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COLLEGE OF ENGINEERING

Department of Chemical and Metallurgical Engineering

Progress Report

A MATHEMATICAL MODEL FOR THE POPPET NOZZLE

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UMRI Project 2931

under contract with:

DELAVAN MANUFACTURING COMPANY

WEST DES MOINES, IOWA

administered by:

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE   ANN ARBOR

September 1959

NOMENCLATURE

Consistent Units

a_o radius of pintle
A area
b_o radius of sleeve
f_N frequency of pintle movement calculated from numerical solution of differential equation
f_L frequency calculated from approximate analytical solution derived from assumption of linear force vs. x relationship
F_o initial spring loading
F force on pintle
k spring constant
m mass of pintle
P_x force exerted on fluid by pintle
p pressure
S dF/dx, slope of F vs. x curve
t time
x position of pintle
x_e equilibrium position of pintle $[F(x_e) = 0]$
u liquid velocity

Subscripts

$_1$ cross section 1 (see Fig. 1)
$_2$ cross section 2 (see Fig. 1)

Greek

θ angle of conical pintle head
ρ density of flowing liquid

I. INTRODUCTION

The poppet nozzle, as is well known, is a variable-area flow device which ideally provides an orifice area proportional to the fluid pressure driving force. This area has been found to vary not only with pressure but also with time, at a fixed pressure, due to the rapid oscillation of the pintle. The purpose of the current research is threefold: (a) to develop a working method of calculation to yield the amplitude and frequency of the pintle vibration as a function of design and operational variables; (b) to evaluate the method by comparing the calculated performance of a nozzle with actual data; and (c) to apply the method in engineering design and simulation of performance of suggested nozzles, thus reducing the necessity for building and testing of new designs.

This report presents results pertinent to item (a) above. The calculated performances for several cases are given and discussed. The derivation of the method of calculation is given in Sections III and IV.

The next phase is experimental testing of several nozzles, both commercial and specially designed, for direct comparison of data and the results of the mathematical model. This will serve to improve the model and at the same time to obtain operational data on mechanical problems, such as sticky operation, resonance, uneven spray, etc.

II. RESULTS AND CONCLUSIONS

Figure 1 is a sketch of the poppet nozzle giving the design dimensions needed in the method of calculation derived in Sections III and IV.

The following values of the geometric and design parameters were used in the calculations:

a_o = .055 in.
b_o = .07 in.
θ = as noted on figures
k = spring constant, as noted on figures
m = mass of pintle, as noted on figures
F_o = spring compression loading when pintle is closed ($x = 0$) = .02 lb force
P_1 = fluid pressure at cross section 1 (see Fig. 1) = 120 psia
P_2 = fluid pressure at cross section 2 = 14.7 psia

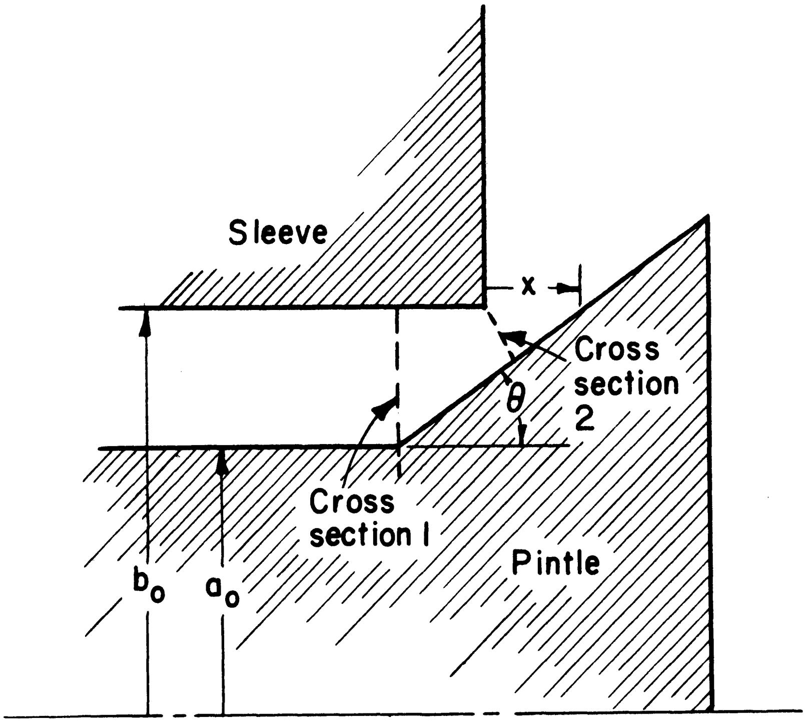

Fig. 1. Sketch of poppet.

Figure 2 shows the calculated force on the pintle as a function of pintle position for the indicated values of k and θ. The differential equation describing the motion of the pintle is given by Newton's Law:

$$m \frac{d^2x}{dt^2} = F(x) \tag{1}$$

where F(x) represents the total force on the pintle at the location defined by the dependent variable x. The quantity m is the mass of the pintle and accessory moving parts in slugs. Figures 3, 4, and 5 give the pintle position as calculated from the numerical solution of the differential equation (1). The instantaneous flow rates (gal./min.) are also plotted. In these calculations all fluid properties used were those of water.

The "lost work" or frictional energy dissipated in the orifice of the nozzle was neglected as a first approximation. Approximate calculations indicate that the lost work is small but not necessarily negligible. Effort is currently being made to take the lost work into account.

Figure 2 shows that for a given geometry of nozzle an increase in spring constant gives a more linear relationship between force on the pintle and distance x. In earlier work, before the force on the pintle was calculated, William Graessley (UMRI 2815-2-P) assumed the force on the pintle to be linear with x and obtained an analytical solution to Eq. (1):

$$x = x_e \left(1 - \cos \sqrt{\frac{-S}{m}}\, t \right) \tag{2}$$

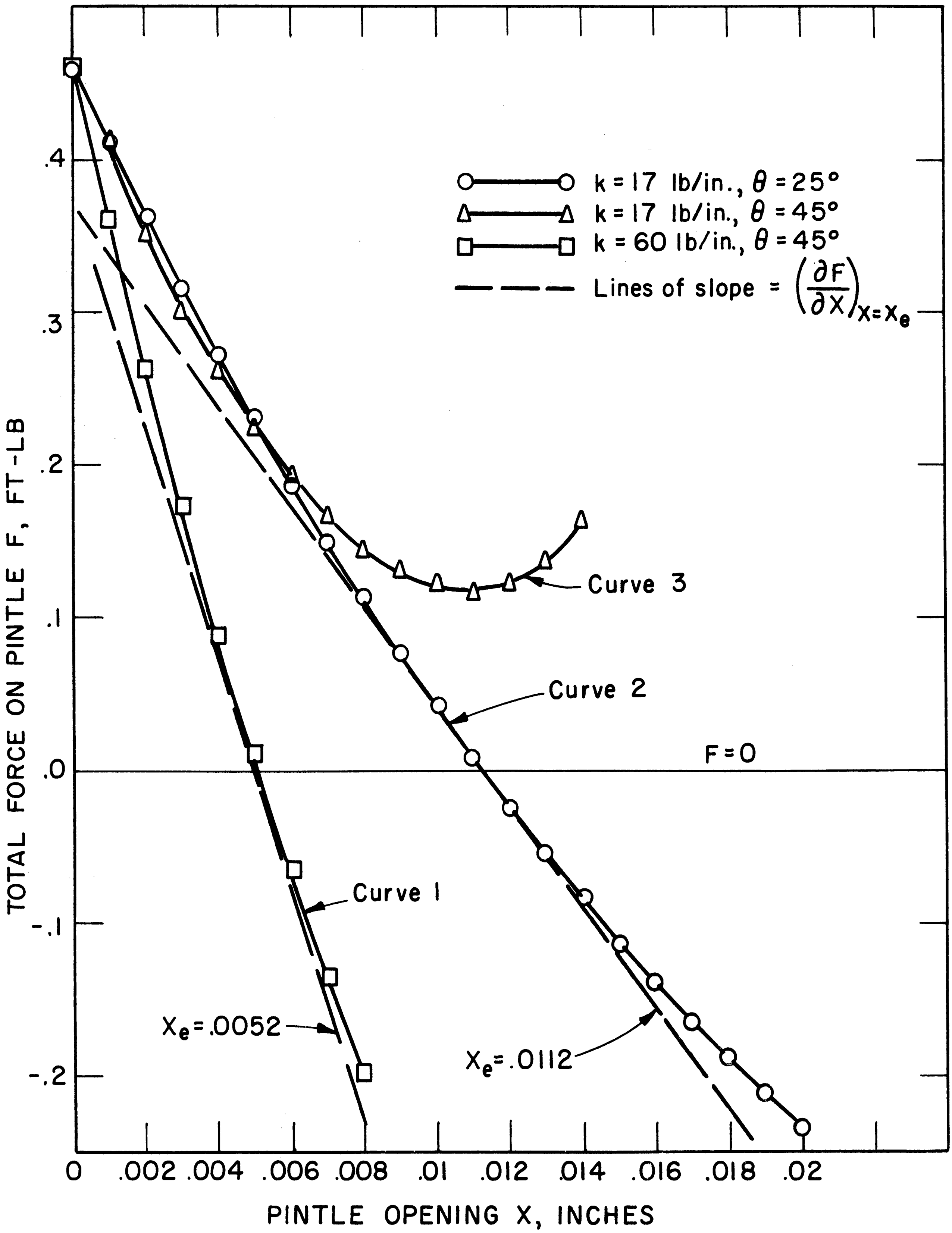

Fig. 2. Force on pintle.

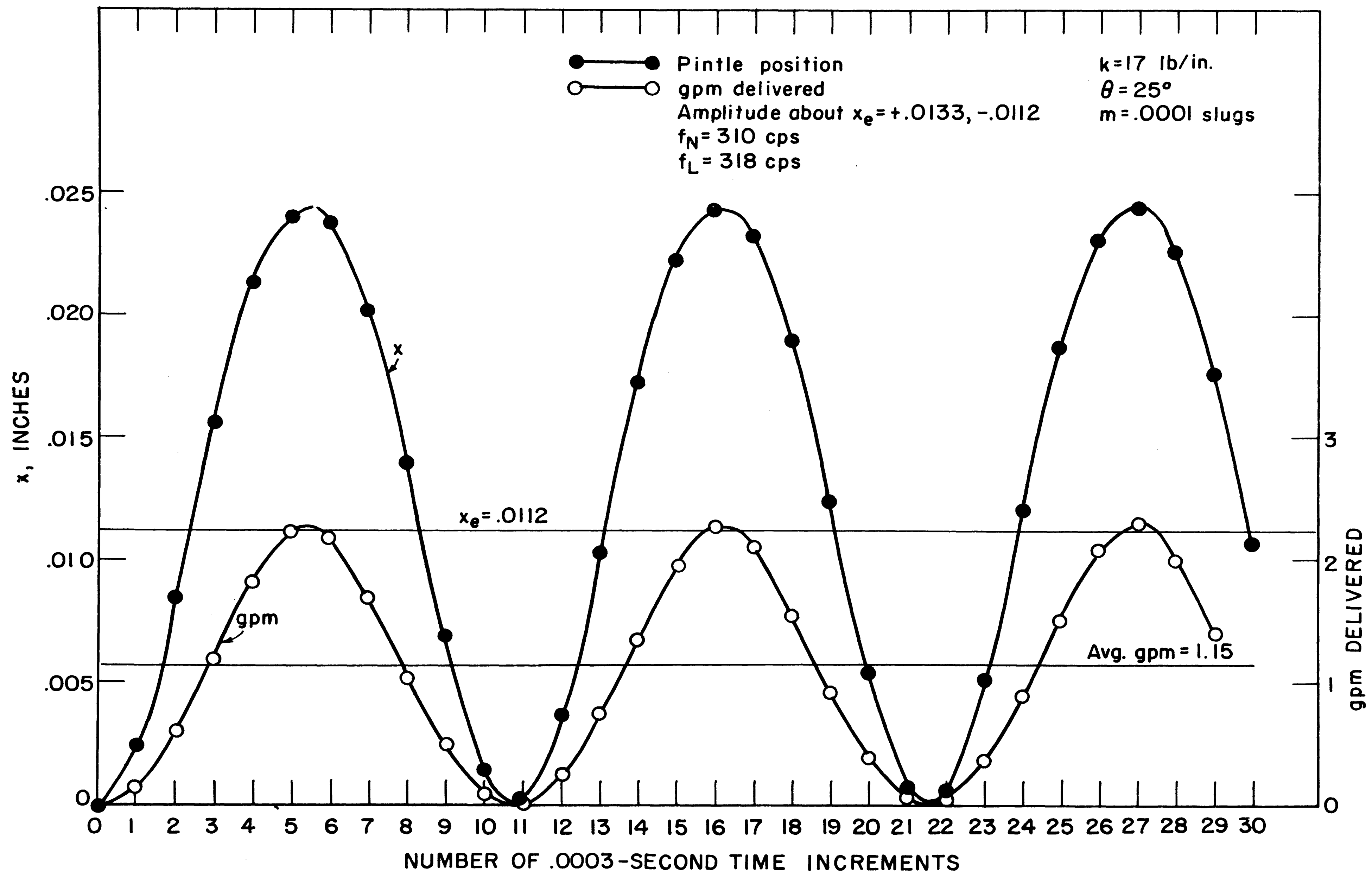

Fig. 3. Pintle movement from numerical solution.

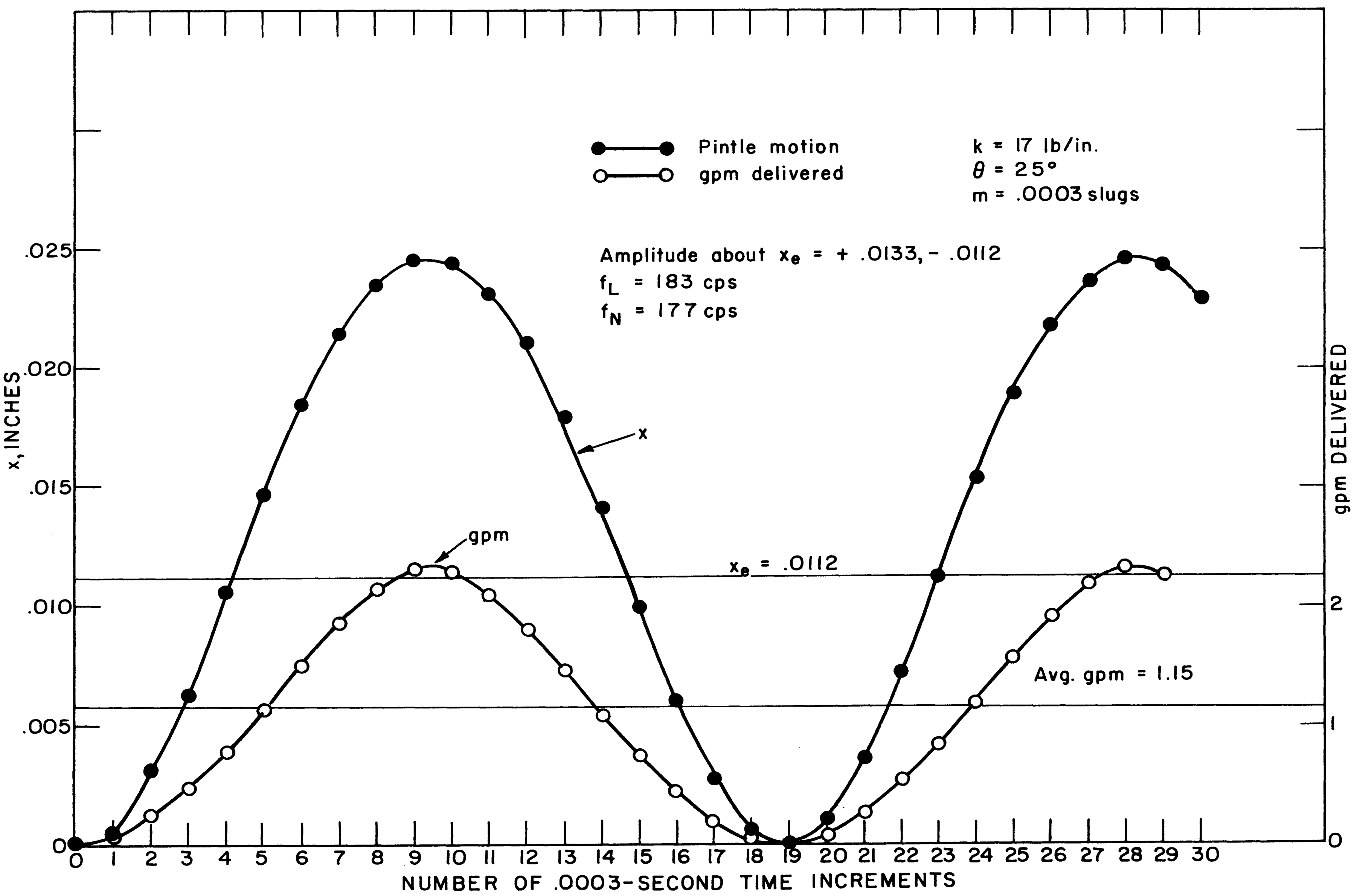

Fig. 4. Pintle movement from numerical solution.

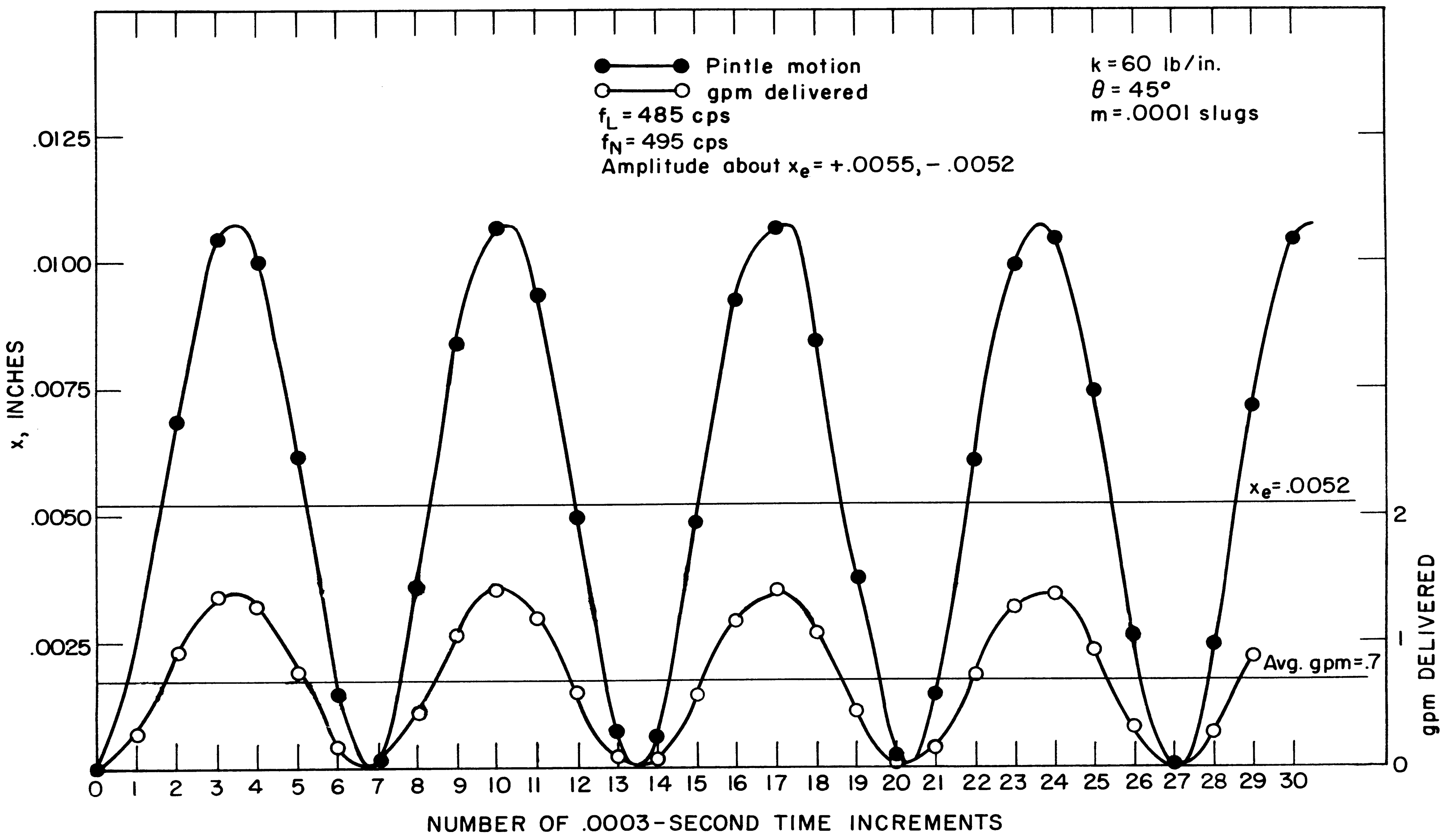

Fig. 5. Pintle movement from numerical solution of Eq. (1).

where S is the slope of the F vs. x relation (dF/dx) and x_e is the equilibrium position of the pintle, i.e., the value of x for which F(x) = 0. Equation (2) yields the frequency of vibration as

$$(f)_L = \frac{1}{2\pi}\sqrt{\frac{-S}{m}} \tag{3}$$

This frequency, f_L, has been calculated for each of the cases shown in Figs. 3, 4, and 5 and is given on those figures. The frequencies determined by numerical solution of the equations developed in Section III and the frequencies of Eq. (3) are seen to lie in close agreement. This agreement is a consequence of the nearly linear relationship between F and x for the cases treated (see Fig. 2). It should be noted that for combinations of parameter values other than those used here, the F vs. x relation may not be as linear and the two solutions may then be in considerable disagreement.

If all parameters except the pintle mass, m, are held constant, then the frequency f is approximately proportional to $1/\sqrt{m}$. That is,

$$\frac{f_1}{f_2} \cong \sqrt{\frac{m_2}{m_1}} \tag{4}$$

Equation (4) follows as an exact relationship from Eq. (3), which in turn is valid only for a linear F vs. x curve. The results plotted in Figs. 3 and 4 are for identical cases except that the pintle mass for the case in Fig. 4 is three times that for the case in Fig. 3. The frequencies 310 cps and 177 cps are in close agreement with Eq. (4):

$$\frac{310}{177} = 1.750 \quad ; \quad \sqrt{\frac{0.0003}{0.0001}} = 1.732$$

Note that for the cases involved, F is not a linear function of x (see curve 2, Fig. 2).

Comparison of curves 2 and 3 of Fig. 2 appears quite interesting, but we hesitate to make any generalizations until more information is available.

The relationship between flow rate and pressure difference was investigated by caluclating the average gpm delivered from a nozzle operating at various overall pressure differences, $p_1 - p_2$. Equation (27) below gives the instantaneous flow rate from which the time-average flow rate can be obtained as shown in Figs. 3, 4, and 5. Figure 6 shows the average flow rates for a nozzle having the indicated parameter values. The flow rate q is seen to increase with Δp at a rate greater than that corresponding to a straight-line curve. It must be remembered that this analysis does not include frictional losses, which will modify the curve somewhat.

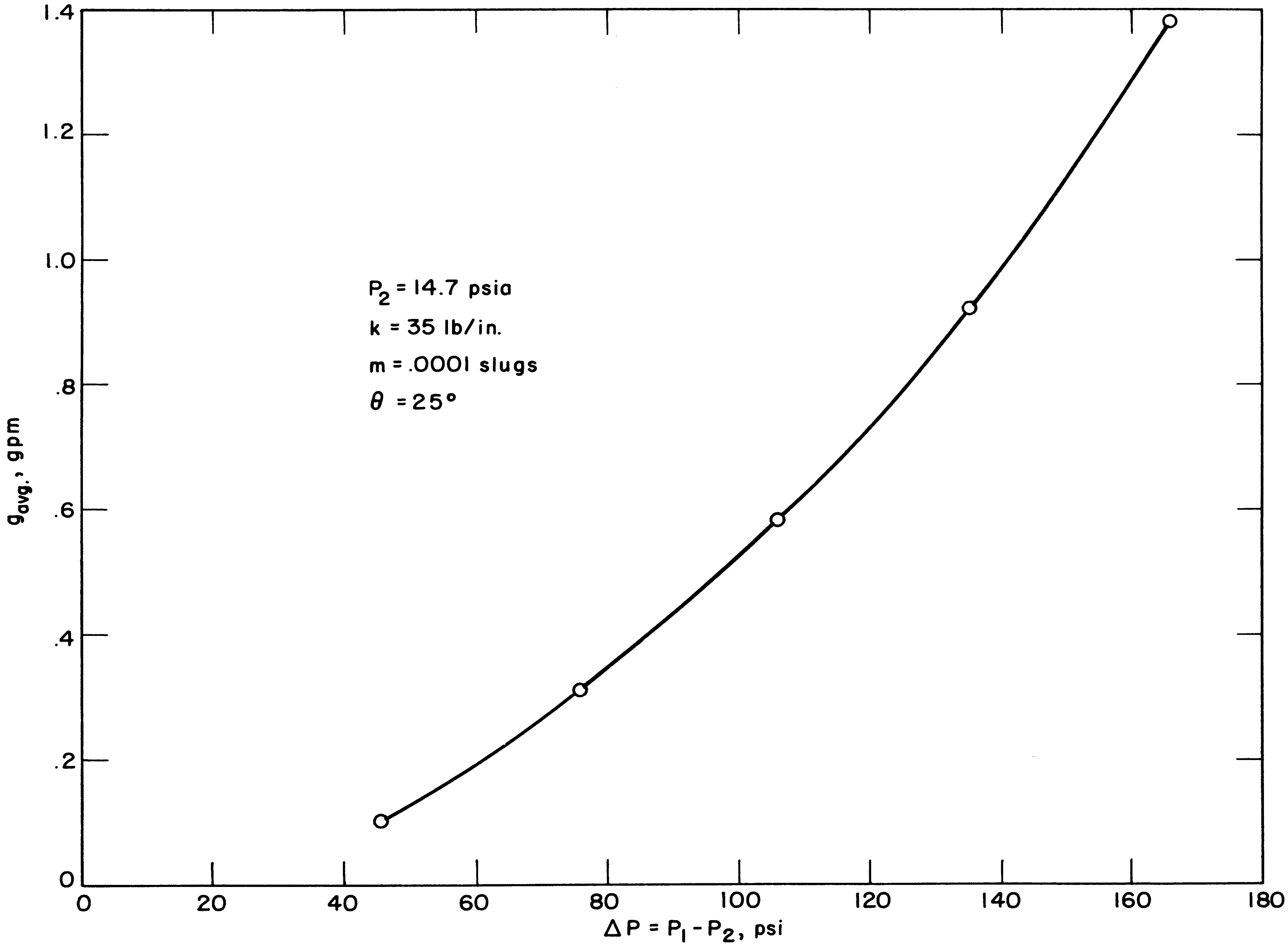

Fig. 6. Flow rate vs. ΔP for poppet nozzle.

SUMMARY OF PROGRESS TO DATE

To summarize the progress to date in the study of variable-area devices, the following may be listed:

(1) An analytical expression has been developed which relates the force on the pintle to various nozzle dimensions, the spring constant, and fluid pressures.

(2) A stable numerical method has been developed which allows solution of the differential equation governing pintle motion, regardless of whether the force is linear with x.

(3) The flow-rate curve was calculated and showed that q increased with Δp at a rate greater than that corresponding to a straight-line curve.

(4) For cases where the force on the pintle is approximately linear with x, the frequency of pintle vibration can be calculated from Eq. (3) above. The amplitude of vibration in these cases is $\pm x_e$ inches on either side of the position $x = x_e$, where x_e is the equilibrium $[F(x_e) = 0]$ position of the pintle.

FUTURE WORK

Future work will involve development of a method of calculation which accounts for frictional energy losses, obtaining amplitude and frequency data on an actual nozzle and comparison of the data with the calculated performance.

III. DEVELOPMENT OF EQUATIONS

An expression giving the force on the pintle as a function of the pintle position, the differential equation governing the pintle motion and the approximate solution to this equation given by W. Graessley are included in the following.

A. FORCE ON THE PINTLE

Figure 7 below shows the forces acting on the free body of liquid between sections 1 and 2. The force P_x is the total force exerted on the liquid by the pintle in the -x direction. P_x includes integrated normal and shear stresses on the lateral sides CC' and BB' of the free body of revolution around the pintle axis.

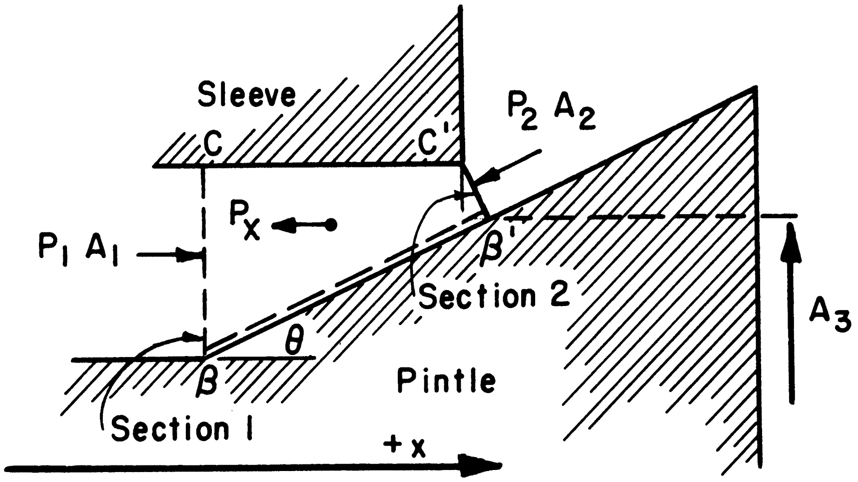

Fig. 7. Forces on free body of liquid.

The momentum equation[1]

$$\sum F_x = \frac{1}{g_c}\frac{dm}{dt}(U_{x_2} - U_{x_1}) \tag{5}$$

equates the sum of the forces in the x direction to the product of the mass flow rate and change in the x component of the velocity. This equation becomes

$$p_1A_1 - p_2A_2 \cos \Theta - P_x = \frac{1}{g_c} U_1A_1\rho\left(\frac{U_1A_1}{A_2}\cos \Theta - U_1\right) \tag{6}$$

where

$\frac{dm}{dt}$ = mass flow rate = $U_1A_1\rho$

$p_2A_2 \cos \Theta$ = component of p_2A_2 force in -x direction

$U_{x_2} = \frac{U_1A_1}{A_2}\cos\Theta$ = velocity component in x direction at section 2.

Equation (6) is valid whether or not frictional drag forces are considered.

The velocity U_1 can be calculated from Bernoulli's equation:

$$\frac{U_1^2}{2g_c} + \frac{p_1}{\rho} = \frac{U_2^2}{2g_c} + \frac{p_2}{\rho} + lw \tag{7}$$

and the continuity equation

[1]See page 105 of Streeter, V. L., Fluid Mechanics.

$$U_2 = \frac{U_1 A_1}{A_2} \tag{8}$$

as

$$U_1 = \sqrt{2g_c \left(\frac{p_1 - p_2}{\rho} - lw\right) \frac{1}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \tag{9}$$

When friction is ignored, lw becomes negligible in Eq. (9) and

$$U_1^2 = 2g_c \frac{p_1 - p_2}{\rho} \frac{1}{\left(\frac{A_1}{A_2}\right)^2 - 1} \tag{10}$$

Equation (6) can be rewritten

$$P_x = p_1 A_1 - p_2 A_2 \cos \theta - \frac{1}{g_c} A_1 \rho\, U_1^2 \left(\frac{A_1}{A_2} \cos \theta - 1\right) \tag{11}$$

which, upon substitution of U_1^2 from (10), becomes

$$P_x = p_1 A_1 - p_2 A_2 \cos \theta - 2A_1 (p_1 - p_2)\left(\frac{A_1}{A_2} \cos \theta - 1\right) \frac{1}{\left(\frac{A_1}{A_2}\right)^2 - 1} \tag{12}$$

The total force, F, on the pintle is P_x plus the spring force plus the down-stream pressure (p_2) force exerted over the area A_3 (see Fig. 7) on the end of the conical pintle head and minus the hydraulic or ambient pressure force on the area πa_o^2 at the back of the pintle. Thus

$$F = P_x - p_2 \pi (b_o - x \sin \theta \cos \theta)^2 - p \pi a_o^2 - F_o - kx \tag{13}$$

where the spring force = $F_o + kx$ and the pressure p at the back of the pintle is either p_1 or p_2. The areas A_1 and A_2 are given by

$$A_1 = \pi (b_o^2 - a_o^2) \tag{14}$$

$$A_2 = \pi \cos \theta \,(2b_o x \tan \theta - x^2 \tan^2 \theta) \tag{15}$$

B. DIFFERENTIAL EQUATION GOVERNING PINTLE MOTION

From Newton's law we have for the pintle

$$F = ma$$

or

$$m \frac{d^2x}{dt^2} = F \tag{16}$$

where m is the mass of the pintle and F is a known function of x [Eq. (13)].

Graessley's approximate solution to (16) involved the assumption of small displacement of the pintle about its equilibrium position x_e,

$$F\,(x_e) \doteq 0 \quad .$$

Thus he represented F(x) by a truncated Taylor's series

$$F\,(x) = F\,(x_e) + (x - x_e)\left(\frac{\partial F\,(x)}{\partial x}\right)_{x\,=\,x_e} + 0\,(x - x_e)^2$$

or, letting $y = x - x_e$ and $S = \left(\frac{\partial F}{\partial x}\right)_{x\,=\,x_e}$,

$$F\,(y) = y\left(\frac{\partial F}{\partial y}\right)_{y\,=\,0} = yS \tag{17}$$

Equation (16) becomes

$$m\,\frac{d^2y}{dt^2} = yS \tag{18}$$

The solution to (18) for the initial conditions

$$y\,(0) = -x_e \qquad \text{(nozzle closed initially)}$$

$$\left(\frac{dy}{dt}\right)_{t\,=\,0} = 0 \qquad \text{(initial 0 velocity)}$$

is

$$y\,(t) = -x_e \cos\sqrt{\frac{-S}{m}}\,t \tag{19}$$

or

$$x = x_e\left(1 - \cos\sqrt{\frac{-S}{m}}\,t\right) \tag{20}$$

The solution (20) yields an oscillation having amplitude $\underline{x_e}$ on either side of the equilibrium position $x = x_e$ and frequency of $1/2\pi\,\sqrt{-S/m}$. The validity of Graessley's solution rests upon the assumption that $\partial F/\partial x$ is constant (i.e., F is a linear function of x) over the entire range of pintle movement.

IV. NUMERICAL SOLUTION OF DIFFERENTIAL EQ. (14)

Equation (16) can be solved numerically without any assumptions concerning the nature of the F vs. x relationship. Letting x_n = the value of x at time $t = n\Delta t$, one can express the derivative d^2x/dt^2 in finite difference form

$$\frac{d^2x}{dt^2} \cong \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2}$$

Equation (16) then becomes

$$x_{n+1} - 2x_n + x_{n-1} = \frac{\Delta t^2}{m}\,F\,(x_n) \tag{21}$$

The two initial conditions

$$x(0) = 0$$

$$\left(\frac{dx}{dt}\right)_{t=0} = 0$$

become, in finite difference form,

$$x_0 = 0 \tag{22}$$

$$x_1 = x_{-1} \tag{23}$$

From (21), (22), and (23) we have

$$x_1 = \frac{(\Delta t)^2}{2m} F(x_0) \tag{24}$$

$$x_{n+1} = 2x_n - x_{n-1} + \frac{(\Delta t)^2}{m} F(x_n) \tag{25}$$

Equation (24) can be solved immediately since the right side can be calculated. Equation (25) can then be solved with n = 1, 2, 3, ... to yield x_2, x_3, x_4, etc. The equations immediately above were programmed in the GAT compiler language and the computations were then performed by the IBM 650 digital computer to yield the results given in Figs. 2-5.

CALCULATION OF FLOW RATE

The gallons per minute delivered by the nozzle can be calculated as

$$\text{gpm} = U_1 \frac{A_1}{144} \frac{\text{ft}^3}{\text{sec}} \cdot 60 \frac{\text{sec}}{\text{min}} \cdot 7.48 \frac{\text{gal}}{\text{ft}^3} = 3.11\, U_1 A_1 \tag{26}$$

Substitution of U_1 from Eq. (9) into (26) and insertion of proper unit conversion factors yields

$$\text{gpm} = 3.11\, A_1 \sqrt{\frac{288\, g_c\, (p_1 - p_2)}{\rho} \frac{1}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \tag{27}$$