THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

DESIGN STUDY OF A LIQUID AND A SOLID ROCKET PROPELLANT SYSTEM

William Whicher
Theodore Petersen

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PREFACE

There are few places in the literature wherein the rocket propulsion system and vehicle performance have been treated together. In addition, a design study of two propulsion systems - one based on a liquid propellant and the other upon a solid propellant is quite interesting for comparison purposes. The two design studies presented here were carried out by the authors as part of the requirements for the Rocket Propulsion course in the Department of Aeronautical and Astronautical Engineering. In the interest of analytical facility, simplified assumptions were made for both preliminary designs. These assumptions, however, would not seriously affect the results.

This report was reproduced in the interest of supplying the Industry Program members with a concise design study of two systems based on the same design objectives and employing the same techniques of construction. It also clears up some of the arguments concerning the relative advantages and disadvantages of solid and liquid rocket motor systems.
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William Whicher

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Theodore Petersen

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LIQUID ROCKET DESIGN

William Whicher
NOMENCLATURE

\begin{align*}
\text{sp gr} & \quad \text{- specific gravity} \\
\text{psi} & \quad \text{- pounds per square inch} \\
A_e & \quad \text{- exit area of rocket nozzle} \\
q & \quad \text{- dynamic pressure} \\
P_c & \quad \text{- rocket chamber pressure} \\
F & \quad \text{- thrust (also} \ T \ \text{is used for thrust)} \\
\lambda & \quad \text{- rocket nozzle correction factor to account for divergence of} \\
& \quad \text{flow at rocket exit} \\
V_e & \quad \text{- gas velocity at exit of rocket nozzle} \\
m & \quad \text{- mass flow} \\
T_c & \quad \text{- exit temperature of rocket gas} \\
P_e & \quad \text{- exit pressure of rocket gas} \\
A_T & \quad \text{- rocket nozzle throat area} \\
P_2/P_1 & \quad \text{- pressure ratio across oblique shock in exit of rocket nozzle} \\
\theta & \quad \text{- cone angle of rocket nozzle at exit} \\
M_1 & \quad \text{- Mach number before oblique shock in exit of rocket nozzle} \\
\Gamma & \quad \text{-} \\
& \quad \frac{2}{\gamma} \left( \frac{\gamma + 1}{2(\gamma - 1)} \right) \\
t & \quad \text{- time} \\
T & \quad \text{- thrust} \\
D & \quad \text{- drag on rocket} \\
M & \quad \text{- mass of rocket at time, } t \ (\text{also} \ M = \text{Mach number}) \\
h & \quad \text{- altitude of the rocket} \\
g & \quad \text{- gravitational constant} \\
\dot{w} & \quad \text{- weight flow}
\end{align*}
$V$ - velocity of the rocket

$L$ - length

$C_o$ - drag coefficient

$R_e$ - Reynolds number

$S$ - area
Problem Number 3

Given the following information, design a rocket.

- Mass ratio = 20/1 (includes payload)
- Payload wt. = 10% dry wt. of bird.
- Fineness ratio = \( \frac{L}{D} = \frac{15}{1} \)
- Make bird out of Fiberglass.
- Sp.gr. Fiberglass = 1.80
- Working stress = \( \sigma = 80000 \text{ psi} \)
- Use a monopropellant sp.gr. = 1.00
- Motor \( \frac{400 \text{ lbs. thrust}}{\text{lb. engine wt.}} \)
- Use a gas pressurization system.
- \( A_e \) in a spherical container.
- \( P_{He} = 3500 \text{ psia initially} \)
- Use a heater charge in Helium tank so that Helium always expands isothermally.
- 25% pressure drop across plumbing.
- 25% pressure drop across injector face.
- Lift off wt. = 25,000 lbs.
- Also make plots of:
  a) head suppression vs. time.
  b) chamber pressure vs. time.
  c) drag vs. time.
  d) acceleration vs. time.
  e) altitude vs. time.
  f) Mach number vs. time (only until \( q \) drops off appreciably)
  g) \( q \) vs. time (\( q = \frac{1}{2} c v^2 \))

-1-
**Additional information**

Use an engine with same geometry as in Problem 2 except chop off nozzle so that flow never separates.

When designing tankage only consider hoop tension.

Assume that a regulator valve is in the Helium line that controls the pressure on top of the propellant at a constant value until the helium runs out.

Assume a head suppression valve is installed in the propellant lines so as to regulate the pressure in the chamber at a constant 150 psia until the helium runs out.

Size rocket

Mass ratio = 20/1

Wt. fuel at lift off = \((0.95)(25000)\)

\[= 23750 \text{ lbs.}\]

Tank volume = \(\frac{\text{Wt. fuel}}{\text{density fuel}}\)

\[= \frac{23750}{62.4} = 381 \text{ ft}^3\]

Calculate tank dimensions

\[V = \frac{L\pi D^2}{4}, \quad \frac{L}{D} = 15, \quad L = 15D\]

\[V = \frac{15D^3\pi}{4}\]

\[D^3 = \frac{(381)(4)}{15\pi} = 32.4\]

\[D = 3.18 \text{ ft.}\]

\[L = 47.7 \text{ ft.}\]

Use \(L/D = 3\) for nose cone

\[L_{\text{Total}} = 47.7 + (3)(3.18)\]

\[= 57.24 \text{ ft.}\]

* Refers to design of a rocket motor chamber and nozzle previously carried out.
Calculate Engine Characteristics

Characteristics for $P_c = \text{const.} = 150 \text{ psia}$

Calculate $F_\infty$

$$F_\infty = \lambda \ m \ V_e + P_e A_e$$

from Problem 2

$$F_\infty = 577 \ M_e T_e^{1/2} + P_e A_e$$

for $\frac{A_e}{A_T} = 3.011; \quad \frac{P_e}{P_c} = 0.06526; \quad \frac{T_e}{T_c} = 0.6345$

$$M_e = 2.40 \ (\text{from gas tables})$$

so


$$= 78000 + 11350$$

$$= 89350 \ \text{lb.}$$

The thrust at any altitude is given by

$$F = F_\infty - A_e P_a$$

Verify that nozzle does not separate

$$\frac{P_2}{P_1} = 1 + \frac{7M_1^2}{(M_1^2 - 1)^{1/2}} = 1 + \frac{(1.2)(2.4)^2(0.1745)}{[(2.4)^2 - 1]^{1/2}}$$

$$= 1 + \frac{1.205}{(4.76)^{1/2}} = 1 + 0.552$$

$$P_2 = (150)(0.06526)(1.552) = 15.19 \text{ psia.}$$

This is above atmospheric pressure of 14.7 psia so separation does not occur.

Calculate thrust after 45.0 seconds

$P_c$ from graph

$$F = 600 \ P_c$$
<table>
<thead>
<tr>
<th>Altitude (ft)</th>
<th>Atmospheric Pressure (lb/ft²)</th>
<th>AePa (lb)</th>
<th>F (lb)</th>
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<td>2116</td>
<td>17060</td>
<td>72290</td>
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<tr>
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<td>1828</td>
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</tr>
<tr>
<td>80000</td>
<td>58.01</td>
<td>467</td>
<td>88883</td>
</tr>
</tbody>
</table>
\[ t = 50.0 \]
\[ F = (600)(130.6) \]
\[ = 78360 \text{ lbs.} \]
\[ t = 55.0 \text{ sec.} \]
\[ F = (600)(118.4) \]
\[ = 71040 \text{ lbs.} \]
\[ t = 60.0 \]
\[ F = (600)(105.0) \]
\[ = 63000 \text{ lbs.} \]
\[ t = 65.0 \]
\[ F = (600)(89.0) \]
\[ = 53400 \text{ lbs.} \]
\[ t = 67.26 = t_{BO} \]
\[ F = (600)(83.3) \]
\[ = 49980 \text{ lbs.} \]

**Determine chamber pressure as a function of time**

A head suppression valve is installed in the propellant lines to keep the pressure in the chamber constant at 150 psia. However due to the limitations on the amount of helium that can be carried, the chamber pressure decreases during the last part of burning. At some time the head suppression term goes to zero and the chamber pressure is determined solely by the helium pressure, the hydraulic head, and the line plus injector impedance. After the critical time (the time at which the chamber pressure starts to fall) the chamber pressure can be determined by an equilibrium equation relating the line impedance, the helium pressure and the hydraulic head.

\[ P_{He} + H - Z P_c = 0 \]
The chamber pressure was constant up to 45 seconds. Therefore it is convenient to take our time origin at 45 seconds. Doing this we can rewrite the equation

$$\frac{P_1 V_1}{V_{45} + \int_0^t \dot{V} \, dt} + \frac{(h_{45} - \int_0^t \dot{h} \, dt) \rho_P F}{W_{45} - \int_0^t \dot{\omega} \, dt} = Z P_c(t)$$

Where

- $P_1 =$ initial helium pressure
- $V_1 =$ initial helium volume
- $V_{45} =$ helium volume at 45 seconds
- $\dot{V} =$ time rate of change of helium volume
- $h_{45} =$ level of the propellant at 45 seconds
- $\dot{\omega} =$ time rate of change of propellant level
- $\rho_P =$ density of propellant
- $W_{45} =$ weight of rocket at 45 seconds
- $\omega =$ weight flow
- $F =$ thrust
- $P_c(t) =$ chamber pressure
- $Z =$ ratio of tank pressure to chamber pressure, previously calculated as 16/9.

The chamber pressure and the chamber temperature are related through the equilibrium constant $K_p$. However since we do not know the chemical composition of the monopropellant we are forced to assume that the reaction products and therefore the chamber temperature are constant. As a consequence of this assumption we note that the mass flow is sensitive only to changes in chamber pressure.
\[
\dot{m} = \frac{\Gamma \, P_c \, A_t}{a_c}
\]
\[
\therefore
\dot{m} = \frac{m_l}{P_{c_1}} \, P_c
\]

By using this relation we can solve for \(\dot{V}, \dot{h},\) and \(\dot{\omega}\) as functions of \(P_c\) alone. At 45 seconds we are at about 100,000 ft. Therefore we can ignore drag, and also assume that we are exhausting into a vacuum. The last assumption allows us to solve for \(F\) as a function of \(P_c\) alone.

\[
F = \lambda \, \dot{m} \, V_e + P_e \, A_e
\]
\[
= \lambda \, \frac{m_l}{P_{c45}} \, P_c \, V_e + \frac{P_e}{P_c} \, P_c \, A_e
\]
\[
= \left[ \lambda \, \frac{m_l}{P_{c45}} \, V_e + \frac{P_e}{P_c} \, A_e \right] \, P_c
\]
\[
\lambda = 0.9924
\]
\[
\frac{m_l}{P_{c45}} = \frac{11.9}{150} = 0.07933
\]

\[
V_e = M_e \sqrt[\gamma]{R \frac{T_e}{T_c}} = 6600 \text{ ft/sec}
\]
\[
\therefore
F = 599.9 \, P_c
\]
\[
\dot{\omega} = g \, \dot{m} = (32.2)(0.07933) \, P_c = 2.554 \, P_c
\]
\[
\dot{V} = \frac{\dot{\omega}}{\rho_p} = 2.554 \, \frac{P_c}{62.4} = 0.04094 \, P_c
\]
\[
\dot{h} = \frac{\dot{V}}{\text{inside tank area}} = \frac{0.04094 \, P_c}{7.94} = 0.005155 \, P_c
\]

Substituting these values into the original equation we obtain

\[
\frac{(3500)(16.83)}{293.4 + 0.04094 \int_0^t P_c(t) \, dt} + \frac{(13.15 - 0.005155 \int_0^t P_c(t) \, dt)(62.4)(599.9 \, P_c(t))}{7765 - 2.554 \int_0^t P_c(t) \, dt(144)} = P_c(t)
\]
The initial values at $t=0$ (45 seconds) have been included in the above equation. This equation gives $P_c$ as a function of time.

$$P_c = P_c(t)$$

We note that this equation is a non-linear integral equation. An analytical solution is not possible. However if we make certain assumptions we can obtain an approximate solution to this equation.

First from the first mean value theorem of integral calculus we note

$$\int_{0}^{t} P_c(t) dt = P_c(\xi)t$$

where $P_c(\xi)$ is some average value in the interval. Now unless we know $\xi$ we cannot locate $P(\xi)$ in the interval. However for calculation purposes we will assume that $\xi$ lies in the middle of the interval. This is true if the interval 0 to $t$ is small. For our purposes we will use 2.5 second intervals.

The equation to be solved is now

$$\frac{P_i V_i}{V_o + 0.04094 P_c(\xi)t} + \frac{[h_o - (0.005155) P_c(\xi)t]}{W_o - 2.554 P_c(\xi)t} = \frac{16}{9} P_c(\xi)$$

where $V_o$, $h_o$, and $W_o$ are the values of the helium volume, the propellant level and the rocket weight at the end of the previous interval. The rest of the terms are as noted above.

The above equation is cubic in $P_c(\xi)$ and can be solved most easily by iteration. The final calculation for each interval is presented below.

$$\frac{58905}{293.4 + 0.10235 P_c} + \frac{(13.15 - 0.1289 P_c)(260)(P_c)}{7765 - 6.385 P_c} = \frac{16}{9} P_c$$
45 \ 0 \leq t \leq 47.5 \quad \text{try } P_c = 142

\frac{58905}{293.4 + 14.53} + \frac{[(13.15) - (1.830)](260)(142)}{7765 - 906} = \frac{16}{9} P_c

\frac{58905}{307.93} + \frac{(11.32)(260)(142)}{6859} = \frac{16}{9} P_c

P_c = \frac{9}{16} (191.2 + 60.9) = \frac{9}{16} (252.1) = 141.9 \quad \text{Use } P_c = 142.0

47.5 \leq t \leq 50.0 \quad \text{try } P_c = 134.5

\frac{58905}{307.93 + 13.78} + \frac{(11.32 - 1.732)(260)(134.5)}{6859 - 858} = \frac{16}{9} P_c

\frac{16}{9} P_c = \frac{58905}{321.71} + \frac{(9.588)(260)(134.5)}{6001}

P_c = \frac{9}{16} (183.0 + 55.9) = \frac{9}{16} (238.9) = 134.5 \quad \text{Use } P_c = 134.5

50.0 \leq t \leq 52.5 \quad \text{try } P_c = 127.5

\frac{58905}{321.71 + 13.07} + \frac{(9.588 - 1.642)(260)(127.5)}{6001 - 814} = \frac{16}{9} P_c

\frac{16}{9} P_c = \frac{58905}{334.78} + \frac{(7.946)(260)(127.5)}{5187}

P_c = \frac{9}{16} (175.9 + 50.7) = \frac{9}{16} (226.6)

= 127.5 \quad \text{Use } P_c = 127.5
52.5 \leq t \leq 55.0

\text{try } P_c = 121.0

\frac{58905}{334.78 + 12.39} + \frac{(7.946 - 1.560)(260)(121)}{5187 - 773} = \frac{16}{9} P_c

\frac{58905}{347.17} + \frac{(6.386)(260)(121)}{44.14} = \frac{16}{9} P_c

P_c = \frac{9}{16} (169.8 + 45.4) = \frac{9}{16} (215.2)

= 121.0 \quad \text{Use } P_c = 121.0 \text{ Psia}

55.0 \leq t \leq 57.5

\frac{16}{9} P_c = \frac{58905}{347.17 + 11.75} + \frac{(6.386 - 1.479)(260)(114.8)}{4414 - 732}

= \frac{58905}{358.92} + \frac{(4.907)(260)(114.8)}{3682}

P_c = \frac{9}{16} (164.0 + 39.8) = \frac{9}{16} (203.8)

P_c = 114.8 \quad \text{Use } P_c = 114.8

57.5 \leq t \leq 60.0

\text{try } P_c = 108.0

\frac{16}{9} P_c = \frac{58905}{358.92 + 11.06} + \frac{(4.907 - 1.391)(260)(108)}{3682 - 689}

= \frac{58905}{369.98} + \frac{(3.516)(260)(108)}{2993}

P_c = \frac{9}{16} (159.1 + 33.0) = \frac{9}{16} (192.4)

= 108.0 \quad \text{Use } P_c = 108.0
\[ 60.0 \leq t \leq 62.5 \]

\[ \text{try } P_c = 101.0 \]

\[ \frac{16}{9} P_c = \frac{58905}{369.99 + 10.35} + \frac{(3.516 - 1.291)(260)(101)}{2993 - 645} \]

\[ = \frac{58905}{380.33} + \frac{(2.225)(260)(101)}{2348} \]

\[ P_c = \frac{9}{16} (154.8 + 24.9) = \frac{9}{16} (179.7) \]

\[ = 101.0 \quad \text{Use } P_c = 101.0 \]

\[ 62.5 \leq t \leq 65.0 \]

\[ \text{try } 93.0 \text{ psi} \]

\[ \frac{16}{9} P_c = \frac{58905}{380.33 + 9.53} + \frac{(2.225 - 1.198)(260)(93)}{2348 - 593} \]

\[ = \frac{58905}{389.86} + \frac{(1.027)(260)(93)}{1755} \]

\[ P_c = \frac{9}{16} (151.0 + 14.1) = \frac{9}{16} (165.1) \]

\[ = 93.0 \text{ psia Use } 93 \]

To determine burn out time we note that the helium volume at burn out is given by

\[ V_{He} = \text{Vol propellant tank} + \text{initial vol He.} \]

\[ = 397.83 \text{ ft}^3 \]

From this we can determine the chamber pressure at burn out.

\[ P_{cBO} = \frac{(3500)(16.83)}{397.83} \]

\[ = 83.287 \times 83.3 \text{ psia} \]
Burnout will occur before 67.5 seconds. We can find the burnout time if the average chamber pressure during the last period is known. Estimate this from the average pressure for the last two times

\[ P_{c65} = 93.0 - \frac{(101 - 93)}{2} \]

\[ = 89.0 \]

The average pressure over the interval 65 → t_{B0} is

\[ P_{cav} = \frac{83.3 + 89.0}{2} \]

\[ = 86.15 \text{ psia} \]

Now since the helium volume must be 397.83 ft^3 at burnout

\[ V_{He65} + (0.04094) P_c(t) t'_b = 397.83 \]

\[ 389.86 + (0.04094)(86.15) t'_b = 397.83 \]

\[ t'_b = \frac{7.97}{(86.15)(0.04094)} \]

\[ = 2.26 \]

\[ t_{B0} = 65.0 + 2.26 \]

\[ = 67.26 \text{ seconds after lift off}. \]
<table>
<thead>
<tr>
<th>Time</th>
<th>Chamber Pressure</th>
<th>Weight</th>
<th>$\frac{w}{\text{lb/sec}}$</th>
</tr>
</thead>
<tbody>
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<td>150.0</td>
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<tr>
<td>63.75</td>
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<tr>
<td>65.00</td>
<td>89.0</td>
<td>1755</td>
<td>223.8</td>
</tr>
<tr>
<td>67.26</td>
<td>83.3</td>
<td>1250</td>
<td></td>
</tr>
</tbody>
</table>
Performance Calculations

Using Newton's second law

\[ M \frac{dV}{dt} = T - D - Mg \]

Now if we consider thrust and drag constant over the time interval and express \( M \) as \( M(t) \) we obtain.

\[ \frac{dV}{dt} = \frac{T - D}{M_0 - m} - g \]

\[ = \frac{g(T - D)}{W_0 - \dot{W}t} - g \]

\[ \int_{V_0}^{V} dV = \int_{0}^{t} g(T-D)dt - \int_{0}^{t} gdt \]

\[ V = V_0 + \frac{T - D}{m} \ln \frac{W_0}{W_0 - \dot{W}t} - gt \]

\[ V = \frac{dh}{dt} \]

\[ \int_{h_0}^{h} dh = \int_{0}^{t} V_0 dt + \frac{T-D}{m} \int_{0}^{t} \ln \frac{W_0}{W_0 - \dot{W}t} - \int_{0}^{t} gdt \]

after consulting an integral table we find

\[ h = h_0 + V_0t - \frac{gt^2}{2} + \frac{\dot{W}t}{m} [t - \ln \frac{W_0 - \dot{W}t}{W_0}] \]

Note that I have integrated from 0 to \( t \). This is equivalent to considering only the time intervals with initial conditions applying from the end of the previous time interval.

\[ 0 \leq t \leq 5 \quad F - D = 72290 \text{ lb} \]
t = 5

\[ V = \frac{72290}{11.9} (\ln 25000 - \ln 23085) - 161 \]

\[ = 6070 \ [10.12663 - 10.04693] - 161 \]

\[ = 6070 \ (0.07970) - 161 \]

\[ = 484 - 161 \]

\[ = 323 \text{ ft/sec}. \]

\[ h = 6070 \ [5 - \frac{(23085)(0.07970)}{383}] - (25)(16.1) \]

Since the \([\ ]\) involve the difference of numbers very close together, use log tables instead of a slide rule so as to improve accuracy.

\[ \log 23085 = 4.36335 \]
\[ \log 0.07970 = \frac{8.90146 - 10}{13.26479 - 10} \]
\[ \log 383 = \frac{2.58320}{10.68159 - 10} = \log 4.8049 \]

\[ h = 6070(5 - 4.8049) - 403 \]

\[ = 1183 - 403 \]

\[ = 780 \text{ ft} \]

\[ \text{Re} = \frac{\rho VL}{\mu} \]

\[ L = 47.7 + (3.18)(3) = 47.7 + 9.54 \]

\[ = 57.24 \]

\[ \text{Re} = \frac{(0.002310)(323)(57.24)}{3.699 \times 10^{-7}} \ (\rho \text{ and } \mu \text{ from NACA TN 3182}) \]

\[ = 105.5 \times 10^6 \]

\[ M = \frac{323}{1114} = 0.29 \ (a \text{ at various altitudes from NACA TN 3182}) \]
From graph

\[ C_D = 0.18 \]

\[ D = C_Dq S \quad (S \text{ based on frontal area}) \]

\[ q = \frac{\rho}{2} v^2 = \frac{(2.310 \times 10^{-3})(323)^2}{2} \]

\[ = 120.3 \text{ lb/ft}^2 \]

Since the estimation of \( C_D \) from the graph is not precise use \( S \) based on the inside area of tank.

\[ S = (1.59)^2 \pi = 7.94 \]

\[ D = (0.18)(120.3)(7.94) \]

\[ = 172.1 \text{ lbs.} \]

\[ 5 \leq t \leq 10 \]

Thrust at 780 ft = 72700 lb

\[ T - D = 72700 - 172 = 72528 \text{ lb} \]

\( t = 10 \)

\[ V = 323 + \frac{72528}{11.9} \ln \frac{23085}{21170} - 161 \]

\[ V = 323 + 6100(10.04693 - 9.96034) - 161 \]

\[ = 162 + 6100(0.08659) \]

\[ = 162 + 528 \]

\[ = 690 \text{ ft/sec} \]

\[ h = 780 + (323)(5) - 403 + 6100 \left[ 5 - \frac{(21170)}{383} \right] (0.08659) \]

\[ = 4.32572 \]

\[ 8.93747 - 10 \]

\[ 13.26319 - 10 \]

\[ 2.58320 \]

\[ 10.67999 - 10 \rightarrow 5.0000 \]

\[ 4.7862 \]
\[ h = 780 + 1212 + 6100 \ (0.2138) \]
\[ = 1992 + 1302 \]
\[ = 3294 \ ft \]

\[ \text{Re} = \frac{(0.002158)(690)(57.24)}{3.669 \times 10^{-7}} \]
\[ = 232 \times 10^6 \]
\[ M = \frac{6.90}{1107} = 0.063 \]

\[ C_D = 0.17 \]
\[ q = \frac{(0.002158)(690)^2}{2} = 514 \ ft^2/\text{lb} \]
\[ D = (0.17)(514)(7.94) \]
\[ = 695 \ lbs \]

\[ 10 < t < 15 \]

\[ T - D = 74200 - 695 = 73505 \ lbs \]

\[ t = 15 \]

\[ V = 690 - 161 + \frac{73505}{11.9} \ [\ln 21170 - \ln 19255] \]
\[ = 529 + 6170 [9.96034 - 9.86552] \]
\[ = 529 + 6170 [0.09482] \]
\[ = 529 + 584 \]
\[ = 1113 \ ft/\text{sec} \]

\[ h = 3294 + (690)(5) - 403 + 6170 \left[5 - \frac{(19255)}{383} \cdot (0.09482)\right] \]
\[ = 4.28454 \]
\[ \frac{8.97690 - 10}{13.26144 - 10} \]
\[ = 2.58320 \]
\[ \frac{10.67824 - 10}{10.67824 - 10} \rightarrow 5.0000 \]
\[ = 4.7671 \]

\[ h = 6341 + (6170)(0.2329) \]
\[ h = 6341 + 1437 \]
\[ = 7778 \ ft \]
\[ \text{Re} = \frac{(0.001881)(1113)(57.24)}{3.561 \times 10^{-7}} \]
\[ = 336 \times 10^6 \]
\[ M = \frac{1113}{1100} = 1.011 \]
\[ C_D = 0.295 \]
\[ q = \frac{(0.001881)(1113)^2}{2} = 1164 \text{ lbs/ft}^2 \]
\[ D = (0.295)(1164)(7.94) \]
\[ = 2726 \text{ lbs} \]

\[ 15 < t \leq 20 \]
\[ T - D = 76600 - 2726 = 73874 \text{ lb} \]

\[ t = 20 \]
\[ V = 1113 + \frac{73874}{11.9} [\ln 19255 - \ln 17340] - 161 \]
\[ = 952 + 6205 [9.86552 - 9.76077] \]
\[ = 952 + (6205)(0.10475) \]
\[ = 952 + 651 \]
\[ = 1603 \text{ ft/sec} \]
\[ h = 7778 + 5565 - 403 + 6205\left\{ 5 - \frac{(17340)(0.10475)}{383} \right\} \]
\[ = 4.23905 \]
\[ = 9.02015 - 10 \]
\[ = 13.25920 \]
\[ = 2.58320 \]
\[ = 10.67600 - 10 \rightarrow 4.7424 \]
\[ h = 12940 + (6205)(0.2576) \]
\[ = 12940 + 1597 \]
\[ = 14537 \text{ ft} \]
\[ \text{Re} = \frac{(1.520 \times 10^{-3})(1603)(57.24)}{3.441 \times 10^{-7}} \]
\[ = 405 \times 10^6 \]
\[ M = \frac{1603}{1060} = 1.512 \]

\[ C_D = 0.275 \]

\[ q = \frac{(1.520 \times 10^{-3})(1603)^2}{2} = 1953 \text{ lb/ft}^2 \]

\[ D = (0.275)(1953)(7.94) = 4260 \text{ lb} \]

\[ 20 \leq t \leq 25 \]

\[ T - D = 79500 - 4260 = 75240 \text{ lb} \]

\[ t = 25 \]

\[ V = 1603 - 161 + \frac{75240}{11.9} [\ln 17340 - \ln 15425] \]

\[ V = 1442 + 6320 (9.76077 - 9.64374) \]

\[ = 1442 + (6320)(0.11703) = 1442 + 740 \]

\[ = 2182 \text{ ft/sec} \]

\[ h = 14537 + (1603)(5) - 403 + 6320 \left[ 5 - \frac{(15425)(0.11703)}{383} \right] \]

\[ = 4.18823 \]

\[ = 9.06830 - 10 \]

\[ = 13.25653 - 10 \]

\[ = 2.58320 \]

\[ = 10.67333 - 10 \rightarrow 4.7133 \]

\[ h = 22149 + (6320)(0.2867) \]

\[ = 22149 + 1812 \]

\[ = 23961 \text{ ft} \]

\[ Re = \frac{(1.103 \times 10^{-3})(2182)(57.24)}{3.234 \times 10^{-7}} \]

\[ = 426 \times 10^6 \]

\[ M = \frac{2182}{1021} = 2.12 \]
\[ C_D = 0.245 \]
\[ q = \frac{(1.103 \times 10^{-3})(2182)^2}{2} = 2620 \text{ lb/ft}^2 \]
\[ D = (0.245)(2620)(7.94) = 5100 \text{ lb} \]
\[ 25 < t \leq 30 \]
\[ T - D = 82700 - 5100 = 77600 \]
\[ t = 30 \]
\[ V = 2182 - 161 + \frac{77600}{11.9} - \left[ \ln 15425 - \ln 13510 \right] \]
\[ = 2021 + 6520 (9.64374 - 9.51118) \]
\[ = 2021 + (6520)(0.13256) \]
\[ = 2021 + 864 \]
\[ = 2885 \text{ ft/sec} \]
\[ h = 23961 + (2182)(5) - 403 + 6520 \left[ 5 - \frac{(13510)(0.13256)}{383} \right] \]
\[ = 34468 + \frac{2885}{968} \]
\[ = 34468 + 2110 \]
\[ = 36578 \text{ ft} \]
\[ Re = \frac{(6.92 \times 10^{-4})(2885)(57.24)}{2.96 \times 10^{-7}} \]
\[ = 386 \times 10^6 \]
\[ M = \frac{2885}{968} = 2.98 \]
\[ C_D = 0.215 \]
\[ q = \frac{(6.92 \times 10^{-4})(2885)^2}{2} = 2880 \text{ lb/ft}^2 \]
\[ D = (0.215)(2880)(7.94) \]
\[ = \frac{4910}{11.9} \text{ lb} \]

\[ 30 < t \leq 35 \]

\[ T - D = 85600 - 4910 = 80690 \]

\[ t = 35 \]

\[ V = 2885 - 161 + \frac{80690}{11.9} \left[ \ln 13510 - \ln 11595 \right] \]
\[ = 2724 + 6790 \left( 9.51118 - 9.35832 \right) \]
\[ = 2724 + (6790)(0.15286) \]
\[ = 2724 + 1038 \]
\[ = 3762 \text{ ft/sec} \]

\[ h = 36578 + (2885)(5) - 403 + 6790 \left[ 5 - \frac{(11595)(0.15286)}{383} \right] \]
\[ = 36578 + (2885)(5) - 403 + 6790 \left[ 5 - \frac{9.18429}{10} \right] \]
\[ = 36578 + (2885)(5) - 403 + 6790 \left[ 5 - \frac{13.24533}{10} \right] \]
\[ = 36578 + (2885)(5) - 403 + 6790 \left[ 5 - \frac{2.58320}{10} \right] \]
\[ = 36578 + (2885)(5) - 403 + 6790 \left[ 5 - 0.258320 \right] \]
\[ = 36578 + (2885)(5) - 403 + 6790 \left[ 4.74168 \right] \]
\[ = 36578 + (2885)(5) - 403 + 32667 \]
\[ = 50600 + (6790)(0.3724) \]
\[ = 50600 + 2524 \]
\[ = 53124 \text{ ft} \]

\[ Re = \left( 3.118 \times 10^{-4} \right)(3762)(5724) \]
\[ = \frac{2.96 \times 10^{-7}}{2.96 \times 10^{-7}} \]
\[ = 226.5 \times 10^6 \]

\[ M = \frac{3762}{968} = 3.88 \]

\[ C_D = 0.19 \]

\[ q = \left( 3.118 \times 10^{-4} \right)(3762)^2 \]
\[ = \frac{2205}{1} \text{ lb/ft}^2 \]

\[ D = (0.19)(2205)(7.94) \]
\[ = 3320 \text{ lb} \]

\[ 35 < t \leq 40 \]

\[ T - D = 87700 - 3320 = 84380 \]
\[ t = 40 \]

\[ V = 3762 - 161 + \frac{84380}{11.9} \ln \frac{11595}{9680} \]

\[ = 3601 + 7100 [9.35833 - 9.17782] \]

\[ = 3601 + (7100)(0.18051) \]

\[ = 3601 + 1281 \]

\[ = 4882 \text{ ft/sec} \]

\[ h = 53124 + (3762)(5) - 403 + 7100 \left[ 5 - \frac{(9680)(0.18051)}{383} \right] \]

\[ = 3.98588 \]

\[ = 9.25650 - 10 \]

\[ = 13.24238 - 10 \]

\[ = 2.58320 \]

\[ = 10.65918 - 10 \rightarrow 4.5622 \]

\[ h = 71531 + (7100)(0.4378) \]

\[ = 71531 + 3104 \]

\[ = 74635 \text{ ft} \]

\[ Re = \frac{(1.115 \times 10^{-4})(4882)(57.24)}{2.96 \times 10^{-7}} \]

\[ = 105 \times 10^6 \]

\[ M = \frac{4882}{971} = 503 \]

\[ C_D = 0.17 \]

\[ q = \frac{(1.115 \times 10^{-4})(4882)^2}{2} = 1328 \text{ lb/ft}^2 \]

\[ D = (0.17)(1328)(7.94) \]

\[ = 1790 \text{ lb} \]

\[ 40 < t < 45 \]

\[ T - D = 88700 - 1790 = 86910 \text{ lb} \]
\[ t = 45 \]

\[ V = 4882 - 161 + \frac{86910}{11.9} \left[ \ln 9680 - \ln 7765 \right] \]
\[ = 4721 + 7300 \left( 9.17781 - 8.95738 \right) \]
\[ = 4721 + 1610 \]
\[ = 6331 \text{ ft/sec} \]

\[ h = 74635 + (4882)(5) - 403 + 7300 \left[ 5 - \frac{(7765)(0.22043)}{383} \right] \]
\[ \frac{3.890141}{9.343271 - 10} \]
\[ \frac{13.233412}{10} \]
\[ \frac{2.583199}{10.650215 - 10} \rightarrow 4.4690 \]

\[ h = 98642 + (7300)(0.5310) \]
\[ = 98642 + 3880 \]
\[ = 102,522 \text{ ft} \]

\[ \text{Re} = \frac{(2.96 \times 10^{-5})(6331)(57.24)}{2.96 \times 10^{-7}} \]
\[ = 36.2 \times 10^6 \]

\[ M = \frac{6331}{971} = 6.52 \]

\[ C_D = 0.18 \]

\[ q = \frac{(2.96 \times 10^{-5})(6331)^2}{2} = 595 \text{ lb/ft}^2 \]

\[ D = (0.18)(595)(7.94) \]
\[ = 848 \text{ lb} \]

An expression for altitude as a function of time can be obtained analytically if an approximate expression is used for chamber pressure as a function of time. However, the altitude integral becomes rather messy, so it was decided to continue the numerical integration using 2.5 second intervals and the average mass flow over each interval.
\[ t = 47.5 \]
\[
V = 6331 - 80 + 7565 \ln \frac{7765}{6859}
\]
\[
= 6251 + 933
\]
\[
= 7184 \text{ ft/sec}
\]
\[
h = 102522 - 100 + (6331)(2.5) + 7565 \left[2.5 - \frac{6859}{362.2} \ln \frac{7765}{6859}\right]
\]
\[
= 118249 + 7565 \left[2.5000 - 2.348\right]
\]
\[
= 118249 + (7565)(0.152)
\]
\[
= 118249 + 1150
\]
\[
= 119399 \text{ ft}
\]
\[ t = 50.0 \]
\[
V = 7184 - 80 + 7565 \ln \frac{6859}{6001}
\]
\[
= 7104 + 1013
\]
\[
= 8117 \text{ ft/sec}
\]
\[
h = 119399 + (7184)(2.5) - 100 + 7565 \left[2.5 - \frac{6001}{343.2} \ln \frac{6859}{6001}\right]
\]
\[
= 137259 + (7565)(2.500 - 2.342)
\]
\[
= 137259 + (7565)(0.158)
= 137259 + 1194
\]
\[
= 138453 \text{ ft}
\]
\[ t = 52.5 \]
\[
V = 8117 - 80 + 7565 \ln \frac{6001}{5187}
\]
\[
= 8037 + 1104
\]
\[
= 9141 \text{ ft/sec}
\]
\[
h = 138453 - 100 + (8117)(2.5) + 7565 \left[2.5 - \frac{5187}{325.6} \ln \frac{6001}{5187}\right]
\]
\[
= 158645 + (7565)(2.500 - 2.324)
\]
\[
= 158645 + (7565)(0.176)
\]
\[
= 158645 + 1332
\]
\[
= 159977 \text{ ft}
\]
\[ t = 55.0 \]
\[ V = 9141 - 80 + 7565 \ln \frac{5187}{4414} \]
\[ = 9061 + 1221 \]
\[ = 10282 \text{ ft/sec} \]
\[ h = 159977 - 100 + (9141)(2.5) + 7565 \left[ 2.5 - \frac{4414}{309.4} \ln \frac{5187}{4414} \right] \]
\[ = 182729 + (7565)(2.500 - 2.304) \]
\[ = 182729 + (7565)(0.196) \]
\[ = 182729 + 1483 \]
\[ = 184212 \text{ ft} \]

\[ t = 57.5 \]
\[ V = 10282 - 80 + 7565 \ln \frac{4414}{3682} \]
\[ = 10202 + 1375 \]
\[ = 11577 \text{ ft/sec} \]
\[ h = 184212 + (2.5)(10282) - 100 + 7565 \left[ 2.5 - \frac{3682}{293} \ln \frac{4414}{3682} \right] \]
\[ = 209817 + 7565 (2.500 - 2.282) \]
\[ = 209817 + (7565)(0.218) \]
\[ = 209817 + 1650 \]
\[ = 211467 \text{ ft} \]

\[ t = 60.0 \]
\[ V = 11577 + 7565 \ln \frac{3682}{2993} - 80 \]
\[ = 11497 + 1570 \]
\[ = 13067 \text{ ft/sec} \]
\[ h = 211467 - 100 + (11577)(2.5) + 7565 \left[ 2.500 - \frac{2993}{275.6} \ln \frac{3682}{2993} \right] \]
\[ = 240309 + (7565)(2.500 - 2250) \]
\[ = 240309 + (7565)(0.250) \]
\[ = 240309 + 1890 \]
\[ = 242199 \text{ ft} \]
\[ t = 62.5 \]

\[
V = 13067 - 80 + \left[ \ln \frac{2993}{2348} \right] 7565 \\
= 12987 + (7565)(2432) \\
= 14829 \text{ ft/sec}
\]

\[
h = 242199 - 100 + (13067)(2.5) + 7565 \left[ 2.5 - \frac{2348}{258} \ln \frac{2993}{2348} \right] \\
= 274766 + 7565 \left( 2.500 - 2.216 \right) \\
= 274766 + (7565)(0.284) \\
= 274766 + 2150 \\
= 276916 \text{ ft}
\]

\[ t = 65.0 \]

\[
V = 14829 - 80 + 7565 \ln \frac{2348}{1755} \\
= 14749 + 2204 \\
= 16953 \text{ ft/sec}
\]

\[
h = 276916 - 100 + (2.5)(14829) + 7565 \left[ 2.5 - \frac{1755}{237.2} \ln \frac{2348}{1755} \right] \\
= 313888 + 7565 \left( 2.500 - 2.158 \right) \\
= 313888 + (7565)(0.342) \\
= 313888 + 2590 \\
= 316478 \text{ ft}
\]

\[ t = 67.26 = t_{80} \]

\[
V = 16953 - (32.2)(2.26) + 7565 \ln \frac{1755}{1250} \\
= 16953 - 73 + 2565 \\
= 19445 \text{ ft/sec}
\]

\[
h = 316478 - (161)(2.26)^2 + (16953)(2.26) + 7565 \left[ 2.26 - \frac{1250}{223.8} \ln \frac{1755}{1250} \right] \\
= 354710 + 7565 \left( 2.26 - 1.895 \right) \\
= 354710 + (7565)(0.365) \\
= 354710 + 2690 \\
= 357400 \text{ ft}
If missile flies in a constant gravitational field then

\[
\frac{Md^2h}{dt^2} = - Mg
\]

or

\[
\frac{d^2h}{dt^2} = - g
\]

\[
h = -\frac{gt^2}{2} + c_1t + c_2
\]

at \( t = 0 \)

\[
h = 19445 \quad h = 357400
\]

\[
h = -\frac{gt^2}{2} + 19445t + 357400
\]

\[
\dot{h} = - gt + 19445 = 0
\]

\[
t_{\text{summit}} = \frac{19445}{32.2} = 604.5 \text{ sec}
\]

\[
h_{\text{summit}} = - (161)(604.5)^2 + (19445)(604.5) + 357400
\]

\[
= - 5875000 + 11740000 + 357400
\]

\[
= 5865000 + 357400
\]

\[
= 6222400 \text{ ft}
\]

\[
= 1180 \text{ miles}
\]

Calculate hydraulic head

\[
H = \text{hydraulic head} = (L - \dot{\theta}t)\rho_p(a + 1)
\]

\[
\dot{\theta} = \frac{\dot{V}}{\pi\delta^2} = \frac{6.15}{\pi(1.59)^2}
\]

\[
= 0.775 \text{ ft/sec}
\]

\[0 \leq t < 5\]

\[
H = \frac{(47.7)(62.4)(2.89)}{144}
\]

\[
= 59.7 \text{ psi}
\]
\[ 5 \leq t < 10 \]
\[ H = \frac{(47.7 - 3.875)(62.4)(3.14)}{144} \]
\[ = \frac{(43.825)(62.4)(3.14)}{144} \]
\[ = 60.4 \text{ psi} \]

\[ 10 \leq t < 15 \]
\[ H = \frac{(43.825 - 3.875)(62.4)(3.47)}{144} \]
\[ = \frac{(39.950)(62.4)(3.47)}{144} \]
\[ = 60.0 \text{ psi} \]

\[ 15 \leq t < 20 \]
\[ H = \frac{(39.950 - 3.875)(62.4)(3.83)}{144} \]
\[ = \frac{(36.075)(62.4)(3.83)}{144} \]
\[ = 59.8 \text{ psi} \]

\[ 20 \leq t < 25 \]
\[ H = \frac{(36.075 - 3.875)(62.4)(4.34)}{144} \]
\[ = \frac{(32.2)(62.4)(4.34)}{144} \]
\[ = 60.5 \text{ psi} \]
\[
25 \leq t < 30
\]
\[
H = \frac{(32.300 - 3.875)(62.4)(5.03)}{144}
\]
\[
= \frac{(28.325)(62.4)(5.03)}{144}
\]
\[
= 61.6 \text{ psi}
\]

\[
30 \leq t < 35
\]
\[
H = \frac{(28.325 - 3.875)(62.4)(5.98)}{144}
\]
\[
= \frac{(24.45)(62.4)(5.98)}{144}
\]
\[
= 63.3 \text{ psi}
\]

\[
35 \leq t < 40
\]
\[
H = \frac{(24.450 - 3.875)(62.4)(7.29)}{144}
\]
\[
= \frac{(20.575)(62.4)(7.29)}{144}
\]
\[
= 65.0 \text{ psi}
\]

\[
40 \leq t < 45
\]
\[
H = \frac{(20.575 - 3.875)(62.4)(8.99)}{144}
\]
\[
= \frac{(16.7)(62.4)(8.99)}{144}
\]
\[
= 65.1 \text{ psi}
\]
45 = t

\[ H = \frac{(13.15)(62.4)(11.48)}{144} \]

= 65.4 psi

For t > 45 sec the hydraulic head was a necessary part of calculating the chamber pressure curve. The values listed below are from those calculations:

<table>
<thead>
<tr>
<th>time, sec</th>
<th>Hydraulic head, psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>46.25</td>
<td>60.9</td>
</tr>
<tr>
<td>48.75</td>
<td>55.9</td>
</tr>
<tr>
<td>51.25</td>
<td>50.7</td>
</tr>
<tr>
<td>53.75</td>
<td>45.4</td>
</tr>
<tr>
<td>56.25</td>
<td>39.8</td>
</tr>
<tr>
<td>58.75</td>
<td>33.0</td>
</tr>
<tr>
<td>61.25</td>
<td>24.9</td>
</tr>
<tr>
<td>53.75</td>
<td>14.1</td>
</tr>
<tr>
<td>67.26</td>
<td>0</td>
</tr>
</tbody>
</table>

Calculate accelerations

\[ F-D-W = \frac{W}{g} a \]

\[ a = \left[ \frac{F-D}{W} - 1 \right] g \]

\[ t = 0 \text{ to } 5 \]

\[ a = \left( \frac{72290}{25000} - 1 \right) g \]

= (2.89 - 1)g

= 1.89 g
\[ t = 5 \text{ to } 10 \]
\[ a = \left( \frac{72528}{23085} - 1 \right) g \]
\[ = (3.14 - 1) g \]
\[ = 2.14 g \]

\[ t = 10 \text{ to } 15 \]
\[ a = \left( \frac{73505}{21170} - 1 \right) g \]
\[ = (3.47 - 1) g \]
\[ = 2.47 g \]

\[ t = 15 \text{ to } 20 \]
\[ a = \left( \frac{73874}{19255} - 1 \right) g \]
\[ = (3.83 - 1) g \]
\[ = 2.83 g \]

\[ t = 20 \text{ to } 25 \]
\[ a = \left( \frac{75240}{17340} - 1 \right) g \]
\[ = (4.34 - 1) g \]
\[ = 3.34 g \]

\[ t = 25 \text{ to } 30 \]
\[ a = \left( \frac{77600}{15425} - 1 \right) g \]
\[ = (5.03 - 1) g \]
\[ = 4.03 g \]
\[ t = 30 \text{ to } 35 \]
\[ a = \left( \frac{80690}{13510} - 1 \right) g \]
\[ = (5.98 - 1) g \]
\[ = 4.98 \text{ g} \]

\[ t = 35 \text{ to } 40 \]
\[ a = \left( \frac{84380}{11595} - 1 \right) g \]
\[ = (7.22 - 1) g \]
\[ = 6.22 \text{ g} \]

\[ t = 40 \text{ to } 45 \]
\[ a = \left( \frac{86910}{9680} - 1 \right) g \]
\[ = (8.99 - 1) g \]
\[ = 7.99 \text{ g} \]

\[ t = 45 \]
\[ a = \left( \frac{89100}{7765} - 1 \right) g \]
\[ = (11.48 - 1) g \]
\[ = 10.48 \text{ g} \]

\[ t = 50 \]
\[ a = \left( \frac{78360}{6001} - 1 \right) g \]
\[ = (13.05 - 1) g \]
\[ = 12.05 \text{ g} \]
t = 55
\[ a = \left( \frac{71040}{4414} - 1 \right) g \]
\[ = (16.1 - 1) g \]
\[ = 15.1 \, g \]

\[ t = 60 \]
\[ a = \left( \frac{63000}{2994} - 1 \right) g \]
\[ = (21.05 - 1) g \]
\[ = 20.05 \, g \]

\[ t = 65.0 \]
\[ a = \left( \frac{53400}{1755} - 1 \right) g \]
\[ = (30.4 - 1) g \]
\[ = 29.4 \, g \]

\[ t = 67.26 \]
\[ a = \left( \frac{49980}{1250} - 1 \right) g \]
\[ = (40.0 - 1) g \]
\[ = 39.0 \, g \]

Calculate Helium Data

Let the inside diameter of the helium tank equal the inside diameter of the propellant tank.

Volume = \[ \frac{4}{3} \pi r^3 = \frac{(4\pi)(1.59)^3}{3} = 16.83 \, ft^3 \]

Use ideal gas law for helium calculations

\[ PV = mRT \quad \text{(use standard temperature)} \]
\[ m = \frac{(3500)(144)(16.83)}{(1545)(518)} = 42.4 \text{ lbs He} \]

Find the required He pressure at the top of the propellant

25\% drop across the injector face

\[ X - 0.25X = 150 \]
\[ X = \frac{4}{3} \cdot 150 = 200 \text{ psia} \]

25\% drop across the plumbing

\[ X - 0.25X = 200 \]
\[ X = \frac{4}{3} \cdot 200 = 266.6 \text{ psia} \]

hydraulic head just before lift off = \( \frac{(47.7)(62.4)}{144} \)

\[ = 20.6 \text{ psi} \]

throttle the helium so that its pressure at the top of the propellant is 246 psia. Find the time at which the pressure of the helium throughout the system is 246 psia.

\[ \frac{\circ V}{\text{sec}} = \frac{\circ mg}{\rho_p} \]
\[ = \frac{(11.9)(32.2)}{62.4} \]
\[ = 6.15 \text{ ft}^3/\text{sec} \]

\[ P_1V_1 = m\rho_T = P_2V_2 \]
\[ V_2 = \frac{(3500)(16.83)}{246} = 239.6 \text{ ft}^3 \]

Vol of He in tank = 239.6 - 16.8 (assume volume of helium pipes negligible)

\[ = 222.8 \]
\[ t_1 = \frac{222.8}{6.15} = 36.3 \text{ sec} \]
Calculate BTU's necessary to make process isothermal

Starting with the first law
\[ \delta q = dU + PdV \]
\[ = mc_v dT + PdV \quad \text{since we assume helium is an ideal gas} \]

Then \( dT = 0 \) so
\[ \delta q = PdV \]

But
\[ P = \frac{RT}{V} \]

so
\[ \ln V_2 - \ln V_1 = \int_{V_1}^{V_2} \frac{RT}{V} \, dV \]
\[ = \frac{1544 \cdot (518)}{4 \cdot 778} \ln \frac{397.83}{16.83} \]
\[ = 814.5 \text{ BTU/lb He} \]
\[ = 34,450 \text{ BTU Total} \]

Calculate head suppression
\[ \Delta P = \text{head suppression} = P_{\text{He}} + H - \frac{16}{9} P_c \]

\[ 0 \leq t < 5 \]
\[ \Delta P = 246 + 59.7 - 266.7 \]
\[ = 39 \text{ psi} \]

\[ 5 \leq t < 10 \]
\[ \Delta P = 246 + 60.4 - 266.7 \]
\[ = 39.1 \text{ psi} \]

\[ 10 \leq t < 15 \]
\[ \Delta P = 246 + 60.0 - 266.7 \]
\[ = 39.3 \text{ psi} \]
$15 \leq t < 20$

\[
\Delta P = 246 + 59.8 - 266.7 = 39.1 \text{ psi}
\]

$20 \leq t < 25$

\[
\Delta P = 246 + 60.5 - 266.7 = 39.8 \text{ psi}
\]

$25 \leq t < 30$

\[
\Delta P = 246 + 61.6 - 266.7 = 40.9 \text{ psi}
\]

$30 \leq t < 35$

\[
\Delta P = 246 + 63.3 - 266.7 = 42.6 \text{ psi}
\]

$t = 35 - 40$

\[
\Delta P = 246 + 65.0 - 266.7 = 44.3 \text{ psi}
\]

At $t = 36.3$ the pressure throughout the helium system is 246 psia. For each time interval ($t > 36.3$) the helium pressure must be calculated by

\[
P_1 V_1 = P_2 V_2
\]

\[
P_{\text{He}} = \frac{P_1 V_1}{V_1 + V_t}
\]

Where

$P_1 =$ initial He pressure

$V_1 =$ initial He volume

$V =$ time rate of change of He volume

$t =$ time
\[ t = 40 \]

\[ P_{He} = \frac{(3500)(16.83)}{16.83 + (6.15)(40)} \]

\[ = \frac{(3500)(16.83)}{262.8} \]

\[ = 233 \text{ psia} \]

\[ \Delta P = 233 + 65.1 - 266.7 \]

\[ = 31.4 \text{ psi} \]

Find time that chamber pressure begins to decrease. The condition for this is:

\[ P_{He} + P_h - \frac{16}{9} P_c = \Delta P = 0 \]

\[ \frac{(3500)(16.83)}{16.83 + 6.15t} + 65.1 = 266.7 \]

\[ 58900 = (16.83 + 6.15t)(201.6) \]

\[ = 3390 + 1240t \]

\[ t = \frac{55510}{1240} \]

\[ = 44.8 \text{ seconds} \]

Since this is very close to 45 seconds consider for calculations that the chamber pressure starts to decrease at 45 seconds.

**Calculate amount of Fiberglas needed**

From the head suppression graph it is seen that the maximum head suppression is 44.6 psi. Therefore, the maximum pressure at the bottom of the tank is 266.6 + 44.6 psia

Maximum pressure = 311.2 psia
\[ F = (311.2)(144)(3.18)(1) \]
\[ = 142,300 \text{ lb/ft} \]

Working Stress = 80,000 lb/in²

Neglect atmospheric pressure in calculating wall thickness. This gives an additional safety factor.

\[ t = \frac{142,300}{(2)(80,000)(12)} \]
\[ = .0741 \text{ inch} \]

\[ V_T = \pi r^2 h + \frac{2}{3} \pi r^3 \]

\[ \Delta V = 2\pi rh\Delta r + 2\pi r^2 \Delta r \]

or

\[ \text{Vol Fiberglas} = (\pi Dh + 2\pi r^2)\Delta r \]
\[ = \pi(Dh + 2r^2)t \]
\[ = \pi[(3.18)(47.7) + (2)(1.59)^2] \frac{0.0741}{12} \]
\[ = \frac{(151.7 + 5.05)(0.0741)\pi}{12} \]
\[
\begin{align*}
\text{Volume} & = \frac{(156.75)(0.0741)\pi}{12} \\
& = 3.04 \text{ ft}^3
\end{align*}
\]

**Helium Tank**

\[F = (3500)(144)(1.59)^2\pi\]
\[= 4,000,000 \text{ lb}\]
\[A = \pi r^2\]
\[dA = 2\pi r dr\]
\[\Delta A = \pi D \Delta r = \pi Dt\]
\[t = \frac{(4,000,000)(12)}{(144)(80,000)(3.18)\pi}\]
\[= (348)(12)\]
\[= 0.417 \text{ inches}\]

\[\Delta V = 4\pi r^2 t\]
\[ V = \frac{(4\pi)(1.59)^2(0.417)}{12} \]
\[ = 1.106 \text{ ft}^3 \]

Total volume of fiberglass = 3.04 + 1.106
\[ = 4.146 \]

Wt. fiberglass = (4.146)(1.8)(62.4)
\[ = 465.0 \text{ lbs.} \]

Wt. engine = 89,350 \textnormal{ lb}
\[ = 223.37 \text{ lb} \]

Total Wt. used

Wt. Helium = 42.4
Wt. fiberglass = 465.0
Wt. engine = 223.4
Total Wt. used = 730.8 lb.

Wt. left over for payload, plumbing, telemetry, etc.
\[ = 519.2 \text{ lb} \]

Payload from problem statement = 125.0 lb.

Wt. left over for plumbing, telemetry, etc.
\[ = 394.2 \text{ lb} \]
The following performance curves and data sheet present a summary of the calculations for the liquid rocket motor system.
Figure 1. Rocket Motor Thrust as a Function of Altitude.
Figure 2. Dynamic Pressure Versus Altitude.
Figure 3. Drag Versus Time
Figure 4. Mach Number as a Function of Time.
Figure 5. Rocket Chamber Pressure as a Function of Time.
Figure 6. Rocket Motor Thrust Versus Time Until Burnout.
Figure 8. Hydraulic Head Versus Time.
Figure 9. Pressure Drop Across the Head Suppression Valve as a Function of Time.
Figure 10. Vehicle Velocity Versus Time During Powered Phase.
Figure 11. Altitude Versus Time During Powered Phase.
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<th>Time (Sec)</th>
<th>Thrust (lb)</th>
<th>Dynamic Pressure (lb/ft^2)</th>
<th>Drag (lb)</th>
<th>Acceleration (ft/sec^2)</th>
<th>Band Suppression (Psi)</th>
<th>Chamber Pressure (Psi)</th>
<th>Static Cond. Band (Psi)</th>
<th>Velocity (ft/sec)</th>
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Summit Altitude is 1800 Miles.
SOLID ROCKET DESIGN

Theodore Petersen
NOMENCLATURE

\( \dot{m} \) - mass flow

\( V_{EX} \) - velocity at exit of rocket nozzle

\( M \) - Mach number

\( T \) - temperature

c - denotes chamber conditions

\( P \) - pressure

\( a \) - speed of sound

\( \gamma \) - ratio of specific heats

\( A \) - area

\( t \) - denotes throat conditions

\( \text{Ex} \) - denotes the conditions at the exit of the rocket nozzle

\( F \) - thrust

\( \rho \) - density

\( n \) - denotes exponent in burning rate law

\( r \) - burning rate

\( q \) - dynamic head

\( \Gamma = \gamma^{\frac{1}{2}} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \)

\( R \) - gas constant
DESIGN SPECIFICATIONS

Assume propellant chamber with internal dimensions as shown:

\[ \begin{array}{c}
\text{3'} \\
\text{40'}
\end{array} \]

Propellant: \( \text{NH}_4\text{NO}_3/\text{C}_2\text{H}_4\text{O}/\text{CATALYST} \)

Properties:
- Adiabatic Flame Temperature = 2700°F
- Average Molecular Weight = 22 lb./mol.
- \( \gamma \) = 1.26
- Typical Sea Level \( I_{sp} \) = 195 Sec.
- Characteristic Velocity = 4000 ft./sec.
- \( r \) @ \( P_c = 1000 \text{ PSI} \) \( 70 \text{ °F} \) = 0.1 in./sec.
- \( r \) Exponent - \( n \) = 0.4
- Specific Weight = 0.056 lb./in.\(^3\)
- Lower Combustion Limit < 100 PSI
- Pressure Limit > 3000 PSI

Specifications:

General:
- Fill up to Base of Nozzle
- Consider Star Grain
- Burning time 20 \( \rightarrow \) 60 seconds
- Payload = 125 lb.
- "Fiberglas" Casing - 80,000 PSI Design Stress
- Protective Heat Material - .050 in. of Gunk at same Sp. Gr. as "Fiberglas."

Motor Weight:
- 70 lb. Thrust
- Lb. motor wt.

Leave out Volume Increase with Respect to Time but Comment on its Effect.

Estimate Velocity at Back End of Grain.

Size Nozzle so \( P_{EX} = 1/2 P_{ATM} \) at Sea Level so Don't Need to Worry About Separation.

Maximum Pressure Fluctuation = 50%

Junk Weight = 150 lb.

Lift Off at 4g.
Weight Estimate:

Assume a loading fraction \( \epsilon = \frac{\text{Total Cross-Sect. Area of Propellant}}{\text{Cross-Sectional Area of Motor}} = .80 \) (Preliminary Assumption)

Volume of propellant charge: \((.80) \cdot \frac{\pi D^2}{4} (40) = 226 \text{ ft.}^3\)

Weight of charge: \((226)(.056)(1728) = 21,850 \text{ lb.}\)

Weight of fiberglas casing: 36 in. x 1000 PSI = 36,000 lb.

Working stress = 80,000 PSI.

So the thickness is:

\[
\frac{36,000}{80,000 \times 2} = .225 \text{ in. thick}
\]

The volume of fiberglas for the walls of the chamber is then given by

\[
(\pi)(3)(40)(.225)(1/12) = 7.07 \text{ ft.}^3
\]

Estimate the same thickness of fiberglas for the top of the chamber and roughly assume 1 ft. of lap joint:

\[
\pi \frac{3^2}{4} + \pi (3)(1) = 3\pi(1 + \frac{3}{4})
\]

Volume = \(3\pi(1.75)(.225)(1/12) = .309 \text{ ft.}^3\)

Total volume = 7.379 ft.\(^3\)

Weight = \(7.379 \times 62.4 \times 1.8 = 828 \text{ lb.}\)

Weight of insulating material:

Volume of insulating material:

\[
(\pi)(3)(40)(.050)(1/12) + (\pi)(9/4)(.050)(1/12)
\]

= 1.6 ft.\(^3\)

Weight = \((1.6) \times (62.4) \times (1.8) = 179.8 \text{ lb.}\)

Payload weight = 125 lb.

Junk weight = 150 lb.

Total weight - Motor weight = 23,133 lb.

Assume motor weight = 1400 lb.

Total Weight = 24,533 lb.

For 4g lift-off thrust - thrust = 98,132 lb. - which corresponds to an engine weight of 1400 lb.
NOZZLE SIZING

It was specified to size the nozzle so that \( P_{\text{EX}} = \frac{1}{2} P_{\text{ATM}} \). This specification, along with a knowledge of the chamber properties and the desired thrust at sea level, enables the nozzle to be sized using isentropic relations and the basic thrust equation; the relations to be used are:

1. \( F = \frac{0}{m} V_{\text{EX}} + (P_{\text{EX}} - P_{\text{ATM}}) A_{\text{EX}} \) (Assuming Ideal Nozzle)

2. \( \frac{0}{m} = \frac{\gamma M}{a} \)

3. \( \frac{T_{\text{c}}}{T} = (1 + \frac{\gamma - 1}{2} M^2) \)

4. \( \frac{P_{\text{c}}}{P} = (1 + \frac{\gamma - 1}{2} M^2)^{\gamma/\gamma-1} \)

5. \( \frac{A}{A_{\text{t}}} = \frac{1}{M} \left[ \left( \frac{2}{\gamma - 1} \right)^{1 + \frac{\gamma - 1}{2} M^2} \right]^{1/2} \)

6. \( V_{\text{EX}} = M_{\text{EX}} a_{\text{EX}} = M_{\text{EX}} \sqrt{\gamma a R T_{\text{EX}}} \)

Now, substituting (2) into (1):

7. \( F = \frac{\gamma M A P_{\text{c}}}{a_{\text{t}}} V_{\text{EX}} + (P_{\text{EX}} - P_{\text{ATM}}) A_{\text{EX}} \)

and substituting in (6) and cancelling constant terms:

8. \( F = \gamma M A P_{\text{c}} M_{\text{EX}} (T_{\text{EX}} / T_{\text{c}})^{1/2} + (P_{\text{EX}} - P_{\text{ATM}}) A_{\text{EX}} \)

and, from the isentropic relationships:

Hence,

9. \( \left( \frac{T_{\text{EX}}}{T_{\text{c}}} \right)^{1/2} = \left[ \left( \frac{\gamma + 1}{\gamma} \right)^{\gamma/\gamma-1} \frac{P_{\text{EX}}}{P_{\text{c}}} \right]^{-1/2} \gamma' \)

Also,

10. \( P_{\text{c}} = \left( \frac{2}{\gamma + 1} \right)^{\gamma/\gamma-1} P_{\text{c}} \)

and,

\( \frac{P_{\text{c}}}{P_{\text{EX}}} = \left( 1 + \frac{\gamma - 1}{2} M_{\text{EX}}^2 \right)^{\gamma/\gamma-1} \)

\( M_{\text{EX}}^2 = \left( \frac{2}{\gamma - 1} \right) \left[ \left( \frac{P_{\text{c}}}{P_{\text{EX}}} \right)^{\gamma/\gamma-1} - 1 \right] \)

11. \( M_{\text{EX}} = \left\{ \frac{2}{\gamma - 1} \left[ \left( \frac{P_{\text{c}}}{P_{\text{EX}}} \right)^{\gamma/\gamma-1} - 1 \right] \right\}^{1/2} \)
Now, divide (8) by \( A_t \):

\[
\frac{F}{A_t} = \gamma P_t M_{EX} \left( \frac{T_{EX}}{T_t} \right)^{1/2} + (P_{EX} - P_{ATM}) A_{EX}/A_t
\]

and the terms on the right of (12) can be evaluated with the known conditions:

\[
\begin{align*}
P_c &= 1000 \text{ PSI} & T_c &= 3160^\circ \text{R} \\
P_{EX} &= 7.35 \text{ PSI} & \gamma &= 1.26
\end{align*}
\]

From (10):

\[
P_t = (0.5532)(1000) = 553.2 \text{ PSI}
\]

From (11):

\[
M_{EX} = \left\{ \frac{7.69 \left[ (136.1)^{2064} - 1 \right]}{2.76 - 1} \right\}^{1/2}
= \left\{ 7.69 \left[ 2.76 - 1 \right] \right\}^{1/2} = 3.678
\]

From (9):

\[
\left( \frac{T_{EX}}{T_t} \right)^{1/2} = \left[ (1.13)^{4.85} \cdot (0.00735) \right]^{1031} = 0.64
\]

From (5):

\[
\frac{A_{EX}}{A_t} = \frac{1}{3.678} \left[ (0.885)(2.758) \right]^{4.345}
= 13.05
\]

Then, from (12)

\[
\frac{98.132}{A_t} = (1.26)(553.2)(3.678)(0.64) - (7.35)(13.05)
98.132 = (1638 + 96) A_t
A_t = 63.6 \text{ Sq. In.}
\]

and now the mass flow can be computed:

\[
\dot{m} = \frac{\gamma M A P}{a_t} = \frac{(1.26)(63.6)(553.2)}{[(1.26)(32.2)(1544)(2160)]^{1/2}}
= 15.7 \text{ slugs/sec.}
\]
Figure 1. Wagon Wheel Grain Configuration.
DASHED LINE REPRESENTS BURNING SURFACE AFTER PROPELLANT HAS BURNED DISTANCE X.
\[ \beta - \theta = \sin^{-1}(\frac{w}{(\frac{D}{2} - w)}) \]

Figure 3.
GRAIN SIZING

Now, in general, the basic differential equation governing continuity for solid propellant rockets is:

\[ A_c r \rho_P \frac{d}{dt} (\rho_c V_c) + \frac{\gamma M_t A_t P_t}{Q_t} \]

- Propellant Mass Burned
- Storage Term
- Mass Flow through Nozzle

and assuming that the storage term is negligible, this equation becomes:

\[ A_c r \rho_P = \frac{\gamma M_t A_t P_t}{a_t} \]

and from the propellant properties:

\[ \rho_P = (0.056)(1728) = 96.8 \text{ lb./ft.}^3 \]
\[ r = a \frac{n}{c} = a(1000)^{\frac{14}{7}} = a(15.9) = 0.1 \text{ in./sec.} = 0.00833 \text{ ft./sec.} \]
\[ a = \frac{1}{15.9} = 0.0629 \]

so,

\[ A_c = \frac{(15.7)(32.2)}{(0.00833)(96.8)} = 626 \text{ ft.}^2 \]

and, as the chamber is 40 ft. long the burning perimeter must be 15.66 ft.

DERIVATION OF EXPRESSIONS RELATING PACKING FRACTION AND BURNING PERIMETER TO CONFIGURATION DIMENSIONS

Burning Perimeter:

The burning perimeter may be found by considering the half spoke shown in the following sketch and multiplying the result by 10.

The perimeter is made up of three portions: The arc subtended by the angle \( \theta \), the line of the spoke inclined at the angle \( \beta \) to the
center-line and the vertical line of length \( m \). The burning perimeter and loading fraction will be analytically determined.

The burning perimeter is initially given by the following relationship:

\[
B_P \cdot_1 = 10(S_1 + S_2 + S_3)
\]

where:

\[
S_1 = \left( \frac{D}{2} - W \right) \cdot \theta
\]

\[
S_2 = L - \frac{Z}{\sin \beta} = \frac{\left( \frac{D}{2} - W - \frac{W}{\sin \beta} \cdot \cos \theta \right)}{\cos (\beta - \theta)} - \frac{Z}{\sin \beta}
\]

\[
S_3 = m = \frac{W}{\sin \beta}
\]

Now, the cross-sectional area of the propellant can be found. This will be done by subtracting the pie-shaped segments from the total area - then adding back in the triangles of base \( m \) and altitude \( L \cdot \sin \beta \) - and finally subtracting out the parallelograms of base \( m \) and altitude \( Z \). Hence the cross-sectional area of the propellant is given by:

\[
A_1 = \frac{\pi D^2}{4} - 5\left( \frac{D}{2} - W \right)^2 \cdot \theta + 5 \frac{W}{\sin \beta} \frac{\left( \frac{D}{2} - W - \frac{W}{\sin \beta} \cdot \cos \theta \right)}{\cos (\beta - \theta)} - 10 mZ
\]

Now, after the propellant has burned a distance \( x \) the burning perimeter is:

\[
B_P \cdot x = 10 \left( S_1 + S_2 + S_3 + S_4 \right)
\]

where:

\[
S_1 = \left( \frac{D}{2} - [W-x] \right) \cdot \theta
\]

\[
S_2 = x \left( \frac{X}{2} - \theta + \beta \right)
\]

\[
S_3 = \frac{\left( \frac{D}{2} - W - \frac{W}{\sin \beta} \cdot \cos \theta \right)}{\cos (\beta - \theta)} - \frac{Z}{\sin \beta} - x \tan \beta/2
\]

\[
S_4 = m - \frac{x}{\sin \beta}
\]
and the cross-sectional area of the propellant after it has burned a distance $x$ can be found as:

$$A_x = A_1 - 5 \left\{ \left( \frac{D}{2} - W + x \right)^2 - \left( \frac{D}{2} - W \right)^2 \right\} \theta - 5 x^2 \left( \frac{x}{2} - \theta + \beta \right)$$

$$- 10 m x - \frac{10 x}{\tan \beta} \cdot \frac{x}{2} - 10 \left( \frac{\frac{D}{2} - W - \frac{W}{\sin \beta} \cdot \cos \theta}{\cos (\beta - \theta)} \right) - \frac{Z}{\sin \beta} - \frac{x}{\tan \beta}$$

DISCUSSION OF CHOICE OF GRAIN CONFIGURATION

A sketch of the grain configuration chosen is shown on Page 60. Preliminary calculations were carried out for several star configurations but these all showed that it would be extremely difficult to meet the 4g lift-off specification for any decent loading factor without a highly regressive burning configuration resulting. It was therefore decided not to consider a star (internal) grain configuration any further. A cylindrical grain was briefly considered as it has the advantage of a uniform burning surface but it was immediately obvious that it would not be possible to meet the 4g lift-off requirement with any sort of loading fraction that was acceptable. Also, as it was specified in class not to use this configuration, its consideration was dropped.

It was then decided to try the wagon wheel configuration, and a five spoke wagon wheel, as shown in the sketch on Page 60, was decided upon as it has only slightly progressive burning as can be seen by the pressure trace on Page 61. An analytical procedure was then determined to size the configuration for a 4g lift-off and an analytic method of determining the burning surface as a function of $x$, the distance burned, was derived.
The actual design would differ slightly from the sketch on Page 60 in that the acute angles of the open or port area should be replaced by small fillets to prevent cracking during storing and handling. Also, although theoretically none are needed, there should be some sort of an inhibitor or structural member (i.e., a fine screen) placed along the center line between the spokes to guard against chunks of the propellant being carried downstream to the nozzle as a result of uneven burning -- i.e., see sketch:

The analytic solution of the optimum grain configuration follows:

**SELECTION OF CORRECT GRAIN CONFIGURATION**

It is first necessary to know the relationship between burning perimeter and loading fraction \( e \) for a 4g lift-off. This relationship may be plotted by proceeding in a manner similar to that used in the preliminary calculations from the calculations on Page 62.

\[
A_t = \frac{F}{1542} \text{ in.}^2
\]

and from the calculations on Page 63:

\[
A_c = \frac{m}{r_0 p_p} = .0064 F \text{ ft.}^2 \text{ when } F \text{ is in units of pounds.}
\]
and to determine the trust necessary for a given loading factor it can be seen from the calculations on Pages 57 and 58 that:

\[
\text{Total weight} = T.W. = 1283 + (27350)\epsilon + \text{motor weight} \\
\text{Motor weight} = F/70; F = 4 \times T.W.
\]

and the values of thrust and loading factor that satisfy these relationships are given below along with the corresponding values of \(A_c\) and burning perimeter:

for \(\epsilon = .60\):

- Total weight = \(18753\) lb.
- Thrust = \(75,012\) lb.
- \(A_c = 460\) ft²
- B. P. = \(12\) ft.

for \(\epsilon = .70\):

- Total weight = \(21648\) lb.
- Thrust = \(86,600\) lb.
- \(A_c = 554\) ft²
- B. P. = \(13.85\) ft.

for \(\epsilon = .75\):

- Total weight = \(23103\) lb.
- thrust = \(92,412\) lb.
- \(A_c = 591\) ft²
- B. P. = \(14.79\) ft.

for \(\epsilon = .80\):

- Total weight = \(24533\) lb.
- Thrust = \(98,132\) lb.
- \(A_c = 627\) ft²
- B. P. = \(15.67\) ft.

for \(\epsilon = .85\):

- Total weight = \(25983\) lb.
- Thrust = \(103932\) lb.
- \(A_c = 665\) ft²
- B. P. = \(16.63\) ft.

**GRAIN CHARACTERISTICS**

Consider a grain with a four inch web and five spokes (the five spoke configuration was chosen as it gives the most neutral progressivity ratio). Then:

\[
W = 4"; \frac{D}{2} = 18; \beta = 36^\circ = .628 \text{ rad.}; \sin \beta = .588; \theta = 36^\circ - \sin^{-1} \frac{4}{14} \\
= 36.0^\circ - 16.6^\circ = 19.4^\circ = .3385 \text{ rad.} \\
\cos \theta = .943; \cos(\beta - \theta) = \cos 16.6^\circ = .959
\]
then, \[ S_1 = (14)(0.3385) = 4.74 \text{ in.} \]
\[ S_2 = \frac{(14 - 6.42)}{0.959} - 1.7z = 7.91 - 1.7z \]
\[ S_3 = 6.80 \text{ in.} \]

Therefore,
\[ B.P._1 = \frac{10}{12} (19.45 - 1.7z) \text{ ft.} \]

and the cross-sectional area is:
\[ A_1 = 1018 - 5 \times 196 \times 0.3385 + 5 \times 6.8 \times 7.91 \times 0.588 - 10 \times 6.8z \]
\[ = 1018 - 332 + 158 - 68z = 844 - 68z \]

so the loading factor is given by:
\[ \epsilon_1 = \frac{844 - 68z}{1018} \]

Then, for \( z = 0 \), \( B.P._1 = 16.20 \text{ ft.} \), \( \epsilon_1 = 0.829 \)
for \( z = 1 \), \( B.P._1 = 14.8 \text{ ft.} \), \( \epsilon_1 = 0.761 \)
for \( z = 2 \), \( B.P._1 = 13.37 \text{ ft.} \), \( \epsilon_1 = 0.695 \)

\[ W = 3.9" \], \[ \frac{D}{2} = 18; \beta = 36^\circ; \theta = 36^\circ - \sin^{-1} \frac{3.9}{14.1} = 36^\circ - 16.05^\circ = 19.95^\circ \]
\[ \cos \theta = 0.940; \cos (\beta-\theta) = 0.961 \]

then,
\[ S_1 = (14.1)(0.3475) = 4.90 \text{ in.} \]
\[ S_2 = \frac{(14.1 - 6.24)}{0.961} - 1.7z = 8.18 - 1.7z \]
\[ S_3 = 6.64 \text{ in.} \]

Therefore,
\[ B.P._1 = \frac{10}{12} (19.72 - 1.7z) \]

and the cross-sectional area is:
\[ A_1 = 1018 - 5 \times 198.8 \times 0.3475 + 5 \times 6.64 \times 8.18 \times 0.588 - 10 \times 6.64z \]
\[ = 1018 - 346 + 159.5 - 66.4z = 832 - 66.4z \]
so, the loading factor is given by:

\[ \epsilon_1 = \frac{832 - 66.4z}{1018} \]

Then, for \( z = 0 \) \( B.P. = 16.42 \text{ ft.}, \) \( \epsilon_1 = .817 \)

for \( z = 1 \) \( B.P. = 15 \text{ ft.}, \) \( \epsilon_1 = .752 \)

for \( z = 2 \) \( B.P. = 13.6 \text{ ft.}, \) \( \epsilon_1 = .687 \)

\[ W = 3.95", \frac{D}{2} = 18, \beta = 36^\circ, \theta = 36^\circ - \sin^{-1} \frac{3.95}{14.05} = 36^\circ - 16.31^\circ = 19.69^\circ \]

\[ \cos \theta = .942, \cos (\beta - \theta) = .960 \]

then,

\[ S_1 = (14.05)(.3435) = 4.83 \text{ in.} \]

\[ S_2 = \frac{(14.05 - 6.32)}{.960} - 1.7z = 8.06 - 1.7z \]

\[ S_3 = 6.71 \text{ in.} \]

Therefore,

\[ B.P. = \frac{10}{12} (19.6 - 1.7z) \]

and the cross-sectional area is:

\[ A_1 = 1018 - 5 \times 197.3 \times .3435 + 5 \times 6.71 \times 8.06 \times .588 - 10 \times 6.71z \]

\[ = 1018 - 339 + 159 - 67.1z = 838 - 67.1z \]

so the loading factor is given by:

\[ \epsilon_1 = \frac{838 - 67.1z}{1018} \]

Then, for \( z = 0 \) \( B.P. = 16.32 \text{ ft.}, \) \( \epsilon_1 = .823 \)

for \( z = 1 \) \( B.P. = 14.91 \text{ ft.}, \) \( \epsilon_1 = .757 \)

for \( z = 2 \) \( B.P. = 13.5 \text{ ft.}, \) \( \epsilon_1 = .691 \)

**DISCUSSION OF INITIAL CONFIGURATION DESIGN PLOT**

The plot on the preceding page was used to analytically size the wagon wheel configuration for a 4g lift-off thrust. It should be noted that the configuration with \( W = 3.95" \) and \( z = 1" \) is a design point and this was the
design point chosen for the configuration as it was felt that \( z = 1'' \) was a reasonable value to keep the internal velocity at a reasonable magnitude and still maintain a fairly high loading factor.

This plot could be extended to cover various g take-off conditions and more values of \( W \) so as to make a complete design chart for a given size propellant chamber and the five spoke wagon wheel configuration.

It should also be noted from the plot that the value of \( W = 3.95'' \) is a very good choice for a \( 4g \) lift-off as it comes very close to giving this for a range of values of \( \epsilon \) and \( z \).

**DETERMINATION OF BURNING PERIMETER AS FUNCTION OF TIME**

Using the five spoke wagon wheel with a web of \( 3.95'' \) and \( z = 1'' \) the burning perimeter as a function of \( x \) is:

\[
B.P. \times = \frac{10}{12} \left\{ (14.05 + x)(.3435) + x\left(\frac{7}{2} - .3435 + .628\right) + (6.36 - x[.3249]) + (6.71) - 1.7x \right\}
\]

\[
= \frac{10}{12} \left\{ 4.825 + .3435x + 1.8553x + 6.36 - .3249x + 6.71 - 1.7x \right\}
\]

\[
= \frac{10}{12} \left\{ 17.9 + .1739x \right\} \text{ ft.}
\]

Now, for \( \epsilon = .757 \) the total propellant weight is 20680 lb. and the mass flow through the nozzle is given as:

\[
\frac{\gamma M_t A_t P_t}{a_t} = \frac{A_t P_t}{a_t} = \frac{\gamma (\frac{2}{\gamma + 1}) \gamma - 1}{a_t} = \frac{\gamma (\frac{2}{\gamma + 1})^{1/2} A_t P_t}{a_c}
\]

the rate of generation of mass is:

\[
A_c^\epsilon \rho_P = B.P. \ (40) \ a_c^{\epsilon} \rho_P
\]

and these must be equal so

\[
(B.P.)^{(40)}(.00629)P_c^{\epsilon} (96.8)(1/12) = \frac{.7408 A_t P_c (144)}{\sqrt{(1.26)(3.2)(1544)}(3160)}
\]
at \( t = 0 \), B.P. = 14.91 ft., \( P_c = 1000 \) PSI, so,

\[
(14.91)(40)(.00629)(96.8)(1/12) = \left(\frac{.7408(144)A_t(1000)}{3000}\right)^{6}(32.2)
\]

\[ A_t = .4195 \text{ ft}^2 \]

and then from the expressions on Page:

\[ F = (1532)(.4195)(144) = 93,200 \text{ lb.} \]

so the motor weight will be 1332 lb and the total weight will be 23,295 lb.

so the lift-off will be at 4g's as expected.

Meanwhile, back at the burning perimeter and hence chamber pressure as a function of time -- the governing equation is:

\[
(B.P.)(40)a_c P_c P = \frac{\Gamma A_t P_c}{a_c}
\]

or, rearranging:

\[
P_c^{1-n} = \frac{a_c \cdot 40 \cdot a \cdot P_c (B.P.)}{\Gamma A_t}
\]

Now, as the geometry is fixed and the combustion is assumed to take place at constant temperature the only variables in the above equation are \( P_c \) and \( (B.P.) \). Taking the logarithms of both sides:

\[
1-n \log P_c = \log \left\{ \frac{a_c \cdot 40 \cdot a \cdot P_c}{\Gamma A_t} \right\} + \log (B.P.)
\]

where the first term on the right hand side is a constant. This constant may be evaluated by substituting in the values for \( (B.P.) \) and \( P_c \) at time \( t = 0 \). Hence

\[
(1-n) \log P_c = C + \log (B.P.)
\]

\[ .6 \log (1000) = C + \log (179) \]
Therefore \[ C = 1.8 - 2.25285 + 10 - 10 \]
\[ = -0.45285 \]
so, \[ 0.6 \log P_c = -0.45285 + \log (\text{B.P.}) \]
\[ \log P_c = -0.75475 + 1.66667 \log (\text{B.P.}) \]

Now the procedure for determining the chamber pressure as a function of time will be as follows. At time \( t = 0 \) \( P_c \) is known and a burning rate can be determined from this pressure. This burning rate will be assumed constant for a five second time interval -- thereby giving the distance, \( x \), that the propellant has burned over the interval. Then a burning perimeter corresponding to that \( x \) can be determined and from the above equation a new value of pressure.

\( t = 0 \rightarrow 5 \) seconds: \( x = 0 \rightarrow .5'' \)

\[ \text{B.P.} = 10[17.9 + .0869] = 179.87 \text{ in.} \]

and

\[ \log P_c = -0.75475 + (1.66667) \log (179.87) \]
\[ = -0.75475 + (1.66667)(2.25496) \]
\[ = -0.75475 + 3.75826 = 3.00351 \]

Therefore: \( P_{c5} = 1008 \text{ PSI} \)

and small pressure change will be assumed to have a negligible effect on \( r \)

\( t = 5 \rightarrow 10 \) seconds: \( x = .5'' \rightarrow 1'' \)

\[ \text{B.P.} = 10[17.9 + .1739] = 180.739 \text{ in.} \]
and

\[ \log P_c = -0.75475 + (1.66667) \log (180.739) \]
\[ = -0.75475 + (1.66667)(2.25706) \]
\[ = -0.75475 + 3.76176 = 3.00701 \]

Therefore: \( P_{c10} = 1016.3 \text{ PSI} \)

and this pressure will again cause negligible change in burning rate.

\( t = 10 \rightarrow 15 \text{ seconds: } x = 1'' \rightarrow 1.5'' \)

\[ \text{B.P.} = 10[17.9 + .2608] = 181.608 \text{ in.} \]

and

\[ \log P_c = -0.75475 + (1.66667) \log (181.608) \]
\[ = -0.75475 + (1.66667)(2.25914) \]
\[ = -0.75475 + 3.75523 = 3.01048 \]

Therefore: \( P_{c15} = 1024.4 \text{ PSI} \)

and this pressure corresponds to an \( r \) of 0.107 in./sec. so \( r = 0.101 \)

in./sec. will be used.

\( t = 15 \rightarrow 20 \text{ seconds: } x = 1.5'' \rightarrow 2.005'' \)

\[ \text{B.P.} = 10[17.9 + .3487] = 182.487 \text{ in.} \]

and

\[ \log P_c = -0.75475 + (1.66667) \log (182.487) \]
\[ = -0.75475 + (1.66667)(2.26123) \]
\[ = -0.75475 + 3.76872 = 3.01397 \]
Therefore: \( P_{c20} = 1032.7 \) PSI \( \rightarrow r = .101 \) in./sec.

\( t = 20 \rightarrow 25 \) seconds: \( x = 2.005" \rightarrow 2.510" \)

B.P. = \( 10[17.9 + .4365] = 183.365 \) in.

and

\[
\log P_c = -.75475 + (1.66667) \log (183.365) \\
= -.75475 + (1.66667)(2.26332)
\]

\[
= -.75475 + 3.77220 = 3.01745
\]

Therefore: \( P_{c25} = 1041.0 \) PSI \( \rightarrow r = .1013 \) in./sec. \( = .101 \) in./sec.

\( t = 25 \rightarrow 30 \) seconds: \( x = 2.510" \rightarrow 3.015" \)

B.P. = \( 10[17.9 + .5243] = 184.243 \) in.

and

\[
\log P_c = -.75475 + (1.66667) \log (184.243) \\
= -.75475 + (1.66667)(2.26539)
\]

\[
= -.75475 + 3.77565 = 3.02090
\]

Therefore: \( P_{c30} = 1049.3 \) PSI \( \rightarrow r = .102 \) in./sec.

\( t = 30 \rightarrow 35 \) seconds: \( x = 3.015" \rightarrow 3.525" \)

B.P. = \( 10[17.9 + .6130] = 185.866 \) in.

and

\[
\log P_c = -.75475 + (1.66667) \log (185.130) \\
= -.75475 + (1.66667)(2.26748)
\]

\[
= -.75475 + 3.77913 = 3.02438
\]
Therefore: \( P_{c35} = 1057.8 \text{ PSI} \quad \Rightarrow \quad r = .1017 \quad \Rightarrow \quad r = .102 \text{ in./sec.} \)

t = 35 \rightarrow 39.150 \text{ seconds: } x = 3.525" \rightarrow 3.948"

B.P. = 10[17.9 + .6866] = 185.866 \text{ in.}

and

\[
\log P_c = -.75475 + (1.66667) \log (185.866)
\]

\[
= -.75475 + (1.66667)(2.26921)
\]

\[
= -.75475 + 3.78202 = 3.02727
\]

Therefore: \( P_{c39.15} = 1064.8 \text{ PSI} \quad \Rightarrow \quad r = .102 \text{ in./sec.} \)

t = 39.150 \rightarrow 39.167: \quad x = 3.948" \rightarrow 3.95"

B.P. = 10[1.8553x] = (18.553)(3.95) = 73.284 \text{ in.}

and

\[
\log P_c = -.75475 + (1.66667) \log (73.284)
\]

\[
= -.75475 + (1.66667)(1.86501)
\]

\[
= -.75475 + 3.10835 = 2.35360
\]

Therefore: \( P_{c39.167} = 225.74 \text{ PSI} \)

and the \( r \) may be computed as:

\[
\log (P_c)^{\frac{1}{4}} = .94144 \quad \Rightarrow \quad (P_c)^{\frac{1}{4}} = 8.7386
\]

\[
r = (.00629)(8.7386) = .054 \text{ in./sec.}
\]

t = 39.167 \rightarrow 42.5: \quad x = 3.95" \rightarrow 4.13"

and from the plot on the following page:

B.P. = 10[(4.13)(1.25)] = 51.625"
and

\[ \log P_c = -0.75475 + (1.66667) \log (51.625) \]
\[ = -0.75475 + (1.66667)(1.71236) \]
\[ = -0.75475 + 2.85477 = 2.10002 \]

Therefore: \( P_{c42.5} = 125.9 \) PSI

and the \( r \) may be computed as:

\[ \log P_c^h = .84001 \Rightarrow P_c^h = 6.9185 \]
\[ r = (.00629)(6.9185) = .044 \text{ in. /sec.} \]
\[ t = 42.5 \rightarrow 44.5; \ x = 4.13" \rightarrow 4.22" \]
\[ \text{B.P.} = 10 [(4.22)(1.098)] = 46.3" \]

and

\[ \log P_c = -0.75475 + (1.66667) \log (46.3) \]
\[ = -0.75475 + (1.66667)(1.66558) \]
\[ = -0.75475 + 2.77596 = 2.02121 \]

Therefore: \( P_{c44.5} = 105.1 \) PSI

and the \( r \) may be computed as:

\[ \log P_c^h = .80848 \quad P_c^h = 6.434 \]
\[ r = (.00629)(6.434) = .040 \]
\[ t = 44.5 \rightarrow 45; \ x = 4.22" \rightarrow 4.24" \]
\[ \text{B.P.} = 10 [(4.24)(1.053)] = 44.65 \]

and

\[ \log P_c = -0.75475 + (1.66667) \log (44.65) \]
\[ = -0.75475 + (1.66667)(1.64982) = 1.99495 \]

Therefore: \( P_c = 98.85 \) PSI
Figure 5. Burning Grain Geometry.
Figure 6. Chamber Pressure with Time.
DETERMINATION OF PROPELLANT CROSS-SECTIONAL AREA AND WEIGHT AS A FUNCTION OF TIME

The cross-sectional area is given by the expression on Page as:

\[
A_x = A_1 - 5 \left\{ \left( \frac{D}{2} - W + x \right)^2 - \left( \frac{D}{2} - W \right)^2 \right\} \theta - 5x^2 \left\{ \frac{1}{2} - \theta + \beta \right\} - 10 \frac{Wx}{\sin \beta} 
- \frac{5x^2}{\tan \beta} - \left\{ \frac{D}{2} - \frac{W}{\sin \beta} \cdot \cos \theta \right\} \frac{Z}{\sin \beta} - \frac{x}{\tan \beta} - x \tan \frac{\theta}{2} \right\} x
\]

and for the particular configuration chosen this reduces to:

\[
A_x = 770.63 - 5 \left\{ (14.05 + x)^2 - (14.05)^2 \right\} \{.3435\} - 5\{1.853\}x^2 - 10(6.71)x
- 10(6.882)x^2 - 10\{8.06 - 1.7 - 1.3764x - .3249x\}x
= 770.63 - 1.7175 \left\{ 28.1x + x^2 \right\} - 9.2765x^2 - 67.1x - 6.882x^2
- (80.6 - 17)x + 17.013x^2
= 770.63 - 178.962x - .862x^2
\]

and evaluating at the following times

at \( t = 0 \) \( A_x = 770.63 \text{ in}^2 \) \( \varepsilon = .757 \)

at \( t = 5 \) \( A_x = 770.63 - 89.481 - .216 = 680.93 \text{ in}^2 \) \( \varepsilon = .669 \)

at \( t = 10 \) \( A_x = 770.63 - 178.962 - .862 = 590.81 \text{ in}^2 \) \( \varepsilon = .580 \)

at \( t = 15 \) \( A_x = 770.63 - 268.443 - 1.94 = 500.25 \text{ in}^2 \) \( \varepsilon = .492 \)

at \( t = 20 \) \( A_x = 770.63 - 358.7 - 3.46 = 408.47 \text{ in}^2 \) \( \varepsilon = .401 \)

at \( t = 25 \) \( A_x = 770.63 - 449.5 - 5.43 = 315.70 \text{ in}^2 \) \( \varepsilon = .310 \)

at \( t = 30 \) \( A_x = 770.63 - 539.5 - 7.83 = 223.3 \text{ in}^2 \) \( \varepsilon = .219 \)

at \( t = 35 \) \( A_x = 770.63 - 630.5 - 10.70 = 129.43 \text{ in}^2 \) \( \varepsilon = .127 \)
at $t = 39.15$ \( A_x = 770.63 - 707.0 - 13.43 = 50.20 \text{ in}^2 \quad \varepsilon = .0493 \)

at $t = 39.167$ \( A_x = 770.63 - 707.0 - 13.43 = 50.20 \text{ in}^2 \quad \varepsilon = .0493 \)

at $t = 44.8$ \( A_x = 50.20 \left[ \frac{(4.23)^2}{2} + \frac{(3.95)^2}{2} \right] \xi - \{4.23 - 3.95\} \{.8\} \times 10 \)

where $\xi = 60.4'' = 1.053$ radians

\( A_x = 50.20 - [(1.5265)(2.29) + .224] 10 = 35.91 \text{ in}^2; \varepsilon = .0353 \)

and similarly the weight as a function of time is given as:

\[ W = 23.295 + (\varepsilon - .757)(27350) \]

so, at \( t = 0 \) seconds \( W = 23,295 \text{ lb.} \)

\( t = 5 \) seconds \( W = 20,888 \text{ lb.} \)

\( t = 10 \) seconds \( W = 18,455 \text{ lb.} \)

\( t = 15 \) seconds \( W = 16,045 \text{ lb.} \)

\( t = 20 \) seconds \( W = 13,555 \text{ lb.} \)

\( t = 25 \) seconds \( W = 11,075 \text{ lb.} \)

\( t = 30 \) seconds \( W = 8,585 \text{ lb.} \)

\( t = 35 \) seconds \( W = 6,055 \text{ lb.} \)

\( t = 39.15 \) seconds \( W = 3,935 \text{ lb.} \)

\( t_{b.o.} = 44.8 \) seconds \( W = 3,545 \text{ lb.} \)

---

**THRUST FOR VARYING \( P_o \) AND \( P_A \)**

We know from Equation (12) on Page 58 that:

\[ \frac{\Delta}{A_t} = \gamma P_t M_{EX} \left( \frac{T_{EX}}{T_c} \right)^{1/2} + \left( P_{EX} - P_A \right) \frac{A_{EX}}{A_t} \]
Figure 7. Weight Versus Time.
now, multiply by $A_t$ and divide by $P_c$

$$\frac{F}{P_c} = \gamma \frac{P_k}{P_c} M_{EX} \left( \frac{T_{EX}}{T_t} \right)^{1/2} A_t + \left( \frac{P_{EX}}{P_c} - \frac{P_A}{P_c} \right) A_{EX}$$

and now the first two terms on the right hand side are known constants from the propellant specifications and nozzle geometry and may be evaluated:

$$\frac{F}{P_c} = (1.26)(.5532)(3.678)(.64)(60.4) + (.00735)(13.05)(60.4)$$

$$- \frac{P_A}{P_0} (13.05)(60.4)$$

so,

$$F = (99.0 + 5.8) P_c - 788.5 P_A$$

$$= 104.8 P_c - 788.5 P_A$$

for a check, consider lift-off:

$$F = (104.8)(1000) - (788.5)(14.7)$$

$$= 104,800 - 11,600 = 93,200 \text{ lb.}$$

and this checks with the previous calculation of this value found in the section dealing with the variation of burning perimeter as a function of time.

**PERFORMANCE EQUATIONS**

Consider a rocket vehicle in a vertical trajectory. Assuming drag negligible as specified for performance calculations, and computing the velocity and altitude relations in time increments, it was shown in the previous design
study that:

\[ V_f = V_o + \frac{gT\Delta t}{W_o - W_f} \ln \frac{W_o}{W_f} - g\Delta t \]

and

\[ h_f = h_o + V_o\Delta t + \frac{gT\Delta t}{W_o - W_f} \left\{ \ln \left( \frac{W_f}{W_o} \right) + \Delta t \right\} - \frac{g}{2} \Delta t^2 \]

Where:  
\( f \) represents values at the end of the interval  
\( o \) represents values at the beginning of the interval  
\( \Delta t \) represents the time interval

It should be noted that the average mass flow over the interval is being used. Although this is more accurate than taking the mass flow at the beginning of the interval the change is very slight due to the almost constant \( P_c \) and the average is being used primarily for ease of computation. The thrust is evaluated at the beginning of the interval.

Because of the nearly constant \( P_c \) 5 second intervals will be used as it is doubtful that any significant amount of accuracy would be added by considering smaller increments:

From \( t = 0 \to 5 \) seconds

\[ V_f = V_o + \frac{gP\Delta t}{W_o - W_f} \left\{ \ln W_o - \ln W_f \right\} - g\Delta t \]

\[ V_5 = 0 + 6230 \left\{ \ln 23.295 - \ln 20.888 \right\} - 161 \]

\[ = 6230 \left\{ 10.05599 - 9.94693 \right\} - 161 = 6230 \left\{ .10906 \right\} - 161 \]

\[ = 680 - 161 = 519 \text{ ft./sec.} \]
\[ h_F = h_o + V_o \Delta t + \frac{gF \Delta t}{W_o - W_F} \left\{ \frac{W_F \Delta t}{W_o - W_F} \ln(W_F) - \ln(W_o) \right\} + \Delta t \right\} \frac{g}{2} \Delta t^2 \]

\[ h_g = 0 + 0 + 6230 \left\{ 43.3 \left\{ -0.10906 \right\} + 5 \right\} - 402.5 \]

\[ = 6230 \left\{ -4.73 + 5 \right\} - 402.5 = 1682 - 402 = 1280 \text{ ft.} \]

and at 1280 ft. \( P_A = 14.03 \) PSI, \( P_c = 1008 \) PSI

\[ F = 104.8 \ (1008) - 788.5 \ (14.03) \]

\[ = 105,700 - 11070 = 94,630 \ \text{lb.} \]

From \( t = 5 \to 10 \) seconds

\[ V_{10} = 519 + \frac{161 \cdot 94,630}{(20,888 - 18,455)} \left\{ \ln 20,888 - \ln 18,455 \right\} - 161 \]

\[ = 519 + 6260 \left\{ 9.94693 - 9.82309 \right\} - 161 \]

\[ = 519 + 775 - 161 = 1133 \ \text{ft./sec.} \]

\[ h_{10} = 1280 + (519)(5) + 6260 \left\{ 37.95 \left\{ -0.12384 \right\} + 5 \right\} - 402.5 \]

\[ = 1280 + 2595 + 6260 \left\{ -4.695 + 5 \right\} - 402.5 = 1280 + 2595 + 1910 - \]

\[ = 5383 \ \text{ft.} \]

and at 5383 ft. \( P_A = 11.99 \) PSI, \( P_c = 1016.3 \)

\[ F = 104.8 \ (1016.3) - 788.5 \ (11.99) \]

\[ = 106,300 - 94.45 = 96,855 \ \text{lb.} \]

From \( t = 10 \to 15 \) seconds

\[ V_{15} = 1133 + \frac{161 \cdot 96,855}{(18,455 - 16,045)} \left\{ \ln 18,455 - \ln 16,045 \right\} - 161 \]

\[ = 1133 + 6470 \left\{ 9.82309 - 9.68315 \right\} - 161 \]

\[ = 1133 + 905 - 161 = 1877 \ \text{ft./sec.} \]
\[ h_{15} = 5383 + 1133(5) + 6470 \left\{ 33.35 \left\{-1.3994\right\} + 5 \right\} -402.5 \]
\[ = 5383 + 5665 + 6470 \left\{-4.67+5\right\} -402.5 = 5383 + 5665 + 2135 - 402 \]
\[ = \underline{12781 \text{ ft.}} \]

and at 12,781 ft. \( P_A = 9.060 \text{ PSI}, \quad P_c = 1024.4 \text{ PSI} \)
\[ F = 104.8 \left(1024.4\right) - 788.5 \left(9.060\right) \]
\[ = 107350 - 7145 = \underline{100,205 \text{ lb.}} \]

From \( t = 15 \to 20 \) seconds

\[ V_{20} = 1877 + \frac{161 \cdot 100,205}{(16,045 - 13,555)} \left\{ \ln 16045 - \ln 13555 \right\} - 161 \]
\[ = 1877 + 6480 \left\{ 9.68315 - 9.51451 \right\} - 161 \]
\[ = 1877 + 1093 - 161 = \underline{2809 \text{ ft./sec.}} \]

\[ h_{20} = 12781 + 1877(5) + 6480 \left\{ 27.2 \left\{-1.6864\right\} + 5 \right\} -402.5 \]
\[ = 12781 + 9385 + \left\{-4.58 + 5\right\} 6480 - 402.0 F + 2781 + 9385 \]
\[ + 2720 - 402 \]
\[ = \underline{24,484 \text{ ft.}} \]

and at 24,484 ft. \( P_A = 5.574 \text{ PSI}, \quad P_c = 1032.7 \text{ PSI} \).

\[ F = 104.8 \left(1032.7\right) - 788.5 \left(5.574\right) \]
\[ = 108,250 - 4395 = \underline{103,855 \text{ lb.}} \]

From \( t = 20 \to 25 \) seconds

\[ V_{25} = 2809 + \frac{161 \cdot 103,855}{(13555 - 11075)} \left\{ \ln 13555 - \ln 11075 \right\} - 161 \]
\[ = 2809 + 6740 \left\{ 9.51451 - 9.31245 \right\} - 161 \]
\[ = 2809 + 1362 - 161 = \underline{4010 \text{ ft./sec.}} \]
\[
\begin{align*}
h_{25} &= 24,484 + (2809)5 + 6740 \{22.34 \{-0.20206\} + 5\} - 402 \\
&= 24,484 + 14045 + 6740 \{-4.515 + 5\} - 402 \\
&= 24,484 + 14045 + 3267 - 402 \\
&= 41,394 \text{ ft.}
\end{align*}
\]

and at 41,394 ft. \( P_A = 2.545 \) PSI. \( P_e = 1041 \) PSI

\[
\begin{align*}
F &= 104.8 (1041) - 788.5 (2.545) \\
&= 109,050 - 2006 = 107,044 \text{ lb.}
\end{align*}
\]

From \( t = 25 \rightarrow 30 \) seconds

\[
\begin{align*}
V_{30} &= 4010 + \frac{161 \cdot 107,044}{(11075 - 8585)} \{\ln 11075 - \ln 8585\} - 161 \\
&= 4010 + 6950 \{9.31245 - 9.05777\} - 161 \\
&= 4010 + 1770 - 161 = 5619 \text{ ft./sec.}
\end{align*}
\]

\[
\begin{align*}
h_{30} &= 41,394 + (4010)5 + 6950 \{17.22 \{-0.25468\} + 5\} - 402 \\
&= 41,394 + 20050 + \{-4.390 + 5\}(6950) - 402 = 41,394 + 20,050 \\
&+ 4240 - 402 \\
&= 65,282 \text{ ft.}
\end{align*}
\]

and at 65,282 ft. \( P_A = .808 \) PSI, \( P_e = 1049.3 \) PSI.

\[
\begin{align*}
F &= 104.8 (1049.3) - 788.5 (.808) \\
&= 109,900 - 637 = 109,263 \text{ lb.}
\end{align*}
\]

From \( t = 30 \rightarrow 35 \) seconds

\[
\begin{align*}
V_{35} &= 5619 + \frac{161 \cdot 109,263}{(8585 - 6055)} \{\ln 8585 - \ln 6055\} - 161 \\
&= 5619 + 6960 \{9.05777 - 8.70864\} - 161 \\
&= 5619 + 6960 \{-0.34913\} - 161 = 7888 \text{ ft./sec.}
\end{align*}
\]
\[ h_{35} = 65,282 + (5619)5 + 6960 \left\{11.97 (-.34913) + 5\right\} - 402 \]
= 65,282 + 28095 + 6960 (.825) - 402 = 98,715 \text{ ft.} 

and at 98,715 ft. \quad P_A = .1641 \text{ PSI} \quad P_c = 1057.8 \text{ PSI.}

\begin{align*}
F & = 104.8 (1057.8) - 788.5 (1.1641) \\
& = 110,900 - 129 = 110,771 \text{ lb.}
\end{align*}

\text{From } t = 35 \to 39.167 \text{ seconds}

\begin{align*}
V_{39.167} & = 7888 + \frac{(4.167)(32.2)(110,771)}{(6055 - 3905)} \left\{\ln 6055 - \ln 3905\right\} - 135 \\
& = 7888 + 6910 \left\{8.70864 - 8.27001\right\} - 135 \\
& = 7888 + 3030 - 135 = 10783 \text{ ft./sec.}
\end{align*}

\[ h_{39.167} = 98,715 + 7888 (4.167) + 6910 \left\{7.57 (-.43863) + 5\right\} - 280 \]
= 98,715 + 32820 + 11620 - 280 = 142,875 \text{ ft.}

and at 142,875 ft. \quad P_A = .026 \text{ PSI,} \quad P_c = 225.74 \text{ PSI}

\begin{align*}
F & = 104.8 (225.74) - 788.5 (.026) \\
& = 23,650 - 20 = 23,630 \text{ lb.}
\end{align*}

\text{From } t = 39.167 \text{ seconds } \to 42.5 \text{ seconds}

\begin{align*}
V_{42.5} & = 10783 + \frac{(3.333)(32.2)(23,630)}{(3905 - 3620)} \left\{\ln 3905 - \ln 3620\right\} - 107 \\
& = 10783 + 8900 \left\{8.27001 - 8.19423\right\} - 107 \\
& = 10783 + 674 - 107 = 11,350 \text{ ft./sec.}
\end{align*}

\[ h_{42.5} = 142,875 + (10783)(3.333) + 8900 \left\{42.3 (-.07578) + 5\right\} - 111 \]
= 142,875 + 35900 + 15930 - 111 = 194,594 \text{ ft.}
Figure 9. Rocket Velocity as a Function of Time.
NOTE: FROM VARYING GRAVITATION CHART
THE SUMMIT ELEVATION WOULD BE 500 MILES

Figure 10. Rocket Altitude Versus Time During Powered Phase.
and at 194,594 $P_A$ is negligible and $P_c = 125.9$

$$F = 104.8 \times (125.9) = 13,200 \text{ lb.}$$

From $t = 42.5 \rightarrow 44.8$ seconds (Burn-out)

$$V_{44.8} = 11,350 + \frac{(2.3)(32.2)(13,200)}{(3620 - 3545)} \{\ln 3620 - \ln 3545\} - 74$$

$$= 11,350 + 9870 \{8.19423 - 8.17329\} - 74$$

$$= 11,350 + 198 - 74 = \frac{11,474}{\text{ft./sec.}}$$

$$h_{44.8} = 194,594 + (11,350)(2.3) + 9870 \{108.8 \{-0.02094\} + 5\} - 85$$

$$= 194,594 + 26100 + 26800 - 85 = 247,409 \text{ ft.}$$

**UTATION OF MACH NUMBER AND DYNAMIC HEAD**

At $t = 0$ seconds

$$M = 0 \quad q = 0$$

At $t = 5$ seconds

$$c = 1112 \text{ ft./sec.} \quad \rho = (.9632)(.002378) = .00229 \text{ lb./sec.}^2$$

$$q = \frac{1}{2}(.00229)(519)^2 = 309 \text{ lb./ft.}^2$$

At $t = 10$ seconds

$$c = 1096 \text{ ft./sec.} \quad \rho = (.8518)(.002378) = .002024 \text{ lb./sec.}^2$$

$$q = \frac{1}{2} (.002024)(1133)^2 = 1300 \text{ lb./ft.}^2$$

At $t = 15$ seconds

$$c = 1067 \text{ ft./sec.} \quad \rho = (.6761)(.002378) = .001608 \text{ lb./sec.}^2$$

$$q = \frac{1}{2} (.001608)(1877)^2 = 2833 \text{ lb./ft.}^2$$
At $t = 20$ seconds

$c = 1018$ ft./sec. \hspace{1cm} \frac{M}{1018} = \frac{2809}{1018} = 2.76$

$\rho = (0.4564)(0.002378) = 0.001085 \frac{\text{lb. sec.}^2}{\text{ft.}^4} \hspace{1cm} q = \frac{1}{2} \cdot (0.001085)(2809)^2 = 4277 \frac{\text{lb.}}{\text{ft.}^2}$

At $t = 25$ seconds

$c = 968.5$ ft./sec. \hspace{1cm} \frac{M}{968.5} = \frac{4010}{968.5} = 4.15$

$\rho = (0.2302)(0.002378) = 0.000547 \frac{\text{lb. sec.}^2}{\text{ft.}^4} \hspace{1cm} q = \frac{1}{2} \cdot (0.000547)(4010)^2 = 4395 \frac{\text{lb.}}{\text{ft.}^2}$

At $t = 30$ seconds

$c = 968.5$ ft./sec. \hspace{1cm} \frac{M}{968.5} = \frac{5619}{968.5} = 5.8$

$\rho = (0.07305)(0.002378) = 0.0001738 \frac{\text{lb. sec.}^2}{\text{ft.}^4} \hspace{1cm} q = \frac{1}{2} \cdot (0.0001738)(5619)^2 = 2745 \frac{\text{lb.}}{\text{ft.}^2}$

At $t = 35$ seconds

$c = 1002$ ft./sec. \hspace{1cm} \frac{M}{1002} = \frac{7888}{1002} = 7.87$

$\rho = (0.01407)(0.002378) = (0.00003343) \frac{\text{lb. sec.}^2}{\text{ft.}^4} \hspace{1cm} q = \frac{1}{2} \cdot (0.00003343)(7888)^2 = 1040 \frac{\text{lb.}}{\text{ft.}^2}$

At $t = 39.167$ seconds

$c = 1086$ ft./sec. \hspace{1cm} \frac{M}{1086} = \frac{10783}{1086} = 9.93$

$\rho = (0.001967)(0.002378) = 0.000004675 \frac{\text{lb. sec.}^2}{\text{ft.}^4} \hspace{1cm} q = \frac{1}{2} \cdot (0.000004675)(10783)^2 = 272 \frac{\text{lb.}}{\text{ft.}^2}$

At $t = 44.8$ seconds

Too high an altitude for values to be significant.

NOTE: The values of density and speed of sound used above were interpolated from the Pratt and Whitney handbook.
Figure 11. Mach Number Versus Time.
Figure 12. Dynamic Head Versus Time.
DRAG CALCULATION

As was specified, drag effects were ignored in the performance evaluation. Using the results of the performance evaluation the drag will now be calculated and the amount of additional fuel necessary to counteract the negative impulse due to the drag determined.

Assuming an \( \ell/D = 3 \) for the nose cone:

\[
L = \ell + 3(\ell) = 49' 
\]

Hence a characteristic dimension of 60' \textbf{will} be assumed in computing the Reynolds Number -- this should be conservative. The drag is given by the relationship:

\[
D = \rho C_D S 
\]

where \( S \) \textbf{will} be conservatively assumed to be 7.1 square feet.

At \( t = 0 \) seconds

\[
D = 0 
\]

At \( t = 5 \) seconds

\[
Re = \frac{\rho VL}{\mu} 
\]

so,

\[
Re = \frac{(0.00229)(519)(60)}{3.693 \times 10^{-7}} = 193.2 \times 10^6 
\]

and from \( C_D \) plot handed out in class it can be seen that for a nose cone with \( \ell/D = 3 \):

\[
C_D = .175 
\]

and then the drag may be evaluated:

\[
D = (.175)(7.10)(309) = 383 \text{ lb}. 
\]
At $t = 10$ seconds

$$R_e = \frac{(0.002024)(1183)(60)}{3.610 \times 10^{-7}} = 381 \times 10^6$$

$C_D = 0.295$

$D = (0.295)(7.10)(1300) = 2720 \text{ lb.}$

At $t = 15$ seconds

$$R_e = \frac{(0.001608)(1877)(60)}{3.453 \times 10^{-7}} = 525 \times 10^6$$

$C_D = 0.255$

$D = (0.255)(7.10)(2833) = 5130 \text{ lb.}$

At $t = 20$ seconds

$$R_e = \frac{(0.001085)(2809)(60)}{3.218 \times 10^{-7}} = 567 \times 10^6$$

$C_D = 0.215$

$D = (0.215)(7.10)(4277) = 6530 \text{ lb.}$

At $t = 25$ seconds

$$R_e = \frac{(0.000547)(4010)(60)}{2.961 \times 10^{-7}} = 445 \times 10^6$$

$C_D = 0.18$

$D = (0.18)(7.10)(4395) = 5610 \text{ lb.}$

At $t = 30$ seconds

$$R_e = \frac{(0.001738)(5619)(60)}{2.961 \times 10^{-7}} = 198 \times 10^6$$

$C_D = 0.155$

$D = (0.155)(7.10)(2745) = 3025 \text{ lb.}$

At $t = 35$ seconds

$$R_e = \frac{(0.0003343)(7888)(60)}{2.961 \times 10^{-7}} = 53 \times 10^6$$

$C_D = 0.16$

$D = (0.16)(7.10)(1040) = 1180 \text{ lb.}$
At \( t = 39.167 \) seconds

\[
R_e = \frac{(0.00004675)(10783)(60)}{4.032 \times 10^{-7}} = 7.48 \times 10^6
\]

\( C_D = 0.19 \)

\[
D = (0.19)(7.10)(272) = 366 \text{ lb.}
\]

The total impulse (drag) is equivalent to 1.34 seconds of sea level thrust or an additional \( (1.34)(481.4) = 645 \text{ lb.} \) of propellant. This impulse was obtained by integrating the area under the following curve.

**ACCELERATION COMPUTATION**

The accelerations may be computed by the second law credited to some obscure physicist. This was used in the derivation of the performance equations and is in general stated as:

\[
T - D - W = \frac{W}{g} \quad a
\]

But as we are neglecting drag:

\[
T - W = \frac{W}{g} \quad a
\]

\[
a = \left[ \frac{T}{W} - 1 \right] g
\]

and computing the accelerations:

At \( t = 0 \) seconds

\[
a = \frac{93,200}{23,295} - 1 \quad g
\]

\[
= [4.00 - 1] \quad g = 3.0 \quad g
\]

At \( t = 5 \) seconds

\[
a = \frac{94,630}{20,888} - 1 \quad g
\]

\[
= [4.53 - 1] \quad g = 3.53 \quad g
\]
Figure 13. Drag Versus Time.

Total Drag Impulse = 125.313 Lb. Sec.
At $t = 10$ seconds

$$a = \left[ \frac{96,855}{18,455} - 1 \right] g = [5.25 - 1] g = 4.25 g$$

At $t = 15$ seconds

$$a = \left[ \frac{100,205}{16,045} - 1 \right] g = [6.25 - 1] g = 5.25 g$$

At $t = 20$ seconds

$$a = \left[ \frac{103,855}{13,555} - 1 \right] g = [7.66 - 1] g = 6.66 g$$

At $t = 25$ seconds

$$a = \left[ \frac{107,044}{11,075} - 1 \right] g = [9.67 - 1] g = 8.67 g$$

At $t = 30$ seconds

$$a = \left[ \frac{109,263}{8585} - 1 \right] g = [12.73 - 1] g = 11.73 g$$

At $t = 35$ seconds

$$a = \left[ \frac{110,771}{6055} - 1 \right] g = [18.3 - 1] g = 17.3 g$$

At $t = 39.15$ seconds

$$a = \left[ \frac{111,400}{3935} - 1 \right] g = [28.3 - 1] g = 27.3 g$$
Figure 14. Rocket Acceleration as a Function of Time.
At \( t = 39.167 \) seconds

\[
\begin{align*}
a &= \frac{23,630}{3930} - 1 \text{ g} \\
&= [6.02 - 1] \text{ g} = 5.02 \text{ g}
\end{align*}
\]

At \( t = 42.5 \) seconds

\[
\begin{align*}
a &= \frac{13,200}{3620} - 1 \text{ g} \\
&= [3.65 - 1] \text{ g} = 2.65 \text{ g}
\end{align*}
\]

At \( t = 44.8 \) seconds

\[
\begin{align*}
a &= \frac{10,480}{3545} - 1 \text{ g} \\
&= [2.96 - 1] \text{ g} = 1.96 \text{ g}
\end{align*}
\]

**DISCUSSION OF STORAGE TERM**

As stated on Page the basic differential equation governing continuity in solid propellant rockets is:

\[
A_c \rho_P \frac{d}{dt} (\rho_c V) = \frac{\gamma M_t A_t P_t}{a_t}
\]

and the mass flow out the nozzle can also be written as:

\[
\frac{\gamma M_t A_t P_t}{a_t} = \gamma' \frac{P_c A_t}{a_c}
\]

where:

\[
\gamma' = \Gamma \sqrt{\gamma} = \gamma \left(\frac{2}{\gamma + 1}\right) \frac{\gamma + 1}{2(\gamma - 1)}
\]

and introducing the equation of state:

\[
P_c = \rho_c^{RT_c}
\]

and defining the abbreviation:

\[
P_P = \rho_P^{RT_c}
\]
The equation becomes:

\[ A_c r \frac{P_p}{RT_c} = \frac{d}{dt} \left( P_c V_c \right) + \Gamma \frac{P_c A_t}{a_c} \]

assuming \( T_c \) is constant:

\[ A_c r P_p = \frac{d}{dt} \left( P_c V_c \right) + \Gamma \frac{P_c A_t RT_c}{a_c} \]

\[ = \frac{d}{dt} \left( P_c V_c \right) + \Gamma \frac{P_c A_t \sqrt{RT_c}}{} * \]

now consider the storage term:

\[ V_c = V_{c0} + \int_0^t r A_c dt \]

\[ \frac{dV_c}{dt} = r A_c \]

and

\[ \frac{d}{dt} \left( P_c V_c \right) = P_c \frac{dV_c}{dt} + V_c \frac{dP_c}{dt} \]

\[ = P_c r A_c + V_c \frac{dP_c}{dt} \]

so the equation can be written:

\[ V_c \frac{dP_c}{dt} = A_c r (P_p - P_c) - \Gamma P_c A_t \sqrt{RT_c} \]

\[ = \frac{A_c}{A_t} \left\{ r (P_p - P_c) A_t - \Gamma P_c \frac{A_t^2}{A_c} \sqrt{RT_c} \right\} \]

\[ = A_c \left\{ r (P_p - P_c) - \Gamma P_c \frac{A_t}{A_c} \sqrt{RT_c} \right\} \]

Now, introduce the expression for the burning rate:

\[ r = a_c^n \]

Then, the equation becomes:

\[ V_c \frac{dP_c}{dt} = A_c \left\{ a_c^n (P_p - P_c) - \Gamma P_c \frac{A_t}{A_c} \sqrt{RT_c} \right\} \]

Now, \( P_c \) can be neglected in comparison to \( P_p \) in the term \( (P_p - P_c) \) as \( P_p \) is of the order of a hundred times as large (i.e., for asphalt potassium
perchlorate propellants it is 125,000 PSI and for ballistite it is 230,000 PSI) - then:

\[ V_c \frac{dP_c}{dt} = A_c \left\{ aP_c^{P_c^n} - \Gamma P_c \frac{A_t}{A_c} \sqrt{RT_c} \right\} \]

and it can be seen that the above equation is not easily solvable for \( P_c \). Consider now the case of equilibrium neutral burning so \( dP_c/dt = 0 \), then:

\[ aA_c^{P_c^n} P_c = \Gamma P_c \frac{A_t}{A_c} \sqrt{RT_c} \cdot A_c \]

\[ P_c^{1-n} = \frac{aA_c^{P_c^n}}{\Gamma A_t \sqrt{RT_c}} \]

now consider the equation when the storage term is neglected:

\[ A_c^{P_c^n} = \Gamma P_c A_t \sqrt{RT_c} \quad \text{(from Equat. * on previous page)} \]

or

\[ A_{c_n} = \Gamma P_c A_t \sqrt{RT_c} \]

so,

\[ P_c^{1-n} = \frac{aA_c^{P_c^n}}{\Gamma A_t \sqrt{RT_c}} \]

so it can be seen that neglecting the storage term will have no effect on the pressure solution as long as there is neutral burning (\( dP_c/dt = 0 \)) -- hence it would have negligible effect on the pressure trace of this design up to the point of web burnout. At web burnout its effect would be as shown in the sketch below:
That is, it would not allow an instantaneous change in pressure - conse-
quently the rocket would maintain a higher thrust level but burn out sooner.

DETERMINATION OF VELOCITY AT END OF GRAIN

The maximum velocity at the back of the grain will occur at the instant of starting as the mass flow is relatively constant, actually increasing slightly for the first 39.167 seconds of burning -- but, the port area is increasing more rapidly so the critical time will be starting.

The mass flow is given by:

\[ \dot{m} = \rho_c A_F V = 481.4 \text{ lb./sec.} \]

where:

\[ A_F = 1018 - 770.63 = 247.37 \text{ in.}^2 = 1.718 \text{ ft.}^2 \]

and from the equation of state:

\[
\rho_c = \frac{P_c}{RT} = \frac{(1000)(144) \text{ lb.} \cdot \text{°R lb.}}{(1544 \div 22)(3160) \text{ ft.} \cdot \text{lb.} \cdot \text{°R ft.}^2} 
\]

\[ = .649 \text{ lb./ft.}^3 \]
Hence,

\[ V = \frac{481.4 \text{ lb. ft}^3}{(649)(1.718) \text{ sec} \cdot \text{lb} \cdot \text{ft}^2} = 432 \text{ ft}./\text{sec.} \]

and this velocity will vary linearly to its value of 0 at the front of grain.

**COMPARISON WITH LIQUID PROPELLANT ROCKET DESIGN**

<table>
<thead>
<tr>
<th>Comparison</th>
<th>Liquid</th>
<th>Solid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payload Weight - lbs.</td>
<td>125 *</td>
<td>125</td>
</tr>
<tr>
<td>Lift-Off Weight - lbs.</td>
<td>25.000</td>
<td>23,295</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>20/1</td>
<td>6.58/1</td>
</tr>
<tr>
<td>Lift-Off g Loading</td>
<td>2.95</td>
<td>4.0</td>
</tr>
<tr>
<td>Maximum Acceleration</td>
<td>38.65 g</td>
<td>27.3 g</td>
</tr>
<tr>
<td>Burn-Out Velocity - ft./sec.</td>
<td>16,151</td>
<td>11,474</td>
</tr>
<tr>
<td>Burn-Out Altitude - ft.</td>
<td>326,298</td>
<td>2,477,409</td>
</tr>
<tr>
<td>Summit Altitude - ft.</td>
<td>(5.73 \times 10^6)</td>
<td>(2.64 \times 10^6)</td>
</tr>
</tbody>
</table>

* Although 125 lb. was the specified payload weight the design analysis showed that this figure could be easily doubled for the performance shown above as there was a weight margin of 404.5 lb.

From the above comparison it is obvious that the liquid propellant design is the better of the two. The two things that hurt the solid design most are high motor weight and unburned propellant. The high motor weight is due to two factors: 1) the high thrust level necessary to attain a 4g lift-off, and 2) the relatively high ratio of motor weight

** The comparison presented above between the Liquid Motor System and Solid Motor System uses a liquid motor design comparable to the one presented here but different in detail. The comments are pertinent.
to thrust produced, which is necessary as there is no way to cool the motor. Although the unburned propellant only represented 3.53% of the volume of the propellant chamber it contributed 965 lb. of dead weight which severely hampers the performance. It would be an interesting study to consider other grain configurations to minimize this factor.

**SUMMARY**

The design of a solid propellant rocket to meet the specifications listed on Page was carried out. Preliminary calculations were based on a loading (or packing) fraction of .80. After consideration of several grain configurations the five spoke wagon wheel was chosen as it provided a nearly neutral burning configuration that would meet the lift-off requirements for a fairly high packing fraction. An analytic procedure was developed to determine the optimum sizing of the wagon wheel to meet the thrust requirements. An analytic procedure was also determined to compute the burning perimeter as it was felt that this would be more accurate than graphical methods. From this procedure the chamber pressure can be determined as a function of time and the performance then analyzed.

The performance analysis was carried out ignoring drag effects as was specified. The total negative impulse due to drag was computed (with aid of the "rambler" $C_D$ plots) and the amount of propellant necessary to counteract this impulse at sea level thrust conditions specified.
The results of the performance calculations were plotted and are included in the previous pages.

Although this is at best a rough analysis as it does not consider such things as heat transfer or propellant stability it is hoped that some of the methods of analysis presented herein would be useful in the more thorough consideration of more sophisticated systems such as grain designs to increase propellant utilization and determination of optimum g-take-off conditions.