THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

# TRANSIENT HEAT TRANSFER IN HEAT EXCHANGERS HAVING ARBITRARY SPACE- AND TIME-DEPENDENT INTERNAL HEAT GENERATION

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September, 1962

IP-582

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# ABSTRACT

The partial differential equations governing transient heat transfer in a heat exchanger are solved analytically for an arbitrary space- and time-variation in internal heat generation, which remains space-time separable as expressed by $\psi(x)\ \phi(\tau)$. The heat exchanger consists of thin solid plates through which a coolant flows. The coolant inlet temperature and all physical properties of the solid plate and coolant are taken as constant. The results include the response of the coolant and solid temperatures. Part I of the paper presents the general mathematical analysis. Part II deals with the limiting cases such as (i) zero solid-coolant heat capacity ratio, (ii) infinite heat transfer coefficient, and (iii) infinite heat transfer coefficient and time-dependent coolant velocity. Heat exchanges to which these results apply include the electrical heater, a chemical reactor in which a chemical reaction occurs within the solid walls and the convection cooled heterogeneous nuclear reactor.

## Introduction

The coupled partial differential equations governing transient heat transfer in heat exchangers having internal heat sources, including nuclear reactors, were solved by many authors. Since a large number of papers and books having important bearing on heat-exchanger dynamics have been reviewed in References 1 and 2, only new literature or other pertinent papers will be discussed.

Yang[3, 4, 5] has studied the step- and frequency-responses of a single-solid, single-fluid heat exchanger resulting from the following disturbances: (1) the uniformly distributed, internal heat generation in the solid wall, (2) the uniform wall temperature, (3) the uniformly distributed internal heat generation in the flowing fluid, (4) the fluid inlet temperature and (5) the appropriate combinations of (1), (2), (3), and (4). The resonance phenomenon in the frequency response is also physically interpreted. Doggett, et al.,[6, 7] have solved the thin plate problem for axially unreflected and reflected cores with sinusoidal space- and exponential time-varying power density. Yang[8, 9] has presented a general analysis for the step- and frequency-responses of the solid and coolant temperatures of heat exchangers having sinusoidal space-dependent internal heat generation. Bonilla, et al.[10], present analytical solutions for the response of a nuclear reactor to step change in time with the spatial distribution in power density arbitrary. All these studies just mentioned and References 1 and 2 have assumed that the heat transfer coefficient between the solid and coolant is constant with axial distance and time. The validity of the assumption has been experimentally verified by Yang[11] for the uniform heat flux distribution and by Hall and Price[12] for the uniform, exponential and sinusoidal heat flux distributions. Hall and Price have reached a conclusion that the consequences of ignoring the effects of heat flux distribution shape on the heat transfer coefficient are not likely to be large in present design of reactor. However, they may become important in smaller reactors having a smaller length to diameter ratio for the cooling channels.

In the present paper the transient heat transfer in heat exchangers having an arbitrary space- and time-dependent internal heat generation is analyzed under the assumption of constant heat transfer coefficient. The following limiting cases are also studied: (i) zero solid-coolant heat capacity ratio, (ii) infinite heat transfer coefficient, and (iii) infinite heat transfer coefficient and time-dependent coolant velocity.

The time-dependent rate of heat generation includes (1) arbitrary time rate of change, (2) step-change, (3) sinusoidal-change, and (4) exponential-change for the uniform and sinusoidal space dependency. The arbitrary space-dependent portion of heat generation $\psi(x)$ in the interval (0, L) is expanded in a Fourier cosine series. Due to the linearity of the governing differential equations, the solution for an arbitrary $\psi(x)$ is obtained by the superposition of the solutions for the uniform and sinusoidal space dependency. This linearity of the problem also suggests the convenience of the use of the principle of superposition, in particular Duhamel's Integral, for both time- and space-dependent heat generations.

Statement of Problem

The physical system analyzed consists of thin solid plates through which a coolant flows with velocity $u$. Initially, both the solid and coolant temperatures and the heat generation rate within the solid are at steady states. At zero time, a certain change in heat generation which may be expressed as $\psi(x)\ \phi(\tau)$ is introduced. As a consequence of this, a transient process is introduced in the temperatures of both the solid and the coolant. For purposes of mathematical convenience, the following assumptions are imposed:

(a) The coolant temperature and velocity are represented by a single value (lumped) at the flow cross section.

(b) The solid temperature does not depend on the distance in the traverse direction, which is valid for sufficiently thin plates.

(c) The axial conduction is negligible in both coolant and solid and heat flows only to the coolant. This is a reasonable assumption when the Pecket number exceeds 100.

(d) The following quantities are constant and uniform throughout: coolant flow area, heat transfer coefficient, inlet coolant temperature, and coolant and solid properties.

Analysis

Application of an energy balance to the system produced the following two differential equations to express the transient behavior of the solid and coolant.

$$\text{Solid} \qquad -(\theta - t) + \frac{p_X''(x, \tau)V_W}{h\ A} = \frac{1}{K_W}\frac{\partial\theta}{\partial\tau} \qquad (1)$$

$$\text{Coolant} \qquad \theta - t = \frac{1}{K}\frac{\partial t}{\partial\tau} + \frac{u}{K}\frac{\partial t}{\partial x} \qquad (2)$$

with the initial and boundary conditions

$$\begin{aligned} \theta\ (x, o) &= 0 \\ t\ (x, o) &= 0 \\ t\ (o, \tau) &= 0 \end{aligned} \qquad (3)$$

The function $p_X''(x, \tau)$ is the arbitrary time- and space-dependent variation of heat generation in the solid and may be expressed as

$$p_X''(x, \tau) = \psi(x)\ \phi(\tau) \qquad (4)$$

Equations (1) and (2) are operated on using the Laplace transformation technique. The transformed equations in solid and coolant temperatures are integrated with the appropriate initial and boundary conditions as outlined in Reference 9. The results for solid and coolant temperatures are as follows:

Solid

$$\overline{\theta}(x,s) = \frac{M\overline{\phi}(s)\psi(x)}{(\rho C_p)_W K(1+s/K_W)} + \frac{M\overline{\phi}(s)e^{-B(s)x}}{(\rho C_p)_W u(1+s/K_W)^2}\left[\Lambda(x,s) - \Lambda(o,s)\right] \qquad (5)$$

Coolant

$$\overline{t}(x,s) = \frac{M\overline{\phi}(s)e^{-B(s)x}}{(\rho C_p)_W u(1+s/K_W)}\left[\Lambda(x,s) - \Lambda(o,s)\right] \qquad (6)$$

where

$$B(s) = \frac{K}{u}\left(1 + \frac{s}{K} - \frac{1}{1 + s/K_w}\right) \quad (7)$$

and

$$\Lambda(x,s) = \int \psi(x) e^{B(s)x} dx \quad (8)$$

with the integration constant omitted.

If $\psi(x)$, the space-dependent portion of heat generation, is specified, then Equation (8) can be integrated and the transient temperatures of the solid and coolant may be obtained by performing the inverse Laplace transformation on Equations (5) and (6). The theory of Fourier series indicates that $\psi(x)$, a bounded function continuous or sectionally continuous in the interval (0, L), may be expanded in the sine or cosine series. Let $\psi(x)$ be expanded in the cosine series as

$$\psi(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \Omega_n x$$

or

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \sin \Omega_n (x_o + x) \quad (9)$$

where

$$a_o = \frac{2}{L} \int_0^L \psi(x) dx \quad (10)$$

$$a_n = \frac{2}{L} \int_0^L \psi(x) \cos \Omega_n x dx \quad (11)$$

$$\Omega_n = \frac{n\pi}{L} \quad (12)$$

and

$$x_o = \left(\frac{4n+1}{2n}\right) L \quad (13)$$

The linearity of the problem suggests that the response to a general transient resulting from $p_x''(x, \tau)$ may be obtained by superposing the solutions to the volumetric rates of heat generation $\frac{a_o}{2} \phi(\tau)$, $a_1 \phi(\tau) \sin \Omega_1 (x_o + x)$, $a_2 \phi(\tau) \sin \Omega_2 (x_o + x)$, ----, $a_n \phi(\tau) \sin \Omega_n (x_o + x)$, ----. Then Equations (5) and (6) may be written as

Solid

$$\frac{\bar{\theta}(x,s)}{\bar{\phi}(s)} = \frac{a_o}{2} \frac{\bar{\theta}_o(x,s)}{\bar{\phi}(s)} + \sum_{n=1}^{\infty} a_n \frac{\bar{\theta}(x,s)}{\bar{\phi}(s)} \tag{14}$$

Coolant

$$\frac{\bar{t}(x,s)}{\bar{\phi}(s)} = \frac{a_o}{2} \frac{\bar{t}_o(x,s)}{\bar{\phi}(s)} + \sum_{n=1}^{\infty} a_n \frac{\bar{t}(x,s)}{\bar{\phi}(s)} \tag{15}$$

where

$$\frac{\bar{\theta}_o(x,s)}{\bar{\phi}(s)} = \frac{K}{(\rho C_p)_w M(1+s/K_w)} + \frac{M\,[1-e^{-B(s)x}]}{(\rho C_p)_w u(1+s/K_w)^2\, B(s)} \tag{16-a}$$

$$\frac{\bar{t}_o(x,s)}{\bar{\phi}(s)} = \frac{M[1-e^{-B(s)x}]}{(\rho C_p)_w u(1+s/K_w)B(s)} \tag{16-b}$$

$$\frac{\bar{\theta}_n(x,s)}{\bar{\phi}(s)} = \frac{M \sin \Omega_n(x_o+x)}{(\rho C_p)_w K(1+s/K_w)} + \frac{M}{(\rho C_p)_w u(1+s/K_w)^2[B^2(s)+\Omega_n^2]} \{B(s)\sin\Omega_n(x_o+x)$$

$$- \Omega_n\cos\Omega_n(x_o+x) - e^{-B(s)x}[-\Omega_n\cos\Omega_n x_o + B(s)\sin\Omega_n\, x_o]\} \tag{17-a}$$

$$\frac{\bar{t}_n(x,s)}{\bar{\phi}(s)} = \frac{M}{(\rho C_p)_w u(1+s/K_w)\,[B^2(s)+\Omega_n^2]} \{B(s)\sin\Omega_n(x_o+x) - \Omega_n\cos\Omega_n(x_o+x)$$

$$- e^{-B(s)x}[-\Omega_n\cos\Omega_n x_o + B(s)\sin\Omega_n\, x_o]\} \tag{17-b}$$

for $n = 1, 2, 3,$ ---- .

Equations (14) to (17) represent the response of the transformed temperature at position $x$ to the transformed time rate of change of heat generation $\phi(\tau)$, that is, the transfer function between the two physical quantities. Equation (16) refers to the case in which heat generation $p_x''(x, \tau)$ is time-dependent but space-independent, and Equation (17) is for the case of a sinusoidal space-dependent variation in heat generation. Therefore the response represented by Equations (14) and (15) may be regarded as the superposition of an infinite number of responses resulting from the variations of heat generation, $\frac{a_o}{2}\phi(\tau)$ , $a_1\phi(\tau)\sin\Omega_1(x_o+x)$ , $a_2\phi(\tau)\sin\Omega_2(x_o+x)$, ---- , $a_n\phi(\tau)\sin\Omega_n(x_o+x)$......

The operation of the inverse Laplace transformation on Equations (14) to (17) yields the response of the solid and coolant temperatures in the physical domain of time and space. Due to the nature of the mathematical attack on this problem, solutionsare obtained for $\theta(x,\tau)$ and $t(x,\tau)$ in two time domains $0 \leq \tau \leq \frac{x}{u}$ and $\tau \geq \frac{x}{u}$ . The results may be expressed as

Solid

$$\theta(x,\tau) = \frac{a_o}{2}\theta_o(x,\tau) + \sum_{n=1}^{\infty} a_n\theta_n(x,\tau) \tag{18}$$

Coolant

$$t(x,\tau) = \frac{a_o}{2}t_o(x,\tau) + \sum_{n=1}^{\infty} a_n t_n(x,\tau) \tag{19}$$

where $\theta_o(x,\tau)$ and $t_o(x,\tau)$ are respectively the transient solid and coolant temperatures resulting from $p_x''(x,\tau) = \phi(\tau)$ , which are given as follows in Reference 1:

(a) $0 \leq \frac{\tau u}{x} \leq 1$

$$\Theta(x,\tau) = \frac{M}{(\rho C_p)_w(M+1)} \int_0^\tau \phi(\xi)[1 + \frac{1}{M} e^{-\frac{K(M+1)}{M}(\xi-\tau)}] \, d\xi \tag{20}$$

$$t(x,\tau) = \frac{M}{(\rho C_p)_w(M+1)} \int_0^\tau \phi(\xi)[1 - e^{-\frac{K(M+1)}{M}(\xi-\tau)}] \, d\xi \tag{21}$$

(b) $\frac{\tau u}{x} \geq 1$

$$\Theta(x,\tau) = \frac{M}{(\rho C_p)_w(M+1)} \{\int_0^\tau \phi(\xi)[1 + \frac{1}{M} e^{-\frac{K(M+1)}{M}(\xi-\tau)}] \, d\xi - \frac{K}{M} e^{-\frac{Kx}{u}} \int_0^{\tau^*} \pi(\tau^*-\xi) \, e^{-\frac{K\xi}{M}} I_o \left(2\sqrt{Kx/u \; K_w\xi}\right) d\xi \} \tag{22}$$

$$t(x,\tau) = \frac{M}{(\rho C_p)_w(M+1)} \{\int_0^\tau \phi(\xi)[1 - e^{\frac{K(M+1)}{M}(\xi-\tau)}] \, d\xi - \frac{K}{M} e^{-\frac{Kx}{u}} \int_0^{\tau^*} \int_0^\xi \phi \, (\eta) d\eta \, e^{-\frac{K\xi}{M}} I_o \left(2\sqrt{Kx/u \; K_w\xi}\right) d\xi\} \tag{23}$$

where

$$\pi(\tau) = \int_0^\tau \phi \, (\xi) d\xi - e^{-\frac{K}{M}(M+1)\tau} \int_0^\tau \phi \, (\xi) e^{\frac{K}{M}(M+1)\xi} \, d\xi \tag{24}$$

and

$$\tau^* = \tau - \frac{x}{u} \tag{25}$$

$\Theta_n(x,\tau)$ and $t_n(x,\tau)$ are respectively the transient solid and coolant temperatures resulting from $p_x''(x,\tau) = \phi(\tau)\sin\Omega_n(x + x)$ . The details of the transform solution are described in Appendix 2.

(a) $1 \geq \frac{\tau u}{x} \geq 0$

$$\Theta_{nI}(x,\tau) = \frac{\phi(\tau)}{(\rho C_p)_w} * \{KK_w[G_1(\tau) + \frac{2(M+1)-a}{\sqrt{4b-a^2}} G_2(\tau)] * [G_3(\tau) - \frac{C}{\sqrt{4d-C^2}} G_4(\tau)]$$

$$* G_6(\tau)\sin\Omega_n(x_o+x) - \frac{4M\Omega_n u}{\sqrt{(4b-a^2)(4d-C^2)}} G_2(\tau)$$

$$* G_4(\tau)\cos\Omega_n(x_o+x) + G_6(\tau)\sin\Omega_n(x_o+x)\} \quad (26)$$

$$t_{nI}(x,\tau) = \frac{\phi(\tau)}{(\rho C_p)_w} * \{K[G_1(\tau) + \frac{2(M+1)-a}{\sqrt{4b-a^2}} G_2(\tau)] * [G_3(\tau)$$

$$- \frac{C}{\sqrt{4d-C^2}} G_4(\tau)]\sin\Omega_n(x_o+x) - M\Omega_n u[G_1(\tau) + \frac{2-a}{\sqrt{4b-a^2}} G_2(\tau)]$$

$$* \frac{2}{\sqrt{4d-C^2}} G_4(\tau) \cos\Omega_n(x_o+x)\} \quad (27)$$

(b) $$\Theta_{nII}(x,\tau) = \Theta_{nI}(x,\tau) + \frac{\phi(\tau^*)}{(\rho C_p)_w} * \{K\Omega_n u e^{-\frac{Kx}{u}} [G_1(\tau^*) + \frac{2-a}{\sqrt{4b-a^2}} G_2(\tau^*)]$$

$$* \frac{2}{\sqrt{4d-C^2}} G_4(\tau^*) * G_5(\tau^*)\cos\Omega_n x_o - KK_w e^{-\frac{Kx}{u}} [G_1(\tau^*)$$

$$+ \frac{2(M+1)-a}{\sqrt{4d-a^2}} G_2(\tau^*)] * [G_3(\tau^*) - \frac{C}{\sqrt{4d-C^2}} G_4(\tau^*)] * G_5(\tau^*)\sin\Omega_n x_o\} \quad (28)$$

$$t_{nII}(x,\tau) = t_{nI}(x,\tau) + \frac{\phi(\tau^*)}{(\rho C_p)_w} * \{K\Omega_n u e^{-\frac{Kx}{u}} [G_1(\tau^*) + \frac{2-a}{\sqrt{4b-a^2}} G_2(\tau^*)]$$

$$* [G_3(\tau^*) + \frac{2-C}{\sqrt{4d-C^2}} G_4(\tau^*)] * G_5(\tau^*)\cos\Omega_n x_o - Ke^{-\frac{Kx}{u}} [G_1(\tau^*)$$

$$+ \frac{2(M+1)-a}{\sqrt{4b-a^2}} G_2(\tau^*)] * [1 - (C-1)K_w G_3(\tau^*)$$

$$- \frac{(2d+C-C^2)K_w}{\sqrt{4d-C^2}} G_4(\tau^*)] * G_5(\tau^*)\sin\Omega_n x_o \} \quad (29)$$

where a, b, C and d are functions defined in Equations (46) and (47) and $G_1(\tau)$ ... $G_5(\tau)$ are functions defined in Table I.

The product of transformed functions, such as $F_1$ $F_2$, may be treated by the method of convolution, a technique of the inverse transformation for product of functions, i.e.,

$$G_i(\tau) * G_j(\tau) = \int_0^\tau G_i(\xi)\, G_j(\tau-\xi) d\xi = \int_0^\tau G_i(\tau-\xi)\, G_j(\xi) d\xi \qquad (30)$$

The convolution integral in Equations (26) to (29) are presented in details in Appendix 2.

The lineality of the equations and boundary conditions also suggests that the system response $\Theta(x,\tau)$ and $t(x,\tau)$ can be obtained employing Duhamel's superposition integral which is derived in Appendix 3. This technique requires the solution for the response to a time-dependent but space-independent heat generation. The results are as follows:

$$R_{X\tau}(x,\tau) = \psi(0) R_\tau(x,\tau) + \int_0^x R_\tau(x-\xi,\tau) \frac{d\psi(\xi)}{d\xi} d\xi \qquad (31\text{-a})$$

or,

$$= \psi(x) R_\tau(o,\tau) + \int_0^x \psi(\xi) \frac{\partial R_\tau(x-\xi,\tau)}{dx} d\xi \qquad (31\text{-b})$$

or,

$$= \psi(x) R_\tau(o,\tau) + \int_0^x \psi(x-\xi) \frac{\partial R_\tau(\xi,\tau)}{\partial \xi} d\xi \qquad (31\text{-c})$$

where $R_{X\tau}$ $(x,\tau)$ and $R_\tau$ $(x,\tau)$ are the response of coolant or solid temperatures to an arbitrary space- and time-dependent heat generation and to an arbitrary time-dependent but space-independent heat generation respectively.

## Special Cases

### 1. Zero Solid-Coolant Heat Capacity Ratio

If the solid-coolant heat capacity ratio is negligibly small, the energy Equation (1) and (2) reduce to

Coolant
$$\frac{\partial t}{\partial x} + \frac{1}{u}\frac{\partial t}{\partial \tau} = \frac{\psi(x)\phi(\tau)V_W}{\rho C_p V u} \qquad (32)$$

Solid
$$\Theta = t + \frac{\psi(x)\phi(\tau)V_W}{hA} \qquad (33)$$

### 2. Infinite Heat Transfer Coefficient

If the heat transfer coefficient from solid to coolant is infinite, the coolant and solid temperatures are always equal.

An energy balance on the system leads to the following equations.

$$\frac{\partial t}{\partial x} + \frac{M+1}{u}\frac{\partial t}{\partial \tau} = \frac{\psi(x)\phi(\tau)V_W}{\rho C_p V u} \qquad (34)$$

$$\Theta = t \qquad (35)$$

### 3. Infinite Heat Transfer Coefficient and Time-Dependent Coolant Velocity

If the heat transfer coefficient between the coolant and the solid is infinite and the coolant velocity varies with respect to time, an energy balance gives

$$u(\tau)\frac{\partial t}{\partial x} + (M+1)\frac{\partial t}{\partial \tau} = \frac{\psi(x)\phi(\tau)V_W}{\rho C_p V} \qquad (36)$$

$$\Theta = t \qquad (37)$$

Let

$$u(\tau) = uf(\tau) \tag{38}$$

and

$$\lambda = \frac{1}{M+1} \int_{0}^{\tau} f(\tau)\, d\tau \tag{39}$$

then

$$\frac{\partial t}{\partial \tau} = \frac{\partial t}{\partial \lambda}\frac{\partial \lambda}{\partial \tau} = \frac{f(\tau)}{M+1}\frac{\partial t}{d\lambda} \tag{40}$$

Substitution of Equation (40) into Equation (36) gives

$$\frac{\partial t}{\partial x} + \frac{1}{u}\frac{\partial t}{\partial \lambda} = \frac{\upsilon(\lambda)\psi(x)V_W}{\rho C_p V u} \tag{41}$$

where

$$\upsilon(\lambda) = \frac{\phi(\tau)}{f(\tau)} \tag{42}$$

Inspection reveals that Equations (32), (34) and (41) are the first order linear partial differential equations of the same type.

Since the same initial and boundary conditions are imposed on each case, the general solutions to Equations (34) and (41) will be in identical form as that of Equation (32). The latter solution may be readily obtained from the general case previously analyzed by merely substituting $M = 0$. The results in analytical form are presented in Table II for various heat generation conditions. The response to an arbitrary space- and time-dependent heat generation may be expressed in similar forms as Equation (19), where $t_o(x,\tau)$ and $t_n(x,\tau)$ refer respectively to the solutions shown in lines 6 and 10 in Table II.

## Conclusion

Solutions for transient behaviors of the solid and coolant temperatures resulting from an arbitrary space- and time-dependent variation of heat generation are obtained. The results for the solid-coolant temperature difference $\Delta t(x,\tau)$ may be formed from $\Theta(x,\tau) - t(x,\tau)$ . Transient local heat flux may be evaluated by multiplying $\Delta t(x,\tau)$ by the heat transfer coefficient.

Those solutions in the second time domain, $\tau \geq \frac{x}{u}$ , might involve two- and three- parameter functions such as $\psi_2^{**}$, $\psi_4$ .... $\psi_{10}$ in References 1 to 8 as the consequences of the convolution integrals. Should $\phi(\tau)$ be a complicated function of time, difficulties might arise from the integration of $\pi_1(\tau)$ , $\pi_2(\tau)$ , $\Delta_1(\tau)$ and $\Delta_2(\tau)$ . For such case it would be convenient to use the Duhamel's superposition integral for $\phi(\tau)$ as described in Appendix 3.

NOMENCLATURE

$A$ = heat transfer area between the solid and the coolant, $ft^2$.

$a$ = function defined by Equation (46).

$a_o$ = Fourier coefficient defined by Equation (10).

$a_n$ = Fourier coefficient defined by Equation (11).

$B(s)$ = $\frac{s}{u} + \frac{K}{u} - \frac{K/u}{1 + s/K_w}$ .

$b$ = function defined by Equation (47).

$C_1 \ldots C_4$ = functions defined by Equation (44).

$C$ = function defined by Equation (46).

$C_p$ = specific heat of coolant, BTU/lbm °F.

$C_{p_w}$ = specific heat of solid, BTU/lbm °F.

$D$ = function defined by Equation (48).

$d$ = function defined by Equation (47).

$E$ = function defined by Equation (49).

$F_1(s) \ldots F_6(s)$, $F_j(s)$ = functions defined in Table I.

$f(\ )$ = function of.

$G_1(\tau) \ldots G_6(\tau), G_j(\tau)$ = functions defined in Table I.

$G_i(\tau) * G_j(\tau)$ = convolution integral of $G_i(\tau)$ and $G_j(\tau)$.

$h$ = heat transfer coefficient between solid and coolant, BTU/hr. $ft^2$ °F.

$I_o$ = Bessel function of first kind and zeroth order.

$K$ = $hA/\rho C_p V$, reciprocal of coolant time constant, 1/hr.

$K_w$ = $hA/\rho C_p V_w)$, reciprocal of solid time constant, 1/hr.

$L$ = axial length of heat exchanger, ft.

$M$ = $(\rho C_p V)_w / \rho C_p V = \frac{K}{K_w}$ , solid-coolant heat capacity ratio, dimensionless.

$p''(x,\tau) = \psi(x) \phi(\tau)$ = volumetric rate of heat generation in the transient state, BTU/hr. $ft^3$.

$R_x(x,\tau)$ = response of coolant or solid temperatures to an arbitrary space-dependent but time-independent heat generation, °F.

$R_\tau(x,\tau)$ = response of coolant or solid temperatures to an arbitrary time-dependent but space-independent heat generation, °F.

$R_{x\tau}(x,\tau)$ = response of coolant or solid temperatures to an arbitrary space- and time-dependent heat generation, °F.

$s$ = Laplace variable, 1/hr.

$t(x,\tau)$ = transient component of coolant temperature, °F.

$t_o(x,\tau)$ = transient coolant temperature resulting from $p_x''(x,\tau) = \phi(\tau)$, °F.

$t_n(x,\tau)$ = transient coolant temperature resulting from $p_x''(x,\tau) = \phi(\tau) \sin \Omega_n (x_o + x)$, °F.

$u$ = coolant velocity, ft/hr.

$V$ = volume of coolant, ft$^3$.

$V_w$ = volume of solid, ft$^3$.

$w$ = reciprocal period of the exponential transient, 1/hr.

$x$ = axial distance, ft.

$x_o$ = $(\frac{4n+1}{2n})L$, ft.

$x_1, x_2, \ldots x_n$ = commencing points of spacewise uniform heat generation element for the derivation of the Duhamel's superposition integral, ft.

$\alpha$ = $\frac{aK_w}{2}$ or $\frac{CK_w}{2}$

$\beta$ = $\frac{\sqrt{4b-a^2}\ K_w}{2}$ or $\frac{\sqrt{4d-C^2}\ K_w}{2}$

$\Delta_1(\tau)$, $\Delta_2(\tau)$, $\Delta_3(\tau)$ = functions defined by Equations (58), (59) and (60).

$\theta(x,\tau)$ = transient component of solid temperature, °F.

$\Theta_o(x,\tau)$ = transient solid temperature resulting from $p_x''(x,\tau) = \phi(\tau)$, °F.

$\Theta_n(x,\tau)$ = transient solid temperature resulting from $p_x''(x,\tau)$

$= \phi(\tau) \text{ Sin } \Omega_n(x_o + x)$, °F.

$\Lambda(x,s)$ = function defined by Equation (8).

$\Lambda_1(x,s)$, $\Lambda_2(x,s)$, $\Lambda_3(x,s)$ = functions defined in Table II.

$\lambda$ = function defined by Equation (39).

$\xi$ = dummy variable.

$\pi(\tau)$, $\pi_1(\tau),\ldots,\pi_4(\tau)$ = functions defined by Equations (54), (55), (56) and (57).

$\rho$ = density of coolant, lbm/ft$^3$.

$\rho_w$ = density of solid, lbm/ft$^3$.

$\tau$ = time, hr.

$\tau^*$ $= \tau - \frac{x}{u}$, hr.

$\upsilon(\lambda)$ $= \frac{\phi(\tau)}{f(\tau)}$ .

$\phi(\tau)$ = time-dependent portion of heat generation variation, BTU/hr.ft.

$\psi(x)$ = space-dependent portion of heat generation variation, dimensionless.

$\Omega_n$ $= \frac{n\pi}{L}$ .

$\omega$ = angular frequency, rad/in.

$(\overline{\phantom{x}})$ = Laplace transformed function.

Subscripts

I: refers to $0 \le \frac{\tau u}{x} \le 1$ in general; $0 \le \frac{\tau}{M+1} \frac{u}{x} \le 1$ for special case (2);

$0 \le \frac{\lambda u}{x} \le 1$ for special case (3).

II: refers to $\frac{\tau u}{x} \ge 1$ in general; $\frac{\tau}{M+1} \frac{u}{x} \ge 1$ for special case (2);

$\frac{\lambda u}{x} \ge 1$ for special case (3).

TABLE I

Laplace Transforms

| F(s) | G(τ) |
|---|---|
| $F_1(s) = \dfrac{s + \frac{aK_W}{2}}{(s + \frac{aK_W}{2})^2 + (\frac{\sqrt{4b-a^2}}{2} K_w)^2}$ | $G_1(\tau) = e^{-\frac{aK_w\tau}{2}} \cos \frac{\sqrt{4b-a^2}}{2} K_w\tau$ |
| $F_2(s) = \dfrac{\frac{\sqrt{4b-a^2}}{2} K_W}{(s + \frac{aK_W}{2})^2 + (\frac{\sqrt{4b-a^2}}{2} K_w)^2}$ | $G_2(\tau) = e^{-\frac{aK_W\tau}{2}} \sin \frac{\sqrt{4b-a^2}}{2} K_w\tau$ |
| $F_3(s) = \dfrac{s + \frac{CK_W}{2}}{(s + \frac{CK_W}{2})^2 + (\frac{\sqrt{4d-C^2}}{2} K_w)^2}$ | $G_3(\tau) = e^{-\frac{CK_W\tau}{2}} \cos \frac{\sqrt{4d-C^2}}{2} K_w\tau$ |
| $F_4(s) = \dfrac{\frac{\sqrt{4d-C^2}}{2} K_W}{(s + \frac{CK_W}{2})^2 + (\frac{\sqrt{4d-C^2}}{2} K_w)^2}$ | $G_4(\tau) = e^{-\frac{CK_W\tau}{2}} \sin \frac{\sqrt{4d-C^2}}{2} K_w\tau$ |
| $F_5(s) = \dfrac{e^{\frac{Kx/u}{1 + s/K_W}}}{s + K_W}$ | $G_5(\tau) = e^{-K_w\tau} I_0(2\sqrt{\frac{Kx}{u}} K_w\tau)$ |
| $F_6(s) = \dfrac{1}{s + K_W}$ | $G_6(\tau) = e^{-K_w\tau}$ |
| $F_j(s)e^{-\frac{sx}{u}}$ | $G_j(\tau) = 0 \quad$ when $\tau \leq \frac{x}{u}$ |
| | $= G_j(\tau - \frac{x}{u}) \quad \tau > \frac{x}{u}$ |

TABLE II

Some Responses of Coolant Temperature for Special Cases (1), (2) and (3)

| | Special Case (1) | Special Case (2) | Special Case (3) |
|---|---|---|---|
| Differential Equation | $\frac{\partial t}{\partial x} + \frac{1}{u}\frac{\partial t}{\partial \tau} = \frac{\psi(x)\phi(\tau)V_w}{\rho C_p V u}$ | $\frac{\partial t}{\partial x} + \frac{M+1}{u}\frac{\partial t}{\partial \tau} = \frac{\psi(x)\phi(\tau)V_w}{\rho C_p V u}$ | $\frac{\partial t}{\partial x} + \frac{1}{u}\frac{\partial t}{\partial \lambda} = \frac{\psi(x)\phi(\tau)V_w}{\rho C_p V u}$, $\upsilon(\lambda) = \frac{\phi(\tau)}{f(\tau)}$ |
| **Initial and Boundary** Conditions | $t(x,0) = 0, \quad \Theta(x,0) = 0$<br>$t(0,\tau) = 0$ | $t(x,0) = 0, \quad \Theta(x,0) = 0$<br>$t(0,\tau) = 0$ | $t(x,0) = 0, \quad \Theta(x,0) = 0$<br>$t(0,\tau) = 0$ |
| General Solution | $\bar{t}(x,s) = \frac{\bar{\phi}(s)V_w}{\rho C_p V u} e^{-\frac{sx}{u}}[\Lambda_1(x,s) - \Lambda_1(0,s)]$ | $\bar{t}(x,s) = \frac{\bar{\phi}(s)V_w}{\rho C_p V u} e^{-\frac{(M+1)sx}{u}}[\Lambda_2(x,s) - \Lambda_2(0,s)]$ | $\bar{t}(x,s) = \frac{\bar{\upsilon}(s)V_w}{\rho C_p V u} e^{-\frac{sx}{u}}[\Lambda_3(x,s) - \Lambda_3(0,s)]$<br>(Laplace transform with respect to $\lambda$) |
| Definition of $\Lambda_1(x,s)$ | $\Lambda_1(x,s) = \int e^{\frac{sx}{u}}\psi(x)dx$ | $\Lambda_2(x,s) = \int e^{\frac{(M+1)sx}{u}}\psi(x)dx$ | $\Lambda_3(x,s) = \int e^{\frac{sx}{u}}\psi(x)dx$ |
| $\psi(x) = 1$<br>arbitrary $\phi(\tau)$<br>or $\upsilon(\lambda)$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V}\int_0^\tau \phi(\xi)d\xi$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V}\int_{\tau-\frac{x}{u}}^{\tau} \phi(\xi)d\xi$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V(M+1)}\int_0^\tau \phi(\xi)d\xi$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V(M+1)}\int_{\tau-(M+1)\frac{x}{u}}^{\tau} \phi(\xi)d\xi$ | $t_I(x,\lambda) = \frac{V_w}{\rho C_p V}\int_0^\lambda \phi(\xi)d\xi$<br>$t_{II}(x,\lambda) = \frac{V_w}{\rho C_p V}\int_{\lambda-\frac{x}{u}}^{\lambda} \phi(\xi)d\xi$ |
| $\psi(x) = 1$<br>$\phi(\tau) = 1$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V}\tau$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V}\frac{x}{u}$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V}\frac{\tau}{M+1}$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V}\frac{x}{u}$ | |
| $\psi(x) = 1$<br>$\phi(\tau) = \sin\omega\tau$ | amplitude-ratio $= \frac{\sqrt{2(1-\cos\frac{\omega x}{u})}}{\frac{\omega x}{u}}$<br>phase-shift $= -\tan^{-1}\frac{(1-\cos\frac{\omega x}{u})}{\sin\frac{\omega x}{u}}$ | amplitude-ratio $= \frac{\sqrt{2[1-\cos\frac{(M+1)\omega x}{u}]}}{(M+1)\frac{\omega x}{u}}$<br>phase-shift $= -\tan^{-1}\frac{[1-\cos(M+1)\frac{\omega x}{u}]}{\sin(M+1)\frac{\omega x}{u}}$ | |
| $\psi(x) = 1$<br>$\phi(\tau) = e^{W\tau}$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V W}(e^{W\tau} - 1)$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V W}(e^{W\tau} - e^{W(\tau - \frac{x}{u})})$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V(M+1)W}(e^{W\tau} - 1)$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V(M+1)W}[e^{W\tau} - e^{W[\tau - (M+1)\frac{x}{u}]}]$ | |
| $\psi(x) = \sin\Omega_n(x_o+x)$<br>arbitrary<br>$\phi(\tau)$ or $\upsilon(\lambda)$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V}\int_0^\tau \phi(\tau-\xi)\sin\Omega_n(x_o+x-\xi u)d\xi$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V}[\int_0^\tau \phi(\tau-\xi)\sin\Omega_n(x_o+x-\xi u)d\xi$<br>$- \int_0^{\tau^*}\phi(\tau^*-\xi)\sin\Omega_n(x_o-\xi u)d\xi]$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V(M+1)}\int_0^\tau \phi(\tau-\xi)\sin\Omega_n(x_o+x-\xi u)d\xi$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V(M+1)}[\int_0^\tau \phi(\tau-\xi)\sin\Omega_n(x_o-\xi u)d\xi$<br>$- \int_0^{\tau^*}\phi(\tau^*-\xi)\sin\Omega_n(x -\xi u)d\xi]$ | $t_I(x,\lambda) = \frac{V_w}{\rho C_p V}\int_0^\lambda \upsilon(\lambda-\xi)\sin\Omega_n(x_o+x-\xi u)d\xi$<br>$t_{II}(x,\lambda) = \frac{V_w}{\rho C_p V}[\int_0^\lambda \upsilon(\lambda-\xi)\sin\Omega_n(x_o+x-\xi u)d\xi$<br>$- \int_0^{\lambda^*}\upsilon(\lambda^*-\xi)\sin\Omega_n(x_o-\xi u)d\xi]$ |
| $\psi(x) = \sin\Omega_n(x_o+x)$<br>$O(\tau) = 1$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V\Omega_n u}[\cos\Omega_n(x_o+x-\tau u)-\cos\Omega_n(x_o+x)]$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V\Omega_n u}[\cos\Omega_n x_o - \cos\Omega_n(x_o+x)]$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V\Omega_n u}[\cos\Omega_n(x_o+x - \frac{\tau}{M+1}u)-\cos\Omega_n(x_o+x)]$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V\Omega_n u}[\cos\Omega_n x_o - \cos\Omega_n(x_o+x)]$ | |
| $\psi(x) = \sin\Omega_n(x_o+x)$<br>$\phi(\tau) = \sin\omega\tau$ | Amplitude ratio $= \frac{1}{[1-(\frac{\omega}{\Omega_n u})^2][\cos\Omega_n x_o-\cos\Omega_n(x_o+x)]}\Big\{[\cos\Omega_n(x_o+x)-\cos\frac{\omega x}{u}\cos\Omega_n x_o \sin\frac{\omega x}{u}]^2$<br>$+ \left\{\frac{\omega}{\Omega_n u}[\sin\Omega_n(x_o+x)-\cos\frac{\omega x}{u}\sin\Omega_n x_o] - \sin\frac{\omega x}{u}\cos\Omega_n x_o\right\}^2\Big\}^{1/2}$<br>Phase shift $= \tan^{-1}\left\{\frac{\frac{\omega}{\Omega_n u}[\sin\Omega_n(x_o+x)-\cos\frac{\omega x}{u}\sin\Omega_n x_o] - \sin\frac{\omega x}{u}\cos\Omega_n x_o}{\cos\Omega_n(x_o+x)-\cos\frac{\omega x}{u}\cos\Omega_n x_o + \frac{\omega x}{u}\sin\Omega_n x_o \sin\frac{\omega x}{u}}\right\}$<br>$\omega = \omega$ for special case (1) | $\omega = (M+1)\omega$ for special case (2) | |
| $\psi(x) = \sin\Omega_n(x_o+x)$<br>$\phi(\tau) = e^{W\tau}$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V\sqrt{W^2+(\Omega_n u)^2}}\Big\{e^{W\tau}\sin[\Omega_n(x_o+x)$<br>$- \tan^{-1}\frac{\Omega_n u}{W}] - \sin[\Omega(x+x-u\tau)-\tan^{-1}\frac{\Omega_n u}{W}]\Big\}$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V\sqrt{W^2+(\Omega_n u)^2}}\Big\{e^{W\tau}\sin[\Omega_n(x_o+x)$<br>$- \tan^{-1}\frac{\Omega_n u}{W}] - e^{W(\tau - \frac{x}{u})}\sin(\Omega_n x_o-\tan^{-1}\frac{\Omega_n u}{W})\Big\}$ | $t_I(x,\tau) = \frac{V_w}{\rho C_p V\sqrt{[W(M+1)]^2 + (\Omega_n u)^2}}\Big\{e^{W\tau}\sin[\Omega_n(x_o+x) - \tan^{-1}\frac{\Omega_n u}{W(M+1)}]$<br>$- \sin[\Omega_n(x_o+x - \frac{\tau u}{M+1}) - \tan^{-1}\frac{\Omega_n u}{W(M+1)}]\Big\}$<br>$t_{II}(x,\tau) = \frac{V_w}{\rho C_p V\sqrt{[W(M+1)]^2 + (\Omega_n u)^2}}\Big\{e^{W\tau}\sin[\Omega_n(x_o+x)$<br>$- \tan^{-1}\frac{\Omega_n u}{W(M+1)}] - e^{W[\tau - (M+1)\frac{x}{u}]}\sin[\Omega_n x_o-\tan^{-1}\frac{\Omega_n u}{W(M+1)}]\Big\}$ | |

TABLE III

Table for Derivation of the Duhamel's Superposition Integral

| Magnitude of spacewise uniform heat generation element | Commencing point of spacewise uniform heat generation element | Effect on the response $R_x(x,\tau)$ at x |
|---|---|---|
| $\psi(0)$ | 0 | $\psi(0)R_x(x,\tau)$ |
| $\psi(x_1) - \psi(0)$ | $x_1$ | $[\psi(x_1) - \psi(0)]\ R_x(x-dx,\tau)$ |
| $\psi(x_2) - \psi(x_1)$ | $x_2$ | $[\psi(x_2) - \psi(x_1)]\ R_x(x-2dx,\tau)$ |
| $\psi(x_3) - \psi(x_2)$ | $x_3$ | $[\psi(x_3) - 2(x_2)]\ R_x(x-3dx,\tau)$ |
| $\psi(x_n) - \psi(x_{n-1})$ | $x_n$ | $[\psi(x_n) - \psi(x_{n-1})]\ R_x(x-ndx,\tau)$ |

APPENDIX 1

Inverse Laplace Transformation of Equation (17)

$B^2(s) + \Omega_n^2$ which appears in the numerators of Equation (17) may be written as

$$B^2(s) + \Omega_n^2 = \frac{1}{KK_w{}^2(1 + \frac{s}{K_w})^2} (s^4+C_1s^3+C_2s^2+C_3s+C_4) \tag{43}$$

in which

$$C_1 = 2(K+K_2) \qquad C_2 = (K+K_2)^2+(\Omega_n u)^2$$
$$C_3 = 2K_w(\Omega u)^2 \qquad C_4 = (K_w\Omega_n u)^2 \tag{44}$$

The fourth order factor $s^4+C_1s^3+C_2s^2+C_3s+C_4$ in Equation (43) may be resolved into factors as

$$s^4+C_1s^3+C_2s^2+C_3s+C_4 = (s^2+aK_w+bK_w^2)(s^2+CK_ws+dK_w^2) \tag{45}$$

where

$$\begin{Bmatrix} a \\ C \end{Bmatrix} = (M+1) \mp \sqrt{\frac{1}{2}[(M+1)^2-M\ (\frac{\Omega_n u}{K})^2] + \frac{E}{2}} \tag{46}$$

$$\begin{Bmatrix} b \\ d \end{Bmatrix} = D + E \mp \sqrt{2DE-(M+1)M^2(\frac{\Omega_n u}{K})^2 + 2D^2} \tag{47}$$

and

$$D = \frac{1}{4}[(M+1)^2 + M^2(\frac{\Omega_n u}{K})^2] \tag{48}$$

$$E = 4\sqrt{D^2 - M^3(\frac{\Omega_n u}{K})^2} \tag{49}$$

The substitution of Equations (7), (43) and (45) into Equation (17) yields

$$\bar{\Theta}(x,s) = \frac{KK_w\bar{\phi}(s)}{(\rho C_p)_w}\Big\{[F_1(s) + \frac{2(M+1)-a}{\sqrt{4b-a^2}} F_2(s)][F_3(s)$$

$$- \frac{C}{\sqrt{4d-C^2}} F_4(s)]\ F_6(s)\sin\Omega_n(x_o+x)$$

$$- \Omega_n u[\frac{2}{K_w\sqrt{4b-a^2}} F_2(s)][\frac{2}{K_w\sqrt{4d-C^2}} F_4(s)]\ \cos\Omega_n(x_o+x)$$

$$+ \Omega_n u e^{-\frac{Kx}{u}} e^{-\frac{sx}{u}} [F_1(s) + \frac{2-a}{\sqrt{4b-a^2}} F_2(s)][\frac{2}{K_w\sqrt{4d-C^2}} F_4(s)]F_5(s)\cos\Omega_n x_o$$

$$- e^{-\frac{Kx}{u}} e^{-\frac{sx}{u}} [F_1(s) + \frac{2(M+1)-a}{\sqrt{4b-a^2}} F_2(s)][F_3(s)$$

$$- \frac{C}{\sqrt{4d-C^2}} F_4(s)]F_5(s)\sin\Omega_n x_o + \frac{F_6(s)}{KK_w} \sin\Omega_n(x_o+x)\Big\} \qquad (50)$$

$$\bar{t}(x,s) = \frac{K\bar{\phi}(s)}{(\rho C_p)_w}\Big\{[F_1(s) + \frac{2(M+1)-a}{\sqrt{4b-a^2}} F_2(s)][F_3(s)$$

$$- \frac{C}{\sqrt{4d-C^2}} F_4(s)]\ \sin\Omega_n(x_o+x)$$

$$- \Omega_n u[F_1(s) + \frac{2-a}{\sqrt{4b-a^2}} F_2(s)][\frac{2}{K_w\sqrt{4d-C^2}} F_4(s)]\ \cos\Omega_n(x_o+x)$$

$$+ \Omega_n u e^{-\frac{Kx}{u}} e^{-\frac{sx}{u}} [F_1(s) + \frac{2-a}{\sqrt{4b-a^2}} F_2(s)][F_3(s)$$

$$+ \frac{2-C}{\sqrt{4d-C^2}} F_4(s)]\ F_5(s)\cos\Omega_n x_o - e^{-\frac{Kx}{u}} e^{-\frac{sx}{u}} [F_1(s)$$

$$+ \frac{2(M+1)-a}{\sqrt{4b-a^2}} F_2(s)][1-(C-1)K_wF_3(s) - \frac{(2d+C-C^2)K_w}{\sqrt{4d-C^2}} F_4(s)]$$

$$\times F_5(s)\ \sin\Omega_n x\ \Big\} \qquad (51)$$

In Equations (50) and (51) the functions $F_1(s)\ldots F_6(s)$ are Laplace transformed functions in the variables which have corresponding original functions $G_1(\tau)\ldots G_6(\tau)$ in the variable $\tau$ obtained by performing an inverse transformation on the function F. The Laplace transformed function F and inverse transformed function G appropriate to Equations (50) and (51) are listed in Table I. The transformed equations in the physical domain of x and $\tau$ are found to be Equations (26) to (29).

## APPENDIX 2

## Convolution Integral in Equations (26) to (29)

Inspection reveals that Equations (26) to (29) consist of the convolution integrals to be discussed in the following. The evaluation of the convolution integrals as defined by Equation (30) is quite straightforward. Let

$$\alpha = \frac{aK_W}{2}, \text{ or } \frac{CK_W}{2} \tag{52}$$

$$\beta = \frac{\sqrt{4b-a^2K_W}}{2}, \text{ or } \frac{\sqrt{4d-C^2K_W}}{2} \tag{53}$$

nd define

$$\pi_1(\tau) = (e^{-\alpha\tau}\sin\beta\tau)*\phi(\tau) = \int_0^\tau \phi(\tau-\xi)e^{-\alpha\xi}\sin\beta\xi d\xi \tag{54}$$

$$\pi_2(\tau) = (e^{-\alpha\tau}\cos\beta\tau)*\phi(\tau) = \int_0^\tau \phi(\tau-\xi)e^{-\alpha\xi}\cos\beta\xi d\xi \tag{55}$$

$$\pi_3(\tau) = 1 * \phi(\tau) = \int_0^\tau \phi(\tau-\xi)d\xi \tag{56}$$

$$\pi_4(\tau) = e^{-K_W\tau} * \phi(\tau) = \int_0^\tau \phi(\tau-\xi)e^{-K_W\xi}d\xi \tag{57}$$

$$\Delta_1(\tau) = \pi_1(\tau)*G_5(\tau) = \pi_1(\tau)*e^{-K_W\tau}I_o(2\sqrt{\frac{Kx}{u}K_W\tau})$$

$$= \int_0^\tau \pi_1(\tau-\xi)e^{-K_W\xi}\, I_o(2\sqrt{\frac{Kx}{u}K_W\xi})d\xi \tag{58}$$

$$\Delta_2(\tau) = \pi_2(\tau)*G_5(\tau) = \int_0^\tau \pi_2(\tau-\xi)e^{-K_W\xi}\, I_o(2\sqrt{\frac{Kx}{u}K_W\xi})d\xi \tag{59}$$

$$\Delta_3(\tau) = \pi_3(\tau)*G_5(\tau) = \int_0^\tau \pi_3(\tau-\xi)e^{-K_W\xi}\, I_o(2\sqrt{\frac{Kx}{u}K_W\xi})d\xi \tag{60}$$

Since the evaluation of convolution integrals indicates that $G_1(\tau)*G_3(\tau)$, $G_1(\tau)*G_4(\tau)$, $G_2(\tau)*G_3(\tau)$ and $G_2(\tau)*G_4(\tau)$ are function of $e^{-\alpha\tau}\sin\beta\tau$ and $e^{-\alpha\tau}\cos\beta\tau$, it is found

$$\left.\begin{aligned} &G_1(\tau)*G_3(\tau)*\phi(\tau) \\ &G_1(\tau)*G_4(\tau)*\phi(\tau) \\ &G_2(\tau)*G_3(\tau)*\phi(\tau) \\ &G_2(\tau)*G_4(\tau)*\phi(\tau) \end{aligned}\right\} \begin{aligned} &= f[(e^{-\alpha\tau}\sin\beta\tau)*\phi(\tau),(e^{-\alpha\tau}\cos\beta\tau)*\phi(\tau)] \\ &= f[\pi_1(\tau),\pi_2(\tau)] \end{aligned} \tag{61}$$

Therefore

$$\left.\begin{aligned} &G_1(\tau)*G_3(\tau)*\phi(\tau)*G_5(\tau) \\ &G_1(\tau)*G_4(\tau)*\phi(\tau)*G_5(\tau) \\ &G_2(\tau)*G_3(\tau)*\phi(\tau)*G_5(\tau) \\ &G_2(\tau)*G_4(\tau)*\phi(\tau)*G_5(\tau) \end{aligned}\right\} \begin{aligned} &= f[\pi_1(\tau)*G_5(\tau),\ \pi_2(\tau)*G_5(\tau)] \\ &= f[\Delta_1(\tau),\ \Delta_2(\tau)] \end{aligned} \tag{62}$$

Similarly one finds that $1 * G_1(\tau) * \phi(\tau)$ and $1 * G_2(\tau) * \phi(\tau)$ are function of $\pi_1(\tau)$, $\pi_2(\tau)$ and $\pi_3(\tau)$. Hence

$$\left.\begin{aligned} &1*G_1(\tau)*\phi(\tau)*G_5(\tau) \\ &1*G_2(\tau)*\phi(\tau)*G_5(\tau) \end{aligned}\right\} \begin{aligned} &= f[\pi_1(\tau)*G_5(\tau),\pi_2(\tau)*G_5(\tau),\pi_3(\tau)*G_5(\tau)] \\ &= f[\Delta_1(\tau),\Delta_2(\tau),\Delta_3(\tau)] \end{aligned} \tag{63}$$

The convolution integrals of $G_1(\tau)*G_3(\tau)$, $G_1(\tau)*G_4(\tau)$, $G_2(\tau)*G_3(\tau)$ and $G_2(\tau)*G_4(\tau)$ with $G_6(\tau)$ yield

$$\left.\begin{aligned} &G_1(\tau)*G_3(\tau)*G_6(\tau) \\ &G_1(\tau)*G_4(\tau)*G_6(\tau) \\ &G_2(\tau)*G_3(\tau)*G_6(\tau) \\ &G_2(\tau)*G_4(\tau)*G_6(\tau) \end{aligned}\right\} \begin{aligned} &= f[(e^{-\alpha\tau}\sin\beta\tau)*G_6(\tau),\ (e^{-\alpha\tau}\cos\beta\tau)*G_6(\tau)] \\ &= f(e^{-\alpha\tau}\sin\beta\tau,\ e^{-\alpha\tau}\cos\beta\tau,\ e^{-K_w\tau}) \end{aligned} \tag{64}$$

Consequently

$$\left.\begin{array}{l} G_1(\tau)*G_3(\tau)*G_6(\tau)*\phi(\tau) \\ G_1(\tau)*G_4(\tau)*G_6(\tau)*\phi(\tau) \\ G_2(\tau)*G_3(\tau)*G_6(\tau)*\phi(\tau) \\ G_2(\tau)*G_4(\tau)*G_6(\tau)*\phi(\tau) \end{array}\right\} = f[(e^{-\alpha\tau}\sin\beta\tau)*\phi(\tau),\ (e^{-\alpha\tau}\cos\beta\tau) *\phi(\tau),\ e^{-K_w\tau}*\phi(\tau)] = f[\pi_1(\tau),\pi_2(\tau),\pi_4(\tau)] \quad (65)$$

The definition of $\pi_4(\tau)$ is nothing but $G_6(\tau)*\phi(\tau)$. Special cases for step change, sinusoidal change and exponential change in $\phi(\tau)$ are respectively given in References 6, 7, 8 and 9.

## APPENDIX 3

### Derivation of the Duhamel's Superposition Integral

The response to an arbitrary space- and time-dependent heat generation $\psi(x)\phi(\tau)$ may be found from the results for an arbitrary time-dependent but space-independent heat generation by the principle of superposition. In terms of differential calculus this principle may be derived as follows. $\psi(x)$ may be resolved into a number of spacewise uniform heat generation elements, each commencing at a different value of x. These elements are shown in Table III.

The appropriate value of $R_{XT}(x,\tau)$ is then the sum of the contributions from the separate cases, each shown in the third column of Table III.

$$R_{XT}(x,\tau) = \psi(0)R_X(x,\tau) + \sum_{n=1}^{\infty} R_X(x-x_n,\tau) \frac{[\psi(x_n)-\psi(x_{n-1})]}{x_n-x_{n-1}} (x_n-x_{n-1}) \quad (66)$$

In the limit, as the number n of elements becomes infinite, the definition of integral results in

$$R_{XT}(x,\tau) = \psi(0)R_X(x,\tau) + \int_0^x R_X(x-\xi,\tau) \frac{d\psi(\xi)}{d\xi} d\xi \quad (67)$$

By using the method of integration by parts an alternative form may be obtained as

$$R_{XT}(x,\tau) = \psi(0)R_X(x,\tau) + [R_X(x-\xi,\tau)\psi(\xi)]_{\xi=0}^{\xi=x} - \int_0^x \psi(\xi) \frac{\partial R_X(x-\xi,\tau)}{\partial\xi} d\xi \quad (68)$$

Since

$$\frac{\partial R_X(x-\xi,\tau)}{\partial \xi} = -\frac{\partial R_X(x-\xi,\tau)}{\partial x} \tag{69}$$

one obtains,

$$R_{XT}(x,\tau) = \psi(x)R_X(0,\tau) + \int_0^x \psi(\xi)\frac{\partial R_X(x-\xi,\tau)}{\partial x}\,d\xi \tag{70}$$

This equation can be re-arranged as follows by using its convolutive property:

$$R_{XT}(x,\tau) = \psi(x)R_X(0,\tau) + \int_0^x \psi(x-\xi)\frac{\partial R_X(\xi,\tau)}{\partial \xi}\,d\xi \tag{71}$$

Similarly if $R_T(x,\tau)$, temperature response of the coolant and solid to a unit step change in heat generation having arbitrary x-dependence, is available, the response of coolant or solid temperatures to an arbitrary time-dependent disturbance $\phi(\tau)$ may be obtained by the Duhamel's Integral which results in

$$R_{XT}(x,\tau) = \phi(0)R_T(x,\tau) + \int_0^\tau R_T(x,\tau-\xi)\frac{d\phi(\xi)}{d\xi}\,d\xi \tag{72}$$

or, in alternative forms

$$R_{XT}(x,\tau) = \phi(\tau)R_T(x,0) + \int_0^\tau \phi(\xi)\frac{\partial R_T(x,\tau-\xi)}{\partial \tau}\,d\xi \tag{73}$$

$$= \phi(\tau)R_T(x,0) + \int_0^\tau \phi(\tau-\xi)\frac{\partial R_T(x,\xi)}{\partial \xi} \tag{74}$$

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