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THE CALCULATION OF TRANSIENT RESPONSE USING THE ANALOG COMPUTER

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NOMENCLATURE

a	reservoir time constant, relating the actual time to dimensionless time[6], [7]
a_0, a_1, a_2, a_3	constants
b_2, b_3	constants
C_e	electrical capacitor, mfd
D	differential operator
f_1	Laplace transform of Q'_{tD}
G	forcing function
g	Laplace transform of G
K	Machine time constant sec/month
L	Laplace transform
P	pressure
P_1, P_2...	potentiometer settings
Q'_{tD}	Unit step response
Q_{tD}	system response for specified G function
R_f, R_i	electrical resistors
r_D	dimensionless radius[6] [7]
s	Laplace transform variable
t_D	dimensionless time
τ	time variable
θ	actual time

INTRODUCTION

The study of transient behavior has become an important subject in chemical engineering applications. When an input to a system fluctuates, one is interested in predicting how the system will respond in order to prepare for the response or to prevent unwanted fluctuations. Because of the dependence upon past history a computational scheme is needed to integrate the effects of past action and thereby determine the present state of the system. Even though the need is frequently encountered in practice, the computational techniques for solving systems with time-varying inputs or time-varying coefficients are not widely known.

For a linear system the separate solutions, obtained by using each driving force alone, can be added to obtain a composite solution with all the driving forces present. This principle of adding solutions is called the Principle of Superposition. The integral expression of the principle is known variously as the convolution integral, superposition integral, Faltung integral, or Duhamel integral.(1) Computation of this classical integral gives a solution for the case where a linear system is subject to a time-varying input. This paper presents a method of direct computation of this integral by simulating the system on an electronic differential analyzer.

In a chemical process, the driving force may be temperature, pressure or concentration. Only one time-varying input is examined at a time while holding the rest of the variables constant.

Theory

When a system is linear, it is possible to describe the system characteristics by a linear differential equation or by its solution. Using the superposition principle (owing to its linearity), the response can be described by the following convolution integral:[2]

$$Q_{tD} = \int_0^{t_D} \frac{dG}{d\tau} Q'_{tD} (t_D - \tau) \, d\tau \qquad (1)$$

where

Q'_{tD} = response to a unit step, assuming initial conditions are zero.

G = forcing function (assumed equal to zero for $t < 0$).

Q_{tD} = system response for a specified G.

Equation (1) can be used to find the response of a linear system with constant coefficients for an arbitrary forcing function when the unit step response is known. The response to a unit step input is a system characteristic and is unique. When the system is nonlinear, the step response is no longer unique in form but depends on the input level. The unit step input and its system response are illustrated in Figures 1 and 2.

The essential feature of analog simulation is to obtain the desired process performance as an output by feeding in known information as an input signal. A pictorial representation of the simulation is shown in Figure 3. The discussion is devoted to the representation of the transfer function.

Upon taking the Laplace transform of Equation (1) we obtain

$$Q_{tD}(s) = s\, g(s) \circ f_1(s)$$

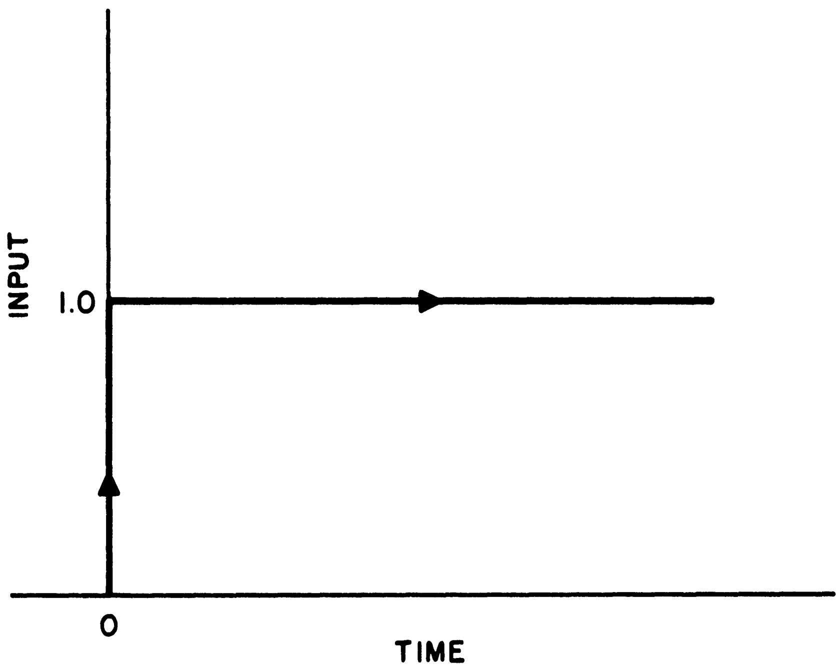

Figure 1. Unit Step Input.

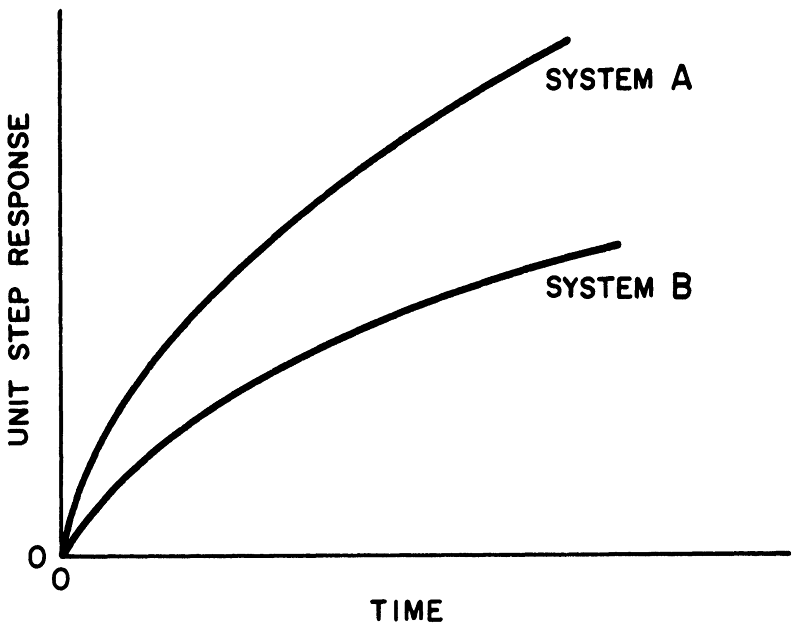

Figure 2. Unit Step Response.

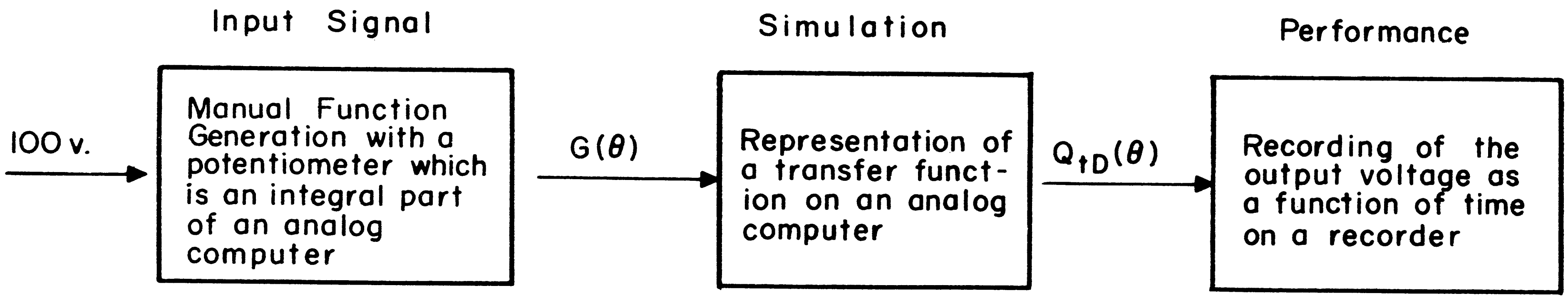

Figure 3 - Overall Block Diagram of the Analog Simulation Arrangement.

or

$$\frac{Q_{tD}(s)}{g(s)} = s\ f_1(s) \tag{2}$$

where

$$g(s) = L\ \{G\}$$

$$f_1(s) = L\ \{Q'_{tD}\}$$

Note that $f_1(s)$ is the Laplace transform of the unit step response. Therefore, $sf_1(s)$ can be approximated from the solution of the constant input case. If the initial value of the unit step response is zero, from the well-known property of the Laplace transform rule for the derivative(2), $sf_1(s)$ can be expressed in the form

$$L \left\{ \frac{dQ'_{tD}}{d\ t_D} \right\} = s\ f_1(s)$$

This relationship can be used to approximate $s\ f_1(s)$ from $\frac{d\ Q'_{tD}}{d\ t_D}$ or directly from Q'_{tD}.

Suppose that $s\ f_1(s)$ is approximated as a sum of a_o, $\frac{a_1}{s}$, $\frac{a_2}{1+\frac{s}{b_2}}$, and $\frac{a_3}{1+\frac{s}{b_3}}$; or

$$\frac{Q_{tD}(s)}{g(s)} = s\ f_1(s) = a_o + \frac{a_1}{s} + \frac{a_2}{1+\frac{s}{b_2}} \quad \frac{a_3}{1+\frac{s}{b_3}} \tag{3}$$

Then one can write

$$Q_{tD}(s) = a_o\ g(s) + \frac{a_1\ g(s)}{s} \quad \frac{a_2\ g(s)}{1+\frac{s}{b_2}} \quad \frac{a_3\ g(s)}{1+\frac{s}{b_3}}$$

The inversion of $Q_{tD}(s)$ to obtain the response is the sum of inverse transforms of the individual terms.

$$Q_{tD}(t_D) = [Q_{tD}(t_D)]_o + [Q_{tD}(t_D)]_1 + [Q_{tD}(t_D)]_2 + [Q_{tD}(t_D)]_3$$

where

$$L\left\{[Q_{tD}(t_D)]_o\right\} = a_o\ g(s) \tag{4}$$

$$L\left\{[Q_{tD}(t_D)]_1\right\} = \frac{a_1\ g(s)}{s} \tag{5}$$

$$L\left\{[Q_{tD}(t_D)]_2\right\} = \frac{a_2\ g(s)}{1 + \frac{s}{b_2}} \tag{6}$$

$$L\left\{[Q_{tD}(t_D)]_3\right\} = \frac{a_3\ g(s)}{1 + \frac{s}{b_3}} \tag{7}$$

These individual terms are to be considered for their theoretical possibilities for analog simulation.

Equation (4) can be written directly in the time domain as

$$\frac{[Q_{tD}(t_D)]_o}{G(t_D)} = a_o$$

The analog simulation for this relationship is represented by amplifier 1 in Figure 4. (Those who are not familiar with analog computers are referred to the Appendix and also to references (3) and (4).

Equation (5) needs to be written in the time domain. Since $Q_{tD}(0) = 0$,

$$sL\left\{[Q_{tD}(t_D)]_1\right\} = L\left\{\frac{d\ Q_{tD}}{dt_D}\right\} = a_1\ g(s)$$

or

$$\left\{\frac{d\ Q_{tD}}{d\ t_D}\right\}_1 = a_1\ G(t_D)$$

or

$$\left[\frac{Q_{tD}(t_D)_1}{G(t_D)}\right] = \frac{a_1}{D}$$

where D is the differential operator $\frac{d}{d\ t_D}$. The analog simulation for this relationship is obtained by amplifier 2.

$$[Q_{tD}(t_D)]_1 = -a_1 \int_0^{t_D} -G(t_D)\ d\ t_D$$

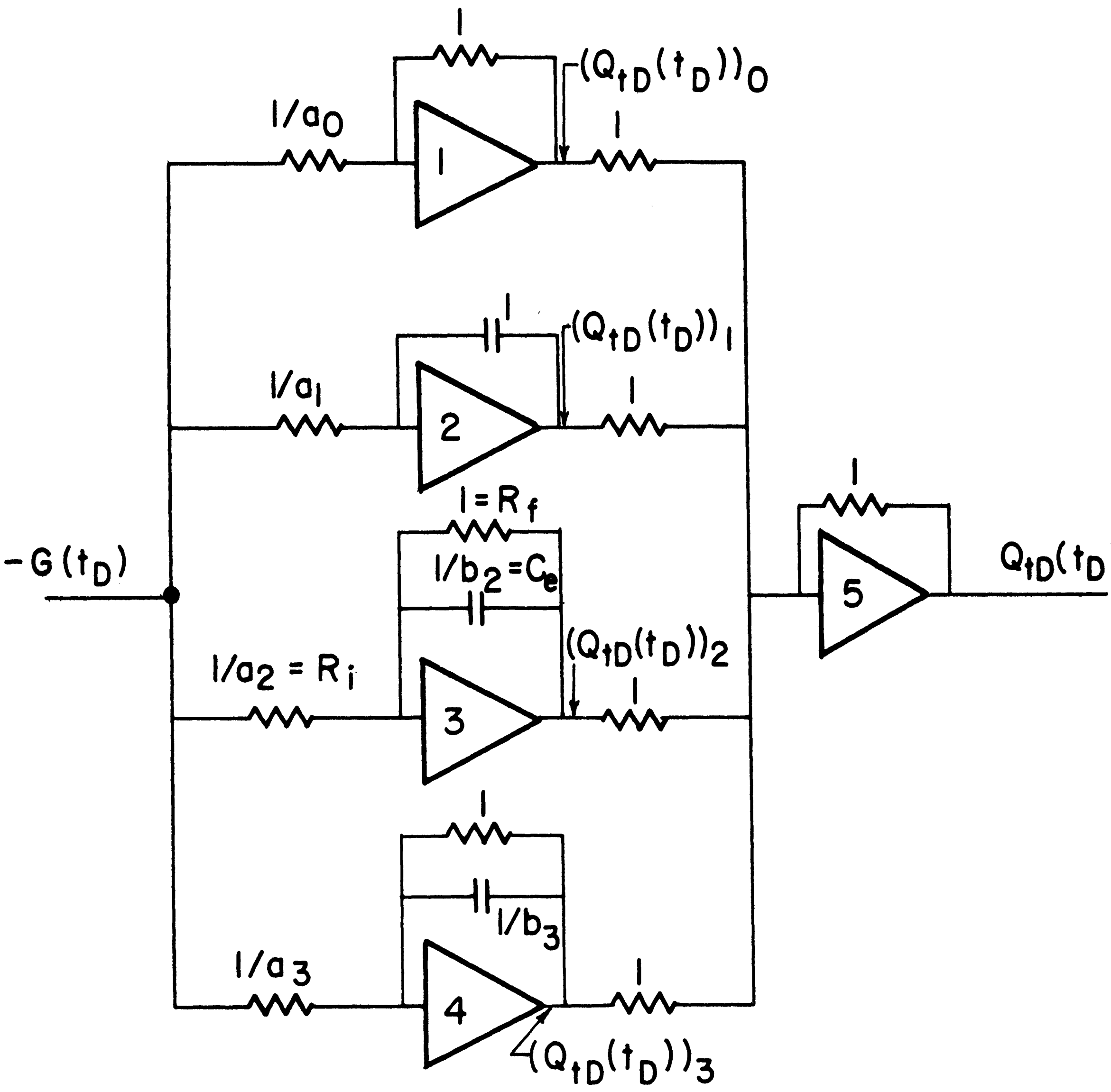

Figure 4 - Analog Simulation Circuit.

or

$$\frac{[Q_{tD}(t_D)]_1}{-G(t_D)} = \frac{-a_1}{D}$$

Similarly, Equation (6) can be written as

$$\frac{s}{b_2}[Q_{tD}(s)]_2 + [Q_{tD}(s)]_2 = a_2\, g(s)$$

Since $Q_{tD}(0) = 0$,

$$\frac{1}{b_2}\frac{d\, Q_{tD}}{dt_D}\bigg|_2 + [Q_{tD}(t_D)]_2 = a_2\, G(t_D)$$

or

$$\frac{[Q_{tD}(t_D)]_2}{G(t_D)} = \frac{a_2}{1 + \frac{D}{b_2}}$$

Amplifier 3 performs this operation.

$$\frac{[Q_{tD}(t_D)]_2}{-G(t_D)} = \frac{-a_2}{1 + \frac{D}{b_2}} = \frac{-R_f/R_i}{1 + R_f C_e\, D}$$

Equation (7) is much the same as Equation (6). Thus, summing up the individual terms with amplifier 5, the system can be simulated on an electronic differential analyzer as shown in Figure 4.

The consequence of this simulation is that Equation (2) can be represented by analogous voltages on the electronic differential analyzer and can be regarded in terms of a system transfer function. The system transfer function is an operator expression that establishes the relationship of output and input variables. Once an adequate check is obtained for a step input to a linear system, the response to an arbitrary input can be obtained with the same circuit by simply applying this input to the circuit.(1)(5)

Example

In order to demonstrate the application of the method to practical problems, an underground reservoir problem is chosen.

The flow of liquid through a homogeneous porous medium can be described by the radial diffusivity equation.(6)(7)

$$\frac{\partial^2 P}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P}{\partial r_D} = \frac{\partial P}{\partial t_D}$$

where

P = pressure

r_D = radius

The solutions to this partial differential equation for constant terminal reservoir conditions are available in tabular form in the literature.(7)(8)(9) For the terminal pressure case, the following approximate transfer function is found:

$$sf_1(s) = \frac{0.75}{1+\frac{s}{0.8a}} + \frac{2.64}{1+\frac{s}{0.108a}} + \frac{18}{1+\frac{s}{0.01a}} + \frac{0.28a}{s} - \frac{2.4\text{x}10^{-5}a^2}{s^2} \qquad (8)$$

where a is the reservoir time constant. In Equation (8) the Laplace transform is taken with respect to the actual time in months. Based on this equation, the circuit shown in Figure 5 is obtained for $a = 4$ and $K = 5$. The result obtained from this circuit is compared with the theoretical constant pressure case solution in Table I.

The circuit of Figure 5 can now be used for the arbitrary input case by simply generating the appropriate input voltage variation. The machine time constant is made large enough (i.e., the computer solution rate is made slow enough) so that the input voltage can be generated manually with a potentiometer. An example of the arbitrary input case is given in Figure 6.

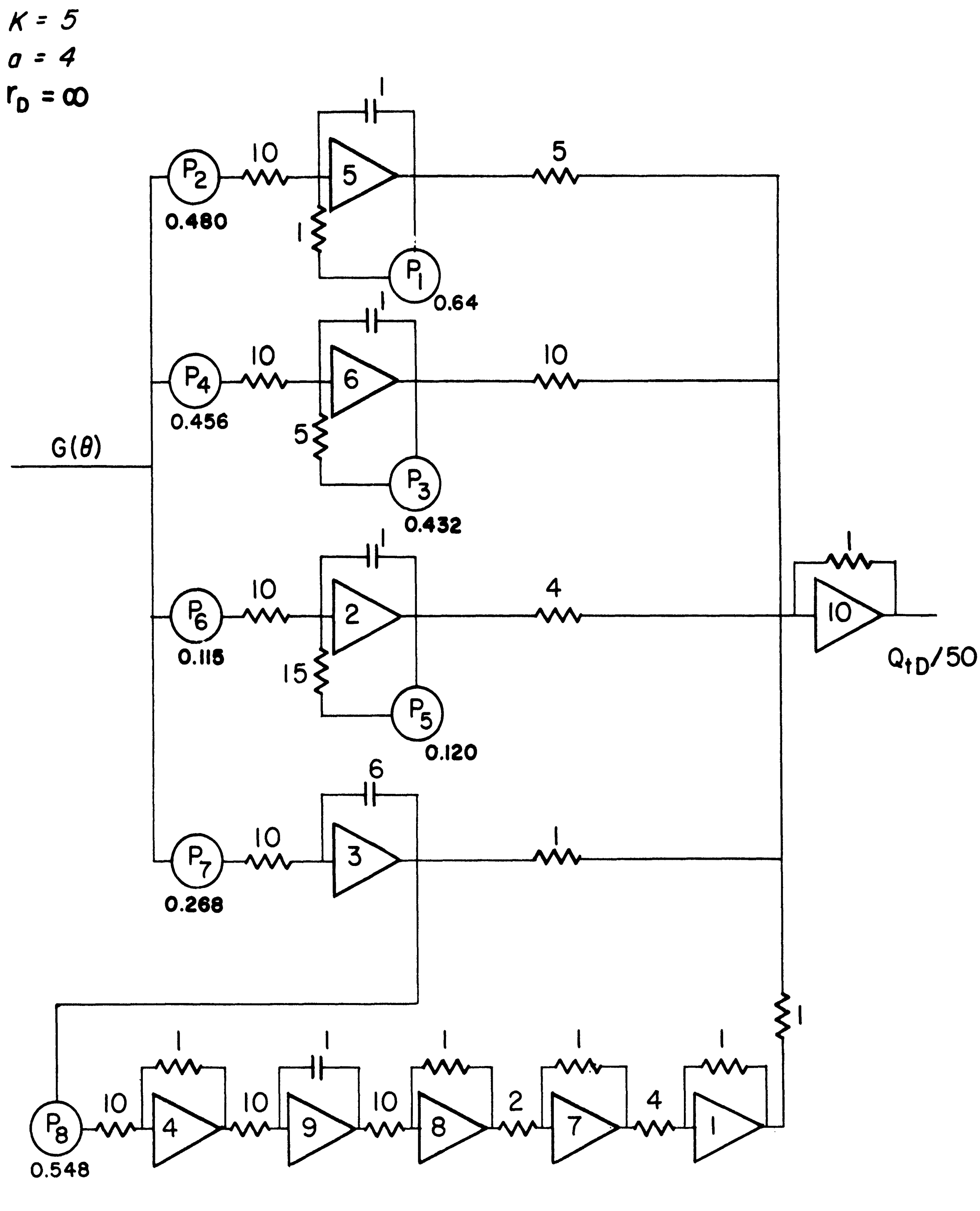

Figure 5 - Analog Computer Circuit Diagram to Get Cumulative Flux Function for Infinite Aquifer.

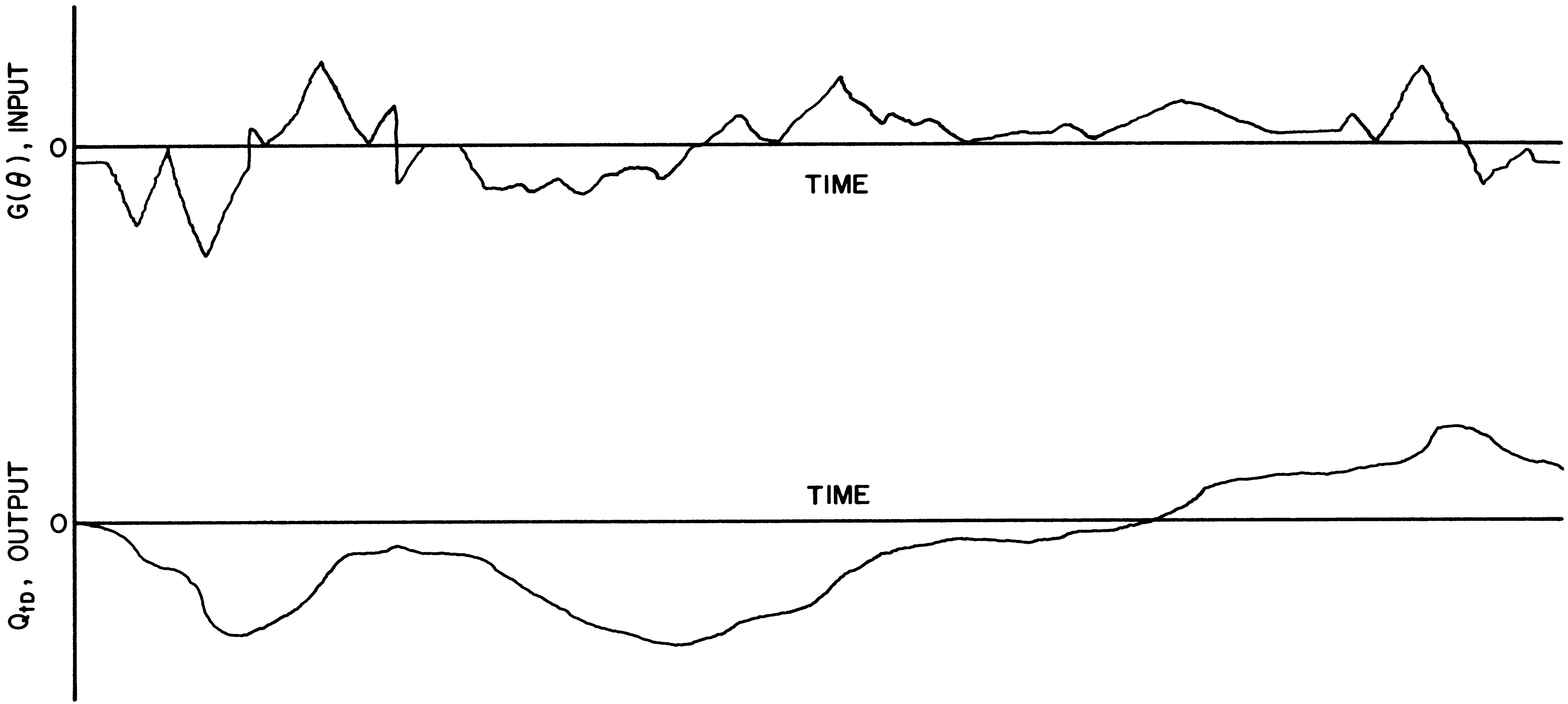

Figure 6. An Arbitrary Input Case Solution.

Complex Systems

We have shown how to simulate a time invariant linear system using approximated transfer function expression. For large number of problems the assumption of constant coefficient results in good approximation. However in some problems the system actually changes its characteristics as a function of time. In other words, the coefficients of the equation change with time in a definite fashion. Therefore, it is necessary to modify the system characteristics to adapt to the new situation at all times. A block diagram of a client adaptive system is shown in Figure 7(a) below:

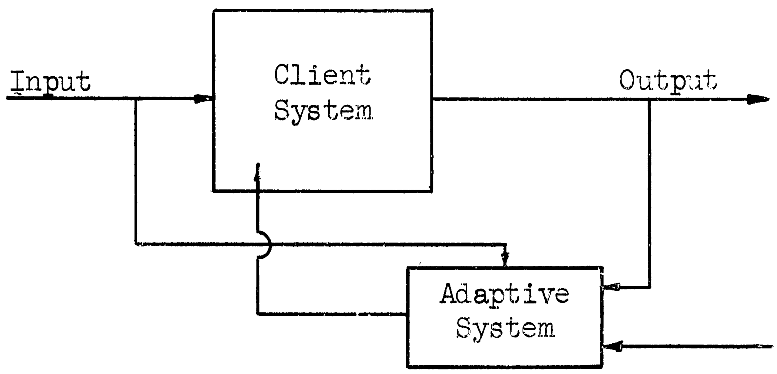

Figure 7 (a)

One probable practical application of the client-adaptive system may be the classical moving boundary problem. The adaptive system will change the characteristics of the client system as the boundary moves. In the analysis the change of coefficients can be handled by the use of servo-multiplier. Once the system is successfully simulated, control schemes can be added on to the process to study its behavior or to improve the transient response. See Figure 7 (b).

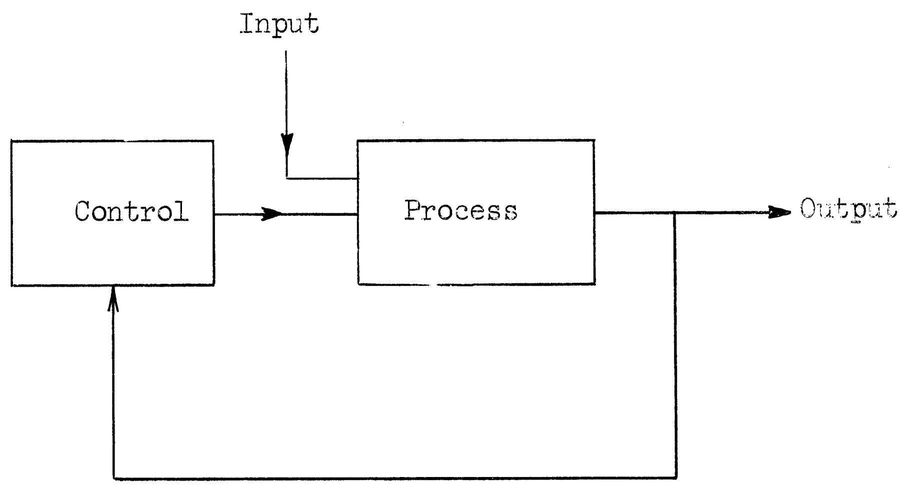

Figure 7 (b)

Conclusion

The proposed method of handling an arbitrary input to a linear system provides a quick and easy way of obtaining an engineering solution. The method can be used to check theoretical model studies or the effects of nonlinearities when experimental data are available. The method can also be used to predict system behavior for the case of an arbitrary input, based on a knowledge of the step response.

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APPENDIX

EXAMPLE OF THE DERIVATION OF A SIMULATION CIRCUIT FROM THE TRANSFER FUNCTION

In order to show the detail derivation for a simulation circuit from the transfer function, an example calculation is shown here.

A. Basic Characteristics of the Electronic Differential Analyzer. Before any attempt to show how numbers are selected in the analog circuit, it would be well to describe some basic features of an electronic differential analyzer in order to understand the symbols used in the circuit. Referring to Figure 8, there are four symbols that must be explained. First, the most important components are the high-gain d-c amplifiers or operational amplifiers which are designated by triangles with the output at the point. It is these amplifiers that become summers or integrators or simulate each term of the transfer function expression. The second symbol is the circle which designates a potentiometer. The potentiometer is used to multiply by a constant less than unity, the value being determined according to the dial setting. This dial setting is written in or by the circle. The third and fourth symbols are for resistors and for capacitors and need no explanation. The input to the computer circuit is at the left and the output is at the right.

Since the independent variable is always time on the analog computer, a machine time constant K plays an important role in relating the actual time and the machine running time. The details are explained as the calculation is carried out.

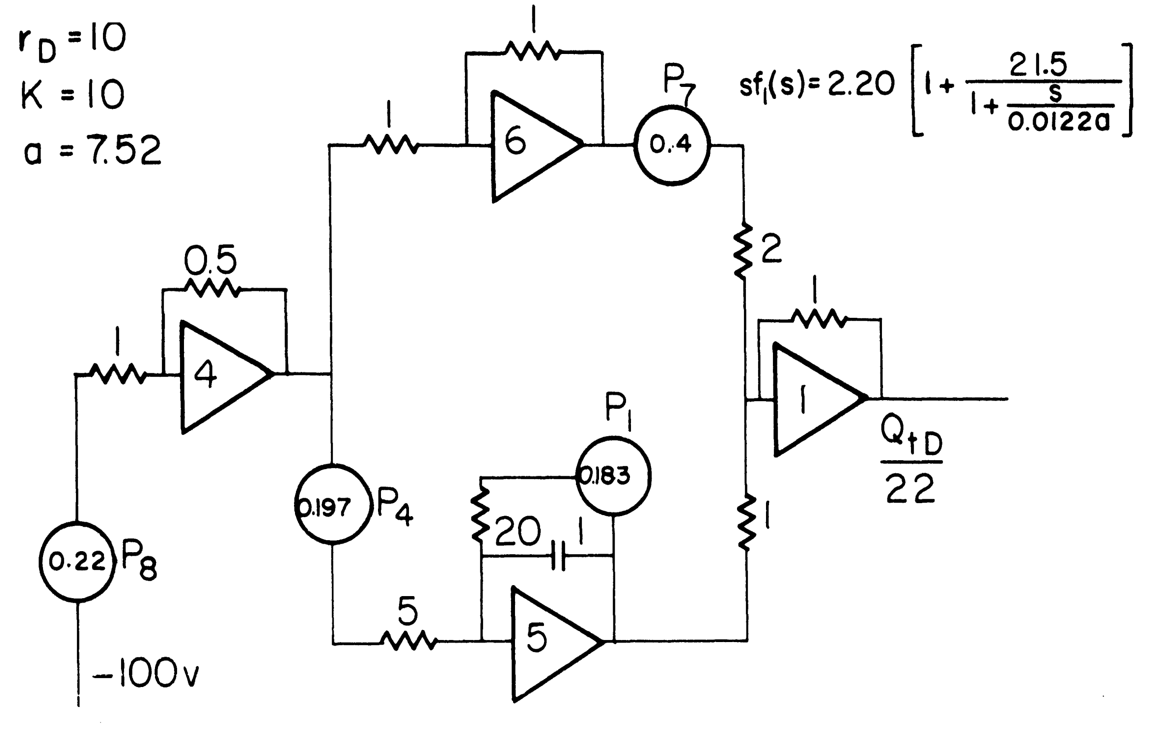

Figure 8 - Analog Computer Circuit Diagram for Simulation of Equation (9).

B. <u>Derivation of the Computer Circuit</u>. Suppose the transfer function of a system is given by

$$s\, f_1(s) = 2.20 \left\{ 1 + \frac{21.5}{1 + \frac{s}{0.0122a}} \right\} \qquad (9)$$

Equation (9) is a combination of Equations (4) and (6). We will show how the numbers are obtained from the transfer function as expressed in Equation (9) to give the circuit of Figure 8.

One of the basic requirements in deriving an analog circuit is to keep the output voltage magnitude for each amplifier less than 100 volts but not too small. Suppose, for example, that a unit magnitude in the equation is represented by one volt on the computer. Then a unit input should produce an output corresponding to the first term of Equation (9) of 2. 20 volts. This can be accomplished by choosing the output of potentiometer 8 to be -22 volts. The potentiometer setting is

$$P_8 = \frac{-22}{-100} = 0.220$$

This -22 volts is fed into amplifier 4 giving an output as follows:

$$\text{Amplifier 4 output} = (-22)\left(\frac{-0.5}{1}\right) = 11 \text{ volts}$$

The 11 volt signal is applied to amplifiers 5 and 6. Amplifier 6 has an output of -11 volts which goes through potentiometer 7 and amplifier 1, being reduced to the required 2.20 volts. The calculation of the required setting of potentiometer 7 to give this voltage is given as follows:

$$2.20 = (-11)\ (P_7)\ (-\tfrac{1}{2})$$

or

$$P_7 = 0.400$$

Next consider the second term. As shown in the simulation of Equation (6),

$$R_f C_e = \frac{1}{0.0122a}$$

or

$$R_f = \frac{K}{(0.0122a)\ (C_e K)}$$

Choosing $C_e K = 1$ and letting $K = 10$ and $a = 7.52$, we find that

$$R_f = 109 \text{ megohm}$$

It is actually represented by a 20 megohm resistance driven by potentiometer 1 set to

$$P_1 \frac{20}{109} = 0.183$$

The steady state output at amplifier 1 due to this second term is (2.2) (21.5) = 47.3 volts. Therefore the dial setting of potentiometer 4 is found from

$$(11)\ (P_4)\ (\frac{109}{5}) = 47.3$$

or

$$P_4 = 0.197$$

The outputs of amplifiers 5 and 6 are summed by amplifier 1 giving the overall transfer function of Equation (9). Since the input to amplifier 4 was set at -22 volts in checking out the step function response it actually corresponded to an input amplitude of -22. So if the negative of the actual input is generated at one volt per unit by potentiometer 8, the output will be $Q_{tD}/22$ as shown in Figure 8.