can be shown that the integrand has a saddle point at $v = 0$. We can now approximate this integral in the vicinity of the saddle point. This can be done in the same manner as is done in the paper by Scott (Ref. 11). We will then obtain a portion of the field corresponding to the incident wave plus the geometrically reflected wave.

Now, omitting the residue terms

$$I_1 \sim \frac{-1}{2(kR)} \left[e^{-ikR} \int_0^{\infty e^{-i\beta}} \frac{v \sin v\pi}{\cos v\pi} e^{\frac{v^2}{2ikR}} dv \right.$$

$$\left. - e^{ikR-2ika} \int_0^{\infty e^{-i\beta}} \frac{v \sin v\pi}{\cos v\pi} e^{v^2\left[\frac{1}{ika} - \frac{1}{2ikR}\right]} dv \right]$$

$$\sim + \frac{i}{2kR} \left[e^{-ikR} \int_0^{\infty e^{-i\beta}} v\, e^{\frac{v^2}{2ikR}} dv - e^{ikR-2ika} \int_0^{\infty e^{-i\beta}} v\, e^{v^2\left[\frac{1}{ika} - \frac{1}{2ikR}\right]} dv \right]$$

$$\sim \frac{e^{-ikR}}{2} - \frac{e^{ikR-2ika}}{2(2R-a)} + 0\left(\frac{1}{ka}\right). \qquad \text{(F.13)}$$

If we evaluate I_2 in the same manner as I_1, then substitute I_1, and I_2 into expression (F.5), we will obtain the diffracted waves (residue terms) which we can ignore, plus the geometrically reflected wave i.e.,

$$\underline{E}^s = - \hat{\underline{i}}_x \, e^{ikR-2ika} \left(\frac{a}{2R-a}\right) \left[1 + 0\left(\frac{1}{ka}\right) + \right]. \qquad \text{(F.14)}$$

We can evaluate the higher order terms in the integrals of I_1 and I_2, by Scott's method, but for the near field it is easier to use the Kline-Luneberg method. We obtain (Appendix A)

$$\underline{E}^s = -\hat{\underline{i}}_x \, e^{ikR-2ika} \frac{a}{(2R-a)} \left[1 - \frac{i}{ka} \frac{2(R-a)^2}{(2R-a)^2} + 0 \left(\frac{1}{(ka)^2} \right) \right].$$

For very large ka, $\underline{E}^s$ behaves like the back-scattered field from a parabola of revolution with semi-latus rectum $\ell = a/2$.

Now when $R \rightarrow a$, $\underline{E}^s$ approaches a plane wave

$$\underline{E}^s \rightarrow -\hat{\underline{i}}_x \, e^{ikR-2ika}.$$

References

1. Levy, B. R. and Keller, J. B., "Propagation of Electromagnetic Pulses Around the Earth", New York University, Research Report No. EM-102, (1957).

2. Friedlander, F.G., Sound Pulses, Cambridge University Press, (1958).

3. Schensted, C., "Electromagnetic and Acoustic Scattering by a Semi-Infinite Body of Revolution", J. Appl. Phys., 26, 306-308(March 1955).

4. Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F., Higher Transcendental Functions, Vol. 2, McGraw-Hill Book Co., Inc. (1953).

5. Watson, G. N., Theory of Bessel Functions, Cambridge University Press, (1952).

6. Sommerfeld, A., Partial Differential Equations, Academic Press, Inc. (1949).

7. Franz, W., "On the Green's Function of the Cylinder and the Sphere," Z. Naturforsch., 9a, 705-716,d(1954).

8. Campopiano, C.N., "Summary of Asymptotic Expansions for Bessel Functions," M.R.I. Report R-582-27, PIB-502, (1957).

9. Kotin, L. and Magnus, W., "Transcendental Equations in Electromagnetic Theory", N.Y.U. Research Report No. BR-27, (1958).

10. Stratton, J., Electromagnetic Theory, McGraw-Hill, (1941).

11. Scott, J.M.C., "An Asymptotic Series for the Radar Scattering Cross Section of a Spherical Target", A.E.R.E. Report T/M.30, (1949).