Plane Curves, Node Polynomials, and Floor Diagrams

by

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To my parents.
And Oma.

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CHAPTER I

Introduction

Enumeration of algebraic curves goes back more than 160 years. In 1848, J. Steiner determined that the number of plane curves of degree d with one node passing through $\frac{d(d+3)}{2} - 1$ points in generic position is $3(d-1)^2$. Much effort has since been put forth towards answering the following question:

How many (possibly reducible) degree d nodal curves with
δ nodes pass through $\frac{d(d+3)}{2} - \delta$ generic points in $\mathbb{CP}^2$?

A more algebro-geometric formulation is as follows. Algebraic plane curves of degree d are zero sets of degree-d homogeneous polynomials with complex coefficients in three indeterminates, and are thus parametrized by the complex projective space $\mathbb{CP}^{\binom{d+2}{2}-1}$. The Zariski closure of the subset of such curves with exactly δ nodes is the *Severi variety*. Its degree $N^{d,\delta}$ (in this embedding) is the *Severi degree* $N^{d,\delta}$, the number that answers the question above.

Severi varieties have received considerable attention since they were introduced by F. Enriques [8] and F. Severi [31] around 1915. Much later, in 1986, J. Harris [20] achieved a celebrated breakthrough by showing their irreducibility. For a detailed history of Severi varieties, see W. Fulton's survey article [10].

A closely related problem concerns the computation of the *Gromov-Witten invariant* of $\mathbb{CP}^2$, the number $N_{d,g}$ of irreducible curves of degree d and genus g in $\mathbb{CP}^2$ which pass through $3d + g - 1$ points in general position. After a great deal of work by many mathematicians and physicists, M. Kontsevich solved this problem for rational curves in 1994. His celebrated formula [23] for the numbers $N_{d,0}$ was obtained in his study of the quantum cohomology ring of the moduli space of stable maps, the work which in part earned him a Fields medal. In 1998, L. Caporaso and J. Harris [6] used deformation theory to develop the first method for computing the Gromov-Witten invariants $N_{d,g}$ for any genus g.

This thesis utilizes and extends a recent successful combinatorial approach to computing Severi degrees and Gromov-Witten invariants of $\mathbb{CP}^2$ as well as some of their generalizations. The approach is based on the *tropical geometry* technique which replaces subvarieties of a complex algebraic torus by piecewise-linear polyhedral complexes (see, for example, [13, 27, 30, 32]). Remarkably, many enumerative invariants survive this degeneration. An instance of this phenomenon is the celebrated Correspondence Theorem of G. Mikhalkin [27] which replaces the computation of the Severi degree $N^{d,\delta}$ (or the Gromov-Witten invariant $N_{d,g}$) by enumeration of certain *tropical plane curves* (cf. Section 4.3). One can go one step further and degenerate tropical plane curves to *floor diagrams*, a class of decorated graphs introduced by E. Brugallé and G. Mikhalkin [4, 5]. These purely combinatorial objects, if counted with appropriate weights, are equinumerous to the tropical curves under consideration, and consequently to the corresponding algebraic curves (cf. Theorem II.10). This allows for a study of the numbers $N^{d,\delta}$ and $N_{d,g}$ by combinatorial methods, and for a development of combinatorial algorithms for their computation.

The thesis is comprised of three parts (Chapters II – IV) each of which follows one of the papers [1, 2, 3]. The logical dependence of the chapters and sections is linear, with the following exceptions. Section 2.5 is independent of Section 2.4. Chapter III depends on Sections 2.1 – 2.2 only. Chapter IV is (mostly) independent of both Chapters II and III. Although floor diagrams are the main technical tool utilized in this thesis, the motivation for their definition and the discussion of their relation to tropical curves are postponed until Chapter IV as these aspects are not essential for understanding the earlier chapters.

Chapter II is devoted to the study of Severi degrees $N^{d,\delta}$ in the regime when δ is fixed while d grows. In 1994, P. Di Francesco and C. Itzykson [7] conjectured that, for sufficiently large d, the numbers $N^{d,\delta}$ are given by a polynomial $N_\delta(d)$ in d (the "node polynomial"). Recently, S. Fomin and G. Mikhalkin [9] established this polynomiality using the combinatorial approach alluded to above.

Polynomiality of Severi degrees and explicit formulas for $N_\delta(d)$ for small values of δ have a long history dating back to the 19th century (see Section 2.1). Building on ideas of S. Fomin and G. Mikhalkin [9], we develop an explicit algorithm for computing the node polynomials for arbitrary δ, and use it to compute $N_\delta(d)$ for $\delta \leq 14$ (see Theorem II.2 and Appendix A), extending earlier results by S. Kleiman and R. Piene [21]. The methods are combinatorial and make heavy use of floor diagrams and their *template decomposition* as introduced by of S. Fomin and G. Mikhalkin [9]. This decomposition recasts the computation of Severi degrees as a (polynomially weighted) lattice point enumeration in certain simplices (see Theorem II.14 and Remark II.15).

We next investigate the *polynomiality threshold* of Severi degrees which tells how large d needs to be for the Severi degree to be given by the evaluations of the node polynomial. We improve on the threshold value of S. Fomin and G. Mikhalkin (see Theorem II.4), and show that L. Göttsche's conjectural threshold [18] holds and is sharp for $3 \leq \delta \leq 14$ (see Proposition II.5).

We then compute the first 9 leading terms of the polynomials $N_\delta(d)$ for general δ (Theorem II.3). This confirms and extends the 1994 prediction made by P. Di Francesco and C. Itzykson [7].

Chapter III is devoted to enumeration of plane curves which, in addition to passing through a collection of points, satisfy tangency conditions to a fixed line. These conditions are encoded by two finite sequences α and β which record tangencies of various orders to the line at fixed (resp. unconstrained) points. The plane curves satisfying all these requirements are enumerated by the *relative Severi degree* $N^{\delta}_{\alpha,\beta}$, the degree of the *generalized Severi variety* introduced by L. Caporaso and J. Harris [6], which parametrizes such curves. Relative Severi degrees satisfy (and are determined by) the rather complicated Caporaso-Harris recursion [6].

We show that, for a fixed number of nodes δ, the relative Severi degrees, appropriately rescaled, are given by combinatorially defined "relative node polynomials" $N_\delta(\alpha;\beta)$ in the tangency orders, provided the latter are large enough (for more details, see Theorem III.1). These polynomials satisfy a "stability condition" (Theorem III.2). We also give an explicit polynomiality threshold that extends the threshold for the non-relative case obtained in Chapter II. We use the combinatorial description to develop a method for computing the polynomials $N_\delta(\alpha;\beta)$ for arbitrary δ, and use it to compute $N_\delta(\alpha;\beta)$ for $\delta \leq 6$ (see Appendix C). We also compute the first few leading terms for general δ (see Theorem III.4).

The methods employed in Chapter III extend those of Chapter II. The argument relies on a combinatorial description of relative Severi degrees in terms of a more refined enumeration of appropriate floor diagrams. The main technical tool is a new template decomposition compatible with the tangency conditions (see Section 3.3).

Chapter IV presents joint work with A. Gathmann and H. Markwig. It is concerned with the combinatorics of a generalization of floor diagrams for *descendant Gromov-Witten invariants* of $\mathbb{CP}^2$. These (possibly fractional) invariants "count" algebraic plane curves which satisfy not only point conditions but also those given by *Psi-classes*. The precise definition is via an intersection product on the moduli space of stable maps to $\mathbb{CP}^2$. Psi-classes are natural cohomology classes which come up frequently in the intersection theory of the moduli space of stable curves or maps. Geometrically, a Psi-condition (together with a point condition) roughly corresponds to a tangency condition to a generic line. Descendant Gromov-Witten invariants are qualitatively different from the relative Severi degrees of Chapter III as different Psi-classes correspond to tangency conditions to different (generic) lines.

We begin Chapter IV by recalling the algebro-geometric definition of (plane) descendant Gromov-Witten invariants and their relative analogues. The latter numbers satisfy (and are determined by) the Caporaso-Harris formula for relative descendant Gromov-Witten invariants (Theorem IV.7). We recall the definition of *tropical descendant plane Gromov-Witten invariants* and their (known) equality to the corresponding classical numbers. We then generalize this definition to *tropical relative plane descendant Gromov-Witten invariants*. We introduce the main combinatorial gadgets of Chapter IV, the *Psi-floor diagrams* and their relative analogues, which further generalize the floor diagrams of E. Brugallé, S. Fomin and G. Mikhalkin [4, 5, 9]. We prove that Psi-floor diagrams count the corresponding tropical curves (Theorem IV.25), and satisfy the same Caporaso-Harris type formula as the corresponding relative descendant Gromov-Witten invariants (Theorem IV.35). It follows that tropical relative descendant Gromov-Witten invariants coincide with their algebraic counterparts. We thus obtain a new correspondence theorem.

CHAPTER II

Computing Node Polynomials for Plane Curves

2.1 Introduction and Main Results

In this chapter we study the degree $N^{d,\delta}$ of the *Severi variety* parametrizing (possibly reducible) nodal algebraic curves in the complex projective plane $\mathbb{CP}^2$ of degree d with δ nodes. Equivalently, the number $N^{d,\delta}$ – called the *Severi degree* – is the number of such curves which pass through $\frac{(d+3)d}{2} - \delta$ points in generic position. Severi varieties have received a lot of attention (see, for instance, [6, 20, 29]) since their introduction by F. Enriques [8] and F. Severi [31] about 100 years ago.

Example II.1. Computing the Severi degree $N^{2,1}$ amounts to counting plane conics with 1 node, i.e., line pairs, through 4 points in the plane in generic position (see Figure 2.1). Thus, we have $N^{2,1} = 3$.

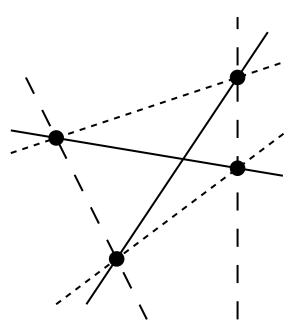


Figure 2.1: Three line pairs through four generic points in the plane.

In 1994, the mathematical physicists P. Di Francesco and C. Itzykson [7] conjectured that the Severi degree $N^{d,\delta}$ is given by a polynomial in d, provided that δ is fixed and d is sufficiently large. It is not hard to see that, if such a polynomial exists, it has to be of degree 2δ.

Recently, S. Fomin and G. Mikhalkin [9] established this polynomiality using the combinatorial approach hinted at in Chapter I. More precisely, they showed that there exists, for every $\delta \geq 1$, a *node polynomial* $N_\delta(d)$ which satisfies $N^{d,\delta} = N_\delta(d)$ for all $d \geq 2\delta$. (The $\delta = 0$ case is trivial as $N^{d,0} = 1$ for all $d \geq 1$.)

For $\delta = 1, 2, 3$, the polynomiality of Severi degrees and the formulas for $N_\delta(d)$ were determined in the 19th century. For $\delta = 4, 5, 6$, this was only achieved by I. Vainsencher [35] in 1995. In 2001, S. Kleiman and R. Piene [21] settled the cases $\delta = 7, 8$. Building on ideas of [9], we develop an explicit algorithm for computing the node polynomials for arbitrary δ, and use it to compute $N_\delta(d)$ for $\delta \leq 14$.

Theorem II.2. *The node polynomials $N_\delta(d)$, for $\delta \leq 14$, are as listed in Appendix A.*

In 1994, P. Di Francesco and C. Itzykson [7] conjectured the first seven terms of the node polynomial $N_\delta(d)$, for arbitrary δ. We confirm and extend their assertion. The first two terms already appeared in [21].

Theorem II.3. *The first nine coefficients of $N_\delta(d)$ are given by*

$$\begin{aligned}
N_\delta(d) = \frac{3^\delta}{\delta!}\Bigg[& d^{2\delta} - 2\delta d^{2\delta-1} - \frac{\delta(\delta-4)}{3} d^{2\delta-2} + \frac{\delta(\delta-1)(20\delta-13)}{6} d^{2\delta-3} + \\
& - \frac{\delta(\delta-1)(69\delta^2-85\delta+92)}{54} d^{2\delta-4} - \frac{\delta(\delta-1)(\delta-2)(702\delta^2-629\delta-286)}{270} d^{2\delta-5} + \\
& + \frac{\delta(\delta-1)(\delta-2)(6028\delta^3-15476\delta^2+11701\delta+4425)}{3240} d^{2\delta-6} + \\
& + \frac{\delta(\delta-1)(\delta-2)(\delta-3)(136288\delta^3-6089\delta^2-295729\delta-24485)}{11340} d^{2\delta-7} + \\
& - \frac{\delta(\delta-1)(\delta-2)(\delta-3)(2828555\delta^4-9311146\delta^3+417490\delta^2+4252029\delta+1141616)}{204120} d^{2\delta-8} + \cdots \Bigg].
\end{aligned}$$

The proofs of Theorems II.2 and II.3 are algorithmic in nature and involve a computer computation. We describe both algorithms in detail in Sections 2.3 and 2.5, respectively.

Let $d^*(\delta)$ denote the *polynomiality threshold* for Severi degrees, i.e., the smallest positive integer $d^* = d^*(\delta)$ such that $N_\delta(d) = N^{d,\delta}$ for $d \geq d^*$. As mentioned earlier S. Fomin and G. Mikhalkin [9, Theorem 5.1] showed that $d^* \leq 2\delta$. We improve this as follows.

Theorem II.4. *For $\delta \geq 1$, we have $d^*(\delta) \leq \delta$.*

In other words, $N^{d,\delta} = N_\delta(d)$ provided $d \geq \delta \geq 1$. L. Göttsche [18, Conjecture 4.1] conjectured that $d^* \leq \lceil \frac{\delta}{2} \rceil + 1$ for $\delta \geq 1$. This was verified for $\delta \leq 8$ by S. Kleiman and R. Piene [21]. By direct computation we can push it further.

Proposition II.5. *For $3 \leq \delta \leq 14$, we have $d^*(\delta) = \lceil \frac{\delta}{2} \rceil + 1$.*

That is, Göttsche's threshold is correct and sharp for $3 \leq \delta \leq 14$. For $\delta = 1, 2$ it is easy to see that $d^*(1) = 1$ and $d^*(2) = 1$.

P. Di Francesco and C. Itzykson [7] hypothesized that $d^*(\delta) \leq \left\lceil \frac{3}{2} + \sqrt{2\delta + \frac{1}{4}} \right\rceil$ (which is equivalent to $\delta \leq \frac{(d^*-1)(d^*-2)}{2}$). However, our computations show that this fails for $\delta = 13$ as $d^*(13) = 8$.

Competing Approaches: Floor Diagrams vs. Caporaso-Harris recursion

An alternative approach to computing the node polynomials $N_\delta(d)$ combines polynomial interpolation with the Caporaso-Harris recursion [6]. Once a polynomiality threshold $d_0(\delta)$ has been established (i.e., once we have proved that $N_\delta(d) = N^{d,\delta}$ for $d \geq d_0(\delta)$), we can use the recursion to determine a sufficient number of Severi degrees $N^{d,\delta}$ for $d \geq d_0(\delta)$, from which we then interpolate.

This approach was first used by L. Göttsche [18, Remark 4.1(1)]. He conjectured [18, Conjecture 4.1] the polynomiality threshold $d_0(\delta) = \lceil \frac{\delta}{2} \rceil + 1$, and combined it with the "Göttsche-Yau-Zaslow formula" [18, Conjecture 2.4] (now a theorem of Y.-J. Tzeng [34]) to calculate the putative node polynomials $N_\delta(d)$ for $\delta \leq 28$. The Göttsche-Yau-Zaslow formula is a stronger version of polynomiality that allows one to compute each next node polynomial by calculating only two additional Severi degrees $N^{d_0(\delta),\delta}$ and $N^{d_0(\delta)+1,\delta}$, which is done via the Caporaso-Harris formula. Since Göttsche's threshold $d_0(\delta) = \lceil \frac{\delta}{2} \rceil + 1$ remains open as of this writing, the algorithm he used to compute the node polynomials is still awaiting a rigorous justification.

The first polynomiality threshold $d_0(\delta) = 2\delta$ was established by S. Fomin and G. Mikhalkin [9, Theorem 5.1]. Using this result, one can compute $N_\delta(d)$ for $\delta \leq 9$ but hardly any further[1]. With the threshold $d_0(\delta) = \delta$ established in Theorem II.4, it should be possible to compute $N_\delta(d)$ for $\delta \leq 16$ or perhaps $\delta \leq 17$.

By contrast, our Algorithm 1 does not involve interpolation nor does it require an *a priori* knowledge of a polynomiality threshold. Our computations verify the results of L. Göttsche's calculations for $\delta \leq 14$. In our implementations, Algorithm 1 is roughly as efficient as the interpolation method discussed above. (We repeat that the latter method depends on the threshold obtained using floor diagrams.)

Gromov-Witten invariants

The *Gromov-Witten invariant* $N_{d,g}$ enumerates irreducible plane curves of degree d and genus g through $3d + g - 1$ generic points in $\mathbb{CP}^2$. Algorithm 1 (with minor adjustments, cf. Theorem II.10(2)) can be used to directly compute $N_{d,g}$, without resorting to a recursion involving relative Gromov-Witten invariants *à la* Caporaso–Harris [6].

[1] We used an efficient C implementation of the Caporaso-Harris recursion by A. Gathmann.

2.2 Floor Diagrams

Floor diagrams – the main combinatorial objects of this thesis – are gadgets which, if counted correctly, enumerate plane curves with certain prescribed properties. E. Brugallé and G. Mikhalkin introduced them in [4] (in slightly different notation) and studied them further in [5]. To keep the presentation self-contained and to fix notation we review them and their markings following [9] where the framework that best suits our purposes was introduced.

Definition II.6. A *floor diagram* $\mathcal{D}$ on a vertex set $\{1, \ldots, d\}$ is a directed graph (possibly with multiple edges) with positive integer edge weights $w(e)$ satisfying:

1. The edge directions respect the order of the vertices, i.e., for each edge $i \to j$ of $\mathcal{D}$ we have $i < j$.

2. (Divergence Condition) For each vertex j of $\mathcal{D}$, we have

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} j}} w(e) \leq 1.$$

This means that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most 1 than the total weight of the incoming edges.

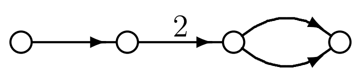
The *degree* of a floor diagram $\mathcal{D}$ is the number of its vertices. It is *connected* if its underlying graph is. Note that in [9] floor diagrams are required to be connected. If $\mathcal{D}$ is connected its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space). The *cogenus* of a connected floor diagram $\mathcal{D}$ of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If $\mathcal{D}$ is not connected, let $d_1, d_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be the degrees and cogenera, respectively, of its connected components. Then the *cogenus* of $\mathcal{D}$ is $\sum_j \delta_j + \sum_{j<j'} d_j d_{j'}$. Via the correspondence

between algebraic curves and floor diagrams ([9, Theorem 3.9]) these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility. Lastly, a floor diagram $\mathcal{D}$ has *multiplicity*[2]

$$\mu(\mathcal{D}) = \prod_{\text{edges } e} w(e)^2.$$

We draw floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

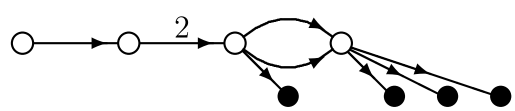
Example II.7. An example of a floor diagram of degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\mu = 4$ is drawn below.



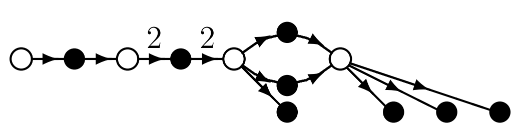
To enumerate algebraic curves via floor diagrams we need the notion of markings of such diagrams.

Definition II.8. A *marking* of a floor diagram $\mathcal{D}$ is defined by the following three step process which we illustrate in the case of Example IV.19.

Step 1: For each vertex j of $\mathcal{D}$ create $1 - div(j)$ many new vertices and connect them to j with new edges directed away from j.

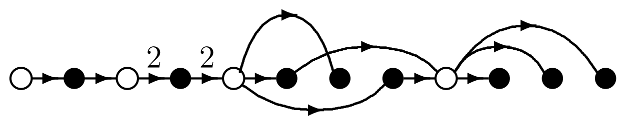


Step 2: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



[2]If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

Step 3: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as before, each edge is directed from a smaller vertex to a larger vertex.



The extended graph $\tilde{\mathcal{D}}$ together with the linear order on its vertices is called a *marked floor diagram*, or a *marking* of the original floor diagram $\mathcal{D}$.

We want to count marked floor diagrams up to equivalence. Two markings $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$. The *number of markings* $\nu(\mathcal{D})$ is the number of marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence.

Example II.9. The floor diagram $\mathcal{D}$ of Example IV.19 has $\nu(\mathcal{D}) = 7$ markings (up to equivalence): In step 3 the extra 1-valent vertex connected to the third white vertex from the left can be inserted in three ways between the third and fourth white vertex (up to equivalence) and in four ways right of the fourth white vertex (again up to equivalence).

Now we can make precise how to compute Severi degrees $N^{d,\delta}$ and Gromov-Witten invariants $N_{d,g}$ in terms of combinatorics of floor diagrams, thereby reformulating the initial question at the beginning of Chapter I. Part 2 is not needed in the chapter and only included for completeness. However, we encounter rational Gromov-Witten invariants and generalizations again in Chapter IV. Part 2 first appeared in [4, Theorem 1].

Theorem II.10. *[9, Corollary 1.9, Theorem 1.6]*

1. *The Severi degree $N^{d,\delta}$, i.e., the number of possibly reducible nodal curves in $\mathbb{CP}^2$ of degree d with δ nodes through $\frac{d(d+3)}{2} - \delta$ generic points, is equal to*

$$N^{d,\delta} = \sum_{\mathcal{D}} \mu(\mathcal{D}) \nu(\mathcal{D}),$$

where $\mathcal{D}$ runs over all possibly disconnected floor diagrams of degree d and cogenus δ.

2. *The Gromov-Witten invariant $N_{d,g}$, i.e., the number of irreducible curves in $\mathbb{CP}^2$ of degree d and genus g through $3d + g - 1$ generic points, is equal to*

$$N_{d,g} = \sum_{\mathcal{D}} \mu(\mathcal{D}) \nu(\mathcal{D}),$$

where $\mathcal{D}$ runs over all connected floor diagrams of degree d and genus g.

2.3 Computing Node Polynomials

In this section we give an explicit algorithm that symbolically computes the node polynomials $N_\delta(d)$, for given $\delta \geq 1$. (As $N^{d,0} = 1$ for $d \geq 1$, we put $N_0(d) = 1$.) An implementation of this algorithm was used to prove Theorem II.2 and Proposition II.5. We mostly follow the notation in [9, Section 5]. First, we rephrase Theorem II.2 in more compact notation. For $\delta \leq 8$ one recovers [21, Theorem 3.1]. For $\delta \leq 14$ this coincides with the conjectural formulas of [18, Remark 2.5].

Theorem II.11. *The node polynomials $N_\delta(d)$, for $\delta \leq 14$, are given by the generating function $\sum_{\delta \geq 0} N_\delta(d) x^\delta$ via the transformation*

$$\sum_{\delta \geq 0} N_\delta(d) x^\delta = \exp\left(\sum_{\delta \geq 0} Q_\delta(d) x^\delta \right),$$

where

$$
\begin{aligned}
Q_0(d) &= 1,\\
Q_1(d) &= 3(d-1)^2,\\
Q_2(d) &= \tfrac{-3}{2}(d-1)(14d-25),\\
Q_3(d) &= \tfrac{1}{3}(690d^2-2364d+1899),\\
Q_4(d) &= \tfrac{1}{4}(-12060d^2+47835d-45207),\\
Q_5(d) &= \tfrac{1}{5}(217728d^2-965646d+1031823),\\
Q_6(d) &= \tfrac{1}{6}(-4010328d^2+19451628d-22907925),\\
Q_7(d) &= \tfrac{1}{7}(74884932d^2-391230216d+499072374),\\
Q_8(d) &= \tfrac{1}{8}(-1412380980d^2+7860785643d-10727554959),\\
Q_9(d) &= \tfrac{1}{9}(26842726680d^2-157836614730d+228307435911),\\
Q_{10}(d) &= \tfrac{1}{10}(-513240952752d^2+3167809665372d-4822190211285),\\
Q_{11}(d) &= \tfrac{1}{11}(9861407170992d^2-63560584231524d+101248067530602),\\
Q_{12}(d) &= \tfrac{1}{12}(-190244562607008d^2+1275088266948600d-2115732543025293),\\
Q_{13}(d) &= \tfrac{1}{13}(3682665360521280d^2-25576895657724768d+44039919476860362),\\
Q_{14}(d) &= \tfrac{1}{14}(-71494333556133600d^2+513017995615177680d-913759995239314452).
\end{aligned}
$$

In particular, all $Q_\delta(d)$, for $1 \le \delta \le 14$, are quadratic in d.

L. Göttsche [18] conjectured that all $Q_\delta(d)$ are quadratic. This theorem proves his conjecture for $\delta \le 14$.

The basic idea of the algorithm (see [9, Section 5]) is to decompose floor diagrams into smaller building blocks. These gadgets are be crucial in the proofs of all theorems in Chapters II and III.

Definition II.12. A *template* Γ is a directed graph (with possibly multiple edges) on vertices $\{0, \ldots, l\}$, for $l \ge 1$, and edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

1. If $i \to j$ is an edge then $i < j$.
2. Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \ge 2$. (No "short edges.")
3. For each vertex j, $1 \le j \le l-1$, there is an edge "covering" it, i.e., there exists an edge $i \to k$ with $i < j < k$.

Every template Γ comes with some numerical data associated with it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of the edge weights

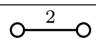
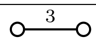
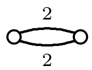
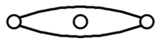
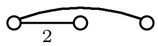
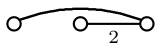
Γ	$\delta(\Gamma)$	$\ell(\Gamma)$	$\mu(\Gamma)$	$\varepsilon(\Gamma)$	$\varkappa(\Gamma)$	$k_{\min}(\Gamma)$	$P_\Gamma(k)$	$s(\Gamma)$
	1	1	4	0	(2)	2	$k-1$	1
	1	2	1	1	(1,1)	1	$2k+1$	0
	2	1	9	0	(3)	3	$k-2$	1
	2	1	16	0	(4)	4	$\binom{k-2}{2}$	2
	2	2	1	1	(2,2)	2	$\binom{2k}{2}$	0
	2	2	4	1	(3,1)	3	$2k(k-2)$	1
	2	2	4	0	(1,3)	2	$2k(k-1)$	1
	2	3	1	1	(1,1,1)	1	$3(k+1)$	0
	2	3	1	1	(1,2,1)	1	$k(4k+5)$	0

Figure 2.2: The templates with $\delta(\Gamma) \leq 2$.

is its *multiplicity* $\mu(\Gamma)$. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) = \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1 \Big].$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \to k$ with $i < j \leq k$ and define

$$k_{\min}(\Gamma) = \max_{1 \leq j \leq l} (\varkappa_j - j + 1).$$

This makes $k_{\min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \ldots\}$ with left-most vertex k. Lastly, set

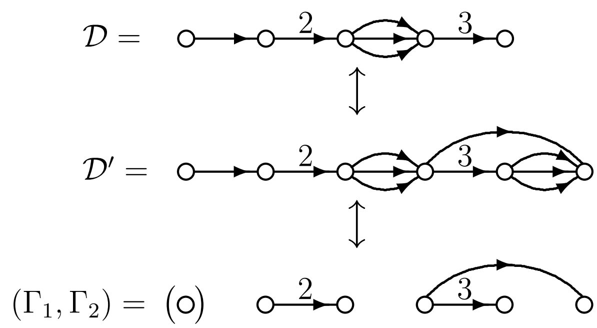
$$\varepsilon(\Gamma) = \begin{cases} 1 & \text{if all edges arriving at } l \text{ have weight } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 2.2 (Figure 10 taken from [9]) lists all templates Γ with $\delta(\Gamma) \leq 2$.

A floor diagram $\mathcal{D}$ on d vertices decomposes into an ordered collection $(\Gamma_1, \ldots, \Gamma_m)$ of templates as follows: First, add an additional vertex $d+1$ $(> d)$ to $\mathcal{D}$ along with, for every vertex j of $\mathcal{D}$, $1 - div(j)$ new edges of weight 1 from j to the new vertex $d+1$.

The resulting floor diagram $\mathcal{D}'$ has divergence 1 at every vertex coming from $\mathcal{D}$. Now remove all *short edges* from $\mathcal{D}'$, that is, all edges of weight 1 between consecutive vertices. The result is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_m)$, listed left to right, and it is not hard to see that $\sum \delta(\Gamma_i) = \delta(\mathcal{D})$. This process is reversible once we record the smallest vertex k_i of each template Γ_i (see Example II.13).

Example II.13. An example of the decomposition of a floor diagram into templates is illustrated below. Here, $k_1 = 2$ and $k_2 = 4$.



To each template Γ we associate a polynomial that records the number of "markings of Γ:" For $k \in \mathbb{Z}_{>0}$ let $\Gamma_{(k)}$ denote the graph obtained from Γ by first adding $k + i - 1 - \varkappa_i$ short edges connecting $i - 1$ to i, for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By [9, Lemma 5.6] the number of linear extensions (up to equivalence) of the vertex poset of the graph $\Gamma_{(k)}$ extending the vertex order of Γ is a polynomial in k, if $k \geq k_{\min}(\Gamma)$, which we denote by $P_\Gamma(k)$ (see Figure 2.2). The number of markings of a floor diagram $\mathcal{D}$ decomposing into templates $(\Gamma_1, \ldots, \Gamma_m)$ is then

$$\nu(\mathcal{D}) = \prod_{i=1}^{m} P_{\Gamma_i}(k_i),$$

where k_i is the smallest vertex of Γ_i in $\mathcal{D}$. The algorithm is based on

Theorem II.14 ([9], (5.13))**.** *The Severi degree $N^{d,\delta}$, for $d, \delta \geq 1$, is given by the* *template decomposition formula*

$$\sum_{(\Gamma_1,\ldots,\Gamma_m)} \prod_{i=1}^{m} \mu(\Gamma_i) \sum_{k_m=k_{\min}(\Gamma_m)}^{d-l(\Gamma_m)+\varepsilon(\Gamma_m)} P_{\Gamma_m}(k_m) \cdots \sum_{k_1=k_{\min}(\Gamma_1)}^{k_2-l(\Gamma_1)} P_{\Gamma_1}(k_1), \tag{2.1}$$

where the first sum is over all ordered collections of templates $(\Gamma_1, \ldots, \Gamma_m)$*, for all* $m \geq 1$*, with* $\sum_{i=1}^{m} \delta(\Gamma_i) = \delta$*, and the sums indexed by* k_i*, for* $1 \leq i < m$*, are over* $k_{\min}(\Gamma_i) \leq k_i \leq k_{i+1} - l(\Gamma_i)$*,*

Remark II.15. Theorem II.14 recasts the calculation of the Severi degree $N^{d,\delta}$ as a lattice point enumeration with polynomial weights. More specifically, each template collection $(\Gamma_1, \ldots, \Gamma_m)$ with $\sum_{i=1}^{m} \delta(\Gamma_i)$ determines a (possibly empty) simplex in $\mathbb{R}^m$ given by

$$\begin{aligned}\{k \in \mathbb{R}^m : k_i \geq k_{\min}(\Gamma_i) \ \ (1 \leq i \leq m), \\ k_i + l(\Gamma_i) \leq k_{i+1} \ \ (1 \leq i \leq m-1), k_m + l(\Gamma_m) \leq d + \varepsilon(\Gamma_m)\}.\end{aligned}$$

We obtain $N^{d,\delta}$ by enumerating the lattice points in all such simplices with the polynomial weight $\prod_{i=1}^{m} \mu(\Gamma_i) P_{\Gamma_i}(k_i)$.

Expression (2.1) can be evaluated symbolically, using the following two lemmata. The first is Faulhaber's formula [22] from 1631 for discrete integration of polynomials. The second treats lower limits of iterated discrete integrals and its proof is straightforward. Here B_j denotes the jth Bernoulli number with the convention that $B_1 = +\frac{1}{2}$.

Lemma II.16 ([22])**.** *Let* $f(k) = \sum_{i=0}^{d} c_i k^i$ *be a polynomial in* k*. Then, for* $n \geq 0$*,*

$$F(n) \stackrel{def}{=} \sum_{k=0}^{n} f(k) = \sum_{s=0}^{d} \frac{c_s}{s+1} \sum_{j=0}^{s} \binom{s+1}{j} B_j n^{s+1-j}. \tag{2.2}$$

In particular, $deg(F) = deg(f) + 1$*.*

Data: The cogenus δ.
Result: The node polynomial $N_\delta(d)$.
begin
  Generate all templates Γ with $\delta(\Gamma) \leq \delta$;
  $N_\delta(d) \leftarrow 0$;
  forall the *ordered collections of templates* $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ *with* $\sum_{i=1}^m \delta(\Gamma_i) = \delta$ **do**
    $i \leftarrow 1$;
    $Q_1 \leftarrow 1$;
    while $i \leq m$ **do**
      $a_i \leftarrow \max\big(k_{\min}(\Gamma_i), k_{\min}(\Gamma_{i-1}) + l(\Gamma_{i-1}), \ldots, k_{\min}(\Gamma_1) + l(\Gamma_1) + \cdots + l(\Gamma_{i-1})\big)$;
    end
    while $i \leq m-1$ **do**
      $Q_{i+1}(k_{i+1}) \leftarrow \sum_{k_i=a_i}^{k_{i+1}-l(\Gamma_i)} P_{\Gamma_i}(k_i) Q_i(k_i)$;
      $i \leftarrow i+1$;
    end
    $Q^{\tilde{\Gamma}}(d) \leftarrow \sum_{k_m=a_m}^{d-l(\Gamma_m)+\varepsilon(\Gamma_m)} P_{\Gamma_m}(k_m) Q_m(k_m)$;
    $Q^{\tilde{\Gamma}}(d) \leftarrow \prod_{i=1}^m \mu(\Gamma_i) \cdot Q^{\tilde{\Gamma}}(d)$;
    $N_\delta(d) \leftarrow N_\delta(d) + Q^{\tilde{\Gamma}}(d)$;
  end
end

Algorithm 1: Algorithm to compute node polynomials.

Lemma II.17. *Let $f(k_1)$ and $g(k_2)$ be polynomials in k_1 and k_2, respectively, and let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 0}$. Furthermore, let $F(k_2) = \sum_{k_1=a_1}^{k_2-b_1} f(k_1)$ be a discrete anti-derivative of $f(k_1)$, where $k_2 \geq a_1 + b_1$. Then, for $n \geq \max(a_1+b_1+b_2, a_2+b_2)$,*

$$\sum_{k_2=a_2}^{n-b_2} g(k_2) \sum_{k_1=a_1}^{k_2-b_1} f(k_1) = \sum_{k_2=\max(a_1+b_1,a_2)}^{n-b_2} g(k_2) F(k_2).$$

Example II.18. An illustration of Lemma II.17 is the following iterated discrete integral:

$$\sum_{k_2=1}^{n} \sum_{k_1=1}^{k_2-2} 1 = \sum_{k_2=1}^{n} \left\{ \begin{array}{ll} k_2 - 2 & \text{if } k_2 \geq 2 \\ 0 & \text{if } k_2 = 1 \end{array} \right\} = \sum_{k_2=3}^{n} \big(k_2 - 2\big).$$

Using these results Algorithm 1 computes node polynomials $N_\delta(d)$ for an arbitrary number of nodes δ. The first step, the template generation, is explained later in this section.

Proof of Correctness of Algorithm 1. The algorithm is a direct implementation of Theorem II.14. The m-fold discrete integral is evaluated symbolically, one sum at a time, using Faulhaber's formula (Lemma II.16). The lower limit a_i of the ith sum is given by an iterated application of Lemma II.17. $\square$

As Algorithm 1 is stated its termination in reasonable time is hopeless for $\delta \geq 8$ or 9. The novelty of this section, together with an explicit formulation, is how to implement the algorithm efficiently. This is explained in Remark II.19.

Remark II.19. The running time of the algorithm can be improved vastly as follows: As the limits of summation in (2.1) only depend on $k_{\min}(\Gamma_i)$, $l(\Gamma_i)$ and $\varepsilon(\Gamma_m)$, we can replace the template polynomials $P_{\Gamma_i}(k_i)$ by $\sum P_{\Gamma_i}(k_i)$, where the sum is over all templates Γ_i with prescribed $(k_{\min}, l, \varepsilon)$. After this transformation the first sum in (2.1) is over all combinations of those tuples. This reduces the computation drastically as, for example, the 167885753 templates of cogenus 14 make up only 343 equivalence classes. Also, in (2.1) we can distribute the template multiplicities $\mu(\Gamma_i)$ and replace $P_{\Gamma_i}(k_i)$ by $\mu(\Gamma_i)P_{\Gamma_i}(k_i)$ and thereby eliminate $\prod \mu(\Gamma_i)$. Another speed-up is to compute all discrete integrals of monomials using Lemma II.16 in advance.

The generation of the templates is the bottleneck of the algorithm. Their number grows rapidly with δ as can be seen from Figure 2.4. However, their generation can be parallelized easily (see below).

Algorithm 1 has been implemented in Maple. Computing $N_{14}(d)$ on a machine with two quad-core Intel(R) Xeon(R) CPU L5420 @ 2.50GHz, 6144 KB cache, and 24 GB RAM took about 70 days.

Remark II.20. Using the combinatorial framework of floor diagrams one can show that also *relative Severi degrees* (i.e., the degrees of *generalized Severi varieties*, see [6, 36]) are polynomial and given by "relative node polynomials" [2, Theorem 1.1].

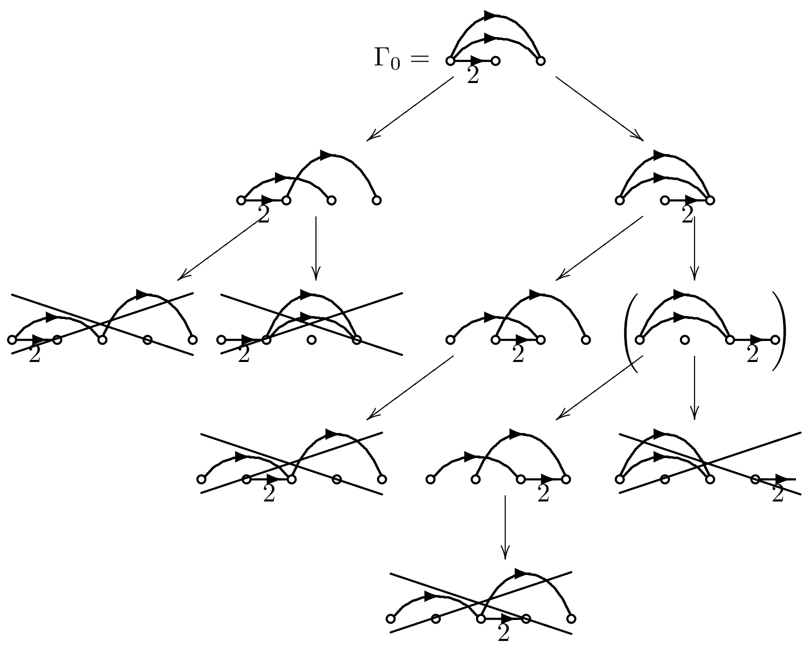


Figure 2.3: Branch-and-bound tree for $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.

This suggests the existence of a generalization of Göttsche's Conjecture [18, Conjecture 2.1] and the Göttsche-Yau-Zaslow formula [18, Conjecture 2.1]. Thus, the combinatorics of floor diagrams lead to new conjectures although the techniques and results seem to be out of reach at this time.

Remark II.21. We can use Algorithm 1 to compute the values of the Severi degrees $N^{d,\delta}$ for prescribed values of d and δ. After we specify a degree d and a number of nodes δ all sums in our algorithm become finite and can be evaluated numerically. See Appendix B for all values of $N^{d,\delta}$ for $0 \leq \delta \leq 14$ and $1 \leq d \leq 13$.

Proof of Proposition II.5. For $1 \leq \delta \leq 14$ we observe, using the data in Appendices A and B, that $N_\delta(d) = N^{d,\delta}$ for all $d_0(\delta) \leq d < \delta$, where $d_0(\delta) = \left\lceil \frac{\delta}{2} \right\rceil + 1$ is Göttsche's threshold. Furthermore, $N_\delta(d_0(\delta) - 1) \neq N^{d_0(\delta)-1,\delta}$ for all $3 \leq \delta \leq 14$. $\square$

2.3.1 Template Generation

To compute a list of all templates of a given cogenus one can proceed as follows. First, we need some terminology and notation. An edge $i \to j$ of a template is said

Data: A graph A with a distinguished edge e_1.
Result: An infinite directed tree of graphs with root A.
begin
  forall the *edges* e_2 *of* A *with* $e_2 \geq e_1$ *(in the fixed order)* **do**
    $B \leftarrow$ graph obtained from A by moving e_2 to the next vertex;
    if *the natural partial order (from left to right) of the edges of* B *that are of the same type as* e_2 *is compatible with the fixed order* **then**
      Insert B as a child of A;
      Execute this procedure with input (B, e_2);
    end
  end
end

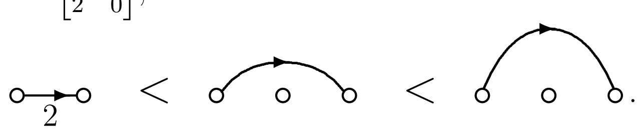
Algorithm 2: A recursion which can generates a tree containing all templates of a given type.

to have *length* $j - i$. A template Γ is of *type* $\alpha = (\alpha_{ij})$, $i, j \in \mathbb{Z}_{>0}$, if Γ has α_{ij} edges of length i and weight j. Every type α satisfies, by definition of cogenus of a template,

$$\sum_{i,j \geq 1} \alpha_{ij}(i \cdot j - 1) = \delta(\Gamma). \tag{2.3}$$

Note that $\alpha_{11} = 0$ as short edges are not allowed in templates. The number of types constituting a given cogenus δ is finite.

We can generate all templates of type α using a branch-and-bound algorithm which slides edges in a suitable order. Let Γ_0 be the unique template of type α with all edges emerging from vertex 0. Fix a linear order on the set of edges of type α. For example, if $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, we could choose:



Algorithm 2 applied to the pair (Γ_0, e_0), where e_0 is the smallest edge of Γ_0, creates an infinite directed tree with root Γ_0 all of whose vertices correspond to different graphs. Eliminate a branch if either

1. no edge of the root of the branch starts at vertex 1, or

δ	# of templates	δ	# of templates	δ	# of templates
1	2	6	1711	11	2233572
2	7	7	7135	12	9423100
3	26	8	29913	13	39769731
4	102	9	125775	14	167885753
5	414	10	529755		

Figure 2.4: The number of templates with cogenus $\delta \leq 14$.

2. condition (3) in Definition II.12 is impossible to satisfy for graphs further down the tree.

See Figure 2.3.1 for an illustration for $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.

A complete, non-redundant list of all templates of type α is then given by all remaining graphs which satisfy condition (3) of Definition II.12 as every template can be obtained in a unique way from Γ_0 by shifting edges in an order that is compatible with the order fixed earlier. Note that it can happen that a non-template graph precedes a template within a branch. For an example see the graph in brackets in Figure 2.3.1. Template generation for different types can be executed in parallel. The number of templates, for $\delta \leq 14$, is given in Figure 2.4.

2.4 Threshold Values

S. Fomin and G. Mikhalkin [9, Theorem 5.1] proved polynomiality of the Severi degrees $N^{d,\delta}$ in d, for fixed δ, provided d is sufficiently large. More precisely, they showed that $N_\delta(d) = N^{d,\delta}$ for $d \geq 2\delta$. In this section we show that their threshold can be improved to $d \geq \delta$ (Theorem II.4).

We need the following elementary observation about robustness of discrete anti-derivatives of polynomials whose continuous counterpart is the well known fact that $\int_{a-1}^{a-s-1} f(x)dx = 0$ if $f(x) = 0$ on the interval $(a-s-1, a-1)$.

Lemma II.22. *For a polynomial $f(k)$ and $a \in \mathbb{Z}_{>0}$ let $F(n) = \sum_{k=a}^{n} f(k)$ be* *the polynomial in n uniquely determined by large enough values of n. ($F(n)$ is a polynomial by Lemma II.16.) If we have $f(a-1) = \cdots = f(a-s) = 0$ for some $0 \leq s < a$ (this condition is vacuous for $s = 0$) then it also holds that $F(a-1) = \cdots = F(a-s-1) = 0$. In particular, $\sum_{k=a}^{n} f(k)$ is a polynomial in n, for $n \geq a-s-1$.*

Even for $s = 0$ the lemma is non-trivial as, in general, $F(a-2) \neq 0$.

Proof. Let $G(n)$ be the polynomial in n defined via $G(n) = \sum_{k=0}^{n} f(k)$ for large n. Then $F(n) = G(n) - \sum_{k=0}^{a-1} f(k)$ for all $n \in \mathbb{Z}_{\geq 0}$. In particular, for any $0 \leq i \leq s$, we have $F(a-i-1) = G(a-i-1) - \sum_{k=0}^{a-1} f(k) = G(a-i-1) - \sum_{k=0}^{a-i-1} f(k) = 0$. $\square$

Recall that for a template Γ, we defined $k_{\min} = k_{\min}(\Gamma)$ to be the smallest $k \geq 1$ such that $k + j - 1 \geq \varkappa_j(\Gamma)$ for all $1 \leq j \leq l(\Gamma)$. Let j_0 be the smallest j for which equality is attained (it is easy to see that equality is attained for some j). Define $s(\Gamma)$ to be the number of edges of Γ from $j_0 - 1$ to j_0 (of any weight). See Figure 2.2 for some examples. The following lemma shows that the template polynomials $P_\Gamma(k)$ satisfy the condition of Lemma II.22.

Lemma II.23. *With the notation from above it holds that*

$$P_\Gamma(k_{\min} - 1) = P_\Gamma(k_{\min} - 2) = \cdots = P_\Gamma(k_{\min} - s(\Gamma)) = 0.$$

Proof. Recall from Section 2.3 that, for $k \geq k_{\min}(\Gamma)$, the polynomial $P_\Gamma(k)$ records the number of linear extension (up to equivalence) of some poset $\Gamma_{(k)}$ which is obtained from Γ by first adding $k + j - 1 - \varkappa_j(\Gamma)$ "short edges" connecting $j-1$ to j, for $1 \leq j \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing a new vertex for each edge.

Using the notation from the last paragraph notice that $k_{\min} + j_0 - 1 = \varkappa_{j_0}(\Gamma)$, and thus $\Gamma_{(k)}$ has $k - k_{\min}$ "short edges" between $j_0 - 1$ and j_0. Every linear extension of

$\Gamma_{(k)}$ can be obtained by first linearly ordering the midpoints of these $k-k_{\min}$ "short edges" and the midpoints of the $s(\Gamma)$ many edges of Γ connecting j_0-1 and j_0 before completing the linear order to all vertices of $\Gamma_{(k)}$. Therefore, the polynomial $(k-k_{\min}+1)\cdots(k-k_{\min}+s(\Gamma))$ divides $P_\Gamma(k)$. $\square$

Before we can prove Theorem II.4 we need a last technical lemma.

Lemma II.24. *Using the notation from above we have, for each template* Γ,

$$k_{\min}(\Gamma)-s(\Gamma)+l(\Gamma)-\varepsilon(\Gamma)\leq\delta(\Gamma)+1.$$

Proof. As before, let j_0 be the smallest j in $\{1,\ldots,l(\Gamma)\}$ with $k_{\min}+j-1=\varkappa_j(\Gamma)$. It suffices to show that $\varkappa_{j_0}(\Gamma)-j_0-s(\Gamma)+l(\Gamma)-\varepsilon(\Gamma)\leq\delta(\Gamma)$.

Let Γ' be the template obtained from Γ by removing all edges $i\to k$ with either $k<j_0$ or $i\geq j_0$. It is easy to see that $l(\Gamma)-\varepsilon(\Gamma)-(l(\Gamma')-\varepsilon(\Gamma'))\leq\delta(\Gamma)-\delta(\Gamma')$. Thus, we can assume without loss of generality that all edges $i\to k$ of Γ satisfy $i<j_0\leq k$. Therefore, as $\varkappa_{j_0}(\Gamma)=\sum_{\text{edges } e \text{ of } \Gamma}\mathrm{wt}(e)$ it suffices to show that

$$l(\Gamma)-\varepsilon(\Gamma)\leq\sum_{\text{edges } e \text{ of } \Gamma}\Big[\mathrm{wt}(e)(\mathrm{len}(e)-1)-1\Big]+s(\Gamma)+j_0, \tag{2.4}$$

where $\mathrm{len}(e)$ is the length of an edge e. The contribution of the $s(\Gamma)$ edges of Γ between j_0-1 and j_0 to the sum is $-s(\Gamma)$, thus the right-hand-side of (2.4) equals

$$\sum\Big[\mathrm{wt}(e)(\mathrm{len}(e)-1)-1\Big]+j_0 \tag{2.5}$$

with the sum now running over all edges of Γ of length at least 2. If there are no such edges, then $l(\Gamma)=1$ and we are done. Otherwise, if $\varepsilon(\Gamma)=1$, expression (2.5) equals $\sum(\mathrm{len}(e)-2)+j_0$, which is $\geq l(\Gamma)-2+j_0$ or $\geq l(\Gamma)-3+j_0$ if $j_0\in\{1,l(\Gamma)\}$ or $1<j_0<l(\Gamma)$, respectively (by considering only edges adjacent to vertices 0 and $l(\Gamma)$ of Γ). In either case the result follows.

If $\varepsilon(\Gamma) = 0$ then expression (2.5) is $\geq l(\Gamma) + (l(\Gamma) - 3 + j_0)$ or $\geq l(\Gamma) - 2 + j_0$ if $j_0 \in \{1, l(\Gamma)\}$ or $1 < j_0 < l(\Gamma)$, respectively. This completes the proof. $\square$

Proof of Theorem II.4. By Lemma II.17 and repeated application of Lemmata II.22 and II.23 it suffices to show that $d \geq \delta$ simultaneously implies

$$\begin{aligned}
d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + k_{\min}(\Gamma_m) - s(\Gamma_m) - 1, \\
d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + k_{\min}(\Gamma_{m-1}) - s(\Gamma_{m-1}) - 2, \\
&\vdots \\
d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + \cdots + l(\Gamma_1) + k_{\min}(\Gamma_1) - s(\Gamma_1) - m,
\end{aligned} \tag{2.6}$$

for all collections of templates $(\Gamma_1, \ldots, \Gamma_m)$ with $\sum_{i=1}^m \delta(\Gamma_i) = \delta$.

The first inequality is a direct consequence of Lemma II.24. For the other inequalities, notice that $l(\Gamma) - \varepsilon(\Gamma) \leq \delta(\Gamma)$ for all templates Γ, hence

$$l(\Gamma_m) - \varepsilon(\Gamma_m) - 1 \leq \delta(\Gamma_m) - 1$$

and

$$l(\Gamma_i) - 1 \leq \delta(\Gamma_i), \quad \text{for } 2 \leq i \leq m - 1.$$

By Lemma II.24 we have

$$l(\Gamma_1) + k_{\min}(\Gamma_1) - s(\Gamma_1) - 1 \leq \delta(\Gamma_1) + 1$$

as $\varepsilon(\Gamma_1) \leq 1$, and the right-hand-side of the last inequality of (2.6) is $\leq \sum_{i=1}^m \delta(\Gamma_i) = \delta \leq d$. The proof of the other inequalities is very similar. $\square$

2.5 Coefficients of Node Polynomials

The goal of this section is to present an algorithm for the computation of the coefficients of $N_\delta(d)$, for general δ. The algorithm can be used to prove Theorem II.3

and thereby confirm and extend a conjecture of P. Di Francesco and C. Itzykson in [7] where they conjectured the 7 terms of $N_\delta(d)$ of largest degree.

Our algorithm should be able to find formulas for arbitrarily many coefficients of $N_\delta(d)$. We prove correctness of our algorithm in this section. The algorithm rests on the polynomiality of solutions of certain polynomial difference equations (see (2.13)).

First, we fix some notation building on terminology of Section 2.3. By Remark II.19 we can replace the polynomials $P_\Gamma(k)$ in (2.1) by the product $\mu(\Gamma)P_\Gamma(k)$, thereby removing the product $\prod \mu(\Gamma_i)$ of the template multiplicities. In this section we write $P^*(\Gamma, k)$ for $\mu(\Gamma)P_\Gamma(k)$. For integers $i \geq 0$ and $a \geq 0$ let $M_i(a)$ denote the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma:\delta(\Gamma)=i} \sum_{k=k_{\min}(\Gamma)}^{n-l(\Gamma)} P^*(\Gamma, k) \cdot f(k), \tag{2.7}$$

where $f(k) = c_0 k^a + c_1 k^{a-1} + \cdots$, a polynomial of degree a, is mapped to the polynomial $M_i(a)(f(k)) = d_0 n^{a+i+1} + d_1 n^{a+i} + \cdots$ in n. (By Lemma II.16 and the proof of Lemma II.25 the image has degree $a+i+1$.) Hence $M_i(a)\mathbf{c} = \mathbf{d}$. Similarly, define $M_i^{\text{end}}(a)$ to be the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma:\delta(\Gamma)=i} \sum_{k=k_{\min}(\Gamma)}^{n-l(\Gamma)+\varepsilon(\Gamma)} P^*(\Gamma, k) \cdot f(k). \tag{2.8}$$

Later we will consider square sub-matrices of $M_i(a)$ and $M_i^{\text{end}}(a)$ by restriction to the first few rows and columns which will be denoted $M_i(a)$ and $M_i^{\text{end}}(a)$ as well. Note that $M_i(a)$ and $M_i^{\text{end}}(a)$ are lower triangular. For example, for a large enough,

$$M_1(a) = \begin{bmatrix} \frac{6}{a+2} & 0 & 0 & 0 & 0 & \cdots \\ -\frac{5a+8}{a+1} & \frac{6}{a+1} & 0 & 0 & 0 & \cdots \\ \frac{5}{2}a+3 & -\frac{5a+3}{a} & \frac{6}{a} & 0 & 0 & \cdots \\ -\frac{1}{4}(4a+1)a & \frac{5}{2}a+\frac{1}{2} & -\frac{5a-2}{a-1} & \frac{6}{a-1} & 0 & \cdots \\ \frac{1}{40}(13a^2-20a+7)a & -a^2+\frac{7}{4}a-\frac{3}{4} & \frac{5}{2}a-2 & -\frac{5a-7}{a-2} & \frac{6}{a-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Lemma II.25. *The first $a+i$ rows of $M_i(a)$ and $M_i^{end}(a)$ are independent of the lower limits of summation in (2.7) and (2.8), respectively.*

Proof. It is an easy consequence of the proof of [9, Lemma 5.7] that the polynomial $P^*(\Gamma, k)$ associated with a template Γ has degree $\leq \delta(\Gamma)$. Equality is attained by the template Γ on vertices $0, 1, 2$ with i edges connecting 0 and 2 (so $\delta(\Gamma) = i$). As discrete integration of a polynomial increases the degree by 1 the polynomial on the right-hand-side of (2.7) has degree $1 + i + a$. $\square$

The basic idea of the algorithm is that templates with higher cogenera do not contribute to higher degree terms of the node polynomial. With this in mind we define, for each finite collection $(\Gamma_1, \ldots, \Gamma_m)$ of templates, its *type* $\tau = (\tau_2, \tau_3, \ldots)$, where τ_i is the number of templates in $(\Gamma_1, \ldots, \Gamma_m)$ with cogenus equal to i, for $i \geq 2$. Note that we do not record the number of templates with cogenus equal to 1.

To collect the contributions of all collections of templates with a given type τ, let $\tau = (\tau_2, \tau_3, \ldots)$ and fix $\delta \geq \sum_{j \geq 2} \tau_j$ (so that there exist template collections $(\Gamma_1, \ldots, \Gamma_m)$ of type τ with $\sum \delta(\Gamma_j) = \delta$). We define two (column) vectors $C_\tau(\delta)$ and $C_\tau^{\text{end}}(\delta)$ as the coefficient vectors, listed in decreasing order, of the polynomials

$$\sum_{(\Gamma_1,\ldots,\Gamma_m)} \sum_{k_m = k_{\min}(\Gamma_m)}^{n - l(\Gamma_m)} P^*(\Gamma_m, k_m) \cdots \sum_{k_1 = k_{\min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} P^*(\Gamma_1, k_1) \tag{2.9}$$

and

$$\sum_{(\Gamma_1,\ldots,\Gamma_m)} \sum_{k_m = k_{\min}(\Gamma_m)}^{n - l(\Gamma_m) + \varepsilon(\Gamma)} P^*(\Gamma_m, k_m) \sum_{k_{m-1} = k_{\min}(\Gamma_{m-1})}^{k_m - l(\Gamma_{m-1})} \cdots \sum_{k_1 = k_{\min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} P^*(\Gamma_1, k_1) \tag{2.10}$$

in the indeterminate n, where the respective first sums are over all ordered collections of templates of type τ.

It might look like $C_\tau(\delta)$ is a product of some matrices $M_i(a)$ applied to the polynomial 1. However, this is not the case. For example, note that

$$C_{(0,0,\dots)}(2) = \begin{bmatrix} \frac{9}{2} \\ -34 \\ 88 \\ -\frac{179}{2} \\ 30 \\ 0 \\ \vdots \end{bmatrix} \neq \begin{bmatrix} \frac{9}{2} \\ -34 \\ 88 \\ -\frac{179}{2} \\ 27 \\ 0 \\ \vdots \end{bmatrix} = M_1(2) \cdot M_1(0) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

This is because, when iterated discrete integrals are evaluated symbolically, the lower limits of integration of the outer sums can change depending on the limits of the inner sums (cf. Lemma II.17). This observation makes it necessary to compute initial values for recursions (described later) up to a large enough δ.

Before we can state the main recursion we need two more notations. For a type $\tau = (\tau_2, \tau_3, \dots)$ and $i \geq 2$ with $\tau_i > 0$ define a new type $\tau{\downarrow_i}$ via $(\tau{\downarrow_i})_i = \tau_i - 1$ and $(\tau{\downarrow_i})_j = \tau_j$ for $j \neq i$. Furthermore, let $\operatorname{def}(\tau) = \sum_{j \geq 2} (j-1)\tau_j$ be the *defect* of τ. The following lemma justifies this terminology.

Lemma II.26. *The polynomials (2.9) and (2.10) are of degree* $2\delta - \operatorname{def}(\tau)$.

Proof. Let $(\Gamma_1, \dots, \Gamma_m)$ be a collection of templates with $\sum_{i=1}^m \delta(\Gamma_i) = \delta$ and type τ. Then, by applying the argument in the proof of Lemma II.25 to each Γ_i, the polynomials (2.9) and (2.10) have degree $\delta + m$. The result follows as

$$\begin{aligned} \delta - \operatorname{def}(\tau) &= \sum_{i=1}^m \delta(\Gamma_i) - \sum_{j \geq 2} (j-1)\tau_j \\ &= \sum_{i=1}^m \delta(\Gamma_i) - \sum_{j \geq 2} \left[\left(\sum_{i : \delta(\Gamma_i) = \tau_j} \delta(\Gamma_i) \right) - \tau_j \right] \\ &= \#\{i : \delta(\Gamma_i) = 1\} + \sum_{j \geq 2} \tau_j = m. \end{aligned}$$

□

The last lemma makes precise which collections of templates contribute to which coefficients of $N_\delta(d)$. Namely, the first N coefficients of $N_\delta(d)$ of largest degree depend only on collections of templates with types τ such that $\operatorname{def}(\tau) < N$. The following recursion is the heart of the algorithm.

Proposition II.27. *For every type τ and integer δ large enough, it holds that*

$$\begin{aligned} C_\tau(\delta) = \sum_{i:\tau_i \neq 0} & M_i\big(2\delta - i - 1 - \operatorname{def}(\tau)\big) C_{\tau\downarrow i}(\delta - i) \\ & + M_1\big(2\delta - 2 - \operatorname{def}(\tau)\big) C_\tau(\delta - 1). \end{aligned} \tag{2.11}$$

More precisely, if we restrict all matrices M_i to be square of size $N - \operatorname{def}(\tau)$ and all C_τ to be vectors of length $N - \operatorname{def}(\tau)$, then recursion (2.11) holds for

$$\delta \geq \max\left(\left\lceil \frac{N+1}{2} \right\rceil, \sum_{j \geq 2} j\tau_j \right).$$

Proof. The coefficient vector $C_\tau(\delta)$ is defined by a sum that runs over all collections of templates $(\Gamma_1, \dots, \Gamma_m)$ of type τ (see (2.9)). Partition the set of such collections by putting $\delta(\Gamma_m) = 1$, or $\delta(\Gamma_m) = 2$, and so forth. This partitioning splits expression (2.9) exactly as in (2.11).

A summand can be written as a product of some matrix M_i and some vector $C_{\tau\downarrow i}$ if δ is large enough, namely if M_i does not depend on the lower limits in (2.9). If we can factor then the polynomials (2.9) defining $C_{\tau\downarrow i}(\delta - i)$ and $C_\tau(\delta - 1)$ have degrees

$$2(\delta - i) - \operatorname{def}(\tau{\downarrow}i) = 2\delta - 2i - \operatorname{def}(\tau) + (i-1) = 2\delta - i - 1 - \operatorname{def}(\tau)$$

by Lemma II.26 and, similarly, $2\delta - 2 - \operatorname{def}(\tau)$, respectively. By Lemma II.25, if the matrix $M_i(2\delta - i - 1 - \operatorname{def}(\tau))$ is of size $N - \operatorname{def}(\tau)$, then it does not depend on the lower limits if and only if $\delta \geq \frac{N+1}{2}$. In order for $C_\tau(\delta)$ to be defined (and the above identity to be meaningful) we need to impose $\delta \geq \sum_{j \geq 2} j\tau_j$. □

Remark II.28. Later, when we formulate the algorithm, we need to solve recursion (2.11) together with an initial condition in order to obtain an explicit formula for

Data: A positive integer N.
Result: The coefficient vector C of the first N coefficients of $N_\delta(d)$.
begin
  Compute all templates Γ with $\delta(\Gamma) \le N$;
  forall the *types* τ *with* $\mathrm{def}(\tau) < N$ **do**
    Compute initial values $C_\tau(\delta_0(\tau))$ using (2.9), with $\delta_0(\tau)$ as in (2.12);
    Solve recursion (2.11) for first $N - \mathrm{def}(\tau)$ coordinates of $C_\tau(\delta)$;
    Set

$$\begin{aligned} C_\tau^{\mathrm{end}}(\delta) \leftarrow & \sum_{i:\tau_i \neq 0} M_i^{\mathrm{end}}\big(2\delta - i - 1 - \mathrm{def}(\tau)\big) C_{\tau\downarrow i}(\delta - i) \\ & + M_1^{\mathrm{end}}\big(2\delta - 2 - \mathrm{def}(\tau)\big) C_\tau(\delta - 1); \end{aligned}$$

  end
  $C \leftarrow 0$;
  forall the *types* τ *with* $\mathrm{def}(\tau) < N$ **do**
    Shift the entries of $C_\tau^{\mathrm{end}}(\delta)$ down by $\mathrm{def}(\tau)$;
    $C \leftarrow C +$ shifted $C_\tau^{\mathrm{end}}(\delta)$;
  end
end

Algorithm 3: Computation of the leading coefficients of the node polynomial.

the first $N - \mathrm{def}(\tau)$ entries of $C_\tau(\delta)$. It suffices to take

$$\delta_0(\tau) \stackrel{\mathrm{def}}{=} \max\left(\left\lceil \frac{N-1}{2} \right\rceil, \sum_{j \ge 2} j\tau_j \right) \tag{2.12}$$

as for any $\delta > \delta_0(\tau)$ the vector $C_\tau(\delta)$ of length $N - \mathrm{def}(\tau)$ can be written in terms of matrices M_i and vectors $C_{\tau'}(\delta')$ for various types τ' and integers $\delta' < \delta$.

We propose Algorithm 3 for the computation of the coefficients of the node polynomial $N_\delta(d)$. We explain how to solve recursion (2.11) below.

Proof of Correctness of Algorithm 3. Proposition II.27 guarantees that the vector $C_\tau(\delta)$ is uniquely determined by recursion (2.9). By a similar argument as in the proof of Proposition II.27 we see that $C_\tau^{\mathrm{end}}(\delta)$ is given by the formula in Algorithm 3. By Lemma II.26 all contributions of template collections of type τ to the node polynomial $N_\delta(d)$ are in degree $2\delta - \mathrm{def}(\tau)$ or less. Hence, after shifting $C_\tau^{\mathrm{end}}(\delta)$ by $\mathrm{def}(\tau)$, their sum is the coefficient vector of $N_\delta(d)$. $\square$

To solve recursion (2.11) for a type τ we make use of the following (conjectural) structure about $C_\tau(\delta)$ which has been verified for all types τ with $\mathrm{def}(\tau) \leq 8$. This refines an observation of L. Göttsche [18, Remark 4.2 (2)] about the first 28 (conjectural) coefficients of the node polynomial $N_\delta(d)$.

Conjecture II.29. *All entries of $C_\tau(\delta)$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ.*

Now, to solve recursion (2.11), we first extend the natural partial order on the types τ given by $|\tau| = \sum_{j\geq 2} \tau_j$ to a linear order with smallest element $\tau = (0, 0, \dots)$. For example, for $N = 4$, we could take

$$(0,0,0) < (1,0,0) < (0,1,0) < (0,0,1) < (1,1,0) < (2,0,0) < (3,0,0).$$

Then solve recursion (2.11) for each τ, in increasing order, using the lowertriangularity of the matrices M_i. For example, to compute the second entry $\frac{3^\delta}{\delta!}p(\delta)$ of $C_{1,1}(\delta)$ (assuming Conjecture II.29), where $p(\delta)$ is a polynomial in δ, we need to solve

$$C_{1,1}(\delta) = M_1(2\delta-5)C_{1,1}(\delta-1) + M_2(2\delta-6)C_{0,1}(\delta-2) + M_3(2\delta-7)C_{1,0}(\delta-3),$$

or, explicitly,

$$\begin{bmatrix} * \\ \frac{3^\delta}{\delta!}p(\delta) \\ \vdots \end{bmatrix} = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ \frac{3^{\delta-1}}{(\delta-1)!}p(\delta-1) \\ \vdots \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \end{bmatrix}.$$

The $*$-entries in the vectors $C_{0,1}$ and $C_{1,0}$ are known by a previous computation. The $*$-entries in M_1, M_2 and M_3 are given by (2.9). The proof of Lemma II.25 implies that all denominators of $M_i(a)$ in row j are $a+i-j+2$ or 1 (after cancellation). To compute $p(\delta)$, or, more generally, the jth entry in $C_\tau(\delta)$, we first clear all denominators and then solve the polynomial difference equation with initial conditions

$$\begin{aligned} (2\delta - \mathrm{def}(\tau) - j + 1)3p(\delta) &= p(\delta-1) + q(\delta), \\ p(\delta_0(\tau)) &= C_\tau(\delta_0(\tau)), \end{aligned} \tag{2.13}$$

where $q(\delta)$ is a rather complicated polynomial depending on earlier calculations and $\delta_0(\tau)$ is as in (2.12). One way to solve (2.13) is to bound the degree of the polynomial $p(\delta)$ and solve the corresponding linear system.

Note that a difference equation of the form (2.13) need not have a polynomial solution in general. Conjecture II.29 is equivalent to all recursions (2.13) appearing in Algorithm 3 to have a polynomial solution.

As in Section 2.3 (Remark II.19), Algorithm 3 can be improved greatly by summing the template polynomials $P_\Gamma(k)$ for templates Γ with fixed $\big(k_{\min}(\Gamma), l(\Gamma), \varepsilon(\Gamma)\big)$ in advance. Algorithm 3 has been implemented in Maple. Once the templates are known the bottleneck of the algorithm is the initial value computation. With an improved implementation this should become faster than the template enumeration. Hence we expect Algorithm 3 to be able to compute the first 14 terms of $N_\delta(d)$ in reasonable time.

CHAPTER III

Relative Node Polynomials for Plane Curves

3.1 Relative Severi Degrees and Main Results

This chapter is devoted to degrees of generalized Severi varieties parametrizing plane curves which, in addition to point conditions, satisfy tangency conditions to a given line. Generalized Severi varieties were first introduced by L. Caporaso and J. Harris [6] in 1998 (see also [36]) in their study of positive genus Gromov-Witten invariants of $\mathbb{CP}^2$, a work which resulted in their famous recursion.

Fix, once and for all, a line $L \subset \mathbb{CP}^2$. The *relative Severi degree* $N^{\delta}_{\alpha,\beta}$ is the number of (possibly reducible) nodal plane curves with δ nodes which have tangency of order i to L at α_i *fixed* points (chosen in advance) and tangency of order i to L at β_i *unconstrained* points, for all $i \geq 1$, and which pass through an appropriate number of generic points (see Figure 3.1). Similar to the non-relative case, the numbers $N^{\delta}_{\alpha,\beta}$ are the degrees of the corresponding *generalized Severi varieties*.

By Bézout's Theorem, the degree of a curve with tangencies of order (α, β) equals $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$. Thus, the number of point conditions (for a potentially finite count) is $\frac{(d+3)d}{2} - \delta - \alpha_1 - \alpha_2 - \cdots$. We recover non-relative Severi degrees by specializing to $\alpha = (0, 0, \ldots)$ and $\beta = (d, 0, 0, \ldots)$.

In this chapter we show that much of the story of (non-relative) node polynomials

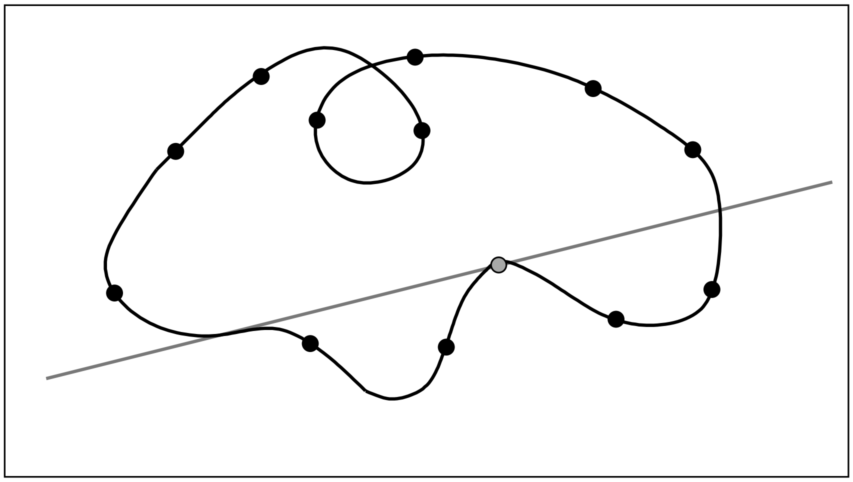


Figure 3.1: An algebraic curve in $\mathbb{CP}^2$ with 1 node and tangency of order 2 at a fixed (grey) point to a line L and tangency of order 1 and 3 at unconstrained points. Hence $\alpha = (0, 1)$ and $\beta = (1, 0, 1)$. Additionally, the curve passes through an appropriate number of points in generic position (only some of which are drawn).

carries over to relative Severi degrees. Our main result is that, up to a simple combinatorial factor and for fixed $\delta \geq 1$, the relative Severi degrees $N^{\delta}_{\alpha,\beta}$ are given by a multivariate *polynomial* in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$, provided that $\beta_1 + \beta_2 + \dots$ is sufficiently large. This is maybe quite surprising as the numbers $N^{\delta}_{\alpha,\beta}$ also satisfy and are determined by the rather complicated Caporaso-Harris recursion [6].

For sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers with finitely many α_i non-zero, we write

$$|\alpha| \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots , \qquad \alpha! \overset{\text{def}}{=} \alpha_1! \cdot \alpha_2! \cdot \cdots .$$

We use the grading $\deg(\alpha_i) = \deg(\beta_i) = 1$ (so that d and $|\beta|$ are homogeneous of degree 1). The following is the main theorem of this chapter.

Theorem III.1. *For every $\delta \geq 1$, there is a combinatorially defined polynomial $N_\delta(\alpha_1, \alpha_2, \dots ; \beta_1, \beta_2, \dots)$ of (total) degree 3δ such that, for all $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ with $|\beta| \geq \delta$, the relative Severi degree $N^{\delta}_{\alpha,\beta}$ is given by*

$$N^{\delta}_{\alpha,\beta} = 1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot N_\delta(\alpha_1, \alpha_2, \dots ; \beta_1, \beta_2, \dots). \tag{3.1}$$

Notice that this theorem generalizes both [9, Theorem 5.1] and Theorem II.4. We call $N_\delta(\alpha;\beta)$ the *relative node polynomial* and use the same notation as in the non-relative case if no confusion can occur. We do not need to specify the number of variables in light of the following stability condition.

Theorem III.2. *For $\delta \geq 1$ and vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_{m'})$ with $|\beta| \geq \delta$, it holds that*

$$N_\delta(\alpha, 0; \beta) = N_\delta(\alpha;\beta) \quad \textit{and} \quad N_\delta(\alpha;\beta,0) = N_\delta(\alpha;\beta)$$

as polynomials. Therefore, there exists a formal power series $N_\delta^\infty(\alpha;\beta)$ in infinitely many variables $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ which specializes to all relative node polynomials under $\alpha_{m+1} = \alpha_{m+2} = \cdots = 0$ and $\beta_{m'+1} = \beta_{m'+2} = \cdots = 0$, for various $m, m' \geq 1$.

Using the combinatorial description we provide a method to compute the relative node polynomials for arbitrary δ (see Sections 3.3 and 3.4). We utilize it to compute $N_\delta(\alpha;\beta)$ for $\delta \leq 6$. Due to spacial constrains we only tabulate the cases $\delta \leq 3$ in this work. The polynomials N_0 and N_1 already appeared (implicitly) in [9, Section 4.2].

Theorem III.3. *The relative node polynomials $N_\delta(\alpha;\beta)$, for $\delta = 0, 1, 2, 3$ (resp., $\delta \leq 6$) are as listed in Appendix C (resp., as provided in the ancillary files to [2]).*

The polynomial $N_\delta(\alpha;\beta)$ is of degree 3δ by Theorem III.1. We compute the terms of $N_\delta(\alpha;\beta)$ of degree $\geq 3\delta - 2$.

Theorem III.4. *The terms of $N_\delta(\alpha;\beta)$ of (total) degree $\geq 3\delta - 2$ are given by*

$$\begin{aligned}
N_\delta(\alpha;\beta) = \frac{3^\delta}{\delta!}\Bigg[& d^{2\delta}|\beta|^\delta + \frac{\delta}{3}\Big[-\frac{3}{2}(\delta-1)d^2 - 8d|\beta| + |\beta|\alpha_1 + d\beta_1 + |\beta|\beta_1\Big] d^{2\delta-2}|\beta|^{\delta-1} + \\
& + \frac{\delta}{9}\Big[\frac{3}{8}(\delta-1)(\delta-2)(3\delta-1)d^4 + 12\delta(\delta-1)d^3|\beta| + (11\delta+1)d^2|\beta|^2 + \\
& - \frac{3}{2}\delta(\delta-1)(d^3\beta_1 + d^2|\beta|\alpha_1) - \frac{1}{2}(\delta+5)(3\delta-2)d^2|\beta|\beta_1 - 8(\delta-1)(d|\beta|^2\alpha_1 + d|\beta|^2\beta_1) + \\
& + \frac{1}{2}(\delta-1)(d^2\beta_1^2 + |\beta|^2\alpha_1^2 + |\beta|^2\beta_1^2) + (\delta-1)(d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + |\beta|^2\alpha_1\beta_1)\Big] d^{2\delta-4}|\beta|^{\delta-2} + \cdots \Bigg],
\end{aligned}$$

where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$.

Theorem III.4 can be extended to terms of $N_\delta(\alpha, \beta)$ of degree $\geq 3\delta - 7$ (see Remark III.20). We observe that all coefficients of $N_\delta(\alpha; \beta)$ in degree $\geq 3\delta - 2$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ. In fact, even more is true. It is conceivable to expect this to hold for arbitrary degrees.

Proposition III.5. *Every coefficient of $N_\delta(\alpha; \beta)$ in degree $\geq 3\delta - 7$ is given, up to a factor of $\frac{3^\delta}{\delta!}$, by a polynomial in δ with rational coefficients.*

The approach in this chapter is similar to the combinatorial methods of Chapter II. We use of version of the Correspondence Theorem of G. Mikhalkin [27, Theorem 1] and the floor diagrams of E. Brugallé and G. Mikhalkin [4, 5] which incorporates tangency conditions (see Theorem III.7). The main new technical tool is an extension of the *template decomposition* of floor diagrams of S. Fomin and G. Mikhalkin which is also suitable in the relative setting (see Section 3.3).

3.2 Relative Markings of Floor Diagrams

The enumeration of plane curves satisfying tangency condition to a fixed line via floor diagrams requires a refinement of the notion of markings of a floor diagram. This extension is due to S. Fomin and G. Mikhalkin [9] who extended the work of E. Brugallé and G. Mikhalkin [4, 5]. Our notation, which is more convenient for our purposes, differs slightly from [9] where the "relative markings" are defined relative to the partitions $\lambda = \langle 1^{\alpha_1} 2^{\alpha_2} \cdots \rangle$ and $\rho = \langle 1^{\beta_1} 2^{\beta_2} \cdots \rangle$. In the sequel, all sequences are sequences of non-negative integers with finite support.

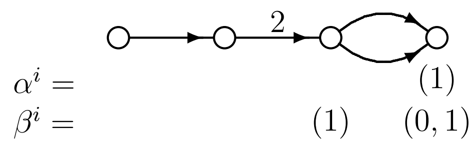
Definition III.6. For two sequences α, β we define an (α, β)-*marking* of a floor diagram $\mathcal{D}$ of degree $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ by the following four step process which we illustrate in the case of Example IV.19 for $\alpha = (1, 0, 0, \ldots)$ and $\beta = (1, 1, 0, 0, \ldots)$.

Step 1: Fix a pair of collections of sequences $(\{\alpha^i\}, \{\beta^i\})$, where i runs over the vertices of $\mathcal{D}$, with:

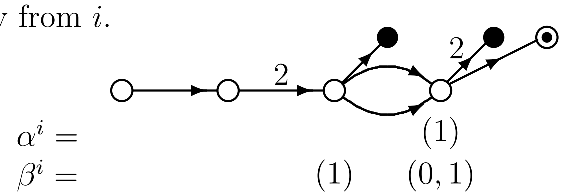
1. The sums over each collection satisfy $\sum_{i=1}^{d} \alpha^i = \alpha$ and $\sum_{i=1}^{d} \beta^i = \beta$.

2. For all vertices i of $\mathcal{D}$ we have $\sum_{j \geq 1} j(\alpha_j^i + \beta_j^i) = 1 - \mathrm{div}(i)$.

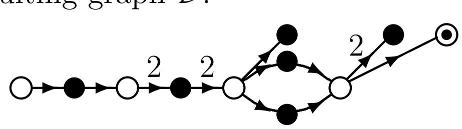
The second condition says that the "degree" of the pair (α^i, β^i) is compatible with the divergence at vertex i. Each such pair $(\{\alpha^i\}, \{\beta^i\})$ is called *compatible* with $\mathcal{D}$ and (α, β). We omit writing down trailing zeros.



Step 2: For each vertex i of $\mathcal{D}$ and every $j \geq 1$ create β_j^i new vertices, called *β-vertices* and illustrated as ●, and connected them to i with new edges of weight j directed away from i. For each vertex i of $\mathcal{D}$ and every $j \geq 1$ create α_j^i new vertices, called *α-vertices* and illustrated as ⦿, and connected them to i with new edges of weight j directed away from i.

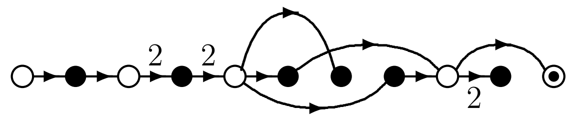


Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



Step 4: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as in $\mathcal{D}$, each edge is directed from a smaller vertex

to a larger vertex. Furthermore, we require that the α-vertices are largest among all vertices, and for every pair of α-vertices $i' > i$, the weight of the i'-adjacent edge is larger than or equal to the weight of the i-adjacent edge.



We call the extended graph $\tilde{\mathcal{D}}$, together with the linear order on its vertices, an (α, β)-*marked floor diagram*, or an (α, β)-*marking* of the floor diagram $\mathcal{D}$.

As before we need to count (α, β)-marked floor diagrams up to equivalence. This notion is verbatim the same as in the non-relative setting: Two (α, β)-markings $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists a weight preserving automorphism of weighted graphs mapping $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$ which fixes the vertices of $\mathcal{D}$. The *number of markings* $\nu_{\alpha,\beta}(\mathcal{D})$ is the number of (α, β)-marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. Furthermore, we write $\mu_\beta(\mathcal{D})$ for the product $1^{\beta_1} 2^{\beta_2} \cdots \mu(\mathcal{D})$. The next theorem follows from [9, Theorem 3.18] by a rather straight-forward extension of the inclusion-exclusion procedure of [9, Section 1] which was used to conclude [9, Corollary 1.9] (the non-relative count of reducible curves via floor diagrams) from [9, Theorem 1.6] (the non-relative count of irreducible curves via floor diagrams).

Theorem III.7. *For any $\delta \geq 1$, the relative Severi degree $N^\delta_{\alpha,\beta}$ is given by*

$$N^\delta_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over all (possibly disconnected) floor diagrams $\mathcal{D}$ of degree $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ and cogenus δ.

Proof. Fix a generic line L in $\mathbb{CP}^2$. Let $N^{\delta,\mathrm{irr}}_{\alpha,\beta}$ be the number of irreducible nodal plane curves with δ nodes and which satisfy tangency conditions to L given by sequences α and β as before. Since an irreducible degree d curve with δ nodes has genus $g = \binom{d-1}{2} - \delta$ we have by the definition of cogenus of a floor diagram and [9, Theorem 3.18] that

$$N^{\delta,\mathrm{irr}}_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

the sum over all connected floor diagrams $\mathcal{D}$ of degree $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$ and cogenus δ.

Now fix a collection of $d = \sum_{j\geq 1} j(\alpha_j + \beta_j)$ generic points Π in $\mathbb{CP}^2$. Let $\Pi_1, \dots, \Pi_t$ be a partition of Π into subsets. Similarly, let $\alpha^1, \dots, \alpha^t$ be sequences which form a partition of α, that is $\alpha = \alpha^1 + \cdots + \alpha^t$, and let the sequences $\beta^1, \dots, \beta^t$ be a partition of β. For each $1 \leq i \leq t$, let C_i be an irreducible nodal plane curve passing through the points in Π_i with tangency conditions (α^i, β^i) to L. By Bézout's theorem, C_i is of degree

$$d_i = \sum_{j\geq 1} j(\alpha^i_j + \beta^i_j) \tag{3.2}$$

and, as C_i is irreducible, we have

$$|\Pi_i| = \frac{(d_i + 3)d_i}{2} - \delta_i, \tag{3.3}$$

where δ_i be the number of nodes of C_i. The curve $C = C_1 \cup \cdots \cup C_t$ is of degree

$$d = d_1 + \cdots + d_t \tag{3.4}$$

and has tangency orders to L given by (α, β) and has, again by Bézout's theorem,

$$\delta = \sum_{i=1}^{t} \delta_i + \sum_{i\leq j} d_i d_j \tag{3.5}$$

nodes. Thus, we can express the relative Severi degree as

$$N^{\delta}_{\alpha,\beta} = \sum_{\Pi = \cup \Pi_i} \sum_{(d_i,\delta_i)} \sum_{(\alpha^i,\beta^i)} \prod_i N^{\delta,\mathrm{irr}}_{\alpha^i,\beta^i},$$

where the first sum is over all partitions of Π, the second sum is over all pairs (d_i, δ_i) satisfying (3.3), (3.4) and (3.5), and the third sum is over all pairs of collections of sequences (α^i, β^i) satisfying (3.2).

A similar analysis holds at the level of floor diagrams and their relative markings. Let $\mathcal{D}$ be a (not necessarily connected) floor diagram. Let $V(\mathcal{D}) = \cup_{i=1}^t V_i$ be the partition of the vertices of $\mathcal{D}$ given by the connected components of $\mathcal{D}$, and let $\mathcal{D}_1, \ldots, \mathcal{D}_t$ be the corresponding (connected) floor diagrams. Then, by definition, $\delta(\mathcal{D}) = \sum_{i=1}^t \delta(\mathcal{D}_i) + \sum_{i<j} d(\mathcal{D}_i) d(D_j)$. Furthermore, the collection $(\mathcal{D}_1, \ldots, \mathcal{D}_t)$ also satisfies equations similar to (3.2), (3.3) and (3.4) and the result follows. $\square$

Relationship between relative marked floor diagrams and tropical plane curves satisfying tangency conditions to a tropical line

Relative marked floor diagrams are closely related to tropical plane curves satisfying tangency conditions to a fixed tropical line. Such tropical plane curves were introduced by A. Gathmann and H. Markwig [15]. As in the non-relative case, we obtain a relative marked floor diagram by a certain contraction of a tropical plane curve (if the tropical curve passes through a vertically stretched point configuration).

The setup is as follows: To define tropical plane curves which are "(α, β)-tangent" to a tropical line and which pass through a generic set of points (whose number we choose to be such that the count is finite) let $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ be the necessary degree of the tropical curve and fix a point configuration $\mathcal{P}$ of $\frac{(d+3)d}{2} - \delta - \alpha_1 - \alpha_2 - \cdots$ points in vertically stretched position (see [9, Definition 3.4]). Furthermore, fix a tropical line L with vertex very far below and to the right of $\mathcal{P}$. Additionally, fix $\alpha_1 + \alpha_2 + \cdots$ points $\mathcal{P}_\alpha$ on L all to the left of $\mathcal{P}$.

Definition III.8 (Definition 3.11 of [9])**.** A *tropical plane curve* (α, β)*-tangent* to L is a weighted graph Γ in $\mathbb{R}^2$ satisfying

- all edges e of Γ have positive integral weights $\omega(e)$,
- all edges of Γ have rational slopes,
- all unbounded edges of Γ are in directions $(-1,0)$, $(0,-1)$ or $(1,1)$,
- all vertices v of Γ are *balanced*, i.e.,

$$\sum_{e:\, v\in\partial e} \omega(e)\cdot \text{primitive}(v,e) = 0,$$

  where the sum is over all v-adjacent edges e of Γ and primitive(v,e) is the primitive direction of e away from v, i.e., the shortest lattice vector pointing in the direction of e away from v,

- all unbounded edges in directions $(-1,0)$ and $(1,1)$ have weight 1,
- there are β_1 unbounded edges with weight 1 in direction $(0,-1)$ which pass through a point in $\mathcal{P}$, β_2 such edges with weight 2, etc.
- there are α_1 unbounded edges with weight 1 in direction $(0,-1)$ which pass through a point in $\mathcal{P}_\alpha$, α_2 such edges with weight 2, etc., and
- the weights of the edges passing through $\mathcal{P}_\alpha$ is weakly increasing from left to right.

An example of a tropical plane curve satisfying tangency conditions given by $\alpha = (1)$ and $\beta = (1,1)$ is shown in Figure 3.2 (the coloring of the marked points is explained later). Notice that for tropical plane curves the tangency order is simply given by the weight of the edge of the tropical curve intersecting the line. Definition III.8 only applies to the very special situation of a vertically stretched point configuration but it can be generalized to arbitrary tropically generic point configurations.

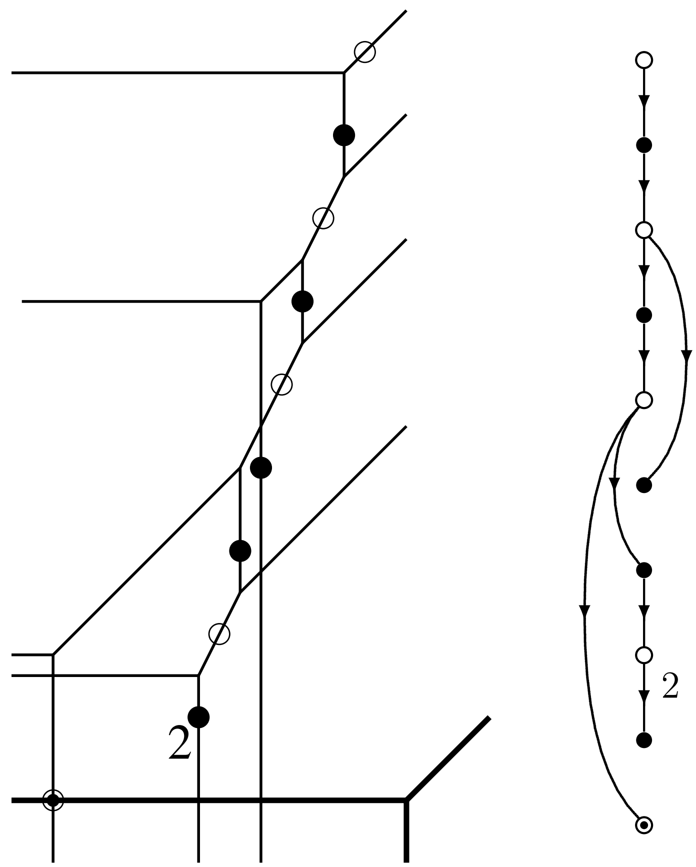


Figure 3.2: Left: A tropical plane curve passing through a vertically stretched point configuration satisfying tangency conditions given by $\alpha = (1)$ and $\beta = (1, 1)$ to the bold tropical line. Edge weights equal to 1 are omitted. Right: The corresponding relative marked floor diagram obtained by floor contraction.

A tropical curve Γ as in Definition III.8 defines an (α, β)-marked floor diagram as follows (c.f. Section 3 of [9]): call a vertical edge of Γ an *elevator*, and each connected component of the complement of the (relative interiors or the) elevators in Γ a *floor*. Then, as we chose a vertically stretched point configuration, every floor and every elevator of Γ contains precisely one point from the point configuration $\mathcal{P} \cup \mathcal{P}_\alpha$ (c.f. Theorem 3.17 of [9]). We obtain the (α, β)-marked floor diagram corresponding to Γ by contracting the floors of Γ to vertices; c.f. Figure 3.2.

In [15], A. Gathmann and H. Markwig consider only *irreducible* tropical plane curves (a tropical plane curve is irreducible if it cannot be written as the union of two tropical plane curves). Their main focus is a combinatorial interpretation of the *relative Gromov-Witten invariants* $N_{d,g}(\alpha, \beta)$, which enumerate irreducible algebraic plane curves of degree d and genus g passing through an appropriate number of generic points, and which satisfy tangency conditions to a fixed line in $\mathbb{CP}^2$ given by two sequences α and β (analogously to the case of relative Severi degrees). One of their main results is that $N_{d,g}(\alpha, \beta)$ is enumerated by certain irreducible tropical plane curves (counted with suitable multiplicity). The idea of the proof is that the enumerations $N_{d,g}^{\text{trop}}(\alpha, \beta)$ of irreducible tropical plane curves of "tropical degree" d and "tropical genus" g which are (α, β)-tangent to a generic tropical line (counted with multiplicity) satisfy the Caporaso-Harris recursion. As this recursion is also satisfied by $N_{d,g}(\alpha, \beta)$, the two sets of numbers agree. The Caporaso-Harris recursion for tropical plan curves can also be shown with floor diagrams. For example, a special case of the main result in Chapter IV (Theorem IV.35) is the Caporaso-Harris recursion of relative genus 0 Gromov-Witten invariants.

The floor diagram technique allows to go well beyond these results. One advantage (besides being a completely combinatorial description of relative Severi degrees) is the notion of a "template decomposition" which we introduce in the next section. The building blocks in this decomposition (the "templates") are the key combinatorial gadgets which will allow to prove the polynomiality of relative Severi degrees (Theorem III.1).

3.3 Relative Decomposition of Floor Diagrams

In this section we introduce a new decomposition of floor diagrams compatible with tangency conditions which we use extensively in Sections 3.4 and 3.5 to prove all our results stated in Section 3.1. This decomposition is an extension of the template decomposition of S. Fomin and G. Mikhalkin [9] discussed in Section2.3.

Recall the definition of a template (see Definition II.12) as it will again be an essential building block. Our new decomposition of a floor diagram $\mathcal{D}$ depends on two (infinite) matrices A and B of non-negative integers. We require both to have only finitely many non-zero entries all of which lie above the respective dth row, where d is the degree of $\mathcal{D}$.

The triple $(\mathcal{D}, A, B)$ decomposes as follows. Let $l(A)$ and $l(B)$ be the largest row indices such that A and B have a non-zero entry in this row, respectively. After we remove all "short edges" from $\mathcal{D}$, i.e., all edges of weight 1 between consecutive vertices, the resulting graph is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_r)$, listed left to right. Let k_s be the smallest vertex in $\mathcal{D}$ of each template Γ_s. Record all pairs (Γ_s, k_s) which satisfy $k_s + l(\Gamma_s) \leq d - \max(l(A), l(B))$. Record the remaining templates together with all vertices i, for $i \geq \max(l(A), l(B))$ in *one* graph Λ on vertices $0, \ldots, l$ by shifting the vertex labels by $d - l$. See Example III.11 for an example of this decomposition. Furthermore, by construction, if m is the number of recorded pairs (Γ_s, k_s), we have

$$
\left\{
\begin{array}{rcll}
k_i & \geq & k_{\min}(\Gamma_i) & \text{for } 1 \leq i \leq m, \\
k_{i+1} & \geq & k_i + l(\Gamma_i) & \text{for } 1 \leq i \leq m-1, \\
k_m + l(\Gamma_m) & \leq & d - l(\Lambda). &
\end{array}
\right.
\tag{3.6}
$$

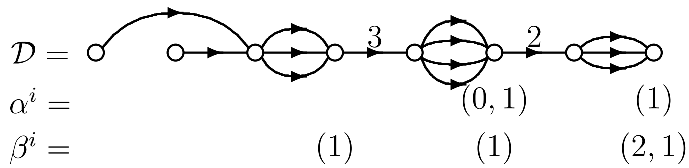
Given a floor diagram $\mathcal{D}$, a partitioning of α and β into a compatible pair of collections $(\{\alpha^i, \beta^i\})$ (see Step 1 in Definition IV.30), where i runs over the vertices of $\mathcal{D}$, determines a pair of matrices A, B, if $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are large enough via the following identification. The ith row vectors a_i and b_i of A and B are given by the sequences α^{d-i} and β^{d-i}, respectively, for $i \geq 1$ (so that a_1 equals the number of α-edges of weight 1 adjacent to the various vertices of Λ, and so on, see Example III.9).

If $d-i \leq 0$ set α^{d-i} to be the zero sequence. The sequences α^d and β^d are given by

$$\alpha^d = \alpha - \sum_{i \geq 1} a_i \quad \text{and} \quad \beta^d = \beta - \sum_{i \geq 1} b_i \tag{3.7}$$

if both expression are (component-wise) non-negative.

Example III.9. For $\alpha = (1,1)$, $\beta = (4,1)$ and the floor diagram $\mathcal{D}$ pictured below, the partitioning



determines the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In light of (3.7) we consider, for given tangency conditions α and β, only the triples $(\mathcal{D}, A, B)$ which satisfy

$$\begin{cases} \displaystyle\sum_{i \geq 1} a_i \leq \alpha \text{ (component-wise)}, \\ \displaystyle\sum_{i \geq 1} b_i \leq \beta \text{ (component-wise)}, \end{cases} \tag{3.8}$$

For fixed d, the decomposition

$$\big(\mathcal{D}, A, B\big) \longrightarrow \big(\{(\Gamma_s, k_s)\}, \Lambda, A, B\big). \tag{3.9}$$

is reversible if the data on the right-hand side satisfies (3.6) and the tuple (Λ, A, B) is an "extended template."

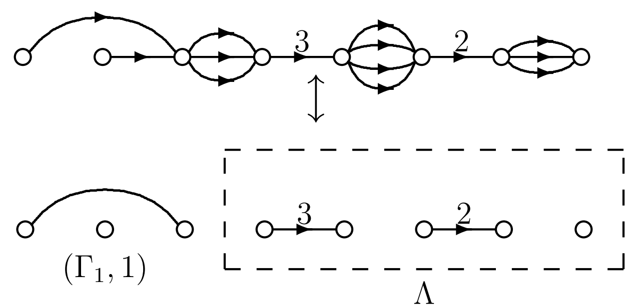
Definition III.10. A tuple (Λ, A, B) is an *extended template* of *length* $l = l(\Lambda) = l(\Lambda, A, B)$ if Λ is a directed graph (possibly with multiple edges) on vertices $\{0, \ldots, l\}$, where $l \geq 0$, with edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

1. If $i \to j$ is an edge then $i < j$.

2. Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")

Moreover, we require A and B to be (infinite) matrices with non-negative integral entries and finite support, and we write $l(A)$ and $l(B)$ for the respective largest row indices of A and B of a non-zero entry. Additionally, we demand that $l(\Lambda) \geq \max(l(A), l(B))$ and that, for each $1 \leq j < l - \max(l(A), l(B))$, there is an edge $i \to k$ of Λ with $i < j < k$.

Example III.11. An example of a decomposition of a floor diagram $\mathcal{D}$ subject to the matrices A and B of Example III.9 is pictured below. Once we fix the degree of the floor diagram the decomposition is reversible (here $d = 8$).



The *cogenus* of an extended template (Λ, A, B) is the sum of the cogenera $\delta(\Lambda)$, $\delta(A)$ and $\delta(B)$, where

$$\delta(\Lambda) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big], \qquad\qquad \delta(A) \stackrel{\text{def}}{=} \sum_{i,j \geq 1} i \cdot j \cdot a_{i,j},$$

and similarly for B. It is not hard to see that the correspondence (3.9) is cogenus preserving in the sense that (compare with Example III.11 (cont'd))

$$\delta(\mathcal{D}) = \left(\sum_{i=1}^{m} \delta(\Gamma_i) \right) + \delta(\Lambda) + \delta(A) + \delta(B).$$

Example III.11 (cont'd). The cogenera of the decomposition are given by

$$\delta(\Gamma_1) + \delta(\Lambda) + \delta(A) + \delta(B) = 1 + 3 + 4 + 6 = 14.$$

(Λ, A, B)	δ	ℓ	μ	$\varkappa$	$d_{\min}$	$q_{(\Lambda,A,B)}(\alpha;\beta)$ of Lemma III.13	s
$\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	0	0	1	()	1	1	0
$\circ\quad\circ$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	1	1	(0)	1	1	0
$\circ\quad\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$	1	1	1	(0)	1	$\beta_1(d+\lvert\beta\rvert-1)$	0
$\circ\overset{2}{\text{——}}\circ$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	4	(2)	4	$(d-3)$	1
$\circ\overset{2}{\text{——}}\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	1	4	(2)	4	$\beta_1(d-3)(d+\lvert\beta\rvert-2)$	1
$\overset{\frown}{\circ\quad\circ\quad\circ}$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	2	1	(1,1)	3	$2(d-2)$	0
$\overset{\frown}{\circ\quad\circ\quad\circ}$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	2	1	(1,1)	3	$\beta_1(d-2)(2d+2\lvert\beta\rvert-3)$	0
$\circ\quad\circ$ $\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	1	0
$\circ\quad\circ$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\beta_1(d+\lvert\beta\rvert-2)$	0
$\circ\quad\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\binom{\beta_1}{2}(d^2+2d\lvert\beta\rvert+\lvert\beta\rvert^2-5d-5\lvert\beta\rvert+6)$	0
$\circ\quad\circ$ $\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	1	0
$\circ\quad\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\beta_2(\lvert\beta\rvert-1)(d+\lvert\beta\rvert-2)$	0
$\circ\quad\circ\quad\circ$ $\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	3	1	(0,0)	3	1	0
$\circ\quad\circ\quad\circ$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}$	2	3	1	(0,0)	3	$\beta_1(\lvert\beta\rvert-1)(2d+\lvert\beta\rvert-3)$	0

Figure 3.3: The extended templates with $\delta(\Lambda, A, B) \leq 2$.

This agrees with the cogenus of $\mathcal{D}$ as $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g = \frac{7\cdot 6}{2} - 7 = 14$.

With an extended template (Λ, A, B) we associate the following numerical data. For $1 \leq j \leq l(\Lambda)$ let $\varkappa_j(\Lambda)$ denote the sum of the weights of edges $i \to k$ of Λ with $i < j \leq k$. Define $d_{min}(\Lambda, A, B)$ to be the smallest positive integer d such that (Λ, A, B) can appear (at the right end) in a floor diagram on $\{1, 2, \ldots, d\}$. We will see later that d_{min} is given by an explicit formula. For a matrix $A = (a_{ij})$ of non-negative integers with finite support define the "weighted lower sum sequence" wls(A) by

$$\mathrm{wls}(A)_i \stackrel{\mathrm{def}}{=} \sum_{i' \geq i, j \geq 1} j \cdot a_{i'j}.$$

We now define the number of "markings" of extended templates and relate them to the number of (α, β)-markings of the corresponding floor diagrams. This definition parallels the number of "markings" of a template Γ "at position k". Recall from

Section 2.3 that this number is given by the polynomial $P_\Gamma(k)$ in k provided $k \geq k_{\min}(\Gamma)$. For details see Section 2.3.

For each pair of sequences (α, β) and each extended template (Λ, A, B) satisfying (3.8) and $d \geq d_{min}$, where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$, we define its number of "markings" as follows. Write $l = l(\Lambda)$ and let $\mathcal{P}(\Lambda, A, B)$ be the poset obtained from Λ by

1. first creating an additional vertex $l+1$ $(> l)$,
2. then adding b_{ij} edges of weight j between $l-i$ and $l+1$, for all $1 \leq i \leq l$ and $j \geq 1$,
3. then adding $\beta_j - \sum_{i\geq 1} b_{ij}$ edges of weight j between l and $l+1$, for $j \geq 1$,
4. then adding

   $$d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} \tag{3.10}$$

   ("short") edges of weight 1 connecting $i-1$ and i, for $1 \leq i \leq l$, and finally
5. subdividing all edges of the resulting graph by introducing a midpoint vertex for each edge.

We denote by $Q_{(\Lambda,A,B)}(\alpha;\beta)$ the number of linear orderings on $\mathcal{P}(\Lambda, A, B)$ (up to equivalence) which extend the linear order on Λ. As $d \geq d_{\min}(\Lambda, A, B)$ if and only if (3.10) is non-negative, for $1 \leq i \leq l$, we have

$$d_{min}(\Lambda, A, B) = \max_{1\leq i\leq l(\Lambda)} (l(\Lambda) - i + 1 + \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}).$$

For sequences $s, t_1, t_2, \ldots$ with $s \geq \sum_i t_i$ (component-wise) we denote by

$$\binom{s}{t_1, t_2, \ldots} \stackrel{\text{def}}{=} \frac{s!}{t_1! t_2! \cdots (s - \sum_i t_i)!}$$

the multinomial coefficient of sequences.

We obtain all (α, β)-markings of the floor diagram $\mathcal{D}$ that come from a compatible pair of sequences $(\{\alpha^i\}, \{\beta^i\})$ by independently ordering the α-vertices and the non-α-vertices. Therefore, the number such markings is given (via the correspondence (3.9)) by

$$\left(\prod_{s=1}^{m} P_{\Gamma_s}(k_s)\right) \cdot \binom{\alpha}{a_1^T, a_2^T, \ldots} \cdot Q_{(\Lambda,A,B)}(\alpha;\beta), \tag{3.11}$$

where $a_1^T, a_2^T, \ldots$ are the *column* vectors of A. We conclude this section by recasting relative Severi degrees in terms of templates and extended templates.

Proposition III.12. *For any $\delta \geq 1$, the relative Severi degree $N^{\delta}_{\alpha,\beta}$ is given by*

$$\sum_{\substack{(\Gamma_1,\ldots,\Gamma_m),\\ (\Lambda,A,B)}} \left(\prod_{s=1}^{m} \mu(\Gamma_s) \sum_{k_1,\ldots k_m} \prod_{s=1}^{m} P_{\Gamma_s}(k_s)\right) \cdot \left(\mu(\Lambda) \prod_{i\geq 1} i^{\beta_i} \binom{\alpha}{a_1, a_2, \ldots} Q_{(\Lambda,A,B)}(\alpha;\beta)\right), \tag{3.12}$$

where the first sum is over all collections $(\Gamma_1, \ldots, \Gamma_m)$ of templates and all extended templates (Λ, A, B) satisfying (3.8), $d \geq d_{min}(\Lambda, A, B)$ and

$$\sum_{i=1}^{m} \delta(\Gamma_i) + \delta(\Lambda) + \delta(A) + \delta(B) = \delta,$$

and the second sum is over all positive integers $k_1, \ldots, k_m$ which satisfy (3.6).

Proof. By Theorem III.7 the relative Severi degree is given by

$$N^{\delta}_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over all floor diagrams $\mathcal{D}$ of degree $d = \sum_{i\geq 1} i(\alpha_i+\beta_i)$ and cogenus δ. The result follows from $\mu_\beta(\mathcal{D}) = \prod_{i\geq 1} i^{\beta_i} \cdot \left(\prod_{s=1}^{m} \mu(\Gamma_s)\right) \cdot \mu(\Lambda)$ and (3.11). $\square$

3.4 Polynomiality of Relative Severi Degrees

We now turn to the proofs of the main result of this chapter by first showing a number of technical lemmata. For a graph G, we denote by $\#E(G)$ the number of edges of G. We write $||A||_1 = \sum_{i,j\geq 1} a_{ij}$ for the 1-norm of a (possibly infinite) matrix $A = (a_{ij})$.

Lemma III.13. *For every extended template* (Λ, A, B) *there is a polynomial* $q_{(\Lambda,A,B)}$ *in* $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ *of degree* $\#E(\Lambda) + ||B||_1 + \delta(B)$ *such that for all* α *and* β *satisfying (3.8) the number* $Q_{(\Lambda,A,B)}(\alpha;\beta)$ *of linear orderings (up to equivalence) of the poset* $\mathcal{P}(\Lambda, A, B)$ *is given by*

$$Q_{(\Lambda,A,B)}(\alpha;\beta) = \frac{(|\beta| - \delta(B))!}{\beta!} \cdot q_{(\Lambda,A,B)}(\alpha;\beta)$$

provided $\sum_{i\geq 1} i(\alpha_i + \beta_i) \geq d_{min}(\Lambda, A, B)$.

Proof. We can choose a linear extension of the order on the vertices of Λ to the poset $\mathcal{P}(\Lambda, A, B)$ in two steps. First, we choose a linear order on the vertices $0, \ldots, l(\Lambda)+1$, the midpoint vertices of the edges of Λ and the midpoint vertices of the edges created in step (2) in the definition of $\mathcal{P}(\Lambda, A, B)$. In a second step, we choose an extension to a linear order on all vertices. Let r_i be the number of vertices between $i-1$ and i after the first extension, for $1 \leq i \leq l(\Lambda)+1$, and let σ_i be the number of equivalent such linear orderings of the interval between $i-1$ and i (σ_i is independent of the particular choice of the linear order). To insert the additional vertices (up to equivalence) between the vertices 0 and $l = l(\Lambda)$ we have

$$\prod_{i=1}^{l} \frac{1}{\sigma_i} \binom{d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} + r_i}{r_i} \tag{3.13}$$

many possibilities where again $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. If $d \geq d_{min}(\Lambda, A, B)$ then expression (3.13) is a polynomial in d of degree $\sum_{i=1}^{l} r_i$, and thus in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$. The number of (equivalent) orderings of the vertices between l and $l+1$ is the multinomial coefficient

$$\binom{|\beta| - ||B||_1 + r_{l+1}}{\beta_1 - |b_1^T|, \beta_2 - |b_2^T|, \ldots}, \tag{3.14}$$

where $|b_j^T|$ denotes the sum of the entries in the jth column of B. As $||B||_1 \leq \delta(B)$, expression (3.14) equals, for all $\beta_1, \beta_2, \dots \geq 0$,

$$\binom{|\beta|}{\beta_1, \beta_2, \dots} \frac{(|\beta| - \delta(B))!}{|\beta|!} P(\beta) = \frac{(|\beta| - \delta(B))!}{\beta!} P(\beta) \tag{3.15}$$

for a polynomial P in $\beta_1, \beta_2, \dots$ of degree $r_{l+1} + \delta(B)$. The product of (3.13) and (3.15) is

$$\frac{(|\beta| - \delta(B))!}{\beta!} P'(\alpha; \beta) \tag{3.16}$$

for a polynomial P' in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ of degree $\#E(\Lambda) + ||B||_1 + \delta(B)$ provided $d \geq d_{min}(\Lambda, A, B)$ where we used that $\sum_{i=1}^{l+1} r_i = \#E(\Lambda) + ||B||_1$. As (3.16) equals the number of linear extensions (up to equivalence) that can be obtained by linearly ordering the vertices in all segments between $i-1$ and i, for $1 \leq i \leq l+1$, the proof is complete. $\square$

Recall that, for an extended template (Λ, A, B), we defined $d_{\min} = d_{\min}(\Lambda, A, B)$ to be the smallest $d \geq 1$ such that $d - l(\Lambda) + i - 1 \geq \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}$ for all $1 \leq i \leq l(\Lambda)$. Let i_0 be the smallest i for which equality is attained (it is easy to see that equality is attained for some i). Define the quantity $s(\Lambda, A, B)$ to be the number of edges of Λ from $i_0 - 1$ to i_0 (of any weight). See Figure 3.3 for examples.

Lemma III.14. *For any extended template* (Λ, A, B) *and any* $\alpha, \beta \geq 0$ *(componentwise) with*

$$d_{\min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \sum_{i \geq 1} i(\alpha_i + \beta_i) \leq d_{\min}(\Lambda, A, B) - 1$$

we have $q_{(\Lambda, A.B)}(\alpha; \beta) = 0$*, where* $q_{(\Lambda, A, B)}$ *is the polynomial of Lemma III.13.*

Proof. Notice that $d_{\min} - l(\Lambda) + i_0 - 1 = \varkappa_{i_0}(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i_0} + \mathrm{wls}(B)_{l(\Lambda)+1-i_0}$ where $d_{\min} = d_{\min}(\Lambda, A, B)$. Therefore, the number of short edges which are added between $i_0 - 1$ and i_0 in step (3) of the definition of the poset $\mathcal{P}(\Lambda, A, B)$ is $d - d_{\min}$, where as before $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. Recall that, up to the factor $\frac{(|\beta|-\delta(B))!}{\beta!}$, the polynomial $q_{(\Lambda,A,B)}$ records the number of linear extension of the poset $\mathcal{P}(\Lambda, A, B)$ (up to equivalence). Every such extension is obtained by first linearly ordering the $d - d_{\min}$ midpoints of the short edges between $i_0 - 1$ and i_0 which were added in step (3) together with the $s(\Lambda, A, B)$ midpoints of the edges of Λ between $i_0 - 1$ and i_0, before extending this to a linear order on all the vertices of $\mathcal{P}(\Lambda, A, B)$. Thus, $q_{(\Lambda,A,B)}$ is divisible by the polynomial $(d - d_{\min} + 1) \cdots (d - d_{\min} + s(\Lambda, A, B))$. $\square$

The next lemma specifies which extended templates are compatible with a given degree.

Lemma III.15. *For every extended template* (Λ, A, B) *we have*

$$d_{min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \delta(\Lambda) + \delta(A) + \delta(B) + 1.$$

Proof. We use the notation from above and write $l = l(\Lambda)$. Notice that

$$d_{\min}(\Lambda, A, B) - l(\Lambda) + i_0 - 1 = \varkappa_{i_0}(\Lambda) + \mathrm{wls}(A)_{l+1-i_0} + \mathrm{wls}(B)_{l+1-i_0}.$$

Therefore, it suffices to show

$$l(\Lambda) \leq \delta(\Lambda) - \varkappa_{i_0}(\Lambda) + s(\Lambda, A, B) + \delta(A) - \mathrm{wls}(A)_{l+1-i_0} + \delta(B) - \mathrm{wls}(B)_{l+1-i_0} + i_0.$$

Let Λ' be the graph obtained from Λ by removing all edges $j \to k$ with either $k < i_0$ or $j \geq i_0$. It is easy to see that $l(\Lambda, A, B) - l(\Lambda', A, B) \leq \delta(\Lambda) - \delta(\Lambda')$. Thus, we can assume without loss of generality that all edges $j \to k$ of Λ satisfy $j < i_0 \leq k$. Therefore, as $\varkappa_{i_0}(\Lambda) = \sum_{\text{edges } e} \mathrm{wt}(e)$, we have

$$\delta(\Lambda) - \varkappa_{i_0} + s(\Lambda, A, B) = \sum_{\text{edges } e} \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big] + s = \sum \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big],$$

where $\mathrm{len}(e)$ is the *length* $k - j$ of an edge $j \xrightarrow{e} k$ and the last sum is over all edges of Λ of length at least 2. It is easy to see that the matrix A satisfies $\delta(A) \geq$

$\mathrm{wls}(A)_i + l(A) - 1$ for all $i \geq 1$, therefore, if $l(A) = l(\Lambda)$, it suffices to show that

$$(3.17)\quad l(A) \leq \sum \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big] + l(A) - 1 + \delta(B) - \mathrm{wls}(B)_{l+1-i_0} + i_0,$$

where the sum again runs over all edges of Λ of length at least 2. But (3.17) is clear as all summands in the sum are non-negative and $\delta(B) \geq \mathrm{wls}(B)_{l+1-i_0}$. The same argument also settles the case $l(B) = l(\Lambda)$.

Otherwise, we can assume that $l(\Lambda) > l(A) \geq l(B)$ and that there exists an edge $0 \to i$ of Λ with $l(\Lambda) - l(A) \leq i - 1$. If, additionally, we have $i_0 \leq l(\Lambda) - l(A)$ then $\mathrm{wls}(A)_{l+1-i_0} = 0$ and, using $\delta(B) \geq \mathrm{wls}(B)_{l+1-i_0}$, it suffices to prove that

$$l(A) + i - 1 \leq i - 2 + \delta(A) + 1$$

which is clear as $l(A) \leq \delta(A)$.

Finally, if $i_0 \geq l(\Lambda) - l(A) + 1$, it remains to show that $l(A) + 1 \leq \delta(A) - \mathrm{wls}(A)_{l+1-i_0} + i_0$. We have (by definition of $\delta(A)$ and $\mathrm{wls}(A)_{l+1-i_0}$) that

$$(3.18)\qquad \delta(A) - \mathrm{wls}(A)_{l+1-i_0} + i_0 = \sum (i-1)ja_{ij} + \sum ija_{ij} + i_0,$$

where the first sum runs over $i \geq l + 1 - i_0$, $j \geq 1$ and the second sum runs over $1 \leq i < l + 1 - i_0$, $j \geq 1$. As $i_0 \geq l(\Lambda) - l(A) + 1$ there exists a non-zero entry $a_{i'j'}$ of A with $i' = l(A) \geq l + 1 - i_0$. Therefore, the index set of the first sum of (3.18) is non-empty and the right-hand side of (3.18) is $\geq i' - 1 + i_0 = l(A) + 1$ as $i_0 \geq l(\Lambda) - l(A) + 1 \geq 2$. $\square$

We now turn to the proof of the main theorem of this chapter.

Proof of Theorem III.1. We first show that (3.1) holds of all α, β with $d \geq \delta + 1$ where we again write $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. This implies (3.1) if at least one of $\alpha_1, \alpha_2, \ldots, \beta_2, \beta_3, \ldots$ is non-zero (note that β_1 is omitted), because in that case $|\beta| \geq \delta$ implies $d \geq \delta + 1$.

Notice that we can remove condition (3.8) from formula (3.12) of Proposition III.12 and still obtain correct relative Severi degrees as $\binom{\alpha}{a_1^T, a_2^T, \dots} Q_{(\Lambda, A, B)}(\alpha; \beta) = 0$ whenever (3.8) is violated. The first factor of (3.12) equals

$$(3.19) \qquad \sum_{k_m = k_{\min}(\Gamma_m)}^{d - l(\Lambda)} \mu(\Gamma_m) P_{\Gamma_m}(k_m) \sum_{k_{m-1} = k_{\min}(\Gamma_{m-1})}^{k_m - l(\Gamma_{m-1})} \cdots \sum_{k_1 = k_{\min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} \mu(\Gamma_1) P_{\Gamma_1}(k_1)$$

and is, therefore, an iterated "discrete integral" of polynomials. By repeated application of [1, Lemma 3.5] (or other means) expression (3.19) is a polynomial in d if $d - l(\Lambda) \geq 2 \sum_{i=1}^m \delta(\Gamma_i)$. Furthermore, as the polynomials $P_{\Gamma_i}(k_i)$ have degrees $\#E(\Gamma_i)$ and each "discrete integration" increases the degree by 1 the polynomial (3.19) is of degree $\sum_{i=1}^m \#E(\Gamma_i) + m$. By a literal application of the argument in Section 4 of [1] one can improve the polynomiality threshold of (3.19) and show that (3.19) is a polynomial in d if $d - l(\Lambda) \geq \sum_{i=1}^m \delta(\Gamma_i) + 1$. Furthermore, we have $l(\Lambda) \leq \delta(\Lambda) + \delta(A) + \delta(B)$. Thus, the first factor of (3.12) is a polynomial in d already if $d \geq \delta + 1 = \sum_i \delta(\Gamma_i) + 1 + \delta(\Lambda) + \delta(A) + \delta(B)$.

The multinomial coefficient $\binom{\alpha}{a_1^T, a_2^T, \dots}$ is a polynomial in $\alpha_1, \alpha_2, \dots$ for fixed sequences of (column) vectors $a_1^T, a_2^T, \dots$, if $\alpha_1, \alpha_2, \dots \geq 0$. Hence, by Lemma III.13, the second factor of (3.12) is of the form

$$(3.20) \qquad \prod_{i \geq 1} i^{\beta_i} \cdot \frac{(|\beta| - \delta)!}{\beta!} \cdot R_{(\Lambda, A, B)}(\alpha; \beta)$$

for a polynomial $R_{(\Lambda, A, B)}(\alpha; \beta)$ in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ of degree $\#E(\Lambda) + ||A||_1 + ||B||_1 + \delta$ provided $d \geq d_{min}(\Lambda, A, B)$, where used that $\delta(B) \leq \delta$. By Lemma III.14 the second factor of (3.12) equals expression (3.20) for all α, β with $d \geq d_{\min}(\Lambda, A, B) - s(\Lambda, A, B)$. Thus, using Lemma III.15, if

$$d \geq \delta + 1 \geq \delta(\Lambda) + \delta(A) + \delta(B) + 1 \geq d_{\min}(\Lambda, A, B) - s(\Lambda, A, B)$$

the second factor in (3.12) is $\prod_{i \geq 1} i^{\beta_i} \cdot \frac{(|\beta| - \delta)!}{\beta!}$ times a polynomial in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ of degree $\#E(\Lambda) + ||A||_1 + ||B||_1 + \delta$. Hence (3.1) holds if $|\beta| \geq \delta$ and at least one

β_i, for $i \geq 2$, or one α_i, for $i \geq 1$, is non-zero. Notice that each summand of (3.12) contributes a polynomial of degree

$$\sum_{i=1}^{m} \#E(\Gamma_i) + m + \#E(\Lambda) + ||A||_1 + ||B||_1 + \delta \tag{3.21}$$

to the relative node polynomial $N_\delta(\alpha;\beta)$. It is not hard to see that expression (3.21) is at most 3δ, and that equality is attained by letting $\Gamma_1, \dots, \Gamma_\delta$ be the unique template on three vertices with cogenus 1 (see Figure 2.2) and (Λ, A, B) be the unique extended template of cogenus 0 (see Figure 3.3).

If $\alpha = 0$ and $\beta = (d, 0, \dots)$ then $N^\delta_{\alpha,\beta}$ equals the (non-relative) Severi degree $N^{d,\delta}$ which, in turn, is given by the (non-relative) node polynomial $N^{\text{abs}}_\delta(d)$ provided $d \geq \delta$ (see [1, Theorem 1.3]). Therefore, we have $N_\delta(0; d) = N^{\text{abs}}_\delta(d) \cdot d(d-1)\cdots(d-\delta+1)$ as polynomials in d. Applying [1, Theorem 1.3] again completes the proof. $\square$

Remark III.16. Expression (3.12) gives, in principle, an algorithm to compute the relative node polynomial $N_\delta(\alpha;\beta)$, for any $\delta \geq 1$. In [1, Section 3] we explain how to generate all templates of a given cogenus, and how to compute the first factor in (3.12). The generation of all extended templates of a given cogenus from the templates is straightforward, as is the computation of the second factor in (3.12).

Remark III.17. The proof of Theorem III.1 simplifies significantly if we relax the polynomiality threshold. More specifically, without considering the quantity $s(\Lambda, A, B)$ and the rather technical Lemmata III.14 and III.15 the argument still implies (3.1) provided $|\beta| \geq 2\delta$ (instead of $|\beta| \geq \delta$).

The immediate conclusion from the proof of Theorem III.1 is two-fold.

Corollary III.18. *For $\delta \geq 1$ the relative node polynomial $N_\delta(\alpha, \beta)$ is a polynomial in d, $|\beta|$, $\alpha_1, \dots, \alpha_\delta$, and $\beta_1, \dots, \beta_\delta$, where $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$*

Proof. Every extended template (Λ, A, B) considered in (3.12) satisfies $\delta(A) \leq \delta$ and $\delta(B) \leq \delta$. Therefore, all rows i in A or B are zero for $i > \delta$. □

Proof of Theorem III.2. By the proof of Lemma III.13 we have, for every extended template (Λ, A, B),

$$R_{(\Lambda,A,B)}(\alpha, 0; \beta) = R_{(\Lambda,A,B)}(\alpha; \beta) \qquad R_{(\Lambda,A,B)}(\alpha; \beta, 0) = R_{(\Lambda,A,B)}(\alpha; \beta).$$

Hence, by the proof of Theorem III.1, the result follows. □

Now it is also easy to prove Theorem III.3.

Proof of Theorem III.3. Proposition III.12 gives a combinatorial description of relative Severi degrees. The proof of Lemma III.13 provides a method to calculate the polynomial $Q_{(\Lambda,A,B)}(\alpha; \beta)$. All terms of expression (3.12) are explicit or can be evaluated using the techniques of [1, Section 3]. This reduces the calculation to a (non-trivial) computer calculation. □

3.5 Coefficients of Relative Node Polynomials

We now turn towards the computation of the coefficients of the relative node polynomial $N_\delta(\alpha; \beta)$ of large degree for any δ. By Theorem III.1 the polynomial $N_\delta(\alpha, \beta)$ is of degree 3δ. In the following we propose a method to compute all terms of $N_\delta(\alpha; \beta)$ of degree $\geq 3\delta - t$, for any given $t \geq 0$. This method was used (with $t = 2$) to compute the terms in Theorem III.4.

The main idea of the algorithm is that, even for general δ, only a small number of summands of (3.12) contribute to the terms of $N_\delta(\alpha; \beta)$ of large degree. A summand of (3.12) is indexed by a collection of templates $\tilde{\Gamma} = \{\Gamma_s\}$ and an extended template (Λ, A, B). To determine whether this summand might contribute to $N_\delta(\alpha; \beta)$ we define the *(degree) defects*

- of the collection of templates $\tilde{\Gamma}$ by

$$\operatorname{def}(\tilde{\Gamma}) \stackrel{\text{def}}{=} \sum_{s=1}^{m} \left[\delta(\Gamma_i)\right] - m, \text{ and}$$

- of the extended template (Λ, A, B) by

$$\operatorname{def}(\Lambda, A, B) \stackrel{\text{def}}{=} \delta(\Lambda) + 2\delta(A) + 2\delta(B) - ||A||_1 - ||B||_1.$$

The following lemma restricts the indexing set of (3.12) to the relevant terms, if only the leading terms of $N_\delta(\alpha;\beta)$ are of interest.

Lemma III.19. *The summand of (3.12) indexed by* $\tilde{\Gamma}$ *and* (Λ, A, B) *is of the form*

$$1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot P(\alpha;\beta),$$

where $P(\alpha;\beta)$ *is a polynomial in* $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ *of degree* $\leq 3\delta - \operatorname{def}(\tilde{\Gamma}) - \operatorname{def}(\Lambda, A, B)$.

Proof. By [1, Lemma 5.2] the first factor of (3.12) is of degree at most

$$2 \cdot \sum_{s=1}^{m} \delta(\Gamma_s) - \sum_{s=1}^{m} (\delta(\Gamma_s) - 1) = \sum_{s=1}^{m} \delta(\Gamma_s) + m.$$

The multinomial coefficient $\binom{\alpha}{a_1^T, a_2^T, \ldots}$ is a polynomial in α of degree $||A||_1$ if a_j^T are the jth column vector of A. Recall from the proof of Theorem III.1 that the second factor of (3.12) is

$$\prod_{i \geq 1} i^{\beta_i} \frac{(|\beta| - \delta)!}{\beta!} \text{ times a polynomial in } \alpha, \beta \text{ of degree } \#E(\Lambda) + ||A||_1 + ||B||_1 + \delta.$$

Therefore, the contribution of this summand to the relative node polynomial is at most of degree

$$\begin{aligned}
&\sum_{s=1}^{m} \delta(\Gamma_s) + m + \#E(\Lambda) + ||A||_1 + ||B||_1 + \delta \\
&= 3\delta - 2\sum_{s=1}^{m} \delta(\Gamma_s) - 2\delta(\Lambda) - 2\delta(A) - 2\delta(B) + \#E(\Lambda) \\
&= 3\delta - \operatorname{def}(\tilde{\Gamma}) - \operatorname{def}(\Lambda, A, B) - \delta(\Lambda) + \#E(\Lambda).
\end{aligned}$$

The result follows as $\delta(\Lambda) \geq \#E(\Lambda)$ $\square$

Therefore, to compute the coefficients of degree $\geq 3\delta - t$ of $N_\delta(\alpha;\beta)$ for some $t \geq 0$, it suffices to consider only summands of (3.12) with $\mathrm{def}(\tilde{\Gamma}) \leq t$ and $\mathrm{def}(\Lambda, A, B) \leq t$.

One can proceed as follows. First, we can compute, for some formal variable $\tilde{\delta}$, the terms of degree $\geq 2\tilde{\delta} - t$ of the first factor of (3.12) to $N_{\tilde{\delta}}(\alpha;\beta)$, that is the terms of degree $\geq 2\tilde{\delta} - t$ of

$$R_{\tilde{\delta}}(d) \overset{\mathrm{def}}{=} \sum \prod_{i=1}^{m} \mu(\Gamma_i) \sum_{k_m = k_{min}(\Gamma_m)}^{d - l(\Gamma_m)} P_{\Gamma_m}(k_m) \cdots \sum_{k_1 = k_min(\Gamma_1)}^{k_2 - l(\Gamma_1)} P_{\Gamma_1}(k_1), \tag{3.22}$$

where the first sum is over all collections of templates $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ with $\delta(\tilde{\Gamma}) = \tilde{\delta}$. (Notice that (3.22) is expression [9, (5.13)] without the "ε-correction" in the sum indexed by k_m.) The leading terms of $R_{\tilde{\delta}}(d)$ can be computed with a slight modification of [1, Algorithm 2] (by replacing, in the notation of [1], C^{end} by C and M^{end} by M). The algorithm relies on the polynomiality of solutions of certain polynomial difference equations, which has been verified for $t \leq 7$, see [1, Section 5] for more details. With a Maple implementation of this algorithm one obtains (with $t = 5$)

$$\begin{aligned} R_{\tilde{\delta}}(d) = \frac{3^{\tilde{\delta}}}{\tilde{\delta}!}\Big[& d^{2\tilde{\delta}} - \frac{8\tilde{\delta}}{3} d^{2\tilde{\delta}-1} + \frac{\tilde{\delta}(11\tilde{\delta}+1)}{3^2} d^{2\tilde{\delta}-2} + \frac{\tilde{\delta}(\tilde{\delta}-1)(496\tilde{\delta}-245)}{6 \cdot 3^3} d^{2\tilde{\delta}-3} \\ & - \frac{\tilde{\delta}(\tilde{\delta}-1)(1685\tilde{\delta}^2 - 2773\tilde{\delta} + 1398)}{6 \cdot 3^4} d^{2\tilde{\delta}-4} + \\ & - \frac{\tilde{\delta}(\tilde{\delta}-1)(\tilde{\delta}-2)(7352\tilde{\delta}^2 + 116611\tilde{\delta} - 25221)}{30 \cdot 3^5} d^{2\tilde{\delta}-5} + \cdots \Big]. \end{aligned}$$

Finally, to compute the coefficients of degree $\geq 3\delta - t$, it remains to compute all extended templates (Λ, A, B) with $\mathrm{def}(\Lambda, A, B) \leq t$ and collect the terms of degree $\geq 3\delta - t$ of the polynomial

$$R_{\tilde{\delta}}(d - l(\Lambda)) \cdot \mu(\Lambda) \binom{\alpha}{a_1^T, a_2^T, \ldots} \prod_{i=\delta(B)}^{\delta-1} (|\beta| - i) \cdot q_{(\Lambda,A,B)}(\alpha;\beta), \tag{3.23}$$

where, as before, $a_1^T, a_2^T, \ldots$ denote the column vectors of the matrix A, $q_{(\Lambda,A,B)}(\alpha;\beta)$ is the polynomial of Lemma III.13, and $\tilde{\delta} = \delta - \delta(\Lambda, A, B)$. Notice that, for an indeterminant x and integers $c \geq 0$ and $\delta \geq 1$, we have the expansion

$$\prod_{i=c}^{\delta-1}(x-i) = \sum_{t=0}^{\delta-c} s(\delta-c, \delta-c-t)(x-c)^{\delta-c-t},$$

where $s(n,m)$ is the *Stirling number of the first kind* [33, Section 1.3] for integers $n, m \geq 0$. Furthermore, with $\delta' = \delta - c$ the coefficients $s(\delta', \delta' - t)$ of the sum equal $\delta'(\delta'-1)\cdots(\delta'-t)\cdot S_t(\delta')$, where S_t is the t-th *Stirling polynomial* [19, (6.45)], for $t \geq 0$, and thus are polynomial in δ'. Therefore, we can compute the leading terms of the product in (3.23) by collecting the leading terms in the sum expansion above.

Proof of Proposition III.5. Using [1, Algorithm 2] we can compute the terms of the polynomial $R_{\tilde{\Gamma}}(d)$ of degree $\geq 2\tilde{\delta} - 7$ (see [1, Section 5]) and observe that all coefficients are polynomial in $\tilde{\delta}$. By the previous paragraph the coefficients of the expansion of the sum of (3.23) are polynomial in δ. This completes the proof. □

Proof of Theorem III.4. The method described above is a direct implementation of formula (3.12), which equals the relative Severi degree by the proof of Theorem III.1.

□

Remark III.20. It is straight-forward to compute the coefficients of $N_\delta(\alpha;\beta)$ of degree $\geq 3\delta - 7$ (and thereby to extend Theorem III.4). Algorithm 3 of [1] computes the coefficients of the polynomials $R_{\tilde{\delta}}(d)$ of degree $\geq 2\tilde{\delta} - 7$, and thus the desired terms can be collected from (3.23). We expect this method to compute the leading terms of $N_\delta(\alpha, \beta)$ of degree $\geq 3\delta - t$ for arbitrary $t \geq 0$ (see [1, Section 5], especially Conjecture 5.5).

CHAPTER IV

Psi-Floor Diagrams and a Caporaso-Harris Type Recursion

4.1 Introduction

In this chapter we study enumeration of rational irreducible algebraic plane curves which, in addition to passing through a collection of points, satisfy conditions given by "Psi-classes" (together with point conditions). Roughly speaking, this means "counting" algebraic plane curves satisfying tangency conditions to *several lines* with one tangency condition for each line. The (possibly fractional) "counts" are defined via an intersection product on the moduli space of stable maps. We begin by recalling some general definitions. All curves in this chapter are *irreducible*.

On the moduli spaces $\overline{M}_{g,r}$ (resp. $\overline{M}_{g,r}(\mathbb{CP}^s, d)$) of r-marked genus-g stable curves (resp. stable maps of degree d to projective space $\mathbb{CP}^s$), the *Psi-class* ψ_i for $i = 1, \ldots, r$ is the first Chern class of the line bundle whose fiber over a point $(C, x_1, \ldots, x_r)$ (resp. $(C, x_1, \ldots, x_r, f)$) is the cotangent space of C at x_i. These Psi-classes are useful to count curves with tangency conditions, for example. To count curves that satisfy incidence conditions (e.g. pass through given points), one defines evaluation maps on the space of stable maps, $\mathrm{ev}_i : \overline{M}_{g,r}(\mathbb{CP}^s, d) \to \mathbb{CP}^s$, which send a stable map $(C, x_1, \ldots, x_r, f)$ to the image $f(x_i)$ of the marked point x_i. Then we can pull back the incidence conditions via the evaluation maps. Finally, we can intersect

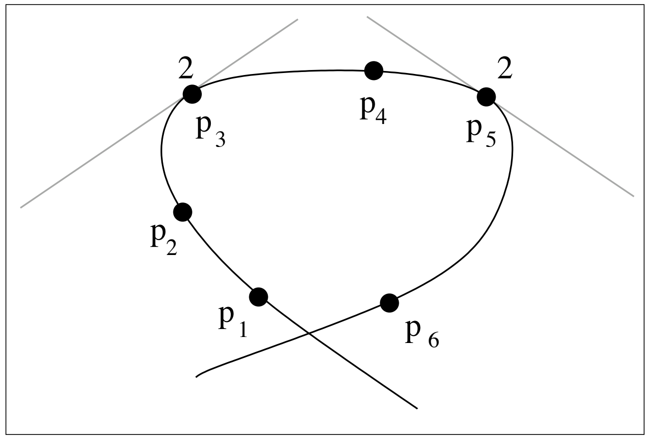


Figure 4.1: A rational degree-3 algebraic curve in $\mathbb{CP}^2$ with tangency of order 2 at two fixed points to two fixed generic lines. Additionally, the curve passes through 4 points in generic position. This curve (fractionally) contributes to the intersection product $\mathrm{ev}_1^*(\mathrm{pt})\, \mathrm{ev}_2^*(\mathrm{pt})\, \mathrm{ev}_3^*(pt)\psi_3\, \mathrm{ev}_4^*(\mathrm{pt})\, \mathrm{ev}_5^*(\mathrm{pt})\psi_5\, \mathrm{ev}_6^*(\mathrm{pt})$ on the moduli space $\overline{M}_{0,6}(\mathbb{CP}^2, 3)$.

pullbacks along the evaluation maps and Psi-classes on $\overline{M}_{g,r}(\mathbb{CP}^s, d)$. The degrees of such zero-dimensional intersection products are called *descendant Gromov-Witten invariants*. They have been studied in detail in Gromov-Witten theory.

Roughly speaking, a condition given by the Psi-classes ψ_i^r together with an incidence condition at the ith point (via the pull back $\mathrm{ev}_i^*(\mathrm{pt})$) corresponds to imposing a tangency of order $r + 1$ to a fixed general line in $\mathbb{CP}^s$ at a fixed point (cf. with Figure 4.1). The contribution can be fractional however (for more details see Section 4.2).

Tropical analogues of moduli spaces of stable curves and maps have been introduced in [26, 14], and tropical intersection theory was used to define tropical enumerative numbers for rational curves analogously to the classical world. A. Gathmann and H. Markwig showed that the famous recursion formulas for the count of plane curves known as *Kontsevich's formula* [16] resp. *the Caporaso-Harris algorithm* [15] also hold in the tropical world and can be proven using purely tropical methods. Tropical analogues of Psi-classes on the space of abstract tropical curves $\mathcal{M}_{0,r}$ have been introduced by G. Mikhalkin [26], and *tropical plane descendant Gromov-Witten invariants* on $\mathcal{M}_{0,r}(\mathbb{R}^2, d)$ by H. Markwig and J. Rau [25]. H. Markwig and J. Rau show that these tropical plane descendant Gromov-Witten invariants for which every Psi-condition ψ_i comes together with a point condition $\mathrm{ev}_i^* \mathrm{pt}$ satisfy the so-called *WDVV equations* which can be thought of as generalizations of Kontsevich's formula [23]. It follows that those numbers are equal to their classical counterparts, i.e. a correspondence theorem holds here as well. Tropical curves contributing to the count of such descendant plane Gromov-Witten invariants have higher-valent vertices at the marked points satisfying the Psi-conditions.

The aim of this chapter is to introduce floor diagrams for plane descendant Gromov-Witten invariants (such that every Psi-condition ψ_i comes together with a point condition $\mathrm{ev}_i^* \mathrm{pt}$) which we call *Psi-floor diagrams*. The count of these diagrams gives exactly the tropical plane descendant Gromov-Witten invariants. Because of the Correspondence Theorem it then follows that they also give the classical plane descendant Gromov-Witten invariants. We generalize our definition to relative Psi-floor diagrams and prove that their count computes tropical relative Gromov-Witten invariants. Afterwards, we show that the numbers of relative Psi-floor diagrams satisfy a Caporaso-Harris formula and we show that our formula coincides with the classical formula by A. Gathmann. It follows that relative Psi-floor diagrams (and thus also tropical relative plane descendant Gromov-Witten invariants) count relative plane descendant Gromov-Witten invariants.

The difficulty in generalizing the definition of floor diagrams to tropical curves satisfying Psi-conditions is that, because of the higher-valent vertices, we cannot necessarily split the curve into single floors. So we have to introduce multiple floors which are harder to deal with combinatorially. As a consequence, there is no longer a bijection between floor diagrams and tropical curves. Rather, there are several tropical curves encoded in one Psi-floor diagram since there are many ways how a multiple floor can look in a tropical curve. Thus, we have to introduce new multiplicities for Psi-floor diagrams that encode how many tropical curves correspond to one diagram.

4.2 Descendant Gromov-Witten Invariants

Let us start by recalling the algebro-geometric construction and computation of the absolute and relative descendant Gromov-Witten invariants whose corresponding tropical version we will study later in this chapter. For details in this section we refer mainly to [11, 24] in the absolute and [12] in the relative case. Throughout this section we will work with the ground field $\mathbb{C}$ of the complex numbers and denote by $A_*(X)$ and $A^*(X)$ the Chow homology and cohomology groups of a scheme (or stack) X. A class $\gamma \in A^i(X)$ will be said to have codimension $\operatorname{codim} \gamma = i$, and the class of a hyperplane in a projective space $\mathbb{CP}^s$ will be denoted $h \in A^1(\mathbb{CP}^s)$.

4.2.1 Absolute Descendant Gromov-Witten Invariants

For $s > 0$ and $r, d \geq 0$ we denote by $\bar{M}_{0,r}(\mathbb{CP}^s, d)$ the moduli space of r-marked rational stable maps of degree d to the projective space $\mathbb{CP}^s$ (see [11, Section 4]). Its points correspond to tuples $(C, x_1, \ldots, x_r, f)$ (modulo automorphisms) such that

- C is a connected, complete rational curve with at most nodes as singularities;
- $x_1, \ldots, x_r$ are distinct smooth points on C;
- $f : C \to \mathbb{CP}^s$ is a morphism of degree d, i.e. such that $f_*[C]$ is the class of d times a line; and
- the tuple $(C, x_1, \ldots, x_r, f)$ has only finitely many automorphisms.

Intuitively, $\bar{M}_{0,r}(\mathbb{CP}^s, d)$ can be thought of as a compactification of the space of all rational degree-d curves in $\mathbb{CP}^s$ with r marked points. It is a smooth, complete, and separated stack of dimension $(s+1)d + s - 3 + r$.

For $i = 1, \ldots, r$ there are so-called evaluation maps $\mathrm{ev}_i : \bar{M}_{0,r}(\mathbb{CP}^s, d) \to \mathbb{CP}^s$ that send a tuple $(C, x_1, \ldots, x_r, f)$ to the image $f(x_i)$ of the i th marked point. Moreover, we denote by $\psi_i \in A^1(\bar{M}_{0,r}(\mathbb{CP}^s, d))$ the i-th cotangent line class (also called the i-th Psi-class), i.e. the first Chern class of the line bundle whose fiber over a point $(C, x_1, \ldots, x_r, f)$ is the cotangent space of C at the (smooth) point x_i.

In general, descendant Gromov-Witten invariants are now obtained by taking degrees of zero-dimensional intersection products of Psi-classes and pull-backs by the evaluation maps on the above moduli spaces. More precisely, pick $a_1, \ldots, a_r \geq 0$ and $\gamma_1, \ldots, \gamma_r \in A^*(\mathbb{CP}^s)$ such that the dimension condition

$$\sum_{i=1}^{r} (a_i + \operatorname{codim} \gamma_i) = \dim \bar{M}_{0,r}(\mathbb{CP}^s, d).$$

holds. Then we define the corresponding Gromov-Witten invariant

$$\langle \tau^{a_1}(\gamma_1) \cdots \tau^{a_r}(\gamma_r) \rangle_d^{\mathbb{CP}^s} \stackrel{\text{def}}{=} \deg \left(\mathrm{ev}_1^* \gamma_1 \cdot \psi_1^{a_1} \cdot \ \cdots \ \cdot \mathrm{ev}_r^* \gamma_r \cdot \psi_r^{a_r} \cdot [\bar{M}_{0,r}(\mathbb{CP}^s, d)] \right) \in \mathbb{Q}.$$

For $a_1 = \cdots = a_r = 0$ we can simply think of this invariant as the number of rational degree-d curves in $\mathbb{CP}^s$ passing through r given generic subvarieties of classes $\gamma_1, \ldots, \gamma_r$. For other choices of $a_1, \ldots, a_r$ these numbers do not have an immediate geometric interpretation, but they occur e.g. in the computation of numbers of curves satisfying tangency conditions in addition to incidence conditions.

The Gromov-Witten invariants above are all well-known; they can be computed e.g. using the WDVV and topological recursion relations (see [23, Section 3] , [24, Corollary 1.3]). In what follows we will need in particular the following invariants of $\mathbb{CP}^1$.

Lemma IV.1. *For all $a, b, c, d \geq 0$ with $a = 2d - 2 + b$ we have*

$$\langle \underbrace{1 \cdots 1}_{b} \underbrace{h \cdots h}_{c} \tau^a(h) \rangle_d^{\mathbb{CP}^1} = \frac{d^c}{d!^2}.$$

Proof. The equation $a = 2d - 2 + b$ is simply the dimension condition. Let us first assume that $d > 0$. By the fundamental class and divisor axioms of Gromov-Witten invariants (see e.g. [17, Proposition 12]) we then know that

$$\langle \underbrace{1 \cdots 1}_{b} \underbrace{h \cdots h}_{c} \tau^a(h) \rangle_d^{\mathbb{CP}^1} = d^c \cdot \langle \tau^{a-b}(h) \rangle_d^{\mathbb{CP}^1}.$$

As the one-point invariant $\langle \tau^a(h) \rangle_d^{\mathbb{CP}^1}$ is equal to $\frac{1}{d!^2}$ by [28, Section 1.4], the result follows.

In the special case $d = 0$ we see first of all that we must have $b \geq 2$ by the dimension condition. Thus we can again use the fundamental class and divisor axioms to reduce the invariant to

$$\langle \underbrace{1 \cdots 1}_{b} \underbrace{h \cdots h}_{c} \tau^a(h) \rangle_d^{\mathbb{CP}^1} = d^c \cdot \langle 1 \; 1 \; \tau^0(h) \rangle_d^{\mathbb{CP}^1} = d^c$$

as stated in the lemma (i.e. to 1 for $c = 0$ and to 0 otherwise). □

Our main concern in this chapter, however, will be the plane[1] Gromov-Witten invariants of the projective plane $\mathbb{CP}^2$ where each of the classes $\gamma_1, \ldots, \gamma_r$ above is the class $\mathrm{pt} = h^2$ of a point. By the dimension condition we then need non-negative integers $a_1, \ldots, a_r$ such that

$$2r + a_1 + \cdots + a_r = \dim \bar{M}_{0,r}(\mathbb{CP}^2, d), \qquad \text{i.e. } a_1 + \cdots + a_r = 3d - 1 - r$$

to get a well-defined number $\langle \tau^{a_1}(\mathrm{pt}) \cdots \tau^{a_r}(\mathrm{pt}) \rangle_d^{\mathbb{CP}^2}$. Note that by the symmetry of the points this number depends only on how often each Psi-power occurs among the numbers $a_1, \ldots, a_r$. Let us therefore introduce a simplified notation that reflects this symmetry and that will be particularly useful when considering floor diagrams later:

[1] To simplify our terminology we suppress the word "plane" in the context of Gromov-Witten invariants if it is clear from context that the ambient space is $\mathbb{CP}^2$.

Notation IV.2 (Sequences)**.** Let $\mathbf{k} = (\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \dots)$ be a sequences of non-negative integers with only finitely many non-zero entries. We set (compatibly with Chapter III)

$$|\mathbf{k}| \stackrel{\text{def}}{=} \mathbf{k}_0 + \mathbf{k}_1 + \mathbf{k}_2 + \cdots ,$$

$$I\mathbf{k} \stackrel{\text{def}}{=} 0\,\mathbf{k}_0 + 1\,\mathbf{k}_1 + 2\,\mathbf{k}_2 + \cdots ,$$

$$I^{\mathbf{k}} \stackrel{\text{def}}{=} 0^{\mathbf{k}_0} \cdot 1^{\mathbf{k}_1} \cdot 2^{\mathbf{k}_2} \cdot \ \cdots ,$$

$$\mathbf{k}! \stackrel{\text{def}}{=} \mathbf{k}_0! \cdot \mathbf{k}_1! \cdot \mathbf{k}_2! \cdot \ \cdots .$$

Moreover, if $\mathbf{k}, \mathbf{k}'$ are two such sequences we define the sequence $\mathbf{k} + \mathbf{k}'$ by componentwise addition and write $\mathbf{k} \le \mathbf{k}'$ if $\mathbf{k}_i \le \mathbf{k}'_i$ for all $i \ge 0$. To simplify notation, we will usually write such sequences as *finite* sequences $(\mathbf{k}_0, \dots, \mathbf{k}_n)$ for some n with the convention that the remaining entries $\mathbf{k}_{n+1}, \mathbf{k}_{n+2}, \dots$ are then equal to zero.

Definition IV.3 ($\tilde{N}_{d,\mathbf{k}}$ and $N_{d,\mathbf{k}}$)**.** Let $d \ge 0$, and let $\mathbf{k} = (\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \dots)$ be a sequence of non-negative integers such that $I\mathbf{k} = 3d - 1 - |\mathbf{k}|$. For $r = |\mathbf{k}|$ let $a_1, \dots, a_r$ be an r-tuple of non-negative integers that contains each number $i \in \mathbb{N}$ exactly $\mathbf{k}_i$ times (in any order), and define

$$\tilde{N}_{d,\mathbf{k}} \stackrel{\text{def}}{=} \langle \tau^{a_1}(\mathrm{pt}) \ \cdots \ \tau^{a_r}(\mathrm{pt}) \rangle_d^{\mathbb{CP}^2};$$

so these are the numbers of rational plane degree-d curves passing through r points and satisfying in addition a ψ^i condition at $\mathbf{k}_i$ chosen marked points for all i. If we do not choose the points for the ψ^i conditions but rather only require that among the r marked points there are $\mathbf{k}_i$ of them at which a ψ^i condition is satisfied (i.e. sum over all tuples $a_1, \dots, a_r$ above containing each i exactly $\mathbf{k}_i$ times) then we get instead the numbers

$$N_{d,\mathbf{k}} \stackrel{\text{def}}{=} \frac{|\mathbf{k}|\,!}{\mathbf{k}!}\, \tilde{N}_{d,\mathbf{k}},$$

which will turn out to be more natural when considering floor diagrams later.

4.2.2 Relative Descendant Gromov-Witten Invariants

Relative invariants are very similar to the absolute invariants of Section 4.2.1, except that we now fix once and for all a line $L \subset \mathbb{CP}^2$ and count curves in $\mathbb{CP}^2$ that have prescribed local intersection multiplicities with L in addition to satisfying the evaluation and Psi-conditions above. In the absence of Psi-conditions this amounts to considering relative Gromov-Witten invariants which are closely related to the relative Severi degree of Chapter III (the former enumerates only irreducible curves whereas the latter includes also reducible curves).

More precisely, choose $d > 0$ and let $\mu_1, \ldots, \mu_r \in \mathbb{N}$ for some $r > 0$ such that $\mu_1 + \cdots + \mu_r = d$. Setting $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$, we denote by $\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d) \subset \bar{M}_{0,r}(\mathbb{CP}^2, d)$ the closure of the subset of all $(C, x_1, \ldots, x_r, f)$ such that C is smooth and $f^*L = \mu_1 x_1 + \cdots + \mu_r x_r$ as divisors on C (see [12, Section 1]). These spaces are called the moduli spaces of stable maps relative to L; they have dimension $2d - 1 + r$.

As in the absolute case, degrees of zero-dimensional intersection products of Psi-classes and pull-backs by the evaluation maps on the moduli spaces of relative stable maps are called relative descendant Gromov-Witten invariants. So if we now fix $a_1, \ldots, a_r \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_r \in A^*(\mathbb{CP}^2)$ such that

$$\sum_{i=1}^{r} (a_i + \operatorname{codim} \gamma_i) = \dim \bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d),$$

we can define in a similar way as above an associated relative Gromov-Witten invariant

$$\langle \tau^{a_1}(\gamma_1) \cdots \tau^{a_r}(\gamma_r) \rangle_{\boldsymbol{\mu}}^{\mathbb{CP}^2} \stackrel{\text{def}}{=} \deg(\operatorname{ev}_1^* \gamma_1 \cdot \psi_1^{a_1} \cdot \ \cdots \ \cdot \operatorname{ev}_r^* \gamma_r \cdot \psi_r^{a_r} \cdot [\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d)]) \in \mathbb{Q}.$$

If $a_1 = \cdots = a_r = 0$ this invariant can be interpreted by construction as the number of plane rational degree-d curves with r marked points that have local intersection multiplicity μ_i and, in addition, pass through a generic subvariety of $\mathbb{CP}^2$ of class γ_i at the i-th marked point, for all $i = 1, \dots, r$. In particular, the marked points x_i with $\mu_i > 0$ will lie on L, whereas the ones with $\mu_i = 0$ in general do not.

As before, we will restrict our attention in this chapter to a certain subset of these invariants. Namely, we will only consider choices of $\mu_1, \dots, \mu_r, a_1, \dots, a_r, \gamma_1, \dots, \gamma_r$ such that, for all $i = 1, \dots, r$, we have one of the following cases:

- $\mu_i > 0$, $a_i = 0$, and $\gamma_i = h^1$ (i.e. a marked point lying on a fixed point of L with a given local intersection multiplicity of the curve to L). For $j \geq 1$ we will denote the number of such i with $\mu_i = j$ by α_j.

- $\mu_i > 0$, $a_i = 0$, and $\gamma_i = h^0$ (i.e. a marked point lying on a non-fixed point of L with a given local intersection multiplicity of the curve to L). For $j \geq 1$ we will denote the number of such i with $\mu_i = j$ by β_j.

- $\mu_i = 0$ and $\gamma_i = h^2$ (i.e. a marked point lying on a fixed generic point of $\mathbb{CP}^2$ and possibly satisfying some Psi-conditions). For $j \geq 0$ we will denote the number of such i with $a_i = j$ by $\mathbf{k}_j$.

By symmetry of the marked points, the three sequences $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots)$, and $\mathbf{k} = (\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \dots)$ determine the invariant under consideration uniquely. So we can make the following definition:

Definition IV.4 ($\tilde{N}_{d,\mathbf{k}}(\alpha, \beta)$ and $N_{d,\mathbf{k}}(\alpha, \beta)$)**.** With notations as above, we set

$$\tilde{N}_{d,\mathbf{k}}(\alpha, \beta) \stackrel{\text{def}}{=} \langle \tau^{a_1}(\gamma_1) \cdots \tau^{a_r}(\gamma_r) \rangle_{\boldsymbol{\mu}}^{\mathbb{CP}^2}.$$

So $\tilde{N}_{d,\mathbf{k}}(\alpha, \beta)$ is the number of plane rational marked degree-d curves $(C, x_1, \dots, x_r, f)$ satisfying the following conditions:

- For each $i \in \mathbb{N}$ fix α_i of the marked points on C and a general point on L for each of them; each of these marked points then has to be mapped by f to the corresponding given point on L, and C must have local intersection multiplicity i to L there.

- For each $i \in \mathbb{N}$ fix β_i of the marked points on C; each of these marked points then has to be mapped by f to L, and C must have local intersection multiplicity i to L there.

- For each $i \in \mathbb{N}$ fix $\mathbf{k}_i$ of the marked points on C and a general point in $\mathbb{CP}^2$ for each of them; each of these marked points then has to be mapped by f to the corresponding given point in $\mathbb{CP}^2$, and C must satisfy in addition a ψ^i condition there.

Note that the dimension condition translates to

$$I(\alpha + \beta + \mathbf{k}) = 3d - 1 + |\beta| - |\mathbf{k}|$$

in these variables, where we use notation IV.2 also for the sequences α and β (although they start at index 1 rather than 0). In the same way, the condition $\mu_1 + \cdots + \mu_r = d$ translates to

$$I(\alpha + \beta) = d.$$

As in Definition IV.3 let us also introduce a slight variant of these invariants where we do not specify which Psi-power condition has to be satisfied at which point x_i with $\mu_i = 0$, and where we do not mark the non-fixed points on L of the curves: we set

$$N_{d,\mathbf{k}}(\alpha, \beta) \stackrel{\text{def}}{=} \frac{1}{\beta!} \cdot \frac{|\mathbf{k}|\,!}{\mathbf{k}!} \cdot \tilde{N}_{d,\mathbf{k}}(\alpha, \beta).$$

Just like their absolute counterparts all relative Gromov-Witten invariants that we have introduced in this section are actually known to be computable recursively. To do so one uses a generalization of the Caporaso-Harris formula of [6] that we will describe now.

4.2.3 The Caporaso-Harris Formula for Descendant Invariants

In this section we want to use relative Gromov-Witten theory to derive a recursive formula for the numbers $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$ (and thus also for $N_{d,\mathbf{k}}(\alpha,\beta)$) of Definition IV.4.

As in the beginning of Section 4.2.2 let $r, d > 0$ and $\mu_1, \dots, \mu_r \geq 0$ with $\mu_1 + \dots + \mu_r = d$. We have then constructed a moduli space $\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d) \subset \bar{M}_{0,r}(\mathbb{CP}^2, d)$ of dimension $2d - 1 + r$ of plane rational degree-d stable maps relative to a fixed line $L \subset \mathbb{CP}^2$, and our invariants $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$ were certain zero-dimensional intersection products on these spaces.

Since $\mu_1 + \dots + \mu_r = d$ there can be at most d marked points x_i with $\mu_i > 0$. Note that our invariants had no Psi-conditions and at most a codimension-1 evaluation condition at all these points. So the conditions at these marked points yield a cycle of codimension at most d — and as the dimension of our moduli space is $2d-1+r > d$ it follows that there must be at least one marked point x_i with $\mu_i = 0$. By symmetry we may assume without loss of generality that x_1 is such a marked point, i.e. that $\mu_1 = 0$. For our invariant this marked point x_1 is then required to map to a given general point in $\mathbb{CP}^2$.

The idea of the proof is now to move this generic chosen point to a special position, namely to a point on L. As we have marked all intersection points of the curves with L already (note that $\mu_1 + \dots + \mu_r = d$) this forces the curves to become reducible and split up into several components of smaller degree, one of which will be mapped completely to L. The curves can then be enumerated recursively over the degree.

To describe this process more formally we follow the notation and results from Section 2 of [12]. Note, however, that our current situation is a little simplified compared to [12] since we have assumed here that $\mu_1 + \cdots + \mu_r = d$.

Construction IV.5 (Moduli spaces $D(A,B)$, see [12, Definition 2.2])**.** Fix $r, d > 0$ and a moduli space $\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d) \subset \bar{M}_{0,r}(\mathbb{CP}^2, d)$ with $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ and $\mu_1 + \cdots + \mu_r = d$ as above.

Choose a partition $A = (A', A^1, \dots, A^t)$ of $\{1, \dots, r\}$ for some $t \geq 0$, and let $\boldsymbol{\mu}^i$ for $i = 1, \dots, t$ be the tuple of all μ_j with $j \in A^i$ (in any order). Moreover, pick a $(t+1)$-tuple $B = (d', d^1, \dots, d^t)$ of non-negative integers with $d^i > 0$ for $i = 1, \dots, t$ and $d' + d^1 + \cdots + d^t = d$. We assume that we have made our choices so that

$$m^i \stackrel{\text{def}}{=} d^i - \sum_{j \in A^i} \mu_j > 0 \tag{4.1}$$

for all $i = 1, \dots, t$, and thus (by adding all these equations up and comparing the sum to $\mu_1 + \cdots + \mu_r = d$) so that

$$d' + m^1 + \cdots + m^t = \sum_{j \in A'} \mu_j. \tag{4.2}$$

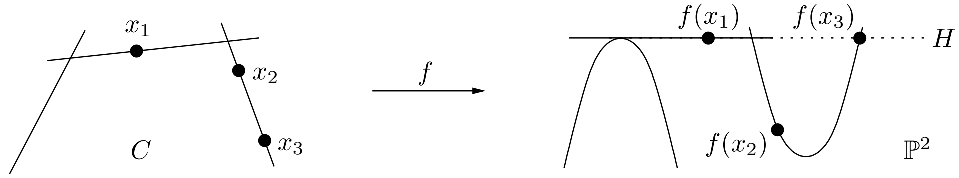
In this case we now define the space $D(A,B)$ to be

$$D(A,B) \stackrel{\text{def}}{=} \bar{M}_{0,t+\#A'}(L, d') \times_{(\mathbb{CP}^1)^t} \prod_{i=1}^{t} \bar{M}_{0,(m^i) \cup \boldsymbol{\mu}^i}(\mathbb{CP}^2, d^i),$$

where $(m^i) \cup \boldsymbol{\mu}^i$ denotes the $(\#A^i + 1)$-tuple obtained by prepending m^i at the beginning of $\boldsymbol{\mu}^i$, and the maps to $(\mathbb{CP}^1)^t$ for the fiber product are the evaluation at the first t marked points of the first factor and at the first marked point of each of the moduli spaces in the second factor. Note that the first factor is a moduli space of absolute stable maps to the line $L \cong \mathbb{CP}^1$, whereas the second factor consists of moduli spaces of stable maps to $\mathbb{CP}^2$ relative to L.

By construction, $D(A,B)$ parameterizes stable maps to $\mathbb{CP}^2$ with (generically) $t+1$ irreducible components: one "central" component in L, and t "external" components in $\mathbb{CP}^2$ all attached to the central one at a point where they have a local intersection multiplicity to L as given by $m^1,\dots,m^t$. The $(t+1)$-tuples A and B simply parameterize how the marked points and the degree split up onto the $t+1$ components. In this way $D(A,B)$ can be considered as a closed subspace of $\bar{M}_{0,r}(\mathbb{CP}^2,d)$.

Note that the case $t=0$ is allowed (i.e. there may be no external components at all), as well as $d'<1$ and $d'>1$ (i.e. the central component may be a contracted one or a multiple cover of L). The following picture shows an example of a general element $(C,x_1,\dots,x_5,f)\in D(A,B)$ for $d=5$, $r=3$, $\boldsymbol{\mu}=(4,0,1)$, $A=(\{1\},\emptyset,\{2,3\})$, $B=(1,2,2)$, and thus $\boldsymbol{\mu}^1=()$, $\boldsymbol{\mu}^2=(0,1)$, $m^1=2$, and $m^2=1$.



The importance of these moduli spaces comes from the fact that they describe precisely the curves appearing when moving a marked point from a general position in $\mathbb{CP}^2$ to L. In fact, all $D(A,B)$ are divisors in $\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2,d)$, and we have the following statement:

Proposition IV.6 ([12, Theorem 2.6]). *With notations as above, we have*

$$\mathrm{ev}_1^* L\cdot \bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2,d)=\sum_{t,A,B}\frac{m^1\cdot\;\cdots\;\cdot m^t}{t!}\,D(A,B)$$

in the Chow group of $\bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2,d)$, where the sum is taken over all $t\geq 0$, A, and B with $1\in A'$ and satisfying condition (4.1) (and thus also (4.2)) as in Construction IV.5.

As usual in Gromov-Witten theory it is now convenient to replace the fiber product in the Construction IV.5 of $D(A,B)$ by the "diagonal splitting" trick: the fiber product $X \times_{\mathbb{CP}^1} Y$ of two spaces X and Y with projections p and q to $\mathbb{CP}^1$ can be rewritten as the pull-back of the diagonal of $\mathbb{CP}^1 \times \mathbb{CP}^1$ by the map $p \times q$, and as this diagonal has class $h \times 1 + 1 \times h$ it follows that

$$X \times_{\mathbb{CP}^1} Y = (p^*h + q^*h) \cdot (X \times Y).$$

Let us apply this formula in the expression for $D(A,B)$ from Construction IV.5 for each of the t factors $\mathbb{CP}^1$ over which we take the fiber product, thus converting $D(A,B)$ into a sum of 2^t terms with no fiber products. By symmetry, we can then always relabel the external t components so that the ones with the ev^*h term in the $\bar{M}_{0,(m^i)\cup\boldsymbol{\mu}^i}(\mathbb{CP}^2, d^i)$ factor come first — if there are $t' \in \{0,\dots,t\}$ of these components we then have $\binom{t}{t'}$ terms in the diagonal splitting that become the same after this relabeling. Hence we can rewrite the formula of Proposition IV.6 in the following form:

$$\begin{aligned}\mathrm{ev}_1^* L \cdot \bar{M}_{0,\boldsymbol{\mu}}(\mathbb{CP}^2, d) = \sum_{t,A,B} \sum_{t'=0}^{t} \frac{m^1 \cdot \; \cdots \; \cdot m^t}{t'!\,(t-t')!} \Big(\mathrm{ev}_{t'+1}^* h \cdot \; \cdots \; \cdot \mathrm{ev}_t^* h \cdot \bar{M}_{0,t+\#A'}(L, d') \Big) \\ \times \prod_{i=1}^{t'} \Big(\mathrm{ev}_1^* h \cdot \bar{M}_{0,(m^i)\cup\boldsymbol{\mu}^i}(\mathbb{CP}^2, d^i) \Big) \times \prod_{i=t'+1}^{t} \bar{M}_{0,(m^i)\cup\boldsymbol{\mu}^i}(\mathbb{CP}^2, d^i).\end{aligned}$$

To get a recursive relation for the invariants

$$\tilde{N}_{d,\mathbf{k}}(\alpha,\beta) = \langle \tau^{a_1}(\gamma_1) \cdots \tau^{a_r}(\gamma_r) \rangle_{\boldsymbol{\mu}}^{\mathbb{CP}^2}$$

of Definition IV.4 we now intersect this equation of cycles with the class

$$\mathrm{ev}_1^* h \cdot \psi_1^{a_1} \cdot \mathrm{ev}_2^* \gamma_2 \cdot \psi_2^{a_2} \cdot \; \cdots \; \cdot \mathrm{ev}_r^* \gamma_r \cdot \psi_r^{a_r}$$

(note that $\gamma_1 = \mathrm{pt}$ by assumption, and thus the two evaluations $\mathrm{ev}_1^* L \cdot \mathrm{ev}_1^* h$ together give the desired condition $\mathrm{ev}_1^* \gamma_1$ at the first point). The left hand side of the equation is then simply $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$. Each summand on the right hand side is a product of one absolute Gromov-Witten invariant of $\mathbb{CP}^1$ and t relative Gromov-Witten invariants of $\mathbb{CP}^2$. The invariant of $\mathbb{CP}^1$ has the condition $\mathrm{ev}_1^* h \cdot \psi_1^{a_1}$ at the first marked point, a condition $\mathrm{ev}_i^* h$ at all gluing points from the last $t-t'$ external components and all x_i with $i \in A'$ such that $\gamma_i = h$, and no condition at all at the other points. On the other hand, the t relative invariants of $\mathbb{CP}^2$ are again of the type of invariants considered in Definition IV.4: we can write them as $N_{d^i,\mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i)$ for the first t' and $N_{d^i,\mathbf{k}^i}(\alpha^i, \beta^i + e_{m^i})$ for the last $t-t'$ invariants, where $\alpha^i, \beta^i, \mathbf{k}^i$ denote the sequences associated to the marked points x_j with $j \in A^i$ according to Definition IV.4. Finally, let us then rewrite the sum over A as a sum over the corresponding sequences α^i, β^i, $\mathbf{k}^i$. If we set (compatibly with Chapter III)

$$\alpha' \stackrel{\text{def}}{=} \alpha - \alpha^1 - \cdots - \alpha^t \quad \text{and} \quad \binom{\alpha}{\alpha^1, \dots, \alpha^t} \stackrel{\text{def}}{=} \prod_{i \geq 1} \frac{\alpha_i!}{\alpha_i^1! \cdot \;\cdots\; \cdot \alpha_i^t! \cdot \alpha_i'!} \tag{4.3}$$

(and similarly for β and $\mathbf{k}$, except that the index of the sequences starts at 0 for $\mathbf{k}$), then exactly

$$\binom{\alpha!}{\alpha^1, \dots, \alpha^t} \cdot \binom{\beta!}{\beta^1, \dots, \beta^t} \cdot \binom{\mathbf{k} - e_a}{\mathbf{k}^1, \dots, \mathbf{k}^t}$$

choices of partitions of A into t subsets will give rise to the same invariants. Here, e_a denotes the sequence with only non-zero entry 1 in the a-th component — we have to write $\mathbf{k} - e_a$ instead of $\mathbf{k}$ since the first marked point is fixed to lie on the central component, so there is no choice here where to put this point. Hence our equation becomes

$$\tilde{N}_{d,\mathbf{k}}(\alpha,\beta) = \sum_{t,t'} \sum_{\alpha,\beta,\mathbf{k}} \sum_{B} \frac{m^1 \cdot \cdots \cdot m^t}{t'!\,(t-t')!} \binom{\alpha}{\alpha^1,\dots,\alpha^t} \binom{\beta}{\beta^1,\dots,\beta^t} \binom{\mathbf{k}-e_a}{\mathbf{k}^1,\dots,\mathbf{k}^t}$$

$$\cdot \langle \underbrace{1 \cdots 1}_{|\beta'|+t'} \underbrace{h \cdots h}_{|\alpha'|+t-t'} \tau^a(h) \rangle_{d'}^{\mathbb{CP}^1}$$

$$\cdot \prod_{i=1}^{t'} N_{d^i,\mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i) \cdot \prod_{i=t'+1}^{t} N_{d^i,\mathbf{k}^i}(\alpha^i, \beta^i + e_{m^i})$$

Note that we must have $\mathbf{k}^1 + \cdots + \mathbf{k}^t = \mathbf{k} - e_a$ in each term since marked points with generic point conditions in $\mathbb{CP}^2$ cannot lie in the central component within L. Moreover, each relative invariant in this expression must of course satisfy the dimension condition

$$\begin{aligned} & I((\alpha^i + e_{m^i}) + \beta^i + \mathbf{k}^i) = 3d^i - 1 + |\beta^i| - |\mathbf{k}^i| && \text{for } i \le t' \\ \text{resp.}\quad & I(\alpha^i + (\beta^i + e_{m^i}) + \mathbf{k}^i) = 3d^i - 1 + |\beta^i + e_{m^i}| - |\mathbf{k}^i| && \text{for } i > t' \end{aligned}$$

of Definition IV.4, as well as condition (4.1)

$$m^i = d^i - I(\alpha^i + \beta^i)$$

of Construction IV.5. We can think of the first of these equations as determining d^i, and of the second as determining m^i from α^i, β^i, and $\mathbf{k}^i$. Finally, inserting the expression of Lemma IV.1 for the absolute Gromov-Witten invariant of $\mathbb{CP}^1$ we get the following result that allows us to compute all numbers $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$ recursively.

Theorem IV.7 (Caporaso-Harris formula for the relative descendant Gromov-Witten invariants $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$)**.** *The relative Gromov-Witten invariants $\tilde{N}_{d,\mathbf{k}}$ of Definition IV.4 satisfy the relations*

$$\tilde{N}_{d,\mathbf{k}}(\alpha,\beta) = \sum \frac{m^1 \cdot \cdots \cdot m^t}{t'!\,(t-t')!} \cdot \frac{d'^{|\alpha'|+t-t'}}{d'!^2} \binom{\alpha}{\alpha^1,\dots,\alpha^t} \binom{\beta}{\beta^1,\dots,\beta^t} \binom{\mathbf{k}-e_a}{\mathbf{k}^1,\dots,\mathbf{k}^t}$$

$$\cdot \prod_{i=1}^{t'} \tilde{N}_{d^i,\mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i) \cdot \prod_{i=t'+1}^{t} \tilde{N}_{d^i,\mathbf{k}^i}(\alpha^i, \beta^i + e_{m^i})$$

for each $a \in \mathbb{N}$ with $\mathbf{k}_a > 0$. Here, the sum is taken over all $0 \leq t' \leq t$ and all sequences $\alpha^1, \dots, \alpha^t$, $\beta^1, \dots, \beta^t$, $\mathbf{k}^1, \dots, \mathbf{k}^t$ such that

- $\alpha' \stackrel{def}{=} \alpha - \alpha^1 - \dots - \alpha^t \geq 0$, $\beta' \stackrel{def}{=} \beta - \beta^1 - \dots - \beta^t \geq 0$, *and* $\mathbf{k}^1 + \dots + \mathbf{k}^t = \mathbf{k} - e_a$;
- $d^i \stackrel{def}{=} \frac{1}{3}(I(\alpha^i + \beta^i + \mathbf{k}^i + e_{m^i}) - |\beta^i| + |\mathbf{k}^i| + 1) \in \mathbb{N}_{>0}$ *for* $i = 1, \dots, t'$, *and* $d^i \stackrel{def}{=} \frac{1}{3}(I(\alpha^i + \beta^i + \mathbf{k}^i + e_{m^i}) - |\beta^i| + |\mathbf{k}^i|) \in \mathbb{N}_{>0}$ *for* $i = t'+1, \dots, t$;
- $d' \stackrel{def}{=} d - d_1 - \dots - d_t \geq 0$;
- $m^i \stackrel{def}{=} d^i - I(\alpha^i + \beta^i) > 0$ *for all* $i = 1, \dots, t$.

It is easy to rewrite this formula so that it computes the invariants $N_{d,\mathbf{k}}(\alpha, \beta)$ instead of $\tilde{N}_{d,\mathbf{k}}(\alpha, \beta)$:

Corollary IV.8 (Caporaso-Harris formula for the relative descendant Gromov-Witten invariants $N_{d,\mathbf{k}}(\alpha, \beta)$)**.** *The invariants $N_{d,\mathbf{k}}(\alpha, \beta)$ of Definition IV.4 satisfy the relations*

$$N_{d,\mathbf{k}}(\alpha, \beta) = \sum_{a:\, \mathbf{k}_a > 0} \sum \frac{m^1 \cdot \,\cdots\, \cdot m^t}{t'!\,(t-t')!} \cdot \frac{d'^{|\alpha'|+t-t'}}{d'!^2} \binom{\alpha}{\alpha^1, \dots, \alpha^t} \frac{1}{\beta'!} \binom{|\mathbf{k}| - 1}{|\mathbf{k}^1|, \dots, |\mathbf{k}^t|}$$
$$\cdot \prod_{i=1}^{t'} N_{d^i,\mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i) \cdot \prod_{i=t'+1}^{t} (\beta^i_{m^i} + 1)\, N_{d^i,\mathbf{k}^i}(\alpha^i, \beta^i + e_{m^i})$$

where the second sum is taken over the same partitions and with the same conditions as in Theorem IV.7.

Proof. Inserting the expression of Definition IV.4 for the numbers $N_{d,\mathbf{k}}(\alpha, \beta)$ in terms of $\tilde{N}_{d,\mathbf{k}}(\alpha, \beta)$ into the formula of Theorem IV.7 gives

$$N_{d,\mathbf{k}}(\alpha, \beta) = \sum \frac{m^1 \cdot \,\cdots\, \cdot m^t}{t'!\,(t-t')!} \cdot \frac{d'^{|\alpha'|+t-t'}}{d'!^2} \binom{\alpha}{\alpha^1, \dots, \alpha^t} \frac{1}{\beta'!} \binom{|\mathbf{k}| - 1}{|\mathbf{k}^1|, \dots, |\mathbf{k}^t|} \frac{|\mathbf{k}|}{\mathbf{k}_a}$$
$$\cdot \prod_{i=1}^{t'} N_{d^i,\mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i) \cdot \prod_{i=t'+1}^{t} (\beta^i_{m^i} + 1)\, N_{d^i,\mathbf{k}^i}(\alpha^i, \beta^i + e_{m^i})$$

for all a with $\mathbf{k}_a > 0$. Multiplying these equations with $\frac{\mathbf{k}_a}{|\mathbf{k}|}$ and summing them up for all a then gives the desired equation since $\sum_a \frac{\mathbf{k}_a}{|\mathbf{k}|} = 1$. $\square$

4.3 Tropical Descendant Gromov-Witten Invariants

In the last section we have introduced several algebro-geometric descendant rational Gromov-Witten invariants of the projective plane:

- the *absolute* invariants $\tilde{N}_{d,\mathbf{k}}$ resp. $N_{d,\mathbf{k}}$ which count degree-d curves through points and Psi-conditions as specified by $\mathbf{k}$ (see Definition IV.3);
- the *relative* invariants $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$ resp. $N_{d,\mathbf{k}}(\alpha,\beta)$ which count degree-d curves through points, Psi-conditions as specified by $\mathbf{k}$, and multiplicity conditions to a fixed line as specified by α and β (see Definition IV.4).

The convention here was that the numbers called $\tilde{N}$ consider all points at which some condition has to be satisfied as marked points, whereas the numbers called N are obtained from these by a simple combinatorial factor dividing out some symmetries in the conditions.

We will now introduce corresponding numbers with a superscript "trop" (e.g. $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}$) arising from the count of *tropical plane curves*, as well as — in the following Section 4.4 — numbers with a superscript "floor" (e.g. $\tilde{N}^{\text{floor}}_{d,\mathbf{k}}$) obtained by counting *floor diagrams*. The convention mentioned above will still hold for these numbers; we will see however that the N numbers seem to be more natural from the point of view of floor diagrams, whereas the $\tilde{N}$ have been more natural in the algebro-geometric setting. In the end however, all corresponding numbers will turn out to be the same, e.g. $\tilde{N}_{d,\mathbf{k}} = \tilde{N}^{\text{trop}}_{d,\mathbf{k}} = \tilde{N}^{\text{floor}}_{d,\mathbf{k}}$ for all d and $\mathbf{k}$. In fact, this is the main result of this chapter: that the (rational plane) absolute and relative descendant Gromov-Witten invariants of algebraic geometry can also be computed using certain counts of floor diagrams.

4.3.1 Absolute Tropical Descendant Gromov-Witten Invariants

As mentioned in the introduction, tropical plane[2] descendant Gromov-Witten invariants can be defined as intersection products on the tropical analogue of the moduli spaces of stable maps [25]. However, in order to avoid introducing too much notation, we choose to define them here purely in terms of the combinatorial properties of the tropical curves which we want to count.

A *(rational) abstract tropical curve* is a connected metric graph Γ of genus 0, such that unbounded edges (with no vertex there) are allowed, and such that each vertex has valence at least 3 (see [14, Definition 3.2]). The unbounded edges will be called *ends*, and the length of a bounded edge e will be denoted $l(e) \in \mathbb{R}_{>0}$. We say that such a curve is an *n-marked abstract tropical curve* if n of the ends are marked by $x_1, \ldots, x_n$. Two (marked) abstract tropical curves are isomorphic (and will from now on be identified) if there is an isometry between them (that respects $x_1, \ldots, x_n$ in the marked case).

We now want to consider maps from marked abstract tropical curves to $\mathbb{R}^2$. For our later purposes it will be convenient to consider some of the *left* ends to be marked ends, whereas the other (non-contracted) ends will be unmarked.

Definition IV.9. Let $m \geq n \geq 0$. A *(parameterized plane) n-marked tropical curve (with $m-n$ marked left ends)* is a tuple $(\Gamma, x_1, \ldots, x_m, h)$, where $(\Gamma, x_1, \ldots, x_m)$ is an m-marked abstract tropical curve and $h : \Gamma \to \mathbb{R}^2$ is a continuous map satisfying the following conditions.

- On each edge e the map h is integer affine linear, i.e. of the form $h(t) = a + t \cdot v$ for $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. If $V \in \partial e$ and we parameterize the edge e starting

[2] Since we only consider tropical curves in the tropical *plane* $\mathbb{R}^2$ we again suppress the word "plane" in the context of tropical (descendant) Gromov-Witten invariants.

at V, the vector v in the above equation will be denoted $v(V,e)$ and called the *direction vector* of e starting at V. If V is understood from the context (e.g. in case e is an end, having only one adjacent vertex) we will also write $v(e)$ instead of $v(V,e)$. The lattice length of $v(V,e)$ will be called the *weight* $\omega(e)$ of e.

- At each vertex V the balancing condition
$$\sum_{e:\, V\in\partial e} v(V,e) = 0$$
is satisfied.

- Each marked end x_i for $i = 1,\dots,n$ is contracted by h (i.e. $v(x_i) = 0$).

- Each marked end x_i for $i = n+1,\dots,m$ is a *left end* (i.e. it is of direction $(-l,0)$ for some $l \in \mathbb{N}_{>0}$).

Two parameterized tropical curves are isomorphic if there is an isomorphism of the underlying marked abstract tropical curves commuting with h. The *degree* of a parameterized n-marked tropical curve is defined to be the multiset consisting of the directions of its non-marked ends, together with the directions of the marked left ends $x_{n+1},\dots,x_m$. If the degree multiset consists of d copies of each of the vectors $(-1,0)$, $(0,-1)$, and $(1,1)$ we say that the curve is *of degree* d (see Example IV.12).

Definition IV.10 (Multiplicity of a curve)**.** Let $C = (\Gamma, x_1,\dots,x_m,h)$ be a marked tropical curve of degree $\Delta = \{v_1,\dots,v_1,v_2,\dots,v_2,\dots,v_r,\dots,v_r\}$ (with $v_1,\dots,v_r$ distinct) such that all vertices that are not adjacent to any of the contracted ends $x_1,\dots,x_n$ are 3-valent. Let $V_1,\dots,V_t$ be the vertices of Γ. For $i = 1,\dots,t$ and $j = 1,\dots,r$ let b_{ij} the number of non-marked ends adjacent to V_i of direction v_j.

Then we set $\nu_C \overset{\text{def}}{=} \prod_{i=1}^{t}\prod_{j=1}^{r}\frac{1}{b_{ij}!}$, and define the *multiplicity* $\operatorname{mult}(C)$ of C to be ν_C times the product of the multiplicities of those vertices without adjacent contracted ends (see [27, Definition 2.16]).

Definition IV.11 ($\tilde{N}_{d,\mathbf{k}}^{\text{trop}}$)**.** Let $d \geq 1$, and let $\mathbf{k}$ be a sequence of non-negative integers with $I\mathbf{k} = 3d - 1 - |\mathbf{k}|$. Furthermore, for $n = |\mathbf{k}|$, fix a vector $(a_1, \ldots, a_n)$ that contains each number $i \in \mathbb{N}$ exactly $\mathbf{k}_i$ times (in any order). Let $p_1, \ldots, p_n \in \mathbb{R}^2$ be points in general position (see Definitions 3.2 and 9.7 of [25]). We define

$$\tilde{N}_{d,\mathbf{k}}^{\text{trop}} \stackrel{\text{def}}{=} \sum_{C} \text{mult}(C),$$

where the sum goes over all tropical curves $C = (\Gamma, x_1, \ldots, x_n, h)$ (with non-marked left ends, i.e. $m - n = 0$) of degree d satisfying

- $h(x_i) = p_i$ for all $i = 1, \ldots, n$, and
- the end x_i is adjacent to a vertex of valence $a_i + 3$ for all $i = 1, \ldots, n$.

It follows from the general position of the points that all other vertices of Γ are then 3-valent.

Example IV.12. Figure 4.2 shows a parameterized 9-marked tropical curve. We have drawn the contracted marked ends as dotted lines. We did not specify the lengths of the bounded edges in the abstract curve since they are determined by the lengths of the images and the (non-zero) direction vectors, which in turn are determined by the directions of the ends using the balancing condition. The direction vectors are all primitive except for the edge with weight 2 in the image.

This curve contributes to $\tilde{N}_{5,\mathbf{k}}^{\text{trop}}$, where $\mathbf{k} = (7, 0, 1, 1)$, and where we chose $a = (0, 0, 0, 0, 0, 2, 0, 3, 0)$. Its multiplicity is $\frac{1}{2} \cdot \frac{1}{2} \cdot 2 \cdot 2 = 2$. The two factors of $\frac{1}{2}$ arise because two non-marked ends of the same direction are adjacent to the end vertex of x_6 and of x_8. The two factors of 2 are the vertex multiplicities of the vertices of the edges of weight 2 (not adjacent to a contracted end). In the future, we want to avoid drawing the abstract curve together with its image. Therefore, we introduce

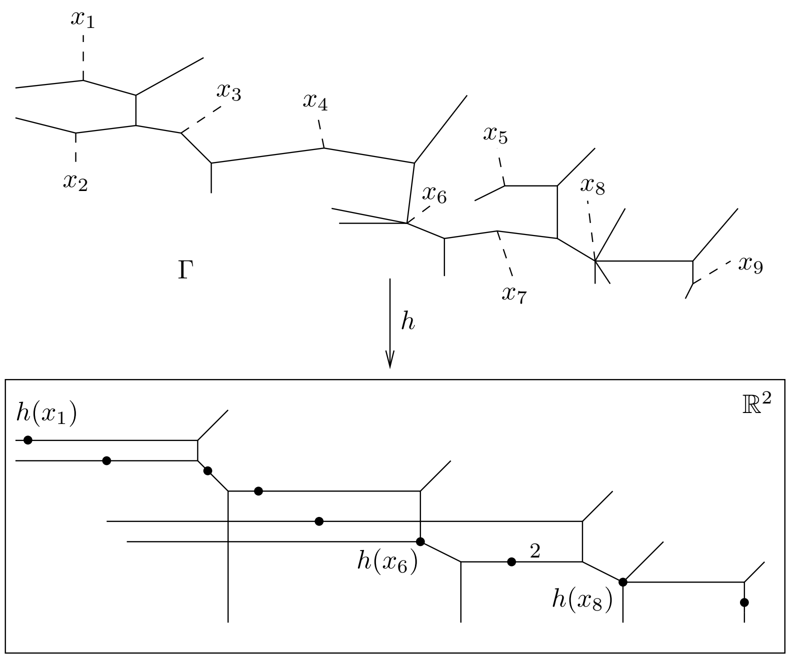
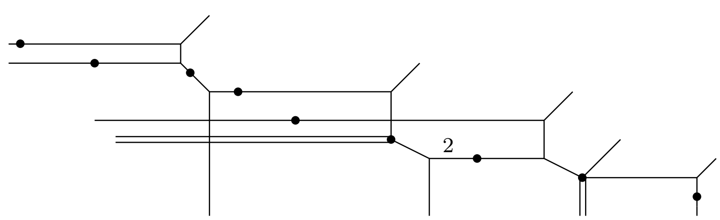


Figure 4.2: A 9-marked tropical curve.

the following shortcut for the picture above. When two edges of the abstract curve are mapped on top of each other in the image, we choose to draw them separately, but close to each other. In this way we can recover the parameterizing abstract curve uniquely (see [25, Lemma 9.9]).



For every vector $(a_1, \ldots, a_n)$ containing i exactly $\mathbf{k}_i$ times for all $i \geq 0$, the number $\tilde{N}_{d,\mathbf{k}}^{\text{trop}}$ equals the tropical intersection product $\prod_{i=1}^{n} \text{ev}_i^*(p_i)\, \psi_i^{k_i}$ on the moduli space $\mathcal{M}_{0,n}(\mathbb{R}^2, d)$ of rational tropical n-marked curves in $\mathbb{R}^2$ of degree d by [25, Remark 3.3], and is thus a *tropical descendant Gromov-Witten invariant.*

Later on, it will be convenient to allow arbitrary orderings of the Psi-powers. This leads to the following invariants.

Definition IV.13 ($N^{\text{trop}}_{d,\mathbf{k}}$)**.** For $d \geq 1$ and $\mathbf{k}$ a sequence of non-negative integers with $I\mathbf{k} = 3d - 1 - |\mathbf{k}|$ we define the number $N^{\text{trop}}_{d,\mathbf{k}} \stackrel{\text{def}}{=} \sum_C \operatorname{mult}(C)$ analogously to Definition IV.11, where now the sum is over all tropical curves C of degree d with non-marked left ends, such that for all i there are $\mathbf{k}_i$ contracted ends whose adjacent vertex has valence $i + 3$.

Obviously, these numbers $N^{\text{trop}}_{d,\mathbf{k}}$ are related to the numbers $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}$ of Definition IV.11 by $N^{\text{trop}}_{d,\mathbf{k}} = \frac{|\mathbf{k}|!}{\mathbf{k}!} \tilde{N}^{\text{trop}}_{d,\mathbf{k}}$.

Remark IV.14 (The equality $\tilde{N}^{\text{trop}}_{d,\mathbf{k}} = \tilde{N}_{d,\mathbf{k}}$). In [25] it was shown that tropical descendant Gromov-Witten invariants $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}$ satisfy the WDVV relations, just as their classical counterparts $\tilde{N}_{d,\mathbf{k}}$ do. As the initial values coincide, we can conclude that $\tilde{N}^{\text{trop}}_{d,\mathbf{k}} = \tilde{N}_{d,\mathbf{k}}$ for all d and $\mathbf{k}$. There is no direct bijection of the corresponding curves known at this point. Since both pairs of numbers $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}$, $N^{\text{trop}}_{d,\mathbf{k}}$ and $\tilde{N}_{d,\mathbf{k}}$, $N_{d,\mathbf{k}}$ differ by the same combinatorial factor, it follows of course that also $N^{\text{trop}}_{d,\mathbf{k}} = N_{d,\mathbf{k}}$. Both equalities also follow as the special case $\alpha = ()$, $\beta = (d)$ from our Caporaso-Harris formulas (see Remark IV.17).

4.3.2 Relative Tropical Descendant Gromov-Witten Invariants

For two sequences α and β with $d = I(\alpha + \beta)$ let

$$\Delta(\alpha,\beta) = \{\underbrace{(-1,0),\ldots,(-1,0)}_{\alpha_1+\beta_1}, \underbrace{(-2,0),\ldots,(-2,0)}_{\alpha_2+\beta_2}, \ldots, \\ \underbrace{(0,-1),\ldots,(0,-1)}_{d}, \underbrace{(1,1),\ldots,(1,1)}_{d}\}$$

and consider parameterized n-marked tropical curves of degree $\Delta(\alpha, \beta)$ with $m - n = |\alpha + \beta|$ marked left ends (i.e. all the left ends are marked).

Definition IV.15 ($\tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ and $N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$)**.** Let $d \geq 1$, and let $\mathbf{k}$ be a sequence with $I(\alpha + \beta + \mathbf{k}) = 3d - 1 + |\beta| - |\mathbf{k}|$. Furthermore, for $n = |\mathbf{k}|$ fix a vector

$(a_1, \ldots, a_n)$ containing each $i \geq 0$ exactly $\mathbf{k}_i$ times. Let $p_1, \ldots, p_n \in \mathbb{R}^2$ be points and $y_{n+1}, \ldots, y_{n+|\alpha|}$ be y-coordinates in general position (analogously to Definitions 3.2 and 9.7 of [25]). For all $i = n+1, \ldots, n+|\alpha|$ choose a weight μ_i such that in total we have chosen each weight $k \geq 1$ exactly α_k times. In the same way, choose weights μ_i for $i = n+|\alpha|+1, \ldots, n+|\alpha+\beta|$ so that in total we have chosen each weight $k \geq 1$ exactly β_k times.

We then define

$$\tilde{N}^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta) \stackrel{\mathrm{def}}{=} \sum_C \frac{1}{I^\alpha} \operatorname{mult}(C),$$

where the sum is taken over all tropical curves $C = (\Gamma, x_1, \ldots, x_m, h)$ with $m - n = |\alpha + \beta|$ marked left ends (i.e. all left ends are marked) of degree $\Delta(\alpha, \beta)$ satisfying

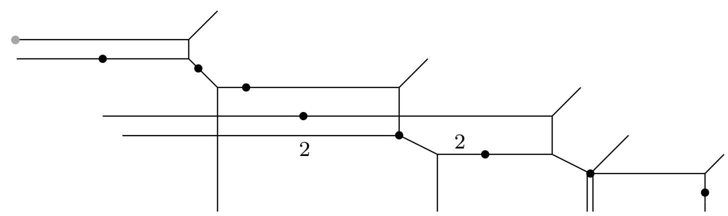
- $h(x_i) = p_i$ for all $i = 1, \ldots, n$;
- the end x_i is adjacent to a vertex of valence $a_i + 3$ for all $i = 1, \ldots, n$;
- for $i = n+1, \ldots, n+|\alpha|$, the y-coordinate of $h(x_i)$ equals y_i;
- for $i = n+1, \ldots, n+|\alpha+\beta|$, the marked end x_i is of weight μ_i, i.e. we have $v(x_i) = (-\mu_i, 0)$.

Again, it follows from the general position of the points that all other vertices of Γ are 3-valent.

We also define the numbers $N^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta)$ analogously to Definition IV.13 as numbers of tropical curves passing through the given points, with $\mathbf{k}_i$ contracted ends whose adjacent vertex has valence $i+3$ for all i, with non-marked left ends of the specified weights, and satisfying that the prescribed set of y-coordinates for a given weight are the y-coordinates of left ends of this weight. The curves are counted with multiplicity $\frac{1}{I^\alpha} \operatorname{mult}(C)$ as above. The numbers $N^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta)$ and $\tilde{N}^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta)$ are related by $N^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta) = \frac{1}{\beta!} \frac{|\mathbf{k}|}{\mathbf{k}!} \cdot \tilde{N}^{\mathrm{trop}}_{d,\mathbf{k}}(\alpha, \beta)$.

Even though tropical descendant Gromov-Witten invariants are defined in [25] only in the non-relative case, a completely analogous argument shows that the numbers $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ can also be interpreted as intersection products of evaluation pull-backs and Psi-classes on a suitable moduli space of tropical curves. Hence we can think of these numbers as *tropical relative descendant Gromov-Witten invariants.*

Example IV.16. The following curve contributes to $N^{\text{trop}}_{5,(6,1,0,1)}((1),(2,1))$ with multiplicity $\frac{1}{2}\cdot 2\cdot 2 = 2$. We have drawn a grey dot at the end of the up most left end in order to indicate that its y-coordinate is fixed.



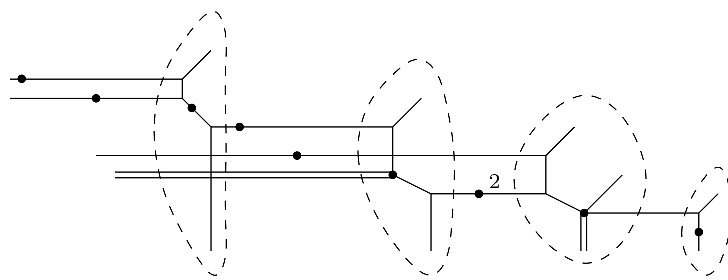
Remark IV.17 (The equality $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta) = \tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$). There is no direct correspondence known between the numbers $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ and $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$. However, we prove in Theorem IV.33 that $N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta)$, and we show in Theorem IV.35 and Corollary IV.8 that the numbers $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta)$ and $N_{d,\mathbf{k}}(\alpha,\beta)$ satisfy the same recursive relation. It follows that $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = N_{d,\mathbf{k}}(\alpha,\beta)$ and thus also that $N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta) = N_{d,\mathbf{k}}(\alpha,\beta)$. Of course, the analogous statements hold for the numbers $\tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ and $\tilde{N}_{d,\mathbf{k}}(\alpha,\beta)$ as well.

4.4 Psi-Floor Diagrams

4.4.1 Absolute Psi-Floor Diagrams

Floor diagrams, introduced by E. Brugallé and G. Mikhalkin [4, 5], are enriched directed graphs which, if counted correctly, enumerate plane curves satisfying certain point and tangency conditions. In the following, we generalize this definition to Psi-floor diagrams, and prove that they enumerate tropical plane curves satisfying point, tangency, and Psi-conditions. Let us begin with an example motivating in which sense floor diagrams extract the combinatorial essence of a tropical curve. For a precise description how to decompose a Psi-marked tropical curve into a Psi-floor diagram see the proof of Theorem IV.25.

Return to Example IV.12. There we have already chosen a horizontally stretched configuration (see [9, Definition 3.1], they use vertically stretched). So we expect the tropical curve to decompose into floors, and the floors are connected by horizontal edges only. Let us point this out in the example:



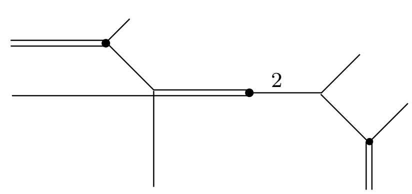
Each floor is fixed by one point, and the horizontal edges which are not adjacent to a Psi-point are also fixed by a point. We can already see that the presence of points satisfying Psi-conditions may lead to multiple floors — the second floor from the right is of degree 2, since it contains two ends of direction $(0, -1)$ resp. $(1, 1)$. The marked Psi-floor diagram of this curve can be found in step 3 of Definition IV.21.

In the original setting of floor diagrams [4, 5, 9] there are only single floors with one end of direction $(0, -1)$ and one of direction $(1, 1)$. There the idea is to shrink each floor to one vertex, and then first consider a weighted graph on the vertex set of all floors (a floor diagram). The weights of the edges correspond to the weights of the corresponding edges of the tropical curve. One obtains the "marking" of the floor diagram by adding in the ends and points on horizontal edges. Since any direction vector of an edge inside a floor has y-coordinate 1, a horizontal edge of weight i has to end at two vertices of multiplicity i each. Therefore, the multiplicity of a floor diagram equals the product over the squares of these weights.

Our setting is similar, but differs in a few features which we address now before giving the precise definition. We have seen already that multiple floors can occur. Consider a contracted end with Psi-condition ψ^a in a multiple floor of degree d' (i.e. d' ends of direction $(0,-1)$ resp. $(1,1)$ belong to the floor). If we remove the contracted end from the abstract graph, we produce $a+2$ connected components. Therefore, we must have $a+2 \geq 2d'$ (the *string inequality*), since otherwise there would be a connected component which contains two ends, and thus a string (see [16, Definition 3.5]), in contradiction to the general position of the points.

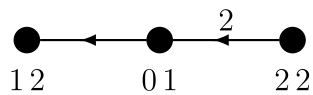
As explained above, a multiple floor of degree d' has d' ends of direction $(0,-1)$ and $(1,1)$. Furthermore, it has some "incoming edges" of directions $(-m,0)$ and some "outgoing edges" of directions $(m,0)$ (for some $m \in \mathbb{Z}_{>0}$). Thus the balancing condition for the x-coordinate implies that the sum of the weights of the incoming edges equals the sum of the weights of the outgoing edges plus d'. This will be called the *divergence condition* of the floor diagram. Note however that we do not draw left ends of the tropical curve in the floor diagram. Therefore the divergence condition will be an inequality (that determines how many left ends are adjacent to a floor) and not an equality.

Psi-points do not need to lie on floors — they can also lie on horizontal edges, as the following picture shows.



Since there may be bounded edges from other floors adjacent to such a Psi-point on a horizontal edge, we have to include these points in the underlying floor diagram.

Therefore, we introduce degree-0 vertices corresponding to these points. As we do not draw ends in the floor diagram, the valence of such a degree zero vertex has to be the correct one after adding the ends. The Psi-floor diagram (for details see below) of the tropical curve above is



Here is the formal definition:

Definition IV.18. A *(rational) Psi-floor diagram* $\mathcal{D}$ is a connected, directed graph (V, E) of genus 0 on a linearly ordered vertex set $(V, <)$ with edge weights $\omega(e) \in \mathbb{Z}_{>0}$ for all edges $e \in E$, together with pairs $(d_v, a_v) \in \mathbb{Z}^2_{\geq 0}$ for each vertex v in V (which we call the *degree* d_v and the *Psi-power* a_v of v), satisfying:

1. The edge directions preserve the vertex order, i.e. for every edge $v \to w$ we have $v < w$.

2. There are no edges between degree-0 vertices, i.e. if $v \to w$ is an edge then $d_v > 0$ or $d_w > 0$.

3. For each $v \in V$ at least one of the numbers d_v and a_v is positive.

4. For each $v \in V$ we have $a_v - 2(d_v - 1) \geq 0$ (string inequality).

5. (Divergence condition) For every vertex v we have

$$\operatorname{div}(v) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ v \xrightarrow{e} w}} \omega(e) - \sum_{\substack{\text{edges } e \\ w \xrightarrow{e} v}} \omega(e) \leq d_v.$$

   This means that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most d_v than the total weight of the incoming edges.

6. If $d_v = 0$ for a vertex v, then $\operatorname{val}(v) = a_v + 2 + \operatorname{div}(v)$ (where $\operatorname{val}(v)$ is the valence of v).

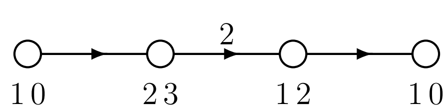
We call $d(\mathcal{D}) = \sum_{v \in V} d_v$ the *degree* of a Psi-floor diagram $\mathcal{D}$. A *floor* of $\mathcal{D}$ is a vertex of positive degree. The *type* of $\mathcal{D}$ is $\mathbf{k}(\mathcal{D}) = (\mathbf{k}_0, \mathbf{k}_1, \dots)$, where $\mathbf{k}_i$ is the number of vertices v of $\mathcal{D}$ with $a_v = i$ for all $i \geq 1$, and $\mathbf{k}_0$ is the number of vertices v with $a_v = 0$ plus $3d - 1 - I\mathbf{k} - \#V$. The number $3d - 1 - I\mathbf{k} - \#V$ that we add to $\mathbf{k}_0$ equals the number of vertices of Psi-power 0 that we will add later and which makes the equality $I\mathbf{k} = 3d - 1 - |\mathbf{k}|$ hold. The *multiplicity* $\mu(\mathcal{D})$ of $\mathcal{D}$ is given by

$$\mu(\mathcal{D}) \stackrel{\text{def}}{=} \prod_{\text{edges } e} \omega(e)^2 \prod_{\substack{v \xrightarrow{e} w \\ \text{s.t. } d_v = 0 \\ \text{or } d_w = 0}} \frac{1}{\omega(e)} \prod_{v:\, d_v = 0} \frac{1}{|\operatorname{div}(v)|\,!}.$$

The first factor in the definition of multiplicity corresponds, as in the original definition of floor diagram, to vertices adjacent to edges of higher weight. If an edge of higher weight is adjacent to a contracted end however (e.g. at a vertex of degree 0), this vertex does not contribute and so we have to divide out by one factor of $\omega(e)$ again. The last factor contributes to the factor ν_C in Definition IV.10 of the multiplicity of a tropical curve, which arises because ends of the same direction are adjacent to a vertex.

We draw Psi-floor diagrams using the convention that vertices in increasing order are arranged left to right, thereby adopting the convention of [9]. Note that in this chapter we draw the corresponding tropical curves in the *opposite* direction. We write the pair (d_v, a_v) below each vertex v. Edge weights of 1 are omitted.

Example IV.19. An example of a Psi-floor diagram $\mathcal{D}$ of degree $d = 5$, type $\mathbf{k} = (2, 0, 1, 1)$, divergences $1, 1, -1, -1$, and multiplicity $\mu(\mathcal{D}) = 4$ is drawn below.



Given a Psi-floor diagram $\mathcal{D}$ we define, for every floor v, the sets $I(v)$ and $O(v)$

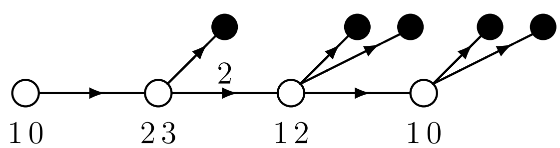
by

$$I(v) \stackrel{\text{def}}{=} \{w \to v : d_w > 0\},$$

$$O(v) \stackrel{\text{def}}{=} \{v \to w : d_w > 0\} \cup \coprod \{v \xrightarrow{1} \circ\},$$

where the latter set is a disjoint union of the outgoing edges of $\mathcal{D}$ at v augmented by $d_v - \operatorname{div}(v)$ many *indistinguishable* edges of weight 1 directed away from v ending in distinct vertices $\circ$. These indistinguishable extra ends correspond to left ends of the tropical curve starting at this floor.

Example IV.19 (continued). We draw the sets $I(v)$ and $O(v)$ by augmenting the Psi-floor diagrams at the respective vertices. If, for example, v is the third black vertex from the left, then $O(v)$ consists of the edge between v and the fourth black vertex and the two edges of weight 1 connecting v with the two adjacent white vertices.



An *edge choice* is a collection $\mathcal{C}(\mathcal{D})$ of subsets $C(v) \subset I(v) \cup O(v)$, one for each floor v of $\mathcal{D}$, satisfying $|C(v)| = a_v - 2(d_v - 1)$, and such that $C(v) \cap C(w) = \emptyset$ for distinct floors v and w. If $d_v = 0$ for a vertex v we set $C(v) = \emptyset$. The *local multiplicity at* v of such a choice is

$$\mu_{v,C(v)} \stackrel{\text{def}}{=} \begin{cases} \frac{d_v^{i(v)}}{d_v!} \cdot \frac{d_v^{o(v)}}{d_v!} & \text{if } d_v > 0, \\ 1 & \text{if } d_v = 0. \end{cases}$$

where $i(v) = |I(v) \backslash C(v)|$ and $o(v) = |O(v) \backslash C(v)|$ are the number of non-chosen edges in $I(v)$ and $O(v)$, respectively.

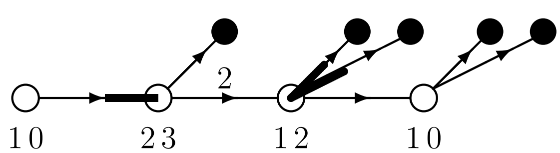
The chosen edges will later correspond to the edges of the tropical curve that are directly adjacent to the Psi-point; the non-chosen edges to those belonging to the floor but not directly adjacent to the Psi-point. We will see later in Lemma IV.26 and the proof of Theorem IV.25 that the local multiplicity at v of an edge choice takes the possibilities for the degree-d_v floor and the contribution to the multiplicity ν_C of Definition IV.10 into account.

The *multiplicity* $\mu(\mathcal{C})$ of the edge choice $\mathcal{C}(\mathcal{D})$ of the Psi-floor diagram $\mathcal{D}$ is

$$\mu(\mathcal{C}) \stackrel{\text{def}}{=} \prod_{v\in V} \mu_{v,C(v)} \frac{1}{|C(v)\cap\{v\to\circ\}|\,!} \prod_{e\in C(v)} \frac{1}{\omega(e)}.$$

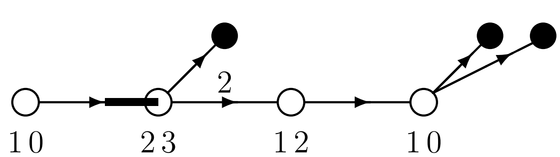
As before, the multiplicity of an edge choice takes for each floor a combination of contributions to ν_C and possibilities for a floor into account, furthermore additional contributions to ν_C and factors of $\frac{1}{\omega(e)}$ that arise because an edge of weight $\omega(e)$ is adjacent to a contracted end.

Example IV.20. We picture an edge choice $\mathcal{C}(\mathcal{D})$ by thickening all edges in $C(v)$ at v, for all vertices v of $\mathcal{D}$. Below is an edge choice for the Psi-floor diagram of Example IV.19. Its multiplicity is $\mu(\mathcal{C}) = \frac{1}{2}$. Notice that $|C(v)| = a_v - 2(d_v - 1)$ for all v since none of the vertices has degree zero.



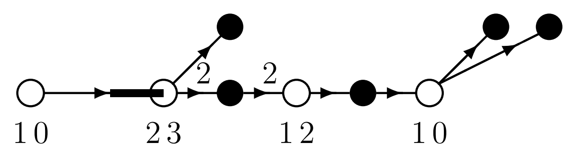
Definition IV.21. A *marking* of a Psi-floor diagram $\mathcal{D}$ with an edge choice $\mathcal{C}$ is defined by the following three-step process which we will illustrate in the case of Example IV.20.

Step 1: For each vertex v of $\mathcal{D}$ create $d_v - \operatorname{div}(v) - |C(v)\cap\{v\to\circ\}|$ many new vertices in $\mathcal{D}$ and connect them to v with new edges directed away from v.



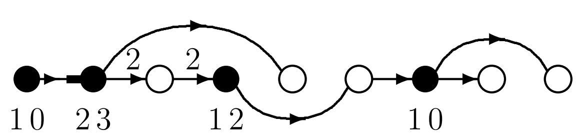
These correspond exactly to the non-chosen edges $v \to \circ$ above, i.e. to the left ends of the tropical curve that are not directly adjacent to the Psi-point in the floor (and therefore have to be fixed later by a point condition).

Step 2: Subdivide each non-chosen edge of the original Psi-floor diagram $\mathcal{D}$ between floors into two directed edges by introducing a new vertex for each such edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



These extra vertices correspond to points on horizontal bounded edges with no Psi-condition.

Step 3: Order the vertices of $\tilde{\mathcal{D}}$ linearly, extending the order of the vertices of the original Psi-floor diagram $\mathcal{D}$, such that (as in $\mathcal{D}$) each edge is directed from a smaller vertex to a larger vertex.



The extended graph $\tilde{\mathcal{D}}$ together with the linear order on its vertices is called a *marked Psi-floor diagram*, or a *marking* of the Psi-floor diagram $\mathcal{D}$.

We added $d_v - \operatorname{div}(v)$ white end vertices for each $v \in V$ before picking the edge choice. It follows by induction that altogether we add d white end vertices. However, in step 1 of Definition IV.21 we really only add the non-chosen ones. In step 2 we subdivide each of the non-chosen edges. There are $\#V - 1$ edges, since the Psi-floor diagram is a rational graph. Thus, altogether we add $d + \#V - 1$ minus the number of chosen edges white vertices, i.e. $d+\#V-1-\sum_{v\in V}(a_v-2(d_v-1)) = 3d-1-I\mathbf{k}-\#V$. It follows that $\mathbf{k}_0$ equals the number of vertices v of the floor diagram with $a_v = 0$ plus the number of white vertices in the marking.

We want to count marked Psi-floor diagrams up to equivalence. Two such $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ are *equivalent* if $\tilde{\mathcal{D}}_1$ can be obtained from $\tilde{\mathcal{D}}_2$ by permuting edges without changing their weights, i.e. if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$.

The *number of markings* $\nu(\mathcal{D},\mathcal{C})$ is the number of marked Psi-floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. In the example, we have $\nu(\mathcal{D},\mathcal{C}) = 7$: the white 1-valent vertex adjacent to the second black vertex (counted from the left) can be inserted in 2 ways between the second and third black vertex, in 2 ways between the third and fourth black vertex, and in 3 ways right of the fourth black vertex.

By specializing to the case $a_v = 0$ for all vertices v of $\mathcal{D}$ we recover the definition of floor diagrams and their markings of S. Fomin and G. Mikhalkin [9]. In this case all floors necessarily have degree $d_v = 1$ and no edges get chosen (so $C(v) = \emptyset$ for all vertices v).

Definition IV.22 ($N^{\text{floor}}_{d,\mathbf{k}}$ and $\tilde{N}^{\text{floor}}_{d,\mathbf{k}}$)**.** Let $d \geq 1$ and $\mathbf{k}$ be a sequence of non-negative integers with $I\mathbf{k} = 3d-1-|\mathbf{k}|$. Set

$$N^{\text{floor}}_{d,\mathbf{k}} \stackrel{\text{def}}{=} \sum_{\mathcal{D}} \mu(\mathcal{D}) \sum_{\mathcal{C}} \mu(\mathcal{C})\, \nu(\mathcal{D},\mathcal{C}),$$

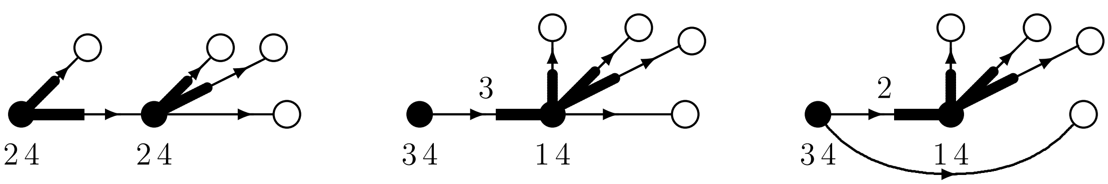
where the first sum is over all Psi-floor diagrams of degree d and type $\mathbf{k}$, and the second sum is over all edge choices $\mathcal{C}$ of $\mathcal{D}$. Correspondingly (see Definition IV.3), we set $\tilde{N}^{\text{floor}}_{d,\mathbf{k}} \stackrel{\text{def}}{=} \frac{\mathbf{k}!}{|\mathbf{k}|!} N^{\text{floor}}_{d,\mathbf{k}}$.

Remark IV.23. We can also define the numbers $\tilde{N}^{\text{floor}}_{d,\mathbf{k}}$ directly using Psi-floor diagrams by requiring that the Psi-powers of the vertices of the marked Psi-floor diagram (the Psi-powers of the white vertices that are not present in the underlying Psi-floor diagram have Psi-power 0) occur in a particular order, and by marking the white end vertices with numbers from 1 to d.

Example IV.24. As an example in degree $d = 4$ we compute the number

$$\tilde{N}^{\text{floor}}_{4,(1,0,0,0,2)} = \frac{1}{4}.$$

There are three markings of Psi-floor diagrams of degree 4 and type $(1, 0, 0, 0, 2)$ which have the Psi-powers in the order $(a_1, a_2, a_3) = (0, 4, 4)$. (Remember that we draw Psi-floor diagrams from left to right and therefore need to invert the order of the a_i.) Every other order of the a_i yields the same answer.



$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2!} = \frac{1}{8}$ $\qquad$ $9 \cdot \frac{1}{3} \cdot \frac{3}{3!} \cdot \frac{3}{3!} \cdot \frac{1}{3!} = \frac{1}{24}$ $\qquad$ $4 \cdot \frac{1}{2} \cdot \frac{1}{3!} \cdot \frac{9}{3!} \cdot \frac{1}{3!} = \frac{1}{12}$

The contribution of the third diagram, for example, arises as follows: The underlying Psi-floor diagram has multiplicity 4. Choosing the weight-2 edge amounts to a factor of $\frac{1}{2}$. The degree-3 vertex has no non-chosen incoming edges and 2 non-chosen outgoing edges, hence the local multiplicity at this vertex is given by the next two factors. Lastly, as all 3 additional edges at the second vertex are chosen, we need to multiply by $\frac{1}{3!}$.

4.4.2 The equality $N^{\text{floor}}_{d,\mathbf{k}} = N^{\text{trop}}_{d,\mathbf{k}}$

Theorem IV.25. *Let $d \geq 1$ and $\mathbf{k}$ be a sequence of non-negative integers with $I\mathbf{k} = 3d - 1 - |\mathbf{k}|$. Then $N^{\text{floor}}_{d,\mathbf{k}} = N^{\text{trop}}_{d,\mathbf{k}}$.*

For the proof of Theorem IV.25 we need the following lemma. For positive integers

a and b, let $S(a, b)$ denote the *Stirling number of the second kind*, i.e. the number of ways of partitioning an a-element set into b non-empty parts.

Lemma IV.26 ([33], (24d))**.** *For integers $e, f \geq 0$ it holds that*

$$\sum_{0 \leq g \leq f} \frac{S(e,g)}{(f-g)!} = \frac{f^e}{f!}.$$

Proof of Theorem IV.25. Pick a horizontally stretched configuration of $|\mathbf{k}|$ points (see [9, Definition 3.1]). Our strategy is as follows: let T be the set of tropical curves of degree d satisfying the conditions, and let F be the set of marked floor diagrams of degree d and type $\mathbf{k}$. We will define a (surjective) map from T to F. Let r be the number of inverse images of a given marked floor diagram $\tilde{\mathcal{D}}$ in F. We will show that each such inverse image is a tropical curve C of the same multiplicity $\mathrm{mult}(C)$, and that $\mathrm{mult}(C) \cdot r = \mu(\mathcal{D}) \cdot \mu(\mathcal{C})$, where $\mathcal{D}$ denotes the underlying floor diagram for $\tilde{\mathcal{D}}$ and $\mathcal{C}$ denotes its choice of edges. Of course, this will then prove the lemma.

Consider a tropical curve in T; we will now how to construct the corresponding marked floor diagram in F. As in [15, Theorem 4.3] resp. of [5, Section 5] it follows that the tropical curve decomposes into floors in the sense that each connected component of Γ minus the horizontal edges (i.e. each floor) is fixed by exactly one point. (A floor can have higher degree here.) For each floor v let d_v denote its number of ends of direction $(0, -1)$ and a_v the power of Psi of the contracted end (i.e. the valence of the adjacent vertex minus 3). Shrink each floor to a vertex labeled with (d_v, a_v). If there is a contracted end with a Psi-condition on a horizontal edge, also keep this as a vertex and set $d_v = 0$, and a_v the power of Psi. Let the edges of the floor diagram be given by the horizontal bounded edges of the tropical curve connecting the floors. We orient the edges towards the left ends of the tropical curve, and reverse the picture (so the left ends are on the right, and edges are oriented to the right). Because of the general position of the points there cannot be two contracted ends mapped to a horizontal line — thus there cannot be any edges between vertices of degree 0. If $d_v = 0$ for a vertex we know that the corresponding contracted end has a Psi-condition, so then $a_v > 0$. Of course, if $a_v = 0$ then we must have $d_v > 0$. If there are horizontal ends adjacent to a contracted end on a floor resp. to a contracted end with higher Psi-condition on a horizontal edge, drop them. The other horizontal ends must be adjacent to a contracted end without a Psi-condition; keep the contracted end as a white end vertex. Also draw white vertices on horizontal edges for contracted ends without a Psi-condition on horizontal edges. Thicken the horizontal edges which are directly adjacent to a contracted end on a floor. A vertex of degree 0 in the floor diagram comes from a contracted end with a Psi-condition, say of power a_v, on a horizontal edge. Since the tropical curve is balanced, the sum of the weights of the incoming horizontal edges must equal the sum of the weights of the outgoing. The divergence condition for degree-0 vertices follows. The valence must be $a_v + 2$ (without counting the contracted end itself). We have dropped the ends adjacent to this vertex however, so we have $\mathrm{val}(v) - \mathrm{div}(v) = a_v + 2$. Now let v be a vertex of the floor diagram with $d_v > 0$. This vertex comes from a floor of the tropical curve which contains a contracted end with Psi-power a_v. If we remove the contracted end from Γ we produce $a_v + 2$ connected components. The floor contains $2d_v$ ends of direction $(0, -1)$ resp. $(1, 1)$. These ends must belong to different connected components since otherwise there would be a string (see [16, Definition 3.5]) in contradiction to the general position of the points. It follows that $a_v + 2 \geq 2d_v$ (string inequality) and that $a_v + 2 - 2d_v$ horizontal edges are directly adjacent to the contracted end, and thus get chosen (including ends, which we drop). For a vertex of the floor diagram with $d_v > 0$, the balancing condition in the x-coordinate tells us that the divergence condition holds. It follows that we have produced a marked Psi-floor diagram in F for the tropical curve in T.

Conversely, let now $\tilde{\mathcal{D}}$ be a marked floor diagram in F; we will construct its inverse images in T. For each white vertex and for each vertex of degree 0 draw horizontal edges of the appropriate weight through the corresponding point p_i. For a vertex of degree $d_v > 0$ there are several possibilities how it can be completed to a floor of a tropical curve. We have seen already that — locally around such a floor of a tropical curve — removing the contracted end produces a_v+2 connected components of which $a_v + 2 - 2d_v$ are horizontal edges and $2d_v$ are connected components containing one of the $2d_v$ ends of direction $(0, -1)$ resp. $(1, 1)$. There are $o(v)$ non-chosen outgoing horizontal edges connected to this floor. Their y-coordinates are fixed by other conditions. Thus they are distinguishable in the tropical curve, even if they are of the same weight. These edges must belong to the connected components containing the ends of direction $(1, 1)$. Assume that g of the d_v connected components containing the ends $(1, 1)$ also contain horizontal edges, whereas $d_v - g$ ends of direction $(1, 1)$ are directly adjacent to the contracted end. Thus we need to partition the set of $o(v)$ horizontal non-chosen edges into g non-empty parts, corresponding to the g connected components. For each such choice there is exactly one possibility to complete the picture to the upper part of a floor of a tropical curve since the y-coordinates of the horizontal edges are fixed by other points. This part of the tropical curve contributes a factor of $\frac{1}{(d_v-g)!}$ to the factor ν_C of the multiplicity of the tropical curve because of the $d_v - g$ ends of direction $(1, 1)$ which are directly adjacent to the contracted end. Thus we can sum up the possibilities with their contribution to ν_C as $\frac{S(o(v),g)}{(d_v-g)!}$ for each g. Summing over all g, we get $\frac{d_v^{o(v)}}{d_v!}$ by Lemma IV.26. This situation is illustrated in Example IV.27.

The analogous statement holds for the lower part of the floor of the tropical curve and the incoming horizontal edges. For any choice of g and a partition (both for the upper and the lower part of each floor) we can complete the picture uniquely to a tropical curve.

The multiplicity of the tropical curve is a product of factors contributing to ν_C and vertex multiplicities. We have taken care of the factors contributing to ν_C inside each floor already. There can still be left ends adjacent to the same vertex that contribute to ν_C. This happens either if left ends are adjacent to vertices of degree 0 in the floor diagram, or if they are directly adjacent to a contracted end inside a floor, i.e. chosen. For the first situation, we get a factor of $\frac{1}{\mathrm{div}(v)!}$, for the second situation we get a factor of $\frac{1}{|C(v)\cap\{v\to\circ\}|\,!}$. Now let us consider the vertex multiplicities. We have seen already that each floor consists of components with one end of direction $(1,1)$ resp. $(0,-1)$, and horizontal edges. The y-coordinate of any direction of an edge of such a component is therefore 1, and thus any vertex adjacent to a horizontal edge of weight $\omega(e)$ is of multiplicity $\omega(e)$. If a horizontal edge is adjacent to a contracted end however, this vertex does not contribute. If this contracted end comes from a white vertex however, there is another horizontal edge of the same weight adjacent to it. Thus, any horizontal edge in the floor diagram (without the marking) will contribute $\omega(e)^2$, unless it is adjacent to a vertex of degree 0, or unless it gets chosen later — in each of these cases it contributes only $\omega(e)$.

It follows that all inverse images of a marked floor diagram are tropical curves of the same multiplicity $\mathrm{mult}(C)$, and if there are r inverse images we have $\mathrm{mult}(C)\cdot r = \mu(\mathcal{D})\mu(\mathcal{C})$. $\square$

Example IV.27. Figure 4.3 illustrates how we can complete a vertex of a marked Psi-floor diagram to floors of a tropical curve. The local picture of $\tilde{\mathcal{D}}$ on the left

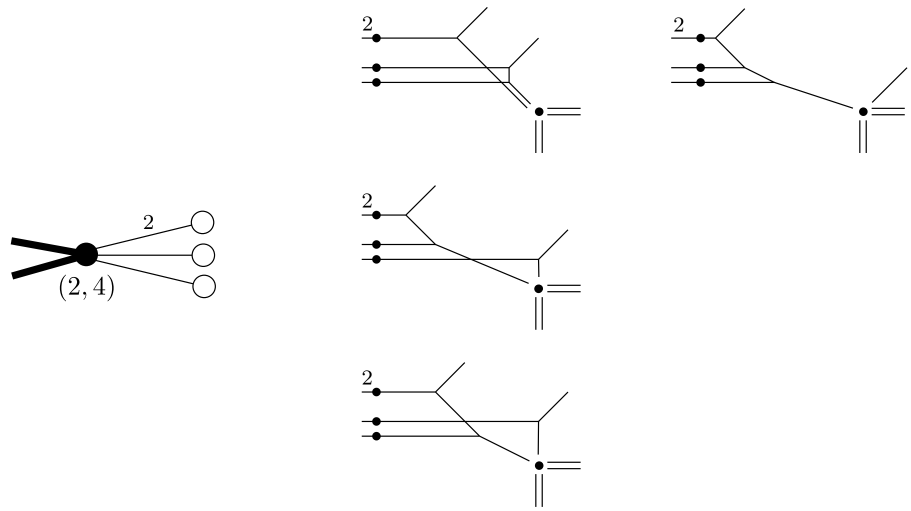


Figure 4.3: An example of a double floor which (locally) corresponds to several tropical curves.

shows 2 chosen incoming edges and 3 non-chosen outgoing edges adjacent to a floor of degree 2. The local multiplicity of this edge choice equals $\frac{2^0}{2!} \cdot \frac{2^3}{2!} = \frac{1}{2!} \cdot 4$. We would like to complete this picture to the floor of a tropical curve. The lower part is unique. The factor of $\frac{1}{2!}$ for the lower part takes care of the two down ends which are adjacent to the contracted end and thus lead to a contribution of $\frac{1}{2!}$ in the factor ν_C. For the upper part there are several possibilities. The middle column shows the $S(3,2) = 3$ possibilities for $g = 2$, i.e. for the case where all components obtained after removing the contracted marked edge also contain horizontal edges. The right column shows the $S(3,1) = 1$ possibility for $g = 1$, i.e. for the case where one of the ends of direction $(1,1)$ is directly adjacent to the contracted end.

Remark IV.28. It follows immediately that also $\tilde{N}^{\text{floor}}_{d,\mathbf{k}} = \tilde{N}^{\text{trop}}_{d,\mathbf{k}}$ by taking the order of the contracted ends resp. vertices into account, both for the tropical curves and the floor diagrams.

4.4.3 Relative Psi-Floor Diagrams

We now define relative analogues of Psi-floor diagrams and their markings. Fix two sequences α and β. Our notation, which is more convenient for our purposes, differs from [9], where relative floor diagrams and their markings were defined relative to partitions $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ and $\rho = (1^{\beta_1} 2^{\beta_2} \cdots)$.

Let $\mathcal{D}$ be a Psi-floor diagram of degree $d = I(\alpha + \beta)$. A pair $(\{\alpha(v)\}, \{\beta(v)\})$ of collections of sequences, where v runs over the vertices of $\mathcal{D}$, is called *compatible* with $\mathcal{D}$ and (α, β), if it satisfies:

1. The sums over each collection satisfy $\sum_{v \in V} \alpha(v) = \alpha$ and $\sum_{v \in V} \beta(v) = \beta$.

2. For all vertices v of $\mathcal{D}$ it holds that $I(\alpha(v) + \beta(v)) = d_v - \operatorname{div}(v)$.

3. If $d_v = 0$ then we require in addition that $|\alpha(v)| = 0$ and $|\beta(v)| = a_v + 2 - \operatorname{val}(v)$.

The sequences $\alpha(v)$ and $\beta(v)$ correspond to the left (fixed and non-fixed) ends adjacent to each floor. For a vertex of degree 0, all adjacent edges are directly adjacent to the contracted end, and thus there cannot be any fixed ends in this case.

In the non-relative case, i.e. when $\alpha = ()$ and $\beta = (d)$, it necessarily follows that $\alpha(v) = ()$ and $\beta(v) = (1 - \operatorname{div}(v))$ for all vertices v of $\mathcal{D}$.

The *(relative) type* $\mathbf{k}(\mathcal{D}) = (\mathbf{k}_0, \mathbf{k}_1, \ldots)$ of a Psi-floor diagram $\mathcal{D}$ is defined as follows: for all $i \geq 1$ let $\mathbf{k}_i$ be the number of vertices v of $\mathcal{D}$ with $a_v = i$. Set $\mathbf{k}_0$ to be the number of vertices with $a_v = 0$ plus $2d + |\beta| - 1 - I\mathbf{k} - \#V$. The latter number equals the number of white vertices that we will add. This makes the equalities $|\mathbf{k}| = 2d + |\beta| - 1 - I\mathbf{k}$, resp. $I(\alpha + \beta + \mathbf{k}) = 3d - 1 + |\beta| - |\mathbf{k}|$ hold, where the latter is equivalent to the former since $d = I(\alpha + \beta)$.

The *relative multiplicity* of a Psi-floor diagram $\mathcal{D}$ together with a collection of sequences $\{\beta(v)\}$ is

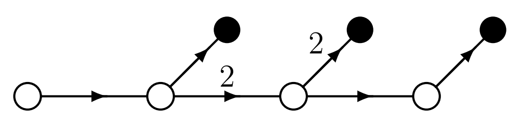
$$\mu^{\mathrm{rel}}(\mathcal{D}) = \mu^{\mathrm{rel}}(\mathcal{D}, \{\beta(v)\}) \overset{\mathrm{def}}{=} I^{\beta} \cdot \prod_{\text{edges } e} \omega(e)^2 \cdot \prod_{\substack{v \xrightarrow{e} w \\ \text{s.t. } d_v = 0 \\ \text{or } d_w = 0}} \frac{1}{\omega(e)} \prod_{v:\, d_v = 0} \frac{1}{\beta(v)!}. \tag{4.4}$$

For a collection of sequences $\{\beta(v)\}$ and a vertex v of $\mathcal{D}$ we define the sets $I^{\mathrm{rel}}(v)$ and $O^{\mathrm{rel}}(v)$ by

$$I^{\mathrm{rel}}(v) \overset{\mathrm{def}}{=} \{w \to v : d_w > 0\},$$
$$O^{\mathrm{rel}}(v) \overset{\mathrm{def}}{=} \{v \to w : d_w > 0\} \cup \coprod \{v \xrightarrow{i} \circ\},$$

where the latter is a disjoint union of the outgoing edges of $\mathcal{D}$ at v augmented by β_i^v *indistinguishable* edges of weight i for all $i \geq 1$, directed away from v and ending in distinct vertices $\circ$. These indistinguishable edges correspond to the non-fixed ends of the tropical curve adjacent to a floor, which a priori could be adjacent to the contracted end, and therefore can be chosen.

Example IV.29. Below we have indicated the sets $I^{\mathrm{rel}}(v)$ and $O^{\mathrm{rel}}(v)$ in the case of the Psi-floor diagram of Example IV.19 with $\alpha = (1)$, $\beta = (2, 1)$, and all $\alpha(v)$ and $\beta(v)$ being the zero sequence unless indicated otherwise. The relative multiplicity $\mu^{\mathrm{rel}}(\mathcal{D}, \{\beta(v)\})$ is $4 \cdot 2 = 8$.



$$\begin{array}{rcccc}
(d_v, a_v) = & 1\,0 & 2\,3 & 1\,2 & 1\,0 \\
\alpha(v) = & & & & (1) \\
\beta(v) = & & (1) & (0,1) & (1)
\end{array}$$

As before, an *edge choice* $\mathcal{C}(\mathcal{D})$ is given by a subset $C(v) \subset I^{\mathrm{rel}}(v) \cup O^{\mathrm{rel}}(v)$ for each floor v of $\mathcal{D}$ such that $|C(v)| = a_v + 2 - 2d_v$ for all v, and $C(v) \cap C(w) = \emptyset$ for distinct floors v and w. If $d_v = 0$, we set $C(v) = \emptyset$. The *local multiplicity at* v of such a choice is

$$\mu^{\mathrm{rel}}_{v,C(v)} \overset{\mathrm{def}}{=} \begin{cases} \frac{d_v^{i(v)}}{d_v!} \cdot \frac{d_v^{o(v)}}{d_v!} & \text{if } d_v > 0, \\ 1 & \text{if } d_v = 0, \end{cases} \tag{4.5}$$

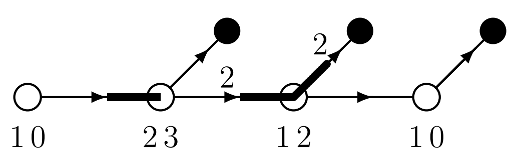
where, similarly to the absolute case, $i(v) = |I^{\mathrm{rel}}(v) \backslash C(v)|$ is the number of non-chosen incoming edges and $o(v) = |O^{\mathrm{rel}}(v) \backslash C(v)| + |\alpha(v)|$ is the number of non-chosen edges in $O^{\mathrm{rel}}(v)$ together with some additional edges (corresponding to tangency conditions at fixed points, resp. to fixed left ends).

The *relative multiplicity* of the edge choice $\mathcal{C}$ of the Psi-floor diagram $\mathcal{D}$ together with a compatible pair of collections of sequences $(\{\alpha(v)\}, \{\beta(v)\})$ is

$$\mu^{\mathrm{rel}}(\mathcal{C}) \overset{\mathrm{def}}{=} \mu^{\mathrm{rel}}(\mathcal{C}, \{\alpha(v)\}, \{\beta(v)\}) \overset{\mathrm{def}}{=} \prod_{v \in V} \mu^{\mathrm{rel}}_{v,C(v)} \prod_{v \in V} \prod_{e \in C(v)} \frac{1}{\omega(e)} \prod_{v \in V} \frac{1}{c(v)!}, \tag{4.6}$$

where $c(v)$ is the sequence given by $c(v)_i \overset{\mathrm{def}}{=} |C(v) \cap \{v \xrightarrow{i} \circ\}|$ for $i \geq 1$.

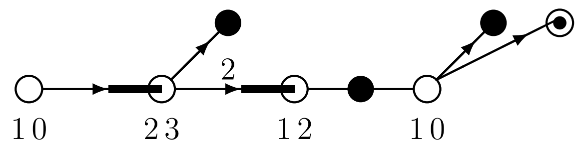
Example IV.29 (continued). An example of an edge choice for the above Psi-floor diagram together with collections $\{\alpha(v)\}$ and $\{\beta(v)\}$ is given below. As before, we indicate chosen edges by thickening edges at the vertices where they are chosen. Notice that $|C(v)| = a_v - 2(d_v - 1)$ at every vertex v since there are no vertices of degree 0. The relative multiplicity of the edge choice is $\mu^{\mathrm{rel}}(\mathcal{C}) = \frac{1}{2}$.



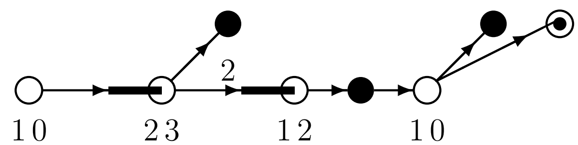
Definition IV.30. An (α, β)-*marking* of a Psi-floor diagram $\mathcal{D}$ with a compatible choice of a pair of collections $(\{\alpha(v)\}, \{\beta(v)\})$ and an edge choice $\mathcal{C}(\mathcal{D})$ is defined by the following three-step process which we illustrate in the case of Example IV.29.

Step 1: For each vertex v of $\mathcal{D}$ and every $i \geq 1$ create $\beta(v)_i - |C(v) \cap \{v \xrightarrow{i} \circ\}|$ new vertices (which we call *β-vertices* and illustrate as $\bigcirc$), and connect them to v

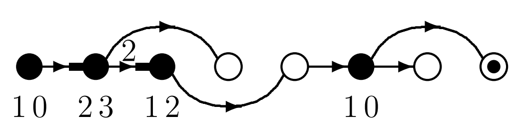
with new edges of weight i directed away from v. Similarly, create $\alpha(v)_i$ new vertices (which we call *α-vertices* and illustrate as ⦿) and connect them to v with new edges of weight i directed away from v.



Step 2: Subdivide each non-chosen edge of the original Psi-floor diagram $\mathcal{D}$ between floors into two edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



Step 3: Order the vertices of $\tilde{\mathcal{D}}$ linearly, extending the order of the vertices of the original Psi-floor diagram $\mathcal{D}$, such that (as in $\mathcal{D}$) each edge is directed from a smaller vertex to a larger vertex. Furthermore, we require that the α-vertices are largest among all vertices, and for every pair of α-vertices $v > w$ the weight of the v-adjacent edge is larger than or equal to the weight of the w-adjacent edge.



The (in this example unique) tropical curve mapping to the floor diagram above can be found in Example IV.16. As in the non-relative case, we call the extended graph $\tilde{\mathcal{D}}$ together with the linear order on its vertices an *(α, β)-marked Psi-floor diagram*, or an *(α, β)-marking* of the Psi-floor diagram $\mathcal{D}$.

In step 1 we added $|\beta|$ white vertices (of which we later remove the chosen ones), and in step 2 we subdivide the non-chosen ones of the $\#V - 1$ bounded edges. That is, altogether we added $|\beta| + \#V - 1 - \sum_{v \in V}(a_v - 2(d_v - 1)) = 2d - 1 + |\beta| - I\mathbf{k} - \#V$ white vertices.

As before, we need to count (α, β)-marked Psi-floor diagrams up to equivalence. Two (α, β)-marked Psi-floor diagrams $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ are *equivalent* if $\tilde{\mathcal{D}}_1$ can be obtained from $\tilde{\mathcal{D}}_2$ by permuting edges without changing their weights, i.e. if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$. The *number of markings* $\nu^{\mathrm{rel}}(\mathcal{D}, \mathcal{C}) = \nu^{\mathrm{rel}}(\mathcal{D}, \{\alpha(v)\}, \{\beta(v)\}, \mathcal{C})$ is the number of (α, β)-marked Psi-floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. In our running example we have $\nu^{\mathrm{rel}}(\mathcal{D}, \mathcal{C}) = 5$: the white vertex attached to the floor labeled $(2, 3)$ can be placed in the linear order at any position to the right of this floor and to the left of the α-vertex.

By specializing to the case $a_v = 0$ for all vertices v of $\mathcal{D}$ we recover the definition of (λ, ρ)-markings of floor diagrams of S. Fomin and G. Mikhalkin [9], for partitions $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ and $\rho = (1^{\beta_1} 2^{\beta_2} \cdots)$. As in the non-relative case, all floors necessarily have degree $d_v = 1$ and no edges get chosen.

Definition IV.31 ($N^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta)$ and $\tilde{N}^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta)$)**.** Let $d \geq 1$ and α, β be two sequences with $I(\alpha + \beta) = d$. Furthermore, let $\mathbf{k}$ be a sequence of non-negative integers with $I(\alpha + \beta + \mathbf{k}) = 3d - 1 + |\beta| - |\mathbf{k}|$. Set

$$N^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta) \stackrel{\mathrm{def}}{=} \sum_{\mathcal{D}, \{\alpha(v)\}, \{\beta(v)\}} \mu^{\mathrm{rel}}(\mathcal{D}) \sum_{\mathcal{C}} \mu^{\mathrm{rel}}(\mathcal{C})\, \nu^{\mathrm{rel}}(\mathcal{D}, \mathcal{C}),$$

where the first sum is over all degree d Psi-floor diagrams of type $\mathbf{k}$ and over all compatible pairs of collections $(\{\alpha(v)\}, \{\beta(v)\})$, and the second sum is over all edge choices $\mathcal{C}$ of $\mathcal{D}$. Correspondingly (see Definition IV.4), we set $\tilde{N}^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta) \stackrel{\mathrm{def}}{=} \beta! \cdot \frac{\mathbf{k}!}{|\mathbf{k}|!} N^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta)$.

Remark IV.32. As in Remark IV.23, we can also define the numbers $\tilde{N}^{\mathrm{floor}}_{d,\mathbf{k}}(\alpha, \beta)$

directly using Psi-floor diagrams. Then we require that Psi-powers of the vertices of the marked Psi-floor diagram are in a fixed order, and we mark the white end vertices.

Theorem IV.33 (The equality $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$)**.** *Let $d \geq 1$ and α, β be two sequences with $I(\alpha+\beta) = d$. Let $\mathbf{k}$ be a sequence of non-negative integers satisfying $I(\alpha+\beta+\mathbf{k}) = 3d-1+|\beta|-|\mathbf{k}|$. Then $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$.*

The proof is analogous to the proof of Theorem IV.25.

Remark IV.34. Again, it follows immediately that the same equality holds for the numbers $\tilde{N}^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = \tilde{N}^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ as well.

4.4.4 The Caporaso-Harris Formula for Floor Diagrams

Now we use Psi-floor diagrams to obtain the Caporaso-Harris type recursion of Corollary IV.8 for the numbers $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta)$. As this recursion formula determines all the numbers it follows that $N_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta)$. As we know by Theorem IV.33 that also $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$ holds, we thus have that

$$N_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta) = N^{\text{trop}}_{d,\mathbf{k}}(\alpha,\beta)$$

for all $d, \mathbf{k}, \alpha, \beta$, as claimed in Remark IV.14. We use Notation IV.2 and the notation in equation (4.3) below.

Theorem IV.35 (Caporaso-Harris formula for Psi-floor diagrams)**.** *The numbers $N^{\text{floor}}_{d,\mathbf{k}}(\alpha,\beta)$ satisfy the Caporaso-Harris recursion in Corollary IV.8.*

Proof. The basic strategy is to examine the possibilities for the largest vertex v' of an (α,β)-marking $\tilde{\mathcal{D}}$ of a Psi-floor diagram $\mathcal{D}$ of degree d and type $\mathbf{k}$ which is not an α-vertex (see step 1 in Definition IV.30 to recall the definition of α-vertices and β-vertices). The idea is to "cut off" the vertex v' and to interpret the contributions of the connected components of the remaining part again in terms of smaller floor diagrams.

The complement of v' and the v'-adjacent edges in $\tilde{\mathcal{D}}$ consists of the markings $\tilde{\mathcal{D}}^1, \dots, \tilde{\mathcal{D}}^t$ of Psi-floor diagrams $\mathcal{D}^1, \dots, \mathcal{D}^t$ and some isolated α-vertices. For $1 \leq i \leq t$ define

1. d^i and $\mathbf{k}^i$ to be the degree and the type of $\mathcal{D}^i$, respectively,
2. $\alpha^i = \sum \alpha(v)$ to be the sequence of multiplicities of edge weights between $\mathcal{D}^i$ and the α-vertices of $\tilde{\mathcal{D}}$, where the sum is over all vertices v in the Psi-floor diagram $\mathcal{D}^i$,
3. $\beta^i = \sum \beta(v)$, the respective count for the β-vertices of $\tilde{\mathcal{D}}$,
4. m^i to be the weight of the edge between v' and $\mathcal{D}^i$.

Of course, $m^i = d^i - I(\alpha^i + \beta^i)$.

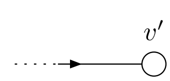
We will see later that all contributions from the components $\mathcal{D}^i$ are of the form $N^{\text{floor}}_{d^i,\mathbf{k^i}}(\alpha^i + e_{m^i}, \beta^i)$ resp. $N^{\text{floor}}_{d^i,\mathbf{k^i}}(\alpha^i, \beta^i + e_{m^i})$. In these cases we necessarily have

$$I\alpha^i + m^i + I\beta^i + I\mathbf{k}^i = 3d^i - 1 + |\beta^i| - |\mathbf{k}^i|, \text{ resp.} \tag{4.7}$$

$$I\alpha^i + I\beta^i + m^i + I\mathbf{k}^i = 3d^i - 1 + |\beta^i| + 1 - |\mathbf{k}^i|. \tag{4.8}$$

Now consider the possibilities for the largest vertex v'. We will distinguish three cases.

Case 1: The vertex v' is not a vertex of the original diagram $\mathcal{D}$. Hence $\tilde{\mathcal{D}}$ looks locally around v' as in the following picture.



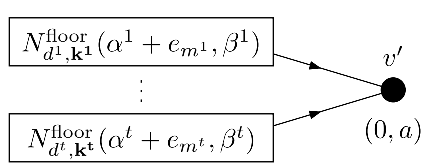
Then $t = 1$, $\alpha^1 = \alpha$ and $\beta^1 = \beta - e_{m^1}$. The (α^1, β^1)-markings of $\mathcal{D}$ with v' maximal among all non-α-vertices are in canonical bijection with $(\alpha^1 + e_{m^1}, \beta^1)$-markings of $\mathcal{D}$ (by making v' an α-vertex and, for example, inserting it to the right of the other α-vertices adjacent to weight m^1 edges). This bijection is weight-preserving up to a factor m^1, as edges of weight m^1 adjacent to β-vertices contribute a factor of m^1 whereas edges adjacent to α-vertices do not (see equation (4.4)). Thus, if v' is not a vertex of the original diagram we get a contribution of

$$\sum_{m^1 : \beta_{m^1} > 0} m^1 \cdot N^{\text{floor}}_{d^1, \mathbf{k}^1}(\alpha^1 + e_{m^1}, \beta^1).$$

This contribution equals the summands with $d' = 0$ and $a = 0$ in the sum of Corollary IV.8: for $d' = 0$ the non-vanishing of $d'^{|\alpha'|+t-t'}$ implies that $|\alpha'| = 0$ and $t = t'$, and equation (4.7) (which can be rearranged to imply a valence and divergence condition on v' as we will show below) implies furthermore that $t' = 1$. This finishes case 1.

Now assume that v' is a vertex of the original diagram $\mathcal{D}$, and denote by d' and a the degree and Psi-power of v', respectively. We need to count the number of ways in which markings of the Psi-floor diagrams $\mathcal{D}^1, \ldots, \mathcal{D}^t$ can be combined to a marking of the Psi-floor diagram $\mathcal{D}$. We need to distinguish whether v' is a floor of $\mathcal{D}$ (i.e. $d' > 0$) or not.

Case 2: v' is a vertex of $\mathcal{D}$, and $d' = 0$. Then we obtain the following local picture for $\tilde{\mathcal{D}}$.



In this case none of the edges between v' and the Psi-floor diagrams $\mathcal{D}^i$ can be chosen. Notice that $(\alpha^i, \beta^i + e_{m^i})$-markings of $\mathcal{D}^i$ with v' largest among all β-vertices (if we consider v' as a β-vertex of $\mathcal{D}^i$) are in canonical bijection with $(\alpha^i + e_{m^i}, \beta^i)$-markings of $\mathcal{D}^i$. This bijection is weight-preserving up to a factor of m^i (see equation (4.4)).

To count the number of ways in which we can combine the markings of the pieces fix an $(\alpha^i + e_{m^i}, \beta^i)$-marking of $\mathcal{D}^i$, one for each $1 \leq i \leq t$. Produce an (α, β)-marking of $\mathcal{D}$ as follows: First, glue the markings by identifying all largest α-vertices in each of the marking of $\mathcal{D}^i$ adjacent to an edge of weight m^i with each other (thereby obtaining the vertex v'). Then order the α-vertices of the markings by extending the partial order on the set of α-vertices given by the markings of the components to a linear order on all vertices. There are $\binom{\alpha}{\alpha^1,\ldots,\alpha^t}$ ways to do this.

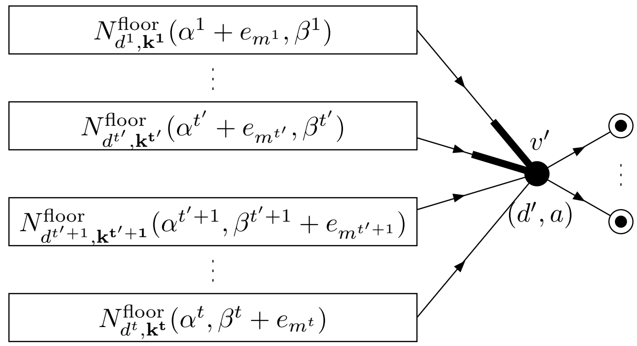
In a second step, we extend the partial order on the vertices that are less than v' to a linear order on all vertices less than v'. As v' is maximal among the non-α-vertices of the marking $\tilde{\mathcal{D}}$ it has $|\mathbf{k}| - 1$ vertices which are less than v'. Using the earlier bijection between (α^i, β^i)-markings of $\mathcal{D}^i$ with v' largest among all β-vertices (if we consider v' as a β-vertex of $\mathcal{D}^i$) and $(\alpha^i + e_{m^i}, \beta^i)$-markings of $\mathcal{D}^i$ we see that there are $|\mathbf{k}^i|$ vertices in component i which are less than v'. Hence there are $\binom{|\mathbf{k}|-1}{|\mathbf{k}^1|,\ldots,|\mathbf{k}^t|}$ linear extensions of the partial order that is induced by the linear orders of the components.

By equation (4.4) the product of the contributions from the t components differs from the contribution of the marking $\tilde{\mathcal{D}}$ by $\frac{1}{\beta(v')!}$, but $\beta(v') = \beta - \sum \beta^i = \beta'$. Moreover, we overcount by $t!$ as we labeled the unlabeled components $1, \ldots, t$. Altogether, we get a contribution of

$$\sum \frac{1}{t!} \frac{1}{\beta'!} \binom{|\mathbf{k}| - 1}{|\mathbf{k}^1|, \ldots, |\mathbf{k}^t|} \cdot \binom{\alpha}{\alpha^1, \ldots, \alpha^t} \prod_{i=1}^{t} m^i \prod_{i=1}^{t} N^{\text{floor}}_{d^i, \mathbf{k}^i}(\alpha^i + e_{m^i}, \beta^i),$$

which equals the summands with $d' = 0$ but $a > 0$ in the recursion of Corollary IV.8. As before, equations (4.7) and (4.8) imply that v' has the correct divergence and valence (see below).

Case 3: v' is a vertex of $\mathcal{D}$, and $d' > 0$. In this case we obtain the following local picture for $\tilde{\mathcal{D}}$.



As then v' is largest among all non-α-vertices we have $C(v') \supset O(v')$. Define the Psi-floor diagrams $\mathcal{D}^1, \ldots, \mathcal{D}^t$ and their markings $\tilde{\mathcal{D}}^1, \ldots, \tilde{\mathcal{D}}^t$ as before, as well as d^i, $\mathbf{k}^i$, α^i, β^i and m^i, for $1 \leq i \leq t$. Without loss of generality we can assume that there is a number $t' \in \{0, \ldots, t\}$ such that the edges between v' and $\tilde{\mathcal{D}}^i$ are chosen at v' for all $i \leq t'$, whereas for $i > t'$ they are not.

Now consider a component $\mathcal{D}^i$. We treat the cases $i \leq t'$ and $i > t'$ separately. If $i \leq t'$ then the $(\alpha^i, \beta^i + e_{m^i})$-markings of $\mathcal{D}^i$ with v' largest among all β-vertices (if we consider v' as a β-vertex of $\mathcal{D}^i$) are in canonical bijection with $(\alpha^i + e_{m^i}, \beta^i)$-markings of $\mathcal{D}^i$ by the same reasoning as for $d' = 0$. As we have seen, this bijection is weight-preserving up to a factor of m^i.

If $i > t'$ then a linear order (up to equivalence) on the vertices of $\tilde{\mathcal{D}}^i$ that can be extended to a marking of $\mathcal{D}$ canonically determines an $(\alpha^i, \beta^i + e_{m^i})$-marking of $\mathcal{D}^i$ together with a distinguished β-vertex adjacent to an edge of weight m^i (namely the image of the edge of $\tilde{\mathcal{D}}^i$ which is closest to v' in $\tilde{\mathcal{D}}$). Conversely, given an $(\alpha^i, \beta^i + e_{m^i})$-marking of $\mathcal{D}^i$ together with a distinguished β-vertex adjacent to an edge of weight m^i, this canonically determines a linear order (up to equivalence) on the vertices of $\tilde{\mathcal{D}}^i$ that can be extended to a marking of $\mathcal{D}$. This $(\beta^i_{m^i} + 1)$-to-1 map is weight-preserving up to a factor of m^i.

Again, to produce an (α, β)-marking of $\mathcal{D}$ we need to extend the partial order on the set of α-vertices given by the markings of the components to a linear order on all α-vertices. There is no difference to the $d' = 0$ case, hence there are $\binom{\alpha}{\alpha^1, \ldots, \alpha^t}$ different extensions. As before, there are $\binom{|\mathbf{k}|-1}{\mathbf{k}^1, \ldots, \mathbf{k}^t}$ ways to extend the partial order on the vertices that are smaller than v' to a linear order.

Also as before, by equation (4.4) the weight of a marking of $\mathcal{D}$ differs from the product of the individual weights of the t components by contributions from the vertex v'. The local multiplicity at v' from equation (4.5) is $\frac{(d')^{t-t'}}{d'!}\frac{(d')^{|\alpha'|}}{d'!}$ as the number of non-chosen incoming vertices is $i(v') = |I(v') \backslash C(v')| = t - t'$ and

$$o(v') = |O(v') \backslash C(v')| + |\alpha(v')| = 0 + |\alpha'| = |\alpha'|$$

since $\alpha(v') = \alpha - \sum_i \alpha^i$. The second contribution from the vertex v' is $\frac{1}{\beta'!}$ (see equation (4.6)), as $\beta' = \beta(v')$, $C(v') \supset O(v')$, and hence $c(v') = \beta(v')$, and these are the only contributions in which the markings of $\mathcal{D}$ and the contributions from its components differ. Moreover, we overcount by $t'! \cdot (t-t')!$ as we labeled the unlabeled components $1, \ldots, t'$ and $t'+1, \ldots, t$.

Divergence and valence conditions for all cases: In all three cases, equations (4.7) and (4.8) imply that v' has the correct divergence and valence: summing up equations (4.7) for $1 \leq i \leq t'$ and (4.8) for $t'+1 \leq i \leq t$ yields

$$I\alpha - I\alpha' + I\beta - I\beta' + I\mathbf{k} - a + \sum m^i = 3d - 3d' - t' + |\beta| - |\beta'| - |\mathbf{k}| + 1.$$

Since $I(\alpha + \beta + \mathbf{k}) = 3d - 1 + |\beta| - |\mathbf{k}|$ we can conclude

$$-I\alpha' - I\beta' - a + \sum m^i = -3d' - t' - |\beta'| + 2. \tag{4.9}$$

Now replace m^i by $d^i - I(\alpha^i + \beta^i)$ and use that $d = I(\alpha + \beta)$ to obtain the valence condition at v':

$$-a = -2d' - t' - |\beta'| + 2, \text{ resp. } 2d' + t' + |\beta'| = a + 2.$$

Together with equation (4.9) the valence condition implies the divergence condition at v':

$$-I\alpha' - I\beta' + \sum m^i = -d', \text{ resp. } d' + \sum m^i = I(\alpha' + \beta').$$

Hence the contributions in the case when v' is a floor equal the summands with $d' > 0$ in the recursion of Corollary IV.8. This completes the proof. $\square$

Of course, one can also prove the recursion in Theorem IV.7 directly using Psi-floor diagrams. We then have to use the numbers $\tilde{N}^{\text{floor}}_{d,\mathbf{k}}(\alpha, \beta)$ of Remark IV.32, where we fix an order for the Psi-powers and mark the white end vertices.

APPENDICES

APPENDIX A

Node Polynomials for $\delta \leq 14$

An explicit list of $N_\delta(d)$, for $\delta \leq 14$, is as below. These polynomials are given implicitly in Theorem II.11. For $\delta \leq 8$ this agrees with [21, Theorem 3.1]. For $\delta \leq 14$ this coincides with the conjectural (implicit) formulas of [18, Remark 2.5].

$$
\begin{aligned}
N_0(d) &= 1,\\
N_1(d) &= 3(d-1)^2,\\
N_2(d) &= \frac{3}{2}(d-1)(d-2)(3d^2-3d-11),\\
N_3(d) &= \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525,\\
N_4(d) &= \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^5 - 642d^4 - 2529d^3 + \frac{37881}{8}d^2 + \frac{18057}{4}d - 8865,\\
N_5(d) &= \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5 + \frac{128859}{8}d^4 + \frac{59097}{2}d^3 - \frac{3528381}{40}d^2\\
&\quad - \frac{946929}{20}d + 153513,\\
N_6(d) &= \frac{81}{80}d^{12} - \frac{243}{20}d^{11} - \frac{81}{20}d^{10} + \frac{8667}{16}d^9 - \frac{9297}{8}d^8 - \frac{47727}{5}d^7 + \frac{2458629}{80}d^6 + \frac{3243249}{40}d^5\\
&\quad - \frac{6577679}{20}d^4 - \frac{25387481}{80}d^3 + \frac{6352577}{4}d^2 + \frac{8290623}{20}d - 2699706,\\
N_7(d) &= \frac{243}{560}d^{14} - \frac{243}{40}d^{13} - \frac{243}{80}d^{12} + \frac{30861}{80}d^{11} - \frac{38853}{40}d^{10} - \frac{802143}{80}d^9 + \frac{3140127}{80}d^8 + \frac{18650493}{140}d^7\\
&\quad - \frac{54903831}{80}d^6 - \frac{72723369}{80}d^5 + \frac{124680069}{20}d^4 + \frac{213537633}{80}d^3 - \frac{3949576431}{140}d^2 - \frac{188754021}{140}d\\
&\quad + 48016791,\\
N_8(d) &= \frac{729}{4480}d^{16} - \frac{729}{280}d^{15} - \frac{243}{140}d^{14} + \frac{35721}{160}d^{13} - \frac{25839}{40}d^{12} - \frac{320841}{40}d^{11} + \frac{11847087}{320}d^{10}\\
&\quad + \frac{170823033}{1120}d^9 - \frac{6685218}{7}d^8 - \frac{1758652263}{1120}d^7 + \frac{1102682031}{80}d^6 + \frac{59797545}{8}d^5 - \frac{510928080111}{4480}d^4\\
&\quad - \frac{3283674393}{1120}d^3 + \frac{558215113803}{1120}d^2 - \frac{3722027733}{56}d - 861732459,\\
N_9(d) &= \frac{243}{4480}d^{18} - \frac{2187}{2240}d^{17} - \frac{729}{896}d^{16} + \frac{121743}{1120}d^{15} - \frac{99549}{280}d^{14} - \frac{824823}{160}d^{13} + \frac{8776593}{320}d^{12} + \frac{74122857}{560}d^{11}\\
&\quad - \frac{2188424421}{2240}d^{10} - \frac{132610923}{70}d^9 + \frac{11404136871}{560}d^8 + \frac{2852923401}{224}d^7 - \frac{3523392270287}{13440}d^6\\
&\quad + \frac{4109675615}{448}d^5 + \frac{261844582229}{128}d^4 - \frac{2156232149611}{3360}d^3 - \frac{29528525065861}{3360}d^2 + \frac{438722045999}{168}d\\
&\quad + 15580950065,
\end{aligned}
$$

$$
\begin{aligned}
N_{10}(d) = {} & \frac{729}{44800}d^{20} - \frac{729}{2240}d^{19} - \frac{729}{2240}d^{18} + \frac{408969}{8960}d^{17} - \frac{746253}{4480}d^{16} - \frac{1932579}{700}d^{15} + \frac{10649961}{640}d^{14} \\
& + \frac{205722099}{2240}d^{13} - \frac{4375229931}{5600}d^{12} - \frac{38815692777}{22400}d^{11} + \frac{30958937073}{1400}d^{10} + \frac{3413568339}{224}d^{9} \\
& - \frac{3624162885799}{8960}d^{8} + \frac{134470136581}{2800}d^{7} + \frac{27023302169081}{5600}d^{6} - \frac{22514488581251}{8960}d^{5} - \frac{811909836973903}{22400}d^{4} \\
& + \frac{253124357071961}{11200}d^{3} + \frac{867510616107447}{5600}d^{2} - \frac{2800250331071}{40}d - 283516631436, \\
N_{11}(d) = {} & \frac{2187}{492800}d^{22} - \frac{2187}{22400}d^{21} - \frac{729}{6400}d^{20} + \frac{150903}{8960}d^{19} - \frac{303993}{4480}d^{18} - \frac{56670273}{44800}d^{17} + \frac{47717667}{5600}d^{16} \\
& + \frac{295979589}{5600}d^{15} - \frac{11410430877}{22400}d^{14} - \frac{4051907631}{3200}d^{13} + \frac{52491198663}{2800}d^{12} + \frac{3418059518271}{246400}d^{11} \\
& - \frac{20587006282467}{44800}d^{10} + \frac{2236636275459}{22400}d^{9} + \frac{49175916627959}{6400}d^{8} - \frac{1464110674563}{256}d^{7} \\
& - \frac{1946239824069277}{22400}d^{6} + \frac{3767687640687823}{44800}d^{5} + \frac{14264414890838423}{22400}d^{4} - \frac{940418544772283}{1600}d^{3} \\
& - \frac{168280746183263029}{61600}d^{2} + \frac{5073050867636909}{3080}d + 5187507215325, \\
N_{12}(d) = {} & \frac{2187}{1971200}d^{24} - \frac{6561}{246400}d^{23} - \frac{2187}{61600}d^{22} + \frac{496449}{89600}d^{21} - \frac{136809}{5600}d^{20} - \frac{1618623}{3200}d^{19} + \frac{674946837}{179200}d^{18} \\
& + \frac{2321658693}{89600}d^{17} - \frac{893195181}{3200}d^{16} - \frac{34334301951}{44800}d^{15} + \frac{289702847403}{22400}d^{14} + \frac{1245724147341}{123200}d^{13} \\
& - \frac{803786361621603}{1971200}d^{12} + \frac{65497548165237}{492800}d^{11} + \frac{16192295343681}{1792}d^{10} - \frac{792669234543351}{89600}d^{9} \\
& - \frac{9506773589164709}{67200}d^{8} + \frac{6296062244021929}{33600}d^{7} + \frac{11029935159768347}{7168}d^{6} - \frac{582428855393100577}{268800}d^{5} \\
& - \frac{5477484616918678589}{492800}d^{4} + \frac{10067756533588172119}{739200}d^{3} + \frac{4454424013895459501}{92400}d^{2} \\
& - \frac{111952943233924509}{3080}d - 95376705265437, \\
N_{13}(d) = {} & \frac{6561}{25625600}d^{26} - \frac{6561}{985600}d^{25} - \frac{19683}{1971200}d^{24} + \frac{1620567}{985600}d^{23} - \frac{88209}{11200}d^{22} - \frac{3212703}{17920}d^{21} + \frac{262066023}{179200}d^{20} \\
& + \frac{494726373}{44800}d^{19} - \frac{673360047}{5120}d^{18} - \frac{35350103511}{89600}d^{17} + \frac{20952637821}{2800}d^{16} + \frac{3013479294723}{492800}d^{15} \\
& - \frac{580214902388013}{1971200}d^{14} + \frac{1666286215401123}{12812800}d^{13} + \frac{16384163286402207}{1971200}d^{12} - \frac{909876952033137}{89600}d^{11} \\
& - \frac{7649416285706767}{44800}d^{10} + \frac{25855007471662161}{89600}d^{9} + \frac{65085797443981191}{25600}d^{8} - \frac{108443195356282427}{22400}d^{7} \\
& - \frac{5299140016292762991 7}{1971200}d^{6} + \frac{1976324604711031517}{39424}d^{5} + \frac{1358075308024310521 9}{70400}d^{4} \\
& - \frac{7327470596743106328 1}{246400}d^{3} - \frac{6817329077609937439 1}{80080}d^{2} + \frac{2813974748454890667}{3640}d + 1761130218801033, \\
N_{14}(d) = {} & \frac{19683}{358758400}d^{28} - \frac{19683}{12812800}d^{27} - \frac{6561}{2562560}d^{26} + \frac{1751787}{3942400}d^{25} - \frac{4529277}{1971200}d^{24} - \frac{562059}{9856}d^{23} \\
& + \frac{398785599}{788480}d^{22} + \frac{5214288411}{1254400}d^{21} - \frac{4860008991}{89600}d^{20} - \frac{63174295089}{358400}d^{19} + \frac{332872084467}{89600}d^{18} \\
& + \frac{3103879378581}{985600}d^{17} - \frac{4913807521304691}{27596800}d^{16} + \frac{899178800016807}{8968960}d^{15} + \frac{279086438050359453}{44844800}d^{14} \\
& - \frac{468967272863997483}{51251200}d^{13} - \frac{318443311640108577}{1971200}d^{12} + \frac{328351365725506869}{985600}d^{11} \\
& + \frac{1120586814080571923}{358400}d^{10} - \frac{944886102844884394 9}{1254400}d^{9} - \frac{3088078521673640614 3}{689920}d^{8} \\
& + \frac{444525313669622586903}{3942400}d^{7} + \frac{114290382216754662 51}{24640}d^{6} - \frac{269709254062572016617}{246400}d^{5} \\
& - \frac{7466063066474887866 5353}{22422400}d^{4} + \frac{1405313594695109830 18159}{22422400}d^{3} + \frac{1686393119515422597 7601}{1121120}d^{2} \\
& - \frac{643144544868253490 85}{4004}d - 32644422296329680.
\end{aligned}
$$

APPENDIX B

Small Severi degrees

Below we list the Severi degrees $N^{d,\delta}$ for $0 \leq \delta \leq 14$ and $1 \leq d \leq 13$, which were obtained by Algorithm 1 (also see Remark II.21). Together with the node polynomials of Appendix A, this is a full description of all Severi degrees $N^{d,\delta}$ for $\delta \leq 14$, see Theorem II.4. The solid line segments indicate the polynomial threshold $d^*(\delta)$ of $N^{d,\delta}$. The dashed line segments illustrate the threshold of our Theorem II.4. The Severi degrees $N^{d,\delta}$ in italic agree with the Gromov-Witten invariants $N_{d,\frac{(d-1)(d-2)}{2}-\delta}$, as for $d \geq \delta + 2$, every plane degree d curve with δ nodes is irreducible.

	$d=1$	2	3	4	5	6	7	8	9
$N^{d,0}$	1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*
$N^{d,1}$	0	3	*12*	*27*	*48*	*75*	*108*	*147*	*192*
$N^{d,2}$	0	0	21	*225*	*882*	*2370*	*5175*	*9891*	*17220*
$N^{d,3}$	0	0	15	675	*7915*	*41310*	*145383*	*404185*	*959115*
$N^{d,4}$	0	0	0	666	36975	*437517*	*2667375*	*11225145*	*37206936*
$N^{d,5}$	0	0	0	378	90027	2931831	*33720354*	*224710119*	*1068797961*
$N^{d,6}$	0	0	0	105	109781	12597900	302280963	*3356773532*	*23599355991*
$N^{d,7}$	0	0	0	0	65949	34602705	1950179922	38232604473	*410453320698*
$N^{d,8}$	0	0	0	0	26136	59809860	9108238023	336507128820	5717863228995
$N^{d,9}$	0	0	0	0	6930	63338881	30777542450	2307156326490	64541125393337
$N^{d,10}$	0	0	0	0	945	40047888	74808824094	12372036675723	595034126865816
$N^{d,11}$	0	0	0	0	0	15580020	129429708147	51941532804912	4504735527185481
$N^{d,12}$	0	0	0	0	0	4361721	157012934283	170460529136614	28096248183844557
$N^{d,13}$	0	0	0	0	0	918918	131024290671	435634878105750	144607321488666150
$N^{d,14}$	0	0	0	0	0	135135	73778495220	861893389007280	614331908473795659

	$d=10$	11	12	13
$N^{d,0}$	*1*	*1*	*1*	*1*
$N^{d,1}$	*243*	*300*	*363*	*432*
$N^{d,2}$	*27972*	*43065*	*63525*	*90486*
$N^{d,3}$	*2029980*	*3939295*	*7139823*	*12245355*
$N^{d,4}$	*104285790*	*257991042*	*579308220*	*1203756165*
$N^{d,5}$	*4037126346*	*12886585236*	*36161763120*	*91629683271*
$N^{d,6}$	*122416062018*	*510681301550*	*1807308075111*	*5622246678741*
$N^{d,7}$	*2983927028787*	*16491272517465*	*74314664917722*	*285826689019395*
$N^{d,8}$	*59546865647151*	*442342707233400*	*2563893146687301*	*12282025653769635*
$N^{d,9}$	985875034961260	*9996104553443766*	*75318503191523715*	*452837863689428040*
$N^{d,10}$	13675748317151382	192382226805469707	*1905520429287295623*	*14494356139317787773*
$N^{d,11}$	160116000544437849	3179784684983704875	41891284347920817345	*406515611290526234685*
$N^{d,12}$	1590895229889323034	45433391588342055421	806014803108359459265	10065752539264069119357
$N^{d,13}$	13467927464624076876	564072316378226731551	13651752114752769885861	221404946495996659008375
$N^{d,14}$	97415160821449934985	6109881487479049410675	204507963635254531972650	4348333391475310314325875

APPENDIX C

Relative node polynomials for $\delta \leq 3$

Below we list the relative node polynomials $N_\delta(\alpha;\beta)$ for $\delta \leq 3$. For $\delta \leq 6$ the polynomials $N_\delta(\alpha;\beta)$ are as provided in the ancillary files accompanying [2]. All polynomials were obtained by a Maple implementation of the formula (3.12). See Remark III.16 for more details. For $\delta \leq 1$ this agrees with [9, Corollary 4.5, 4.6]. As before, we write $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. By Theorem III.1 the relative Severi degrees $N^\delta_{\alpha,\beta}$ are given by $N^\delta_{\alpha,\beta} = 1^{\beta_1}2^{\beta_2}\cdots\frac{(|\beta|-\delta)!}{\beta!}N_\delta(\alpha,\beta)$ provided $|\beta| \geq \delta$.

$$
\begin{aligned}
N_0(\alpha,\beta) &= 1,\\
N_1(\alpha,\beta) &= 3d^2|\beta| - 8d|\beta| + d\beta_1 + |\beta|\alpha_1 + |\beta|\beta_1 + 4|\beta| - \beta_1,\\
N_2(\alpha,\beta) &= \tfrac{9}{2}d^4|\beta|^2 - \tfrac{9}{2}d^4|\beta| - 24d^3|\beta|^2 + 3d^3|\beta|\beta_1 + 3d^2|\beta|^2\alpha_1 + 3d^2|\beta|^2\beta_1 + 24d^3|\beta| - 3d^3\beta_1 + 23d^2|\beta|^2\\
&\quad - 3d^2|\beta|\alpha_1 - 14d^2|\beta|\beta_1 + \tfrac{1}{2}d^2\beta_1^2 - 8d|\beta|^2\alpha_1 - 8d|\beta|^2\beta_1 + d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + \tfrac{1}{2}|\beta|^2\alpha_1^2 + |\beta|^2\alpha_1\beta_1\\
&\quad + \tfrac{1}{2}|\beta|^2\beta_1^2 - 23d^2|\beta| + \tfrac{21}{2}d^2\beta_1 + \tfrac{3}{2}d|\beta|^2 + 8d|\beta|\alpha_1 + 11d|\beta|\beta_1 + d|\beta|\beta_2 - d\alpha_1\beta_1 - \tfrac{5}{2}d\beta_1^2 - \tfrac{1}{2}|\beta|^2\alpha_1\\
&\quad + |\beta|^2\alpha_2 - \tfrac{1}{2}|\beta|^2\beta_1 + |\beta|^2\beta_2 - \tfrac{1}{2}|\beta|\alpha_1^2 - 3|\beta|\alpha_1\beta_1 - \tfrac{5}{2}|\beta|\beta_1^2 - \tfrac{83}{2}d|\beta| - \tfrac{3}{2}d\beta_1 - d\beta_2 - 48|\beta|^2 + \tfrac{1}{2}|\beta|\alpha_1\\
&\quad - |\beta|\alpha_2 + \tfrac{29}{2}|\beta|\beta_1 - 3|\beta|\beta_2 + 2\alpha_1\beta_1 + 3\beta_1^2 + 48|\beta| - 15\beta_1 + 2\beta_2,
\end{aligned}
$$

$$
\begin{aligned}
N_3(\alpha,\beta) = {} & \tfrac{9}{2}d^6|\beta|^3 - \tfrac{27}{2}d^6|\beta|^2 - 36d^5|\beta|^3 + \tfrac{9}{2}d^5|\beta|^2\beta_1 + \tfrac{9}{2}d^4|\beta|^3\alpha_1 + \tfrac{9}{2}d^4|\beta|^3\beta_1 + 9d^6|\beta| + 108d^5|\beta|^2 - \tfrac{27}{2}d^5|\beta|\beta_1 \\
& + 51d^4|\beta|^3 - \tfrac{27}{2}d^4|\beta|^2\alpha_1 - 42d^4|\beta|^2\beta_1 + \tfrac{3}{2}d^4|\beta|\beta_1^2 - 24d^3|\beta|^3\alpha_1 - 24d^3|\beta|^3\beta_1 + 3d^3|\beta|^2\alpha_1\beta_1 + 3d^3|\beta|^2\beta_1^2 \\
& + \tfrac{3}{2}d^2|\beta|^3\alpha_1^2 + 3d^2|\beta|^3\alpha_1\beta_1 + \tfrac{3}{2}d^2|\beta|^3\beta_1^2 - 72d^5|\beta| + 9d^5\beta_1 - 153d^4|\beta|^2 + 9d^4|\beta|\alpha_1 + 93d^4|\beta|\beta_1 - 3d^4\beta_1^2 \\
& + \tfrac{1243}{6}d^3|\beta|^3 + 72d^3|\beta|^2\alpha_1 + 92d^3|\beta|^2\beta_1 + 3d^3|\beta|^2\beta_2 - 9d^3|\beta|\alpha_1\beta_1 - \tfrac{35}{2}d^3|\beta|\beta_1^2 + \tfrac{1}{6}d^3\beta_1^3 + \tfrac{19}{2}d^2|\beta|^3\alpha_1 \\
& + 3d^2|\beta|^3\alpha_2 + \tfrac{19}{2}d^2|\beta|^3\beta_1 + 3d^2|\beta|^3\beta_2 - \tfrac{9}{2}d^2|\beta|^2\alpha_1^2 - 23d^2|\beta|^2\alpha_1\beta_1 - \tfrac{37}{2}d^2|\beta|^2\beta_1^2 + \tfrac{1}{2}d^2|\beta|\alpha_1\beta_1^2 \\
& + \tfrac{1}{2}d^2|\beta|\beta_1^3 - 4d|\beta|^3\alpha_1^2 - 8d|\beta|^3\alpha_1\beta_1 - 4d|\beta|^3\beta_1^2 + \tfrac{1}{2}d|\beta|^2\alpha_1^2\beta_1 + d|\beta|^2\alpha_1\beta_1^2 + \tfrac{1}{2}d|\beta|^2\beta_1^3 + \tfrac{1}{6}|\beta|^3\alpha_1^3 \\
& + \tfrac{1}{2}|\beta|^3\alpha_1^2\beta_1 + \tfrac{1}{2}|\beta|^3\alpha_1\beta_1^2 + \tfrac{1}{6}|\beta|^3\beta_1^3 + 102d^4|\beta| - 54d^4\beta_1 - \tfrac{1243}{2}d^3|\beta|^2 - 48d^3|\beta|\alpha_1 - \tfrac{199}{2}d^3|\beta|\beta_1 \\
& - 9d^3|\beta|\beta_2 + 6d^3\alpha_1\beta_1 + \tfrac{45}{2}d^3\beta_1^2 - 458d^2|\beta|^3 - \tfrac{57}{2}d^2|\beta|^2\alpha_1 - 9d^2|\beta|^2\alpha_2 + 116d^2|\beta|^2\beta_1 - 23d^2|\beta|^2\beta_2 \\
& + 3d^2|\beta|\alpha_1^2 + \tfrac{95}{2}d^2|\beta|\alpha_1\beta_1 + \tfrac{105}{2}d^2|\beta|\beta_1^2 + d^2|\beta|\beta_1\beta_2 - d^2\alpha_1\beta_1^2 - 2d^2\beta_1^3 + \tfrac{155}{2}d|\beta|^3\alpha_1 - 8d|\beta|^3\alpha_2 \\
& + \tfrac{155}{2}d|\beta|^3\beta_1 - 8d|\beta|^3\beta_2 + 12d|\beta|^2\alpha_1^2 + \tfrac{61}{2}d|\beta|^2\alpha_1\beta_1 + d|\beta|^2\alpha_1\beta_2 + d|\beta|^2\alpha_2\beta_1 + \tfrac{37}{2}d|\beta|^2\beta_1^2 + 2d|\beta|^2\beta_1\beta_2 \\
& - \tfrac{3}{2}d|\beta|\alpha_1^2\beta_1 - \tfrac{11}{2}d|\beta|\alpha_1\beta_1^2 - 4d|\beta|\beta_1^3 - \tfrac{5}{2}|\beta|^3\alpha_1^2 + |\beta|^3\alpha_1\alpha_2 - 5|\beta|^3\alpha_1\beta_1 + |\beta|^3\alpha_1\beta_2 + |\beta|^3\alpha_2\beta_1 - \tfrac{5}{2}|\beta|^3\beta_1^2 \\
& + |\beta|^3\beta_1\beta_2 - \tfrac{1}{2}|\beta|^2\alpha_1^3 - 3|\beta|^2\alpha_1^2\beta_1 - \tfrac{9}{2}|\beta|^2\alpha_1\beta_1^2 - 2|\beta|^2\beta_1^3 + \tfrac{1243}{3}d^3|\beta| + \tfrac{70}{3}d^3\beta_1 + 6d^3\beta_2 + 1374d^2|\beta|^2 \\
& + 19d^2|\beta|\alpha_1 + 6d^2|\beta|\alpha_2 - \tfrac{845}{2}d^2|\beta|\beta_1 + 48d^2|\beta|\beta_2 - 27d^2\alpha_1\beta_1 - 40d^2\beta_1^2 - 2d^2\beta_1\beta_2 - \tfrac{842}{3}d|\beta|^3 \\
& - \tfrac{465}{2}d|\beta|^2\alpha_1 + 24d|\beta|^2\alpha_2 - 396d|\beta|^2\beta_1 + 29d|\beta|^2\beta_2 + d|\beta|^2\beta_3 - 8d|\beta|\alpha_1^2 - 33d|\beta|\alpha_1\beta_1 - 3d|\beta|\alpha_1\beta_2 \\
& - 3d|\beta|\alpha_2\beta_1 + 2d|\beta|\beta_1^2 - 11d|\beta|\beta_1\beta_2 + d\alpha_1^2\beta_1 + 7d\alpha_1\beta_1^2 + \tfrac{47}{6}d\beta_1^3 - \tfrac{92}{3}|\beta|^3\alpha_1 - 6|\beta|^3\alpha_2 + |\beta|^3\alpha_3 \\
& - \tfrac{92}{3}|\beta|^3\beta_1 - 6|\beta|^3\beta_2 + |\beta|^3\beta_3 + \tfrac{15}{2}|\beta|^2\alpha_1^2 - 3|\beta|^2\alpha_1\alpha_2 + \tfrac{87}{2}|\beta|^2\alpha_1\beta_1 - 6|\beta|^2\alpha_1\beta_2 - 6|\beta|^2\alpha_2\beta_1 + 36|\beta|^2\beta_1^2 \\
& - 9|\beta|^2\beta_1\beta_2 + \tfrac{1}{3}|\beta|\alpha_1^3 + \tfrac{11}{2}|\beta|\alpha_1^2\beta_1 + 13|\beta|\alpha_1\beta_1^2 + \tfrac{47}{6}|\beta|\beta_1^3 - 916d^2|\beta| + 303d^2\beta_1 - 28d^2\beta_2 + 842d|\beta|^2 \\
& + 155d|\beta|\alpha_1 - 16d|\beta|\alpha_2 + \tfrac{1237}{2}d|\beta|\beta_1 - 31d|\beta|\beta_2 - 3d|\beta|\beta_3 + 8d\alpha_1\beta_1 + 2d\alpha_1\beta_2 + 2d\alpha_2\beta_1 - \tfrac{103}{2}d\beta_1^2 \\
& + 14d\beta_1\beta_2 + 706|\beta|^3 + 92|\beta|^2\alpha_1 + 18|\beta|^2\alpha_2 - 3|\beta|^2\alpha_3 - 46|\beta|^2\beta_1 + 48|\beta|^2\beta_2 - 6|\beta|^2\beta_3 - 5|\beta|\alpha_1^2 \\
& + 2|\beta|\alpha_1\alpha_2 - \tfrac{197}{2}|\beta|\alpha_1\beta_1 + 11|\beta|\alpha_1\beta_2 + 11|\beta|\alpha_2\beta_1 - \tfrac{271}{2}|\beta|\beta_1^2 + 26|\beta|\beta_1\beta_2 - 3\alpha_1^2\beta_1 - 12\alpha_1\beta_1^2 - 10\beta_1^3 \\
& - \tfrac{1684}{3}d|\beta| - \tfrac{808}{3}d\beta_1 + 10d\beta_2 + 2d\beta_3 - 2118|\beta|^2 - \tfrac{184}{3}|\beta|\alpha_1 - 12|\beta|\alpha_2 + 2|\beta|\alpha_3 + \tfrac{1184}{3}|\beta|\beta_1 - 102|\beta|\beta_2 \\
& + 11|\beta|\beta_3 + 63\alpha_1\beta_1 - 6\alpha_1\beta_2 - 6\alpha_2\beta_1 + 150\beta_1^2 - 24\beta_1\beta_2 + 1412|\beta| - 362\beta_1 + 60\beta_2 - 6\beta_3.
\end{aligned}
$$

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