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WEAKLY NONLINEAR ANALYSIS OF DISPERSIVE WAVES IN MIXTURES OF LIQUID AND GAS BUBBLES BASED ON A TWO-FLUID MODEL

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ABSTRACT

One-dimensional nonlinear dispersive waves in liquids containing a number of microbubbles are theoretically studied based on two-fluid averaged equations derived by the present authors. The set of equations consists of the conservation laws of mass and momentum for gas and liquid phases, and the equation of motion of the bubble wall. The compressibility of liquid is taken into account, and this leads to the wave attenuation due to bubble oscillations. By using the method of multiple scales, two types of equations for nonlinear wave propagation in long ranges are derived. In a moderately low frequency band, the behavior of weakly nonlinear waves is described by the Korteweg–de Vries–Burgers equation. On the other hand, in a moderately high frequency band, the nonlinear modulation of quasi-monochromatic wave train is described by the nonlinear Schrödinger equation with an attenuation term.

INTRODUCTION

The characteristics of sound waves in bubbly liquids are considerably different from those in single phase fluids [1–10]. Especially, the dispersion in the sense that waves of different wavelengths propagate with different phase velocities is an important property, which is caused by bubble oscillations.

Egashira *et al.* [11] have derived a set of averaged equations based on a two-fluid model. On the basis of these equations, we have analyzed one-dimensional linear dispersive waves in bubbly liquids [11–13]. By considering the compressibility of the liquid phase, we have shown the existence of the two propagation modes of pressure waves, i.e., slow mode and fast mode.

In the present paper, we shall extend the previous studies [11–13] to nonlinear wave motions. The one-dimensional nonlinear dispersive waves in liquids containing a number of small spherical gas bubbles of slow mode are theoretically investigated. The compressibility of the liquid phase is taken into account in the same way as the previous studies, and this leads to an attenuation effect due to acoustic radiation caused by bubble oscillations.

Figure 1 shows the linear dispersion relation of slow mode [11]. Here, Band A and Band B in Fig. 1 correspond to the moderately low and high frequency bands, and these are regarded as the weakly and strongly dispersive bands, respectively. We

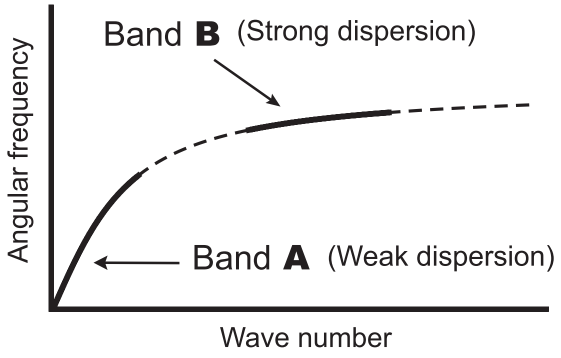

Figure 1. The dispersion relation of the slow mode in a bubbly quiescent liquid [11]. Band A and Band B correspond to the weakly and strongly dispersive bands, respectively.

shall prescribe both Band A and Band B by an appropriate scaling of parameters. The weakly nonlinear propagations of pressure waves in both Band A and Band B are studied by the use of the method of multiple scales. As a result, the behaviors of waves in Band A and Band B are described by the Korteweg–de Vries–Burgers equation [3, 14] and the nonlinear Schrödinger equation [5, 14] with an attenuation term, respectively.

In this paper, we shall demonstrate that appropriate scalings of a set of physical parameters enable us to do systematic derivations of the Korteweg–de Vries–Burgers equation and the nonlinear Schrödinger equation from a set of basic equations for bubbly flows.

FORMULATION OF THE PROBLEM

We shall analyze one-dimensional nonlinear dispersive waves in mixtures of a liquid and a number of small spherical gas bubbles on the basis of the averaged equations. At an initial state, the mixtures are assumed to be uniform and at rest. The pressure waves are generated by oscillations of a sound source in the bubbly liquid.

The compressibility of liquid phase is taken into account. For the simplicity, we neglect the viscosity of gas phase, the thermal conductivity of gas and liquid phases, the phase change across the gas-liquid interface, and the Reynolds stress.

Governing equations

The system of governing equations of bubbly flows is composed of the mass and momentum conservation laws, the equation of motion for the bubble wall, the equations of state for gas and liquid, and so on [11–13]. For the one-dimensional waves, the conservation laws of the mass and momentum for gas and liquid phases based on a two-fluid model by Egashira *et al.* [11] are written as follows:

$$\frac{\partial}{\partial t^*}(\alpha\rho_G^*) + \frac{\partial}{\partial x^*}(\alpha\rho_G^* u_G^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}[(1-\alpha)\rho_L^*] + \frac{\partial}{\partial x^*}[(1-\alpha)\rho_L^* u_L^*] = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_G^* u_G^*) + \frac{\partial}{\partial x^*}\left(\alpha\rho_G^* {u_G^*}^2\right) + \alpha\frac{\partial p_G^*}{\partial x^*} = F^*, \tag{3}$$

$$\frac{\partial}{\partial t^*}[(1-\alpha)\rho_L^* u_L^*] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_L^* {u_L^*}^2\right] + (1-\alpha)\frac{\partial p_L^*}{\partial x^*} + P^*\frac{\partial\alpha}{\partial x^*} = -F^*, \tag{4}$$

where t^* is the time, x^* is the space coordinate normal to the wave front, α is the volume fraction of the gas phase $(0 < \alpha < 1)$, ρ^* is the density, u^* is the fluid velocity, p^* is the pressure, and the subscripts G and L denote volume-averaged variables in gas and liquid phases, respectively. In addition to the volume-averaged pressures p_G^* and p_L^*, P^* is introduced as the liquid pressure averaged on the gas-liquid interface. Here and hereafter, the superscript "$*$" denotes dimensional quantities.

As the interfacial momentum transport F^*, we employ the following model of the virtual mass force [12, 15, 16]

$$F^* = -\beta_1\alpha\rho_L^*\left(\frac{D_G u_G^*}{Dt^*} - \frac{D_L u_L^*}{Dt^*}\right) - \beta_2\rho_L^*(u_G^* - u_L^*)\frac{D_G\alpha}{Dt^*} - \beta_3\alpha(u_G^* - u_L^*)\frac{D_G\rho_L^*}{Dt^*}, \tag{5}$$

where the values of coefficients β_1, β_2, and β_3 may be set as $1/2$. Here, the operators D_G/Dt^* and D_L/Dt^* are defined as

$$\frac{D_G}{Dt^*} \equiv \frac{\partial}{\partial t^*} + u_G^*\frac{\partial}{\partial x^*}, \tag{6}$$

$$\frac{D_L}{Dt^*} \equiv \frac{\partial}{\partial t^*} + u_L^*\frac{\partial}{\partial x^*}. \tag{7}$$

The Keller equation for oscillations of the spherical bubble in the compressible liquid is introduced as follows [17]:

$$\left(1 - \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)R^*\frac{D_G^2 R^*}{{Dt^*}^2} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\left(\frac{D_G R^*}{Dt^*}\right)^2 = \left(1 + \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*}\frac{D_G}{Dt^*}(p_L^* + P^*), \tag{8}$$

where R^* is the bubble radius, c_{L0}^* and ρ_{L0}^* are, respectively, the speed of sound and the density in liquid phase at the initial unperturbed state. The second term in the right-hand side of Eq. (8) is responsible for the wave attenuation due to the acoustic radiation due to bubble oscillations.

In order to close the system of equations (1)–(5) and (8), the following equations are used: (i) the polytropic equation of state for gas,

$$\frac{p_G^*}{p_{G0}^*} = \left(\frac{\rho_G^*}{\rho_{G0}^*}\right)^{\gamma}, \tag{9}$$

(p_{G0}^* and ρ_{G0}^* are, respectively, the pressure and density inside the bubble in the unperturbed state, and γ is the polytropic exponent), (ii) the Tait equation of state for liquid,

$$p_L^* = p_{L0}^* + \frac{\rho_{L0}^* c_{L0}^{*2}}{n}\left[\left(\frac{\rho_L^*}{\rho_{L0}^*}\right)^n - 1\right], \tag{10}$$

(p_{L0}^* is the liquid pressure in the unperturbed state, and $n = 7.15$ is used if the liquid is water), (iii) the conservation law of mass inside the bubble,

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left(\frac{R_0^*}{R^*}\right)^3, \tag{11}$$

(R_0^* is the bubble radius in the unperturbed state), (iv) the pressure balance at the gas-liquid interface,

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*}\frac{D_G R^*}{Dt^*}, \tag{12}$$

where σ^* is the surface tension, and μ^* is the liquid viscosity. Note that all the variables in the initial unperturbed state, c_{L0}^*, ρ_{L0}^*, ρ_{G0}^*, p_{L0}^*, p_{G0}^*, and R_0^*, are constants.

The liquid viscosity μ^* in Eq. (12) has dropped in Eq. (4). This is because the perturbation of the liquid density is regarded as significantly small compared with that of other variables, although the liquid compressibility is taken into account in the present study. Therefore, we neglect the term of coupling of the liquid viscosity and compressibility in the momentum equation with the assumption of spherical symmetry.

Perturbation expansions

We shall use the method of multiple scales (see, e.g., [14]), to derive the so-called far-field equations, which describe slow variations of behavior in the propagation process of long ranges of weakly nonlinear waves.

Firstly, the time t^* and the space coordinate x^* are, respectively, normalized by

$$t = \frac{t^*}{T^*}, \qquad x = \frac{x^*}{L^*}, \tag{13}$$

where T^* and L^* are the characteristic time and length, respectively. We introduce the new independent variables defined by t, x, and a small nondimensional parameter $\varepsilon\,(\ll 1)$:

$$t_m = \varepsilon^m t, \quad x_m = \varepsilon^m x \quad (m = 0, 1, 2, \cdots), \tag{14}$$

where $t_0 = t$ and $x_0 = x$ represent fast scales, whereas $t_1 = \varepsilon t$, $x_1 = \varepsilon x$, and so on, represent slow scales, and are called as slow variables. The small parameter ε denotes a typical amplitude of waves. By using chain rules and Eq. (14), the differential operators can be expanded as follows [14]:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + O(\varepsilon^3), \tag{15}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon\frac{\partial}{\partial x_1} + \varepsilon^2\frac{\partial}{\partial x_2} + O(\varepsilon^3). \tag{16}$$

The four dependent variables, α, u_G^*, u_L^*, and R^*, are nondimensionalized and expanded in a power series of ε, as follows:

$$\alpha/\alpha_0 = 1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + O(\varepsilon^3), \tag{17}$$

$$u_G^*/U^* = \varepsilon u_{G1} + \varepsilon^2 u_{G2} + O(\varepsilon^3), \tag{18}$$

$$u_L^*/U^* = \varepsilon u_{L1} + \varepsilon^2 u_{L2} + O(\varepsilon^3), \tag{19}$$

$$R^*/R_0^* = 1 + \varepsilon R_1 + \varepsilon^2 R_2 + O(\varepsilon^3), \tag{20}$$

where α_0 is the initial volume fraction and U^* is the characteristic velocity. In Eqs. (17)–(20) and the following equations, all expansion coefficients are of $O(1)$. The characteristic velocity U^* is a typical propagation speed of waves, the characteristic time T^* is a typical period of the incident wave, and the characteristic length $L^* \equiv U^* T^*$ is a typical wavelength.

Furthermore, the expansion of the liquid density ρ_L^* is defined as

$$\rho_L^*/\rho_{L0}^* = 1 + \varepsilon^a \rho_{L1} + \varepsilon^{a+1}\rho_{L2} + O(\varepsilon^{a+2}), \tag{21}$$

where $a\,(> 1)$ is an integer number, whose explicit values are to be determined in the following sections, by considering the conditions of each problem. We shall remark that the expansion of the liquid density begins with $O(\varepsilon^a)$ in Eq. (21), which is because the compressibility of liquid is very small compared with that of gas.

Substituting Eq. (21) into the Tait equation (10) gives the expansion of the liquid pressure p_L^* as

$$\begin{aligned} p_L &\equiv \frac{p_L^*}{\rho_{L0}^* U^{*2}} \\ &= \frac{p_{L0}^*}{\rho_{L0}^* U^{*2}} + \varepsilon^{a-2b}\frac{\rho_{L1}}{V^2} + \varepsilon^{a-2b+1}\frac{\rho_{L2}}{V^2} + O(\varepsilon^{a-2b+2}), \end{aligned} \tag{22}$$

where $V\varepsilon^b$ is introduced as a measure of the ratio of the characteristic velocity and the speed of sound in liquid in the unperturbed state,

$$\frac{U^*}{c_{L0}^*} \equiv V\varepsilon^b \equiv O(\varepsilon^b). \tag{23}$$

The parameter V is of $O(1)$ and b is a real number to be determined. Since the perturbation in the liquid pressure should begin with the $O(\varepsilon)$ term in Eq. (22), as in the expansions (17)–(20), the following condition is required

$$a - 2b = 1. \tag{24}$$

Hence Eq. (22) may be rewritten as

$$p_L = p_{L0} + \varepsilon\frac{\rho_{L1}}{V^2} + \varepsilon^2\frac{\rho_{L2}}{V^2} + O(\varepsilon^3). \tag{25}$$

In addition, the nondimensional pressures for gas and liquid in the unperturbed state, p_{G0} and p_{L0}, are introduced as

$$p_{G0} \equiv \frac{p_{G0}^*}{\rho_{L0}^* U^{*2}} \equiv O(1), \qquad p_{L0} \equiv \frac{p_{L0}^*}{\rho_{L0}^* U^{*2}} \equiv O(1), \tag{26}$$

respectively. The ratio of initial densities of gas and liquid is assumed to be of $O(\varepsilon^3)$:

$$\frac{\rho_{G0}^*}{\rho_{L0}^*} \equiv O(\varepsilon^3). \tag{27}$$

The nondimensional liquid viscosity μ is introduced as

$$\mu \equiv \frac{\mu^*}{\varepsilon^2 \rho_{L0}^* R_0^* U^*} \equiv O(1). \tag{28}$$

NONLINEAR WAVE OF WEAK DISPERSION

In this section, we shall analyze the weakly nonlinear propagation of pressure waves in the moderately low frequency band, i.e., Band A in Fig. 1. Band A is regarded as the weakly dispersive band.

To characterize the problem, we shall choose the scalings of three parameters, U^*, L^*, and T^*, as follows:

$$\frac{U^*}{c_{L0}^*} \equiv O(\sqrt{\varepsilon}) \equiv V\sqrt{\varepsilon}, \tag{29}$$

$$\frac{R_0^*}{L^*} \equiv O(\sqrt{\varepsilon}) \equiv \Delta\sqrt{\varepsilon}, \tag{30}$$

$$\frac{\omega^*}{\omega_B^*} \equiv \frac{1}{T^*\omega_B^*} \equiv O(\sqrt{\varepsilon}) \equiv \Omega\sqrt{\varepsilon}, \tag{31}$$

where Δ and Ω are constants of $O(1)$ (Ω corresponds to a normalized frequency of waves), $\omega^* \equiv 1/T^*$ is a frequency of the sound source, and ω_B^* is the eigenfrequency of linear spherical symmetric oscillation of single bubble,

$$\omega_B^* \equiv \sqrt{\frac{3\gamma p_{G0}^* - 2\sigma^*/R_0^*}{\rho_{L0}^* R_0^{*2}}}. \tag{32}$$

For the simplicity, the effects of liquid compressibility and viscosity are neglected in Eq. (32), i.e., ω_B^* is the same as the eigenfrequency obtained from the linearized Rayleigh–Plesset equation.

The set of scalings (29)–(31) shows that the focused wave motion is of low frequency compared with the eigenfrequency of bubble, of large wavelength compared with the initial bubble radius, and of small propagation speed compared with the speed of sound in the liquid phase.

Comparison of Eq. (24) with Eq. (29) yields $a = 2$, and hence the expansions of the liquid density (21) and the liquid pressure (25), respectively, may be rewritten as

$$\rho_L^*/\rho_{L0}^* = 1 + \varepsilon^2\rho_{L1} + \varepsilon^3\rho_{L2} + O(\varepsilon^4), \tag{33}$$

$$p_L = p_{L0} + \varepsilon p_{L1} + \varepsilon^2 p_{L2} + O(\varepsilon^3), \tag{34}$$

where the expansion coefficients in Eq. (34) are defined as

$$p_{L1} = \frac{\rho_{L1}}{V^2}, \quad p_{L2} = \frac{\rho_{L2}}{V^2}. \tag{35}$$

The following analysis in this section requires only the slow variables, $t_1 = \varepsilon t$, and therefore Eq. (15) may be simplified into

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1}. \tag{36}$$

First-order equations

We substitute expansions (17)–(20), (33) and (34) into basic equations (1)–(4) and (8)–(12), and use scalings (29)–(31), derivative expansion (36), and so on. As a result, we firstly obtain the following set of linearized equations as the first-order equations: the mass conservation law in gas phase,

$$\frac{\partial\alpha_1}{\partial t_0} - 3\frac{\partial R_1}{\partial t_0} + \frac{\partial u_{G1}}{\partial x} = 0, \tag{37}$$

the mass conservation law in liquid phase,

$$\alpha_0\frac{\partial\alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{L1}}{\partial x} = 0, \tag{38}$$

the momentum conservation law in gas phase,

$$\beta_1 \frac{\partial u_{G1}}{\partial t_0} - \beta_1 \frac{\partial u_{L1}}{\partial t_0} - 3\gamma p_{G0} \frac{\partial R_1}{\partial x} = 0, \tag{39}$$

the momentum conservation law in liquid phase,

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{L1}}{\partial t_0} - \beta_1\alpha_0 \frac{\partial u_{G1}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{L1}}{\partial x} = 0, \tag{40}$$

and the Keller equation,

$$p_{L1} + \frac{\Delta^2}{\Omega^2} R_1 = 0. \tag{41}$$

Eliminating α_1, u_{G1}, u_{L1}, and p_{L1} from Eqs. (37)–(41), the linear wave equation can be derived as

$$\frac{\partial^2 R_1}{\partial t_0^2} - v_p^2 \frac{\partial^2 R_1}{\partial x^2} = 0, \tag{42}$$

where the phase velocity v_p is

$$v_p = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{G0} + \beta_1(1-\alpha_0)\Delta^2/\Omega^2}{3\beta_1\alpha_0(1-\alpha_0)}}. \tag{43}$$

Similarly to the well-known speed of sound in the incompressible liquid containing gas bubbles [2, 3], v_p is in proportion to $1/\sqrt{\alpha_0(1-\alpha_0)}$. This is the reflection of the fact that the expansion of the liquid density starts with $O(\varepsilon^2)$ in Eq. (33), i.e., the compressibility of liquid may be regarded as so weak. Now, we choose the characteristic velocity U^* as

$$U^* = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p^*_{G0}/\rho^*_{L0} + \beta_1(1-\alpha_0)R_0^{*2}\omega_B^{*2}}{3\beta_1\alpha_0(1-\alpha_0)}}, \tag{44}$$

and this leads to $v_p \equiv 1$. Rewriting R_1 into f in Eq. (42), we have

$$\frac{\partial^2 f}{\partial t_0^2} - \frac{\partial^2 f}{\partial x^2} = 0. \tag{45}$$

That is, the near field is described by the linear wave equation (45), and the dispersion and dissipation effects of waves due to bubble oscillations do not appear there.

Second-order equations

Let us derive the second-order equations. By the use of the same procedure as the derivation of Eq. (42), we have

$$\frac{\partial^2 R_2}{\partial t_0^2} - \frac{\partial^2 R_2}{\partial x^2} = H(x, t_0, t_1), \tag{46}$$

where the inhomogeneous term, H, is composed of the partial derivatives of the first-order expansion coefficients (e.g., u_{G1}, R_1) with respect to x, t_0, and t_1.

Focusing on the right-running wave, the phase function $\hat{\varphi}$ is introduced as

$$\hat{\varphi} \equiv x - t_0. \tag{47}$$

Rewriting Eqs. (37)–(41) by $\hat{\varphi}$ and integrating them with respect to $\hat{\varphi}$, we can express α_1, u_{G1}, u_{L1}, and p_{L1}, as multiples of $f = R_1$, and hence the inhomogeneous term H in Eq. (46) may be regarded as a function of $\hat{\varphi}$ and t_1.

The solvability condition or the non-secular condition requires

$$H(\hat{\varphi}, t_1) = 0. \tag{48}$$

From Eq. (48), we can derive a far-field equation

$$\frac{\partial f}{\partial t_1} + C_0 \frac{\partial f}{\partial \hat{\varphi}} + C_1 f \frac{\partial f}{\partial \hat{\varphi}} + C_2 \frac{\partial^2 f}{\partial \hat{\varphi}^2} + C_3 \frac{\partial^3 f}{\partial \hat{\varphi}^3} = 0, \tag{49}$$

where the coefficients of advective, dissipative, and dispersive terms, C_0, C_2, and C_3, are given by

$$C_0 = -\frac{(1-\alpha_0)\Delta^2 V^2}{6\alpha_0\Omega^2}, \tag{50}$$

$$C_2 = -\frac{\Delta^3 V}{6\alpha_0\Omega^2}, \tag{51}$$

$$C_3 = \frac{\Delta^2}{6\alpha_0}, \tag{52}$$

respectively. Clearly, C_0 and C_2 have negative values, while C_3 has a positive value. In the present paper, the explicit representation of the coefficient of the nonlinear term C_1 is not shown because of the complexity of expression.

Introducing a modified phase function φ

$$\varphi \equiv \hat{\varphi} - C_0 t_1 = x - t_0 - C_0 t_1, \tag{53}$$

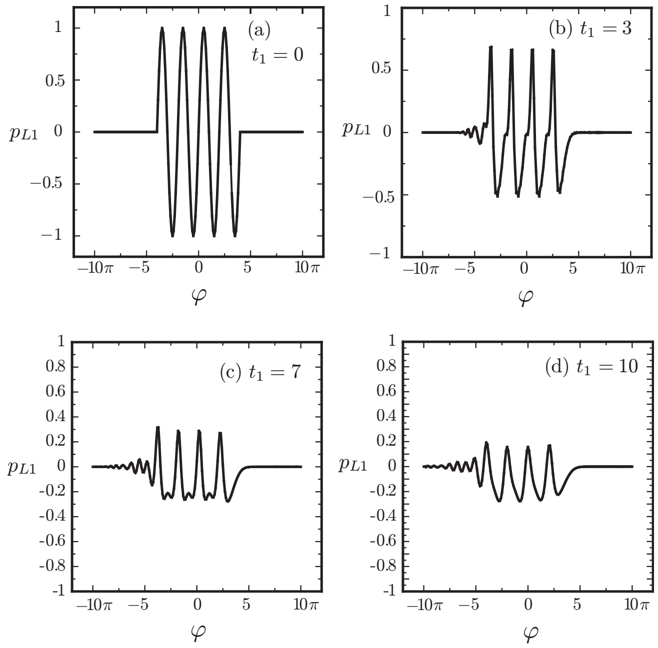

Figure 2. Wave profiles of the first-order liquid pressure p_{L1} in the case of $\alpha_0 = 0.02$, $\Delta = 0.1$, $\Omega = 0.4$, $V = 0.3$, $\beta_1 = \beta_2 = 0.5$, and $-10\pi \le \varphi \le 10\pi$: (a) $t_1 = 0$, (b) $t_1 = 3$, (c) $t_1 = 7$, (d) $t_1 = 10$.

we readily rewrite Eq. (49) into

$$\frac{\partial f}{\partial t_1} + C_1 f \frac{\partial f}{\partial \varphi} + C_2 \frac{\partial^2 f}{\partial \varphi^2} + C_3 \frac{\partial^3 f}{\partial \varphi^3} = 0. \tag{54}$$

Equation (54) is the so-called Korteweg–de Vries–Burgers equation (KdV–Burgers equation) [3]. The KdV–Burgers equation (54) describes the behavior of waves in the far field characterized by t_1 and φ, where the weak dissipation and weak dispersion effects appear and compete with the weak nonlinear effect.

In Fig. 2, we show some examples of time evolutions of wave profiles of the liquid pressure p_{L1}, evaluated by the numerical calculation of KdV–Burgers equation (54) with a finite difference method.

NONLINEAR WAVE OF STRONG DISPERSION

In this section, we focus on the moderately high frequency band, i.e., Band B in Fig. 1. Band B can be regarded as the strongly dispersive band. We shall study the nonlinear modulation of a quasi-monochromatic wave train in the long-range propagation of weakly nonlinear waves with a strong dispersion effect.

We shall define the scalings of parameters, as

$$\frac{U^*}{c^*_{L0}} \equiv O(\varepsilon^2) \equiv V\varepsilon^2, \tag{55}$$

$$\frac{R^*_0}{L^*} \equiv O(1) \equiv \Delta, \tag{56}$$

$$\frac{\omega^*}{\omega^*_B} \equiv T^*\omega^* \equiv O(1) \equiv \Omega, \tag{57}$$

where we determine $T^* \equiv 1/\omega^*_B$. The set of scalings (55)–(57) shows that the frequency is comparable with the eigenfrequency of bubble, the wavelength is also comparable with the initial bubble radius, and the propagation speed of waves is very small compared with the speed of sound in the liquid phase.

Although the method of averaged equation is usually prohibited to be applied to such short waves, the plane wave problem may be excluded from the restriction because the average volume can be sufficiently large along the plane parallel to the wave front [13]. Nevertheless, the assumption of spherical symmetry of bubble oscillations should be validated. We will address this problem in a future work.

Comparison of Eq. (24) with Eq. (55) yields $a = 5$, and therefore, in the present analysis, the liquid compressibility is assumed to be very small compared with the analysis in the previous section. Hence the expansions of the liquid density and pressure are, respectively, given as

$$\rho^*_L/\rho^*_{L0} = 1 + \varepsilon^5 \rho_{L1} + \varepsilon^6 \rho_{L2} + O(\varepsilon^7), \tag{58}$$

$$p_L = p_{L0} + \varepsilon p_{L1} + \varepsilon^2 p_{L2} + O(\varepsilon^3). \tag{59}$$

The expansion coefficients in Eq. (59) are defined by $p_{Lj} = \rho_{Lj}/V^2\,(j = 1, 2, 3, 4, 5)$. We therefore emphasis that the liquid compressibility cannot be neglected.

The analysis in this section requires t_1, t_2, x_1, and x_2 as the slow variables, and hence Eqs. (15) and (16), respectively, may be rewritten into

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2}, \tag{60}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_1} + \varepsilon^2 \frac{\partial}{\partial x_2}. \tag{61}$$

First-order equations

By using the same procedure as the derivation of Eq. (42) from Eqs. (37)–(41) in the previous section, we have the following set of linear equations,

$$\frac{\partial\alpha_1}{\partial t_0} - 3\frac{\partial R_1}{\partial t_0} + \frac{\partial u_{G1}}{\partial x_0} = 0, \tag{62}$$

$$\alpha_0\frac{\partial\alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{L1}}{\partial x_0} = 0, \tag{63}$$

$$\beta_1\frac{\partial u_{G1}}{\partial t_0} - \beta_1\frac{\partial u_{L1}}{\partial t_0} - 3\gamma p_{G0}\frac{\partial R_1}{\partial x_0} = 0, \tag{64}$$

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{L1}}{\partial t_0} - \beta_1\alpha_0\frac{\partial u_{G1}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{L1}}{\partial x_0} = 0, \tag{65}$$

$$\frac{\partial^2 R_1}{\partial t_0^2} + R_1 + \frac{p_{L1}}{\Delta^2} = 0, \tag{66}$$

and these can be reduced to the single equation

$$\mathcal{L}_1[R_1] = 0, \tag{67}$$

where the linear differential operator $\mathcal{L}_1$ is defined as

$$\mathcal{L}_1 \equiv \frac{\partial^2}{\partial t_0^2} - \left[\frac{\Delta^2}{3\alpha_0} + \frac{(1-\alpha_0+\beta_1)\gamma p_{G0}}{\beta_1(1-\alpha_0)}\right]\frac{\partial^2}{\partial x_0^2} - \frac{\Delta^2}{3\alpha_0}\frac{\partial^4}{\partial x_0^2\partial t_0^2}. \tag{68}$$

Equation (67) corresponds to the linear wave equation with the dispersion term.

We take the solution of Eq. (67) as the form of a quasi-monochromatic wave train which evolves into a slowly modulated wave packet [18]:

$$R_1 = A(t_1, t_2, x_1, x_2)e^{\mathrm{i}\theta} + \text{c.c.}, \tag{69}$$

with the phase function

$$\theta = k^* x_0^* - \omega^*(k^*)t_0^* = kx_0 - \Omega(k)t_0, \tag{70}$$

where A is the slowly varying complex amplitude which depends on t_1, t_2, x_1, and x_2, $k \equiv k^* L^*$ is the normalized wave number, i denotes the imaginary unit, and c.c. denotes the complex conjugate.

Note that the solution (69) describes a monochromatic wave train when A is independent of either t_1 and t_2 or x_1 and x_2.

The linear dispersion relation is obtained by substituting Eq. (69) into Eq. (67), as follows:

$$D(k,\Omega) = \frac{\Delta^2 k^2(1-\Omega^2)}{3\alpha_0} + \frac{(1-\alpha_0+\beta_1)\gamma p_{G0}}{\beta_1(1-\alpha_0)}k^2 - \Omega^2 = 0, \tag{71}$$

or

$$k = \pm\Omega\sqrt{\frac{3\alpha_0(1-\alpha_0)\beta_1}{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{G0} - \Delta^2\beta_1(1-\alpha_0)(\Omega^2-1)}}. \tag{72}$$

The cutoff frequency Ω_C is given as

$$\Omega_C = \sqrt{1 + \frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{G0}}{\beta_1(1-\alpha_0)\Delta^2}}. \tag{73}$$

The dispersion relations calculated from Eq. (71) are shown in Fig. 3.

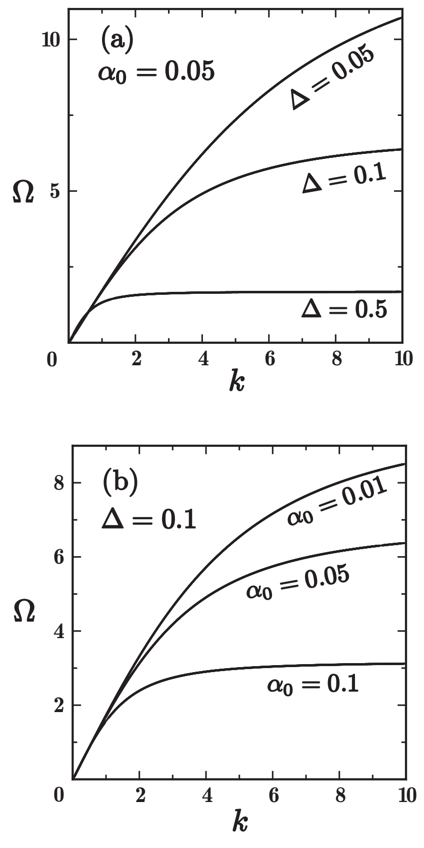

Figure 3. The dispersion relations in the bubbly liquid in the case of $\beta_1 = 0.5$, $\gamma = 1$, $p_{G0} = 1$, and typical values as follows: (a) $\alpha_0 = 0.05$: $\Delta = 0.05, 0.1, 0.5$; (b) $\Delta = 0.1$: $\alpha_0 = 0.01, 0.05, 0.1$.

Second-order equations

Substituting the solution (69) into the set of equations (62)–(66) and integrating them with respect to t_0 and x_0, we have α_1, u_{G1}, u_{L1}, and p_{L1} expressed by multiples of R_1. Then, the second-order equation is obtained as

$$\mathcal{L}_1[R_2] = H_2(x_0, x_1, t_0, t_1), \tag{74}$$

with the inhomogeneous term

$$H_2 = \Gamma A^2 e^{2\mathrm{i}\theta} + \mathrm{i}\left(-\frac{\partial D}{\partial \Omega}\right)\left(\frac{\partial A}{\partial t_1} + v_g \frac{\partial A}{\partial x_1}\right) e^{\mathrm{i}\theta} + \text{c.c.}, \tag{75}$$

where Γ is a real constant.

From the non-secular condition of inhomogeneous equation (74), the second term in the right-hand side of Eq. (75) should vanish. We therefore obtain the following solvability condition

$$\frac{\partial A}{\partial t_1} + v_g \frac{\partial A}{\partial x_1} = 0. \tag{76}$$

Here, the normalized group velocity v_g is calculated by linear dispersion relation (71), as follows:

$$v_g = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = \frac{3\alpha_0 \Omega(k)}{k(3\alpha_0 + \Delta^2 k^2)}. \tag{77}$$

The particular solution of Eq. (74) is given as

$$R_2 = \frac{\Gamma}{D(2k, 2\Omega)} A^2 e^{2\mathrm{i}\theta} + \text{c.c.}. \tag{78}$$

As in the case of first-order equations, substituting the solution (78) into the set of equations of $O(\varepsilon^2)$ gives the explicit representations of α_2, u_{G2}, u_{L2}, and p_{L2}.

Third-order equations

Let us proceed to the next-order problem in order to determine the behavior of the slowly modulated wave packet as a result of long range propagation with weak nonlinear and strong dispersion effects.

The slightly lengthy calculations give the third-order equation

$$\mathcal{L}_1[R_3] = H_3(x_0, x_1, x_2, t_0, t_1, t_2), \tag{79}$$

with the inhomogeneous term

$$H_3 = \Gamma_1 e^{3\mathrm{i}\theta} + \Gamma_2 e^{2\mathrm{i}\theta} + \Gamma_3 e^{\mathrm{i}\theta} + \Gamma_4 + \text{c.c.}, \tag{80}$$

where Γ_j $(j = 1, 2, 3, 4)$ are the complex variables including A and its derivatives. The explicit representation of Γ_3 is

$$\Gamma_3 = \left(-\frac{\partial D}{\partial \Omega}\right)\left[\mathrm{i}\left(\frac{\partial A}{\partial t_2} + v_g \frac{\partial A}{\partial x_2}\right) + \frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial x_1^2} + \nu_1 |A|^2 A + \mathrm{i}\nu_2 A\right]. \tag{81}$$

From the solvability condition of Eq. (79), we have

$$\mathrm{i}\left(\frac{\partial A}{\partial t_2} + v_g \frac{\partial A}{\partial x_2}\right) + \frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial x_1^2} + \nu_1 |A|^2 A + \mathrm{i}\nu_2 A = 0, \tag{82}$$

where the derivative $\mathrm{d}v_g/\mathrm{d}k$ is calculated by the expression of group velocity (77), as

$$\frac{\mathrm{d}v_g}{\mathrm{d}k} = -\frac{9\alpha_0 \Delta^2 \Omega(k)}{(3\alpha_0 + \Delta^2 k^2)^2}, \tag{83}$$

and the real coefficient ν_2 is given as

$$\nu_2 = \frac{\Delta k^2}{2(3\alpha_0 + \Delta^2 k^2)}(4\mu + V\Delta^2), \tag{84}$$

and ν_2 has a positive value. The explicit representation of the real coefficient of the nonlinear term ν_1 is not shown here because of the complexity of expression.

By making use of the solvability conditions of the second-order (76) and the third-order (82), and the definitions of derivative expansions (60) and (61), we obtain

$$\mathrm{i}\left(\frac{\partial A}{\partial t} + v_g \frac{\partial A}{\partial x}\right) + \frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial x^2} + \varepsilon^2(\nu_1 |A|^2 A + \mathrm{i}\nu_2 A) = 0. \tag{85}$$

Furthermore, Eq. (85) can be rewritten into

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial \xi^2} + \nu_1 |A|^2 A + \mathrm{i}\nu_2 A = 0, \tag{86}$$

through the variable transformations

$$\tau \equiv \varepsilon^2 t = t_2, \quad \xi \equiv \varepsilon(x - v_g t) = x_1 - v_g t_1. \tag{87}$$

Equation (86) agrees with the well-known nonlinear Schrödinger equation [5, 8] if the coefficient ν_2 equals to zero. The fourth term in the left-hand side of Eq. (86) describes the attenuation effect of waves due to bubble oscillations. Therefore, Eq. (86) describes the behavior of the slowly modulated wave packet with the weak nonlinear, strong dispersion, and weak dissipation effects.

Let us put $A = ge^{ih}$, where g is the amplitude and h is the phase. Substituting it into Eq. (86) gives the following set of equations:

$$g\frac{\partial h}{\partial \tau} = \frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\left[\frac{\partial^2 g}{\partial \xi^2} - g\left(\frac{\partial h}{\partial \xi}\right)^2\right] + \nu_1 g^3, \tag{88}$$

$$\frac{\partial g}{\partial \tau} = -\frac{1}{2}\frac{\mathrm{d}v_g}{\mathrm{d}k}\left[g\frac{\partial^2 h}{\partial \xi^2} + 2\left(\frac{\partial g}{\partial \xi}\right)\left(\frac{\partial h}{\partial \xi}\right)\right] - \nu_2 g. \tag{89}$$

We shall solve the set of equations (88) and (89) with a finite difference method. Figures 4 and 5 show some examples of the time evolutions of the amplitude g and phase h, where the coefficients ν_1 and ν_2 are selected as specific values.

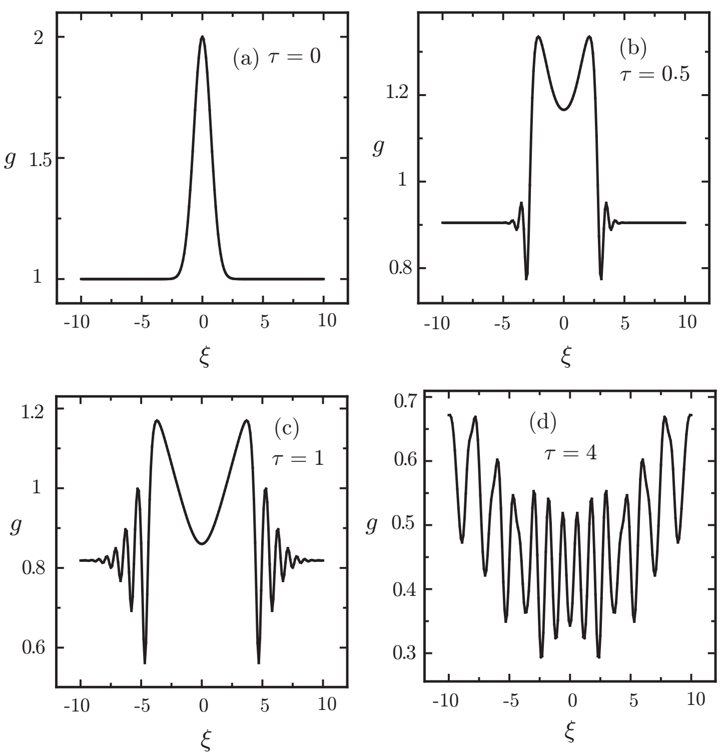

Figure 4. The profiles of the amplitude g in the case of $\nu_1 = 4$, $\nu_2 = 0.2$, $\mathrm{d}v_g/\mathrm{d}k = -1$, and $-10 \le \xi \le 10$: (a) $\tau = 0$, (b) $\tau = 0.5$, (c) $\tau = 1$, (d) $\tau = 4$.

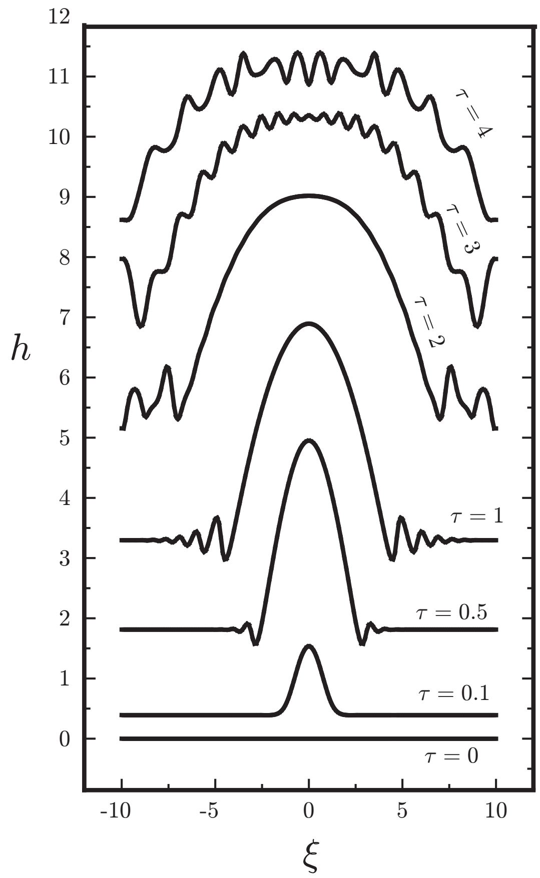

Figure 5. The profiles of the phase h in the case of $\nu_1 = 4$, $\nu_2 = 0.2$, $\mathrm{d}v_g/\mathrm{d}k = -1$, and $-10 \le \xi \le 10$: $\tau = 0, 0.1, 0.5, 1, 2, 3, 4$.

CONCLUSION

The weakly nonlinear analyses of pressure waves in bubbly liquids have been carried out based on the two-fluid averaged equations. We have derived the equations describing wave motions in far fields by the use of the method of multiple scales and the appropriate scaling of parameters.

In the moderately low frequency band, the KdV–Burgers equation describes the behavior of waves in the far field, where the weak dissipation and weak dispersion effects appear and compete with the weak nonlinear effect.

In the moderately high frequency band, the nonlinear Schrödinger equation with the attenuation term can be derived as the far field equation of the quasi-monochromatic wave train. It describes the weakly nonlinear modulation with the weak dissipation effect, caused by the strong dispersion effect.

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