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THE INFLUENCE OF AERODYNAMIC PRESSURE ON THE WATER-ENTRY CAVITIES FORMED BY HIGH-SPEED PROJECTILES

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ABSTRACT

We present the results of a theoretical investigation of the vertical impact of high-speed projectiles onto a water surface. A model is developed to describe the evolution of the resulting air cavity. Expressions for the cavity profile and pinch-off time are obtained in the limit where collapse is caused primarily by aerodynamic pressure. Theoretical predictions compare favorably with experimental observations reported in the literature.

INTRODUCTION

When a solid object strikes a water surface with sufficient speed, it creates an air cavity whose eventual collapse leads to a vigorous jet and an entrained bubble. Accurate models of this phenomenon are essential for the effective design of air-to-sea projectiles as may be used to target under-water mines, torpedoes, or enemy vessels [1]. A question of particular interest is how to design a supercavitating projectile that fits entirely within a sustained vapor cavity in order to achieve a drag-reduced state [2]. In other situations, including the operation of propellers and pumps, one tries to avoid the creation of cavities, whose implosion causes noise, damage, and loss of efficiency [3, 4]. The water-entry problem is also relevant to applications in ship slamming [5], stone skipping [6], and the locomotion of water-walking creatures [7]. For a review of the water-entry literature, see Seddon & Moatamedi [8], Aristoff & Bush [9], and references therein.

Consider a solid sphere with radius R_0 vertically impacting a horizontal water surface with speed U_0 as depicted in figure 1. Let g be the gravitational acceleration and ρ the liquid density. The flow of air into the cavity behind the sphere gives rise to a characteristic pressure drop of $\rho_a U_0^2$, where ρ_a is the air density. Provided that the Weber number $\mathcal{W} = \rho U_0^2 R_0/\sigma \gg \rho/\rho_a$ and the Froude number $\mathcal{F} = U_0^2/(gR_0) \gg \rho/\rho_a$, one may neglect curvature pressure and hydrostatic pressure in favor of aerodynamic pressure, respectively[1]. For very high impact speeds, $U_0 \gtrsim U_s$, where $U_s \approx 340$ m sec^{-1} is the speed of sound in air, the pressure drop cannot be estimated using the incompressible Bernoulli equation. Instead, one expects the flow of air to become choked, so that the pressure drop reaches a maximum value that is independent of the flow speed. Following Lee *et al.* [10], and based on the cavity pressures recorded by Wolfe & Gutierrez [11], we take the maximum pressure drop to be 1 atmosphere (p_{atm}). Thus, in this high-$\mathcal{W}$, high-$\mathcal{F}$ limit, one should observe two distinct types of cavity collapse depending on whether $\rho_a U_0^2 \ll p_{atm}$ or $\rho_a U_0^2 \gg p_{atm}$. In the first regime, the impact may be characterized by the air-liquid density ratio $\tilde{\mathcal{D}} = \rho_a/\rho$, and in the second, by the product of the air-liquid density ratio and the Euler number $\mathcal{E} = p_{atm}/(\rho_a U_0^2)$.

The influence of aerodynamic pressure on the evolution of water-entry cavities has been considered by several authors. Gilbarg & Anderson [12], Richardson [13], Birkhoff & Isaacs [14], May [15], and Abelson [16] investigated experimentally the cavity dynamics of high-speed projectiles, and offered some

[1] $\rho/\rho_a \approx 800$ at standard temperature and pressure

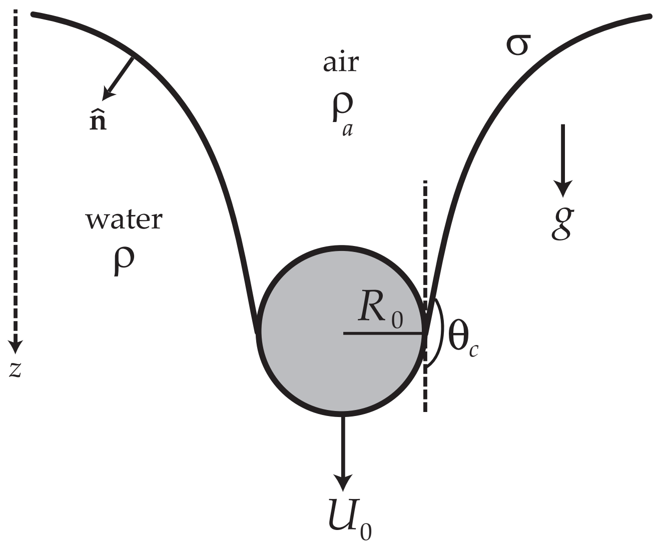

Figure 1. Schematic of the impact parameters. The angle at which the cavity detaches from the sphere, θ_c, is referred to as the cone angle.

explanation for the observed cavity shapes and pinch-off characteristics. A sophisticated model for the cavity dynamics of high $\mathcal{F}$ impacts ($\mathcal{F} > 150$) was developed by Lee *et al.* [10] by approximating the combined effect of the projectile and the cavity on the fluid motion using distributed point sources along the vertical axis. At a given depth, their model predicts the cavity evolution only when the cavity diameter exceeds that of the projectile. Nonetheless, Lee *et al.* rationalize the observations of Gilbarg & Anderson [12] regarding the apparent independence of the dimensionless pinch-off time on the impact speed.

An alternative model for the cavity dynamics, based on the solution to the Rayleigh-Besant problem [17, 18] was introduced by Duclaux *et al.* [19], and extended by Aristoff & Bush [9] to account for aerodynamic pressure in the limit $\rho_a U_0^2 \ll p_{atm}$. A similar approach will be adopted in the present study, where we shall reexamine this limit as well as its opposite, $\rho_a U_0^2 \gg p_{atm}$. In particular, we shall rationalize the observations of May [15] regarding the dependence of the pinch-off time on the air density.

THEORETICAL MODEL

When a projectile, say a sphere, is shot vertically into water, it creates an axisymmetric cavity that expands radially before closing under the combined influence of hydrostatic pressure, surface tension, and aerodynamic pressure. The evolution of the water-entry cavity is amenable to analytical treatment if one assumes a purely radial motion, $ru = R\dot{R}$, initiated by the passing of the sphere and prescribed by that of the cavity walls having radial speed $\dot{R}(t,z)$, where r is the radial coordinate and u the radial component of the liquid velocity. Using the corresponding velocity potential, together with the Bernoulli equation, Duclaux *et al.* [19] obtained an approximate expression for the evolution of the cavity wall $R(t,z)$ at depth z:

$$\frac{\rho}{2}\left(\frac{d^2(R^2)}{dt^2}\right) = -p(R,z), \tag{1}$$

where $p(R,z)$ is the pressure in the liquid at the cavity boundary that resists the inertial expansion of the cavity and eventually leads to its collapse.

The pressure at the cavity boundary may be separated into three components. The first is the hydrostatic pressure, $\rho g z$, that increases with depth. The second is the curvature pressure, $\sigma(\nabla \cdot \hat{\mathbf{n}})$. The third is the aerodynamic pressure that is due to the flow of air into the cavity behind the projectile. By neglecting any unsteadiness in the air flow, we may approximate the aerodynamic pressure as $C_a \rho_a U_0^2$, where C_a is assumed to be a constant. This assumption is consistent with previous experiments that found that $7.5 < C_a < 10$ and no appreciable pressure gradients arose within the cavity over a substantial range of impact speeds [16]. However, this expression for the aerodynamic pressure is valid only when it does not exceed 1 atmosphere. If $C_a \rho_a U_0^2$ exceeds p_{atm}, the aerodynamic pressure should be limited to this value according to the assumed choked-flow condition. By explicitly including these pressures in (1), we obtain

$$\frac{\rho}{2}\left(\frac{d^2(R^2)}{dt^2}\right) = -\rho g z - \sigma(\nabla \cdot \hat{\mathbf{n}}) - \min\left(C_a \rho_a U_0^2, p_{atm}\right). \tag{2}$$

In what follows we non-dimensionalize lengths by R_0 and time by R_0/U_0, so that (2) reduces to

$$\frac{1}{2}\left(\frac{d^2(R^2)}{dt^2}\right) = -\frac{z}{\mathcal{F}} - \frac{(\nabla \cdot \hat{\mathbf{n}})}{\mathcal{W}} - \mathcal{P}, \tag{3}$$

where $\mathcal{P} = \min\left(C_a \tilde{\mathcal{D}}, \mathcal{E}\tilde{\mathcal{D}}\right)$.

The boundary conditions for (3) are provided by the sphere trajectory: $R(t=0) = 1$ and $\dot{R}(t=0) = \sqrt{\alpha}\, U(z)$, where $U(z)$ is the dimensionless sphere speed when its center is at depth z. The parameter α is related to the cavity cone angle, θ_c (see figure 1), by geometry, $\alpha = \cot^2\left(\theta_c - \frac{\pi}{2}\right)$, and is taken to be constant. This is consistent with our experimental observations at low $\mathcal{F}$ [20], as well as those of May [15] at high $\mathcal{F}$. The pinch-off time is the minimum time over depths $0 < z < \infty$ of the cavity collapse:

$$t_{pinch} = \min_{0<z<\infty} \left(t(z) + t_c(z)\right) \tag{4}$$

where $t(z)$ is the time taken for the sphere to arrive at depth z, and $t_c(z)$ is the collapse time for a particular depth. We note that $t = 0$ corresponds to $z = 0$.

Using (3), and (4), Duclaux *et al.* [19] obtained expressions for the pinch-off time and depth in the limit where cavity collapse is influenced primarily by hydrostatic pressure, corresponding to $\mathcal{B} = \mathcal{W}/\mathcal{F} \gg 1$ and $1 \ll \mathcal{F} \ll \tilde{\mathcal{P}}^{-1}$. In this limit, cavity collapse is favored at depth. Aristoff & Bush [9] obtain analogous expressions in the limit where cavity collapse is influenced primarily by surface tension, corresponding to $\mathcal{B} = \mathcal{W}/\mathcal{F} \ll 1$ and $1 \ll \mathcal{W} \ll \tilde{\mathcal{P}}^{-1}$, where pinch-off is shallow. The influence of aerodynamic pressure on the cavity dynamics was briefly considered by Aristoff & Bush [9] using a similar theoretical model, but only for the case in which the pressure drop did not exceed 1 atmosphere.

Here we consider in detail the regime in which collapse is influenced primarily by aerodynamic pressure, corresponding to the limit $\mathcal{W} \gg \mathcal{P}^{-1}$ and $\mathcal{F} \gg \mathcal{P}^{-1}$, in which surface seal precedes deep seal. Since the parameter that we use to characterize the aerodynamic pressure, $\mathcal{P} = \min\left(C_a\tilde{\mathcal{D}}, \mathcal{E}\tilde{\mathcal{D}}\right)$, does not depend on the evolution of the cavity walls, we may consider both cases $C_a\rho_a U_0^2 < p_{atm}$ and $C_a\rho_a U_0^2 > p_{atm}$ simultaneously. Integrating (3) gives an expression for the evolution of the cavity radius:

$$R(t,z) = \sqrt{1 + 2\sqrt{\alpha}Ut - \mathcal{P}t^2}. \tag{5}$$

At time τ, the cavity profile is thus defined parametrically by

$$R(t) = \sqrt{1 + 2\sqrt{\alpha}U(\tau - t) - \mathcal{P}(\tau - t)^2}, \tag{6}$$

$$z(t) = \int_0^t U(t')dt', \tag{7}$$

for $0 \leq t \leq \tau$. Using (5), we find the maximum radial extent of the cavity

$$R_{max} = \sqrt{1 + \frac{\alpha U^2}{\mathcal{P}}}, \tag{8}$$

and the collapse time,

$$t_c(z) = \frac{\sqrt{\alpha}U + \sqrt{\alpha U^2 + \mathcal{P}}}{\mathcal{P}}. \tag{9}$$

Further insight into the cavity dynamics may be obtained by taking the sphere speed to be constant over the time scale of cavity collapse. Provided that $|U_0 - U_0 \cdot U(z)|/U_0 \ll 1$, we may approximate $U(z) \approx 1$ and combine (6) and (7) to obtain an expression for the cavity profile:

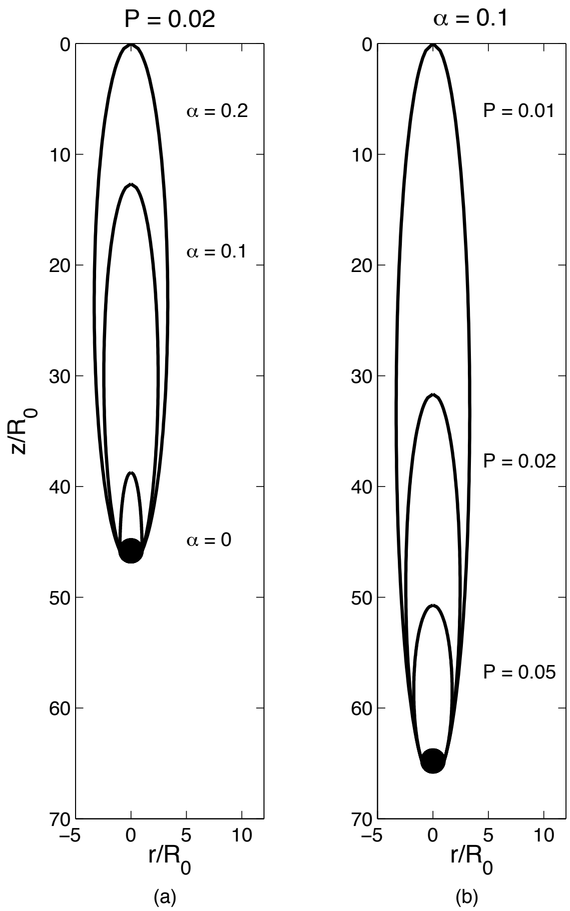

Figure 2. Theoretically predicted cavity profiles, given by (10) for (a) fixed $\mathcal{P} = 0.005$, varying α, and (b) fixed $\alpha = 0.1$, varying $\mathcal{P}$.

$$R(z) = \sqrt{1 + 2\sqrt{\alpha}(\tau - z) - \mathcal{P}(\tau - z)^2}. \tag{10}$$

In figure 2, we plot the predicted cavity profile, given by (10), for different values of α and $\mathcal{P}$.

In this constant speed limit, the collapse time given by (9) is depth-independent, so pinch-off occurs where the cavity was first initiated, at $z = 0$, and the dimensionless pinch-off time may be written as

$$t_{pinch} = \frac{\sqrt{\alpha} + \sqrt{\alpha + \mathcal{P}}}{\mathcal{P}} = \begin{cases} 2\sqrt{\alpha}\mathcal{P}^{-1} & \text{for } \alpha/\mathcal{P} \gg 1 \\ \mathcal{P}^{-1/2} & \text{for } \alpha/\mathcal{P} \ll 1. \end{cases} \tag{11}$$

The theoretically predicted pinch-off time is found without considering the dynamics of the splash curtain, that may seal the cavity from above prior to its pinching off. Nevertheless, we expect (11) to be an upper bound for the pinch-off time. We note that a simple balance between inertia and atmospheric pressure, $\rho U^2 \sim \min(C_a \rho_a U_0^2, p_{atm})$, leads to the dimensionless pinch-off time scaling $t_{pinch} \sim \mathcal{P}^{-1/2}$, which is retained when $\alpha \ll \mathcal{P}$ in (11). In this limit, the cavity collapses without initially expanding. Impacting projectiles, however, transfer momentum into the cavity, that necessarily expands owing to fluid inertia.

DISCUSSION

To test the applicability of our expression for the pinch-off time (11) in the limit for which $C_a \rho_a U_0^2 \ll p_{atm}$, we refer to a previous experimental study on the influence of the air-liquid density ratio on the water-entry cavity. May [15] recorded the time of surface closure (either by the splash doming over or by the cavity pinching off) for the water entry of half-inch diameter steel spheres for the range $2 \cdot 10^3 < \mathcal{F} < 10^6$. In figure 3, we recast May's data alongside (11), given by the black curve, where we estimate $\alpha = 0.05$ from published photographs. Good agreement is obtained by choosing $C_a = 40$, a value that is roughly consistent with those measured by Abelson [16]. Our model is not highly sensitive to the choice of C_a, as evidenced by the upper and lower dash-dotted curves, that are given, respectively, by (11) for $C_a = 20$ and $C_a = 75$. A scaling proposed by Birkhoff & Isaacs [14], $t_{pinch} \sim \tilde{\mathcal{D}}^{-1}$, based on a purely dimensional argument, is also shown. The variation of the pinch-off time for a given density ratio is small relative to the variation in $\mathcal{F}$. This observation suggests that the pinch-off time is roughly independent of the impact speed, and is consistent with (11). Experimental data is not available to test our theoretical predictions in the limit for which $C_a \rho_a U_0^2 \gg p_{atm}$.

Owing to the pressure drop inside the cavity being limited to one atmosphere, we expect two distinct dependencies of the dimensional pinch-off time, t'_{pinch}, on the impact speed. For $U_0 < U_s$, (11) predicts the dependence $t'_{pinch} \sim U_0^{-1}$, and for $U_0 > U_s$, the dependence $t'_{pinch} \sim U_0$. We note that these trends are compatible with those predicted by Lee *et al.* [10], and we have taken the limit relevant to impacting spheres: $\alpha/\mathcal{P} \gg 1$.

In our comparison between experiment and theory, we have directly measured the cone angle θ_c from available photographs, and so inferred the value of α. Alternatively, one may express α in terms of the drag coefficient, C_d, by equating the energy lost via form drag, $\frac{1}{2}\rho U^2 C_d \pi R^2 dz$, to the energy of the radially expanding fluid layer, which is given by:

$$\frac{1}{2}\int_R^{R_\infty} 2\pi r \rho u^2 dr dz = \rho \dot{R}^2 \pi R^2 dz. \tag{12}$$

In writing (12), we have followed Duclaux *et al.* [19] by assuming that the radial fluid motion extends over a region comparable

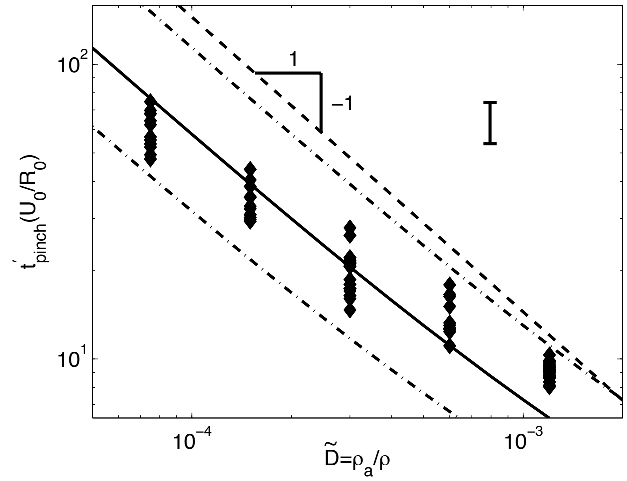

Figure 3. Dimensionless pinch-off time versus air-liquid density ratio $\tilde{\mathcal{D}}$ for the water entry of half-inch diameter steel spheres for the range $2 \cdot 10^3 < \mathcal{F} < 10^6$. Data reproduced from May [15], where we estimate $\alpha = 0.05$ from published photographs and take $\rho_a = 1.2$ kg m^{-3}. The solid curve denotes the theoretically predicted pinch-off time, and is given by (11) for $C_a = 40$. The upper and lower dash-dotted curves are given, respectively, by (11) for $C_a = 20$ and $C_a = 75$. The scaling proposed by Birkhoff & Isaacs [14] is given by the dashed line. A characteristic error bar is shown.

to the size of the cavity ($R_\infty \approx 2.7R$). Since $\dot{R} = \sqrt{\alpha}U$ when $R = 1$, we find that α is proportional to the drag coefficient:

$$\alpha = \frac{C_d}{2}. \tag{13}$$

Therefore, our assumption of constant α is consistent with the choice of velocity potential, provided that C_d is also constant. For a discussion of the drag on an impacting body, see Aristoff *et al.* [20], where reasonable agreement between experiment and theory is obtained by taking C_d to be a constant, corresponding to its mean value over the time scale of cavity collapse.

CONCLUSION

We have presented the results of a theoretical investigation of the cavity dynamics of water entry. Particular attention has been given to the regime in which the cavity evolution is influenced primarily by aerodynamic pressure that has characteristic magnitude $10\rho_a U_0^2$, but is limited to one atmosphere owing to the air flow becoming choked. A theoretical model, developed to describe the cavity dynamics, yields expressions for the cavity profile and pinch-off time in this high-$\mathcal{F}$, high-$\mathcal{W}$ limit. Additional comparisons with experiments are needed to determine the range of validity of our theoretical model. Discrepancies between experimental observations and theoretical predictions may arise from the neglect of the dynamics of the splash curtain, that may seal the cavity from above prior to pinch-off, thereby altering the cavity evolution. In addition, the two-dimensional geometry of the cavity obliged us to approximate the radial extent of the fluid motion [19]; shortcomings of this approximation are discussed by Bergmann *et al.* [21]. Finally, cavitation has been observed during water entry [22], and is known to affect the drag on an underwater body [23], yet its role in the water-entry problem remains unexplored.

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