THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

GENERALIZATION OF A CLASS OF SOLUTIONS OF THE LAMINAR, INCOMPRESSIBLE BOUNDARY LAYER EQUATIONS

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LIST OF FIGURES

NOMENCLATURE

$a, a', a_1, a_2, a'_1, a'_2$	constants
b, b'	constants
C_p	constant pressure specific heat
C_i	constant ($i = 0, 1, \ldots\ldots\ldots\ldots 10$)
$F(\eta)$	function of similarity parameter η, $u = UF'(\eta)$
$G(\eta)$	function of similarity parameter η, $w = WG'(\eta)$
$g(x,z)$	function of coordinates, x and z
κ	thermal conductivity
m	constant
n, n'	constants
Pr	Prandtl number, $Pr = \frac{C_p \mu}{\kappa}$
r	constant
s	constant
T^*	scaled surface temperature, $T^*(x,z) = T_w - T_\infty$
T_∞	temperature of mainstream
T_w	temperature of surface
u, v, w	boundary layer velocity components in x, y, z directions respectively
$\bar{u}, \bar{v}, \bar{w}$	boundary layer velocity components in X, Y, Z directions respectively
U, W	mainstream velocity components in x, z directions respectively
x, y, z	skew-linear coordinate system
X, Y, Z	rectangular coordinate system

NOMENCLATURE (continued)

η	similarity space variable,
ν	coefficient of kinematic viscosity
μ	coefficient of absolute viscosity
ρ	density
θ	angle between X and x in surface plane
θ^*	dimensionless temperature, $\theta^* = \frac{T - T_\infty}{T_W - T_\infty}$
Θ	function of similarity parameter η, $\theta^* = \Theta(\eta)$

susperscript prime denotes differentiation with respect to η

INTRODUCTION

Exact solutions of the laminar, incompressible boundary-layer equations are generally obtained by the use of similarity transformations. By means of such transformations, the equations of the boundary layer as well as the associated energy equation are reduced to ordinary differential equations which can then be solved by standard numerical methods. The price that is paid for obtaining such solutions is that the method is applicable only for rather restricted classes of mainstream flows (Reference 1). This might well be expected since one of the characteristics of similarity solutions is that the computed boundary layer velocity profiles are allowed to vary only in scale along coordinate directions. This limitation is reflected at once in the permissible types of mainstream flows which give rise to the boundary layers. It can be stated, therefore, that the choice of the coordinate system is one of the key factors in obtaining physically significant similar solutions of the boundary layer equations. Unfortunately, there are also restrictions on the type of coordinate systems which may be employed in similarity analyses. For example, if flow over a plane surface is considered, the permissible orthogonal coordinate systems are rectangular, polar, and spiral (Ref. 1).

In the analysis which follows, it will be shown that a considerable degree of generalization can be achieved in formulating similarity solutions of the boundary-layer equations by the simple expedient of relaxing the requirement that the coordinate system be orthogonal.

The system to be considered will be a linear skew system. It will further be shown that similar solutions of the boundary layer referred to a rectangular coordinate system are carried over directly as solutions in the linear skew systems.

Analysis of the Boundary Layer Equations
in Linear Skew Coordinates

Consider a linear skew coordinate system embedded in a surface as shown in Figure 1. The skew coordinates x, z are related to the rectangular coordinates X, Z by the equations.

$$x = X \sec\theta \qquad X = x\cos\theta$$
$$z = Z - X\tan\theta \qquad Z = z + x\sin\theta \tag{1}$$

The y-coordinate remains normal to the surface and hence is unchanged by the above transformation.

If mainstream velocity components in the x- and z- directions are designated by U and W, respectively, and the velocity components of the boundary layer in the x-, z- and y- directions by u, w and v respectively, the three-dimensional boundary-layer equations become (c.f. Ref. 2 Chap. 18).

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2a}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} + \nu\frac{\partial^2 w}{\partial y^2} \tag{2b}$$

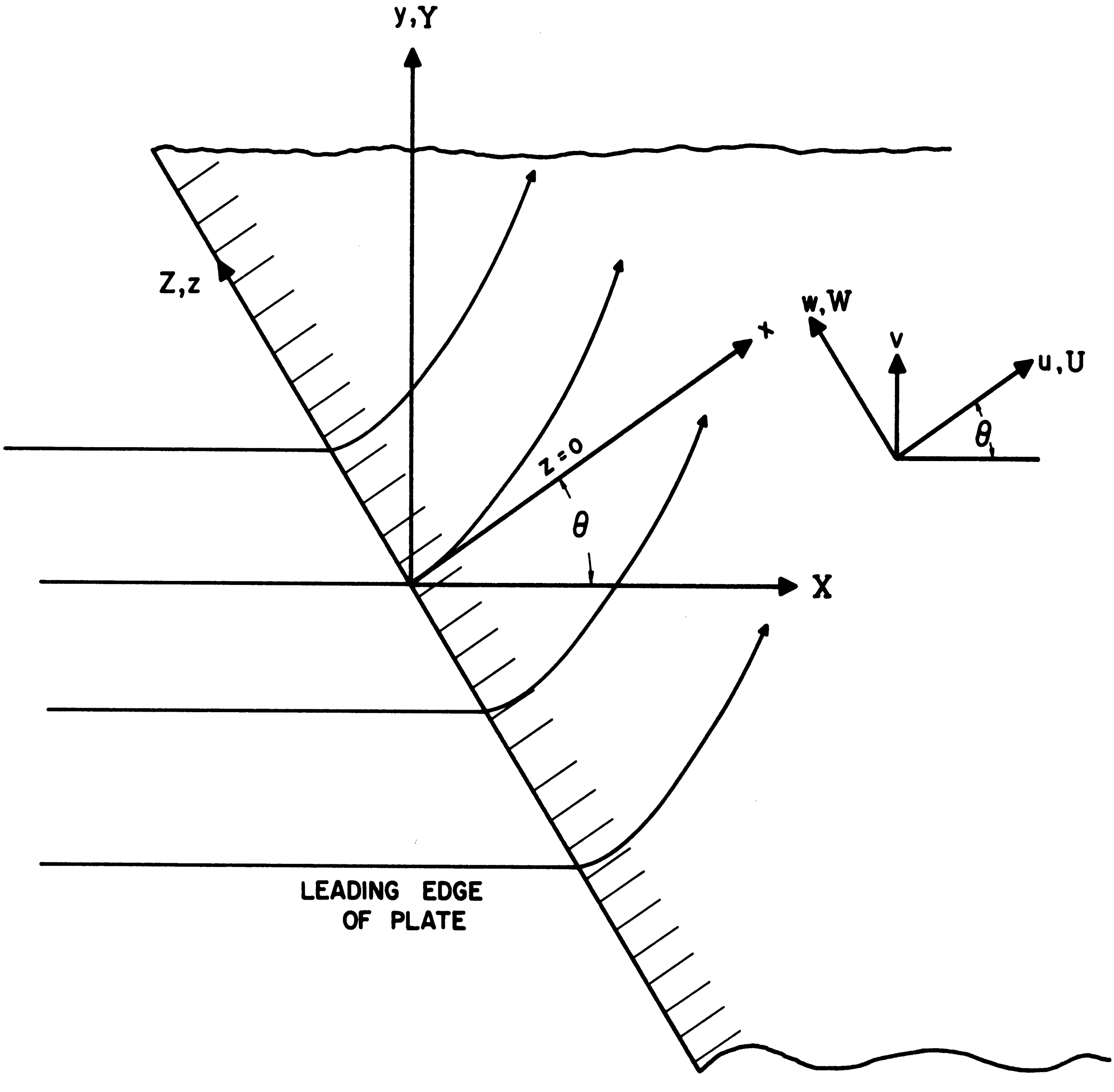

Figure 1. Orientation of skew coordinates

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2c}$$

Equations (2a) - (2c) are clearly identical to the equations which one would write for a system of rectangular coordinates. It follows that solutions of the boundary layer equations in rectangular coordinates are valid for the equations expressed in linear skew coordinates.

The requirements for similarity solutions of the boundary layer equations in rectangular coordinates have been determined in Reference (1). The results can be summarized as follows.

$$\eta = \frac{y}{\sqrt{\nu}} \; g(x,z) \tag{3a}$$

$$\frac{u}{U} = F'(\eta) \tag{3b}$$

$$\frac{w}{W} = G'(\eta) \tag{3c}$$

$$v = -\frac{\sqrt{\nu}}{g}\left[\frac{\partial U}{\partial x}F + U\frac{\partial \ell n g}{\partial x}(\eta F' - F) + \frac{\partial W}{\partial z}G + W\frac{\partial \ell n g}{\partial z}(\eta G' - G)\right] \tag{3d}$$

Substitution of expressions (3b) - (3d) into Equations (2a) - (2c) gives

$$F''' - C_1 F'^2 + (C_1 - C_2)F''F + (C_3 - C_4)F''G - C_5 G'F' + C_1 + C_5 = 0 \tag{4a}$$

$$G''' - C_3 G'^2 + (C_3 - C_4)G''G + (C_1 - C_2)G''F - C_6 F'G' + C_6 + C_3 = 0 \tag{4b}$$

where

$$C_1 = \frac{1}{g^2}\frac{\partial U}{\partial x} \qquad C_2 = \frac{U}{g^2}\frac{\partial \ell n g}{\partial x}$$

$$C_3 = \frac{1}{g^2}\frac{\partial W}{\partial x} \qquad C_4 = \frac{W}{g^2}\frac{\partial \ell n g}{\partial z}$$

$$C_5 = \frac{W}{Ug^2}\frac{\partial U}{\partial z} \qquad C_6 = \frac{U}{Wg^2}\frac{\partial W}{\partial x}$$

The boundary conditions on F and G are

$$F(0) = G(0) = 0$$

$$F'(0) = G'(0) = 0$$

$$\lim_{\eta\to\infty} F'(\eta) = \lim_{\eta\to\infty} G'(\eta) = 1$$

If it is now required that the C_i $(i = 1, 2, \ldots, 6)$ be constant, (in order that Equations (4a) and (4b) become ordinary differential equations in (η) four sets of solutions for U, W and g^2 can be found, they are: (Reference 1)

Case I

$$U = ae^{nx}z^{m-1}$$

$$W = be^{nx}z^{m}$$

$$g^2 = cU = \frac{ca}{b}\frac{W}{z}$$

Case II

$$U = ax^{n}z^{m-1}$$

$$W = bx^{n-1}z^{m}$$

$$g^2 = \frac{cU}{x} = \frac{cb}{a}\frac{W}{z}$$

Case III

$U = ax^n$

$W = bx^m$

$g^2 = \frac{cU}{x}$

Case IV

$U = ae^{nx}$

$W = be^{mx}$

$g^2 = cU$

(Four other cases could also be listed by obvious symmetrical interchange of variables)

It is interesting at this point to investigate the differences between the requirements on the mainstream flow as referred to rectangular coordinates and those referred to the linear skew coordinates. This can be done readily by expressing the results for the two systems in terms of rectangular coordinates by use of Equations (1). Table I lists expressions for mainstream velocity components and streamlines:

Table I - Comparison between mainstream characteristics in rectangular and skew-linear coordinate systems.

	Case I		Case II	
	Rectangular	Skew-Linear	Rectangular	Skew-Linear
U	$ae^{nX}Z^{m-1}$	$ae^{n'X}(Z-X\tan\theta)^{m-1}$	aX^nZ^{m-1}	$a'X^n(Z-\tan\theta X)^{m-1}$
W	$be^{nX}Z^m$	$be^{n'X}(Z-X\tan\theta)^m$	$bX^{n-1}Z^m$	$b'X^{n-1}(Z-\tan\theta X)^m$
Mainstream Streamlines	$Z = a_1e^{a_2X}$	$Z=a_1e^{a_2'X}+X\tan\theta$	$Z = a_1X^n + a_2$	$Z=a_1'X^n + X\tan\theta$

Table I (continued)

	Case III		Case IV	
	Rectangular	Skew-Linear	Rectangular	Skew-Linear
U	aX^n	$a'X^n$	ae^{nX}	$ae^{n'X}$
W	bX^m	$b'X^m$	be^{mX}	$be^{m'X}$
Mainstream Streamlines	$Z=a_1X^{(m-n+1)}$	$Z=a_1'X^{m-n+1}+X\tan\theta$	$Z=a_1e^{(m-n)X}$	$Z=a_1e^{(m'-n)X}+X\tan\theta$

In order to illustrate more completely the distinctions between solutions expressed in rectangular and skew-linear systems the following example has been chosen from Case III. Let $U = U_o = \text{const}$ and $W = 2U_o x$. The corresponding mainstream streamlines are then given by

$$z = x^2 + \text{const} \tag{5}$$

The similarity equations for the boundary layer are

$$\frac{FF''}{2} + F''' = 0 \tag{6a}$$

$$G''' + G''F - F'G' + 1 = 0 \tag{6b}$$

$$F(0) = G(0) = F'(0) = G'(0) = 0$$

$$\lim_{\eta\to\infty} F' = \lim_{\eta\to\infty} G' = 1$$

Solutions for Equations (6a) and (6b) are given in Reference (3). The functions $F'(\eta)$ and $G'(\eta)$ are plotted in Figure 2. It is shown in

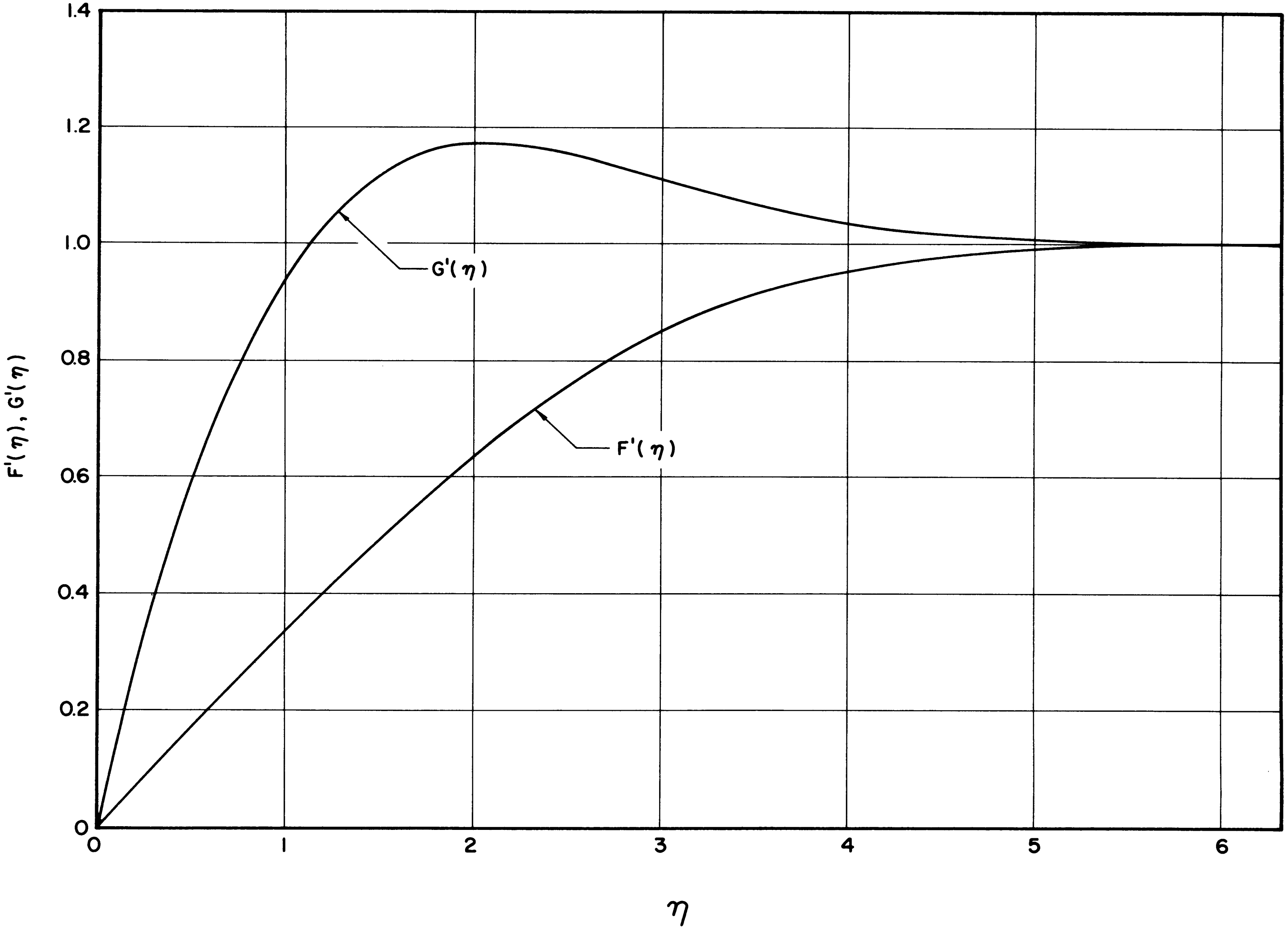

Figure 2. Functions F' (η) and G' (η) plotted against η

Reference (3) that the so-called "limiting streamline" (i.e. the envelope of boundary layer streamlines as $y \to 0$) is given by

$$z = X^2 \frac{G''(0)}{F''(0)} + \text{const.} = 4.270x^2 + \text{const.} \tag{7}$$

Plots of Equations (5) and (7) appear in Figures 3a - 3c for a rectangular system and linear-skew systems in which the angle inclination of the x-axis and the X-axis is 30° and 60° respectively. It can be seen from these plots that when rectangular coordinates are employed the mainstream flow crosses the surface leading edge at right angles while in the skew system the mainflow crosses the leading edge at the angle of inclination of the x-axis. In the latter cases, the surfaces can be considered as inclined to a uniform flow.

The relationship between boundary-layer velocity profiles can be obtained readily from Equation (1):

$$\bar{u} = u \cos \theta \tag{8a}$$

$$\bar{v} = v \tag{8b}$$

$$\bar{w} = w + u \sin \theta \tag{8c}$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ have been used to denote the boundary-layer velocity components of a flow in a skew system referred to the X, Z - axes. Substituting the relations for u and w from (3b) and (3c) into (8a) and (8c) gives

$$u = U \cos \theta \, F'(\eta) \tag{9a}$$

$$w = WG'(\eta) + U \sin \theta \, F'(\eta) \tag{9b}$$

The velocity component normal to the surface remains identical to that of the orthogonal coordinate system. Thus it is seen that the velocity profiles for a skew system exhibit similarity in the X-direction but not in the Z-direction unless W = U.

Requirements for Similar Temperature Profiles

Requirements for similar temperature profiles in a skew-linear system can be determined from an analysis of the boundary-layer energy equations under the assumption of constant fluid properties, (c.f. Reference 4). The equation in terms of the skew system variables is:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + 2 \sin \theta \frac{\partial u}{\partial y} \frac{\partial w}{\partial y}\right] \quad (10)$$

A non-dimensional temperature ratio is now defined as (c.f. Reference 4)

$$\theta^* = \frac{T - T_\infty}{T^*} = \frac{T - T_\infty}{T_w - T_\infty}$$

where

T = temperature in the boundary layer

T_∞ = temperature of the mainstream

T_w = surface temperature

$T^* = T^*(x,z) = T_w - T_\infty$

If θ^* is expressed as a function of the similarity variable η, $\theta^* = \Theta(\eta)$ Equations (3a), (3b), (3c), (3d) and (11) can be used to transform Equation (10) into the form

$$\Theta'' - C_7F'\Theta - C_8G'\Theta + P_r(C_1-C_2)\Theta'F + P_r(C_3-C_4)\Theta'G + C_9F''^2 + C_{10}G''^2 + C_{11}F''G'' = 0 \quad (12)$$

$$\Theta(0) = 1 \qquad \lim_{\eta \to \infty} \Theta(\eta) = 0$$

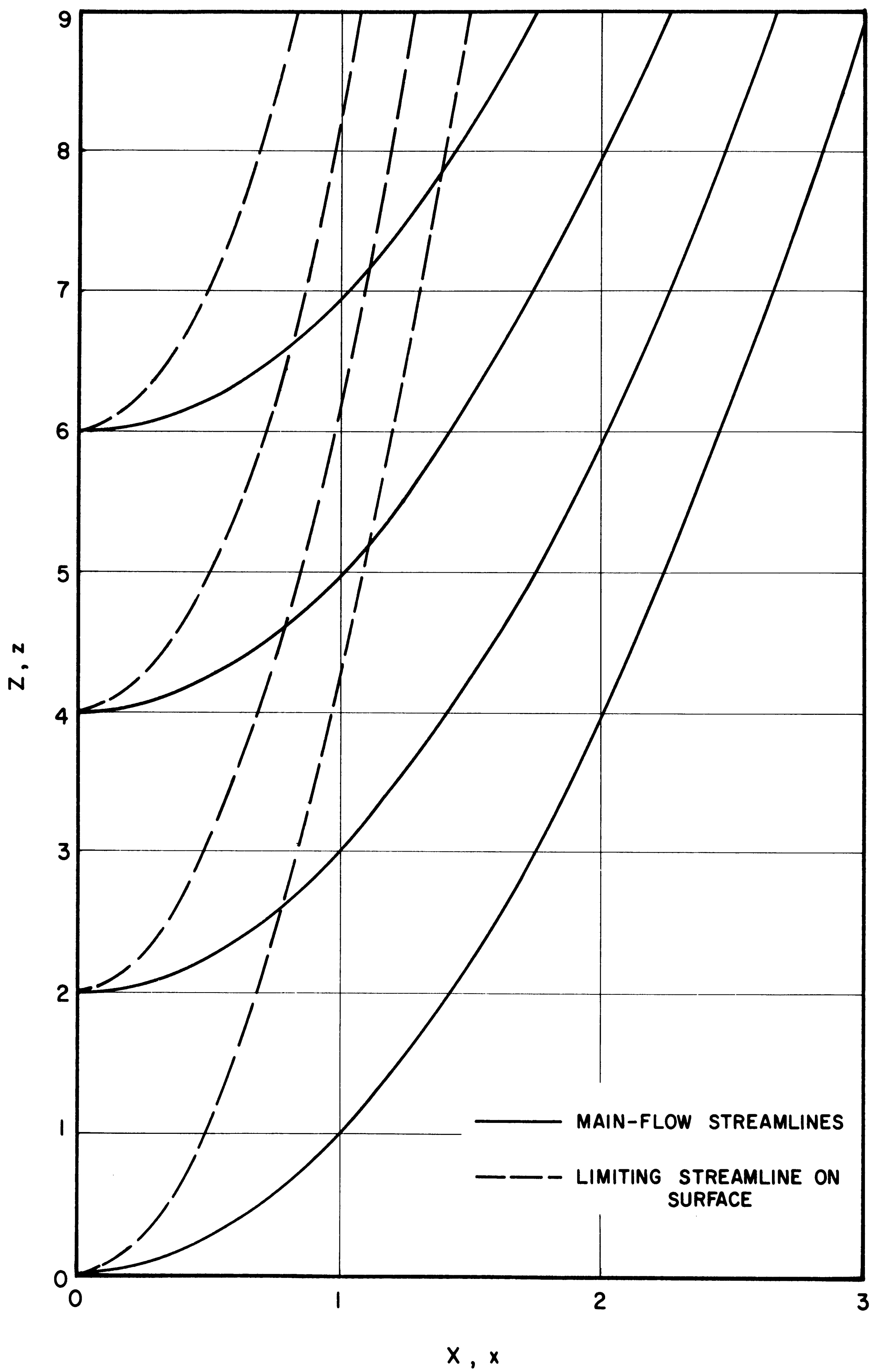

Figure 3a. Streamlines of flow in skew-linear coord.
$\theta = 0°$

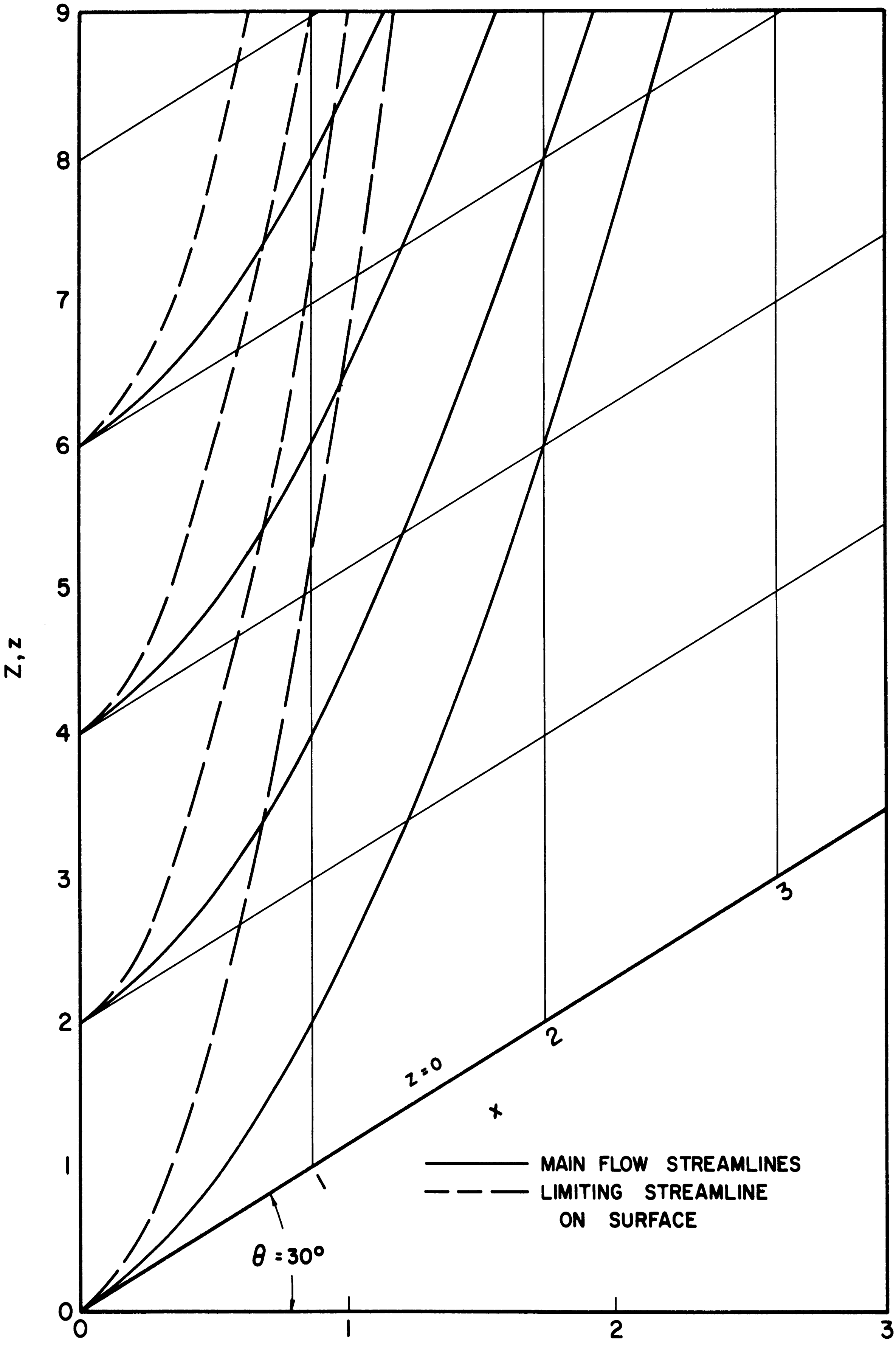

Figure 3b. Streamlines of flow in skew-linear coord, $\theta = 30°$

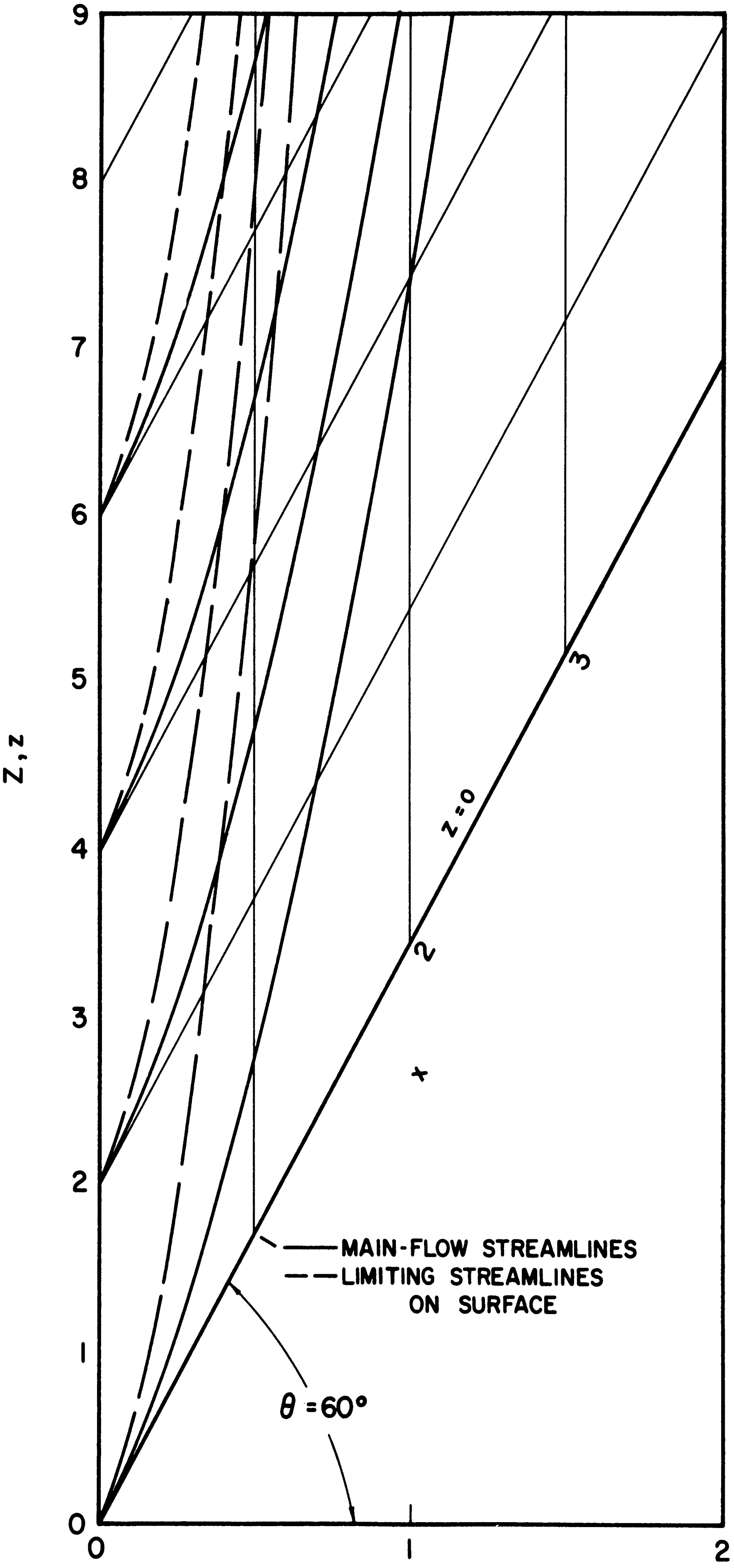

Figure 3c. Streamlines of flow in skew-linear coord, $\theta = 60°$

where

$$C_7 = \frac{P_r U}{g^2} \frac{\partial \ln T^*}{\partial x} \qquad C_9 = \frac{P_r U^2}{T^* C_p}$$

$$C_8 = \frac{P_r W}{g^2} \frac{\partial \ln T^*}{\partial z} \qquad C_{10} = \frac{P_r W^2}{T^* C_p}$$

$$C_{11} = 2 \sin \theta \frac{P_r}{T^* C_p} UW = 2 \sin \theta \sqrt{C_{10}} \sqrt{C_9}$$

In order that Equation (12) be an ordinary differential equation C_7, C_8, C_9 and C_{10}(and hence, C_{11}) are required to be constant. From the expressions for C_9 and C_{10} it is readily seen that U must be proportional to W, i.e. the streamlines of the mainstream are straight. However, the coefficients C_9 and C_{10} arise from the viscous dissipation term in the energy equation and if it is assumed that viscous heating is negligible, the requirements imposed by C_9 and C_{10} can be disregarded. If it is also assumed that T* = constant, Equation (12) becomes

$$\Theta '' + P_r(C_1 - C_2)\Theta ' F + P_r(C_3 - C_4)\Theta ' G = 0 \qquad (13)$$

Thus the requirements for similar temperature profiles are precisely those given for the mainstream velocity components and the function g. A similar result is given in Reference [5] for two - dimensional flows over a body of revolution rotating in a fluid at rest and a body of revolution in a rotating fluid.

If the case of negligible viscous heating but variable wall temperature is considered, the two equations

$$C_7 = \text{const.} = \frac{P_r U}{g^2 T^*} \frac{\partial T^*}{\partial x}$$

$$C_8 = \text{const.} = \frac{P_r W}{g^2 T^*} \frac{\partial T^*}{\partial z}$$

constitute a system of equations for T*.

The solutions for T* corresponding to the four sets of requirements on U, W and g are as follows:

Case I: $T^* = (\text{const.})\ e^{rx} z^{s}$

Case II: $T^* = (\text{const.})\ x^{r} z^{s}$

Case III: $T^* = (\text{const.})\ x^{r} e^{sz}$

Case IV: $T^* = (\text{const.})\ e^{rx} e^{sz}$

(Again, four additional cases can be obtained by appropriate interchange of independent and dependent variables in all expressions of any of the four cases)

At present solutions of Equation (12) corresponding to computed functions F and G have not been carried out by the authors. Obtaining such solutions should be a rather straightforward numerical procedure.

In conclusion, it can be said that the requirements for all possible similarity solutions of the boundary layer and energy equations in skew-linear coordinates have been found and presented. The usual Cartesian system is simply a special case.

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