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Technical Report

ON INERTIAL FLOW ON THE SPHERE

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ABSTRACT

The general problem of three-dimensional inertial flow on the spherical earth is analysed and solved in Section 2. The inertial trajectory is of a helical nature on a circular cylinder. The axis of the cylinder is perpendicular to the equatorial plane on the earth. The radius of the cylinder is $V_{EO}/2\Omega$, where V_{EO} is the component of the initial velocity parallel to the equatorial plane. The period of the motion is 12 hr.

Section 3 contains an analysis of horizontal inertial motion on the spherical earth. The equations of motion for horizontal flow are integrated in such a way that one obtains differential equations for the rate of change of the longitude and the latitude. These equations can be integrated in terms of elliptic integrals. Several special cases are considered in detail such as a trajectory starting from the North Pole. The period of the motion is found whenever such a period exists. The maximum and minimum latitudes of the inertial trajectory are found.

A comparison between the present analysis and an earlier analysis by the author is made in Section 4 with emphasis on the effects of the so-called metrical terms on the inertial trajectory.

Section 5 contains a detailed analysis of the periods of the inertial motion showing considerable differences from the elementary periods computed for a constant value of the Coriolis parameter. In Section 6 a number of inertial trajectories, computed numerically, are shown, while Section 7 contains an analysis of a motion influenced by both a constant gravity and a constant Coriolis parameter.

1. INTRODUCTION

In a recent report (Wiin-Nielsen, 1970) the author investigated inertial flow on a beta-plane and formulated the common inertial flow problem for the sphere although no general solution was given in the latter case. It is the purpose of the present paper to report on some additional studies of more general types of inertial flows. Investigations of this kind are of interest because of the recent attempts to identify observed winds in the ionosphere (Rosenberg, 1968) with a type of inertial flow analyzed by Moses (1970).

The ordinary problem of inertial flow is to find the motion (velocity and trajectory) of a particle which is forced to move horizontally ($w = 0$ in the local coordinate system) under the influence of the horizontal component of the Coriolis force. If the Coriolis parameter, $f = 2\Omega \sin \varphi$, where Ω is the angular velocity of the earth and φ is latitude, is assumed to be a constant $f = f_0$, we find the well-known inertia circle where the period is half a pendulum day. The report by Wiin-Nielsen (1970) had as its purpose to generalize this elementary solution and to provide a detailed discussion of the various possibilities for inertial trajectories on the beta-plane.

It is somewhat artificial to restrict the inertial motion to be horizontal ($w = 0$) at all times. We shall therefore in this paper consider the more general case in which inertial motion is defined as the motion which results when the Coriolis force is the only acting force. The general solution of this case will be given in the next section. Furthermore, in Section 3 we shall show some examples of horizontal inertial flow on the sphere. When we consider the even more general case of motion influenced by the Coriolis force <u>and</u> gravity, we shall still be able to obtain approximate solutions given in Section 4 of this paper.

2. THREE-DIMENSIONAL INERTIAL MOTION

The equation for the general inertial flow is according to the definition given in the introduction

$$\frac{d\vec{v}}{dt} = -2\vec{\Omega} \times \vec{v} \tag{2.1}$$

where $\vec{v}$ is the three-dimensional velocity and $\vec{\Omega}$ is the rotation vector of the earth. We note that the Coriolis force acts in a plane perpendicular to the axis of rotation. It follows therefore that the component along the axis of rotation has a constant velocity because no component of the acceleration acts in this direction. In order to solve (2.1) it is most convenient to introduce a coordinate system (ξ,η,ζ) fixed relative to the earth in which the (ξ,η) plane coincides with the equatorial plane while the ζ-axis points along the axis of rotation. In this coordinate system we have $\vec{\Omega} = (0,0,\Omega)$, and if we define $\vec{v} = (u_1,u_2,u_3)$ we get the following scalar equations:

$$\frac{du_1}{dt} = 2\Omega\, u_2, \qquad \frac{du_2}{dt} = -2\Omega\, u_1, \qquad \frac{du_3}{dt} = 0 \tag{2.2}$$

The solutions of (2.2) are elementary, and we write them in the form

$$\begin{aligned} u_1 &= u_{10} \cos 2\Omega t + u_{20} \sin 2\Omega t \\ u_2 &= u_{20} \cos 2\Omega t - u_{10} \sin 2\Omega t \\ u_3 &= u_{30} \end{aligned} \tag{2.3}$$

in which the subscript 0 indicates the initial values. We may immediately integrate (2.3) to obtain the inertial trajectory. We find

$$\begin{aligned} \xi &= \xi_o + \frac{u_{10}}{2\Omega} \sin 2\Omega t + \frac{u_{20}}{2\Omega}(1 - \cos 2\Omega t) \\ \eta &= \eta_o + \frac{u_{20}}{2\Omega} \sin 2\Omega t - \frac{u_{10}}{2\Omega}(1 - \cos 2\Omega t) \\ \zeta &= \zeta_o + u_{30}\, t \end{aligned} \tag{2.4}$$

The trajectory described by (2.4) is, for $u_{30} = 0$, a circle in the plane through (ξ_0, η_0) and parallel to the equatorial plane. The equation for the circle is obtained from the first two equations in (2.4) by an elimination of t. We get:

$$\left[\xi - \left(\xi_0 + \frac{u_{20}}{2\Omega}\right)\right]^2 + \left[\eta - \left(\eta_0 - \frac{u_{10}}{2\Omega}\right)\right]^2 = \frac{u_{10}^2 + u_{20}^2}{4\Omega^2} \tag{2.5}$$

The trajectory is shown in Figure 1, which also shows the position of the center of the circle and its radius $(V_0/2\Omega)$ where V_0 is the initial windspeed. From (2.3) and (2.4) it is furthermore seen that the period of the motion is

$$T = \frac{\pi}{\Omega} = \frac{\pi}{2\pi} \times 24 \text{ hr} = 12 \text{ hr} \tag{2.6}$$

When $u_{30} \neq 0$ we find that the inertial trajectory is a helical spiral on the cylinder whose cross section is the circle in Figure 1 and where the axis is along the ζ-axis.

While it is easy to describe the general inertial trajectory as a helix in the (ξ,η,ζ)-system, it is an inconvenient system in practice. We would then prefer to use spherical coordinates (λ,φ,r) where λ is longitude, φ latitude, and r the distance from the center (see Figure 2).

We find

$$\xi = r \cos\varphi \cos\lambda$$

$$\eta = r \cos\varphi \sin\lambda$$

$$\zeta = r \sin\varphi \tag{2.7}$$

and the corresponding formulas with subscript zero for the initial conditions. Normally, we will give the initial condition as $(\lambda_0,\varphi_0,r_0)$. (2.7) may then be used to calculate (ξ_0,η_0,ζ_0), and (2.4) gives the functions $\xi(t)$, $\eta(t)$, $\zeta(t)$. When we, at a given time, want to come from (ξ,η,ζ) to (λ,φ,r) we must use the formulas

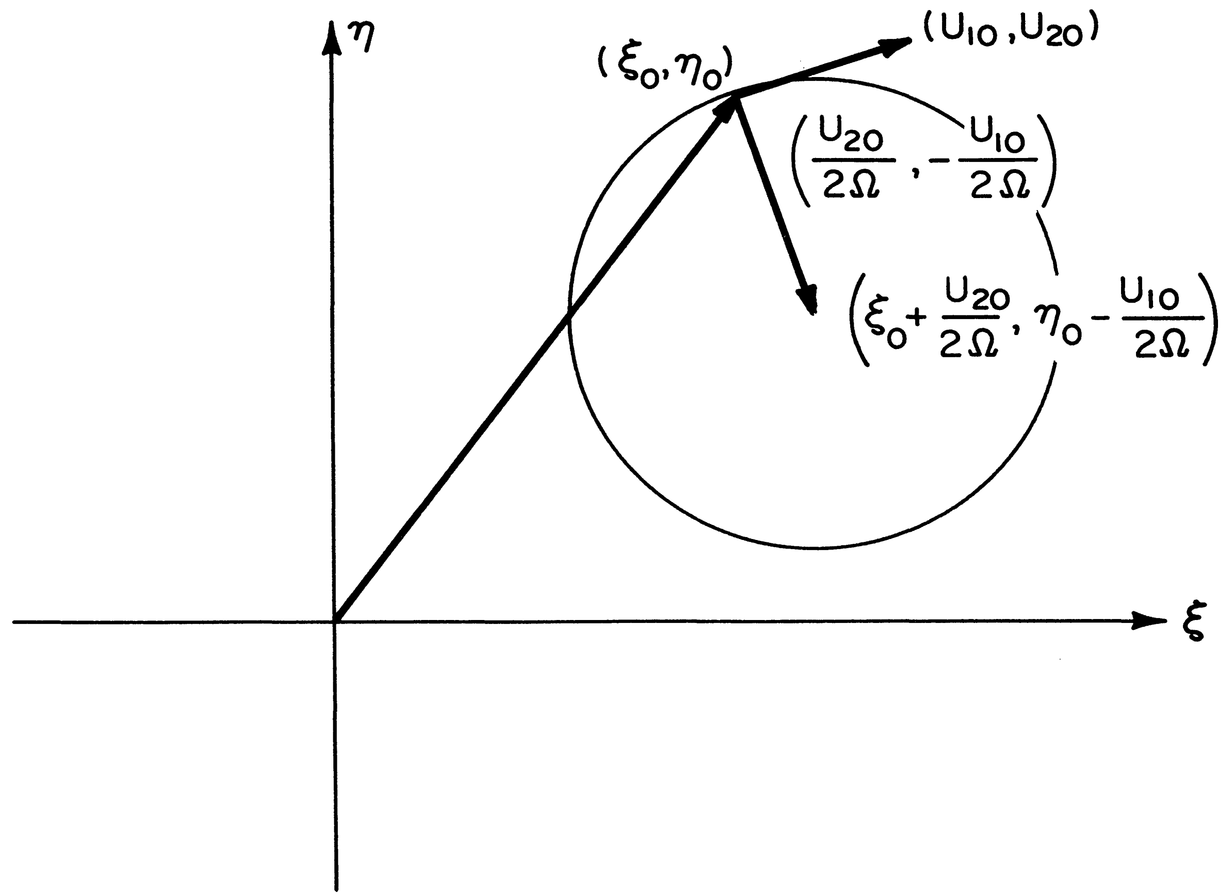

Figure 1. Projection on the equatorial plane (ξ, η) of the inertial trajectory. (ξ_0, η_0) is the initial position, (u_{10}, u_{20}) the initial velocity, and the circle is the projection of the helical trajectory on the (ξ, η) plane.

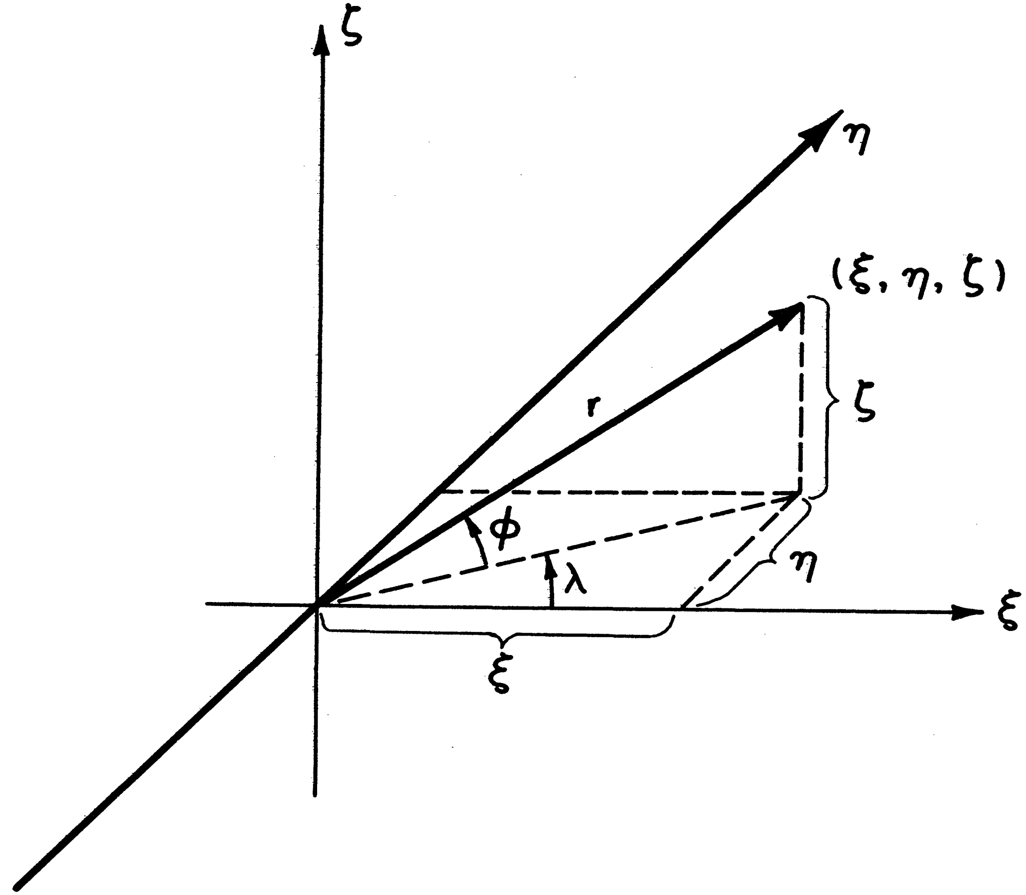

Figure 2. The notations used in the transformations from the (ξ,η,ζ) coordinate system to the spherical coordinates. r is the distance from the center of the earth, λ is longitude, and φ is latitude.

$$r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$$

$$\tan \lambda = \frac{\eta}{\xi}$$

$$\tan \varphi = \frac{\zeta}{\sqrt{\xi^2 + \eta^2}} \tag{2.8}$$

The systems (2.4), (2.7), and (2.8) are sufficient to calculate the inertial trajectory in spherical coordinates except for the fact that (2.4) requires a knowledge of (u_{10}, u_{20}, u_{30}) in terms of the ordinary velocity components

$$u = r \cos \varphi \, (d\lambda/dt)$$

$$v = r \, (d\varphi/dt)$$

$$w = (dr/dt)$$

We find using (2.7)

$$u_1 = \frac{d\xi}{dt} = - u \sin \lambda - v \sin \varphi \cos \lambda + w \cos \varphi \cos \lambda$$

$$u_2 = \frac{d\eta}{dt} = + u \cos \lambda - v \sin \varphi \sin \lambda + w \cos \varphi \sin \lambda$$

$$u_3 = + v \cos \varphi + w \sin \varphi \tag{2.9}$$

(2.9) is used to compute (u_{10}, u_{20}, u_{30}) from given initial values of $(\lambda_o, \varphi_o, r_o)$ and (u_o, v_o, w_o). The three-dimensional inertial trajectory described in this section is entirely different from the usual horizontal trajectory obtained when the horizontal component of the Coriolis force is the only force considered. We shall return to this question in Section 3.

3. HORIZONTAL INERTIAL MOTION

This problem is the description of the trajectory of a particle which is influenced by the horizontal component of the Coriolis force. It follows that the entire trajectory will be horizontal. The problem is treated in most text-books for the case of a constant Coriolis parameter and was analyzed by the author (Wiin-Nielsen, 1970) for the beta-plane case. The equations for the spherical case are:

$$\frac{du}{dt} = 2\Omega \sin\varphi\, v + \frac{uv}{a}\tan\varphi \tag{3.1}$$

$$\frac{dv}{dt} = -2\Omega \sin\varphi\, u - \frac{u^2}{a}\tan\varphi \tag{3.2}$$

in which Ω is the angular velocity of the earth, a the radius of the earth, and

$$u = a\cos\varphi\frac{d\lambda}{dt}, \qquad v = a\frac{d\varphi}{dt} \tag{3.3}$$

Introducing (3.3) in (3.1) and (3.2) we get

$$\frac{d}{dt}\left[\cos\varphi\frac{d\lambda}{dt}\right] = 2\Omega\sin\varphi\frac{d\varphi}{dt} + \sin\varphi\frac{d\lambda}{dt}\frac{d\varphi}{dt} \tag{3.4}$$

$$\frac{d^2\varphi}{dt^2} = -2\Omega\sin\varphi\cos\varphi\frac{d\lambda}{dt} - \sin\varphi\cos\varphi\left(\frac{d\lambda}{dt}\right)^2 \tag{3.5}$$

The integration of (3.4) and (3.5) is by no means straightforward because the equations are nonlinear. It is, however, possible to obtain first and second integrals in this case as outlined below. We note first of all that (3.1) and (3.2) imply the conservation of kinetic energy. Multiplying (3.1) by u, (3.2) by v and, adding the resulting equations, we get:

$$\frac{d}{dt}\left[\frac{1}{2}(u^2 + v^2)\right] = 0 \tag{3.6}$$

which may be written in the form

$$\cos^2\varphi\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2 = c^2 \tag{3.7}$$

where

$$C^2 = \cos^2\varphi_o \left(\frac{d\lambda}{dt}\right)_o^2 + \left(\frac{d\varphi}{dt}\right)_o^2 \tag{3.8}$$

The subscript zero indicates initial conditions. (3.7) is a first integral.

We shall next write (3.4) in the form

$$\cos\varphi \frac{d^2\lambda}{dt^2} - \sin\varphi \frac{d\varphi}{dt}\frac{d\lambda}{dt} = 2\Omega \sin\varphi \frac{d\varphi}{dt} + \sin\varphi \frac{d\lambda}{dt}\frac{d\varphi}{dt} \tag{3.9}$$

or

$$\cos\varphi \frac{d^2\lambda}{dt^2} + 2\Omega \frac{d(\cos\varphi)}{dt} + 2\frac{d\lambda}{dt}\frac{d(\cos\varphi)}{dt} = 0 \tag{3.10}$$

Separating the variables we find

$$\frac{1}{\Omega + \frac{d\lambda}{dt}} \frac{d}{dt}\left[\Omega + \frac{d\lambda}{dt}\right] + 2\frac{1}{\cos\varphi}\frac{d(\cos\varphi)}{dt} = 0 \tag{3.11}$$

or, by integration for $\Omega + d\lambda/dt > 0$,

$$\ln\left(\Omega + \frac{d\lambda}{dt}\right) + \ln(\cos^2\varphi) = \ln D \tag{3.12}$$

which finally gives

$$\Omega + \frac{d\lambda}{dt} \cos^2\varphi = D \tag{3.13}$$

where

$$D = \left(\Omega + \left(\frac{d\lambda}{dt}\right)_o\right)\cos^2\varphi_o \tag{3.14}$$

Combining (3.7) and (3.14) we may write the first integrals in the form:

$$\frac{d\lambda}{dt} = \frac{D}{\cos^2\varphi} - \Omega \tag{3.15}$$

and

$$\frac{d\varphi}{dt} = \pm\sqrt{C^2 - \cos^2\varphi\left(\frac{d\lambda}{dt}\right)^2} \tag{3.16}$$

The initial conditions are given in terms of a position (λ_o,φ_o) on the spherical earth and an initial angular velocity $((d\lambda/dt)_o,(d\varphi/dt)_o)$.

The system (3.15), (3.16) may be integrated in terms of elliptic integrals. In order to see this we write (3.16) in the form

$$dt = \pm\frac{\cos\varphi\ d\varphi}{(-D^2 + (2\Omega D + C^2)\cos^2\varphi - \Omega^2\cos^4\varphi)^{1/2}} \tag{3.17}$$

Introducing the substitution $\mu = \sin\varphi$ in (3.17) we find

$$dt = \pm\frac{d\mu}{\sqrt{R}} \tag{3.18}$$

where

$$R = (2\Omega D + C^2 - D^2 - \Omega^2) + (2\Omega^2 - 2\Omega D - C^2)\mu^2 - \Omega^2\mu^4 \tag{3.19}$$

The form of (3.18) combined with the fact that R, as expressed in (3.19), is a polynomial of the fourth degree shows that (3.18) can be integrated in terms of standard elliptic integrals, see Jahnke and Emde (1945). Substituting from (3.18) into (3.15) we find that

$$d\lambda = \pm D\frac{d\mu}{(1-\mu^2)\sqrt{R}} \mp \Omega\frac{d\mu}{\sqrt{R}} \tag{3.20}$$

The form of (3.20) indicates that λ is expressible in terms of elliptic integrals. The details of the transformations necessary to reduce (3.19) and (3.20) to the canonical forms can be found in Abramowitz and Stegun (1964). Since we are interested mainly in some specific examples and in a comparison with the β-plane cases, we shall integrate (3.15) and (3.16) numerically using finite difference methods and the initial conditions. Examples of inertial trajectories computed in this way will be described in Section 6. It may, however, be interesting to consider some special cases in which the reduction to canonical form is simple and straightforward.

As our first example we shall select the case where the particle starts at the North Pole ($\varphi_o = \pi/2$) with an initial velocity vector of magnitude

$a(d\varphi/dt)_o$ along the Greenwich meridian ($\lambda_o = 0$). We have in this case $D = 0$ and $C = (d\varphi/dt)_o$. (3.15) and (3.16) become

$$\frac{d\lambda}{dt} = -\Omega \, ; \qquad \frac{d\varphi}{dt} = \pm \sqrt{\left(\frac{d\varphi}{dt}\right)_o^2 - \Omega^2 \cos^2\varphi} \tag{3.21}$$

Introducing the nondimensional variable $\tau = \Omega t$ and the notation $q = \Omega^{-1}(d\varphi/dt)_o$, we may write (3.21) in the form

$$d\lambda = \mp \frac{d\varphi}{\sqrt{\sin^2\varphi - (1-q^2)}} \tag{3.22}$$

A new variable θ is introduced through the relation

$$1 - q^2 \sin^2\theta = \frac{1 - q^2}{\sin^2\varphi} \tag{3.23}$$

which upon substitution in (3.22) leads to

$$\lambda = \mp \int_{\pi/2}^{\theta} \frac{d\theta}{\sqrt{1 - q^2 \sin^2\theta}} \tag{3.24}$$

or

$$\lambda = \pm \left(F\left(q,\frac{\pi}{2}\right) - F(q,\theta)\right) \tag{3.25}$$

where

$$F(q,\theta) = \int_{o}^{\theta} \frac{d\theta}{\sqrt{1 - q^2 \sin^2\theta}} \tag{3.26}$$

is the elliptic integral of the first kind. The reduction of (3.22) to (3.25) is valid as long as $q \leq 1$ which is the same as $v_o \leq \Omega a$. This condition will hold in the cases of meteorological interest because Ωa is equal to the velocity of the earth's rotation at the equator (≈ 462 msec^{-1}). Let the angle α be defined through the relation $\sin \alpha = q$. We see from (3.23) that the minimum latitude of the trajectory is $\pi/2 - \alpha$. α is therefore the co-latitude of the southernmost point of the inertial trajectory. Three inertial trajectories corresponding to $\alpha = 30°$, $60°$, and $90°$ are shown in Figure 3 on a polar stereographic projection true at the North Pole. The initial velocities at the North Pole for these trajectories are 232, 402, and 464 msec^{-1}, respectively. It is

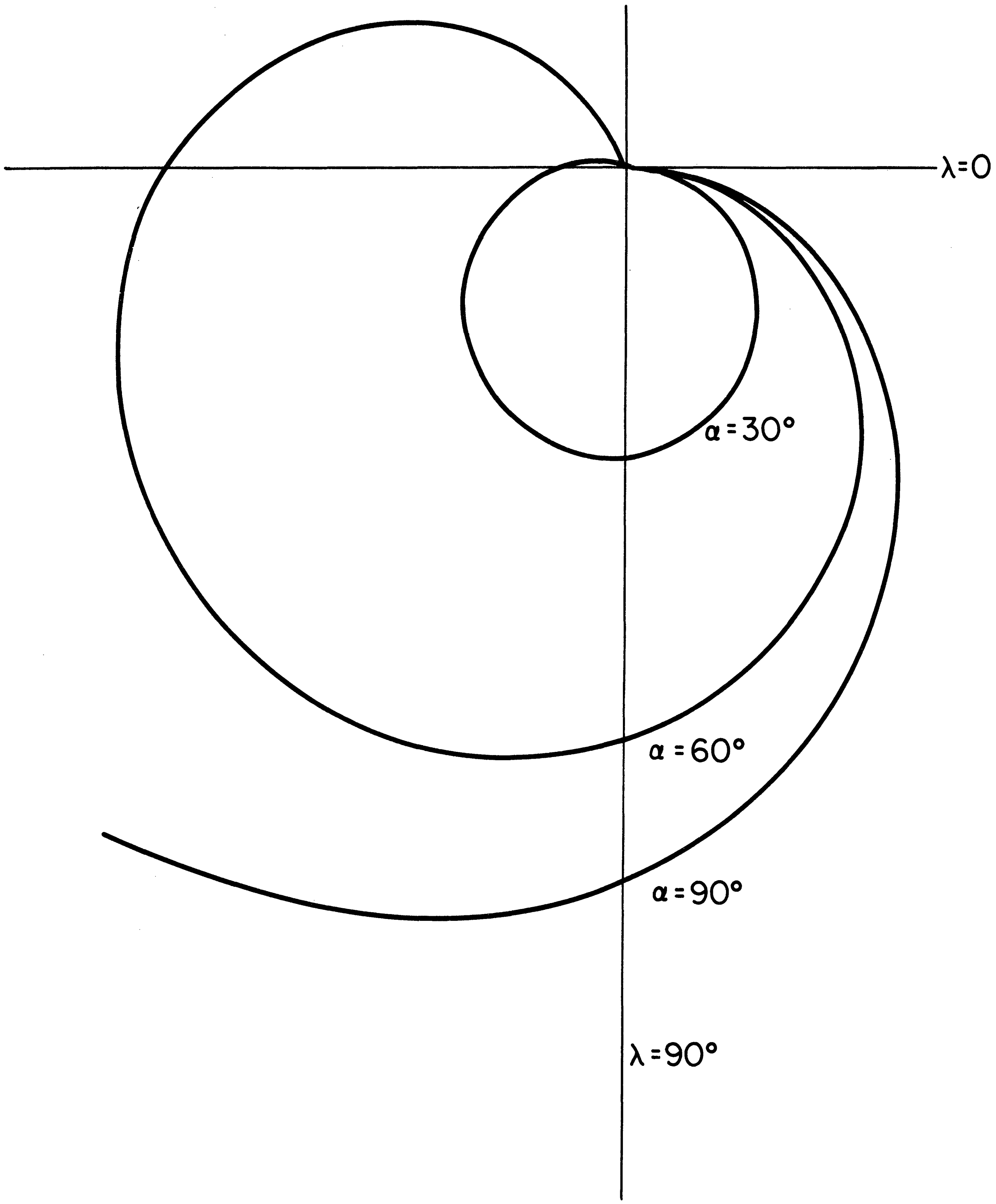

Figure 3. Inertial trajectories starting from the North Pole on a polar stereographic map. α indicates the maximum colatitude of the trajectory. The trajectory marked $\alpha = 90°$ will approach the equator asymptotically.

also of interest to calculate the time it takes to return to the North Pole in the trajectories given in Figure 3. The period in τ, i.e., τ_P, is according to (3.25) and recalling that $\tau = -\lambda$:

$$\tau_P = 2K(q) \tag{3.27}$$

where K is the complete elliptic integral, i.e., $K(q) = F(q,\pi/2)$. The value given in (3.27) is obtained by noting that the initial point corresponds to $\theta = \pi/2$ while the value of θ when the particle returns is $\theta = -\pi/2$. The period in hours is

$$T = \frac{\tau_P}{\Omega} = \frac{24}{\pi} K(q) \tag{3.28}$$

We notice that the limiting value for $q \to 0$ is T = 12 hr which is obtained because $K(0) = \pi/2$. Increasing values of q give increasing values of T with $T \to \infty$ as $q \to 1$. The values of T corresponding to $\alpha = 30°$ and $\alpha = 60°$ are 12.88 hr and 16.47 hr, respectively.

A different special case, corresponding to $(d\lambda/dt)_o = -\Omega$, was solved in the previous report, Wiin-Nielsen (1970). We shall comment on the previous solutions in the next section.

It is also of interest to consider some general properties of the solution. Without solving the problem completely it is possible to find the extrema in latitude of the inertial trajectory. At such a point we have $d\varphi/dt = 0$, and it follows from (3.16) that an extremum point is characterized by

$$\cos\varphi \frac{d\lambda}{dt} = \pm C \tag{3.29}$$

Since $C > 0$ we find that the plus sign will correspond to $d\lambda/dt > 0$ which is a maximum point in the northern hemisphere, while the minus sign will apply to $d\lambda/dt < 0$ or a minimum point in the northern hemisphere. Substituting from (3.15) in (3.29) we find that the extreme values of φ for the inertial trajectory are determined by the expression:

$$\cos\varphi_e = \left\{\frac{1}{4}\left(\frac{V_o}{V_E}\right)^2 + \cos^2\varphi_o + \frac{u_o}{V_E}\cos\varphi_o\right\}^{1/2} \mp \frac{1}{2}\frac{V_o}{V_E} \tag{3.30}$$

where $V_E = a\Omega$ is the velocity of the earth at equator and $V_o = aC = \{a^2 \cos^2\varphi_o\, (d\lambda/dt)_o^2 + a^2\, (d\varphi/dt)^2\}^{1/2}$ is the initial speed. (3.30) is very useful in connection with numerical integrations of (3.15) and (3.16) to be described later, because we may compute the northernmost and southernmost points of an inertial trajectory. It is possible that the lower sign in (3.30) will give a value of the right-hand side of (3.30) larger than unity. This means that the inertial trajectory will cross the equator. If so, we must have complete symmetry around the equator as seen from (3.15) and (3.16). Table 1 contains the maximum and minimum latitudes for inertial trajectories starting at φ_o with an initial meridional velocity $v_o = V_o = 100$ msec^{-1}.

TABLE 1

VALUES OF MAXIMUM AND MINIMUM LATITUDES FOR INERTIAL TRAJECTORIES STARTING AT φ_o WITH AN INITIAL MERIODIONAL VELOCITY OF 100 msec^{-1}

φ_o	90	80	70	60	50	40	30	20	10	0
φ_{max}	90	84.47	75.48	66.19	57.04	48.25	40.10	33.06	29.23	26.08
φ_{min}	77.55	71.81	62.20	51.74	40.57	28.19	11.37	-33.06	-29.23	-26.08
$\varphi_{max} - \varphi_o$	0.00	4.47	5.48	6.19	7.04	8.25	10.10	13.06	19.23	26.08
$\varphi_o - \varphi_{min}$	12.45	8.19	7.80	8.26	9.43	11.81	18.63	53.06	39.23	26.08

Equation (3.30) becomes particularly simple if we start the particle with a zonal velocity in which case we have $V_o = u_o$. It is seen that (3.30) reduces to

$$\cos \varphi_e = \begin{cases} \cos \varphi_o \\ \cos \varphi_o + \dfrac{u_o}{V_E} \end{cases} \tag{3.31}$$

(3.31) shows that the initial latitude is a minimum latitude if $u_o < 0$ and a maximum latitude if $u_o > 0$. The maximum latitude for $u_o = -100$ msec^{-1} and the minimum latitude for $u_o = +100$ msec^{-1} are given in Table 2 for some selected values of φ_o.

TABLE 2

MAXIMUM AND MINIMUM LATITUDES FOR A PARTICLE STARTING AT φ_o WITH INITIAL VELOCITY u_o = -100 msec^{-1} AND u_o = +100 msec^{-1}, RESPECTIVELY

φ_o	10	20	30	40	50	60	70	80	msec^{-1}
φ_{max}	39.67	43.57	49.39	56.58	64.68	73.45	82.71	90.00	u_o = -100
φ_{min}	-10.00	-20.00	-30.00	11.17	30.92	44.35	56.15	67.13	u_o = +100

Equation (3.30) may naturally also be used for other purposes. If we, for example, are interested in the limiting case where the equator is an asymptote to the inertial trajectory we may characterize this case from (3.30) by the relation

$$\left\{\frac{1}{4}\left(\frac{V_o}{V_E}\right)^2 + \cos^2\varphi_o + \frac{u_o}{V_E}\cos\varphi_o\right\}^{1/2} + \frac{1}{2}\frac{V_o}{V_E} = 1 \qquad (3.32)$$

For a given initial velocity (u_o, v_o), $V_o = (u_o^2 + v_o^2)^{1/2}$ we may use (3.32) to find the latitude where we have to start the particle in order to reach the equator in an asymptotic way. If, for example, $u_o = 0$, $v_o = 100$ msec^{-1}, we find from (3.32) that we must start the particle at 27.66°, with the corresponding value is 8.43° if $u_o = 0$, $v_o = 10$ msec^{-1}. (3.32) may also, in special cases, be used to calculate the initial velocity at a given latitude necessary to reach the equator in an asymptotic way. For example, if $u_o = 0$, $v_o \neq 0$, we find from (3.32) the initial value of v_o necessary to reach the equator is

$$v_o = V_E \sin^2\varphi_o \qquad (3.33)$$

while the value of $u_o > 0$, $v_o = 0$ to obtain the same limiting case is

$$u_o = V_E (1 - \cos\varphi_o) \qquad (3.34)$$

The values of u_o and v_o calculated from (3.34) and (3.33), respectively, are given in Table 3 as a function of latitude.

TABLE 3

VALUES OF $(u_o,0)$ AND $(0,v_o)$ NECESSARY TO REACH THE EQUATOR BY INERTIAL TRAJECTORY AS A FUNCTION OF LATITUDE

φ_o	90	80	70	60	50	40	30	20	10	0
u_o	464.0	383.4	305.3	232.0	165.7	108.6	62.2	28.0	7.1	0.0
v_o	464.0	450.0	409.7	348.0	272.3	191.7	116.0	54.3	14.0	0.0

The limiting case where $v_o = (u_o,0)$ and u_o is such that the equator is an asymptote may be solved in terms of elementary functions. When we introduce $\mu = \sin\varphi$ in (3.17) we can write the equation in the form:

$$dt = \pm \frac{d\mu}{\{(C^2 + 2\Omega D - D^2 - \Omega^2) - (C^2 + 2\Omega D - 2\Omega^2)\mu^2 - \Omega^2\mu^4\}^{1/2}} \tag{3.35}$$

where

$$C^2 = \cos^2\varphi_o \left(\frac{d\lambda}{dt}\right)_o^2 ; \qquad D = \left(\Omega + \left(\frac{d\lambda}{dt}\right)_o\right)\cos^2\varphi_o \tag{3.36}$$

Introducing (3.36) in (3.35), we find after some reduction that (3.35) may be written in the form

$$d\tau = \pm \frac{d\mu}{\{(\mu^2 - \mu_o^2)([1 - (1 + p/\Omega)^2(1 - \mu_o^2)] - \mu^2\}^{1/2}} \tag{3.37}$$

where

$$\tau = \Omega t \tag{3.38}$$

and

$$p = \left(\frac{d\lambda}{dt}\right)_o \tag{3.39}$$

The integration of (3.37) will of course lead to incomplete elliptic integrals in the general case, but in the special case where u_o is given by (3.34) or

$$p = \Omega \frac{1 - \cos \varphi_o}{\cos \varphi_o}$$

we find that (3.37) reduces to

$$d\tau = \pm \frac{d\mu}{\mu \sqrt{\mu_o^2 - \mu^2}} \qquad (3.40)$$

We change the variable μ to a new variable θ defined through

$$\mu = \mu_o \sin \theta \qquad (3.41)$$

and (3.40) becomes after integration

$$\tau = \frac{1}{\mu_o} \ln \left(\tan \frac{\theta}{2} \right) \qquad (3.42)$$

where we have selected the minus sign in (3.37) in order to have $\tau > 0$.

The solution for λ can now be obtained from (3.15) giving

$$d\lambda = - \left[\frac{D}{\Omega} \frac{d\tau}{1 - \mu^2} - d\tau \right] \qquad (3.43)$$

which can be integrated to give

$$\lambda = \lambda_o + \tan \frac{\varphi_o}{2} \ln \left(\tan \frac{\theta}{2} \right) + \text{arc tan } (\tan \varphi_o \cos \theta) \qquad (3.44)$$

The solution for the latitude φ is given by (3.41) or

$$\varphi = \text{arc sin } (\sin \varphi_o \sin \theta) \qquad (3.45)$$

(3.42), (3.44), and (3.45) give the solution for the asymptotic case. An example of a trajectory computed from these equations is shown in Figure 4 where we have selected $\varphi_o = 30°$ (i.e., $u_o = 62.2$ msec^{-1}). The numbers on the curve give elapsed time in hours since the start of the trajectory.

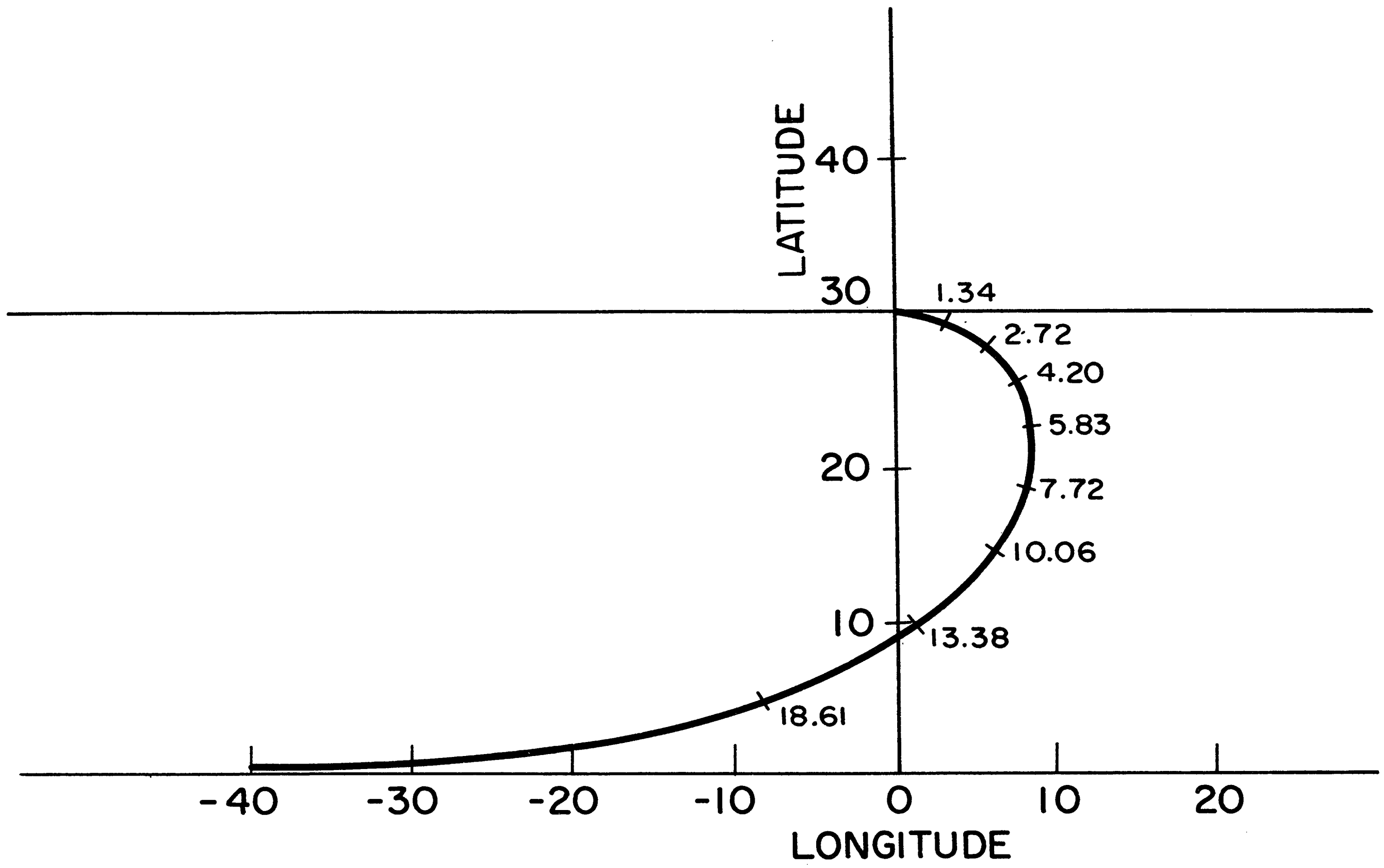

Figure 4. An inertial trajectory starting at 30°N with a zonal velocity (62.17 $msec^{-1}$) necessary to reach the equator as an asymptote. The marks on the trajectory indicate elapsed time in hours.

4. COMPARISON WITH PREVIOUS CALCULATIONS

The present analysis is a generalization of the previous case treated by the author (1970). The spherical case was considered in the first analysis, but the considerations were limited to the equations

$$\frac{du}{dt} = 2\Omega \sin \varphi \; u \tag{4.1}$$

$$\frac{dv}{dt} = 2\Omega \sin \varphi \; v \tag{4.2}$$

where the so-called metrical terms due to the convergence of the meridians on the spherical earth were neglected as presumably small. The equations (3.1), (3.2) and (4.1), (4.2) were thought to give essentially the same results as long as the initial velocities are small and the inertial trajectories do not enter the very high latitudes where tan φ becomes very large. However, test calculations have shown that the metrical terms are very important at high latitudes even for small values of the velocity components.

The qualitative effects of the metrical terms on an inertial trajectory may be seen by writing (3.1) and (3.2) in the form

$$\frac{du}{dt} = \left(2\Omega + \frac{u}{a \cos \varphi}\right) \sin \varphi \; v \tag{4.3}$$

$$\frac{dv}{dt} = -\left(2\Omega + \frac{u}{a \cos \varphi}\right) \sin \varphi \; u \tag{4.4}$$

or

$$\frac{d\vec{v}}{dt} = \left(2\Omega + \frac{u}{a \cos \varphi}\right) \sin \varphi \; \vec{v} \times \vec{k} \tag{4.5}$$

where $\vec{k}$ is a vertical unit vector. On the other hand, the previous calculations (Wiin-Nielsen, 1970) were based on the equation

$$\frac{d\vec{v}}{dt} = 2\Omega \sin \varphi \; \vec{v} \times \vec{k} \tag{4.6}$$

(4.5) and (4.6) express the fact that the curvature of the inertial trajectory is created by a force acting perpendicularly to the right in the northern hemisphere (and to the left in the southern hemisphere). We may now compare (4.5) and (4.6). Let us assume that we have carried out a calculation satisfying (4.6) for the case $u_o = 0$, $v_o > 0$ at $\varphi = \varphi_o$. The result of this calculation is given by the full curve on Figure 5 which was taken from a numerical integration of (4.1) and (4.2) using $u_o = 0$, $v_o \doteq 100$ msec^{-1} and $\varphi_o = 70°$N. An integration of (4.3) and (4.4)—or rather (3.15) and (3.16)—will differ from the full curve in Figure 5 in the following way. During the upper branch $\varphi > \varphi_o$ we will have a deviating force larger than before according to (4.5). The new inertial trajectory will have a greater curvature than before as indicated by the dashed line on Figure 5. On the other hand, when the particle has passed the initial latitude and starts to turn westward we will have a deviating force which is smaller than in the previous case as long as the quantity in the parenthesis on the right-hand side of (4.5) is positive, i.e., as long as $u > -2\Omega a \cos \varphi$. This condition will normally be fulfilled for meteorological motion. The result is that the curvature of the trajectory will be smaller than before and the particle will move along the dashed curve. The net effect of the metrical terms on an inertial trajectory is therefore that they cause a net westward displacement per period. This result holds as long as the metrical terms are small compared to the Coriolis force.

One may naturally also be interested in the trajectory which a particle would have if it were under influence of the metrical terms as the only apparent force. This problem is expressed by the equations

$$\frac{du}{dt} = \frac{uv}{a} \tan \varphi \qquad (4.6)$$

$$\frac{dv}{dt} = -\frac{u^2}{a} \tan \varphi \qquad (4.7)$$

The trajectory in this case is naturally obtained as a special case ($\Omega = 0$) of the general solution of (3.15) and (3.16). We have now:

$$\frac{d\lambda}{dt} = \frac{D}{\cos^2 \varphi} \qquad (4.8)$$

$$\frac{d\varphi}{dt} = \pm \sqrt{C^2 - \frac{D^2}{\cos^2 \varphi}} \qquad (4.9)$$

where

$$D = \left(\frac{d\lambda}{dt}\right)_o \cos^2 \varphi_o \; ; \qquad C^2 = \cos^2 \varphi_o \left(\frac{d\lambda}{dt}\right)_o^2 + \left(\frac{d\varphi}{dt}\right)_o^2 \qquad (4.10)$$

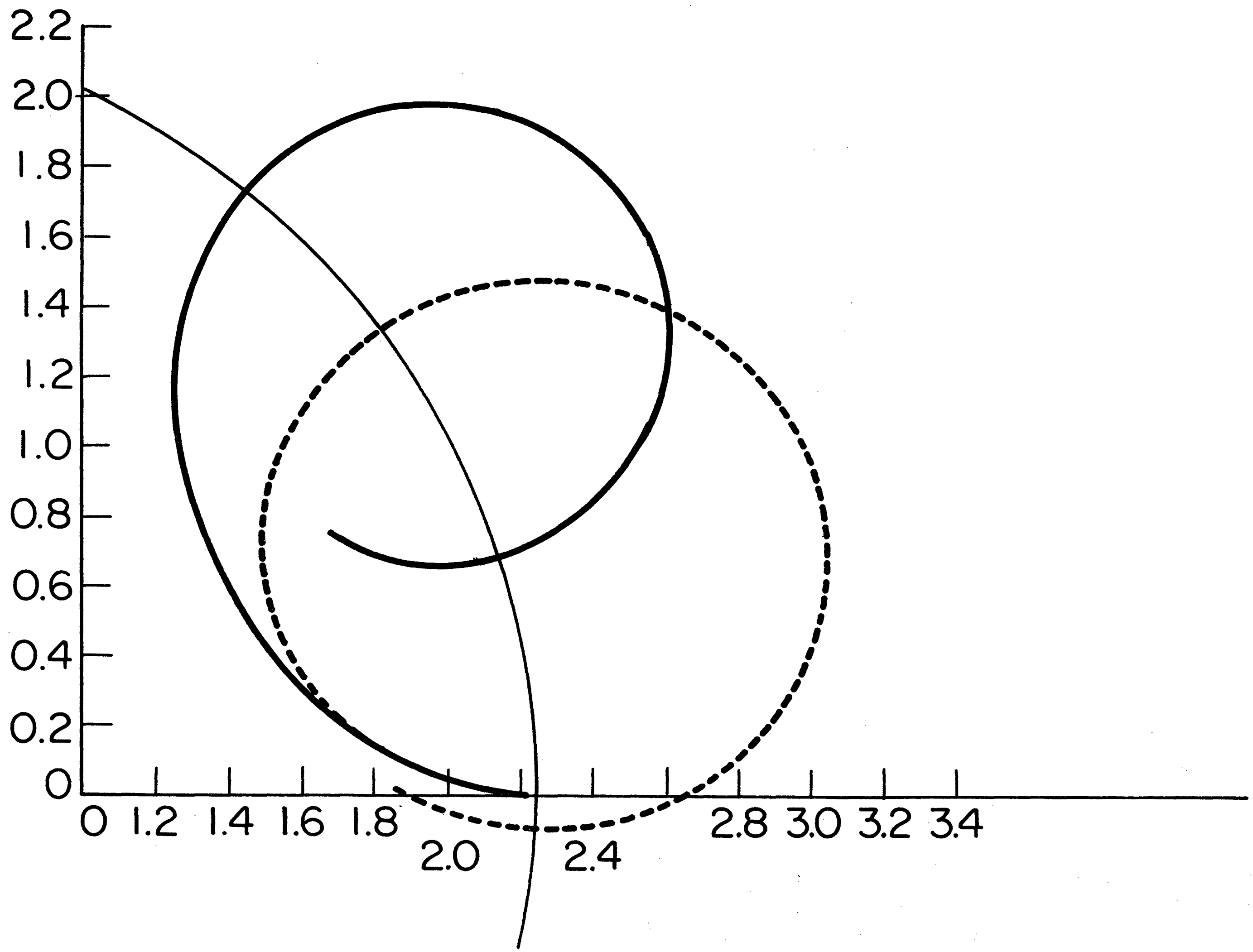

Figure 5. The thin solid curve is the latitude circle $\varphi = 70°N$. The thick solid curve is the inertial trajectory computed without the metrical terms from an initial position at $\lambda = 0°$, $\varphi = 70°N$ with an initial velocity of $u_o = 0$, $v_o = 100 \text{ msec}^{-1}$, while the dashed curve is the trajectory computed from the same initial position and velocity but including the metrical terms.

The problem, posed by (4.8) and (4.9), can be solved in a straightforward way. Let

$$\mu = \sin\varphi \qquad \text{and} \qquad S^2 = \frac{C^2 - D^2}{C^2} \tag{4.11}$$

Introducing (4.11) in (4.9) we find

$$dt = \pm \frac{d\mu}{C\sqrt{S^2 - \mu^2}} \tag{4.12}$$

In (4.12) we introduce the new variable θ defined through

$$\mu = S \sin\theta \tag{4.13}$$

and we get:

$$dt = \pm \frac{1}{C}\, d\theta \tag{4.14}$$

We may therefore write the solution in terms of the variable θ as follows:

$$t = \pm \frac{1}{C}(\theta - \theta_o)\,; \qquad \theta_o = \text{arc}\sin\left(\frac{\sin\varphi_o}{S}\right) \tag{4.15}$$

and

$$\varphi = \text{arc}\sin[S \sin\theta] \tag{4.16}$$

The solution for λ is obtained from (4.8) which after substitution of (4.11) and (4.13) becomes:

$$d\lambda = \pm \frac{D}{C} \frac{d\theta}{1 - S^2 \sin^2\theta} \tag{4.17}$$

(4.17) may be integrated directly with the result that

$$\lambda = \lambda_o \pm \frac{D}{C} \frac{1}{\sqrt{1 - S^2}} \left[\text{arc}\tan\left\{\sqrt{1 - S^2}\tan\theta\right\} - \text{arc}\tan\left\{\sqrt{1 - S^2}\tan\theta_o\right\}\right] \tag{4.18}$$

(4.15), (4.16), and (4.18) give the complete solution to the problem. We notice from (4.15) that the motion is periodic with a period given by

$$T = \frac{2\pi}{C} = \frac{2\pi a}{V_o} \tag{4.18}$$

where V_o is the initial speed and a is the radius of the earth. The motion has a large period for ordinary meteorological velocities. For V_o = 10 msec^{-1} we find a period of T = 46.3 days. Let us next consider a couple of examples. Let the particle start with the conditions $\lambda_o = 0$, $\varphi = \varphi_o$, $(d\lambda/dt)_o > 0$, and $(d\varphi/dt)_o = 0$. For these initial conditions we find $S = \sin\varphi_o$ and $\theta_o = \pi/2$. Figure 6 shows two trajectories, one starting at 60°N and the other at 30°N. The trajectories cross the equator at 90° of longitude and reach their minimum points at 60°S and 30°S. Each of the trajectories cover a half period.

We notice that the solution given by (4.15), (4.16), and (4.18) breaks down if the initial velocity has $(d\lambda/dt)_o = 0$, because in that case D = 0 and S = 1, leading to $\sqrt{1 - S^2} = 0$. It is, however, easy to see from (4.8) and (4.9) that D = 0 results in the equations:

$$\frac{d\lambda}{dt} = 0 ; \qquad \frac{d\varphi}{dt} = C = \left(\frac{d\varphi}{dt}\right)_o \tag{4.19}$$

The solutions of (4.19) are

$$\lambda = \lambda_o ; \qquad \varphi = \varphi_o + \left(\frac{d\varphi}{dt}\right)_o \cdot t \tag{4.20}$$

indicating that the particle in this special case will travel along the meridian $\varphi = \varphi_o$ with a constant speed $v_o = a(d\varphi/dt)_o$.

In order to complete the comparison between the previous and the present solutions we shall finally consider a case where it is relatively easy to obtain analytical solutions in both cases. As the example we shall select the case treated in Section 3 (Eqs. (3.21) to (3.28)). If the metrical terms are neglected we have the equations

$$\frac{d}{dt}\left(\cos\varphi \frac{d\lambda}{dt}\right) = 2\Omega \sin\varphi \frac{d\varphi}{dt} \tag{4.21}$$

which can be integrated in a straightforward way to give

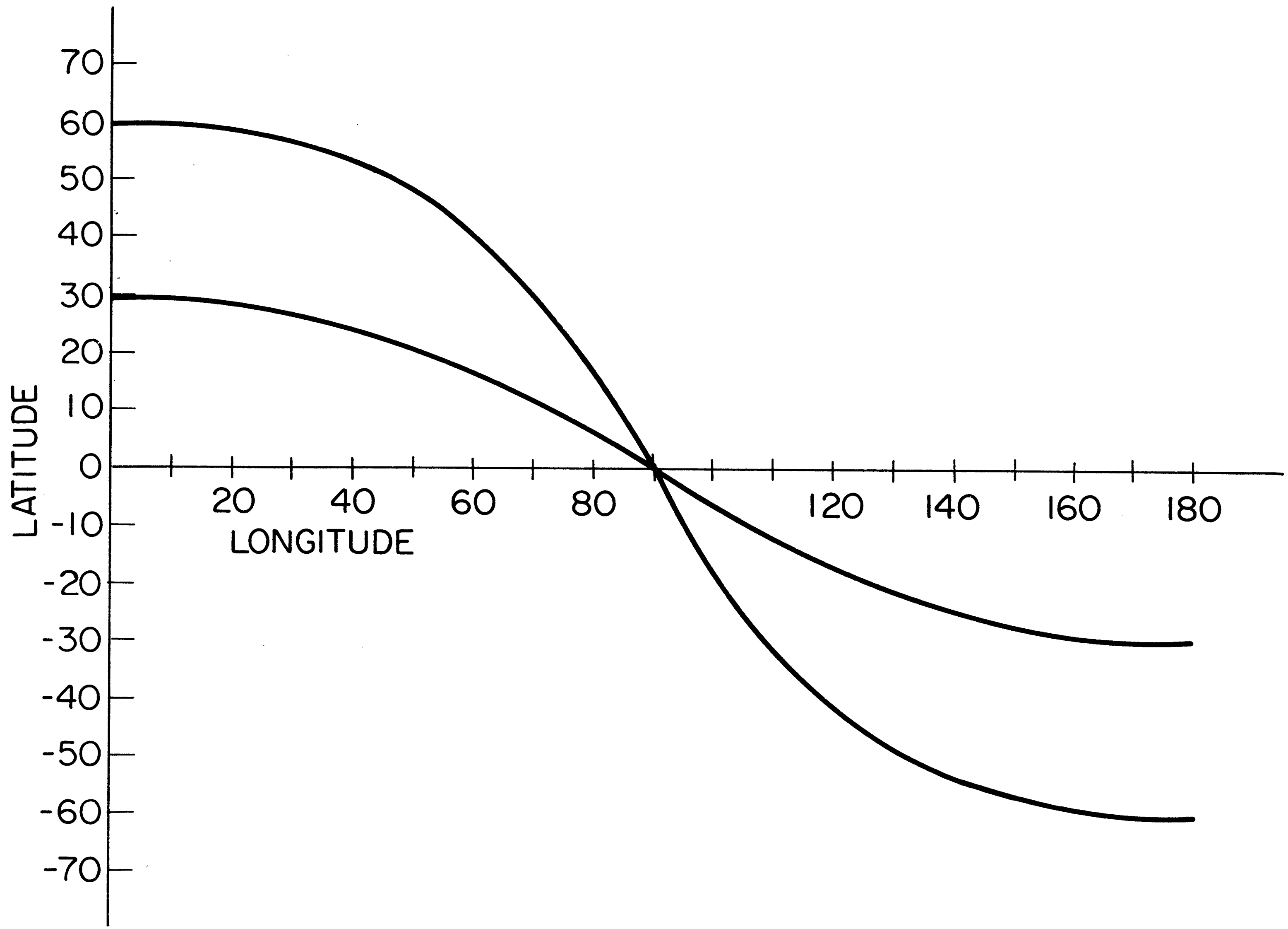

Figure 6. An illustration of the effect of the metrical terms. The figure shows two trajectories, starting at 30°N and 60°N, respectively, computed with $\Omega = 0$. The trajectories intersect the equator at 90° of longitude and reach the minimum points (30°S and 60°S, respectively) at 180° of longitude.

$$\frac{d\lambda}{dt} = \frac{D_*}{\cos\varphi} - 2\Omega ; \qquad D_* = \cos\varphi_o \left(\left(\frac{d\lambda}{dt}\right)_o + 2\Omega\right) \tag{4.22}$$

which is the equivalent to (3.15). Corresponding to (3.16) we have exactly the same equation because it expresses the conservation of kinetic energy which is true in both cases. Considering now the case when the initial position is the North Pole with a meridional velocity along the Greenwich meridian we find corresponding to (3.21):

$$\frac{d\lambda}{dt} = -2\Omega ; \qquad \frac{d\varphi}{dt} = \pm\sqrt{\frac{d\varphi}{dt}_o^2 - 4\Omega^2 \cos^2\varphi} \tag{4.23}$$

Introducing the nondimensional variables $\tau_* = 2\Omega t$ and the notation $q_* = 1/2\Omega\ (d\varphi/dt)_o = (d\varphi/d\tau_*)_o$ we find

$$\frac{d\lambda}{d\tau_*} = -1 ; \qquad \frac{d\varphi}{d\tau_*} = \pm\sqrt{q_*^2 - \cos^2\varphi} \tag{4.24}$$

and corresponding to (3.22), we get the identical equation. It follows therefore that the solution of (4.24) is expressed by (3.25) and (3.26) with q replaced by q_* and τ by τ_*. We notice in particular that the period now is

$$T_* = \frac{24}{\pi} K(q_*) = \frac{24}{\pi} K\left(\frac{q}{2}\right) \tag{4.25}$$

corresponding to (3.28). We notice that the periods expressed in (3.28) and (4.25) agree in the limiting case of q and q_* approaching zero. However, for values of q different from zero there will be significant differences. Corresponding to the two periods of 12.88 hr and 16.47 hr given in connection with Figure 3 we find now the periods 12.19 hr and 12.62 hr, respectively. A neglect of the metrical terms is thus not justified.

5. THE PERIOD OF INERTIAL MOTION

The period of inertial motion is usually quoted as

$$T = \frac{2\pi}{f_o} = \frac{2\pi}{2\Omega \sin \varphi_o} = \frac{12}{\sin \varphi_o} \text{ (hours)} \tag{5.1}$$

where f_o is the value of the Coriolis parameter at the initial latitude. (5.1) comes from the case of constant f. The purpose of this section is to discuss the period of the inertial motion in the general spherical case as discussed in Section 3 of this paper. We base the discussion on equation (3.16). In order to derive an expression for the period in the general case, it is necessary to bring (3.16) into canonical form. We note that (3.16) may be written in the form

$$\frac{d\varphi}{dt} = \pm \sqrt{\left(C + \cos \varphi \frac{d\lambda}{dt}\right) \left(C - \cos \varphi \frac{d\lambda}{dt}\right)} \tag{5.2}$$

Substituting from (3.15) for $d\lambda/dt$ we find after some calculations that we may write (5.2) in the form

$$dt = \pm \frac{\cos \varphi \, d\varphi}{\Omega \sqrt{-\left[\cos^2\varphi - \left(\frac{1}{2}\frac{V_o}{V_E} + R\right)^2\right]\left[\cos^2\varphi - \left(\frac{1}{2}\frac{V_o}{V_E} - R\right)^2\right]}} \tag{5.3}$$

where V_o is the initial wind speed, $V_E = \Omega a$ the speed of the rotating earth at the equator, and

$$R = \sqrt{\frac{1}{4}\left(\frac{V_o}{V_E}\right)^2 + \cos^2\varphi_o + \frac{u_o}{V_E}\cos\varphi_o} \tag{5.4}$$

in which φ_o is the initial latitude, and u_o is the zonal component of the initial wind.

Considering the form of (5.3), it is now natural to introduce the independent variables

$$\mu = \sin \varphi \tag{5.5}$$

and

$$\tau = \Omega t \tag{5.6}$$

and we get:

$$d\tau = \pm \frac{d\mu}{\sqrt{\left(\mu^2 - \left\{1 - \left(\frac{1}{2}\frac{V_o}{V_E} + R\right)^2\right\}\right)\left(\left\{1 - \left(\frac{1}{2}\frac{V_o}{V_E} - R\right)^2\right\} - \mu^2\right)}} \tag{5.7}$$

It is seen from (5.7) that there are the following cases to consider:

① $\mu_1^2 = 1 - \left(R + \frac{1}{2}\frac{V_o}{V_E}\right)^2 > 0, \qquad \mu_2^2 = 1 - \left(R - \frac{1}{2}\frac{V_o}{V_E}\right)^2 > 0:$

In this case we have

$$d\tau = \pm \frac{d\mu}{\sqrt{\left(\mu^2 - \mu_1^2\right)\left(\mu_2^2 - \mu^2\right)}} \tag{5.8}$$

A use of the transformation

$$\mu = \frac{\mu_1}{\sqrt{1 - k^2 \sin^2\theta}}; \qquad k^2 = \frac{\mu_2^2 - \mu_1^2}{\mu_2^2} \tag{5.9}$$

leads to

$$\tau = \frac{1}{\mu_2}\int_o^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} - \frac{1}{\mu_2}\int_o^{\theta_o} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{5.10}$$

(5.10) tells us that the period expressed in terms of the dimensional quantity is

$$T = \frac{2}{\mu_2 \Omega} K(k) \tag{5.11}$$

where $K(k)$ is the complete elliptic integral of the first kind.

(2) $\nu_1^2 = \left(R + \frac{1}{2}\frac{V_o}{V_E}\right)^2 - 1 > 0 , \qquad \nu_2^2 = 1 - \left(R - \frac{1}{2}\frac{V_o}{V_E}\right) > 0 :$

In this case we have

$$d\tau = \pm \frac{d\mu}{\sqrt{\left(\mu^2 + \nu_1^2\right)\left(\nu_2^2 - \mu^2\right)}} \tag{5.12}$$

The transformation

$$\mu = \nu_2 \cos \theta \tag{5.13}$$

leads to

$$\tau = \pm \frac{1}{\sqrt{\nu_1^2 + \nu_2^2}} \left\{ \int_o^{\theta_o} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} - \int_o^{\theta} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \right\} \tag{5.14}$$

(5.13) and (5.14) tells us that the period is

$$T = \frac{4}{\Omega \sqrt{\nu_1^2 + \nu_2^2}} K(k) \tag{5.15}$$

(3) $1 - \left(R + \frac{1}{2}\frac{V_o}{V_E}\right)^2 = 0 , \qquad \mu_2^2 = 1 - \left(R - \frac{1}{2}\frac{V_o}{V_E}\right)^2 > 0 :$

In this case we have:

$$d\tau = \frac{d\mu}{\mu\sqrt{\mu_2^2 - \mu^2}} \tag{5.16}$$

We note from the conditions in this limiting case that

$$R = 1 - \frac{1}{2}\frac{V_o}{V_E} \tag{5.17}$$

We find therefore

$$\mu_2^2 = \frac{V_o}{V_E}\left(2 - \frac{V_o}{V_E}\right) \tag{5.18}$$

The transformation

$$\mu = \mu_2 \sin \theta \tag{5.19}$$

leads to

$$d\tau = \frac{1}{\mu_2} \frac{d\theta}{\sin \theta} \tag{5.20}$$

and

$$\tau = \frac{1}{\mu_2}\left\{\ln\left(\tan \frac{\theta}{2}\right) - \ln\left(\tan \frac{\theta_o}{2}\right)\right\} \tag{5.21}$$

Case 3 is therefore nonperiodic and is indeed the case where the trajectory approaches the equator asymptotically.

We may use the formulas (5.11) and (5.15) to compute the periods for a couple of examples. As the first example we have selected a case where $\varphi_o = 45°N$ and $V_o = 100 \text{ msec}^{-1}$. Expressing u_o as $u_o = V_o \cos \psi$, where ψ is the angle between the wind vector and the zonal direction we find the results listed in Table 4, computed from (5.11).

TABLE 4

PERIOD OF INERTIAL MOTION, MEASURED IN HOURS, FOR $\varphi_o = 45°N$, $V_o = 100 \text{ msec}^{-1}$, AND $u_o = V_o \cos \psi$

ψ	0	30	60	90	120	150	180
T	22.5	21.6	19.6	17.7	16.3	15.5	15.2

It is seen from Table 4 that a considerable variation exists in the computed periods depending on the initial direction of the velocity vector. The values in Table 4 should be compared with the elementary period computed from (5.1). Using φ_o = 45°N we find from (5.1) that the period is about 17.0 hr. This estimate of the period is thus almost 25% too small compared with the period when $\psi = 0$.

The second example is selected in such a way that (5.15) applies. We use $\varphi_o = 0°$, $V_o = 100$ msec^{-1}. Defining ψ in the same way as before we get the results listed in Table 5, where the period is given in hours.

TABLE 5

PERIOD OF INERTIAL MOTION, MEASURED IN HOURS,
FOR $\varphi_o = 0°$, $V_o = 100$ msec^{-1}, AND $u_o = V_o \cos \psi$

ψ	0	30	60	90	120	150	180
T	34.7	35.2	37.7	42.0	49.8	65.1	∞

We note first of all that we have no elementary period to compare with in this case because T in (5.1) goes to infinity when $\varphi_o \to 0$. We may, however, compare with the period given by Wiin-Nielsen (1970) using a beta-plane approximation centered at the equator. These results, computed with the same parameters as in Table 5, are given in Table 6. We notice that the beta-plane approximation gives a slight overestimate of the period for small values of ψ and an underestimate for large values of ψ.

TABLE 6

PERIOD OF INERTIAL MOTION, MEASURED IN HOURS,
PARAMETERS AS IN TABLE 5, BUT USING A BETA-PLANE GEOMETRY

ψ	0	30	60	90	120	150	180
T	36.4	37.1	39.1	43.0	50.0	64.2	∞

We have already found the period in the special case where the particle starts at the North Pole, see Eq. (3.28), and where $V_o/V_E < 1$. Some additional values are given in Table 7 for various values of V_o/V_E.

TABLE 7

VALUES OF THE PERIOD, MEASURED IN HOURS,
FOR A PARTICLE STARTING AT THE NORTH POLE WITH $V_o/V_E < 1$

V_o/V_E	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
T	12.00	12.03	12.12	12.28	12.53	12.88	13.38	14.10	15.24	17.43	∞

It is also of interest to tabulate some values for the same case, where the initial windspeed is so large that the particle goes into the Southern Hemisphere. The formula for the period is a special case of (5.15), which for $\varphi_o = \pi/2$ becomes

$$T = \frac{48}{\pi\left(\frac{V_o}{V_E}\right)} K\left(\frac{V_E}{V_o}\right) \tag{5.22}$$

Some values are given in Table 8.

TABLE 8

VALUES OF THE PERIOD, MEASURED IN HOURS,
FOR A PARTICLE STARTING AT THE NORTH POLE WITH $V_o/V_E > 1$

V_o/V_E	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	3.0	4.0	5.0	6.0
T	32.2	26.3	22.8	20.3	18.4	16.9	15.6	14.6	13.7	12.9	8.2	6.1	4.8	4.0

6. EXAMPLES OF INERTIAL TRAJECTORIES

As pointed out in Section 3 it is most convenient to use numerical methods to calculate actual inertial trajectories. The equations used for this purpose are (3.15) and the equation which results when (3.15) is substituted in (3.5). They are:

$$\frac{d\lambda}{dt} = \frac{D}{\cos^2 \varphi} - \Omega$$

$$\frac{d\varphi}{dt} = \mu$$

$$\frac{d\mu}{dt} = \Omega^2 \cos \varphi \sin \varphi - D^2 \frac{\sin \varphi}{\cos^3 \varphi} \tag{6.1}$$

where the variable μ has been introduced in order to obtain a system of first-order equations which are well suited for numerical integration. The quantity D is given by (3.14) and is determined completely by the initial conditions.

The system (6.1) was integrated using a fourth-order Runge-Kutta method with a timestep of 5 min during all of the calculations.

Figures 7a-d show inertial trajectories computed with the initial conditions that $u_o = 0$ and $v_o = 100$ msec^{-1}. The starting latitudes are 0°, 20°, 40°, and 60°N, respectively. The trajectories are plotted on a polar stereographic map with a standard latitude of 90°N. The first two trajectories cross the equator where the curvature changes because of the Coriolis parameter, while the last two trajectories (φ_o = 40°N and 60°N) stay entirely in one hemisphere. They show the typical spiral motion with a net westward displacement as discussed in Section 4 due to the combined effect of the Coriolis and metrical terms in the basic equations.

The next example consists of a group of trajectories computed with an initial velocity of 100 msec^{-1} and an initial position on the equator. The angles which the initial velocity forms with the equator were 10°, 30°, 50°, 70°, and 90°, respectively, for the five trajectories shown in Figures 8a-e. These types of trajectories were studied many years ago by Whipple (1917) using mathematical approximations which are equivalent to what we today would call an equatorial beta-plane approximation.

It is seen from Figures 8a-e that the inertial trajectory shows a net westward displacement of the particle when the angle between the initial velocity and the equator is sufficiently small as in Figures 8a-b, while the

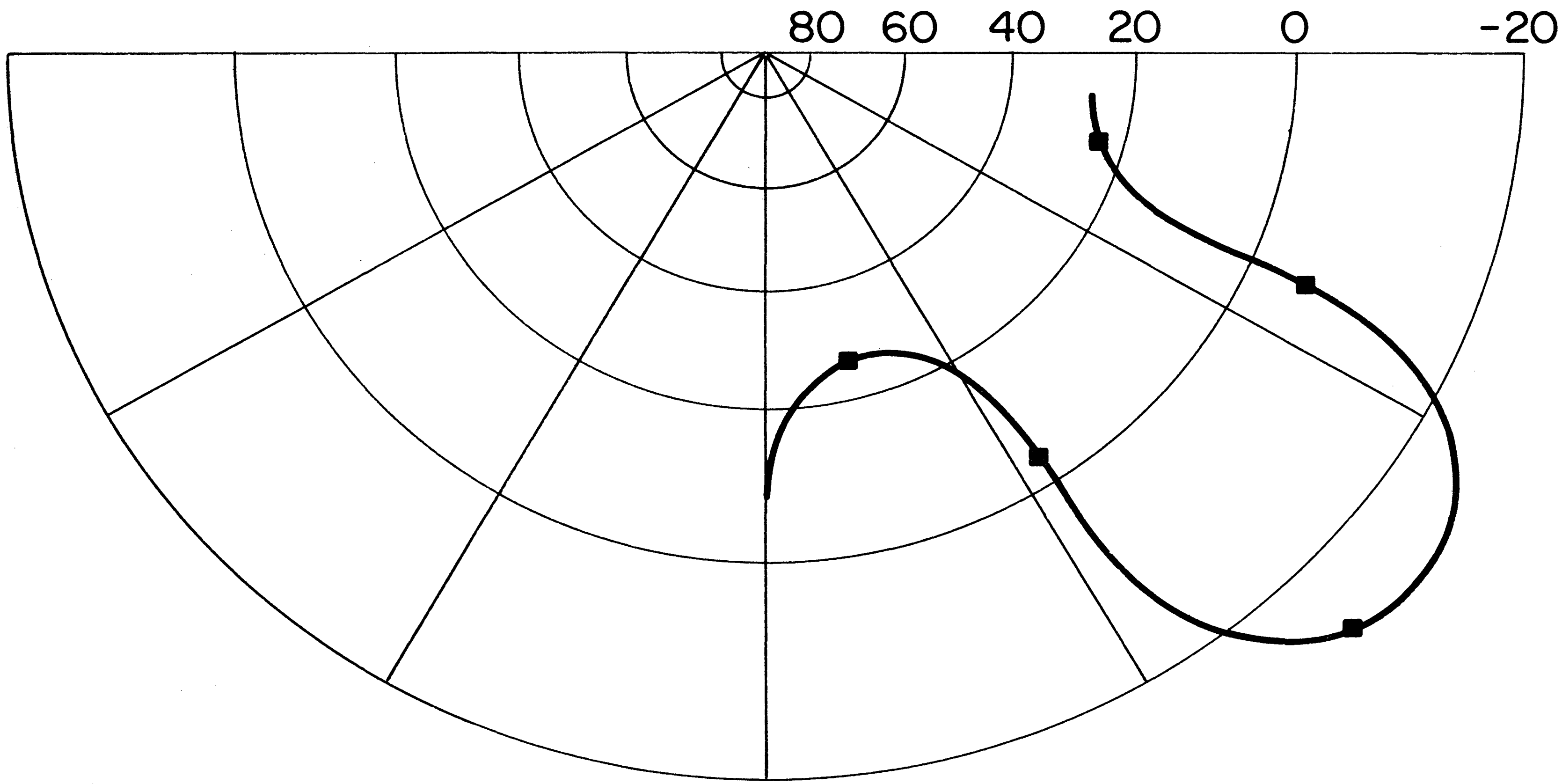

Figure 7a. Inertial trajectory starting at the equator with an initial velocity $u_o = 0$, $v_o = 100$ msec^{-1}.

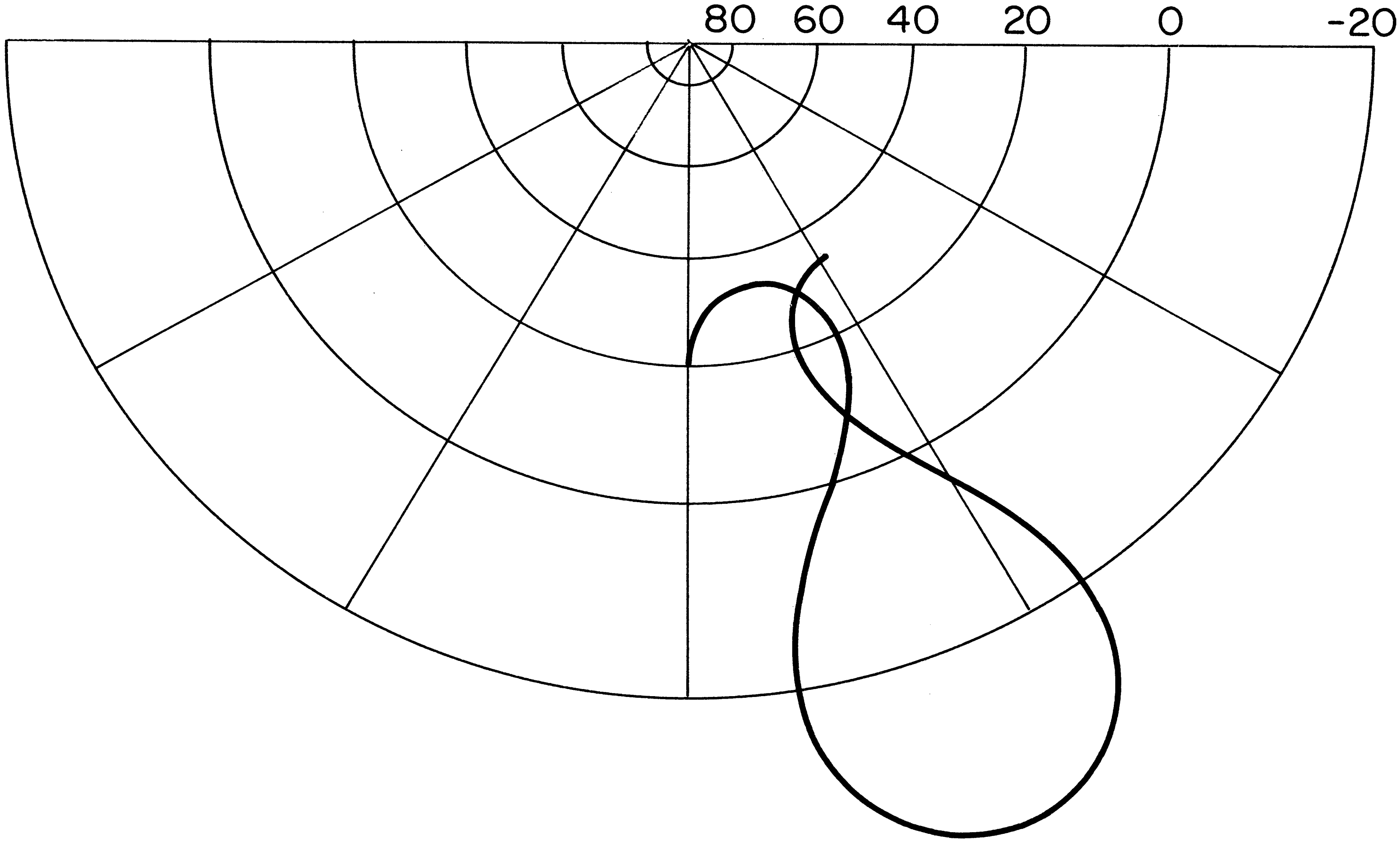

Figure 7b. Inertial trajectory starting at 20°N with an initial velocity $u_o = 0$, $v_o = 100$ msec^{-1}.

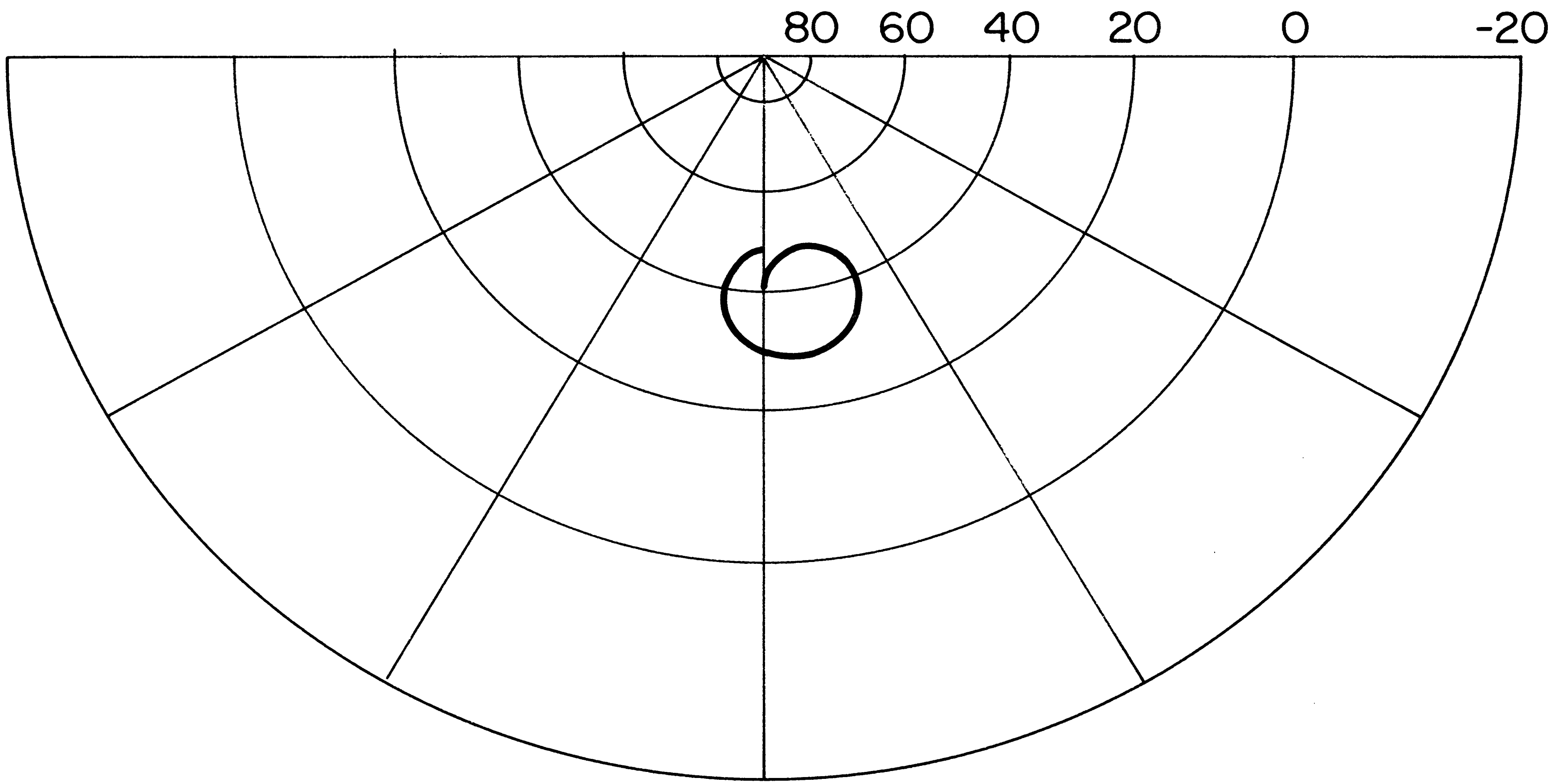

Figure 7c. Inertial trajectory starting at 40°N with an initial velocity $u_o = 0$, $v_o = 100\ msec^{-1}$.

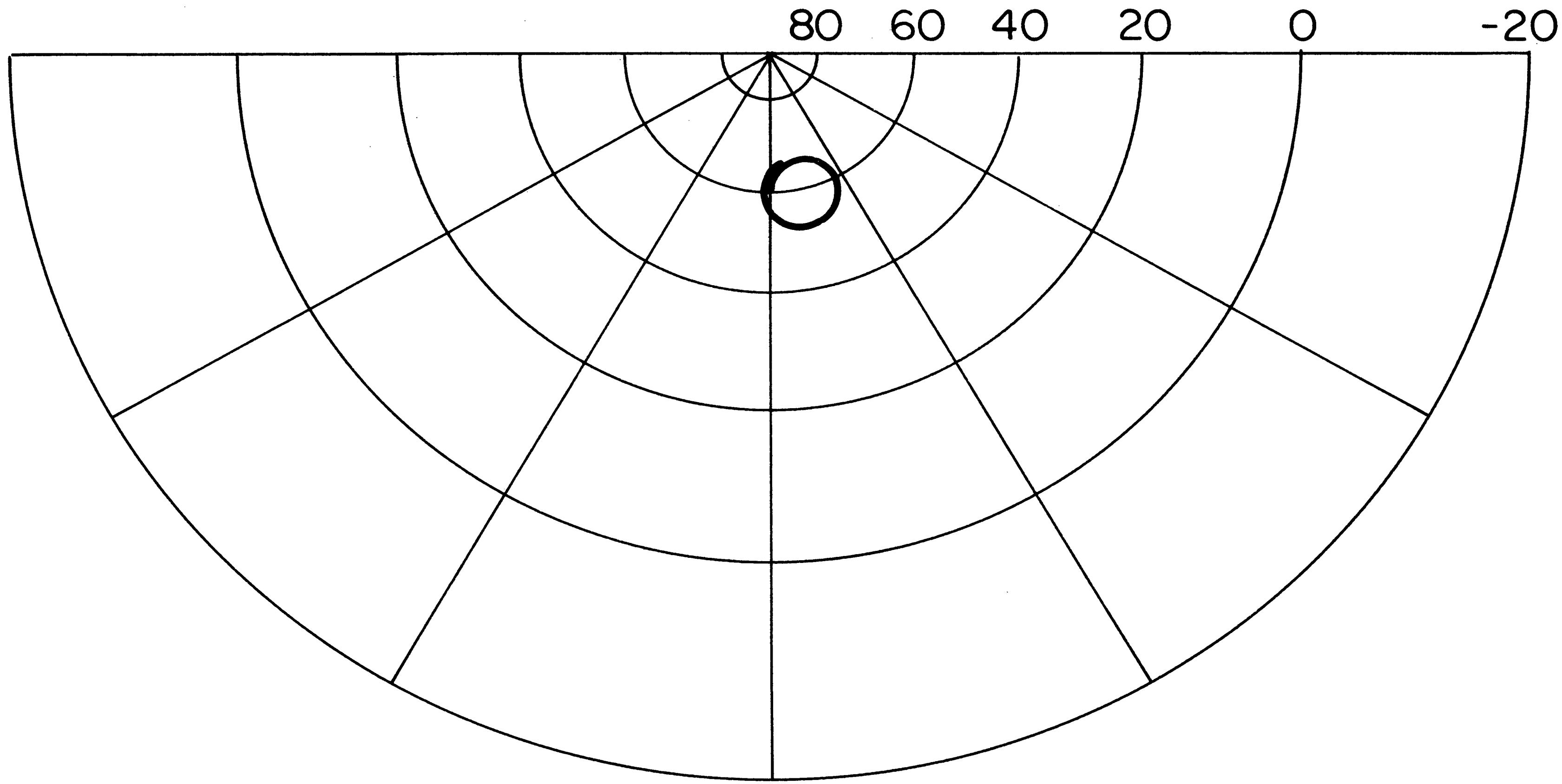

Figure 7d. Inertial trajectory starting at 60°N with an initial velocity $u_o = 0$, $v_o = 100$ msec^{-1}.

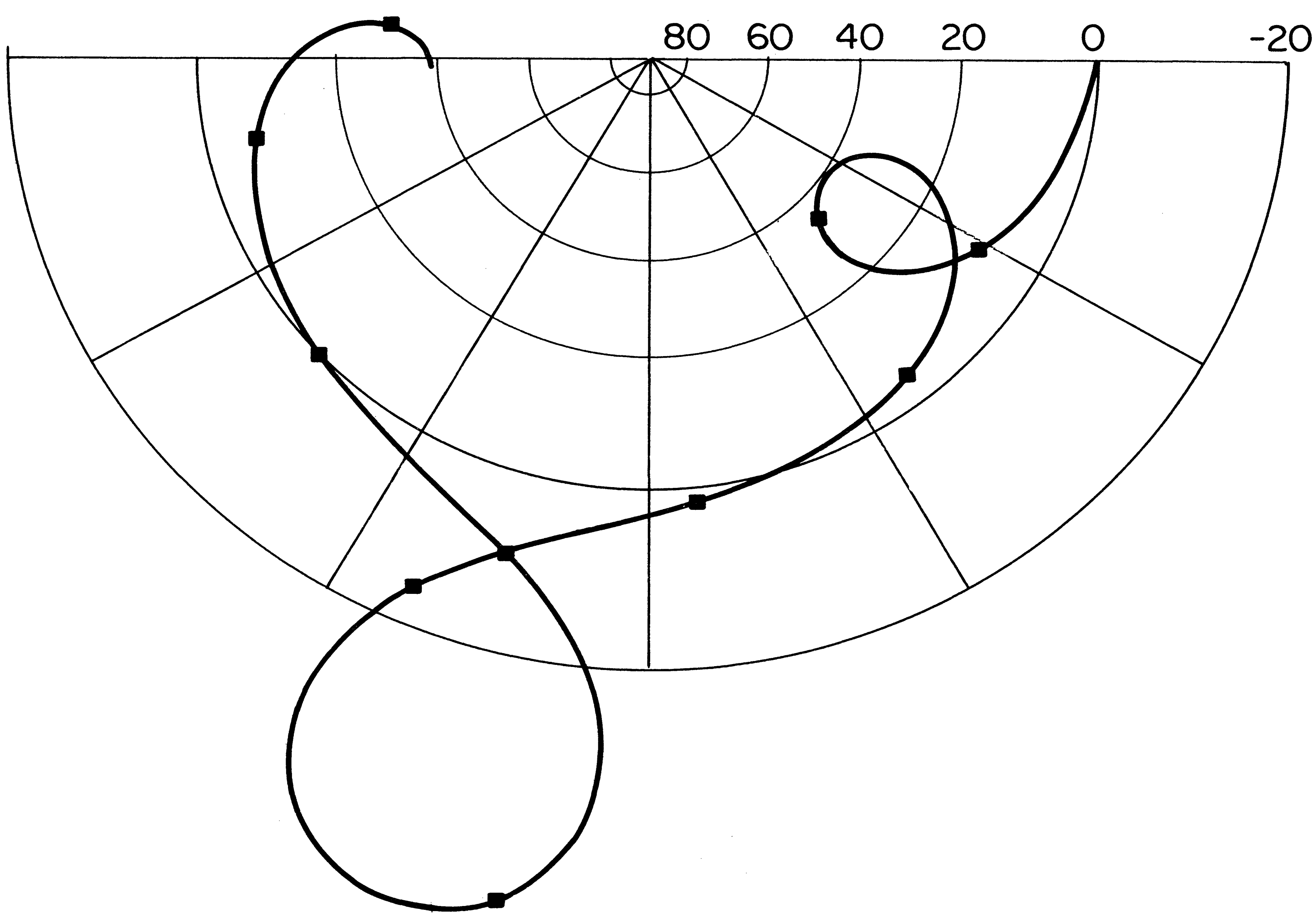

Figure 8a. Inertial trajectory starting at the equator with an initial speed of $V_o = 100$ msec^{-1}. The angle θ between the initial velocity and the equator is equal to 170°, i.e., $u_o = V_o \cos 170°$, $v_o = V_o \sin 170°$. Marks on the trajectory indicate elapsed time for each 10 hr.

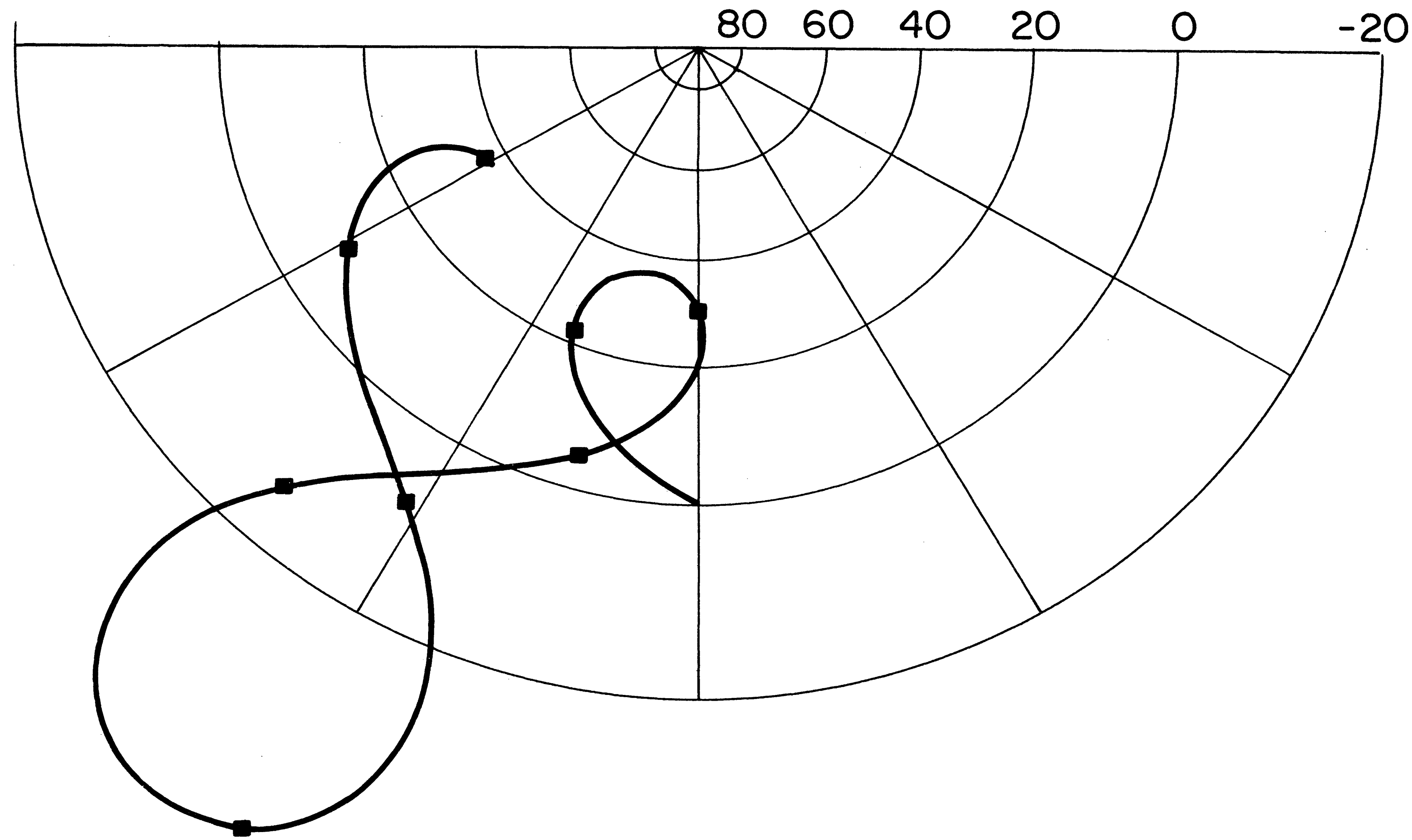

Figure 8b. As Figure 8a, but with $\theta = 150°$.

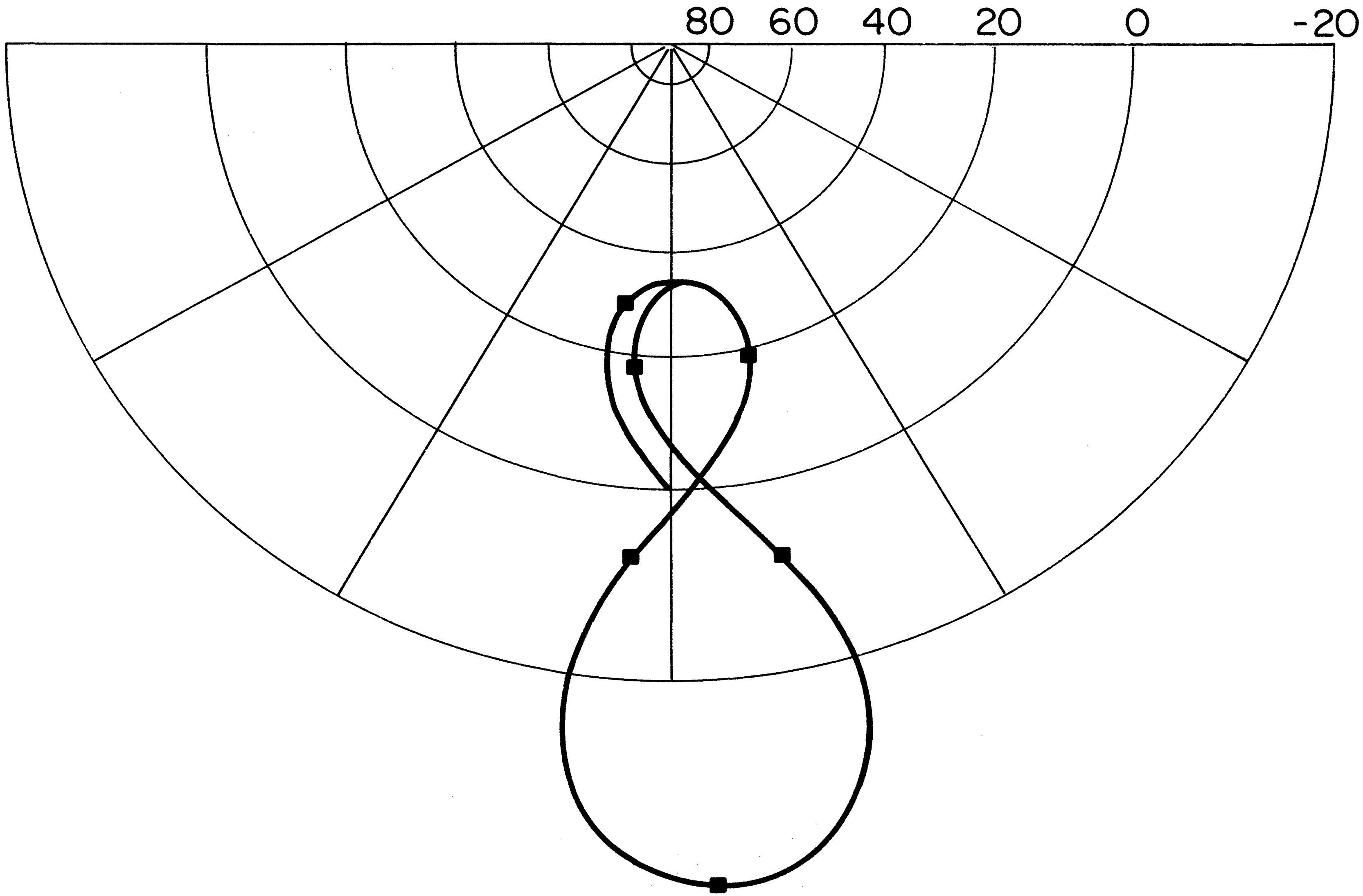

Figure 8c. As Figure 8a, but with $\theta = 130°$.

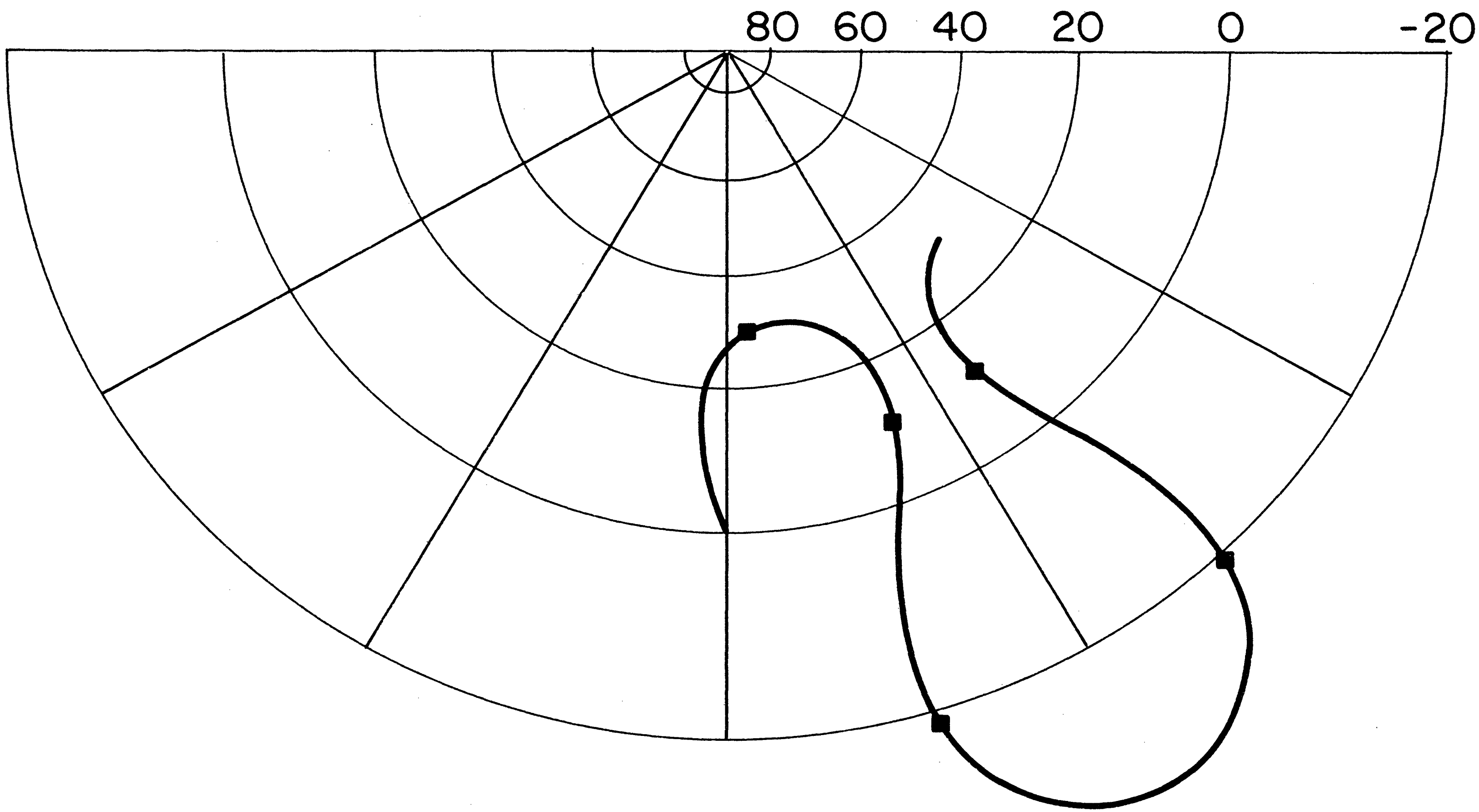

Figure 8d. As Figure 8a, but with $\theta = 110°$.

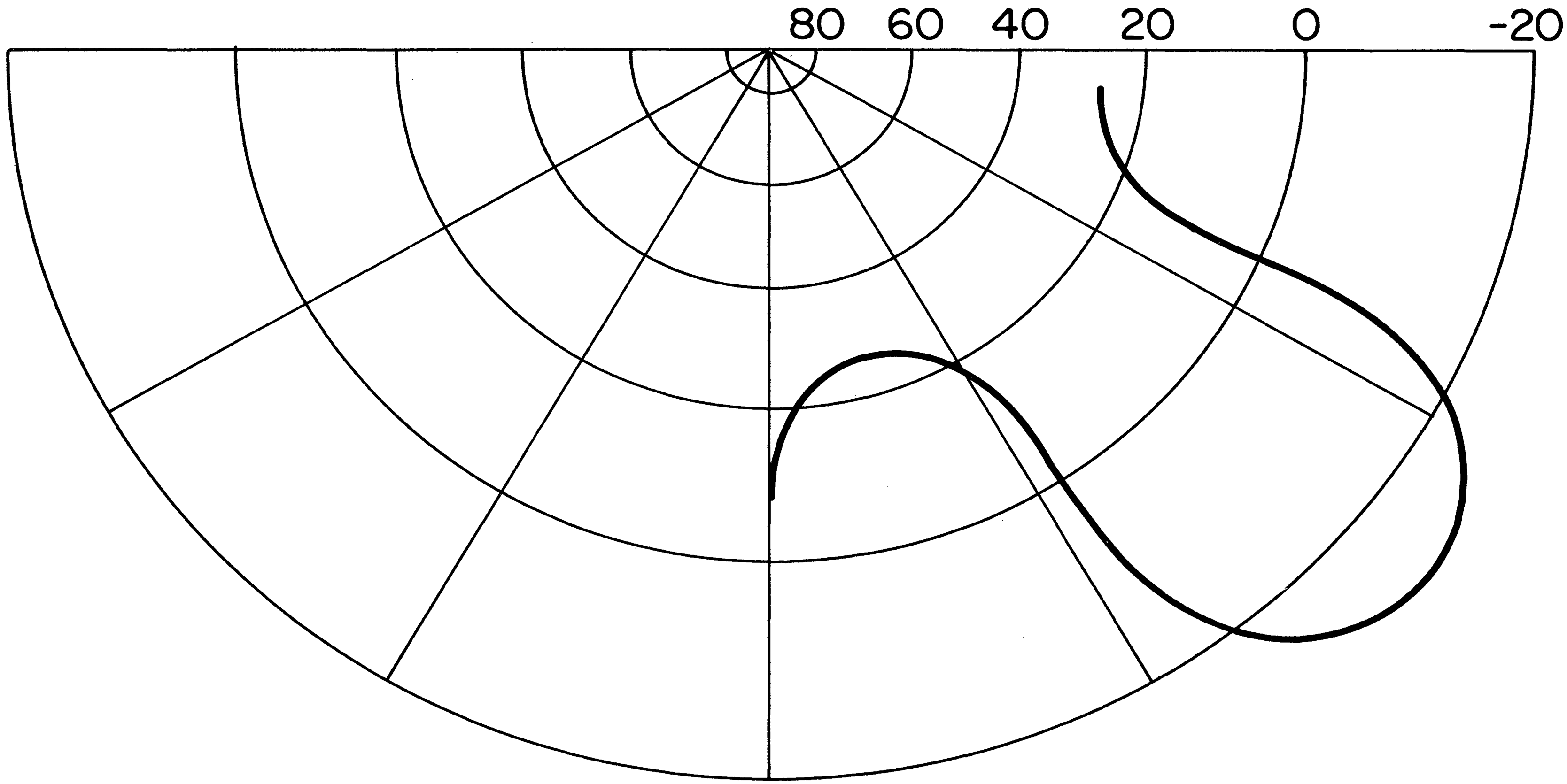

Figure 8e. As Figure 8a, but with $\theta = 90°$.

remaining cases (Figures 8c-e) shows a net eastward displacement. Using the equatorial beta-plane approximation one can show that the initial angle has to be 49.32° (Wiin-Nielsen, 1970) to give a vanishing net displacement.

7. MOTION INFLUENCED BY CORIOLIS' FORCE AND GRAVITY

We shall in this section consider some examples of motion influenced by a (constant) acceleration of gravity and by the Coriolis' force. It is obvious that such motion will be of greater complexity than the types of motion considered in the previous sections of this paper. To compensate for this fact we shall restrict ourselves to rather approximate methods.

The three equations of motion for the kind of motion considered in this section are

$$\frac{du}{dt} = fv - ew \tag{7.1}$$

$$\frac{dv}{dt} = -fu \tag{7.2}$$

$$\frac{dw}{dt} = -g + eu \tag{7.3}$$

where $f = 2\Omega \sin \varphi$ and $e = 2\Omega \cos \varphi$. Note that we have used the ordinary orientation of the coordinate system in writing the scalar Eqs. (7.1) - (7.3). The common problem is an initial value problem where we know the initial position (x_o, y_o, z_o) and the initial velocity (u_o, v_o, w_o), and where we want to calculate the trajectory.

The most simple problem is the one which results when the Coriolis' force can be disregarded altogether. In that case we obtain as the solution the ordinary parabola which most easily can be obtained by selecting the x-direction along the direction of the initial horizontal velocity such that $\vec{v}_o = u_o \vec{i}$.

The more interesting case is the one where the Coriolis' force must be taken into consideration. We shall treat the case where we will assume from the outset that f and e can be assumed to be constants. The justification of this assumption must be made a posteriori, but it is clear that the lateral region covered by the motion must be small. In any case, let $f = f_o$ and $e = e_o$ where the numerical values are determined by the initial position of the particle, i.e., $f = 2\Omega \sin \varphi_o$ and $e = 2\Omega \cos \varphi_o$. Under this condition it is rather straightforward and elementary to solve the problem posed by (7.1) - (7.3). These equations may for example be solved by forming a single equation for the time-dependent u-component. We get from (7.1) by a differentiation with respect to time:

$$\frac{d^2u}{dt^2} + 4\Omega^2 u = ge \tag{7.4}$$

The formal solution of (7.4) is

$$u = \frac{e}{4\Omega^2} g + C_1 \cos 2\Omega t + C_2 \sin 2\Omega t \tag{7.5}$$

where C_1 and C_2 are integration constants.

Let us for simplicity select the initial state by setting $x_o = y_o = z_o = u_o = v_o = 0$, but $w_o > 0$. Using the condition that $u_o = 0$ at $t = 0$ we find

$$C_1 = - \frac{e}{4\Omega^2} g \tag{7.6}$$

The initial conditions imply that

$$\frac{du}{dt} = - ew_o \qquad \text{at } t = 0 \tag{7.7}$$

(7.7) leads to a determination of C_2 with the value

$$C_2 = - \frac{e}{2\Omega} w_o$$

and the complete solution $u = u(t)$ is

$$u = \frac{e}{2\Omega} \frac{g}{2\Omega} (1 - \cos(2\Omega t)) - \frac{e}{2\Omega} w_o \sin(2\Omega t) \tag{7.8}$$

The solution for v is obtained from (7.2) which upon integration gives

$$v = \frac{ef}{4\Omega^2} \left[- gt + \frac{g}{2\Omega} \sin(2\Omega t) + w_o (1 - \cos 2\Omega t) \right] \tag{7.9}$$

The vertical velocity is obtained by an integration of (7.3) using the initial conditions leading to

$$w = \sin^2\varphi_o\,(w_o - gt) - \cos^2\varphi_o\left(\frac{g}{2\Omega}\sin 2\Omega t - w_o\cos 2\Omega t\right) \qquad (7.10)$$

The actual trajectory of the particle, i.e., x(t), y(t), and z(t) are obtained from integrations of (7.8), (7.9), and (7.10) using $x_o = y_o = z_o = 0$. We obtain

$$x = \frac{\cos\varphi_o}{2\Omega}\left\{gt - \frac{g}{2\Omega}\sin 2\Omega t - w_o\,(1 - \cos 2\Omega t)\right\}$$

$$y = \sin\varphi_o\cos\varphi_o\left\{w_o t - \frac{1}{2}gt^2 - \frac{w_o}{2\Omega}\sin(2\Omega t) + \frac{g}{2\Omega^2}(1 - \cos 2\Omega t)\right\}$$

$$z = \sin^2\varphi_o\left(w_o t - \frac{1}{2}gt^2\right) + \cos^2\varphi_o\left(\frac{w_o}{2\Omega}\sin 2\Omega t - \frac{g}{4\Omega^2}(1 - \cos 2\Omega t)\right)$$

(7.11)

It is seen from (7.11) that the only places on the earth where the Coriolis' force has no effect on this type of motion is at the Poles. At these places we have $\cos\varphi_o = 0$ and $\sin^2\varphi_o = 1$, and (7.11) reduces to

$$z = w_o t - \frac{1}{2}gt^2 \qquad (7.12)$$

which is the solution which is obtained if the Coriolis' force is disregarded.

The particular case chosen for investigation will result in a motion where the particle will return to the surface of the earth after a relatively short time. One may see this from (7.12) which applies as a lowest degree of approximation. It follows from (7.12) that the maximum time for the particle in the air is approximately

$$t_{max} \approx \frac{2w_o}{g} \qquad (7.13)$$

which is small, in general, compared to the period of the inertial motion which is $T = (2\pi)/(2\Omega) = 12$ hr. Using this fact it is possible to simplify (7.11) by using the approximations

$$\sin 2\Omega t \approx 2\Omega t - \frac{(2\Omega t)^3}{6} \qquad (7.14)$$

and

$$(1 - \cos 2\Omega t) \approx \frac{(2\Omega t)^2}{2} - \frac{(2\Omega t)^4}{24} \qquad (7.15)$$

Including the first terms of (7.14) and (7.15) we will get the influence of the Coriolis term in the lowest approximation. Introducing (7.14) and (7.15) in (7.11) we get after some manipulations:

$$x = -\cos\varphi_o \cdot w_o \Omega t^2$$

$$y = 0$$

$$z = w_o t - \frac{1}{2} g t^2 \qquad (7.16)$$

As (7.16) shows we get a displacement of the particle to the west to the first degree of approximation. One would indeed expect such a result from an inspection of (7.1) - (7.3) which shows at the initial time that $(du/dt)_o < 0$ while $(dv/dt)_o = 0$. (7.16) may be used to obtain an estimate of the displacement in the westward direction at the time when the particle reaches the ground. From the third equation we find that the elapsed time is

$$t_{max} = \frac{2w_o}{g} \qquad (7.17)$$

and we find

$$x_{max} = -4 \cos\varphi_o \, \Omega \frac{w_o^3}{g^2} \qquad (7.18)$$

while the maximum height is

$$z_{max} = \frac{w_o^2}{2g} \qquad (7.19)$$

Table 9 shows values of t_{max}, x_{max}, and z_{max} computed for various values of w_o using $\varphi_o = 45°N$, $g = 9.8$ msec^{-2}, and $\Omega = 7.3 \times 10^{-5}$ sec^{-1}. As expected we find that the displacement in the x-direction is negligible when w_o is small.

TABLE 9

VALUES OF t_{max}, z_{max}, AND x_{max} FOR VARIOUS VALUES OF w_o.
PARAMETERS: $\varphi_o = 45°N$, $g = 9.8$ msec^{-2}, AND $\Omega = 7.3 \times 10^{-5}$ sec^{-1}

w_o, msec^{-1}	1	10	100	1000
t_{max}, sec	0.204	2.04	20.4	204.1
z_{max}, m	0.05	5.10	510.2	51020.4
x_{max}, m	-0.2×10^{-5}	-0.2×10^{-2}	- 2.15	- 2149.9

As shown by (7.11) there will be a displacement in the y-direction. In order to calculate this distance, we must go to a higher degree of approximation. Incorporating both terms in the approximate expressions (7.14) and (7.15) we obtain from the second expression in (7.11) that

$$y = \frac{1}{6} \sin \varphi_o \cos \varphi_o \Omega^2 (4w_o t^3 - gt^4) \tag{7.20}$$

Assuming that the value of t_{max} given in (7.17) applies as a good first approximation in this case we find that

$$y_{max} = \frac{8}{3} \sin \varphi_o \cos \varphi_o \Omega^2 \frac{w_o^4}{g^3} \tag{7.21}$$

(7.21) shows that y_{max} will be extremely small for small values of w_o. As an example we find that $y_{max} = 7.5$ m for $w_o = 10^3$ m using the same numerical values for the other parameters as before. This result is in agreement with the tendency calculated from (7.2) which shows that v will be positive when u has become negative as it will be as seen from (7.1). The second-order correction to x_{max} is not negligible, but can be calculated from substituting (7.14) and (7.15) in the first equation of (7.11). We find that the corrected value of x_{max} is - 716.7 m.

The motion described here is very easy to analyze from an energetical point of view. Initially, all the energy exists as kinetic energy in the vertical velocity component. As the particle leaves the initial position the kinetic energy of the vertical component is transformed into potential energy and into kinetic energy of the horizontal motion. When the particle reaches its highest position, it will have its maximum potential energy. During the trajectory down to the earth the potential energy will decrease, while the kinetic energy will increase with a partitioning of the energy between the horizontal and the vertical motion. Using $k_h = 1/2\ (u^2 + v^2)$, $k_w = 1/2\ w^2$, and $p = gz$ we find from (7.1) - (7.3) that

$$\frac{dk_h}{dt} = - euw$$

$$\frac{dk_w}{dt} = - gw + euw$$

$$\frac{dp}{dt} = g \frac{dz}{dt} = gw \qquad (7.22)$$

(7.22) shows that the energy conversion from k_w to p is $C(k_w,p) = gw$, while the conversion from k_w to k_h is $C(k_w,k_h) = - euw$. $C(k_w,p)$ will thus be positive during the upward part of the trajectory and negative on the downward part. Since $w > 0$ and $u < 0$ during the upward part of the trajectory we will get a conversion from k_w to k_h during the rising part of the trajectory, but opposite after the maximum height has been reached. Figures 9a-d show the time variation of u, v, x, and y during a particle trajectory starting at $x = y = z = 0$ with the velocity $u_o = v_o = 0$, $w_o = 10^3$ m, and $\varphi_o = 45°$N. The trajectory has been plotted to the point where the particle reaches the ground again. Note, that $|u|$ is about two orders of magnitude larger than v during the whole trajectory and consequently $|x| \gg y$. The curves in the figures were computed from the exact Eqs. (7.11). The kinetic energy k_h is generally small compared to the potential energy, which increases from zero to about 5×10^5 m^2sec^{-2} at the top of the trajectory (z = 51 km). At that point we have $k_h = 14$ m^2sec^{-2} which is the maximum value during the whole trajectory. On the other hand, k_w decreases from 5×10^5 m^2sec^{-1} initially to zero at the highest point of the trajectory. As expected we find $C(k_w,p)$ and $C(k_w,k_h)$ positive as long as $w > 0$ and negative when $w < 0$, but these simple relationships would not hold at a later stage as can be seen from Figure 9a which shows that u will turn to negative values.

The value of $\varphi_o = 45°$N was selected because we should expect the largest meridional displacements in the middle latitudes. We have already remarked that $x = y = 0$ for all t at the Poles. It is furthermore seen from the second equation that $y = 0$ for all t at the equator, when the displacement will be entirely in a direction from east to west. However, even here we find the maximum displacement to be no larger than about 1 km.

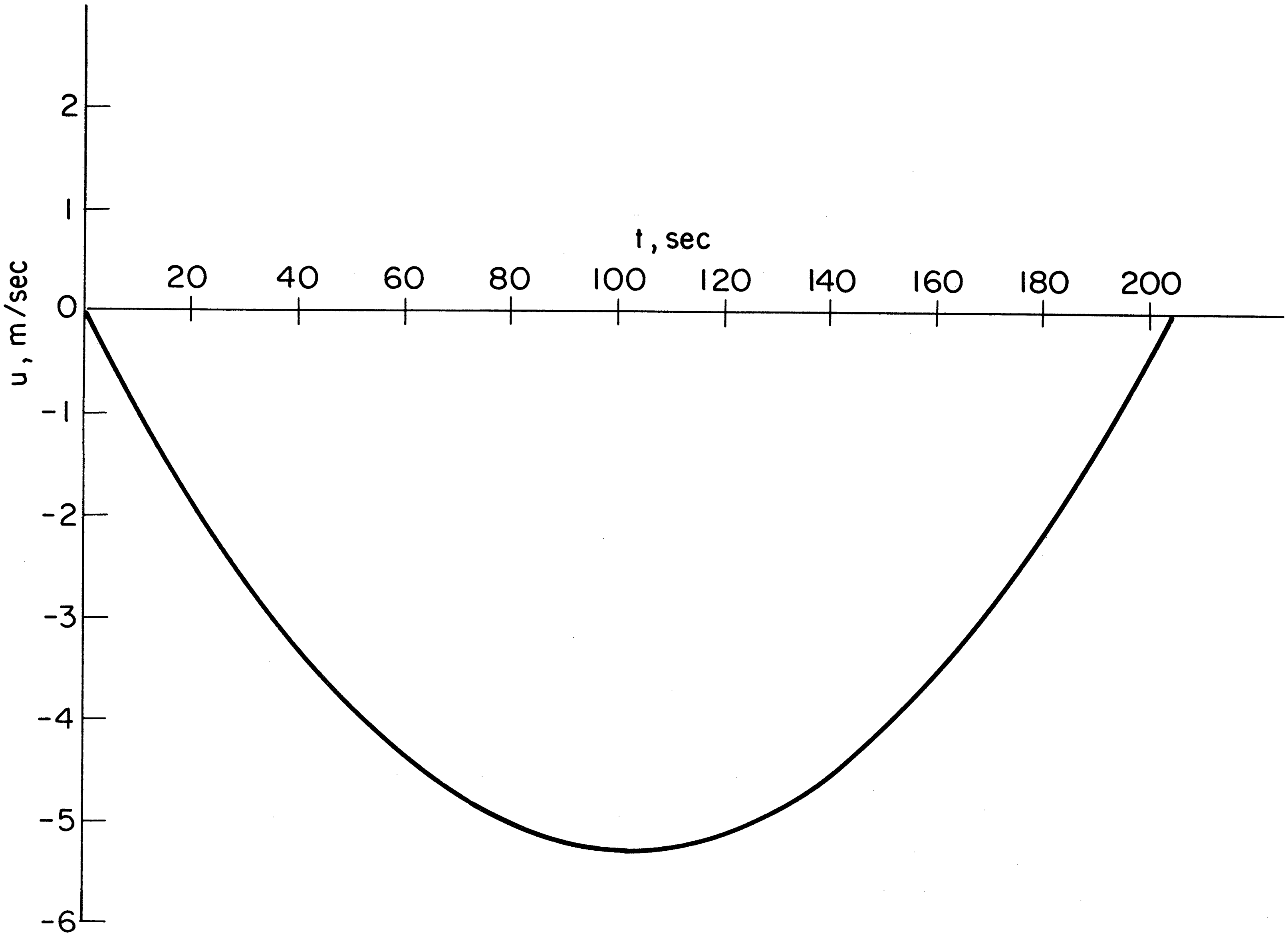

Figure 9a. Zonal velocity as a function of time for a trajectory determined by the acceleration of gravity and the Coriolis force using $\varphi_o = 45°$, $u_o = v_o = 0$, and $w_o = 10^3$ msec^{-1}.

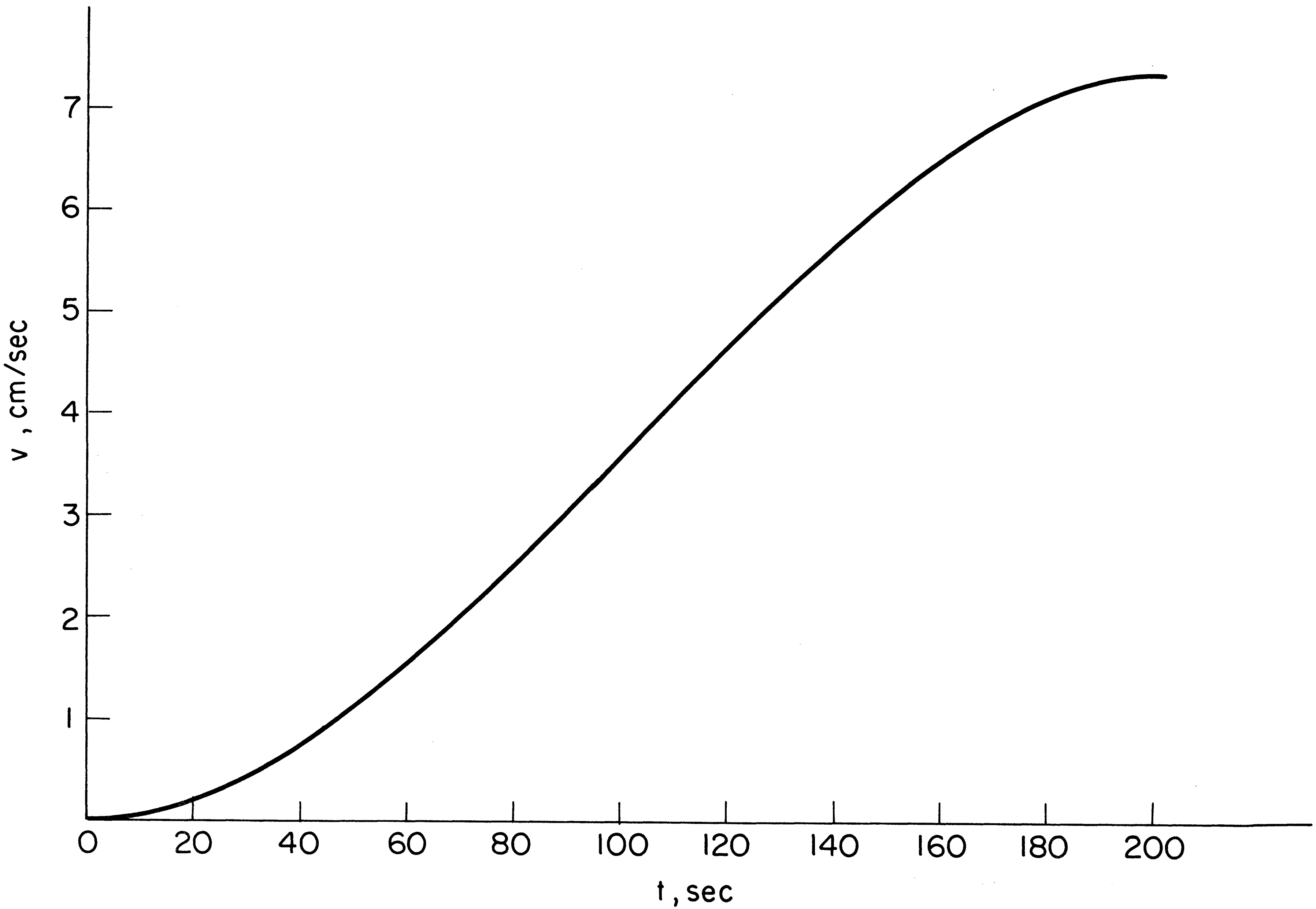

Figure 9b. Meridional velocity as a function of time for the same trajectory as in Figure 9a.

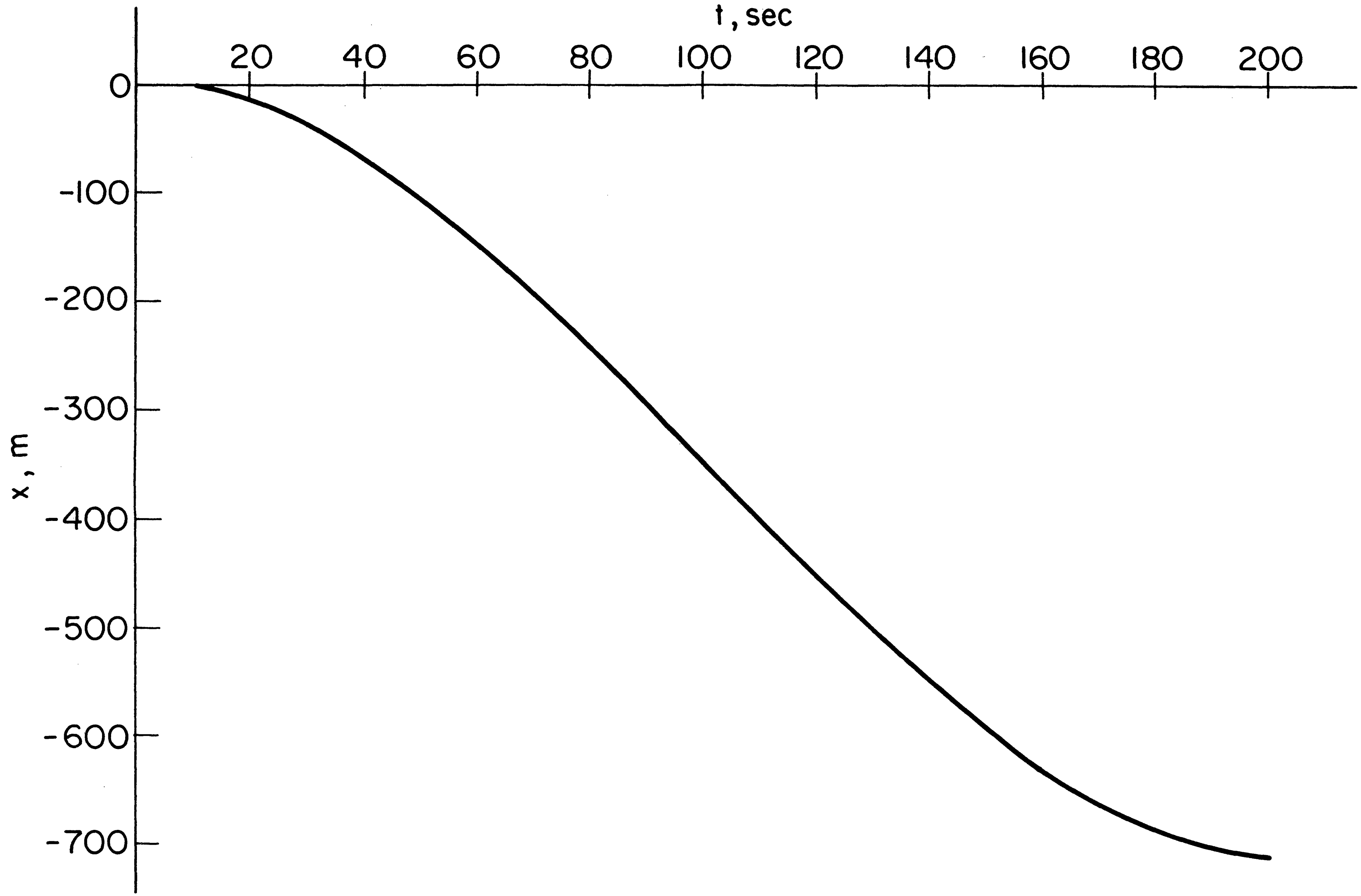

Figure 9c. Displacement in the zonal direction as a function of time for the same trajectory as in Figure 9a.

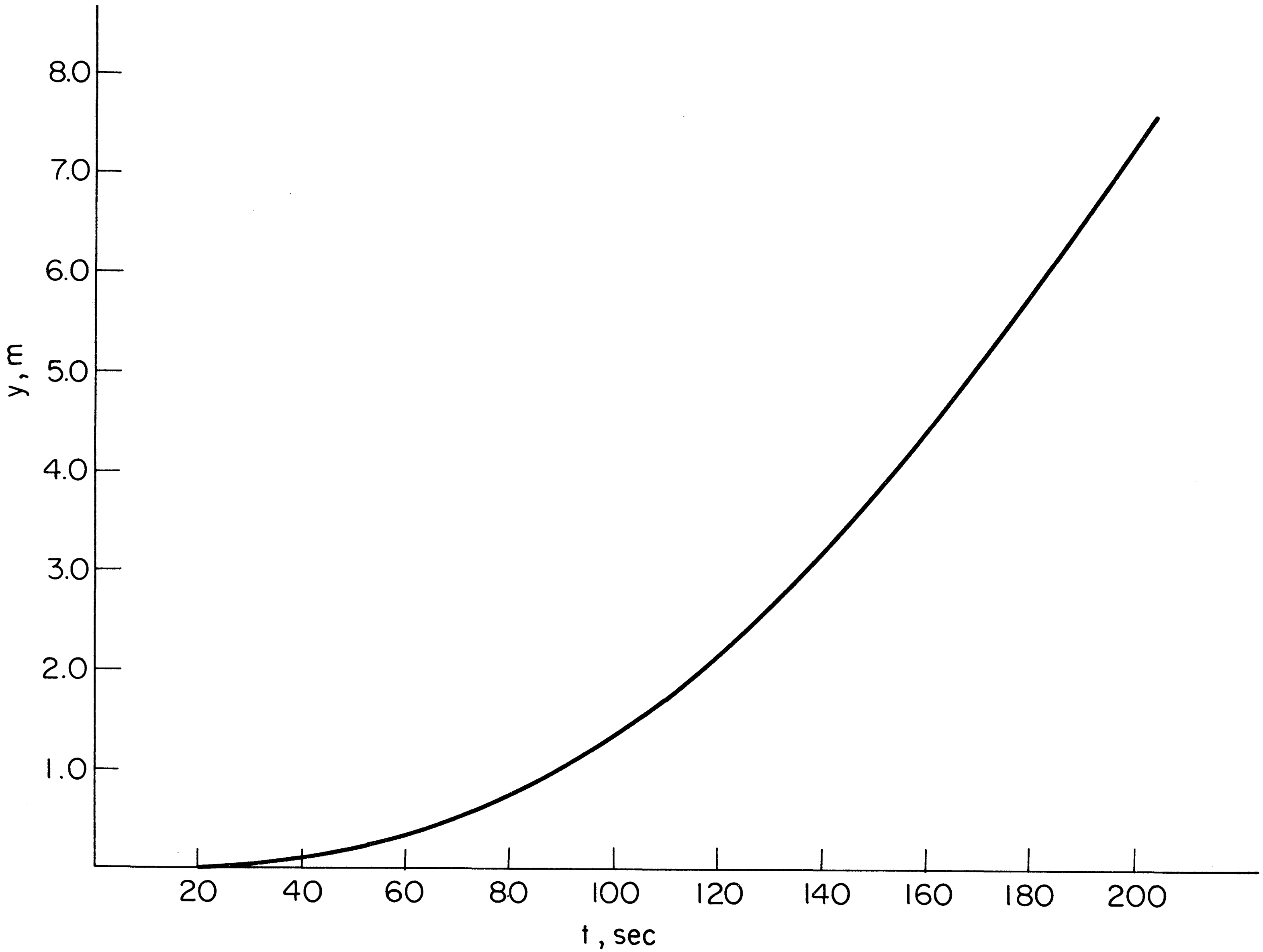

Figure 9d. Displacement in the meridional direction as a function of time for the same trajectory as in Figure 9a.

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