THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

DESIGN PROCEDURES FOR DYNAMICALLY LOADED FOUNDATIONS

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February, 1967

IP-766

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INTRODUCTION

The necessity for developing effective and economical designs for footings subjected to dynamic loads has become more important in recent years. This has been caused primarily by the trend toward larger machines and the detrimental effects of vibrations emanating from them in industrial installations, and by the requirements for safety and stability of structures in regions affected by blasts or earthquakes. The problems include the dynamic behavior of the footing supporting the source of dynamic energy, the process whereby this energy is transmitted through soils to adjacent structures, and the response of the structure to the dynamic energy acting on its foundation. A foundation for a precision tracking radar, for example, has design requirements based on the dynamic force outputs developed by motions of the radar antenna as well as requirements to protect the entire structure from the effects of energy arriving from external sources. The foundation of a building subjected to earthquake loadings must accommodate the response of the structure to external energy arriving through the soil.

The design of foundations subjected to dynamic loads is a trial and error procedure. Initial dimensions are selected considering such factors as the dimensions of the equipment or structure to be supported, the space available for the foundation, and the normal static bearing stress. The trial design must be analyzed to determine its response to the design dynamic loading, and then be adjusted and re-analyzed if necessary.

This paper is specifically concerned with practical methods for carrying out the required dynamic analysis. These methods employ a system of lumped masses, springs and dashpots which is approximately equivalent to the actual foundation-soil system. In such "lumped" systems, the mass represents all of the inertia present in the actual system, while the springs and dashpots respectively represent all of the flexibility and damping present in the actual system. The key step, of course, is evaluation of the parameters of the equivalent lumped system. Once this has been done, available mathematical solutions of the lumped system can be used to estimate the response of the actual system.

This approach is by no means new; it goes back to the first organized studies of foundation dynamics during the 1930's. What is new is an increased confidence in the validity of the approach and in the validity of methods for choosing numerical values for the parameters of the equivalent lumped system. One key to the improved knowledge and methods has been provided by the theory for the dynamic motion of a rigid disk resting upon a homogeneous elastic half-space. This theory, which is discussed in the companion paper by Richart and Whitman (1967)* has sorted out the factors which influence dynamic behavior and has provided the only satisfactory method for evaluating certain of these factors.

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* References are listed alphabetically at the end of the paper.

The second recent development of great importance has been the large scale field tests upon machine foundations (see the companion paper by Maxwell, Fry and Ballard, 1967). These field tests validated both the theoretical procedures and the laboratory and field tests needed to measure the so-called elastic constants required by the theory.

The main emphasis in this paper is upon foundations for reciprocating and rotating machinery and radar towers, but the methods of analysis and the suggestions for evaluating the parameters of the equivalent lumped systems are generally applicable to all of the problems mentioned in the first paragraph.

DESIGN CRITERIA FOR MACHINE FOUNDATIONS

Steady state vibrations. Machines may have rotating or reciprocating parts which develop dynamic forces varying periodically with time. Under steady state operation, forces are developed which have a frequency equal to the operating frequency of the machine. Reciprocating machinery also produces dynamic forces (called the secondary unbalanced forces) which act at a frequency twice that of the machine operation. Typical operating frequencies range from 200 cycles/minute for large reciprocating air compressors to about 12,000 cycles/minute for turbines and high speed rotary compressors.

Vibrations developed by operating machinery produce several effects which must be considered in the design of foundations. The motion of the foundation and the machine it supports must be of sufficiently low magnitude that no structural damage occurs to the machine and its various connections. The vibration of the foundation must also be of a proper magnitude and frequency that the resulting motion of the machinery supported on the foundation does not interfere with the prescribed function of the machine itself. As an example, the foundation of a radar tracking tower must be extremely stable for the mechanical and electronic tracking system to function satisfactorily. Finally, the vibrating machinery transmits energy through the foundation into the soil and this vibratory energy is then transmitted to adjacent machinery or buildings. In the process of transmitting energy the soil's internal structure may be altered, resulting in a progressive settlement of the surface, the foundations of adjacent machinery may be set into oscillations which aggravate their settlements, or the transmitted energy may set buildings or other personnel enclosures into oscillations which may be noticeable, uncomfortable, or intolerable to the occupants.

The design criteria, varying from a consideration of damage to the machinery to annoyance of persons, must include both amplitude of motion and frequency of vibration. The combination of amplitude and frequency of a particular vibration is often expressed in terms of the maximum particle velocity or acceleration developed during the periodic motion. Generally, to avoid damage to machines or machine foundations, the maximum velocity of the vibration should not exceed 1 in/sec., or the maximum acceleration should not exceed 0.5 times the acceleration of gravity (Rausch, 1943). Vibrations begin to be troublesome to persons when the maximum velocity exceeds 0.1 in/sec. and they are noticeable to persons when the velocity exceeds about 0.01 in/sec. (Reiher and Meister, 1931). At a frequency of 1000 cycles/min. these velocity criteria correspond to amplitudes of motion of 0.01, 0.001, and 0.0001 inches, respectively.

It should be noted that the motion which may be noticed by persons is of the order of 1/100 of that which is likely to cause damage to machines. Finally, to provide a stable base to support precision machinery or calibration equipment, it is sometimes necessary to restrict the acceleration to the order of 10^{-4} g at relatively low frequencies.

Transient loadings. Machines which develop intermittent force pulses, for example punch presses or forging hammers, apply a large pulse of energy to the foundation during each load application. The resulting motion of the foundation transmits a primary pulse of energy into the supporting soil, followed by periodic pulses which die out after a few cycles. The frequency and manner of decay of these pulses depends upon the dynamic characteristics of the machine foundation and its soil support. Thus the timing of a series of intermittent machine loads and the damping of the foundation system determine the resulting motions of the foundation as well as the characteristics of the energy transmitted away from the foundation.

One of the most difficult parts of the design procedure for transient loadings is the evaluation of the force-time pulse which is delivered to the foundation. Often this must be established from strain-time or acceleration-time records obtained from measurements on similar machines. After the input pulse has been established, the calculation for the response of the foundation can be obtained by analyzing the equivalent lumped system.

Foundations for radar towers. Often radar towers are supported by spread footings of approximately circular plan. These foundations must provide stability against excessive motions, in the rocking and torsional modes of oscillation, which may be caused by wind loads, inertia forces developed by motions of the antenna, or external vibratory energy transmitted through the soil. Thus, the response of the foundation must be evaluated for a combination of steady-state and transient loadings.

The foundation for a radar tower is only one part of the total system which provides structural support for the receiving system. The antenna dish itself is a flexible structure with low damping which may respond to transient pulses by vibrating at its own natural frequency. The antenna drive system has some structural flexibility, particularly in the bearings, and the structural tower between the drive system and footing has low damping and some flexibility. The footing-soil system is subjected to rotational modes of oscillation which may be analyzed after the spring and damping constants for the equivalent lumped system are established. From this discussion of the various flexibilities involved in components of a radar tower, it is evident that an evaluation of its dynamic response must be based on a consideration of an equivalent system having several degrees of freedom (Fu and Jepson, 1959). Consequently, the criteria which are applied to the design of the foundation can be understood only through consideration of the tower as a unit, and the engineers responsible for the design of the foundation should participate in the development of design criteria. However, the basic criterion is still one of permissible motions and some typical values for actual installations have been given by Horn (1964).

Effect of dynamic loading on soils. Under sustained vibratory loads or repeated impacts the internal structure of soils may change, thereby producing settlement of the surface or possibly a reduction in strength. It is well known that loose, saturated cohesionless soils are particularly susceptible to compaction by impacts or vibrations. To prevent settlements of loose cohesionless deposits by vibrations from machinery, it is necessary to pre-compact the construction site either by blasting (Lyman, 1942, Prugh, 1963), by vibroflotation (Steuerman, 1939, D'Appolonia, 1953), or by the use of a vibrating surface roller (D'Appolonia, 1966). By each method it is planned to introduce a more severe dynamic loading to the soil during site preparation than is expected during the operating life of the installed machines.

Laboratory tests (Florin and Ivanov, 1961, Seed and Lee, 1966) and small scale field tests on saturated cohesionless materials have indicated that the significant parameters governing compaction at any point in the soil mass are (a) the initial void ratio, (b) the confining pressure, (c) the intensity of the dynamic loading, and (d) the duration of loading or the stress history of loadings with variable amplitudes.

There is a need for studies of compaction under conditions of sustained dynamic loads, including random load applications, to evaluate the effects of the four variables noted in the preceding paragraph.

The designer does have some control, however, on the amount of vibratory energy transmitted through the soil in the neighborhood of operating machinery. By proper design of the machine-foundation system he can minimize the amplitude of motion of the foundation, thereby controlling the amplitude of the transmitted vibrations.

LUMPED SYSTEM PARAMETERS

In the companion paper by Richart and Whitman (1967) it has been demonstrated that the dynamic behavior of an actual foundation can be represented by a lumped parameter system. For a single degree of freedom system this is described by

$$m\ddot{z} + c\dot{z} + kz = Q(t) \tag{1}$$

in which z, $\dot{z}$, and $\ddot{z}$ represent the displacement, velocity, and acceleration, respectively, of the mass in the z-direction. The lumped parameters are the equivalent mass, m, the effective damping, c, and the effective spring constant, k, which must be evaluated to represent the dynamic motions in the z-direction of the real system as induced by the external force $Q(t)$. As illustrated in Figure 1, the values of the equivalent masses, and effective damping and spring constants will generally be different for each mode of motion which is excited. However, when these parameters are established for each possible motion, then the effects of coupled oscillations may be studied.

Under steady state oscillation, the exciting function $Q(t)$ in Equation 1 may be expressed either as $Q_o \sin \omega t$, where Q_o is a constant, or as $m_1 e \omega^2 \sin \omega t$ when the force is developed by a mass m_1 with an eccentricity e rotating at an angular velocity ω. The frequency of induced dynamic force is designated by

$$f = \frac{\omega}{2\pi} \tag{2}$$

and the influence of variations in exciting frequency on the response of the system is shown by relating it to the "natural frequency" of the

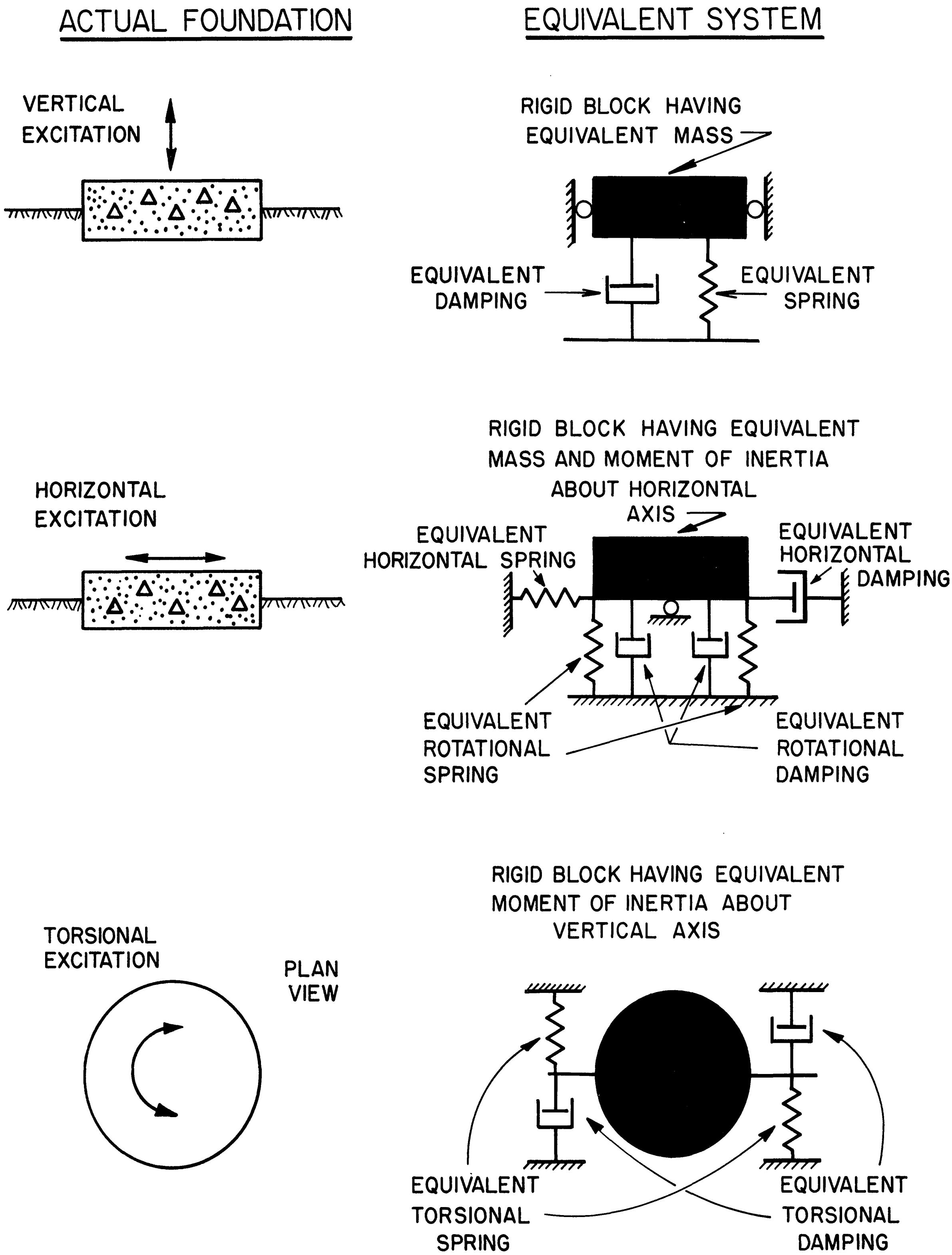


Figure 1. Typical Equivalent Lumped Systems.

system, f_o , where

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad (3)$$

Figure 2a shows a typical dimensional plot of amplitude versus frequency ratio for a single degree of freedom system excited by a rotating mass exciter. The maximum amplitude of motion is governed by the damping constant, c , in Equation 1, but this influence is usually represented in terms of the damping ratio, D , which relates c to the "critical damping"

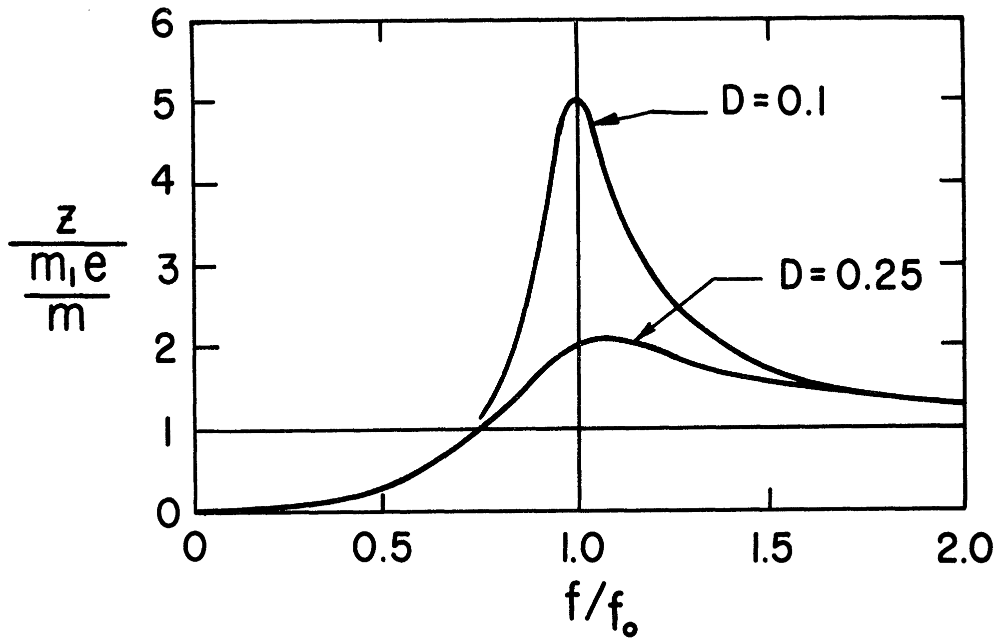
$$c_c = 2\sqrt{km} \quad (4)$$

Thus

$$D = \frac{c}{c_c} \quad (5)$$

Figure 2b illustrates the effect of the damping ratio, D , on the maximum amplitude of oscillation. A complete discussion of the damped single degree of oscillation system can be found in any textbook on mechanical vibrations and does not need further elaboration here.

The next three sections provide guidance for the selection of actual numerical values for the lumped parameters. It is important to keep in mind the roles which these parameters play in dynamic analyses: these roles are summarized in Figure 3. The approach to dynamic analysis and to design differs depending upon the amount of damping which is present in the system. Hence the magnitude of damping in actual foundations is considered first in the following section. When damping is so small that resonance must be avoided, it becomes necessary to estimate the



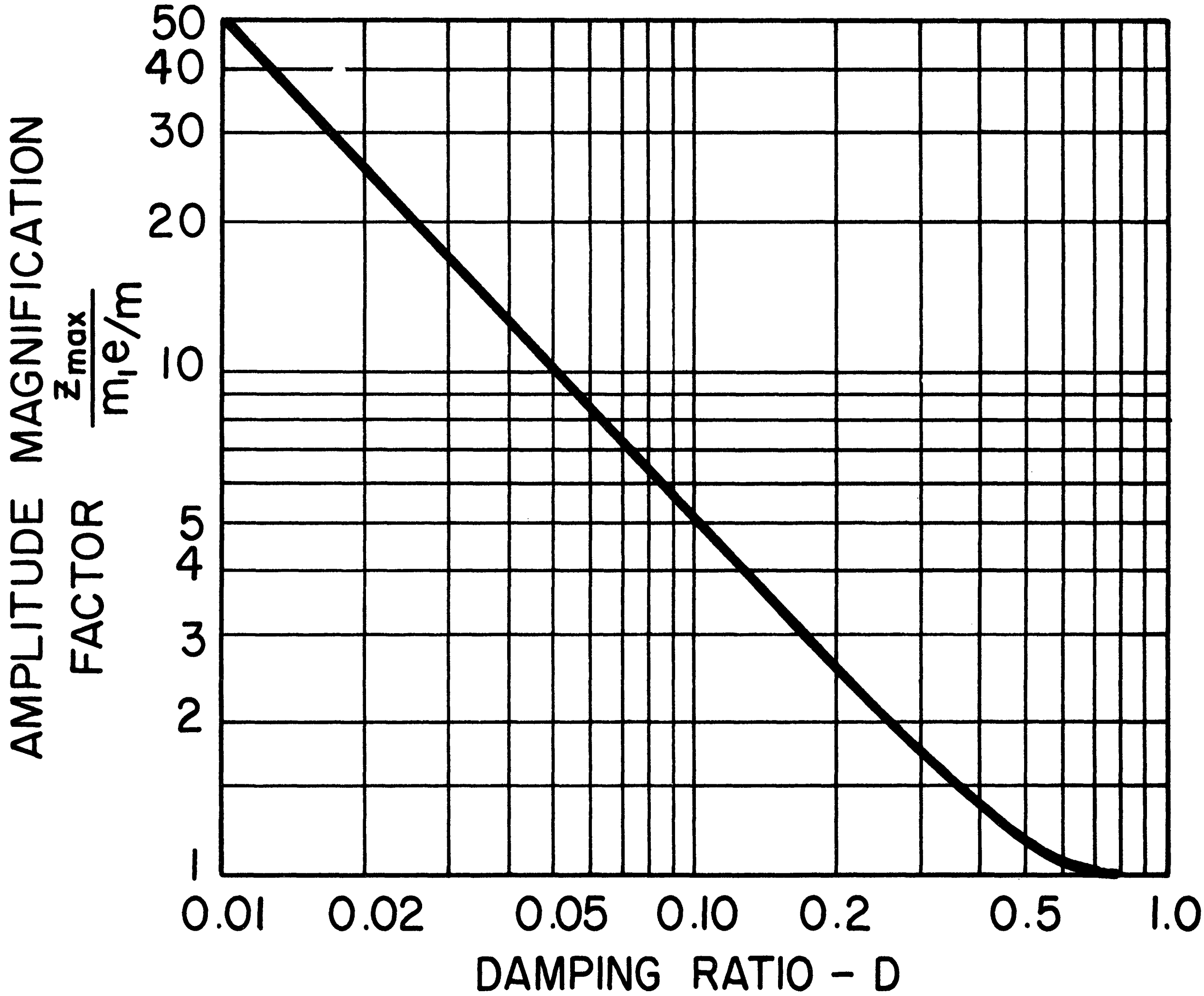


Figure 2. Influence of Frequency and Damping on Mass-Spring-Dashpot System Loaded by Rotating Mass Exciter.

Analysis		Factors Required
Approximate estimate for resonant frequency		k and m
Approximate estimate for motions at frequencies well away from resonance	$\ll f_o$	k
	$\gg f_o$	m
Upper limit for motion at frequencies near resonant frequency		D and k or m

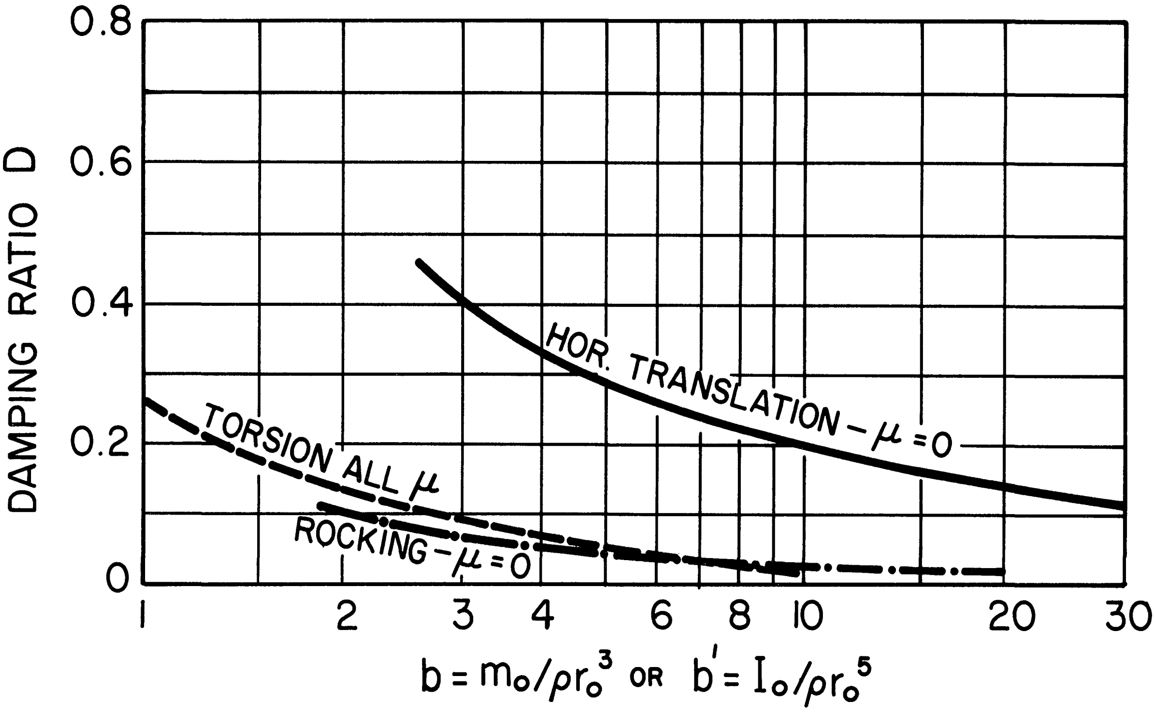
Figure 3. Summary of Parameters Required For Dynamic Analysis.

natural frequency, which requires that the spring constant and mass must be known. Since it is easier to make a reasonable estimate for the mass, it is considered after damping. Finally, the spring constant, which is at the same time the most important and the most difficult parameter for the engineer to evaluate, is discussed.

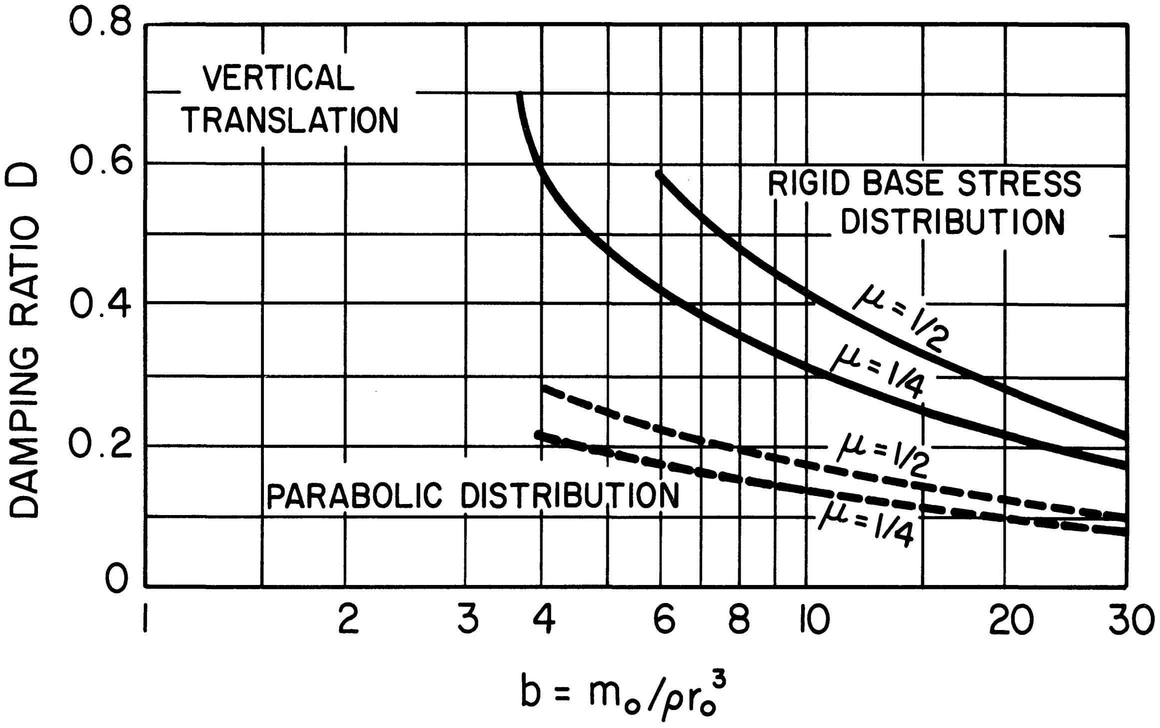
CHOICE OF DAMPING FOR EQUIVALENT LUMPED SYSTEMS

The dashpots of the lumped system represent the damping of the soil. There are two types of damping: the loss of energy through propagation of waves away from the immediate vicinity of the footing, and the internal energy loss within the soil due to hysteretic and viscous effects. The use of dashpots in the lumped system does not necessarily imply that the engineer believes that soil has viscous properties. Linear dashpots are used in order to derive simple, useful mathematical expressions for the response of the lumped system. The damping ratios are chosen to represent an equivalent amount of damping, and not to represent a particular type of damping. In general, the damping ratios are different for each mode of motion.

Damping due to radiation. The theory for the elastic half-space can be used to provide estimates of the magnitude of the radiation damping which should be included in the equivalent lumped system for a foundation. Figure 4 gives one set of curves for equivalent damping ratio, obtained by equating the maximum amplitudes of motion as determined by the elastic half-space theory to that given by the damped, one-degree-of-freedom method. The mass ratio b from the elastic half-space theory is the key parameter, and is defined as



(a) Amplitude-Frequency Relations



(b) Peak Amplitude-Damping Relations

Figure 4. Charts for Equivalent Damping Ratio (from Whitman, 1966).

$$b = \frac{m_o}{\rho r_o^3} \quad \text{for translation, and} \tag{6a}$$

$$b' = \frac{I_o}{\rho r_o^5} \quad \text{for rotation,} \tag{6b}$$

where m_o = the mass of the foundation block plus machinery

I_o = the mass moment of inertia of the foundation block plus machinery, evaluated about the vertical axis through the center of gravity for torsional motion, or about a horizontal axis through the centroid of the foundation base in the case of rocking

ρ = mass density of the soil

r_o = radius of the soil contact area at the foundation base.

Figure 4 shows clearly that the damping due to radiation is most important when the mass ratio b is small.

The foregoing equations and charts can be used to provide estimates of the radiation damping for rectangular bases by converting the rectangular shape into an equivalent circular base. The radius of the equivalent circular base is given by:

$$r_o = \begin{cases} \sqrt{\dfrac{BL}{\pi}} & \text{for translation} \\ \sqrt[4]{\dfrac{BL^3}{3\pi}} & \text{for rocking} \\ \sqrt[4]{\dfrac{BL(B^2+L^2)}{6\pi}} & \text{for twisting} \end{cases} \tag{7}$$

where B = width of foundation (along axis of rotation for case of rocking)

L = length of foundation (in plane of rotation for case of rocking)

<u>Internal damping of soil.</u> For dry or relatively dry cohesionless soils an energy loss during stress reversals is developed by sliding between mineral particles. This loss is exhibited by a hysteresis loop in the stress-strain relations obtained under either static or

dynamic conditions. The magnitude of this energy loss, or damping, is primarily a function of the amplitude of strain and the initial confining pressure. Hardin (1965a) has developed an empirical relation for the logarithmic decrement, δ , of dry sands undergoing shearing deformations. This may be interpreted in the form of the damping ratio as

$$D \approx \frac{\delta}{2\pi} = 4.5\, \gamma_{xz}^{0.2}\, \sigma_o^{-0.5} \qquad (8)$$

in which γ_{xz} is the shearing strain and $\sigma_o(\mathrm{lb/ft}^2)$ is the confining pressure. Equation 8 is applicable within the limits of shear strain amplitudes of 10^{-6} to 10^{-4} , for confining pressures between 500 and 3000 psf and for frequencies less than 600 cps. Below 600 cps, the effect of frequency is apparently negligible, and the effect of changes in void ratio are relatively unimportant.

Test on saturated sands (Hall and Richart, 1963) have indicated that the effect of amplitude of strain is less important than it is for the dry condition. However, the log decrement for saturated sands was found to be consistently greater than that for dry sands under the same confining pressure, indicating the effect of damping resulting from relative motion between the soil skeleton and pore fluid.

Table 1 summarizes some of the available information relating to the internal damping of soils at the level of stress changes occurring under machine foundations. Test results given as damping capacity or log decrement have been expressed in terms of an equivalent damping ratio, D . From this table, it is evident that a typical value of D is of the order of 0.05 for internal damping in soils.

Combined effects of radiation and internal damping. In order to give some crude estimate of the combined effects of radiation and

TABLE 1

INTERNAL DAMPING IN SOILS

Type Soil	Equivalent D	Reference
Dry sand and gravel	0.03 to 0.07	Weissmann and Hart (1961)
Dry and saturated sand	0.01 to 0.03	Hall and Richart (1963)
Dry sand	0.03	Whitman (1963)
Dry and saturated sands and gravels	0.05 to 0.06	Barkan (1962)
Clay	0.02 to 0.05	Barkan (1962)
Silty sand	0.03 to 0.10	Stevens (1966)
Dry sand	0.01 to 0.03	Hardin (1965)

internal damping, the typical value of D = 0.05 for internal damping may be added to the values of D given on Figure 4. For horizontal translation, and especially for vertical translation, internal damping appears to be relatively unimportant when compared to radiation damping. For rotational motions, however, the radiation damping is low and the internal damping then becomes a significant part of the total damping.

It is particularly significant that the radiation damping for the rocking mode of oscillation (See Figure 4) has such low damping because this type of oscillation, coupled with one or more of the translatory modes of vibration, commonly occurs in machine foundations. Because of this low damping, the resonant frequency is an important factor to be considered in design against rocking oscillations, whereas the resonant frequency may be of little significance for vertical oscillations which develop high values of radiation damping. Thus for the analysis of coupled vibrations for which one mode of vibration is associated with high damping and the other with low damping, it becomes necessary to evaluate the amplitude-frequency response curve over the entire possible frequency range.

The foregoing comparison between the effectiveness of radiation damping and that for internal friction damping of the soils has demonstrated the value of the elastic half space theory in establishing radiation damping for various modes of oscillation of a simplified footing. The theory treats only footings which rest on the surface of the half space, whereas most foundations are partially embedded. Barkan (1962), Pauw (1952), and Maxwell, et al (1967) have reported on tests of footings partially embedded as well as for footings at the surface. In general, partial embedment reduces the magnitude of motions at the resonant peaks, thus indicating greater damping. However, the influence of embedment on the amplitude and frequency of oscillation depends upon the mode of vibration and the magnitude of the motion. For motions within the range of design criteria for machinery, it appears that this reduction in amplitude by partial embedment is of the order of 10 to 25 per cent. Therefore, the design calculations will be on the conservative side if the footing is considered to rest on the surface. Further field tests are needed to establish the influence of partial embedment on the dynamic behavior of footings, particularly in the rocking mode.

A second major discrepancy between the theoretical assumptions and real installations lies in the assumption of a uniform elastic homogeneous supporting body. Often the subgrade is layered, with a hard stratum of soil or rock at a shallow depth below the footing. This stratum will impede the radiation of energy from the footing, by reflecting part of it back to the footing, thereby reducing the radiation damping. Warburton (1957) has indicated that the influence of a rigid layer may be significant if it is within about 3 footing diameters of the surface. However, if such a layer is close enough to the surface to cause important magnification of the footing vibrations through reflections, this proximity may be used to advantage in providing a rigid support for a pile foundation for the footing. Each time the prototype conditions differ from the assumptions upon which the theoretical solution was based requires the exercise of engineering judgement on the part of the designer.

CHOICE OF MASS FOR EQUIVALENT LUMPED SYSTEM

Clearly the mass of the equivalent lumped system should at least include the mass of the foundation block plus the mass of the machinery. At first glimpse, it might appear that an additional mass term should also be used represent the inertia of the soil underlying the foundation block.

Actually, there is no such thing as an identifiable mass of soil which moves with the same amplitude and in phase with the foundation block. At any instant of time, various points within the soil are moving in different directions with different magnitudes of acceleration. The use of an "effective mass" is justified only to the extent that a mass larger than that of the foundation block plus machinery is needed to make the response curve of the lumped mass fit the response curve of the actual system. If an "effective mass" is used, it must be regarded as a totally fictictious quantity which cannot meaningfully be related to any actual mass of soil.

The simplest assumption which can be made when choosing the mass of the lumped system is simply to take this mass equal to that of the foundation and machinery, and to ignore any "effective mass" of the soil. In the companion paper by Richart and Whitman (1967) it was indicated that for vertical oscillations the lumped system with zero effective soil mass established amplitudes of motions which were for all practical purposes the same as those given by tests on actual foundations. A comparison of response curves from the lumped system with zero effective soil mass and from the elastic half space theory (Lysmer and Richart, 1966) showed good agreement between these curves for frequencies from zero to well above resonance.

Thus, for most engineering analyses it is prudent to make the simple assumption of zero effective mass of the soil to act with the footing, and to concentrate upon making the best possible estimates for the damping and spring constants.

Estimates for the "effective mass" using the theory for the half-space. Occasionally engineers may feel that they have achieved very accurate estimates for the spring constant and damping, such that it becomes worthwhile to choose an "effective mass". Table 2 gives values of "effective mass" as developed from Hsieh (1962). Note that the "effective mass" is different for each mode of motion.

The mass ratio plays an important role when assessing the possible importance of "effective mass". For actual foundations, b typically has values between 2 to 10 for translation and between 1 and 5 for rotation. Thus the "effective masses" given in Table 2 are small compared to the mass of the foundation block unless the values of b are unusually small. Moreover, as discussed previously, foundations involving small values of b do not have sharp resonant peaks, (especially for translational motions), so that the phenomenon of resonance is of little importance in such systems. Since one of the main reasons for choosing the mass of a dynamic system is to provide an estimate for the natural frequency of the system, the "effective mass" is of significant magnitude only when the mass is of little practical consequence.

Other estimates for "effective mass": Various writers in the past have attempted to estimate the "effective mass" on intuitive grounds or by fitting response curves for lumped systems to the response curves of actual foundations: for example, see Crockett and Hammond (1949),

TABLE 2

EFFECTIVE MASS AND MASS MOMENT OF INERTIA FOR
SOIL BELOW A VIBRATING FOOTING
(from Hsieh, 1962)

Mode of Vibration	Effective mass or mass moment of inertia of soil		
	$\mu=0$	$\mu=1/4$	$\mu=1/2$
Vertical translation	$0.5\rho r_o^3$	$1.0\rho r_o^3$	$2.0\rho r_o^3$
Horizontal translation	$0.2\rho r_o^3$	$0.2\rho r_o^3$	$0.1\rho r_o^3$
Rocking	$0.4\rho r_o^5$	Not computed	
Torsion (about vert. axis)	$0.3\rho r_o^5$	$0.3\rho r_o^5$	$0.3\rho r_o^5$

Lorenz (1953) and Heukelom (1959). The results obtained thereby have scattered widely, and sometimes have appeared to be unreasonable.

In the face of these difficulties, it is best to fall back upon the conclusions derived from the theory for the half-space. This theory correctly takes the inertia of the body into account without intuitive assumptions, and has been shown to correctly predict the behavior of actual foundations. To repeat the main conclusion from this theory: the ficticious "effective mass" is small and it usually is best to ignore it completely.

CHOICE OF SPRING CONSTANTS, USING THEORETICAL EQUATIONS AND MEASURED STRESS-STRAIN RELATIONS

The spring constant is the most important of the three parameters involved in a lumped system. The value of the spring constant affects the frequency of the resonant peak, the magnitudes of the motions which occur at frequencies well below the natural frequency, and (except in the special circumstance of the eccentric mass machine) the magnitude of motions at resonance. While much the same can be said regarding the importance of the mass of the lumped system, it was seen in the previous paragraph that the proper value for this mass is known within rather narrow limits. However, the spring constant can seldom be pinned down this closely. It can be said that an estimate of dynamic response can be no better than the estimate of the spring constant.

Any method for evaluating the spring constant for a particular problem must in some way account for the following factors, some of which are interrelated: (1) the effect of partial embedment of the footing, (2) the dependence of the spring constant upon the initial static stress as well as upon the magnitude of the dynamic stress increment, (3) the distribution of stresses over the contact area between the foundation block and the soil, and (4) the dependence of the spring constant upon the size of the contact area, which in turn depends upon the variation of the modulus of the soil with depth or with the presence of a layered soil structure.

The outline below indicates several methods which are available for establishing reasonable values of the spring constant.

<ins>Method A</ins> - Use formulas for spring constants derived from the theory of elasticity and evaluate the elastic constants either from in-situ shear wave velocity measurements or from laboratory tests.

<ins>Method B</ins> - Determine spring constants from small-scale plate bearing tests using static repeated loadings.

<ins>Method C</ins> - Deduce spring constants from the results of small-scale vibrator tests.

<ins>Method D</ins> - Use the concept of an elastic subgrade modulus along with tables or charts correlating subgrade modulus to soil type.

None of these methods is necessarily better than the others because each involves approximations and assumptions, and considerable engineering judgement is required to take into account the factors listed in the preceding paragraph. This section considers Method A of the general methods indicated above, while Methods B , C , and D are considered in the next section.

Except for Method C, all of these approaches are basically the same as those which would be used to estimate load-settlement relationships for conventional, static problems. Hence the soil engineer is already acquainted with the problems involved in the application of these approaches. The only different feature of the problem is that the spring constants must be evaluated for the ranges of strain corresponding to the dynamic rather than static loadings.

Formulas for spring constants: These formulas relate the spring constants to the basic stress-strain behavior of the soil. At the present time, such formulas must inevitably be based upon the theory of elasticity - a reasonable assumption as long as the live load stresses are less than one-half of the dead load stresses.

Table 3 gives formulas for the spring constants in the case of a rigid circular base resting upon the surface of an elastic half-space. Table 4 gives formulas applicable to rigid rectangular bases. The formula for the vertical spring constant is rigorous. The formula for horizontal spring constant was obtained by assuming a uniform distribution of shear stress over the contact area and computing the average horizontal displacement of the contact area. The formula for rocking is rigorous. The authors know of no solution for torsional motion involving a rectangular foundation. An approximate evaluation may be made by using an equivalent circular base having the same second moment for the contact area.

In many problems, the distribution of stress over the contact area will be different from that for a rigid base upon an elastic medium. For example, it is often assumed that a vertically loaded footing resting on the surface of a sand layer develops a parabolic distribution of contact

TABLE 3

SPRING CONSTANTS FOR RIGID CIRCULAR BASE RESTING ON ELASTIC HALF-SPACE

Motion	Spring Constant	Reference
Vertical	$k_z = \dfrac{4Gr_o}{1-\mu}$	Timoshenko and Goodier (1951)
Horizontal	$k_x = \dfrac{32(1-\mu)Gr_o}{7-8\mu}$	Bycroft (1956)
Rocking	$k_\phi = \dfrac{8Gr_o^3}{3(1-\mu)}$	Borowicka (1943)
Torsion	$k_\theta = \dfrac{16}{3} Gr_o^3$	Reissner and Sagoci (1944)

(Note: $G = \dfrac{E}{2(1+\mu)}$)

TABLE 4

SPRING CONSTANTS FOR RIGID RECTANGULAR BASE RESTING ON ELASTIC HALF-SPACE

Motion	Spring Constant	Reference
Vertical	$k_z = \dfrac{G}{1-\mu}\beta_z \sqrt{BL}$	Barkan (1962)
Horizontal	$k_x = 2(1+\mu)G\beta_x \sqrt{BL}$	Barkan (1962)
Rocking	$k_\phi = \dfrac{G}{1-\mu}\beta_\phi BL^2$	Gorbunov-Possadov (1961)

(Note: values for β_z, β_x, and β_ϕ are given in Figure 5 for various values of L/B)

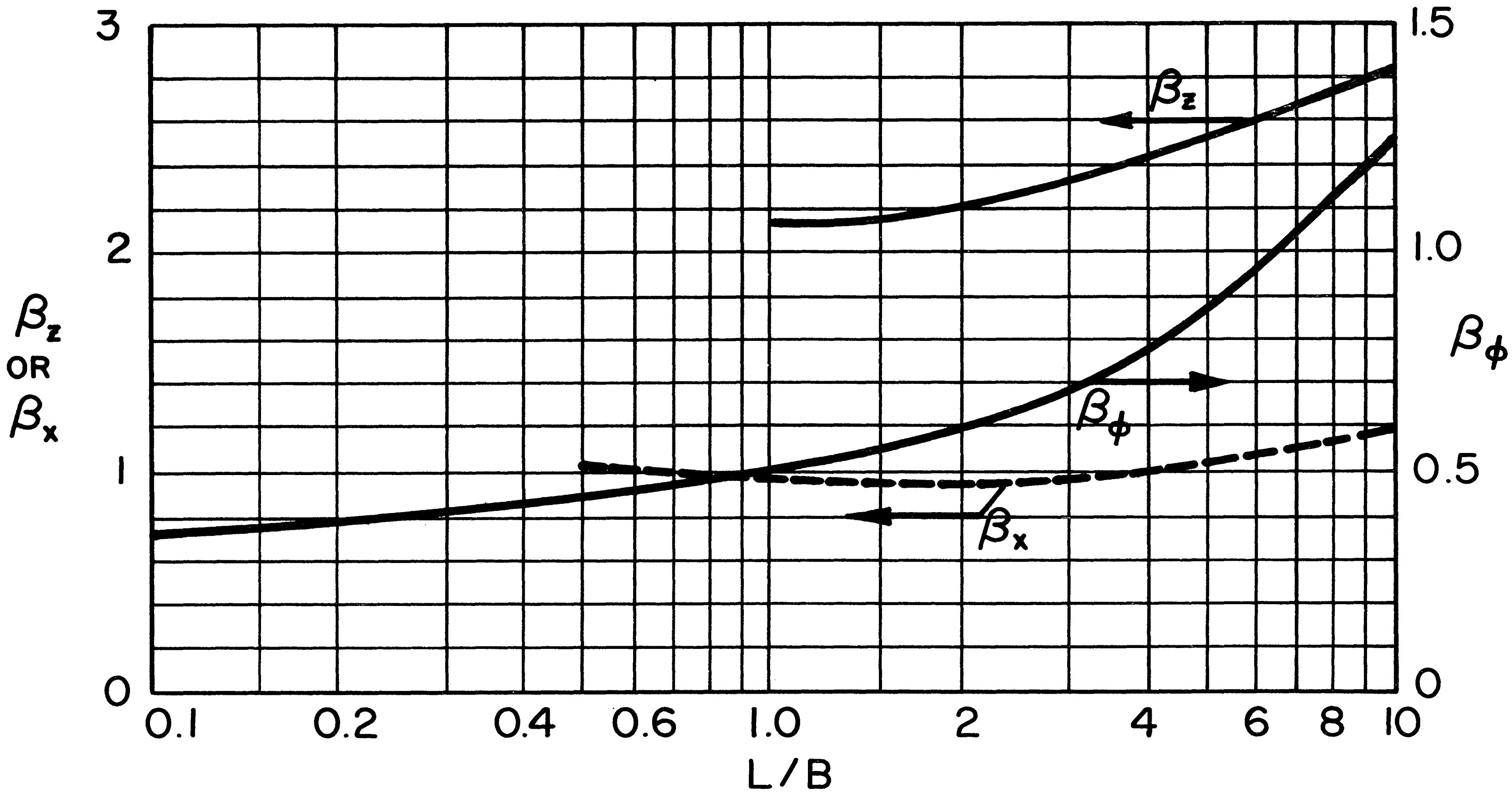


Figure 5. Coefficients β_z, β_x, and β_ϕ for Rectangular Footings.

pressure on the base. For a circular loaded area, a parabolic distribution of contact pressure gives a spring constant (based upon settlement at the center of loaded area) only 59% of that for the rigid base distribution. However, this parabolic distribution corresponds to failure conditions, and with lower stress levels and an embedment of the footing, the pressure distribution approaches that for the rigid base condition. The stress distribution existing under dynamically loaded foundations are discussed in the companion paper by Richart and Whitman (1967).

Poissons's ratio, μ, for soils usually varies only between 0.35 and 0.5 for the stress ranges encountered in dynamic loadings. Hence uncertainty as to Poisson's ratio introduces relatively little uncertainty into the estimate of the spring constant. The major problem is to estimate either the shear modulus G or Young's modulus E.

Use of dynamic laboratory tests to obtain elastic constants. The theory and practice of dynamic triaxial tests have been discussed by Wilson and Dietrich (1960) and by Hardin and Richart (1963). Triaxial specimens are vibrated at various frequencies until a resonant condition is obtained. From the resonant frequency and sample length it is possible to determine the wave velocity and the elastic modulus. By using longitudinal vibrations, one obtains Young's modulus E, or by using torsional vibrations one obtains the shear modulus G. Knowing these two moduli, it is possible to compute Poisson's ratio μ, although the computed value of μ is very sensitive to slight errors in E and G.

Shear wave velocities and hence values of shear modulus may also be conveniently obtained from laboratory tests using the ultrasonic pulse technique (Lawrence, 1965).

In general, the modulus of soil is a function of the void ratio of the soil and of the confining pressure. Thus good undisturbed samples must be obtained in the case of cohesive soils, and some measurement of in-place void ratio must be obtained in the case of sands. Tests must generally be carried out at several confining pressures. Lawrence (1965) and Hardin and Black (1966) have both found that the shear modulus is related to the mean of the three principal stresses acting upon the soil and is independent of the amount of static shear stress which is present. Hall and Richart (1963) have shown that the modulus measured in dynamic triaxial tests is influenced by the magnitude of the dynamic strains. Consequently, the strains used in these tests should be similar to the magnitudes of strains expected just below the foundation in the prototype situation.

Because in actual soil deposits the shear modulus generally varies with depth, it is necessary to choose an effective value of G for use with the theoretical analysis. For the computations described in the paper by Richart and Whitman (1967) for correlation with the field tests by Maxwell, et al. (1967) the following procedure gave useful values of G : (a) the distribution of vertical pressure caused by the weight of the footing and machinery was computed along a vertical line extending downward from the periphery of the footing, (b) the vertical pressure caused by the soil overburden was also computed along this line, (c) the minimum value of total pressure determined by adding these two effects was determined, and this often occurred at a depth of about one radius, and (d) this minimum value of pressure was used with the dynamic soil data to establish the effective value of shear modulus.

Use of static laboratory tests. In the case of sands, there is impressive evidence that the modulus measured during ordinary triaxial tests with repeated static loadings (which develop strains comparable to dynamic strains) agrees well with the modulus as determined from dynamic triaxial tests: see Shannon et al (1959), Whitman et al (1964), and Hardin (1965 b).

As for cohesive soils, Wilson and Dietrich (1960) provided some evidence of the same nature. More study is needed of this subject, but events may prove that the elastic modulus needed to compute the spring constants for dynamic loading can be measured satisfactorily using ordinary triaxial testing equipment but with special techniques for evaluating small strains. In principle, both longitudinal and torsional repeated loadings are possible in a triaxial cell. In practice, it will prove easier to run only longitudinal static loading tests, but with minor modification to standard triaxial equipment, it is possible to incorporate the dynamic torsional tests (Hardin, 1966).

Use of seismic velocities measured in-situ. It is relatively easy to measure the seismic dilatational wave velocity through soil. However, it is difficult to determine shear wave velocity (and hence shear modulus) from dilatational wave velocity unless Poisson's ratio is known very accurately. Only recently have satisfactory techniques been developed for measuring the shear wave velocity directly: see Maxwell, et al (1966).

Typical values for modulus and Poisson's ratio. Figure 6 shows the shear wave velocity through quartz sand as a function of void ratio and confining pressure. These relations apply at least approximately for both uniform and well-graded sands and for both dry and saturated sands.

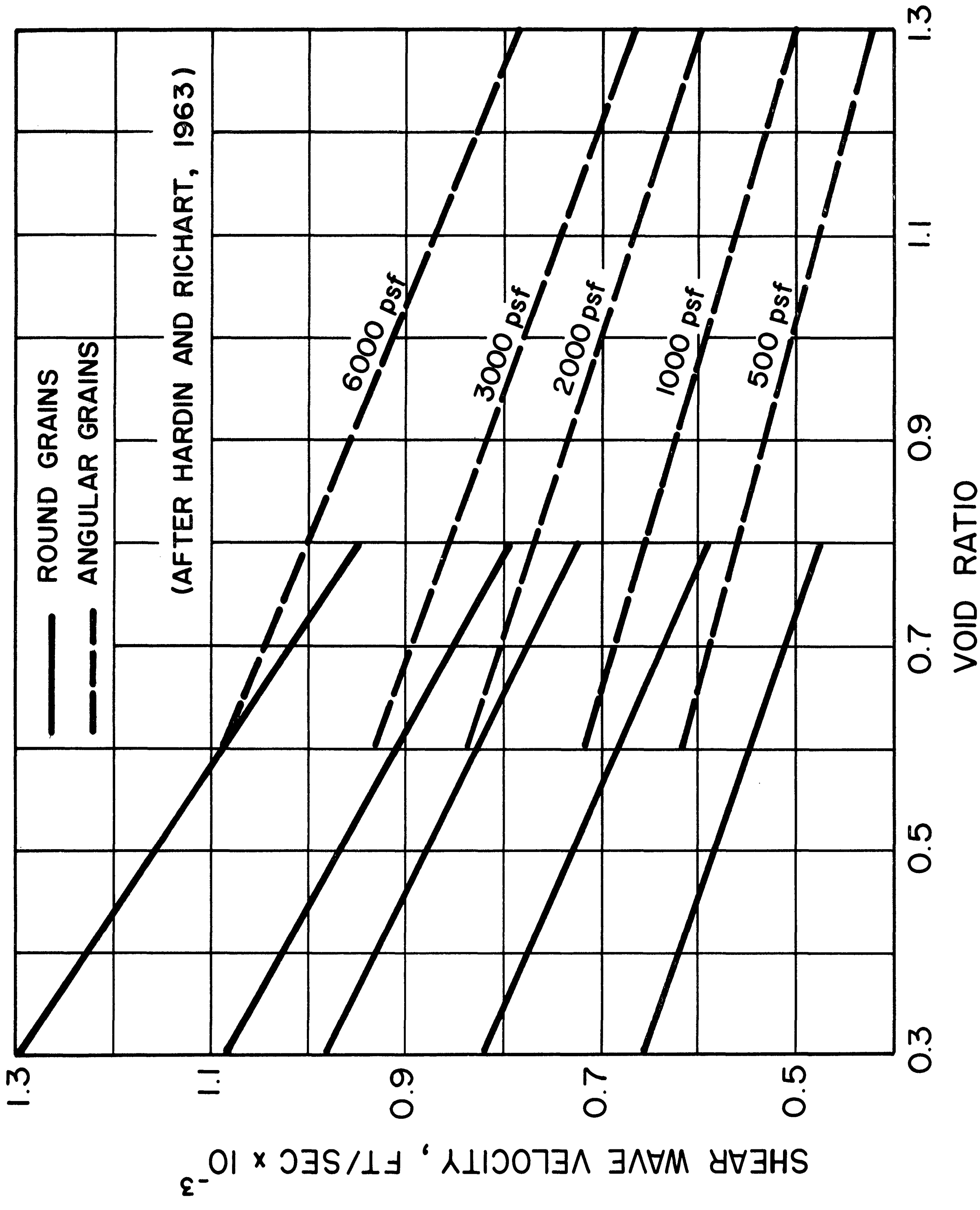


Figure 6. Shear Wave Velocity vs. Void Ratio and Confining Stress, for Quartz Sand.

The shear wave velocity may be converted to shear modulus using the relation

$$G = \rho(v_s)^2 \tag{9}$$

where

v_s = shear wave velocity

and

ρ = mass density, based upon the total unit weight of soil.

Figure 6 and the Equation 9 can be used to provide at least preliminary estimates of the shear modulus through any sand. Data by Lawrence (1965) suggest that the same curves will apply approximately for clays as well.

Figure 7 gives Young's modulus for cohesive soils as a function of the undrained compressive strength; i.e., the maximum deviator stress during an undrained triaxial test. The solid curve applies except for the bentonitic clays. While much more work is needed to establish the limits of validity of this correlation, it can at least be used for preliminary estimates of Young's modulus for cohesive soils.

For a sand, whether dry or saturated, Poisson's ratio is generally between 0.35 and 0.4 (Whitman and Lawrence, 1964). This ratio is very close to 0.5 for saturated clays (Wilson and Dietrich, 1960). A value of 0.4 is a good average value for most partially saturated soils.

CHOICE OF SPRING CONSTANT, USING FIELD TESTS AND SUBGRADE MODULUS

Plate bearing tests. Barkan (1962) cites numerous field tests which prove that the spring constant applicable to periodic dynamic motion is essentially equal to the ratio of increment of load to increment of

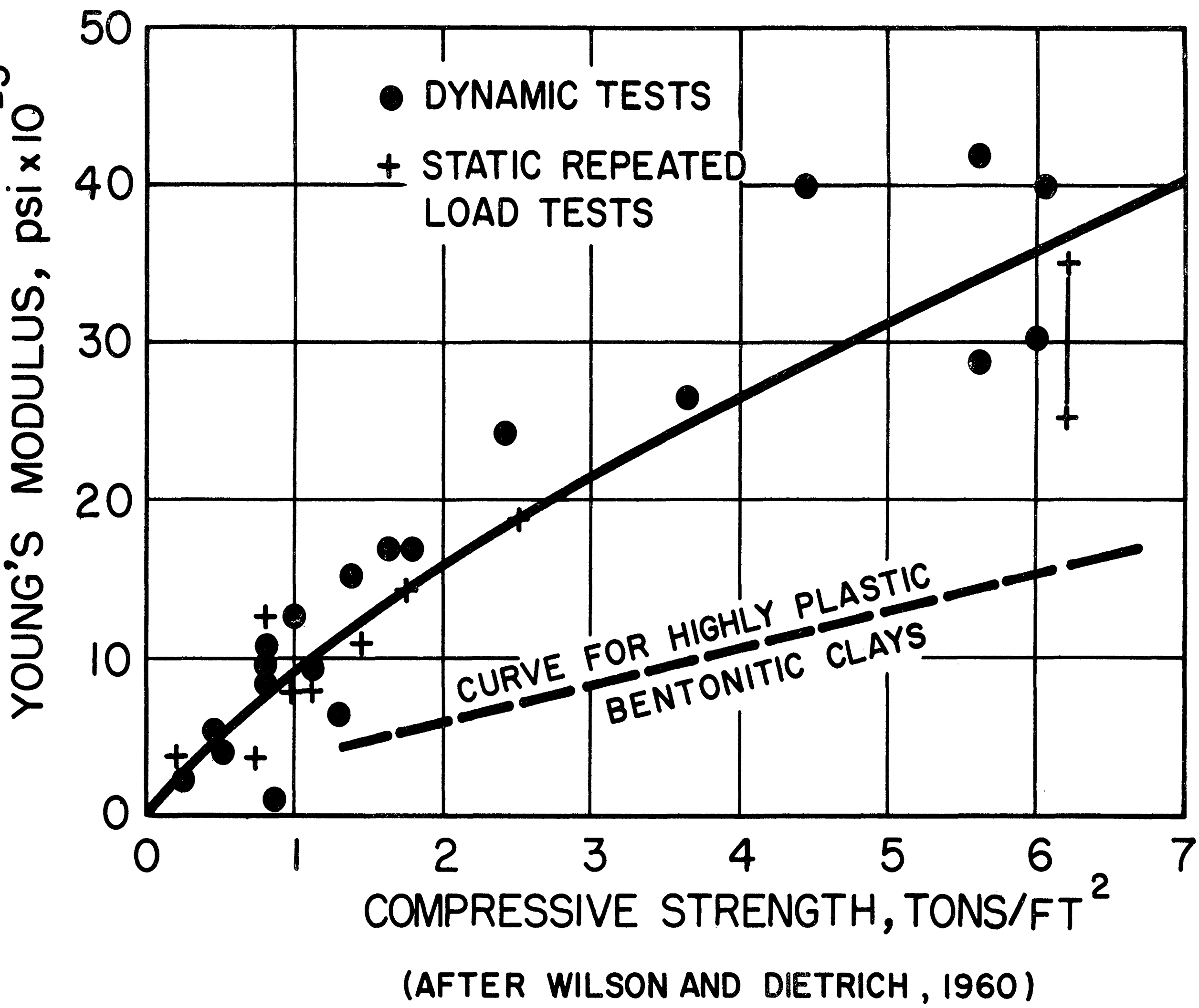


Figure 7. Correlation Between Young's Modulus and Compressive Strength for Saturated Clays.

deflection (or moment to rotation) during static repeated loading tests. Table 5 compares the resonant frequency observed during dynamic tests upon a foundation block with the undamped natural frequency computed using just the mass of the foundation block plus machinery, and using the value of k measured during a static repeated loading test upon the same foundation block.

The key here is the use of repeated loadings. The magnitude of the "dead load" and "live load" stresses should be similar to those expected under the actual foundation. The loads should not be applied so slowly that consolidation and secondary compression effects are present; it suffices to apply and remove the loads only as quickly as the necessary reading may be taken - say a cycle every 10 to 20 minutes.

The chief difficulty with this approach lies in the need to extrapolate the spring constants as measured using small bearing areas so as to obtain values applicable to the bearing area actually to be used. The discussion by Terzaghi (1955) and others concerning the choice of a subgrade modulus for static loading problems applies to the machine foundation problem as well. Depending upon whether the soil is thought to behave as a highly cohesive or cohesionless material, one has the choice of extrapolating k_z (and presumably the other coefficients as well) according to one of the following rules:

highly cohesive soil: $$k_z = k_{z_1} C \qquad (10a)$$

cohesionless soil: $$k_z = k_{z_1} \frac{(C+1)^2}{4} \qquad (10b)$$

TABLE 5

COMPARISON OF RESONANT FREQUENCY
WITH NATURAL FREQUENCY COMPUTED ON
BASIS OF REPEATED LOADINGS

(after Barkan, 1962)

Soil	Areas of Foundations (ft^2)	Ratio of Frequency at Peak Motion $f_{obs.}/f_{comp.}$
Clay	21-86	0.86-1.21
Clay	5-16	0.99-1.04
Sand	11-161	0.93-1.03
Loess	9-43	0.85-0.98
Avg. from 15 data points		0.97±0.062

where

$$k_{z_1} = k_z \text{ for bearing test on a plate one foot square}$$

and

$$C = \text{least dimension of foundation, in feet.}$$

The ratio k_z/k_{z_1} is plotted in Figure 8 and clearly indicates that the extrapolation can introduce considerable uncertainty.

As a final comment, it should be noted that it is not easy to conduct the loading tests. The movements during the repeated loadings will be small, and special instrumentation and care are needed for the satisfactory measurement of such small movements.

Small-scale vibrator tests. In this approach, a small vibrator is set upon a small plate (perhaps 12 to 30 inches in diameter) and the frequency of excitation is varied until a resonance condition is achieved. The spring constant is then computed from the measured resonant frequency.

The conversion from observed resonant frequency to spring constant involves some uncertainties. The effective mass of the soil must be either neglected or estimated. If a constant force type of oscillator is used, the resonant frequency may be significantly less than the undamped natural frequency. With very small bases, it is difficult to keep the accelerations to less than 0.5 that of gravity, but unless this is done the spring constant will not correspond to prototype conditions because of non-linear effects.

Having obtained the spring constant for the small base, it still is necessary to extrapolate this result to the prototype base. Thus, the approach using a small vibrator suffers from all the difficulties

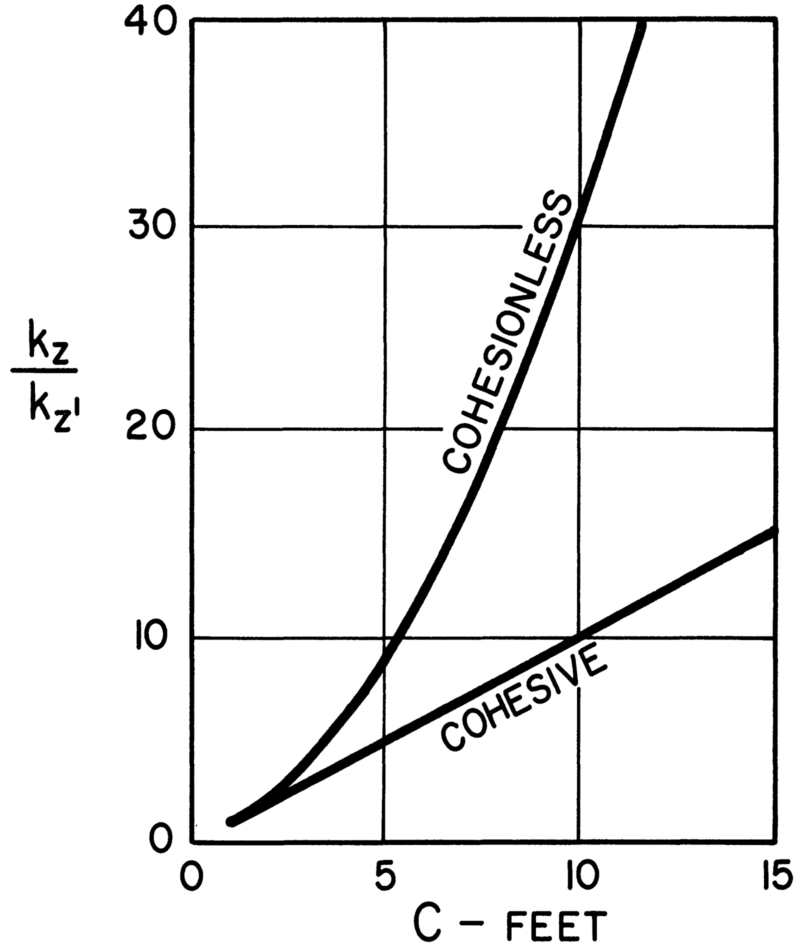


Figure 8. Extrapolation of Spring Constant from Plate Bearing Test.

described in connection with plate bearing tests. The same method that is used to deduce a spring constant from the small vibrator results should be used to calculate the natural frequency of the prototype foundation.

In most such tests, only vertical excitations are used. Spring constants applicable to other modes can be estimated by taking the ratios of the formulas in Tables 3 and 4.

Tables for spring constants. Using the concept of subgrade modulus, the spring constants for the various modes can be written in the following form:

$$\begin{aligned} &\text{vertical motion:} && k_z = c_z S \\ &\text{horizontal motion:} && k_x = c_x S \\ &\text{rocking motion:} && k_\phi = c_\phi I' \\ &\text{torsional motion:} && k_\theta = c_\theta I'' \end{aligned} \qquad (11)$$

where

S = area of horizontal contact between foundation and soil

I' = second moment of contact area about horizontal axis through centroid of area and normal to plane of rocking

I" = second moment of contact area about vertical axis through centroid of area.

The coefficients c_z , c_x , c_ϕ , and c_θ are subgrade moduli, and are functions of soil type, size, and to a certain extent, of geometry of the foundation. However, these coefficients are often assumed to be functions only of soil type. Barkan (1962) has provided the data in Table 6 for c_z, and has suggested that the remaining coefficients can be evaluated approximately as follows:

$$c_x \approx c_z/2$$
$$c_\phi \approx 2c_z \qquad (12)$$
$$c_\theta \approx 1.5\ c_x$$

The spring constants computed on the basis of the above equations and Table 6 can be used for preliminary design when circumstances prevent a more thorough investigation of the stiffness of the foundation soil.

CONCLUSIONS

The objective of this paper has been to discuss the principles and procedures which may be employed for the analysis and design of dynamically loaded foundations. The following statements summarize the key points from the paper.

(a) The basic criterion regarding the satisfactory performance of a dynamically loaded foundation is the permissible dynamic motion. It is important that soil engineers understand the origin of the criteria given for any particular problem in order to apply these criteria properly and to be able to advise clients of their significance.

(b) In order to ensure that the dynamic motions meet the criteria for situations involving low damping, it is necessary to ensure that the natural frequencies of the foundation do not coincide with the operating frequency. However, for cases involving high damping the frequency corresponding to the maximum of the response curve has relatively little significance.

(c) Because soil is a complex material, engineering judgement is required to arrive at a satisfactory estimate for the dynamic motions

TABLE 6

RECOMMENDED DESIGN VALUES FOR COEFFICIENT c_z

(from Barkan, 1962)

Soil Group	Allowable static bearing stress ton/ft^2	Coefficient c_z ton/ft^3
Weak soils (clay and silty clays with sand, in a plastic state; clayey and silty sands; also soils of categories II and III with laminae of organic silt and of peat)...	1.5	95
Soils of medium strength (clays and silty clays with sand, close to the plastic limit; sand)........	1.5 - 3.5	95 - 155
Strong soils (clays and silty clays with sand, of hard consistency; gravels and gravelly sands;loess and loessial soils.....	3.5 - 5	155 - 310
Rocks	5	310

of a foundation. These judgements can be made most satisfactorily, keeping the clearest line from assumption to result, within the framework of evaluating the parameters of a lumped system which is equivalent to the actual foundation. In addition, the use of equivalent lumped systems is the most convenient way for analyzing coupled motions.

(d) The theory for a rigid disk resting upon an elastic half-space has led to a major breakthrough in our understanding of the foundation dynamics problem. This theory has shown the proper role of the inertia of the soil, and has pointed out the important phenomenon of radiation damping. This theory is the primary tool for evaluating the mass and damping for the equivalent lumped system.

(e) Damping is provided in part by radiation damping and in part by the internal damping of the soil. When translational movements occur, the damping due to radiation often overshadows the internal damping and such motions are highly damped. When rotational motions occur, the two types of damping are likely of about equal value, thus systems with such motions have relatively little damping.

(f) The mass of the equivalent lumped system can, with an accuracy which is adequate for engineering purposes, simply be taken as the mass of the foundation block plus machinery. The "effective mass" of the soil, which has caused so much controversy and confusion in the past, is so small as to be of little consequence.

(g) Engineers should concentrate most of their attention upon obtaining suitable values for the spring constants. Several methods of obtaining spring constants have been described, and the approximations involved in each were discussed.

ACKNOWLEDGEMENT

The development of the approaches and procedures outlined in this paper was stimulated and aided by the foundation dynamics investigation undertaken by the office of the Chief of Engineers, U. S. Army and by the U. S. Army Engineer Waterways Experiment Station. The writers wish to acknowledge the cooperation and assistance provided by Messrs. Frank Hennion and T.R. Wathen of OCE, and of Messrs. W.J. Turnbull, A.A. Maxwell, Z.B. Fry, and others of WES. The general approach outlined herein is very similar to that which has been published in Engineer Circular 1110-345-310. Special acknowledgement is also due to the other members of the Panel of Consultants who acted as advisers to OCE and WES during this program of tests: Dean R.E. Fadum, Professors N.M. Newmark, R.B. Peck, and W.J. Hall, Dr. Lydik Jacobsen, and Mr. S.D. Wilson. Finally, the writers wish to thank their colleagues at their respective universities for their helpful discussions and suggestions.

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APPENDIX

LIST OF SYMBOLS

B	Width of foundation, along axis of rotation for rocking or normal to direction of horizontal force.
b	Parameter in theory for mass supported by elastic body.
C	Least dimension of foundation, in feet.
c_z	Subgrade modulus for vertical motion.
c_x	Subgrade modulus for horizontal motion.
c_ϕ	Subgrade modulus for rocking.
c_θ	Subgrade modulus for torsional rotation.
D	Damping ratio, i.e. of damping to critical damping.
E	Young's modulus.
e	Eccentricity of unbalanced mass.
f	Frequency (cycles per second).
f_o	Undamped natural frequency, defined by Equation 3.
G	Shear modulus.
I_o	Moment of inertia of foundation block plus machinery about axis of rotation.
I'	Second moment of contact area about horizontal axis.
I"	Second moment of contact area about vertical axis.
k	Spring constant for mode of motion under consideration.
k_z	Spring constant for vertical translation.
k_{z_1}	Value of k_z in bearing test on plate 1 foot square.
k_x	Spring constant for horizontal translation.
k_ϕ	Spring constant for rocking.
k_θ	Spring constant for torsional rotation.

L Length of rectangular foundation, in plane of rotation for rocking or in direction of horizontal force.

m Mass.

m_1 Eccentric mass.

m_o Mass of foundation block plus machinery.

r_o Radius of circular foundation.

S Contact area between foundation and soil.

z Displacement.

β_z Influence coefficient for vertical spring constant.

β_x Influence coefficient for horizontal spring constant.

β_ϕ Influence coefficient for rocking spring constant.

μ Poisson's ratio.

ρ Mass density of soil.