The Azukawa Metric and the Pluricomplex Green Function

by

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CHAPTER I

Introduction

The two main topics we study in this thesis are the Azukawa metric and the pluricomplex Green function. These two topics are closely related because the Azukawa metric is defined in terms of the pluricomplex Green function.

The Azukawa metric is an example of a biholomorphically invariant pseudometric. Recall that an invariant pseudometric on a domain Ω in $\mathbb{C}^n$ is a function $F^{\Omega}(p,\xi)$: $\Omega \times \mathbb{C}^n \to [0,\infty)$ which satisfies

$$F^{\Omega}(p,\lambda\xi) = |\lambda| F^{\Omega}(p,\xi)$$

for all $\lambda \in \mathbb{C}$, and under a biholomorphism $\Phi : \Omega \to \Omega'$ has the following behavior

$$F^{\Omega}(p,\xi) = F^{\Omega}(\Phi(p), \Phi'(p) \cdot \xi).$$

Some of the most well known examples of invariant metrics are the Carathéodory, Kobayashi and Bergman metrics. The Azukawa and Sibony metrics that we study are most closely related to the Carathéodory and Kobayashi metrics, whereas the Skwarczyński metric that we study is most closely related to the Bergman metric.

Invariant metrics are an important tool in Several Complex Variables, and more recently they have also been used in Complex Dynamics. In Section 2.5 we discuss many examples of applications of invariant metrics.

At the end of the thesis we have a short section where we use the Skwarczyński metric to study the Bergman space, which is the space of holomorphic L^2 functions on a domain. The Bergman space is a central class of functions and is widely studied.

The second main topic of this thesis, the pluricomplex Green function, is the analog in Several Complex Variables of the classical Green function related to the Laplace equation. The classical Green function is a tool used to solve the Laplace equation. It is also closely related to harmonic and subharmonic functions, and in particular it is a reproducing kernel for harmonic functions. The pluricomplex Green function is closely related to the complex Monge-Ampère operator and plurisubharmonic functions. One way to define the pluricomplex Green function is as a solution to the extremal problem

$$g_\Omega(z,a) = \sup\left\{u(z) : u \in \mathcal{K}(\Omega, a)\right\},$$

where $u \in \mathcal{K}(\Omega, a)$ if it satisfies the following three properties:

1. u is plurisubharmonic on Ω,
2. $u \leq 0$ on Ω,
3. and $u(z) - \log ||z - a|| < c(u)$ in a neighborhood of a, where $c(u)$ is a constant depending on u.

It can also be defined as the solution to the following Monge-Ampère problem: with Ω a domain in $\mathbb{C}^n$ and $z \in \Omega$, does there exist a function $u_z(w) : \overline{\Omega} \to [-\infty, 0]$ satisfying the conditions

1. $(dd^c u_z)^n = 0$ on $\Omega \setminus \{z\}$,

2. $u_z(w) \sim \log ||z - w||$ in a neighborhood of z,

3. u_z is continuous on $\overline{\Omega}$ and plurisubharmonic on Ω,

4. $u_z = 0$ on $\partial\Omega$.

We will study a particular generalization of the pluricomplex Green function proposed by Al Taylor. In the process we study the behavior of the Monge-Ampère operator on $\log ||F||$ near the singularities for a holomorphic function F, which is an interesting topic in its own right.

We will now discuss the specifics of each chapter. In Chapter II we have some background topics. First, we discuss the motivation behind generalizing the classical Green function to the pluricomplex Green function, and we define a the further generalized pluricomplex Green function that we will study in this thesis. Second, we outline the background for Resolution of Singularities, which is a difficult theorem we use in one of our proofs. Third, we discuss the basics of invariant metrics. Fourth, we discuss some necessary background results from pluripotential theory. Finally, we have a section listing many applications of invariant metrics.

In Chapter III we study the Azukawa and Sibony metrics. Our main theorem in Chapter III is that the Azukawa metric is upper semicontinuous.

Theorem I.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $u(z, w)$ be the pluricomplex Green function on Ω. Then $u(z, w)$ is upper-semicontinuous.*

We proved this result independently before discovering that Jarnicki and Pflug already published it in [20], and we still include it in this thesis because it represents work completed during the PhD. Next, we study the relationship between the closely related Azukawa and Sibony metrics on some explicit examples. These metrics are known to be different, and we construct an explicit example of a domain on the metrics are different.

Theorem I.2. *The Azukawa and Sibony metrics are different on the domain* $\Omega \subset \mathbb{C}^2$ *defined by*

$$f(z,w) := \frac{1}{4}\log|z| + \frac{1}{4}\log|w| + \frac{1}{4}\log|z-w| + \frac{1}{4}\log|z-iw|$$

$$\Omega = \{(z,w) : f(z,w) < 0\}$$

Finally, we study the hyperbolicity of the two metrics, and we construct a domain on which pointwise hyperbolicity of the metrics is different.

Theorem I.3. *There exists a pseudoconvex domain* $\Omega \subset \mathbb{C}^2$ *that is Azukawa pointwise hyperbolic but not Sibony pointwise hyperbolic. Define* Ω *to be*

$$\left\{(z_1,z_2) \in \mathbb{C}^2 : |(z_1,z_2)| < f(z_1,z_2)\right\},$$

where f *is defined as*

$$f(z_1,z_2) = f(z_1/z_2) = \begin{cases} (1+|z_1/z_2|^2)^{1/2} e^{-v(z_1/z_2)} & z_2 \neq 0 \\ 1 & z_2 = 0 \end{cases} \tag{1.1}$$

and $v(\lambda)$ *is defined as*

$$v(\lambda) = \max\left\{\log|\lambda|, \sum_{k=2}^{\infty} k^{-2}\log|\lambda - 1/k|\right\}$$

An interesting further direction would be to study non-pointwise hyperbolicity.

In Chapter IV we study the regularity properties of a generalized pluricomplex Green function. Recall that the pluricomplex Green function is the solution to an extremal problem using plurisubharmonic functions with logarithmic singularities, or the solution to a certain Monge-Ampère equation. There is a significant body of research on a number of different generalized Green functions. The author is grateful to Al Taylor for suggesting the particular generalized Green function in this thesis, along with the question of whether it is plurisubharmonic. The main result of Chapter IV is that this generalized Green function is plurisubharmonic when the zeros of its related holomorphic function are isolated.

Theorem I.4. *Let Ω be a domain in $\mathbb{C}^n$, and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros. Then the generalized Green function U is plurisubharmonic.*

In this section we also study a three simple cases where one can compare the generalized Green function to the usual pluricomplex Green function. The first two are based on the linear approximation of a holomorphic function.

Theorem I.5. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Then we can compare the generalized Green function U to the usual pluricomplex Green functions with poles at the zeros of $||F||$ as follows:*

$$U(z) \leq \inf_{\{w:||F(w)||=0\}} g_\Omega(z, w).$$

Theorem I.6. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Suppose that the rank of the complex Jacobian is n at each point w with $||F(w)|| = 0$. If there is a function u satisfying $u \in \cap_{\{w:||F(w)||=0\}} \mathcal{K}(\Omega, w)$ then $u \in \mathcal{T}(\Omega, F)$. In particular in the case where $||F||$ has a single zero, $g_\Omega(z, w) \leq U(z)$.*

The third is for the case of homogeneous polynomials.

Theorem I.7. *Let Ω be a domain in $\mathbb{C}^n$ that includes the origin, and let $F = (f_1, ..., f_m)$ be a polynomial function where f_i are all homogeneous polynomials of the same degree, d, and which have common zero only at the origin. Then $U(z) = d \cdot g_\Omega(z, 0)$, where $g_\Omega(z, 0)$ is the usual pluricomplex Green function with pole at the origin.*

We have one other relatively simple theorem where we construct an invariant $\nu^{GL}(\Omega)$ for the behavior near the singularity.

Theorem I.8. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Suppose there exists $u \in \cap_{\{w:||F(w)||=0\}} \mathcal{K}(\Omega, w)$. Suppose that the Green-Lelong numbers exist and are positive at each point w where $||F(w)|| = 0$, and that for each w there is a number $\epsilon(w) > 0$ such that there exists a function u satisfying $\frac{1}{\nu_w^{GL}(\log ||F||)+\epsilon(w)} \cdot u(z) \in \mathcal{K}(\Omega, w)$ for each w, with u independent of w. Then $u \in \mathcal{T}(\Omega, F)$. In particular if $||F||$ has a single zero, then $\nu^{GL}(\log ||F||) \cdot g_\Omega(z, w) \leq U(z)$.*

There is some hope that one could use generalized Green functions to define generalized Azukawa metrics. One potential application of generalized Azukawa metrics would be to get better estimates on the boundary behavior of the Kobayashi metric, since the Azukawa metric is a lower bound for the Kobayashi metric. The chapter is organized into four sections: two comparing the generalized and usual Green functions and two with different proofs of the main plurisubharmonicity result. Other than constructing generalized Azukawa metrics, further directions for this research include studying the same question where the zeros of the holomorphic function are not isolated, studying the relationship between this generalized Green function and the Green function with many poles in the case of isolated zeros, and checking whether theorems known for the pluricomplex Green function will follow analogously for this generalized Green function now that plurisubharmonicity is known.

In Chapter V we include two examples which came up in the course of studying the problems of Chapter III. First, we study balanced domains with holes. One can write formulas for the Green function of balanced domains, so we extend these formulas to balanced domains with holes.

Theorem I.9. *Let $\Omega \subset \mathbb{C}^n$ be a balanced pseudoconvex domain and let 0 denote the origin in $\mathbb{C}^n$. Let $K \subset\subset \Omega$ so that $0 \notin K$. Let $D(z) = \Omega \cap \{\alpha z : \alpha \in \mathbb{C}\}$ be the disk through z and the origin. If $D(z) \cap K$ is a closed subset of $D(z)$ which is the $-\infty$ set for some plurisubharmonic function on $D(z)$, then*

$$u_{\Omega \setminus K}(0, z) = u_{\Omega}(0, z) = \log \frac{||z||}{r},$$

where r is the radius of $D(z)$. Also,

$$F_A^{\Omega \setminus K}(0, \xi) = F_A^{\Omega}(0, \xi) = \frac{||\xi||}{r}.$$

Fornaess and Lee recently studied the boundary behavior of various metrics on the ring domain, which is a simple example of a balanced domain with holes, in [12]. Second, we study the boundary behavior of the pluricomplex Green function, which came out of a larger study of the regularity of the Green function and Azukawa metric. These were already known to not always be continuous, and we construct an example where the Green function cannot be continuously extended to the boundary.

Theorem I.10. *There exists a domain $\Omega \subset\subset \mathbb{C}^2$ with $(0, 0) \in \Omega$ so that the pluricomplex Green's function with singularity at zero, $u_{\Omega,0}$, is not in $C(\overline{\Omega} \setminus 0)$. The domain Ω has the following defining function*

$$\log|(z,w)| + \max\left\{-\sum_{i=1}^{\infty}\epsilon_i \log\frac{1}{i} + \sum_{i=1}^{\infty}\epsilon_i \log\left|\frac{1}{i}w - \left(1-\frac{1}{i}\right)z\right|, -1\right\} \tag{1.2}$$

and ϵ_i are chosen so that

$$-\sum_{i=1}^{\infty}\epsilon_i \log\frac{1}{i} \tag{1.3}$$

converges to a finite number [14, p.30].

In Chapter VI study the Skwarczyński metric, which is closely related to the Bergman metric. Recall that the Bergman space of a domain is the space of L^2 holomorphic functions on that domain, and that completeness of a metric roughly means that boundary points are infinite distance from interior points. The main theorem in Chapter VI is that Skwarczyński complete domains have infinite dimensional Bergman space.

Theorem I.11. *If Ω is Skwarczyński complete, then the Bergman space, $L^2_h(\Omega)$, is infinite dimensional.*

Our motivating question here is the well known problem of whether there exists a pseudoconvex domain with finite dimensional Bergman space. Wiegerinck constructed an example of a non-pseudoconvex domain with finite dimensional Bergman space in [34], but so far there are only partial results for pseudoconvex domains [21]. The main theorem in Chapter VI is joint work with Lina Lee, but I am including it in this thesis because we don't yet have plans to publish it elsewhere. Interesting further directions for this research would be whether Bergman complete domains have infinite dimensional Bergman space, and the related question of whether Skwarczyński completeness implies Bergman completeness.

CHAPTER II

Background

2.1 Definition of a Generalized Pluricomplex Green Function

In this section we will discuss the pluricomplex Green function and a generalization of it that we study in this thesis. For a detailed discussion of the pluricomplex Green function and related problems, see [23].

The classical Green function is a tool from partial differential equations used to solve the Laplace or Poission equations, which find harmonic functions or respectively functions with prescribed Laplacians given given boundary values. This Green function can be described either as a solution to a certain differential equation or as the upper envelope of subharmonic functions with the given boundary values. It can be used as a reproducing kernel to recover harmonic functions with given boundary values, and it can also be used to give a proof of the Riemann Mapping Theorem.

Harmonic and subharmonic functions can be on open subsets of $\mathbb{R}^n$, and thus can also be defined on open subsets of $\mathbb{C}^n$. However, they do not transform well under holomorphic maps because the composition of a harmonic or subharmonic function with a holomorphic map does not necessarily remain harmonic or subharmonic. Here's a simple example: consider the function $\phi = |z|^2 - |w|^2$ on $\mathbb{C}^2$. To get $\Delta\phi$ we calculate the following derivatives:

$$\frac{\partial^2}{\partial z \partial \overline{z}} \phi = 1$$
$$\frac{\partial^2}{\partial w \partial \overline{w}} \phi = -1,$$

so that ϕ is harmonic on $\mathbb{C}^2$. Now consider the funciton $F : \mathbb{C}^2 \to \mathbb{C}^2$ which sends $(z, w) \mapsto (zw, w)$. F is a biholomorphism off the line $w = 0$. But $F \circ \phi$ is not harmonic off the line $w = 0$. Calculating the derivatives we get:

$$\frac{\partial^2}{\partial z \partial \overline{z}} \phi \circ F = |w|^2$$
$$\frac{\partial^2}{\partial w \partial \overline{w}} \phi \circ F = |z|^2 - 1.$$

Thus $\Delta \phi \circ F = |z|^2 + |w|^2 - 1$ so that $\phi \circ F$ is not even subharmonic, much less harmonic.

However, if we work in one dimesion, i.e. on a domain in $\mathbb{C}$, harmonic and subharmonic functions do transform well under holomorphic maps. In particular, if ϕ is subharmonic or harmonic and $F : \Omega \to \Omega'$ is holomorphic with $\Omega, \Omega' \subset \mathbb{C}$, then $\phi \circ F$ is subharmonic or harmonic.

To replace subharmonic and harmonic functions, in complex analysis people study plurisubharmonic and pluriharmonic functions. These are required to be subharmonic or harmonic, respectively, on each complex line rather than on open sets. Like harmonic and subharmonic functions in one dimension, these transform well under holomorphic maps.

The analog of the classical Green function in complex analysis again has the property of transforming well under holomorphic maps, because it is defined in terms of plurisubharmonic functions or in terms of a Monge-Ampère equation.

The pluricomplex Green function on a domain $\Omega \subset \mathbb{C}^n$ with pole at a is defined by

$$g_\Omega(z,a) = \sup \left\{u(z) : u \in \mathcal{K}(\Omega, a)\right\},$$

where $u \in \mathcal{K}(\Omega, a)$ if it satisfies the following three properties:

1. u is plurisubharmonic on Ω,
2. $u \leq 0$ on Ω,
3. and $u(z) - \log ||z-a|| < c(u)$ in a neighborhood of a, where $c(u)$ is a constant depending on u.

It turns out that $g_\Omega(z,a) \in \mathcal{K}(\Omega, a)$. The proof proceeds by considering the upper semicontinuous regularization g^*_Ω of g_Ω, which is the smallest upper semincontinuous function majorizing g_Ω, or equivalently

$$g^*_\Omega(z,a) = \limsup_{\zeta \to z} g_\Omega(\zeta, a).$$

One shows that $g^*_\Omega(z,a) \in \mathcal{K}(\Omega, a)$, and it follows that $g_\Omega = g^*_\Omega$. To show that g^*_Ω satisfies condition (1), one applies the following theorem of Lelong, see [27, p. 56]:

Theorem II.1 (Lelong)**.** *Let Ω be a domain in $\mathbb{C}^n$ and let $(f_i)_{i\in I}$ be a family of plurisubharmonic functions on Ω which is bounded above on compacts. Let $F(z) = \sup_{i\in I} f_i(z)$. Then the upper semicontinuous regularization $F^*(z)$ is plurisubharmonic on Ω.*

It is clear that $g^*_\Omega(z,a)$ satisfies condition (2). To show that g^*_Ω satisfies condition (3) one compares $g_\Omega(z,a)$ to $g_{B_\epsilon(a)}(z,a)$, where $B_\epsilon(a) \subset \Omega$ is a small ball centered at a. On a ball, we can calculate the pluricomplex Green function by restricting to disks – they key point being that $\log ||z||$ is harmonic on disks. This calculation shows that the Green function on a ball centered at the origin is just $\log ||z/r||$, were r is the radius of the ball. It is easy to show that a smaller domain has a larger Green function, so that

$$g_\Omega(z,a) \le g_{B_\epsilon(a)}(z,a) = \log ||z-a||/\epsilon,$$

whenever g_Ω and $g_{B_\epsilon(a)}$ are both defined, which is on $B_\epsilon(a)$. Since $\log ||z-a||/\epsilon$ is already upper semicontinuous, we get that

$$g^*_\Omega(z,a) \le g^*_{B_\epsilon(a)}(z,a) = \log ||z-a||/\epsilon,$$

and rearranging this equation shows that $g^*_\Omega(z,a)$ satisfies condition (3).

We define a generalized pluricomplex Green function by replacing $\log ||z||$ with $\log ||F||$, where $F : \Omega \to \mathbb{C}^m$ is a holomorphic function, following a suggestion of B. A. Taylor.

Definition II.2.

$$U_\Omega(z) = \sup \{u(z) : u \in \mathcal{T}(\Omega, F)\},$$

where $u \in \mathcal{T}(\Omega, F)$ if it satisfies the following three properties:

1. u is plurisubharmonic on Ω,
2. $u \le 0$ on Ω,
3. and $u(z) - \log ||F(z)|| < c(u,w)$ in a neighborhood of each point w where $F(w) = 0$, where $c(u,w)$ is a constant depending on u and w.

Also following a suggestion of B. A. Taylor, we investigate the question of whether $U_\Omega \in \mathcal{T}(\Omega, F)$. Our main result is the following:

Theorem II.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated singularities. Then $U_\Omega(z) \in \mathcal{T}(\Omega, F)$.*

2.2 Statement of Resolution of Singularities

In this section, we will outline the statement and definitions involved in Hironaka's resolution of singularities, which we will later use in our proof of Theorem II.3. Our sources are Grauert, Peternell, and Remmert [15] and Rudin [30]. It is not necessary for our result to use resolution of singularities in full generality, so we will refer the interested reader to [15] for a more complete picture.

Definition II.4 (Complex Analytic Subvariety)**.** Let Ω be an open set in $\mathbb{C}^n$. A set $V \subset \Omega$ is said to be an complex analytic subvariety, or subvariety, of Ω if

1. V is relatively closed in Ω

2. and every point $p \in \Omega$ has a neighborhood $N(p)$ such that

$$V \cap N(p) = Z(f_1) \cap \cdots \cap Z(f_r)$$

   for some $f_1, ..., f_r$ which are holomorphic on $N(p)$, where $Z(f_i)$ denotes the zero variety of f_i.

Definition II.5 (Holomorphic map from a complex manifold to a complex analytic subvariety)**.** Let M be a complex manifold and let V be a complex analytic subvariety in $\mathbb{C}^n$. A map $f : M \to V$ is holomorphic if f is holomorphic when considered as a map from charts in M to $\mathbb{C}^n$.

Definition II.6 (Weak Resolution of Singularities)**.** Let X be a complex analytic subvariety. A weak resolution of singularities of X is a proper surjective holomorphic map $f : \tilde{X} \to X$ such that $\tilde{X}$ is a complex manifold.

Hironaka proved the first major results on the existence of a resolution of singularities [17], and the following statement is part of a general theorem that was proven by Arcoa-Hironaka-Vincente [1] and Bierstone-Milman [5] [6].

Theorem II.7. *Every complex analytic subvariety has a weak resolution of singularities.*

From now on we will abbreviate by just saying resolution of singularities. The additional hypotheses in the full resolution of singularities hold in our case, but we only need the hypotheses above for the proof of our theorem.

2.3 Basics of Invariant Metrics

In this section, we give discuss some of the common invariant metrics and their properties. For more detailed discussion of the metrics, see [19].

Definition II.8. Let $\Omega \subset \mathbb{C}^n$ be a domain, $p \in \Omega$, and $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$. Let Δ be the unit disk in $\mathbb{C}$, and let $\mathbb{B}^n(p, r) \subset \mathbb{C}^n$ be the ball of radius r centered at p.

- Carathéodory pseudometric $F_C^\Omega(p, \xi)$ is defined as

$$F_C^\Omega(p, \xi) = \sup\{|f'(p) \cdot \xi| : f \in \mathcal{O}(\Omega, \Delta), f(p) = 0\},$$

  where $\mathcal{O}(\Omega, \Delta)$ is the set of holomorphic functions from Ω to Δ.

- Kobayashi pseudometric $F_K^\Omega(p, \xi)$ is defined as

$$F_K^\Omega(p, \xi) = \inf\left\{|\alpha| \ : \ f \in \mathcal{O}(\Delta, \Omega), f(0) = p, \exists\alpha > 0, \alpha f'(0) = \xi\right\},$$

where $\mathcal{O}(\Delta,\Omega)$ is the set of holomorphic functions from Δ to Ω.

- Sibony pseudometric $F_S^\Omega(p,\xi)$ is defined as

$$F_S^\Omega(p,\xi)=\sup\left\{(\partial\overline{\partial}u(p)(\xi,\overline{\xi}))^{1/2}=\left(\sum_{i,j=1}^{n}\frac{\partial^2 u(p)}{\partial z_i\partial\overline{z_j}}\xi_i\overline{\xi_j}\right)^{1/2}:u\in\mathcal{S}_\Omega(p)\right\},$$

where $\mathcal{S}_\Omega(p)$ is the set of functions u such that $u:\Omega\to[0,1)$ vanishes at p, $\log u$ is plurisubharmonic, and u is C^2 near p.

- Azukawa pseudometric $F_A^\Omega(p,\xi)$ is defined as

$$F_A^\Omega(p,\xi)=\sup\left\{\limsup_{\lambda\searrow 0}\frac{1}{|\lambda|}u(p+\lambda\xi):u\in\mathcal{K}_\Omega(p)\right\},$$

where $\mathcal{K}_\Omega(p)$ is the set of functions u such that $u:\Omega\to[0,1)$, $\log u$ is plurisubharmonic, and there exits $M>0, r>0$ such that $\mathbb{B}^n(p,r)\subset\Omega$ and $u(z)\le M\|z-p\|$ for all $z\in\mathbb{B}^n(p,r)$.

Let K_Ω be the Bergman kernel of Ω.

Definition II.9. The Bergman metric $F_B^\Omega(p,\xi)$ is defined by

$$F_B^\Omega(p,\xi)=\left(\sum_{\nu,\mu=1}^{n}\frac{\partial^2}{\partial z_\nu\partial\overline{z_\mu}}\log K_\Omega(z,z)\xi_\nu\overline{\xi_\mu}\right)^{1/2}$$

provided that K_Ω is nonvanishing on Ω.

A metric closely related to the Bergman metric is the Skwarczyński distance. It is defined using the Bergman kernel as follows.

Definition II.10. The Skwarczyński distance $\rho_\Omega(z',z'')$ is defined by

$$\rho_\Omega(z',z'')=\left(1-\frac{|K_\Omega(z',z'')|}{\sqrt{K_\Omega(z',z')}\sqrt{K_\Omega(z'',z'')}}\right)^{1/2}.$$

Except for the Bergman and Skwarczyński metrics the other four metrics are non-increasing with respect to holomprhic mappings, that is, if $\Phi : \Omega_1 \to \Omega_2$ is holomprhic, then $F^{\Omega_1}(p,\xi) \geq F^{\Omega_2}(\Phi(p), \Phi_*(\xi))$ where F^{Ω_i} is one of $F_C^{\Omega_i}, F_S^{\Omega_i}, F_A^{\Omega_i}$, and $F_K^{\Omega_i}$. Moreover, they satisfy the following relationship:

$$F_C^{\Omega}(p,\xi) \leq F_S^{\Omega}(p,\xi) \leq F_A^{\Omega}(p,\xi) \leq F_K^{\Omega}(p,\xi) \tag{2.1}$$

for all p and ξ. The Bergman metric behaves differently from the rest of metrics in the sense that it does not have non-increasing property, nor does it fit in the comparison (2.1). Between the Carathéodory and the Bergman metric the following is known.

Theorem II.11 (K. Hahn, [16])**.** *In any complex manifold Ω, the Bergman metric F_B^{Ω} is always greater than or equal to the Carathéodory differential metric F_C^{Ω} if M admits them:*

$$F_C^{\Omega}(p,\xi) \leq F_B^{\Omega}(p,\xi). \tag{2.2}$$

However, Kobayashi and Bergman metrics do not have any such relation and they are in fact incomparable.

2.4 A Few Concepts from Pluripotential Theory

In this section we include statements of basic results from pluripotential theory that we will use in this thesis. A more complete reference would be [23].

Maximum principles are results that say certain kinds of function must take their maximums on the boundary. Below is the maximum principle for plurisubharmonic functions.

Theorem II.12 (The Maximum Principle for Plurisubharmonic Functions)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $u \in \mathcal{PSH}(\Omega)$. Then either u is constant or for each $z \in \Omega$*

$$u(z) < \sup_{w \in \partial\Omega} \left\{ \limsup_{y \to w, y \in \Omega} u(y). \right\}$$

A related concept is that of maximal plurisubharmonic functions.

Definition II.13 (Maximal Plurisubharmonic Function)**.** Let u be a plurisubharmonic function on a domain Ω in $\mathbb{C}^n$. We say u is maximal if for every $G \subset\subset \Omega$ and for every function v which is upper semicontinuous on $\overline{G}$ and plurisubharmonic on G and satisfies $v \leq u$ on ∂G, then $v \leq u$ on G as well.

The following theorem classifies maximal plurisubharmonic functions in terms of the Monge-Ampère operator, see [23].

Theorem II.14. *Let Ω be a domain in $\mathbb{C}^n$ and let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{PSH}(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$ in Ω.*

In parts of this thesis we will be working with hyperconvex domains, which are a special kind of pseudoconvex domain defined by certain plurisubharmonic functions.

Definition II.15 (Hyperconvex)**.** We say that a bounded domain $\Omega \subset \mathbb{C}^n$ is hyperconvex if there is a continuous plurisubharmonic exhaustion function $\rho : \Omega \to (-infty, 0)$.

We also need to define the upper semicontinuous regularization of a function.

Definition II.16. Let Ω be a domain in $\mathbb{C}^n$ and let $u : \Omega \to [-\infty, \infty)$ be a function that is locally bounded above near each point in $\overline{\Omega}$. Then define the upper semicontinuous regularization u^* of u by the formula

$$u^*(x) = \limsup_{y \to x, y \in \Omega} u(y).$$

The upper semicontinuous regularization is the smallest upper semicontinuous function that majorizes the original function.

2.5 Applications of Invariant Metrics

Complex analysis in several varaibles turns out to be quite different from complex analysis in one variable. There are many elegant tools in one complex variable that fail catastrophically in more than one variable, and as a result there are many basic questions still unanswered in higher dimensions. The Riemann Mapping Theorem is an example of such a failure. The Riemann Mapping Theorem says that all simply connected domains in $\mathbb{C}$, other than $\mathbb{C}$ itself, are biholomorphic to the disk. Intuitively such domains are just the sets with no holes, so this represents quite a large class of domains. The existence of the biholomorphism means that we can answer all questions about such domains by answering them for the disk, and it is relatively simple to do do complex analysis on the disk because of its symmetries. In contrast, even the unit ball $B(0,1)$ and polydisk $\Delta(0,1)$ are not biholomorphic in higher dimesions. Roughly, one can think of the ball as a circle and the polydisk as a square, and they can be described precisely as:

$$B(0,1) = \{z \in \mathbb{C}^n : ||z|| < 1\}$$

and

$$\Delta(0,1) = \{z \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1\}.$$

The proof that there is no biholomorphism from the ball to the polydisk is not simple, and was quite surprising when it was first discovered. This proof relies on invaraint metrics, which turn out to play a large role in several complex variables.

An invariant metric can be thought of as a way to measure distance that is invariant under biholomorphism, which means that if f is a biholomorphism from Ω_1 to Ω_2 then the distance between p and q in Ω_1 is the same as the distance between $f(p)$ and $f(q)$ in Ω_2. Formally, we say that $d_\Omega : \Omega \times \mathbb{C}^n \to [0, \infty)$ is a pseudometric if for all $p \in \Omega, \lambda \in \mathbb{C}$, and $\xi \in \mathbb{C}^n$

$$d_\Omega(p; \lambda\xi) = |\lambda| d_\Omega(p; \xi).$$

A pseudometric is distance decreasing if for any $f \in \mathcal{O}(\Omega_1, \Omega_2)$ we have

$$d_{\Omega_2}(F(p); F'(p)\xi) \leq d_{\Omega_1}(p; \xi).$$

Distance decreasing metrics are an important subclass of invariant metrics, and most of the metrics in this thesis will be distance decreasing. The above equation with biholomorphic maps and equality would give the formal definition of biholomorphic invariance. These pseudometrics do not directly give distances between two points, but in most cases they can be integrated to give such distances.

Invariant metrics have many important applications in several complex variables. One of the most obvious questions they can be used to answer is under what conditions is a holomorphic map biholomorphic. There are a number of positive results in this direction, including the following well known result of Cartan.

Theorem II.17 (Cartan [25]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, $f : \Omega \to \Omega$ be a holomorphic map, and p be a fixed point. Then f is a biholomorphism if and only if f is a Carathéodory or Kobayashi isometry at p.*

There are also more general results which give conditions for when a holomorphic map between distinct domains is a biholomorphism [19]. One can also go the opposite direction and get results which give necessary conditions for biholomorphisms. One such result is the following theorem of Lempert.

Theorem II.18 (Lempert [19])**.** *If $\Omega \subset \mathbb{C}^n$ is biholomorphic to a bounded convex domain, then the Carathéodory and Kobayashi metrics must be the same.*

One of the most well known uses of invariant metrics to the problem of extending holomorphic functions to the boundary. Fefferman proved a well known result regarding this problem using the Bergman metric.

Theorem II.19 (Fefferman, [11])**.** *Let $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be strictly pseudoconvex with smooth boundaries and let $F : \Omega_1 \to \Omega_2$ be a biholomorphism. Then F extends to a diffeomorphism from $\overline{\Omega}_1$ to $\overline{\Omega}_2$.*

Invariant metrics have also started finding applications in complex dynamics, and in particular following result of Ueda has become a useful tool.

Theorem II.20 (Ueda [33])**.** *Suppose $f : \mathbb{P}^k \to \mathbb{P}^k$ is holomorphic of degree $d \geq 2$. Then each Fatou component of f is Kobayashi hyperbolic.*

An interesting result that can be proved with the Carathéodory metric is a theorem of Reiffen about fixed points.

Theorem II.21 (Reiffen, [29])**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then a holomorphic mapping $f : \Omega \to \Omega$ with $f(\Omega) \subset\subset \Omega$ has a unique fixed point.*

There has also been much research on automorphism groups using invariant metrics. An example of such a result is the following classification by Stanton.

Theorem II.22 (Stanton, [32])**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth $\mathcal{C}^1$ boundary, and assume that the Carathéodory and Kobayashi metrics are $\frac{1}{\sqrt{n+1}}$ times the Bergman metric. Then Ω is biholomorphic to the ball.*

This classification has a number of important consequences, including that any strongly pseudoconvex domain is biholomorphically equivalent to the ball if and only iff its automorphism group is not compact, and that a bounded domain with twice differentiable boundary is biholomorphic to the ball if and only iff its automorphism group acts transitively [19].

In addition to being useful in several complex variables, invariant metrics provide relatively simple proofs to a number of difficult theorems from one complex variable, including the Riemann Mapping theorem and Picard's theorems. Also, the construction of the Carathéodory and Kobayashi metric can be viewed as a generalization of the Schwarz lemma, and indeed one can use them to prove this lemma as well [26].

CHAPTER III

Results on the Azukawa and Sibony Pseudometrics

3.1 Upper-Semicontinuity of the Azukawa Metric and Green Function

The results from this section are results that I proved independently during my PhD, but later found out that the results had already been proven by Jarnicki and Pflug in [20].

In this section we study upper-semicontinuity of the Azukawa metric and the pluricomplex Green function. Whenever the Green function $u(z, w)$ is upper-semicontinuous, the Azukawa metric is also upper-semicontinuous, [22], [19, p. 120]. We will prove that the Green function is upper-semicontinuous in Theorem III.7, and the following corollary follows from that result.

Corollary III.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $F_A^{\Omega}(z, \xi)$ be the Azukawa pseudometric on Ω. Then $F_A^{\Omega}(z, \xi)$ is upper-semicontinuous.*

An application of Corollary III.1 is that the Azukawa metric is integrable and can be used to define a related a pseudodistance on all domains.

A number of regularity results for the pluricomplex Green function are already known. On bounded domains $u(\cdot, w)$ is locally uniformly continuous and $u(z, \cdot)$ is plurisubharmonic, c.f. [10], [27], [23]. On hyperconvex domains u is continuous, and on pseudoconvex domains it is upper-semicontinuous, c.f. [10]. We show that u is upper semicontinuous on arbitrary domains. This answers a question asked by Marek Jarnicki and Peter Pflug in [19].

We will provide a proof of local uniformity of $u(\cdot, w)$ on bounded domains for the convenience of the reader.

Theorem III.2 (Demailly Lemma 4.13 [1], [10])**.** *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain and let $a \in \Omega$. Given $\epsilon > 0$ and a neighborhood U of a, there exists a neighborhood U' of a with $U' \subset\subset U$ so that for all $w_1, w_2 \in U'$ and for all $z \in \Omega \setminus \{U\}$ we have the following inequality*

$$(1+\epsilon)^{-1} \leq \frac{u(z, w_1)}{u(z, w_2)} \leq 1 + \epsilon.$$

Proof. We would like to construct a candidate function $c(z)$ for $u(z, w_2)$ based on $u(z, w_1)$. Let $B(a, \alpha) \subset U$ with $\alpha > 0$ such that $B(a, \alpha) \subset\subset \Omega$. Define

$$c(z) = \begin{cases} \log \frac{||z-w_2||}{R} & ||z-a|| < \eta \\ \max\left\{(1+\epsilon)u(z, w_1), \log \frac{||z-w_2||}{R}\right\} & ||z-a|| > \eta. \end{cases}$$

where $\eta \in (0, \alpha)$ is yet to be determined. In order to show that $c(z)$ is plurisubharmonic, we need to show that for w_1, w_2 in some neighborhood of a, when $||z-a|| = \eta$

$$(1+\epsilon)u(z, w_1) \leq \log \frac{||z-w_2||}{R}. \tag{3.1}$$

Since Ω is bounded we can find r and $R \in (0, \infty)$ so that for all $w \in B(a, \alpha)$

$$B(r, w) \subset \Omega \subset B(R, w).$$

[1]Note the hypothesis of a bounded domain is written at the beginning of section 4 in Demailly's paper.

Because larger domains have smaller Green functions, when $w \in B(a,\alpha)$ and $z \in \Omega$

$$\log\left(\frac{||z-w_1||}{R}\right) \leq u(z,w_1) \leq \log\left(\frac{||z-w_1||}{r}\right). \tag{3.2}$$

First let us work with the left hand side of equation 3.1. Using equation 3.2 we have that when $w \in B(a,\alpha)$ and $z \in \Omega$

$$(1+\epsilon)u(z,w_1) \leq (1+\epsilon)\log\left(\frac{||z-w_1||}{r}\right). \tag{3.3}$$

If we pick $w_1, w_2 \in B(a,\eta/2)$ then for $||z-a|| = \eta$

$$\eta/2 < ||z-w_i|| < 3\eta/2. \tag{3.4}$$

Combining equations 3.3 and 3.4 when $w_1 \in B(a,\eta/2)$ and $||z-a|| = \eta$

$$(1+\epsilon)u(z,w_1) \leq (1+\epsilon)\log\left(\frac{3\eta}{2r}\right).$$

Now working with the right hand side of equation 3.1 and using equation 3.4 we get when $w_2 \in B(a,\eta/2)$ and $||z-a|| = \eta$

$$\log\left(\frac{\eta}{2R}\right) \leq \log\left(\frac{||z-w_2||}{R}\right)$$

If we can pick η such that

$$(1+\epsilon)\log\left(\frac{3\eta}{2r}\right) \leq \log\left(\frac{\eta}{2R}\right)$$

then equation 3.1 will hold when $w_1, w_2 \in B(a,\eta/2)$ and $||z-a|| = \eta$. We can algebraically rearrange this equation to find out exactly how small η must be. So now we have shown that $c(z)$ is in fact plurisubharmonic for the appropriate choice of η.

Furthermore, it is clear that $c(z) < 0$ on Ω and that on the neighborhood $B(a, \nu)$ of $a, c(z) - \log ||z - w_2|| = -\log R$. So $c(z)$ is a candidate for the Green function on Ω with singularity at w_2. Thus on $\Omega, c(z) \leq u(z, w_2)$. Then when $w_1, w_2 \in B(a, \eta/2)$ and $||z - a|| > \eta$ we have

$$(1 + \epsilon)u(z, w_1) \leq u(z, w_2) \tag{3.5}$$

We can make the same argument switching w_1 and w_2 to get

$$(1 + \epsilon)u(z, w_2) \leq u(z, w_1) \tag{3.6}$$

Note that the inequality flips when we divide since the Greens functions are negative. Let $U' = B(a, \eta/2)$. Then the two equations 3.5 and 3.6 give us the theorem.

□

Corollary III.3. *On unbounded domains $u(z, \cdot)$ is upper-semicontinuous for fixed z.*

Proof. If $\cup_n \Omega_n = \Omega$ with $\Omega_1 \subset \Omega_2 \subset \cdots$, then $u_{\Omega_n} \searrow u_\Omega$. □

The following theorems are well known.

Theorem III.4. *The function $u(\cdot, w)$ is upper-semicontinuous for fixed w on bounded domains $\Omega \subset \mathbb{C}^n$.*

Theorem III.5. *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be such that $\Omega_n \subset \Omega_{n+1}$ for all n, and let $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$. Then $u_{\Omega_n}(z, w) \searrow u_\Omega(z, w)$.*

Now we will prove a lemma which shows that the Green function is upper-semicontinuous on bounded domains.

Lemma III.6. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. Then the pluricomplex Green function $u(z,w)$ on Ω is upper-semicontinuous.*

Proof. Let (a,b) be a point in $\Omega \times \Omega$. If $a = b$ then given a sequence $(z_i, w_i) \in \Omega \times \Omega$ converging to (a,a) and given $\epsilon > 0$ small, we can find I_1 so that for all $i > I, ||z_i - w_i|| < \epsilon$. Let $\mathrm{dist}(a, \partial\Omega) = d$. Then there exists I_2 such that for all $i > I_2, ||z_i - a|| < d/2$ so that $B(z_i, d/2) \subset \Omega$. Then for $i > \max I_1, I_2$, $u(z_i, w_i) < \log(2\epsilon/d)$. Letting $\epsilon \to 0$, we see that

$$\limsup_{(z_i,w_i)\to(a,a)} u(z_i, w_i) = -\infty,$$

giving us upper-semicontinuity on the diagonal.

So now assume that that $a \neq b$ and let $\mathrm{dist}(a,b) = \delta$. Let $(z_i, w_i) \in \Omega \times \Omega$ such that $(z_i, w_i) \to (a,b)$. We can apply Theorem III.2 with $U = B(a, \delta/2)$. We get a neighborhood $U' \subset\subset U$ of a, and since $z_i \to a$ and $w_i \to b$ we can find I such that for all $i > I, z_i \in U'$ and $w_i \notin U$. We then get for $i > I$

$$(1+\epsilon)^{-1} u(a, w_i) \geq u(z_i, w_i) \geq (1+\epsilon) u(a, w_i). \tag{3.7}$$

Taking lim sup of both sides of the left hand inequality and applying Theorem III.4, which says that that $u_w(z)$ is upper-semicontinuous in z, we get

$$\limsup_{i\to\infty} u(z_i, w_i) \leq \limsup_{i\to\infty} (1+\epsilon)^{-1} u(a, w_i) \leq (1+\epsilon)^{-1} u(a,b).$$

Letting $\epsilon \to 0$ we are done.

□

The main theorem of this section is the following.

Theorem III.7. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $u(z, w)$ be the pluricomplex Green function on Ω. Then $u(z, w)$ is upper-semicontinuous.*

Proof of Theorem III.7. Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $\Omega_n = \Omega \cap B(0, N)$. Then $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$. Therefore by Theorem III.5

$$u_{\Omega_n}(z, w) \searrow u_{\Omega}(z, w).$$

By Lemma III.6 $u_{\Omega_n}(z, w)$ is upper-semicontinuous, and the decreasing limit of upper-semicontinuous functions is again upper-semicontinuous.

□

3.2 Comparison of the Metrics

The Azukawa and Sibony metrics have similar definitions, with the Sibony metric requiring more regularity of its candidate functions. The relationship between these metrics was studied in [22], and it turns out that the two metrics are different in some cases. Looking at the regularity of these metrics we see that the Azukawa metric is upper-semicontinuous, but there is an example of a pseudoconvex domain on which the Sibony metric is not upper-semicontinuous, c.f. [18]. Another way to see that the two metrics are different is to look at their indicatrices, which are the unit balls in the tangent space for a fixed point in the domain. The indicatrices for the Sibony metric are always convex, c.f. [31], but for balanced domains the Azukawa indicatrix at the origin is equal to the holomorphic hull of the domain, c.f. [3]. In this section we will construct an explicit example of a domain where the two metrics are different.

Theorem III.8. *The Azukawa and Sibony metrics are different on the domain* $\Omega \subset \mathbb{C}^2$ *defined by*

$$f(z,w) := \frac{1}{4}\log|z| + \frac{1}{4}\log|w| + \frac{1}{4}\log|z-w| + \frac{1}{4}\log|z-iw|$$

$$\Omega = \{(z,w) : f(z,w) < 0\}$$

Proof. Notice that Ω contains the complex lines $L_1 = \{z=0\}, L_2 = \{w=0\}, L_3 = \{z=w\}, L_4 = \{z=iw\}$. So if $u \in \mathcal{A}_\Omega(0,0)$, then u is bounded on L_i, so constant on L_i for i between 1 and 4. Since u is constant on L_1 and L_2 we have that

$$\frac{\partial^2 u}{\partial z \partial \overline{z}}(0,0) = \frac{\partial^2 u}{\partial w \partial \overline{w}}(0,0) = 0 \tag{3.8}$$

The definition of complex partial differentiation is that for $z = x+iy$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

So for real valued u with $w = s+it$

$$\begin{aligned}
\frac{\partial^2 u}{\partial w \partial \overline{z}} &= \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) - i\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial^2 u}{\partial s \partial x} + i\frac{\partial^2 u}{\partial s \partial y} - i\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial t \partial y}
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial z \partial \overline{w}} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial s} + i\frac{\partial u}{\partial t}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s} + i\frac{\partial u}{\partial t}\right) - i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s} + i\frac{\partial u}{\partial t}\right)\\ &= \frac{\partial^2 u}{\partial x \partial s} + i\frac{\partial^2 u}{\partial x \partial t} - i\frac{\partial^2 u}{\partial y \partial s} + \frac{\partial^2 u}{\partial y \partial t}\end{aligned}$$

Since u is C^2 in a neighborhood of $(0,0)$ we have that

$$\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) = \overline{\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)} \tag{3.9}$$

Since u is constant on L_3, by taking the derivative in the direction (ξ_1, ξ_1) at the point $(0,0)$ we get that

$$\begin{aligned}0 &= \frac{\partial^2 u}{\partial z \partial \overline{z}}(0,0)|\xi_1|^2 + \frac{\partial^2 u}{\partial w \partial \overline{w}}(0,0)|\xi_1|^2\\ &+ \frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 + \frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)|\xi_1|^2\\ &= \frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 + \frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)|\xi_1|^2\end{aligned}$$

Combining this with observation 3.9 we get that

$$0 = 2\mathrm{Re}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 \tag{3.10}$$

Now since u is constant on L_4, by taking the derivative in the direction $(i\xi_1, \xi_1)$ at the point $(0,0)$ we get that

$$\begin{aligned}
0 &= \frac{\partial^2 u}{\partial z \partial \overline{z}}(0,0)|\xi_1|^2 + \frac{\partial^2 u}{\partial w \partial \overline{w}}(0,0)|\xi_1|^2 \\
&+ i\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 - i\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)|\xi_1|^2 \\
&= i\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 - i\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)|\xi_1|^2
\end{aligned}$$

Combining this with observation 3.9 we get that

$$0 = -2\mathrm{Im}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)|\xi_1|^2 \tag{3.11}$$

Now combining observations 3.9, 3.10, and 3.11 gives us that

$$\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) = \frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0) = 0 \tag{3.12}$$

Since equations 3.8 and 3.12 hold for each $u \in \mathcal{A}_\Omega(0,0)$, we have that $F_S^\Omega(0,\xi) = 0$ for every $\xi \in \mathbb{C}^2$.

Next, we would like to get a lower bound on the Azukawa metric that is greater than zero in a direction ξ. We'd like to show that the defining function for Ω is a competitor for the Green function, or in other words that $f(z,w) \in PS_\Omega(0,0)$. By definition $f(z,w) < 0$ on Ω. It is clear that $f(z,w)$ is plurisubharmonic on Ω. Finally, to show that $f(z,w) - \log|(z,w)|$ is bounded above on a neighborhood of $(0,0)$ we can split up $\log|(z,w)|$ to rewrite $f(z,w) - \log|(z,w)|$ as

$$\frac{1}{4}\log\frac{|z|}{|(z,w)|} + \frac{1}{4}\log\frac{|w|}{|(z,w)|} + \frac{1}{4}\log\frac{|z-w|}{|(z,w)|} + \frac{1}{4}\log\frac{|z-iw|}{|(z,w)|}$$

which is bounded above on a neighborhood of $(0,0)$ since

$$\frac{|z|}{|(z,w)|}, \frac{|w|}{|(z,w)|} \le 1$$

$$\frac{|z-w|}{|(z,w)|}, \frac{|z-iw|}{|(z,w)|} \le \frac{2\max\{|z|,|w|\}}{|(z,w)|} < 2$$

Thus $f(z,w) \le g_{\Omega,0}$. It follows that for $\xi = (1,\alpha)$

$$\limsup_{\lambda \to 0} \frac{e^{f(\xi\lambda)}}{|\lambda|} \le \limsup_{\lambda \to 0} \frac{e^{g_{\Omega,0}(\xi\lambda)}}{|\lambda|} = F_A^{\Omega}(\xi, 0)$$

Now we can simplify e^f to get

$$e^{f(\xi\lambda)} = |\lambda||\alpha|^{1/4}|1-\alpha|^{1/4}|1-i\alpha|^{1/4}$$

Thus for $\alpha \ne 0, 1, -i$,

$$\limsup_{\lambda \to 0} \frac{e^{f(\xi\lambda)}}{|\lambda|} = |\alpha|^{1/4}|1-\alpha|^{1/4}|1-i\alpha|^{1/4} > 0$$

□

Lemma III.9. *If $e^{2g_{\Omega,P}}$ is C^2 in a neighborhood of P, then*

$$F_S^{\Omega}(P,\xi) = \left(\sum_{i,j} \frac{\partial^2 e^{2g_{\Omega,P}}}{\partial z_i \partial \overline{z_j}} \xi_i \overline{\xi_j}\right)^{1/2}$$

Proof. Let $\xi = (\xi_1, \xi_2)$ and let f be a competitor for the Green's function. Now suppose $e^{2f(z,w)}$ is C^2 in a neighborhood of P. We will show that e^{2f} is a competitor for the Sibony metric. We have the four conditions

1. $e^{2f(P)} = 0$ since $f(P) = -\infty$

2. $e^{2f(P)}$ is C^2 in a neighborhood of P by assumption

3. $\log e^{2f} = 2f$ is plurisubharmonic since f is plurisubharmonic

4. $0 \leq e^{2f} \leq 1$ since $f < 0$

Conversely, we will show that given any u which is a competitor for the Sibony metric, the function $\frac{1}{2}\log u(z,w)$ is a competitor for the Green's function. We have the three conditions

1. $\frac{1}{2}\log u$ is plurisubharmonic since $\log u$ is assumed to be plurisubharmonic for u to be a competitor function for the Sibony metric.

2. $\frac{1}{2}\log u \leq 0$ on Ω since $0 \leq u \leq 1$. Follow the argument for the pluricomplex Green function: Suppose there is a point Q in Ω so that $u(Q) = 1$. Then since u is plurisubharmonic, we can follow any path in Ω and use the maximum principle to show that $u \equiv 1$. But then $u(P) \neq 0$. So in fact we have that $0 \leq u < 1$, so $\frac{1}{2}\log u < 0$.

3. We would like to show that

$$\frac{1}{2}\log u(z,w) - \log|(z,w)| = \frac{1}{2}\log\left(\frac{u(z,w)}{|(z,w)|^2}\right) \tag{3.13}$$

is bounded above on a deleted neighborhood of P. For small ϵ let

$$M = \max_{i,j}\left\{\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(z) : |(z_1,z_2) - P| < \epsilon\right\}$$

For small ϵ such an M exists since u is C^2 in a neighborhood of P. Note also that $u(P) = 0$ and all the first derivatives of u are zero at P because P is a local minimum. Then for $|(z,w) - P| < \epsilon$ we can write

$$u(z,w) = |u(z,w)| \leq 16M|(z,w)|^2$$

Which gives us that equation 3.13 is bounded above on a neighborhood of P since

$$\frac{u(z,w)}{|(z,w)|^2} \leq 16M$$

We now cliam that there is a bijective correspondence between competitors for the Sibony metric (u), and competitors for the Green function (f) which are C^2 in a neighborhood of P. This is shown by the work above and the following two equations

$$e^{2\frac{1}{2}\log u} = u$$

$$\frac{1}{2}\log\left(e^{2f}\right) = f$$

Now suppose that

$$\sup\left\{\left(\sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(P)\xi_i\overline{\xi_j}\right)^{1/2} : u \in \mathcal{A}_\Omega(P)\right\} > \left(\sum_{i,j}\frac{\partial^2 e^{2g_{\Omega,P}}}{\partial z_i \partial \overline{z_j}}\xi_i\overline{\xi_j}\right)^{1/2} \tag{3.14}$$

Note that the $\geq$ version of the inequality holds trivially because $e^{2g_{\Omega,P}}$ is assumed to be C^2 in a neighborhood of P and hence is a competitor function. So if we show that $>$ does not hold we will have equality. Let x, y be real vectors that span the complex subspace generated by ξ. Then equation 3.14 is equivalent to

$$\frac{\partial^2 u^*}{\partial^2 x}(P) + \frac{\partial^2 u^*}{\partial^2 y}(P) > \frac{\partial^2 e^{2g_{\Omega,P}}}{\partial^2 x}(P) + \frac{\partial^2 e^{2g_{\Omega,P}}}{\partial^2 y}(P)$$

for some competitor for the Sibony metric u^*. Then either

$$\text{(3.15)} \qquad \frac{\partial^2 u^*}{\partial^2 x}(P) > \frac{\partial^2 e^{2g_{\Omega,P}}}{\partial^2 x}(P)$$

or

$$\text{(3.16)} \qquad \frac{\partial^2 u^*}{\partial^2 y}(P) > \frac{\partial^2 e^{2g_{\Omega,P}}}{\partial^2 y}(P)$$

Again we note that $u(P) = e^{2g_{\Omega,P}}(P) = 0$ and the first derivatives of each of these are all zero. Then if either of equation 3.15 or equation 3.16 hold, $u^* > e^{2g_{\Omega,P}}$ in a neighborhood of P in either the x or the y direction. But $\frac{1}{2}\log u^*$ is a competitor for the Green function, so this is impossible.

□

Theorem III.10. *There exists a bounded domain $\Omega \subset \mathbb{C}^2$ on which the Sibony and Azukawa pre-metrics are different. Let $f(z,w)$ be as in Theorem III.8. Let $h_R(z,w) = \max\{f(z,w), |(z,w)| - R\}$. Let*

$$\Omega_R = \{(z,w) : h_R(z,w) < 0\}$$

Then for large enough R, $\Omega := \Omega_R$ will work.

Proof. Notice that the disks $D_1 = \{(0,w) : |w| < R\}, D_2 = \{(z,0) : |z| < R\}, D_3 = \{(z,z) : |z| < R/\sqrt{2}\}, D_4 = \{(iw,w) : |w| < R/\sqrt{2}\}$ are contained in Ω_R and have modulus R. Thus we get the inequality

$$g_{\Omega_R,0}|_{D_i} \leq \log \frac{|(z,w)|}{R}$$

But since $\Omega_R \subset \mathbb{B}_R$, where $\mathbb{B}_R$ is the ball of radius R, we get the inequality

$$g_{\Omega_R,0} \geq \log \frac{|(z,w)|}{R}$$

Thus

$$g_{\Omega_R,0}|_{D_i} = \log \frac{|(z,w)|}{R}$$

for $i = 1..4$. We can calculate the derivatives for $e^{2g_{\Omega_R,0}}$ in some of these directions to get estimates on the derivatives of u, for u in the admissible class for the Sibony metric.

By the proof of Lemma III.9, $\frac{1}{2}\log u$ is a competitor for the Green function so $u \leq e^{2g_{\Omega_R,0}}$. Note that $u(0,0) = e^{2g_{\Omega_R,0}}(0,0) = 0$. Also, u and $e^{2g_{\Omega_R,0}}$ have a local minimum at $(0,0)$, so all their first derivatives are zero at the origin. Thus

$$\frac{\partial^2 u}{\partial^2 x}(0,0) \leq \frac{\partial^2 e^{2g_{\Omega_R,0}}}{\partial^2 x}(0,0)$$

And the same statements hold when replacing x with y, s, or t. Notice that

$$\begin{aligned}
\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) - i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial^2 u}{\partial^2 x} + i\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial^2 y} \\
&= \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y}
\end{aligned}$$

This is a result about the mixed second derivative on D_1, and the analogous result will hold for D_i with $i = 2..4$. So we have that for s in the direction of D_i

$$\frac{\partial^2 u}{\partial s \partial \overline{s}}(0,0) \leq \frac{\partial^2 e^{2g_{\Omega_R,0}}}{\partial s \partial \overline{s}}(0,0)$$

By noticing that mixed second derivative on D_i is $1/R^2$ for $i = 1..4$, we get that for D_1 and D_2

$$\frac{\partial^2 e^{2g_{\Omega_R,0}}}{\partial z \partial \overline{z}}(0,0) = \frac{\partial^2 e^{2g_{\Omega_R,0}}}{\partial w \partial \overline{w}}(0,0) = \frac{1}{R^2}$$

To calculuate the mixed second derivative on D_3 in the direction $\xi = (\xi_1, \xi_1)/|\sqrt{2}\xi_1|$, we get that

$$\begin{aligned}\frac{\partial^2 u}{\partial s \partial \overline{s}}(0,0) &= \frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{z}}(0,0) + \frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{w}}(0,0) + \frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) + \frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0) \\ &\leq \frac{1}{2R^2} + \frac{1}{2R^2} + \frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) + \frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)\end{aligned}$$

Nothing that the second partial in the direction D_i is also smaller than $1/R^2$, this shows that

$$\left|\mathrm{Re}\frac{\partial^2 e^{2g_{\Omega_R,0}}}{\partial z \partial \overline{w}}(0,0)\right| \leq 2/R^2$$

Similarly if we calculate the mixed second derivative on D_4 in the direction $\xi = (i\xi_2, \xi_2)/|\sqrt{2}\xi_2|$, we get that

$$\begin{aligned}\frac{\partial^2 u}{\partial s \partial \overline{s}}(0,0) &= \frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{z}}(0,0) + \frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{w}}(0,0) + i\frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) - i\frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0) \\ &\leq \frac{1}{2R^2} + \frac{1}{2R^2} + i\frac{1}{2}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0) - i\frac{1}{2}\frac{\partial^2 u}{\partial w \partial \overline{z}}(0,0)\end{aligned}$$

This shows that

$$\left|\mathrm{Im}\frac{\partial^2 u}{\partial z \partial \overline{w}}(0,0)\right| \leq 2/R^2$$

Thus for u a competitor for the Sibony metric we have that

$$\sum_{i,j} \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(0,0)\xi_i\overline{\xi_j} \leq \frac{1}{R^2}(|\xi_1|^2 + 4\mathrm{Re}\xi_1\overline{\xi_2} + |\xi_2|^2)$$

Which gives us that

$$F_S^{\Omega_R}(0,\xi) \leq \frac{1}{R}(|\xi_1|^2 + 4\mathrm{Re}\xi_1\overline{\xi_2} + |\xi_2|^2)^{1/2}$$

Next we would like to show that $f(z,w)$ is a competitor for the Green function on Ω_R. We have the following

1. $f(z,w)$ is clearly plurisubharmonic on Ω_R.

2. $f(z,w) < 0$ on Ω_R since by definition of Ω_R both $f(z,w)$ and $|(z,w)| - R$ are negative on Ω_R.

3. In Theorem III.8 we showed that it is a competitor on Ω, giving us that $f(z,w) - \log|(z,w)|$ was bounded above in a neighborhood of the origin. This carries over to Ω_R.

We can now follow the same proof as Theorem III.8 to get that in the direction $\xi^* = (1,\alpha)$

$$F_A^{\Omega_R}(0,\xi^*) = |\alpha|^{1/4}|1-\alpha|^{1/4}|1-i\alpha|^{1/4} > 0$$

The estimate on the Sibony metric in the direction ξ^* would be

$$F_S^{\Omega_R}(0,\xi^*) \leq \frac{1}{R}(1 + 4\mathrm{Re}\alpha + |\alpha|^2)^{1/2}$$

Thus for large enough R when $\alpha \neq 0, 1, i$

$$F_S^{\Omega_R}(0,\xi^*) < F_A^{\Omega_R}(0,\xi^*)$$

Now Ω_R is hyperconvex because $h_R(z,w)/R$ is a a continuous plurisubharmonic exhaustion that takes values in $[-1,0)$.

1. $h_R(z,w)/R$ is clearly plurisubharmonic.

2. $h_R(z,w)/R$ is continuous because $f(z,w)$ is countinuous outside of D_i, $|(z,w)|-R$ is continuous everywhere, and $|(z,w)|-R$ dominates on D_i.

3. Both $f(z,w)$ and $|(z,w)|-R$ are always negative so their maximum is negative on Ω_R. $|(z,w)|-R$ is larger than $-R$ so our function takes values in $[-R,0)$

4. Let $S(c)=\{(z,w) : h_R(z,w)/R < -c\}$. Then $S(c)=S_1(c)\cap S_2(c)$ where $S_1(c)=\{(z,w) : f(z,w) < -c\}$ and $S_2(c)=\{(z,w) : |(z,w)|-R < -c\}$. Looking at $S_2(c)$ gives us that $S(c)\subset \mathbb{B}_{R-c}$, so $S(c)$ is compactly contained away from the boundary of $\mathbb{B}_R$. Now notice that since $f(z,w)$ is continuous away from L_i, so $S_1(c)$ must be compactly contained away from the boundary of Ω. Thus $S(c)$ is contained away from the boundary of Ω_R and $h_R(z,w)/R$ is an exhaustion.

□

3.3 Azukawa and Sibony Hyperbolicity

The Azukawa and Sibony metrics are closely related but different, so an interesting question is whether Azukawa and Sibony hyperbolicity are equivalent. There are two notions of hyperbolicity that one could study, which we define now.

Definition III.11 (Pointwise Hyperbolicity)**.** We say that $\Omega\subset\mathbb{C}^n$ is pointwise hyperbolic with respect to a pseudometric $F^{\Omega}_m(z,\xi)$ if for all $\xi\neq 0, F^{\Omega}_m(z,\xi)>0$.

Definition III.12 (Hyperbolicity)**.** We say that $\Omega\subset\mathbb{C}^n$ is hyperbolic with respect to a pseudometric $F_m^\Omega(z,\xi)$ if for all $z \in \Omega$ there exists a neighborhood $U \subset \Omega$ and a constant $C > 0$ such that $F_m^W(z,\xi) \geq C||\xi||$ for all $z \in U$ and $\xi \in \mathbb{C}^n$.

Metric hyperbolicity as defined above is closely related to distance hyperbolicity. Here distance refers to the corresponding distance given by integrating the metric, which gives a distance between two points rather than assigning a length to each tangent vector. Distance hyperbolicity means that any two non-equal points have non-zero distance. For the Kobayashi metric, metric and distance hyperbolicity are equivalent [19, p. 207]. For the Carathéodory metric, metric and distance hyperbolicity are equivalent for domains in $\mathbb{C}$, but in higher dimensions the question is still open [19, p. 207, p. 28]. On the other hand, pointwise hyperbolicity does not in general imply distance hyperbolicity. Even though integrating over any given curve results in a positive distance, the infimum over all curves may still be zero. For an example of a domain that is pointwise hyperbolic with respect to the Kobayashi metric but which is not Kobayashi hyperbolic, see [19, p. 98].

In order to find a relationship among hyperbolicities of various metrics, we first look at how the metrics themselves are related. It is known that any distance decreasing map which agrees with the Poincare metric on the disk lies between the Carathéodory and Kobayashi metrics [19, p. 17], so we get the following inequality

$$F_C^\Omega(z,\xi) \leq F_S^\Omega(z,\xi) \leq F_A^\Omega(z,\xi) \leq F_K^\Omega(z,\xi).$$

Thus Carathéodory (pointwise) hyperbolicity $\Rightarrow$ Sibony (pointwise) hyperbolicity $\Rightarrow$ Azukawa (pointwise) hyperbolicity $\Rightarrow$ Kobayashi (pointwise) hyperbolicity. The domain $\mathbb{C} \setminus \{0,1\}$ shows that Kobayashi (pointwise) hyperbolicity is not equivalent to Carathéodory, Sibony, or Azukawa (pointwise) hyperbolicity in general. For hyperbolicity of this domain, one just notes that it is covered by the disk, c.f. [19, p.

206]. For pointwise hyperbolicity, one can apply Landau's theorem. A special case of Landau's theorem that will suffice is the following.

Theorem III.13 (Landau)**.** *Let $f : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ be a holomorphic mapping with expansion*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where $a_1 \neq 0$. Let τ denote the inverse of the elliptic modular function. Then

$$1 \leq \frac{2\mathrm{Im}\,(\tau(a_0))}{|a_1||\tau'(a_0)|}$$

A proof of Landau's theorem can be found in [8, p. 196-197]. We will now use Landau's theorem to show that $\mathbb{C} \setminus \{0, 1\}$ is pointwise Kobayashi hyperbolic.

Proposition III.14. *The domain $\mathbb{C} \setminus \{0, 1\}$ is pointwise Kobayashi hyperbolic.*

Proof. Let $p \in \Omega$ and let $f : \mathbb{D} \longrightarrow \Omega$ be a holomorphic mapping with $f(0) = p$. Then by Theorem III.13, we have

$$\frac{1}{|f'(0)|} \geq \frac{|\tau'(p)|}{2\mathrm{Im}\,\{\tau(p)\}}$$

As τ is conformal, τ' does not vanish. Since this estimate is independent of f, the Kobayashi metric does not vanish. $\square$

For our theorem about Azukawa and Sibony pointwise hyperbolicity we will need the following definition.

Definition III.15. Let λ be in the unit disk. We say that $\Omega \subset \mathbb{C}^n$ is a balanced domain if $z \in \Omega \Rightarrow \lambda z \in \Omega$. The Minkowski function $h(z) : \mathbb{C}^n \to \mathbb{R}$ of a balanced domain Ω is given by

$$h(z) = \inf \{t > 0 : z/t \in \Omega\}. \tag{3.17}$$

Notice that $\Omega = \{z \in \mathbb{C}^n : h(z) < 1\}$. It is known that if Ω is a balanced domain, then the pluricomplex Green function is given by

$$u(0, w) = h(w), \tag{3.18}$$

and the Azukawa metric is given by

$$F_A^{\Omega}(0, \xi) = h(\xi), \tag{3.19}$$

where h is the Minkowski function on Ω, c.f. [19, p. 119].

One might suspect that Azukawa and Sibony hyperbolicity could be equivalent since the two metrics are so closely related. We will give an example of a pseudoconvex domain which is pointwise Azukawa hyperbolic but not pointwise Sibony hyperbolic. The domain we use was constructed by Azukawa in [2].

Theorem III.16. *There exists a pseudoconvex domain $\Omega \subset \mathbb{C}^2$ that is Azukawa pointwise hyperbolic but not Sibony pointwise hyperbolic. Define Ω to be*

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : |(z_1, z_2)| < f(z_1, z_2)\right\},$$

where f is defined as

$$f(z_1, z_2) = f(z_1/z_2) = \begin{cases} (1 + |z_1/z_2|^2)^{1/2} e^{-v(z_1/z_2)} & z_2 \neq 0 \\ 1 & z_2 = 0 \end{cases} \tag{3.20}$$

and $v(\lambda)$ is defined as

$$v(\lambda) = \max\left\{\log|\lambda|, \sum_{k=2}^{\infty} k^{-2} \log|\lambda - 1/k|\right\}$$

Proof. Azukawa showed in [2] that Ω is a balanced pseudoconvex domain, so we can apply equation 3.19 to get that

$$F_A^{\Omega}((0,0),\xi) = h(\xi), \tag{3.21}$$

where h is the Minkowski function defined in equation 3.17.

Complex lines through the origin are of the form $(a\tau, b\tau)$ with $a, b \in \mathbb{C}$ constants and $\tau \in \mathbb{C}$ a complex parameter. So when $z_2 \neq 0$, $|z_1/z_2| = a/b$ is well defined and finite. We want to check that no complex lines through the origin are contained in Ω. If Ω contained a complex line through the origin, we would have $f(a\tau, b\tau) = \infty$ for some (a, b). In the case $z_2 \neq 0$, since a/b is finite we don't need to worry about the term $(1 + |z_1/z_2|^2)^{1/2}$. The exponential term could only cause us trouble if $v(z_1/z_2) = -\infty$. Since v is defined as a maximum, both terms in it would have to be $-\infty$. The first term being $-\infty$ would mean that $z_1/z_2 = 0$, i.e. $a = 0$. Once $a = 0$, the second term becomes $\sum_{k=2}^{\infty} k^{-2} \log(1/k)$, which converges to a finite number. Thus Ω contains no complex lines through the origin. Then by formula 3.21, Ω is Azukawa pointwise hyperbolic at the origin.

On the other hand, candidates for the Sibony metric must lie underneath the Green function. Since Ω is a balanced pseudoconvex domain, we can apply equation 3.18 to get $u(0, w) = \log h(w)$. Again considering complex lines through the origin, $(a\tau, b\tau)$, we see that if $a = 1/k$ and $b = 1$ then the second term in $v(\lambda)$ is $-\infty$. Since the first term is $\log|1/k|$, we see that a disk of radius $k(1+|1/k|^2)^{1/2}$ in that direction is contained in Ω. As $k \to \infty$ we get disks of arbitrarily large radius through the origin, and the directions of these disks approach the direction $z_1 = 0$. Since Ω contains arbitrarily large disks near $z_1 = 0$ and since candidates v for the Sibony metric are C^2 near the origin, we must have $v|_{\{z_1=0\}} = 0$ near the origin. Thus the

Sibony metric in this direction is zero.

□

Note that this provides an example of a domain that is Kobayashi pointwise hyperbolic at the origin but not Kobayashi hyperbolic, since balanced domains are Kobayashi hyperbolic if and only if they are bounded, [24].

This result contrasts the result for distance hyperbolicity. Kodama showed that for balanced domains, Carathéodory distance hyperbolicity and Kobayashi distance hyperbolicity are equivalent, and that these two properties are also equivalent to the domain being bounded, [24]. For Kobayashi distance hyperbolicity the metric must be locally uniformly bounded away from zero, which our example is not.

It would be an interesting question to see whether Azukawa and Sibony hyperbolicity are equivalent.

CHAPTER IV

A generalized pluricomplex Green function

4.1 Some Simple Cases

In this section we will study the generalized Green function U from Definition II.2 in Section 2.1 in certain cases that have simple proofs. We will prove three theorems which compare the generalized Green function to the usual pluricomplex Green function.

Theorem IV.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Then we can compare the generalized Green function U to the usual pluricomplex Green functions with poles at the zeros of $||F||$ as follows:*

$$U(z) \leq \inf_{\{w:||F(w)||=0\}} g_{\Omega}(z, w).$$

Proof. Let $F = (f_1, ..., f_m)$. Then each f_i is totally differentiable since the partial derivatives exist and are continuous. Assume that $w = 0$. So we get

$$\limsup_{z\to 0}\frac{|f_i(z)|}{||z||} \le \limsup_{z\to 0}\frac{|f_i(z)-f_i'(0)\cdot z|+|f_i'(0)\cdot z|}{||z||}$$
$$\le \limsup_{z\to 0}\frac{|f_i(z)-f_i'(0)\cdot z|}{||z||}+\limsup_{z\to 0}\frac{|f_i'(0)\cdot z|}{||z||}$$

$$= \limsup_{z\to 0}\frac{|f_i'(0)\cdot z|}{||z||}$$
$$\le \limsup_{z\to 0}\frac{||f_i'(0)||\cdot ||z||}{||z||}$$
$$= ||f_i'(0)||.$$

Thus

$$\limsup_{z\to 0}\frac{||F(z)||^2}{||z||^2} = \limsup_{z\to 0}\sum_{i=1}^{m}\frac{|f_i(z)|^2}{||z||^2} \le \sum_{i=1}^{m}||f_i'(0)||^2.$$

So for any competitor $u \in \mathcal{T}(\Omega, F)$, in a neighborhood of the origin

$$u(z) \le \log ||F(z)|| + c(u)$$
$$= \log ||z|| + \log \frac{||F(z)||}{||z||} + c(u)$$
$$\le \log ||z|| + c'(u),$$

which shows that $u \in \mathcal{K}(\Omega, 0)$, i.e. u is also a candidate for the usual pluricomplex Green function, $g_\Omega(z,0)$ on Ω. Taking the supremum over $u \in \mathcal{T}(\Omega, F)$ we get that $U(z) \le g_\Omega(z,0)$. We were assuming $w = 0$ but this argument would work for any $w \in \Omega$.

□

An easy example in which the generalized Green function is strictly less than the Green function even when F has a single isolated zero is the unit disk with the function $F(z) = z^2$. Then Green function on the disk is $\log|z|$, but $\log|z| - \log|z|^2$ is unbounded as $z \to 0$.

Theorem IV.2. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Suppose that the rank of the complex Jacobian is n at each point w with $||F(w)|| = 0$. If there is a function u satisfying $u \in \cap_{\{w:||F(w)||=0\}}\mathcal{K}(\Omega, w)$ then $u \in \mathcal{T}(\Omega, F)$. In particular in the case where $||F||$ has a single zero, $g_\Omega(z, w) \leq U(z)$.*

Proof. Again assume that $w = 0$. We will do something similar to the last proof, except now we will consider the reciprocal of $\frac{||F(z)||}{||z||}$. We can use the triangle inequality and the fact that F is totally differentiable to show that

$$\begin{aligned}
\liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i(z)|^2}{||z||^2} &\geq \liminf_{z\to 0} \sum_{i=1}^{m} \frac{(-|f_i(z) - f_i'(0)\cdot z| + |f_i'(0)\cdot z|)^2}{||z||^2} \\
&\geq \liminf_{z\to 0} \Bigg(\sum_{i=1}^{m} \frac{|f_i'(0)\cdot z|^2}{||z||^2} \\
&+ \sum_{i=1}^{m} \frac{|f_i(z) - f_i'(0)\cdot z|^2 - 2|f_i(z) - f_i'(0)\cdot z|\cdot|f_i'(0)\cdot z|}{||z||^2} \Bigg) \\
&\geq \liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i'(0)\cdot z|^2}{||z||^2} \\
&+ \liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i(z) - f_i'(0)\cdot z|^2 - 2|f_i(z) - f_i'(0)\cdot z|\cdot|f_i'(0)\cdot z|}{||z||^2} \\
&= \liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i'(0)\cdot z|^2}{||z||^2},
\end{aligned}$$

so that

$$\limsup_{z\to 0} \frac{||z||^2}{||F(z)||^2} = \frac{1}{\liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i|^2}{||z||^2}} \leq \frac{1}{\liminf_{z\to 0} \sum_{i=1}^{m} \frac{|f_i'(0)\cdot z|^2}{||z||^2}}. \tag{4.1}$$

Equation 4.1 will be bounded above if

$$\sum_{i=1}^{m} \frac{|f_i'(0) \cdot z|^2}{||z||^2} = \sum_{i=1}^{m} \left| f_i'(0) \cdot \frac{z}{||z||} \right|^2$$

is bounded away from zero in a neighborhood of the origin, which occurs if and only if the kernel of the complex Jacobian is zero.

So for any competitor $u \in \mathcal{K}(\Omega, 0)$, in a neighborhood of the origin

$$\begin{aligned} u(z) &\leq \log ||z|| + c(u) \\ &= \log ||F(z)|| + \log \frac{||z||}{||F(z)||} + c(u) \\ &\leq \log ||F(z)|| + c'(u). \end{aligned}$$

We were assuming that w was the origin, but this argument would work for any $w \in \Omega$. So if we have a function

$$u \in \cap_{\{w:||F(w)||=0\}} \mathcal{K}(\Omega, w),$$

then $u \in \mathcal{T}(\Omega, F)$, i.e. u is also a candidate for the generalized Green function, $U(z)$ on Ω.

□

Another case with a simple proof is the case where F is a homogeneous polynomial.

Theorem IV.3. *Let Ω be a domain in $\mathbb{C}^n$ that includes the origin, and let $F = (f_1, ..., f_m)$ be a polynomial function where f_i are all homogeneous polynomials of the same degree, d, and which have common zero only at the origin. Then $U(z) = d \cdot g_\Omega(z, 0)$, where $g_\Omega(z, 0)$ is the usual pluricomplex Green function with pole at the origin.*

Proof. We can scale the domain to assume that it contains the unit ball. Let $M = \max\{||F(z)|| : ||z|| = 1\}$ and let $m = \min\{||F(z)|| : ||z|| = 1\}$. Note that by assumption $m \neq 0$. By homogeneity when $z \neq 0$

$$F(z) = ||z||^d F(z/||z||),$$

so that

$$m||z||^d \leq ||F(z)|| \leq M||z||^d.$$

Thus in a neighborhood of the origin any candidadate $u \in \mathcal{T}(\Omega, F)$ satisfies

$$\begin{aligned} u(z) &\leq \log ||F(z)|| + c(u) \\ &\leq d \log ||z|| + \log M + c(u), \end{aligned}$$

so that $\frac{1}{d}u \in \mathcal{K}(\Omega, 0)$, i.e. it is a candidate for the usual pluricomplex Green function. Furthermore, given a candidate $u' \in \mathcal{K}(\Omega, 0)$ we have that

$$\begin{aligned} du'(z) &\leq d \log ||z|| + d \cdot c(u') \\ &\leq d \log ||z|| + \log m - \log m + d \cdot c(u') \\ &\leq \log ||F(z)|| + C'(u'), \end{aligned}$$

so that $du' \in \mathcal{T}(\Omega, F)$.

□

4.2 A Lelong Type Number

In this section our motivating question to see under what conditions the generalized Green function can be compared to a multiple of the usual pluricomplex Green function.

The Lelong number is a number associated to a singularity of a plurisubharmonic function, which measures in some sense the strength of the singularity.

Definition IV.4 (Lelong Number)**.** Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a plurisubharmonic function on Ω. Let w be a singularity of ϕ, i.e. a point in Ω so that $\phi(w) = -\infty$. The Lelong number of ϕ at w is given by

$$\nu^L(\phi) = \lim_{r \to 0} \frac{sup_{B(w,r)}\phi}{\log r}.$$

We will define another number associated to a plurisubharmonic function, motivated by the Lelong number but tailored to our particular situation. We will call this the Green Lelong number because of Theorem IV.6.

Definition IV.5 (Green Lelong Number)**.** Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a plurisubharmonic function on Ω . Let w be a singular point of ϕ. Then we define the Green-Lelong number to be

$$\nu_w^{GL}(\phi) = \lim_{r \to 0} \frac{\inf_{\partial B(0,r)} \phi}{\log r},$$

if such a limit exists.

If there is no chance of confusion we will drop the index w on the Green Lelong number and just say $\nu^{GL}(\phi)$.

Theorem IV.6. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be holomorphic. Suppose that the Green-Lelong numbers exist and are finite at each point w where $||F(w)|| = 0$, and that for each w there is a number $\epsilon(w) > 0$ such that there exists a function u satisfying $\frac{1}{\nu_w^{GL}(\log ||F||)+\epsilon(w)} \cdot u(z) \in \mathcal{K}(\Omega, w)$ for each w, with u*

independent of w. Then $u \in \mathcal{T}(\Omega, F)$. In particular if $||F||$ has a single zero, then $\nu^{GL}(\log ||F||) \cdot g_\Omega(z,w) \leq U(z)$.

Proof. Let us start with the case of a single isolated zero $w = 0$. Since we are assuming $\nu^{GL}(\log ||F||) > 0$, the limit in the Green Lelong number exists. Given $\epsilon > 0$ we can find $r^*(\epsilon)$ so that for all $r < r^*(\epsilon)$, on $\partial B(0,r)$,

$$\frac{\inf_{\partial B(0,r)} \log ||F||}{\log r} - \epsilon \leq \nu^{GL}(\log ||F||),$$

so that for $||z|| = r < r^*(\epsilon)$

$$\begin{aligned} \nu^{GL}(\log ||F||) \cdot \log ||z|| &= \nu^{GL}(\log ||F||) \cdot \log r \\ &\leq \inf_{\partial B(0,r)} \log ||F|| - \epsilon \log r, \\ &\leq \log ||F(z)|| - \epsilon \log r, \\ &= \log ||F(z)|| - \epsilon \log ||z||, \end{aligned}$$

so that for $||z|| < r^*(\epsilon)$

$$(\nu^{GL}(\log ||F||) + \epsilon) \log ||z|| \leq \log ||F(z)||.$$

This means that if $\frac{1}{\nu^{GL}(\log ||F||)+\epsilon} u \in \mathcal{K}(\Omega, 0)$ then $u \in \mathcal{T}(\Omega, F)$. Thus

$$\begin{aligned} U(z) &= \sup_{u \in \mathcal{T}(\Omega,F)} u(z) \\ &\geq \sup_{\frac{1}{\nu^{GL}(\log ||F||)+\epsilon} u \in \mathcal{K}(\Omega,0)} u(z) \\ &= \sup_{v \in \mathcal{K}(\Omega,0)} (\nu^{GL}(\log ||F||) + \epsilon) v(z) \\ &= (\nu^{GL}(\log ||F||) + \epsilon) g_\Omega(z, 0). \end{aligned}$$

Letting $\epsilon \longrightarrow 0$ we get the result in the case of a single zero.

In the case of more zeros, at each zero w we can work out the inequality given by the the Green Lelong number at w. Following an argument analogous to the case of a signle zero, given $\epsilon(w) > 0$ we can find a neighborhood $N(w)$ of w where

$$(\nu_w^{GL}(\log ||F||) + \epsilon(w)) \log ||z - w|| \leq \log ||F(z)||.$$

The result now follows.

□

Now we will work out an example where one can calculate the Green Lelong number explicitly.

Example IV.7. Let $F = (z, z + w^2)$ be defined on the unit ball in $\mathbb{C}^2$. Then the only zero of F is at the origin.

First we calculate the Lelong number of $\log ||F||$, which turns out to be 1. By picking a particular point, $z = r$ and $w = 0$, we know that the supremum lies above that and we get the following inequality.

$$\begin{aligned} &\limsup_{r \to 0} \frac{\sup_{B(0,r)} \frac{1}{2} \log(|z|^2 + |z + w^2|^2)}{\log r} \\ &\geq \limsup_{r \to 0} \frac{\frac{1}{2} \log(2r^2)}{\log r} \\ &= 1 \end{aligned}$$

Notice that the same inequalities hold for $\liminf$. On the other hand using the fact that $|z|$ and $|w|$ are each necessarily less than r, we can show the other inequality.

$$
\begin{aligned}
&\limsup_{r\to 0}\frac{\sup_{B(0,r)}\frac{1}{2}\log(|z|^2+|z+w^2|^2)}{\log r}\\
&=\limsup_{r\to 0}\frac{\sup_{B(0,r)}\frac{1}{2}\log(2|z|^2+|w|^4+z\overline{w}^2+w^2\overline{z})}{\log r}\\
&=\limsup_{r\to 0}\frac{\sup_{B(0,r)}\frac{1}{2}\log(2|z|^2+(r^2-|z|^2)^2+2\mathrm{Re}(w^2\overline{z}))}{\log r}\\
&=\limsup_{r\to 0}\frac{\sup_{B(0,r)}\frac{1}{2}\log(2(1-r^2)|z|^2+r^4+|z|^4+2\mathrm{Re}(w^2\overline{z}))}{\log r}\\
&\le\limsup_{r\to 0}\frac{\frac{1}{2}\log(2r^2+2r^3)}{\log r}\\
&=\limsup_{r\to 0}\frac{\log(r^2)+\log(2+2r)}{2\log r}\\
&=1
\end{aligned}
$$

Again, the same inequalities hold for $\liminf$, so in fact the limit exists and is 1.

Next we will calculate the Green Lelong number of $\log||F||$, which turns out to be two. By picking a particular point: $z=0$ and $w=r$, we know that the infimum will lie below that and we get the following inequality.

$$
\begin{aligned}
&\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(|z|^2+|z+w^2|^2)}{\log r}\\
&\le\limsup_{r\to 0}\frac{\frac{1}{2}\log(r^4)}{\log r}\\
&=2
\end{aligned}
$$

Again, the same inequality holds with $\liminf$. On the other hand on the sphere $||(z,w)||=r$, we get the following inequalities. The first inequality below follows from the fact that $2\mathrm{Re}(\mathrm{w}^2\mathrm{z})\ge -2(\mathrm{r}^2-|\mathrm{z}|^2)|\mathrm{z}|$ and the second inequality follows from dropping some positive terms.

$$\begin{aligned}
&\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(|z|^2+|z+w^2|^2)}{\log r}\\
&=\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(2|z|^2+|w|^4+z\overline{w}^2+w^2\overline{z})}{\log r}\\
&=\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(2|z|^2+(r^2-|z|^2)^2+2\mathrm{Re}(w^2\overline{z}))}{\log r}\\
&=\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(2(1-r^2)|z|^2+r^4+|z|^4+2\mathrm{Re}(w^2\overline{z}))}{\log r}\\
&\geq\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(2(1-r^2)|z|^2+r^4+|z|^4-2(r^2-|z|^2)|z|)}{\log r}\\
&=\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(|z|^4+2|z|^3+2(1-r^2)|z|^2-2r^2|z|+r^4)}{\log r}\\
&\geq\limsup_{r\to 0}\frac{\inf_{\partial B(0,r)}\frac{1}{2}\log(2(1-r^2)|z|^2-2r^2|z|+r^4)}{\log r}
\end{aligned}$$

Notice that the equation $f(x)=2(1-r^2)x^2-2r^2x$ has critical point at x^* when $4(1-r^2)x^*-2r^2=0$, or

$$x^*=\frac{r^2}{2(1-r^2)},$$

and $f(x^*)$ takes the value

$$f(x^*)=\frac{2(1-r^2)r^4}{4(1-r^2)^2}-\frac{2r^4}{2(1-r^2)}=\frac{r^4}{2(1-r^2)}-\frac{2r^4}{2(1-r^2)}=\frac{-r^4}{2(1-r^2)},$$

which is negative for r small enough. We claim that this is a minimum of $f(x)$: the second derivative is $4(1-r^2)$ which is positive; and at the endpoints we have $f(0)=0$, and for $f(r)$ we calculate

$$2(1-r^2)r^2-2r^4=2r^2-4r^4=2r^2(1-2r^2),$$

which is positive for r small enough. So we get

$$\begin{aligned}
&\limsup_{r\to 0} \frac{\inf_{\partial B(0,r)} \frac{1}{2}\log(|z|^2+|z+w^2|^2)}{\log r} \\
&\geq \limsup_{r\to 0} \frac{\frac{1}{2}\log\left(\frac{-r^4}{2(1-r^2)}+r^4\right)}{\log r} \\
&= \limsup_{r\to 0} \frac{\frac{1}{2}\log\left(\frac{(1-2r^2)r^4}{2(1-r^2)}\right)}{\log r} \\
&= \limsup_{r\to 0} \frac{\frac{1}{2}\left(\log(1-2r^2)+\log r^4-\log(2-2r^2)\right)}{\log r} \\
&= 2
\end{aligned}$$

Again, the same inequalities hold with $\liminf$, so the limit exists and is 2.

4.3 Behavior under Holomorphic Maps

Let Ω be a domain in $\mathbb{C}^n$ and let $F:\Omega\to\mathbb{C}^m$ be holomorphic. Consider a second domain Ω' in $\mathbb{C}^l$ and a holomorphic function $h:\Omega'\to\Omega$. We would like to compare the generalized Green function on Ω associated to the function F, call this U_Ω^F, to the generalized Green function on Ω' associated to the function $F\circ h$, call this $U_{\Omega'}^{F\circ h}$.

Let $u\in\mathcal{T}(\Omega,F)$ be a candidate function on Ω. We would like to show that $u\circ h$ is a candiate function on Ω', i.e. $u\circ h\in\mathcal{T}(\Omega',F\circ h)$. It is clear that $u\circ h$ is negative and plurisubharmonic on Ω', so all we need to do is check its behavior near singular points. Let $w\in\Omega'$ be a point where $||F(h(w))||=0$. We know that in a neighborhood N of $h(w)$ in Ω

$$u(z)\leq \log||F(z)||+c(u).$$

Let $N'=h^{-1}(N)$. Then N' is an open set in Ω' that contains w. Since for each $z'\in N'$ we know that $h(z')\in N$, we get that for $z'\in N'$

$$u(h(z')) \leq \log ||F(h(z'))|| + c(u)$$

Thus $u \circ h \in \mathcal{T}(\Omega', F \circ h)$.

Lemma IV.8. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be a holomorphic function on Ω. Let Ω' in $\mathbb{C}^l$ be a second domain and let $h : \Omega' \to \Omega$ be a holomorphic function. Then the generalized Green function related to F on Ω is related to the generalized Green function related to $F \circ h$ on Ω' by*

$$U_{\Omega'}^{F \circ h}(z') \geq U_{\Omega}^{F}(h(z')).$$

for $z' \in \Omega'$.

Proof. In the argument above, we have shown that if $u \in \mathcal{T}(\Omega, F)$ then $u \circ h \in \mathcal{T}(\Omega, F \circ h)$. Thus

$$U_{\Omega'}(z') \geq \sup_{u \in \mathcal{T}(\Omega, F)} u \circ h(z') = U_{\Omega}(h(z')).$$

$\square$

In the case of a biholomorphism, we get equality in the lemma above.

Corollary IV.9. *Let Ω and Ω' be domains in $\mathbb{C}^n$ and let $B : \Omega' \to \Omega$ be a biholomorphism. Let $F : \Omega \to \mathbb{C}^m$ be a holomorphic function. Then the generalized Green functions related to $F \circ B$ on Ω' and F on Ω are related by*

$$U_{\Omega'}^{F \circ B}(z') = U_{\Omega}^{F}(B(z')).$$

for $z' \in \Omega'$.

Proof. We immediately get that

$$U_{\Omega'}^{F\circ B}(z') \geq U_{\Omega}^{F}(B(z')).$$

Now if we consider $B^{-1} : \Omega \to \Omega'$ together with the function $F \circ B$ on Ω' we get

$$U_{\Omega}^{F\circ B\circ B^{-1}}(z) \geq U_{\Omega'}^{F\circ B}(B^{-1}(z)).$$

for $z \in \Omega$. Setting $z = B(z')$ we get the desired result. □

If we specialize to the case where $h = i$, the inclusion map, we get the following result that we will make use of in this thesis.

Corollary IV.10. *Let $\Omega' \subset \Omega$ be domains in $\mathbb{C}^n$ and let $i : \Omega' \to \Omega$ be the inclusion map. Let $F : \Omega \to \mathbb{C}^m$ be a holomorphic function. Then the generalized Green functions related to F on Ω' and Ω are related by*

$$U_{\Omega'}(z') \geq U_{\Omega}(z').$$

for $z' \in \Omega'$.

Using this corollary about the behavior of the generalized Green function under inclusion, we will now prove the key lemma for our main result in the special case of one dimension. We will start by defining the domain where we will ultimately be able to calculate the Taylor Green function in both one and higher dimensions.

Definition IV.11 (The domain $B_{\epsilon}^{F}(w)$)**.** Let $\Omega \subset \mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with $w \in \Omega$ being a point where $||F(w)|| = 0$. We use the notation $B_{\epsilon}^{F}(w)$ to denote the connected component containing w of the open set

$$\{z \in \Omega : ||F(z)|| < \epsilon\}.$$

Now we will work with this domain in one dimension, and in sections 4.4 and 4.5 we will work with this domain in higher dimensions.

Theorem IV.12. *Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be holomorphic with isolated zeros. Then $U(z) \leq \log|f| + c(w)$ near points w where $f(w) = 0$.*

Proof. We claim that we can pick ϵ small enough so that $B_\epsilon(w) \subset\subset \Omega$ and so that w is the only zero of f in $B_\epsilon(w)$. Take a Euclidean ball of small radius r around w which is compactly contained in Ω and whose closure contains no zeros of f other than w. If for all ϵ, $B_\epsilon(w)$ is not contained in this Euclidean ball then by connectedness of $B_\epsilon(w)$ we can find a point of length equal to r in $B_\epsilon(w)$, which we will call z_ϵ. The sequence $z_{1/n}$ will have a convergent subsequence, converging to z of length r, and by continuity $f(z) = 0$, which violates our assumption that the closure of the Euclidean ball had no zeros other than w. Consider now instead the domain $B = B_\epsilon(w)$. We claim that $U_B = \log|f/\epsilon|$. Suppose that there exists a candidate function u_B^* that is larger than $\log|f/\epsilon|$ at some point. By taking $\max\{u_B^*, \log|f/\epsilon|\}$, we may assume $u_B^* \geq \log|f/\epsilon|$. Then u_B^* extends continuously to the boundary and is zero there. We know that $\log|f|$ is harmonic off of w and $u_B^* - \log|f|$ is bounded above in a neighborhood of w, so that $u_B^* - \log|f/\epsilon|$ extends to be subharmonic on B. Then $u_B^* - \log|f/\epsilon| \equiv 0$ by the maximum principle because it is nonnegative and zero on the boundary, which shows that $U_B = \log|f/\epsilon|$. Since $U_\Omega \leq U_B$ the result follows.

□

4.4 Plurisubharmonicity with Isolated Zeros using Resolution of Singularities

In this section we will study the generalized Green function U defined in definition II.2 in a case that is more complicated than the examples from the previous chapter.

We present our first proof of Theorem II.3, which follows the outline of the proof in the case of the pluricomplex Green function. In this case however, we must work with varieties and ultimately use Resolution of Singularities to get the result.

Recall the definition of the domain $B_\epsilon(w)$ from definition IV.11.

Lemma IV.13. *Let $\Omega \subset \mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros, and let $w \in \Omega$ be such that $||F(w)|| = 0$. For small enough ϵ we get the following three properties:*

1. *$B_\epsilon^F(w)$ contains no zeros of F other than w.*

2. *$B_\epsilon^F(w) \subset\subset \Omega$.*

3. *The sets $\{E^n\}_{n\in\mathbb{N}}$, defined by*

$$E^n = \{||F(z)|| < \epsilon - 1/n\} \cap B_\epsilon^F(w),$$

are open and their closures, $\left\{\overline{E^n}\right\}$, give a compact exhaustion of $B_\epsilon^F(w)$. We also get the slightly stronger result that $\overline{E^n} \subset E^{n_1}$ as opposed to the usual $\overline{E^n}$ being a subset of the interior of $\overline{E^{n+1}}$. Also,

$$\partial E^n \subset \{||F(z)|| = \epsilon - 1/n\}.$$

and

$$\partial B_\epsilon^F(w) \subset \{||F(z)|| = \epsilon\}.$$

Proof. (1) Since F has isolated zeros there is some euclidean ball around w of radius r, $B_r(w)$, which contains no zeros of F other than w and so that $B_r(w) \subset\subset \Omega$. We claim that for small enough ϵ, $B_\epsilon^F(w) \subset B_r(w)$. Suppose that for every $\epsilon > 0$, there is a point $p(\epsilon) \in B_\epsilon^F(w)$ so that $p(\epsilon) \notin B_r(w)$. Then there is a path $\gamma_{p(\epsilon)}$ from w to $p(\epsilon)$ so that $\gamma_{p(\epsilon)} \subset B_\epsilon^F(w)$ since $B_\epsilon^F(w)$ is path connected. Since $||\gamma_{p(\epsilon)}(0)|| = 0$, $||\gamma_{p(\epsilon)}(1)|| \geq r$, and $||\gamma_{p(\epsilon)}||$ is continuous, there is some $t^* \in (0,1)$ such that $||\gamma_{p(\epsilon)}(t^*)|| = r/2$. Consider $\epsilon = 1/n$. Then since $B_{r/2}(w)$ is compact, the sequence $\gamma_{p(1/n)}(t^*)$ has a subsequence converging to the point $\gamma^* \in B_{r/2}(w)$. Since $||F(\gamma_{p(1/n)}(t^*))|| < 1/n$ and F is continuous, $||F(\gamma^*)|| = 0$. But we assumed that F has no zeros in $B_r(w)$ other than w, and $||\gamma^* - w|| = r/2$. Thus there exists $\epsilon > 0$ so that $B_\epsilon^F(w) \subset B_r(w)$.

(2) Let $B_r(w)$ be defined as above. Since $B_r(w) \subset\subset \Omega$, we get that $B_\epsilon^F(w) \subset\subset \Omega$ as well.

(3) It is clear from the definition, since $||F||$ is continuous, that E^n is open.

By saying that $\{\overline{E^n}\}$ are a compact exhaustion of $B_\epsilon^F(w)$ we mean that

1. $\overline{E^n}$ are compact in $B_\epsilon^F(w)$

2. $\overline{E^n}$ is contained in the interior of $\overline{E^{n+1}}$

3. and $\cup_n \overline{E^n} = B_\epsilon^F(w)$.

To show (1), let us first consider $\overline{E^n}$ to be the closure of E^n in $\mathbb{C}^n$, rather than Ω. We will later show that both closures are the same. Clearly in this sense $\overline{E^n} \subset \overline{B_\epsilon^F(w)}$, which is contained in Ω by (2). Now F is defined on all of Ω, so in particular it is continuous across the boundary of $B_\epsilon^F(w)$. We claim that

$$(4.2) \qquad \partial B_\epsilon^F(w) \subset \{||F(z)|| = \epsilon\}\,.$$

If $b \in \partial B_\epsilon^F(w)$ then there is a sequence $b_i \to b$ with $b_i \in B_\epsilon^F(w)$. In particular $||F(b_i)|| < \epsilon$, so $||F(b)|| \leq \epsilon$ by continuity of $||F||$. On the other hand, if $||F(b)|| < \epsilon$ then write $||F(b)|| = \epsilon - c$ with $c > 0$. The set $\{||F(z)|| < \epsilon - c/2\}$ is open and contains b, meaning that on some ball N_c centered at b, $||F(z)|| < \epsilon - c/2$. Since $b \in \partial B_\epsilon^F(w)$, $N_c \cap B_\epsilon^F(w)$ is nonempty. But then $N_c \subset B_\epsilon^F(w)$ since N_c is connected, intersects $B_\epsilon^F(w)$, and $||F|| < \epsilon$ on N_c. This contradicts the assumption that b is in the boundary of $B_\epsilon^F(w)$, so it is impossible for $||F(b)||$ to be strictly smaller than ϵ.

We would like to use equation (4.2) to show that $\overline{E^n} \subset B_\epsilon^F(w)$. For every point $a \in \partial E^n$ there is a sequence $a_i \to a$ with $a_i \in E^n$. In particular, $||F(a)|| \leq \epsilon - 1/n$ by continuity of $||F||$. This means that $a \notin \partial B_\epsilon^F(w)$, so that $a \in B_\epsilon^F(w)$ since $\overline{E^n} \subset \overline{B_\epsilon^F(w)}$. Thus $\overline{E^n} \subset B_\epsilon^F(w)$. By part (2) $B_\epsilon^F(w) \subset\subset \Omega$, so $\overline{E^n}$ is bounded, and thus compact, in $\mathbb{C}^n$.

It follows that $\overline{E^n}$ is compact in $B_\epsilon^F(w)$ since it is contained in $B_\epsilon^F(w)$, and any cover of open sets in $B_\epsilon^F(w)$ is a cover of open sets in $\mathbb{C}^n$ as well.

Note that while we initially were using the notation $\overline{E^n}$ to be the closure of E^n in $\mathbb{C}^n$, now that we showed $\overline{E^n} \subset B_\epsilon^F(w)$ it is equivalent to consider $\overline{E^n}$ to be the closure of E^n in $B_\epsilon^F(w)$.

To show (2), we claim that

$$(4.3) \qquad \partial E^n \subset \{||F(z)|| = \epsilon - 1/n\}\,.$$

Using the same notation as above, we know that $||F(a)|| \leq \epsilon - 1/n$. Since $a \in B_\epsilon^F(w)$, if $||F(a)|| < \epsilon - 1/n$ then $a \in E^n$ so (4.3) follows. This means that $\overline{E^n} \subset$

E^{n+1}, and E^{n+1} is open so it is a subset of the interior of $\overline{E^{n+1}}$.

For condition (3), clearly $\cup_n E^n = B_\epsilon^F(w)$.

□

Lemma IV.14. *Let $K \subset B_\epsilon^F(w)$ be compact, and let $V \subset B_\epsilon^F(w)$ be a complex analytic variety. Then $K \cap V$ is compact in V.*

Proof. The topology on V is the subspace topology coming from $B_\epsilon^F(w)$. Let $\{U_i\}_{i\in I}$ be open sets in V whose union covers $K \cap V$. Then there are open sets $U_i' \subset B_\epsilon^F(w)$ so that $U_i' \cap V = U_i$. We would like ot add more open sets to form a cover of K. We know that V is closed in $B_\epsilon^F(w)$ since it is the intersection of the zero sets of finitely many holomorphic functions. For each point $x \in K \setminus V$ pick a neighborhood $\mathcal{U}_x$ so that $\mathcal{U}_x \subset B_\epsilon^F(w) \setminus V$. Then

$$\{U_i'\}_{i\in I} \cup \{\mathcal{U}_x\}_{x\in K\setminus V}$$

is an open cover of K. Since K is compact there is a finite subcover which after reindexing includes $\{U_1', ..., U_n'\}$ and no other U_i'. Then $\{U_1, ..., U_n\}$ covers $K \cap V$ since $\mathcal{U}_x \cap V = \emptyset$ for all $x \in K \setminus V$. □

For a reference for the following lemma about extending plurisubharmonic functions across closed negative infinity sets of plurisubharmonic functions, see [23][p. 71].

Lemma IV.15. *Let $\Omega \subset \mathbb{C}^n$ be an open subset, and let F be a closed subset of Ω of the form $F = \{z \in \Omega : v(z) = -\infty\}$, where v is plurisubharmonic on Ω. If u is plurisubharmonic on $\Omega \setminus F$ and u is bounded above, then the function $\tilde{u}$ defined by*

$$\tilde{u}(z) = \begin{cases} u(z) & : z \in \Omega \setminus F \\ \limsup_{y \to z, y \notin F} u(y) & : z \in F \end{cases}$$

is plurisubharmonic on Ω. If Ω is connected, then so is $\Omega \setminus F$.

We will use the following elementary lemma in the course of proving Theorem IV.19. A proof can be found in [28][p. 97].

Lemma IV.16. *Let A be a subset of a topological space X; let A' be the set of all limit points of A, where $x \in X$ is a limit point of A if every neighborhood of x intersects A in some point other than x itself. Then $\overline{A} = A \cup A'$, where $\overline{A}$ is the intersection of all closed sets containing A.*

We will use the following two easy lemmas in the course of proving Theorem IV.19.

Lemma IV.17. *Let $S_1 \subset S_2 \subset M$, where M is a topological manifold, and let S_2 have the subspace topology given by M. Notice if $S_2 \subset \Omega$ where Ω is a domain in $\mathbb{C}^n$, then S_2 having the subspace topology given by Ω is the same as S_2 having the subspace topology given by $\mathbb{C}^n$. Let $\overline{S_1}$ be the closure of S_1 in S_2, i.e. the smallest closed set in S_2 containing S_1, which is the intersection of all closed sets in S_2 which contain S_1. Then if $x \in \overline{S_1}$ there exist $x_i \in S_1$ so that $x_i \to x$.*

Proof. If not, there exists an open set U containing x which is disjoint from S_1. Let $N = U|_{S_2}$, so that N is open in S_2 and N^c, the complement of N in S_2, is closed in S_2. Now $S_1 \subset N^c$, so we have that $S_1 \subset N' := N^c \cap \overline{S_1}$. But then N' is closed in S_2, contains S_1 and is stricly smaller than $\overline{S_1}$ since it does not contain x. This contradicts the assumption that $\overline{S_1}$ is the smallest closed set in S_2 containing S_1. So every ball centered at $x \in \overline{S_1}$ must contain a point from S_1. $\square$

Lemma IV.18. *Let $S_1 \subset S_2 \subset \mathbb{C}^n$ be sets so that S_2 is given the subspace topology from $\mathbb{C}^n$ and S_1 is closed in S_2. Let $x_i \subset S_1$ and $x \in S_2$ be such that $x_i \to x$. Then $x \in S_1$.*

Proof. If $x \notin S_1$, there would be an open set $U \subset \mathbb{C}^n$ containing x so that $U \cap S_1 = \emptyset$. But every such U contains the tail end of the sequence x_i. $\square$

For an explanation of the domain $B_\epsilon^F(w)$, see definition IV.11.

Theorem IV.19. *Let $\Omega \subset \mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with an isolated zero at w. Then for ϵ small enough, $U_{B_\epsilon^F(w)} = \log ||F(z)/\epsilon||$.*

Proof. Our strategy will be to show that $U_{B_\epsilon^F(w)} = \log ||F(z)/\epsilon||$ by restricting to varieties on which $-\log ||F||$ is subharmonic. Along these varieties we will be able to apply the maximum principle to $U_{B_\epsilon^F(w)} - \log ||F/\epsilon||$ to show that $U_{B_\epsilon^F(w)} \leq \log ||F/\epsilon||$. Since $\log ||F/\epsilon||$ is itself a candidate function for $U_{B_\epsilon^F(w)}$, this inequality will be enough to prove the result.

Pick $\epsilon > 0$ as in Lemma IV.13. Let $p \in B_\epsilon^F(w)$ with $p \neq w$, so that $f_i(p) \neq 0$ for some $1 \leq i \leq n$. We can renumber so that $f_1(p) \neq 0$. We define a complex analytic variety V_p passing through p as follows:

$$V_p = B_\epsilon^F(w) \cap \bigcap_{i=2}^{n} \left\{ \frac{f_i(p)}{f_1(p)} f_1 = f_i \right\}.$$

Notice that V_p is dimension at least one since F maps into $\mathbb{C}^n$, and is nonempty because it contains at least p and w. Also, every point in Ω is in such a subvariety because if $p \neq w$ then $p \in V_p$, whereas w is in every such subvariety.

The purpose of defining V_p in this way is so that on V_p we can write F in terms of f_1. This leads us to the equation

$$\log ||F/\epsilon|| = \log |f_1/\epsilon| + c(V_p) \text{ on } V_p, \tag{4.4}$$

where

$$c(V_p) = \log \sqrt{1 + \sum_{i=2}^{n} \left| \frac{f_i(p)}{f_1(p)} \right|^2}.$$

Actually the variety V_p is not so dependent on the point p in the sense that if $p' \in V_p$ with $p' \neq w$, then $V_p = V_{p'}$. The first point to note is that, according to the defining equations for V_p, if $f_1(p') = 0$ then $||F(p')||$ would be zero, which is impossible. Plugging p' into the defining equations for V_p and keeping in mind that $f_1(p') \neq 0$ shows that for each $i = 2, ...n$ we have

$$\frac{f_i(p)}{f_1(p)} = \frac{f_i(p')}{f_1(p')}.$$

Thus the defining equations for V_p and $V_{p'}$ are identical. Also, the constant $c(V_p)$ only depends on the variety V_p and not the point p used to define it, as the notation suggests. Finally, notice that it makes no sense to talk about V_w since $w \in V_p$ for every $p \in B_\epsilon^F(w)$.

We will now work on the desingularization manifold for V_p, see Theorem II.7. Our notation for the resolution of singularities for V_p will be $\pi : \tilde{V_p} \to V_p$. Let $u \in T(\Omega, F)$ be a candidate for $U_{B_\epsilon^F(w)}$, as defined in the introduction, and consider the function h on $\tilde{V_p}$ defined by

$$h := u \circ \pi - \log ||F \circ \pi/\epsilon|| \tag{4.5}$$

$$= u \circ \pi - \log |f_1 \circ \pi/\epsilon| - c(V_p). \tag{4.6}$$

The function u is plurisubharmonic on $B_\epsilon^F(w)$ and π is holomorphic on $\tilde{V}_p$, so $u \circ \pi$ is plurisubharmonic on $\tilde{V}_p$. On $\tilde{V}_p$, $\log|f_1/\epsilon|$ is harmonic or pluriharmonic away from $\pi^{-1}(w)$ since f_1 maps into $\mathbb{C}$. Thus, h is plurisubharmonic away from $\pi^{-1}(w)$. If we show that h is bounded above, since the zero set of a holomorphic function is closed, we can use Lemma IV.15 locally on $\tilde{V}_p$ to show that h extends to be be subharmonic or plurisubharmonic on $\tilde{V}_p$. Let $N(w)$ be a neighborhood of w in $B_\epsilon^F(w)$ on which $u - \log||F||$ is bounded above. Then $\tilde{N} := \pi^{-1}(N(w))$ is open in $\tilde{V}_p$ and contains $\pi^{-1}(w)$, and h is bounded above on $\tilde{N}$. Suppose there is a sequence of points $t_i \in \tilde{V}_p \setminus \tilde{N}$ so that $F(\pi(t_i)) \to 0$. Since $\overline{B_\epsilon^F(w)}$ is compact, a subsequence $\pi(t_{i_j})$ converges to a point in $\overline{B_\epsilon^F(w)}$. By Lemma IV.13 parts (1) and (3), w is the only zero of F in $\overline{\Omega}$, $\pi(t_{i_j})$ converges to w. However, then t_{i_j} are arbitrarily close to the set $\pi^{-1}(w)$, but this contradicts the assumption that $t_i \in \tilde{V}_p \setminus \tilde{N}$. Thus h is bounded above on $\tilde{V}_p$ and so extends to be plurisubharmonic.

Thus the maximum principle holds for h on $\tilde{V}_p$ for each p. In order to finish the proof of Theorem IV.19 we need to construct a compact exhaustion of $\tilde{V}_p$, so that the values of h on the boundary of these compact sets approach zero. Then we will be able to apply the maximum principle to get that $u \circ \pi \leq -\log||F \circ \pi/\epsilon||$ on $\tilde{V}_p$. Since π is surjective onto V_p, and since the V_p's cover $B_\epsilon^F(w)$, this will mean that $u \leq \log||F/\epsilon||$ on $B_\epsilon^F(w)$.

By Lemma IV.13 the sets E^n are open and $\left\{\overline{E^n}\right\}_{n\in\mathbb{N}}$ is a compact exhaustion of $B_\epsilon^F(w)$. Let

$$E_p^n = E^n \cap V_p.$$

We would like to show that the sets E_p^n are open in V_p and that $\left\{\overline{E_p^n}\right\}_{n\in\mathbb{N}}$ is a compact exhaustion of V_p. Notice here that $\overline{E_p^n}$ is the closure of E_p^n in V_p. It is clear that E_p^n are open in V_p since V_p has the subspace topology from $B_\epsilon^F(w)$ and E^n is open in $B_\epsilon^F(w)$. To show that a family of sets $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of a domain, manifold, or variety X, we need to show

1. K_n are compact in X,

2. K_n are contained in the interior of $\overline{K_{n+1}}$,

3. and $\cup_n K_n = X$.

To show (1) for $K_n = \overline{E_p^n}$ in $X = V_p$, we claim that

$$\overline{E_p^n} \subset \overline{E^n} \cap V_p.$$

First, we claim that we can apply Lemma IV.16 with $A = E_p^n$ and $X = V_p$ to show that if $x \in \overline{E_p^n}$ then there exist $x_i \in E_p^n$ with $x_i \to x$. If $x \in E_p^n$ then this is clear because we can set $x_i = x$ for all i. If $x \notin E_p^n$ then by Lemma IV.16, x is a limit point of E_p^n in V_p. Thus every neighborhood of x intersects E_p^n in some point other than x itself. Since V_p's topology is the subspace topology from $\mathbb{C}^n$, neighborhoods are given by euclidean balls intersected with V_p, so that inside each ball of radius $1/i$ centered at x we can find a point in E_p^n. These points will be x_i. Since $x_i \in E_p^n$, $x_i \in E^n$ as well. Next we will apply Lemma IV.16 with $A = E^n$ and $X = B_\epsilon^F(w)$ to show that $x \in \overline{E^n}$. Since $x \in V_p$, $x \in B_\epsilon^F(w)$ as well. Furthermore, x is a limit point of E^n since $x_i \in E^n$. It follows that $x \in \overline{E^n}$. Finally, we can verify our original claim because Lemma 2 shows that $\overline{E_p^n}$, being a closed subset of the compact set $\overline{E^n} \cap V_p$, is compact in V_p.

To show (2) for $K_n = \overline{E_p^n}$ in $X = V_p$, notice the following:

$$\overline{E_n^p} \subset (\overline{E^n} \cap V_p) \subset (E^{n+1} \cap V_p) = E_p^{n+1}.$$

Since E_p^{n+1} is open in V_p, it is a subset of the interior of $\overline{E_p^{n+1}} = \overline{E^{n+1} \cap V_p}$, so (2) follows.

To show (3) for $K_n = \overline{E_p^n}$ in $X = V_p$, let $x \in V_p$. Thus $x \in B_\epsilon^F(w)$, so $x \in \overline{E^n}$ for some n, which is contained in E^{n+1} by Lemma IV.13. This means that $x \in E_p^{n+1}$, so in particular $x \in \cup_n E_p^n$, which is contained in $\cup_n \overline{E_p^n}$. The result $V_p \subset \cup_n \overline{E_p^n}$ follows.

Now we want to pull $\left\{\overline{E_p^n}\right\}$ back to a compact exhaustion on $\tilde{V_p}$. Consider the sets

$$\tilde{E_p^n} = \pi^{-1}(E_p^n). \tag{4.7}$$

We would like to show that $\tilde{E_p^n}$ are open and that $\left\{\overline{\tilde{E_p^n}}\right\}_{n\in\mathbb{N}}$ is a compact exhaustion of $\tilde{V_p}$, where again we need to verify properties (1) - (3) for $K_n = \tilde{E_p^n}$ and $X = \tilde{V_p}$. It is clear that $\tilde{E_p^n}$ are open because $\pi : \tilde{V_p} \to V_p$ is continuous and E_p^n are open in V_p.

To show (1) we claim that

$$\overline{\tilde{E_p^n}} \subset \pi^{-1}(\overline{E_p^n}). \tag{4.8}$$

Take $x \in \overline{\tilde{E_p^n}}$. By Lemma IV.17, there exist $x_i \in \tilde{E_p^n}$ so that $x_i \to x$. Then $\pi(x_i) \in E_p^n$ and by continuity of π and $\pi(x_i) \to \pi(x)$. By Lemma IV.18 this means $\pi(x) \in \overline{E_p^n}$. Now since π is proper by equation (4.8) we see that the sets $\overline{\tilde{E_p^n}}$ are closed subsets of compacts sets and thus also compact, so (1) holds.

To show (2), we can expand equation (4.8) as follows:

(4.9) $$\overline{\tilde{E_p^n}} \subset \pi^{-1}(\overline{E_p^n}) \subset \pi^{-1}(E_p^{n+1}),$$

which is in turn a subset of the interior of $\overline{\pi^{-1}(E_p^{n+1})} = \overline{E_p^{\tilde{n+1}}}$.

To show (3), let $x \in \tilde{V_p}$. Then $\pi(x) \in V_p$, so for some n

$$\pi(x) \in \overline{E_p^n} \subset E_p^{n+1},$$

and thus

$$x \in \pi^{-1}(E_p^{n+1}) = E_p^{\tilde{n+1}} \subset \overline{E_p^{\tilde{n+1}}}.$$

So $\left\{\overline{\tilde{E_p^n}}\right\}$ is indeed a compact exhaustion of $\tilde{V_p}$.

Next we would like to examine how the function h, which was defined earlier in this proof, behaves on the boundary of $\tilde{E_p^n}$. Combining equations 4.7 and 4.8 shows that

$$\pi(\partial\tilde{E_p^n}) \subset \partial(E_p^n).$$

Since $\overline{E_p^n} \subset \overline{E^n}$, we claim it follows from the above equation that that

$$\pi(\partial\tilde{E_p^n}) \subset \partial(E^n).$$

This is because by Lemma IV.17 if $b \in \partial E_p^n$ then there exist $b_i \in E_p^n$ so that $b_i \to b$. Since $b_i \in E^n$, we have that $b \in \overline{E^n}$ by Lemma IV.18. On the other hand if $b \in E^n$ then since $b \in V_p$ by definition, $b \in E_p^n$.

So by part (3) of Lemma IV.13 we have on $\partial\tilde{E_p^n}$ that

$$h \circ \pi \leq -\log\left(1 - \frac{1}{n\epsilon}\right). \tag{4.10}$$

We will argue that every connected component C' of $\tilde{E}_p^n$ has nonempty boundary in $\tilde{V}_p$. Every such C' is an open set inside a connected component C of $\tilde{V}_p$. If $C' \neq C$ and if C' has empty boundary, every point in $C \setminus C'$ is surrounded by a neighborhood that does not intersect C'. The union of these neighborhoods would form an open set which together with C' itself would disconnect C. So if C' has empty boundary, then $C' = C$. This would mean that $\pi(C) \subset E^n$, and $\pi(C)$ is again a variety since π is proper [30]. This means that $\pi(C)$ of V_p is compact, which is impossible [30]. Thus, every C' has nonempty boundary. Recall as well that the dimension of C' is always greater than or equal to one because F maps into $\mathbb{C}^n$.

We can apply the maximum principle on $\tilde{E}_p^n$ to the function h, because $\tilde{E}_p^n$ is an open subset of $\tilde{V}_p$ and h is plurisubharmonic on $\tilde{V}_p$. Since $\overline{\tilde{E}_p^n}$ is compact, every sequence has a convergent subsequence, so that if the maximum doesn't occur on the boundary (which exists), we can find a subsequence of points converging to an interior maximum, which is a contradiction. So equation 4.10 holds on all of $\tilde{E}_p^n$. Since $\tilde{E}_p^n$ is a compact exhaution of $\tilde{V}_p$, we have that $h \circ \pi \leq 0$ on $\tilde{V}_p$. Since π is surjective, on V_p we get

$$u \leq \log ||F/\epsilon||. \tag{4.11}$$

Since V_p cover $B_\epsilon^F(w)$ as we vary p, we have that equaton 4.11 holds on $B_\epsilon^F(w)$ as well. Since $\log ||F/\epsilon||$ is a candidate for $U_{B_\epsilon^F(w)}$, we get that

$$U_{B_\epsilon^F(w)} = \log ||F/\epsilon||.$$

Since $\log ||F/\epsilon||$ is its own upper semicontinuous regularization, we also get that

$$U^*_{B^F_\epsilon(w)} = \log ||F/\epsilon||.$$

$\square$

4.5 Plurisubharmonicity with Isolated Zeros using Monge-Ampère Methods

In this section we will again study the generalized Green function U defined in definition II.2. We present another proof of Theorem II.3, this time using methods developed to study the Monge-Ampère operator.

In this section we will again be studying the domain $B_\epsilon(F)$ which we defined earlier in definition IV.11.

Lemma IV.20. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be a holomorphic function with the following properties: $||F||$ has finitely many zeros, $z_1, ..., z_n$, in Ω; and $||F||$ has a nonzero minimum in a neighborhood of $\partial\Omega$. Then for each candidate $u \in \mathcal{T}(\Omega, F)$, there is a constant such that $u \leq \log ||F|| + c(u)$ in Ω.*

Proof. Consider a domain $G \subset \Omega \setminus \{z_1, ..., z_n\}$. Then F has no zeros in G, so that the Monge-Ampère operator applied to $\log ||F||$ is zero there, see for example [4]. This means that u is maximal in G, see [23, p. 131]. Let N_i be a neighborhood of z_i on which $u < \log ||F|| + c_i$ and let N_0 be a neighborhood of $\partial\Omega$ on which $u < \log ||F|| + c_0$. Choose G that $\partial G \subset \cup_{i=0}^n N_i$. Then $u \leq \log ||F|| + \max\{c_i : i = 0, ..., n\}$ on ∂G. Since $\log ||F||$ is maximal on G, we get that $u \leq \log ||F|| + c(u)$ on G. Since the constant depends only on u and not on G, the inequality holds in all of $\Omega \setminus \{z_1, ...z_n\}$. But since $z_i \in N_i$ the inequality holds at each z_i as well. $\square$

Recall the following two well known theorems regarding approximations of plurisubharmonic functions by smooth functions.

Theorem IV.21 (Fornaess-Narasimhan, [13]). *Let $\psi : X \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function on a Stein space X. Then there exists a sequence of C^∞ strongly plurisubharmonic functions $\psi_n : X \to \mathbb{R}$ such that $\psi_n \searrow \psi$ pointwise.*

This second theorem can be found in Klimek [23, p. 63].

Theorem IV.22. *Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in \mathcal{PSH}(\Omega)$. Define $\Omega_\epsilon := \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\}$ and define $\chi_\epsilon(x) = \frac{1}{\epsilon^m}\chi(\frac{x}{\epsilon})$. If $\epsilon > 0$ is such that $\Omega_\epsilon \neq \emptyset$, then $u * \chi_\epsilon \in C^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$. Moreover, $u * \chi_\epsilon$ monotonically decreases with decreasing ϵ, and $\lim_{\epsilon \to 0} u * \chi_\epsilon(z) = u(z)$ for each $z \in \Omega$.*

Note the following lemma, which can be found in [23, p. 69].

Lemma IV.23. *Let $\Omega \subset \mathbb{C}^n$, and let ω be a non-empty proper open subset of Ω. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(\omega)$, and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then the formula*

$$w = \begin{cases} \max\{u, v\} & : t \in \omega \\ u & : t \in \Omega \setminus \omega \end{cases}$$

defines a plurisubharmonic function in Ω.

Proof. By the assumption $\limsup_{x \to y} v(x) \leq u(y)$, w is upper semicontinuous in Ω. Off of $\partial\omega$, w automatically satisfies the subaveraging property on complex lines. So take $y \in \partial\omega$. Then on any complex line through y with $\Delta(y, r)$ the disk of radius r centered at y in that line,

$$u(y) \leq \int_{\Delta(y,r)} u(x)dx \leq \int_{\Delta(y,r)} w(x),$$

and the result follows.

□

We can also get the following very similar lemma, with the proof being completely analogous.

Lemma IV.24. *Let $\Omega \subset \mathbb{C}^n$, and let ω be a non-empty proper open subset of Ω. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(\Omega \setminus \overline{\omega})$, and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then the formula*

$$w = \begin{cases} u & : t \in \omega \\ \max\{u, v\} & : t \in \Omega \setminus \omega \end{cases}$$

defines a plurisubharmonic function in Ω.

Note that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic on $\Omega \subset \mathbb{C}^n$ with isolated zeros, there is always ϵ such that $B_\epsilon(w) \subset\subset \Omega$, and such that w is the only zero of F in $\overline{B_\epsilon(w)}$. The argument is the same in higher dimensions as in one dimension, and the one dimensional argument can be found in Lemma IV.12.

The proof of the next lemma is complicated, drawing its motivation from work Demailly did on regularity of the pluricomplex Green function in [10]. I suspect that there may be a different method which would yield a more streamlined proof.

Lemma IV.25. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros, and let ϵ be such that $B_\epsilon(w) \subset\subset \Omega$, and such that w is the only zero of F in $\overline{B_\epsilon(w)}$. If there exists a negative plurisubharmonic function ϕ on $B_\epsilon(w)$ so that $\phi - \log ||F|| < C$ in a neighborhood of w and $\phi(p) > \log ||F(p)/\epsilon||$ for some $p \in B_\epsilon(w)$, then there exists a continuous negative plurisubharmonic function ϕ_c on $B_\epsilon(w)$ so that $\phi_c - \log ||F|| < C$ in a neighborhood of w and $\phi_c(p) > \log ||F(p)/\epsilon||$.*

Proof. We claim that the following inequalities hold for ϕ for some constant $a > 0$ and for all $z \in B_\epsilon(w)$.

$$\log ||F(z)/\epsilon|| \leq \phi(z) \leq \log ||F(z)|| + a. \tag{4.12}$$

The first inequality follows because we may assume that $\phi \geq \log ||F/\epsilon||$ by considering instead $\max \{\phi, \log ||F/\epsilon||\}$. The second inequality follows from Lemma IV.20.

There is a decreasing sequence of smooth plurisubharmonic functions ϕ_j so that $\lim_{j\to\infty} \phi_j = \phi$. We can justify this claim using Theorem IV.21. To verify the hypotheses, we will show that $B_\epsilon^F(w)$ is hyperconvex (see definition II.15). It is bounded, and we claim that $\log ||F/\epsilon||$ is a continuous negative plurisubharmonic exhaustion function. To show that $\log ||F/\epsilon||$ is exhausting, we consider the sets in $B_\epsilon(w)$ where $\{\log ||F/\epsilon|| < c\}$ for $c < 0$. Suppose z_i is a sequence of points in such a set which is not contained in any compact of $B_\epsilon(w)$. Then a subsequence converges to a point $z^* \in \partial B_\epsilon(w)$. By continuity $\log ||F(z^*)/\epsilon|| \leq c$. Again by continuity, there will be a ball around z^* on which $\log ||F/\epsilon|| < c/2 < 0$. But then that ball is contained in $B_\epsilon(w)$, so z^* could not have been a boundary point. Note that we could also justify the decreasing sequence of smooth plurisubharmonic functions using Theorem IV.22.

Let us consider the set $\{z \in B_\epsilon(w) : ||F(z)|| = \delta\}$, and choose $\delta \in (0, \epsilon)$ so that on this set

$$\phi_j(z) \geq \phi(z) \geq \log(\delta/\epsilon) > (1 - \delta)\log(\delta^2). \tag{4.13}$$

Here we are using equation 4.12 and choosing δ small enough so that in the last inequality the dominating terms are $\log(\delta)$ and $\log(\delta^2)$.

Now let us consider the set $\{z \in B_\epsilon(w) : ||F(z)|| = \eta\}$ and choose $\eta \in (0, \delta)$ so that on this set

$$(1-\delta)\log(\delta\eta) > \log(\eta) + a \geq \phi(z). \tag{4.14}$$

Here again we are using equation 4.12 and picking η small enough so that in the last inequality the dominating terms are $(1-\delta)\log(\eta)$ and $\log(\eta)$. We claim that we can use equation 4.14 to choose j large enough so that

$$\phi_j < (1-\delta)\log(\delta\eta)$$

on the set $\{z \in B_\epsilon(w) : ||F(z)|| = \eta\}$; and we claim that we can choose j large enough so that

$$\phi_j(z) < 0$$

on the set $\left\{z \in B_\epsilon(w) : ||F(z)|| = \epsilon e^{-\delta^3}\right\}$. If these inequalities hold at a point, they hold in a neighborhood of that point since ϕ_j is smooth. Since the sets we are looking at are compact, finitely many such neighborhoods are enough.

We can now use lemmas IV.23 and IV.24 to show that with the above choices of δ, η and j, the following formula defines a continuous negative plurisubharmonic function on $B_\epsilon(w)$:

$$u_c = \begin{cases} (1-\delta)\log(\delta||F(z)||) - \delta & : ||F(z)|| \leq \eta \\ \max\{(1-\delta)\log(\delta||F(z)||) - \delta, \phi_j - \delta\} & : \eta \leq ||F(z)|| \leq \delta \\ \phi_j - \delta & : \delta \leq ||F(z)|| \leq \epsilon e^{-\delta} \\ \max\{\phi_j - \delta, \delta^{-2}\log||F(z)/\epsilon||\} & : \epsilon e^{-\delta} \leq ||F(z)|| \leq \epsilon e^{-\delta^3} \\ \delta^{-2}\log||F(z)/\epsilon|| & : \epsilon e^{-\delta^3} \leq ||F(z)|| \leq \epsilon \end{cases}$$

Let us comment about what happens on the set $\left\{z \in B_\epsilon(w) : ||F(z)|| = \epsilon e^{-\delta}\right\}$. On this set

$$\delta^{-2}\log||F(z)/\epsilon|| = -1/\delta.$$

We also have $\phi_j > \phi > \log||F/\epsilon|| = -\delta$, and $-2\delta > -1/\delta$ for δ small enough.

We will now show that hypotheses of lemmas IV.23 and IV.24 are satisfied. One can check using the arguments above that the appropriate inequalities hold on each boundary. We need to check that each of our sets is open and proper within the next open set. Openness is clear. To show proper containment, we will show that inside $B_\epsilon(w)$

$$\{||F|| < \eta\} \subset\subset \{||F|| < \delta\} \subset\subset \left\{||F|| < e^{-\delta}\right\} \subset\subset \left\{||F|| < e^{-\delta^3}\right\} \subset\subset B_\epsilon(w).$$

Since $B^F_\epsilon(w) \subset\subset \Omega$, we have that $\partial B^F_\epsilon(w) \subset \{||F(z)|| = \epsilon\}$. This is because for every point b in $\partial B^F_\epsilon(w)$, there is a sequence $b_i \in B^F_\epsilon(w)$ converging to b. By continuity of F, $||F(b)|| \leq \epsilon$. But since $b \in \Omega$, if $||F(b)|| < \epsilon$ we would have $b \in B^F_\epsilon(w)$. So for any $\epsilon' < \epsilon$ the set $\{||F|| < \epsilon'\}$ that is inside $B_\epsilon(w)$ is relatively compact in $B^F_\epsilon(w)$. As such the boundary of any such set is contained in $B_\epsilon(w)$. The argument for relative compactness of each set in the next now follows the subsequence argument above by showing that the boundary of each of these sets is contained in the set $\{||F|| = \epsilon'\}$.

Now we know that u_c is plurisubharmonic, and it is clear that u_c is continuous and negative, so we just need to show it has the correct behavior near the singularity. In fact, what we will need to do instead is consider the function $u_c/(1-\delta)$, and it is clear that this function has the correct behavior in a neighborhood of the singularity.

Let $0 < ||F(p)|| = P < \epsilon$. Then we can pick δ small enough so that $\delta < P < \epsilon e^{-\delta}$, so that

$$u_c(p) = \phi_j(p) - \delta.$$

Let $b > 0$ be such that

$$\phi(p) = \log ||P/\epsilon|| + b.$$

If we pick

$$\delta < \max\left\{\frac{b}{1-\log ||P/\epsilon||}, 1\right\},$$

then we get

$$\begin{aligned} u_c(p)/(1-\delta) - \log ||P/\epsilon|| &= \frac{\phi_j(p)-\delta}{1-\delta} - \frac{\log ||P/\epsilon||(1-\delta)}{(1-\delta)} \\ &\geq \frac{\phi(p) - \log ||P/\epsilon|| - \delta + \delta \log ||P/\epsilon||}{1-\delta} \\ &= \frac{b + \delta(\log ||P/\epsilon|| - 1)}{1-\delta} \end{aligned}$$

which will be positive if $\delta < 1$ and if

$$\delta(\log ||P/\epsilon|| - 1) > -b.$$

Since $\log||P/\epsilon|| < 0, \log||P/\epsilon|| - 1 < 0$ as well. So we get the condition

$$\delta < \frac{-b}{(\log||P/\epsilon|| - 1)},$$

which was the condition we required of b. Thus $\phi_c = \frac{u_c}{1-\delta}$ has the desired properties.

$\square$

The following lemma can be found in [23, p. 230], see also [10] and [9].

Lemma IV.26 (Demailly)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$, let $a \in \Omega$, and let $u, v \in \mathcal{PSH} \cap \mathcal{C}(\Omega, [-\infty, \infty))$ be such that $u^{-1}(-\infty) = v^{-1}(-\infty) = \{a\}$, $u < v$ in $\Omega \setminus \{a\}$, and*

$$\limsup_{z \to a} \frac{u(z)}{v(z)} = 1.$$

Then $(dd^c u)^n(\{a\}) \leq (dd^c v)^n(\{a\})$.

The following lemma can be found in [10].

Lemma IV.27 (Demailly)**.** *Let $\phi, \psi : \Omega \to [-\infty, 0)$ be continuous plurisubharmonic exhaustion functions so that $\phi \leq \psi \leq 0$ and $\int_\Omega (dd^c \phi)^n < \infty$. Then $\int_\Omega (dd^c \psi)^n < \int_\Omega (dd^c \phi)^n$.*

Lemma IV.28. *Let Ω be a domain in $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros, and consider a particular zero w. Let ϵ be small enough so that $B_\epsilon(w) \subset\subset \Omega$ and $\overline{B_\epsilon(w)}$ contains no zeros other than w. Then the generalized Green function $U_{B_\epsilon(w)} = \log||F/\epsilon||$.*

Proof. Suppose that there exists u which is plurisubharmonic and nonpositive on $B_\epsilon(w)$, with that $u \leq \log||F|| + c$ in a neighborhood of w, so that $u \geq \log||F/\epsilon||$

and $u(p) > \log ||F(p)/\epsilon||$ at some point $p \in B_\epsilon(w)$. By Lemma IV.25 we can assume that u is continuous. Let γ be a strictly plurisubharmonic function defined on a neighborhood of $\overline{B_\epsilon(w)}$ such that $\gamma \le -1$ in $B_\epsilon(w)$. Choose $\epsilon' > 0$ small enough that

$$u(p) + \epsilon'\gamma(p) > \log ||F(p)/\epsilon||$$

in a neighborhood of p, and define the function

$$\omega(z) = \max\left\{u(z) + \epsilon'\gamma(z), \log ||F(z)/\epsilon||\right\}.$$

Then there are constants C_1 and C_2 with $C_1 < C_2$ so that in a neighborhood of w

$$\log ||F(z)|| + C_1 < \omega(z) < \log ||F(z)|| + C_2.$$

Thus

$$\limsup_{z\to w}\left(1 + \frac{C_1}{\log ||F(z)||}\right) \ge \limsup_{z\to w}\frac{\omega(z)}{\log ||F(z)||} \ge \limsup_{z\to w}\left(1 + \frac{C_2}{\log ||F(z)||}\right),$$

so that

$$\limsup_{z\to w}\frac{\omega(z)}{\log ||F(z)||} = 1.$$

Similarly,

$$\limsup_{z\to w}\frac{1}{1+\frac{C_1}{\log ||F(z)||}} \le \limsup_{z\to w}\frac{\log ||F(z)||}{\omega(z)} \le \limsup_{z\to w}\frac{1}{1+\frac{C_2}{\log ||F(z)||}}.$$

so that

$$\limsup_{z\to w}\frac{\log ||F(z)||}{\omega(z)} = 1.$$

By applying lemma IV.26 twice we get

$$(dd^c\omega)^n(\{w\}) = (dd^c \log ||F(z)||)^n(\{w\}). \tag{4.15}$$

Note that ω and $\log ||F/\epsilon||$ are negative continuous plurisubharmonic exhaustion functions on $B_\epsilon^F(w)$ with $\log ||F/\epsilon|| \leq \omega$, so we can use lemma IV.27 to show that

$$\begin{aligned}(dd^c\omega)^n(\{w\}) &\leq \int_{B_\epsilon(w)} (dd^c\omega)^n \\ &\leq \int_{B_\epsilon(w)} (dd^c \log ||F/\epsilon||)^n = (dd^c \log ||F(z)||)^n(\{w\}).\end{aligned}$$

This equation, together with 4.15 and Lemma IV.32, imply that

$$(dd^c\omega)^n|_{B_\epsilon^F(w)\setminus\{w\}} \equiv 0.$$

But this contradicts the fact that $\omega = u + \epsilon\gamma$ in a neighborhood of p, since $u + \epsilon\gamma$ is strictly plurisubharmonic.

$\square$

Before proving the next lemma, we will need to make a small detour to show that the Monge-Ampère mass of $\log ||F||$ is finite at zeros of $||F||$.

Definition IV.29 (Radon Measure)**.** A Radon measure on Ω is a continuous $\mathbb{C}$-linear functional on $\mathcal{C}_0(\Omega, \mathbb{C})$.

The Riesz representation theorem says that for every Radon measure μ there is a corresponding unique complex Borel measure μ on Ω so that

$$\mu(\phi) = \int_\Omega \phi d\mu.$$

One can show that for any compact set $K \subset \Omega$,

$$\mu(K) = \inf\left\{\mu(\phi) : \phi \in \mathcal{C}_0(\Omega, [0,1)), K \subset \phi^{-1}(1)\right\}. \tag{4.16}$$

Let $\psi \in \mathcal{PSH} \cap L^\infty_{loc}$. Since $(dd^c\psi)^n$ is a positive (n,n) current, it is in fact a Radon measure so we can define $(dd^c\psi)^n(\{a\})$ using Equation 4.16 for a point $a \in \Omega$. We can also define the Monge-Ampère operator on wider classes of functions, and in particular for the class

$$\mathcal{PSH}(\Omega; a) = \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{a\}).$$

The following theorem is then used to define the Monge-Ampère operator for functions in $\mathcal{PSH}(\Omega; a)$, [23].

Theorem IV.30. *Let Ω be a domain in $\mathbb{C}^n$ and let $u \in \mathcal{PSH}(\Omega; a)$. Then there exists a positive Borel measure μ on Ω so that, for any decreasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap L^\infty_{loc}(\Omega)$ convergent to u pointwise in Ω, the sequence $\{(dd^c u_j)^n\}_{j\in\mathbb{N}}$ is weak*-convergent to μ.*

Thus for $\psi \in \mathcal{PSH}(\Omega; a), (dd^c\psi)^n$ is again a positive Radon measure and we can define $(dd^c\psi)^n(\{a\})$ using Equation 4.16 for a point $a \in \Omega$. In particular we will use this for $\psi = \log ||F||$, where $F : \Omega \to \mathbb{C}^m$ is holomorphic with isolated singularities.

In what follows, we will also need the Chern-Levine-Nirenberg estimate, [23].

Theorem IV.31 (Chern-Levine-Nirenberg estimate)**.** *Let Ω be an open neighborhood of a compact set $K \subset \mathbb{C}^n$. Then there exist a constant $C > 0$ and a compact set $L \subset \Omega \setminus K$, which depend on K and Ω, such that for all $u_1, ...u_n \in \mathcal{PSH} \cap L^\infty(\Omega)$*

$$\int_K dd^c u_1 \wedge ... \wedge dd^c u_n \leq C||u_1||_L \cdot ... \cdot ||u_n||_L.$$

Now we can show that the Monge-Ampère mass of $\log ||F||$, for $F : \Omega \to \mathbb{C}^n$ a holomorphic function, is finite at isolated zeros of $||F||$. We will need this lemma in a later proof.

Lemma IV.32. *Let $\Omega \subset \mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$ be such that $||F||$ has isolated zeros. Then the Monge-Ampère mass of $\log ||F(z)||$ at each of the zeros of $||F||$ is finite.*

Proof. Let $u = \log ||F||$ and consider the functions $u_j = \max\{\log ||F||, -j\}$. Then $u_j \to u$ pointwise and the u_j are decreasing. By Theorem IV.30

$$\begin{aligned}(dd^c \log ||F||)^n(\{w\}) &= \inf\left\{\mu(\phi) : \phi \in \mathcal{C}_0(\Omega, [0,1)), w \in \phi^{-1}(1)\right\} \\ &= \inf\left\{\lim_{j\to\infty} \mu_j(\phi) : \phi \in \mathcal{C}_0(\Omega, [0,1)), w \in \phi^{-1}(1)\right\} \\ &= \inf\left\{\lim_{j\to\infty} \int_\Omega \phi d\mu_j : \phi \in \mathcal{C}_0(\Omega, [0,1)), w \in \phi^{-1}(1)\right\}.\end{aligned}$$

Suppose that $\phi \in \mathcal{C}_0(\Omega, [0,1)), w \in \phi^{-1}(1)$. Then if we pick $\Omega = B_\epsilon^F(w)$ and $K = Supp(\phi)$, using IV.31 we get

$$\lim_{j\to\infty} \int_\Omega \phi(dd^c u_j)^n \leq \lim_{j\to\infty} C||u_j||_L^n.$$

Since $||u_j||_L$ is finite for each j and eventually the same for all j, we are done.

□

Lemma IV.33. *Let Ω be a domain in $\mathbb{C}^n$, and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros. Then the upper semicontinuous regularization U^* (see definition II.16) of the generalized Green function U satisfies $U^* < \log ||F|| + C(w)$ in a neighborhood of each zero w.*

Proof. According to Corollary IV.10 and Lemma IV.28 for each zero w there is a corresponding neighborhood $B_\epsilon(w)$ so that

$$U^*_\Omega|_{B_\epsilon(w)} \leq U^*_{B_\epsilon(w)} = \log ||F/\epsilon||.$$

So pick $C(w) = -\log(\epsilon)$.

□

The following theorem can be found in [27, p. 54].

Theorem IV.34 (Lelong)**.** *Let Ω be a domain in $\mathbb{C}^n$. Let $(f_i)_{i\in I}$ be a family of plurisubharmonic function on Ω that are bounded above on all compacts. Define $\mathcal{F}(z) = \sup_{i\in I} f_i(z)$. Then the upper semicontinuous regularization $\mathcal{F}^*(z)$ is plurisubharmonic on Ω.*

Theorem IV.35. *Let Ω be a domain in $\mathbb{C}^n$, and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with isolated zeros. Then the generalized Green function U is plurisubharmonic.*

Proof. We will show that the upper semicontinuous regularization U^* is a candidate function. Clearly, $U^* \leq 0$. Also, Lemma IV.33 shows that $U^* - \log ||F|| < C(w)$ in a neighborhood of each zero w. By Theorem IV.34, we also know that U^* is plurisubharmonic. Thus $U^* = U$, so U itself is plurisubharmonic. □

Recall the maximal plurisubharmonic functions defined in definition II.13. The following well known theorem relates maximal plurisubharmonic functions to certain solutions of a Monge-Ampère equation, see [23, p. 158] and [4].

Theorem IV.36. *Let Ω be an open subset onf $\mathbb{C}^n$, and let u be a locally bounded plurisubharmonic function on Ω. Then u is maximal if and only if it satisfies the homogeneous Monge-Ampère equation $(dd^c u)^n = 0$.*

The following theorem now follows in the same way that it did for the Green function, see [23, p. 222].

Theorem IV.37. *Let Ω be a domain in $\mathbb{C}^n$, and let $F : \Omega \to \mathbb{C}^n$ be holomorphic with finitely many isolated zeros. Let Z be the zero set of $||F||$ on Ω. Then U is maximal in $\Omega \setminus Z$, or equivalently*

$$(dd^c U)^n \equiv 0$$

in $\Omega \setminus Z$.

Proof. Let $G \subset\subset \Omega \setminus Z$, and let $v \in \mathcal{PSH}(G)$ that is upper semicontinuous on $\overline{G}$ be such that $v \leq U$ on ∂G. Then define

$$u(z) = \begin{cases} \max\{v(z), U(z)\} & : z \in G \\ U & : z \in \Omega \setminus G. \end{cases}$$

$u(z)$ is plurisubharmonic by Lemma IV.23.

Since $v < 0$ on ∂G, by the maximum principle $v < 0$ on G. Thus u is a candidate for U, which means that that $v \leq U$ in G. Thus U is maximal. By Theorem IV.36, its Monge-Ampère mass is zero off Z.

□

There are now a number of theorems which should follow directly in analogy to how they were proved for the standard Green function. We give an example of one such theorem.

Theorem IV.38. *Let $(\Omega_j)_{j\in\mathbb{N}}$ be an increasing sequence of domains in $\mathbb{C}^n$ and let F be a holomorphic function defined from $\Omega = \cup\Omega_j$ to $\mathbb{C}^n$ such that F has isolated zeros. Then*

$$U_{\Omega}^F = \lim_{j\to\infty} U_{\Omega_j}^F.$$

Proof. If for any Ω_j, $U_{\Omega_j}^F \equiv -\infty$ the result is trivial by Corollary IV.10. If not, then by Theorem IV.35 we get $U_{\Omega_j}^F \in \mathcal{PSH}(\Omega_j)$. Define

$$g_{\Omega}^F := \lim_{j\to\infty} U_{\Omega_j}^F.$$

Since $U_{\Omega_j}^F$ is a decreasing sequence of plurisubharmonic functions, g_{Ω}^F is either identically $-\infty$ or plurisubharmonic. By Corollary IV.10 we have $g_{\Omega}^F \geq U_{\Omega}^F$, so if g_{Ω}^F is identically $-\infty$ then we are done. If g_{Ω}^F is plurisubharmonic, then $g \leq 0$ and by Lemma IV.28 and Corollary IV.10 $g_{\Omega}^F \leq \log||F|| + C(w)$ in a neighborhood of each zero w of F. Thus $g_{\Omega}^F \leq U_{\Omega}^F$.

$\square$

CHAPTER V

Other Examples

5.1 Balanced Domains with Holes

In general, it is very hard to calculate the Green function or Azukawa metric of a given domain. One notable exception is balanced pseudoconvex domains, where the term balanced indicates that $\Delta \cdot \Omega = \Omega$ for Δ the unit disk in $\mathbb{C}$. On balanced pseudoconvex domains, the Green function and Azukawa metric behave as they do on disks.

If Ω is a balanced pseudoconvex domain, then $u(0, w) = \log h(w)$, where $h(w)$ was defined in equation 3.17, [19, p. 119]. This means that on balanced pseudoconvex domains, the Green function with pole at the origin is the same as the Green function of each circular slice. One inequality is given by inclusion argument, which shows that the Green function would be smaller than or equal to the behavior on a slice. One can then show the other inequality by checking that $\log h(z) \in \mathcal{K}_\Omega(0)$. This follows because, for a balanced domain, $\log h(z)$ is plurisubharmonic if and only if Ω is pseudoconvex.

We study the Green function on balanced pseudoconvex domains Ω with a compact set K removed. One example of a domain of this type is the ring domain $\Omega_r \subset \mathbb{C}^n$, where

$$\Omega_r = \left\{ z \in \mathbb{C}^n : r^2 < ||z||^2 < 1 \right\}.$$

Fornaess and Lee showed that, in directions not touching the inner boundary, the Sibony metric on the ring domain is the same as the Sibony metric on the ball, [12]. It is reasonable to suspect that in these directions the Green function is the same as the Green function on the ball. More generally, we will show that the Green function on $\Omega \setminus \{K\}$ is the same as the Green function on Ω in directions not cutting through K. It follows that the Azukawa metric on $\Omega \setminus \{K\}$ is also the same as the Azukawa metric on Ω in these directions.

Theorem V.1. *Let $\Omega \subset \mathbb{C}^n$ be a balanced pseudoconvex domain and let 0 denote the origin in $\mathbb{C}^n$. Let $K \subset\subset \Omega$ so that $0 \notin K$. Let $D(z) = \Omega \cap \{\alpha z : \alpha \in \mathbb{C}\}$ be the disk through z and the origin. If $D(z) \cap K$ is a closed subset of $D(z)$ which is the $-\infty$ set for some plurisubharmonic function on $D(z)$, then*

$$u_{\Omega\setminus K}(0, z) = u_\Omega(0, z) = \log \frac{||z||}{r},$$

where r is the radius of $D(z)$. Also,

$$F_A^{\Omega\setminus K}(0, \xi) = F_A^{\Omega}(0, \xi) = \frac{||\xi||}{r}.$$

Proof of Theorem V.1. Let $S(z)$ denote the slice through z in Ω_K, so that $S(z) = D(z) \cap K^c$. Consider the function $f(z)$ defined on $S(z)$ given by

$$f(z) = u(0, z) - \log \frac{||z||}{r},$$

where $u(0, z)$ is the pluricomplex Green function on $\Omega \setminus \{K\}$ with pole at the origin and where r is the radius of $D(z)$. Although the funciton $\log ||z||$ is not pluriharmonic on $\Omega \setminus \{0\}$, it is harmonic on $D(z) \setminus \{0\}$. So $f(z)$ is subharmonic on $S(z) \setminus \{0\}$. Furthermore, by the following calculation $f(z)$ is bounded near the origin, so $f(z)$ extends to be subharmonic on $S(z)$.

$$\begin{aligned} f(z) &= u(0,z) - \log ||z|| + \log ||z|| - \log \frac{||z||}{r} \\ &\leq C + \log r \end{aligned}$$

Notice that $f(z)$ is bounded above on $S(z) = D(z) \cap K^c$ since $\log ||z||$ is bounded there. So if $D(z) \cap K$ is a closed subset of $D(z)$ which is the $-\infty$ set of a plurisubharmonic function, then $f(z)$ extends to a subharmonic function on $D(z)$. We will call this extended function $\tilde{f}(z)$.

Since u_Ω is given by the logarithmic Minkowski function, we know that $u_\Omega|_{\partial\Omega} = 0$. Since $\Omega_K \subset \Omega$ we know that $u_{\Omega_K} \geq u_\Omega$. Thus $u_{\Omega_K}|_{\partial\Omega_K} = 0$, so $\tilde{f}(z)|_{\partial D(z)} = 0$.

Also, since

$$\log \frac{||z||}{r} = u_\Omega(0,z)|_{\Omega_K},$$

and since $u_{\Omega_K} \geq u_\Omega$, $\tilde{f}(z) \geq 0$. Now we can apply the maximum principle on $D(z)$ to get that $\tilde{f}(z) = 0$. $\square$

5.2 Boundary Behavior of the Green Function

The pluricomplex Green function is known to be continuous on hyperconvex domains, [10]. By exhausting a pseudoconvex domain with hyperconvex domains, it directly follows that on pseudoconvex domains the Green function is upper-semicontinuous. In fact, the Green function is upper-semicontinuous on all domains as we proved in Section 3.1. It is known that there are cases in which the Green function is only upper-semicontinuous and not not continuous. In this section we construct an example of a pseudoconvex domain where the Green function cannot be extended continuously to the boundary.

Lemma V.2. *There is a neighborhood of* $(0,0)$ *on which the function*

$$\max\left\{-\sum_n \epsilon_n \log\frac{1}{n} + \sum_n \epsilon_n \log\left|\frac{1}{n}w - (1-\frac{1}{n})z\right|, -1\right\}$$

is identically -1.

Proof. We will show that

$$f(z,w) = -\sum_n \epsilon_n \log\frac{1}{n} + \sum_n \epsilon_n \log\left|\frac{1}{n}w - (1-\frac{1}{n})z\right|$$

is less than $\log|(z,w)/2| + C$, which is less than -1 in a neighborhood of $(0,0)$. One can check that $f(z,w)$ is well defined.

$$\log\left|\frac{1}{n}w - (1-\frac{1}{n})z\right|$$

$$= \frac{1}{2}\log\left(\frac{1}{n^2}|w|^2 + \frac{2}{n}\left(1-\frac{1}{n}\right)\mathrm{Re}(w\overline{z}) + \left(1-\frac{1}{n}\right)^2|z|^2\right)$$

If $|w| \geq |z|$ then $\mathrm{Re}(w\overline{z}) \leq |w\overline{z}| \leq |w|^2$. This gives us

$$\frac{1}{2}\log\left(\frac{1}{n^2}|w|^2 + \frac{2}{n}\left(1-\frac{1}{n}\right)\mathrm{Re}(w\overline{z}) + \left(1-\frac{1}{n}\right)^2|z|^2\right)$$

$$\leq \frac{1}{2}\log\left(\frac{2n-1}{n^2}|w|^2 + \left(1-\frac{1}{n}\right)^2|z|^2\right)$$

$$\leq \frac{1}{2}\log\left(|w|^2 + |z|^2\right)$$

If $|z| \geq |w|$ then $\mathrm{Re}(w\bar{z}) \leq |w\bar{z}| \leq |z|^2$. This gives us

$$\frac{1}{2}\log\left(\frac{1}{n^2}|w|^2 + \frac{2}{n}\left(1-\frac{1}{n}\right)\mathrm{Re}(w\bar{z}) + \left(1-\frac{1}{n}\right)^2|z|^2\right)$$

$$\leq \frac{1}{2}\log\left(\frac{1}{n^2}|w|^2 + \left(1-\frac{1}{n^2}\right)|z|^2\right)$$

$$\leq \frac{1}{2}\log\left(|w|^2 + |z|^2|\right)$$

So

$$\begin{aligned} f(z) &= -\sum_n \epsilon_n \log\frac{1}{n} + \sum_n \epsilon_n \log\left|\frac{1}{n}w - (1-\frac{1}{n})z\right| \\ &\leq C + \left(\sum_n \epsilon_n\right)\log|(z,w)| \end{aligned}$$

we can pick

$$\left(\sum_n \epsilon_n\right) = 1$$

. $\square$

Theorem V.3. *There exists a domain $\Omega \subset\subset \mathbb{C}^2$ with $(0,0) \in \Omega$ so that the pluri-complex Green's function with singularity at zero, $u_{\Omega,0}$, is not in $C(\overline{\Omega} \setminus 0)$. The domain Ω has the following defining function*

$$\log|(z,w)| + \max\left\{-\sum_{i=1}^{\infty}\epsilon_i \log\frac{1}{i} + \sum_{i=1}^{\infty}\epsilon_i \log\left|\frac{1}{i}w - \left(1-\frac{1}{i}\right)z\right|, -1\right\} \tag{5.1}$$

and ϵ_i are chosen so that

(5.2)
$$-\sum_{i=1}^{\infty} \epsilon_i \log \frac{1}{i}$$

converges to a finite number [14, p.30].

Proof. Let $\Omega = \{(z,w) : \rho(z,w) < 0\}$ where ρ is given by 5.1. The point $(0,0)$ is in Ω because $\log(0) = -\infty$. Also, $\rho(z,w) \geq \log|(z,w)| - 1$, so we have for all $(z,w) \in \Omega$, that $|(z,w)| < e$. Consider the sequence

$$t_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$$

$$|t_n|^2 = 1 - \frac{2}{n} + \frac{2}{n^2}$$

We can check that $t_n \in \Omega$ for all n. In fact, we have that Clearly $t_n \to t = (0,1)$ as $n \to \infty$. Pick ϵ_n so that

$$\sum_n \epsilon_n \log \frac{1}{n} > -\infty$$

Then $t = (0,1) \in \partial\Omega$. Consider $D_{t_n} = \{ct_n : |ct_n| < e\}$ for $c \in \mathbb{C}$. Since for $(z,w) \in D_{t_n}$

$$\max\left\{-\sum_n \epsilon_n \log \frac{1}{n} + \sum_n \epsilon_n \log \left|\frac{1}{n}w - (1 - \frac{1}{n})z\right|, -1\right\} = -1$$

we have that $D_{t_n} \in \Omega$. This means that $u_{\Omega,0}(z) \leq u_{D_n,0}(z)$ for $z \in D_n$. This follows from noticing that if $g < 0$ is a plurisubharmonic function on Ω so that $g(z) \to \log|z| + a$ as $z \to 0$, then $g|_{D_n} < 0$ is a subharmonic function on D_n so that $g|_{D_n} \to \log|z| + a$ as $z \to 0$. Now consider $u_{D_n,0}(t_n)$. Since $|t_n| < 1$ and since $u_{D_n,0}(z,w) = \log|(z,w)/e|$ it follows that $u_{D_n,0}(t_n) < \log(1/e)$ for all t_n. Thus

$$\lim_{n\to\infty} u_{\Omega,0}(t_n) \leq \lim_{n\to\infty} u_{D_n,0}(t_n) \leq \log(1/e)$$

Now consider the sequence

$$a_n = \left(0, 1 - \frac{1}{n}\right)$$

$$\begin{aligned}
\rho(a_n) &= \log\left(1 - \frac{1}{n}\right) + \max\left\{-\sum_n \epsilon_n \log\frac{1}{n} + \sum_n \epsilon_n \log\left|\frac{1}{n}\left(1 - \frac{1}{n}\right)\right|, -1\right\} \\
&= \log\left(1 - \frac{1}{n}\right) + \max\left\{\sum_n \epsilon_n \log\left|\left(1 - \frac{1}{n}\right)\right|, -1\right\} \\
&< 0
\end{aligned}$$

So $a_n \in \Omega$. Now we want to show that ρ is in the admissible class. By definition, $\rho < 0$ on Ω. In addition, we will show in a calculation below that in a neighborhood of $(0,0)$

$$\max\left\{-\sum_n \epsilon_n \log\frac{1}{n} + \sum_n \epsilon_n \log\left|\frac{1}{n}w - (1 - \frac{1}{n})z\right|, -1\right\} = -1$$

From this fact, is clear that $\rho \to \log|(z,w)| - 1$ as $(z,w) \to (0,0)$. Finally, it is clear from the definition that ρ is plurisubharmonic. The only point worth mentioning is that

$$\log\left|\frac{1}{n}w - (1 - \frac{1}{n})z\right|$$

is negative and plurisubharmonic, so the partial sums of

$$\sum_n \epsilon_n \log\left|\frac{1}{n}w - (1 - \frac{1}{n})z\right|$$

are a decreasing sequence of plurisubharmonic functions. Since ρ is in the admissible class for Ω we get that

$$\lim_{n\to\infty} u_{\Omega,0}(a_n) \geq \lim_{n\to\infty} \rho(a_n) = 0$$

$\square$

CHAPTER VI

Skwarczyński Metric

6.1 Skwarczyński Completeness and the Bergman Space

An interesting open question that many people are interested in is whether there is a pseudoconvex domain that has finite dimensional Bergman space, which is the space of L^2 holomorphic functions. Wiegerinck has an example of a non-pseudoconvex domain with finite dimensional Bergman space in [34], but the example cannot be easily modified to the pseudoconvex case.

A related question which motivates this section was whether Bergman completeness implies that the Bergman space is infinite dimensional. We prove that statement for a closely related metric, called the Skwarczyński distance, which was defined in Section 2.3. We will now define what it means for a domain to be Skwarczyński complete and prove the result.

Definition VI.1. Let Ω be a domain in $\mathbb{C}^n$. The distance induced by ρ is continuous and induces a topology on Ω. Ω is said to be ρ-complete if any ρ-Cauchy sequence $\{z_\nu\}_{\nu=1}^{\infty} \subset \Omega$ converges to a point $z_0 \in \Omega$ with respect to the usual topology.

Theorem VI.2. *If Ω is ρ-complete, then $dim(L_h^2(\Omega)) = \infty$.*

Proof. Suppose that $L_h^2(\Omega)$ is finite dimesional. Then its unit ball is compact. Consider the following functions of norm one:

$$f_{z'}(\cdot) := \frac{K(\cdot, z')}{\sqrt{K(z', z')}}$$

where $z' \in \Omega$. To check their norm we have

$$\begin{aligned}
\left(\int_\Omega f_{z'}\overline{f_{z'}}\right)^{1/2} &= \left(\int_\Omega \frac{K(\cdot, z')}{\sqrt{K(z', z')}}\overline{\frac{K(\cdot, z')}{\sqrt{K(z', z')}}}\right)^{1/2} \\
&= \frac{1}{\sqrt{K(z', z')}}\left(\int_\Omega K(\cdot, z')\overline{K(\cdot, z')}\right)^{1/2} \\
&= \frac{1}{\sqrt{K(z', z')}}\sqrt{K(z', z')} \\
&= 1
\end{aligned}$$

Take a sequence $\{z_\nu\}_{\nu=1}^\infty$ where $z_\nu \to z_0$ for some $z_0 \in \partial\Omega$. Then since the unit ball is compact, the sequence $\{f_{z_\nu}\}_{\nu=1}^\infty$ has a subsequence which converges in the L^2 norm to a function $f \in L_h^2(\Omega)$ with $||f||_{L_h^2} = 1$. Let us re-index this convergent subsequence as $\{f_{z_j}\}_{j=1}^\infty$. We claim that the sequence of points $\{z_j\}_{j=1}^\infty$ is a Cauchy sequence in the norm induced by ρ. Since $z_j \to z_0$, we have a Cauchy sequence converging to a boundary point, so Ω will not be ρ-complete. Given $\epsilon > 0$ we need to show there exists $N \in \mathbb{N}$ such that for all $j_1, j_2 > N$, $\rho_\Omega(z_{j_1}, z_{j_2}) < \epsilon$. Notice that

$$\begin{aligned}
| < f_{z'}, f_{z''} > | &= \left|\int_\Omega f_{z'}\overline{f_{z''}}\right| \\
&= \left|\int_\Omega \frac{K(\cdot, z')}{\sqrt{K(z', z')}}\overline{\frac{K(\cdot, z'')}{\sqrt{K(z'', z'')}}}\right| \\
&= \frac{1}{\sqrt{K(z', z')}\sqrt{K(z'', z'')}}\left|\int_\Omega K(\cdot, z')\overline{K(\cdot, z'')}\right| \\
&= \frac{|K_\Omega(z'', z')|}{\sqrt{K_\Omega(z', z')}\sqrt{K_\Omega(z'', z'')}} \\
&= \frac{|K_\Omega(z', z'')|}{\sqrt{K_\Omega(z', z')}\sqrt{K_\Omega(z'', z'')}}
\end{aligned}$$

So that

$$\rho_\Omega(z', z'') = (1 - | < f_{z'}, f_{z''} > |)^{1/2}$$

Let $f_{z_j} = f + g_j$. Pick N such that for all $j > N$, $||g_j||_{L^2_h} < \epsilon^2/3$. This is possible since $g_j = f_{z_j} - f$ converges to zero in the L^2 norm. Then if $j_1, j_2 > N$

$$\begin{aligned}
| < f_{z_{j_1}}, f_{z_{j_2}} > | &= | < f + g_{j_1}, f + g_{j_2} > | \\
&= | < f, f > + < f, g_{j_2} > + < g_{j_1}, f > + < g_{j_1}, g_{j_2} > | \\
&= |1 + < f, g_{j_2} > + < g_{j_1}, f > + < g_{j_1}, g_{j_2} > | \\
&\geq 1 - | < f, g_{j_2} > | - | < g_{j_1}, f > | - | < g_{j_1}, g_{j_2} > | \\
&\geq 1 - ||f||_{L^2_h} ||g_{j_2}||_{L^2_h} - ||g_{j_1}||_{L^2_h} ||f||_{L^2_h} - ||g_{j_1}||_{L^2_h} ||g_{j_2}||_{L^2_h} \\
&= 1 - ||g_{j_2}||_{L^2_h} - ||g_{j_1}||_{L^2_h} - ||g_{j_1}||_{L^2_h} ||g_{j_2}||_{L^2_h} \\
&> 1 - \epsilon^2/3 - \epsilon^2/3 - \epsilon^4/9 \\
&> 1 - \epsilon^2
\end{aligned}$$

Thus for the N chosen, we have that for all $j_1, j_2 > N$, $\rho_\Omega(z_{j_1}, z_{j_2}) < \epsilon$, which proves our claim that the sequence $\{z_j\}_{j=1}^{\infty}$ is ρ-Cauchy.

□

We note also that the converse of Theorem VI.2 is not true. It is known that the Skwarczynski metric is smaller than the Bergman metric so Skwarczynski complete domains are Bergman complete [19], and it is known that Bergman complete domains are pseudoconvex [7]. If domains with infinite dimensional Bergman space were always Skwarczynski complete, then such domains would always be pseudoconvex. But all bounded domains, including non-pseudoconvex domains, have infinite dimensional Bergman space because for bounded domains all polynomials are in the Bergman space.

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