ANALYSIS AND CONTROL OF TRANSIENT FLOW
IN NATURAL GAS PIPING SYSTEMS

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Civil Engineering)
in The University of Michigan
1971

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To my wife Hsiukang

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his doctoral committee. In particular, the author appreciated the guidance of Professor V. L. Streeter and Professor E. B. Wylie, the co-chairmen of the doctoral committee, whose active interest and helpful suggestions were a major factor in the completion of this work.

The author is indebted to all those persons of the Consumers Power Company, Jackson, Michigan, who assisted in the gathering of the field data used in this investigation.

The author gratefully acknowledges the financial assistance from the Michigan Gas Association for awarding him the research fellowship for two years, and from the National Science Foundation for the support of his first year of study. Without this support, the work would have been virtually impossible.

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NOMENCLATURE

Symbol	Meaning
A	Cross-sectional area of pipe--ft^2
AO	A real variable representing the combination of the known quantities of PC, PS and QC, QS
B	Isothermal wave speed - ft/sec
C	The constant of a valve or a regulator
C11,C12...	Elements of a field transfer matrix
D	Pipe diameter - ft
E	A constant representing $e^{i\omega k}$
$\{F\}$	The forcing vector in a non-homogeneous equation
f	Darcy-Weisbach friction factor
g	Acceleration of gravity - ft/sec^2
G,H	Collections of non-linear oscillatory terms used in the second-order approximation
h	Dimensionless parameter representing $\Delta x/L$
HP	Horsepower of a compressor
I	Index denoting the discrete variable of space
i	Unit of complex number being $\sqrt{-1}$
J	Index denoting the discrete variable of time
K	A constant representing $\frac{10^6}{86400} \frac{M_w P_b}{gRT_b}$
k	Time increment in dimensionless form
k_1,k_2,k_3	Constants of a compressor unit
L	Length of pipe - ft
L_1,L_2	Linear difference operators
M	Mass flow rate - slugs/sec
m	Dimensionless parameter representing $\frac{KBQ_o}{AP_o}$

Symbol	Meaning
M_w	Gas molecular weight - lb/mole
N	Number of reaches per pipe
P	Absolute pressure - psia
P	Point subscript, conditions unknown
p	Absolute pressure - lb/ft^2
p	Oscillatory pressure, dimensionless
$\overline{P}$	Mean or average pressure, dimensionless
$\tilde{P}$	The average of the mean pressure at Ith reach
PC	The modulus of the complex variable ξ
PS	The argument of the complex variable ξ
P_b	The base pressure - psia
P_d	Discharge pressure of a compressor - psia
P_d	Downstream pressure of a regulator - psia
P_d	Dimensionless parameter representing the steady state outlet pressure
P_s	Suction pressure of a compressor - psia
P_o	Fixed pressure at the upstream end of a pipe - psia
P_u	Upstream pressure of a regulator - psia
Q	Flow rate - mmcfd
q	Oscillatory flow rate, dimensionless
$\overline{Q}$	Mean or average flow rate, dimensionless
QC	The modulus of the complex variable η
QS	The argument of the complex variable η
Q_t	The steepest slope of a flow demand curve - mmcfd/sec
Q_o	The steady state flow rate - mmcfd
R	Gas constant being 1545 ft-lb/^{o}R - mole

Symbol	Meaning
R	Point subscript, conditions known
R1,R2,R3	Complex variables representing the combination of the known quantities PC, PS and QC, QS
R11,R12...	Elements of the over-all transfer matrix
S	Point subscript, conditions known
s	A constant representing $2g \Delta x \sin \theta / B^2$
S1,S2,S3	Complex variables representing the combination of the known quantities PC, PS and QC, QS
[SN]	The over-all transfer matrix
T	Gas temperature - ^{o}R
t	Time - seconds
T_b	The base temperature - ^{o}R
[TN]	The field transfer matrix
V	Mass transport velocity - ft/sec
x	Distance - ft
z	Gas compressibility factor
ZP,ZQ	Elements of the forcing vector $\{F\}$
$\{Z\}$	State vector denoting the conditions of ξ and η at a section
α	The inertial multiplier
δx	Control volume length
ΔQ	The amplitude of the flow variation - mmcfd
Δq	Dimensionless parameter representing $\Delta Q / Q_o$
Δt	Time increment - seconds
Δx	Reach length - ft
η	Complex variable representing the discrete space function of the oscillatory flow rate
θ	Angle of the inclination of pipe measured upward from the horizontal

Symbol	Meaning
λ	Multiplier in the method of characteristics
ξ	Complex variable representing the discrete space function of the oscillatory pressure
ρ	Mass density - slugs/ft^3
σ	Dimensionless parameter representing $fL/2D$
τ_o	Wall shear stress - lb/ft^2
Φ	Dimensionless parameter representing the error bound in flow solution per steady state flow rate
ψ	Dimensionless parameter representing the error bound in pressure solution per inlet pressure
ω	Frequency of the flow variation - rad/sec

I. INTRODUCTION

Transients in a natural gas system are initiated by the time variational nature of demands at the distribution points and the reactions taken by the system operator in order to meet these demands. The study of transient flow in gas systems, therefore, considers the following problems: (1) analysis of the system under given time variant loading conditions, and (2) control of the system to meet the time variant gas deliveries and also satisfy the contract pressure specifications. The first problem concerns the analysis of transients that took place in the past or evaluation of the system capability with future forecasted loads. The second problem deals with the optimal operation of the system which is of considerable practical value since tremendous operating costs can be saved by following a good control procedure.

The duration of transients in a normal operating gas system is very long, perhaps one or several days. The computation of such a slow transient in a complicated system will be very time-consuming and uneconomical if no special treatment is given regarding the selection of a large time step for a solution method. The motivation for the present study stems from the desirability of developing a method suitable for solution of the above-mentioned problems under various transient conditions in a simple and economical way even though the transient is of long duration.

Although natural gas systems have been operating under transient flow conditions, the design and the operation of a system are primarily based upon the steady state calculations. It was not until recently that the desirability of transient investigations has been brought to the attention of gas industry. In 1961, Batey et al.[2] solved the natural gas transient flow equations by explicit finite difference procedures. Their work was limited to the consideration of simple pipeline sections. In 1965, Wilkinson et al.[24] studied various techniques for the simulation of transient flow in natural gas piping systems. The implicit finite difference technique and an analytical solution by the use of a power series were developed. Agreement of the calculated results with field data was observed. In extending the technique for transient solution, they also proposed the transfer function techniques to accomplish a solution from the linearized equations. However, the solution revealed that the linearized system may not adequately describe the non-linear transient behavior in a natural gas pipeline. Recently, the algorithm identified as PIPETRAN[15] was developed for the simulation of transient flow in general pipeline systems. The algorithm employs the explicit procedures combined with certain empirical restraints to aid in achieving stability. The method of characteristics for transient simulation[17,18] has also been developed and the results proved to be excellent including experimental confirmation. The only disadvantage with the method of characteristics is its costly computation for slow transients. For stability reasons, the time increment cannot exceed the reach length divided by the isothermal wave speed. Most recently, an implicit finite difference procedure combined with the sparse matrix algebra has also been used successfully in complex systems.[25]

From the above algorithms, it can be realized that the finite difference approximation is the most effective means of simulating the natural gas transient flow in complicated systems. However, the discretization error with such an approximation may be appreciable in the solution. Although the investigations of the problems of stability and convergence in the numerical approximation have been made in a number of references (1, 5, 13, 16), no investigation has been set forth so far to evaluate quantitatively the error bound in the finite difference solution. A limitation on the allowable magnitude of the reach length used in a specific problem is highly desirable. The quantification of the discretization error in the numerical solution by the method of characteristics is investigated in this study.

The mathematical model that simulates the natural gas transient flow is formed by considering the dynamic equilibrium and mass conservation in the system. The parameters involved in this model include frictional, inertial, gravitational and pressure forces and system storage and compliance. It has been realized that the inertia of gas is negligible compared with other parameters in the system. By enlarging the inertial effect within certain limits, one may observe very little influence on the dynamic equilibrium or on the conservation of mass in the system. However, with this treatment, a large time increment may be allowed in the numerical computation with the characteristics method. In view of this fact, the concept of an inertial multiplier is initiated in this study. The use of the inertial multiplier in natural gas systems not only permits an optimum time increment for numerical computation to be used under various transient flow conditions, but also enables the direct calculation of a feasible control solution by use of the valve-stroking principles applied to the characteristic equations. The selection of the magnitude of the inertial multiplier is dependent upon the parameters involved in the particular problem. The allowable magnitude of the multiplier corresponding to a particular set of system parameters is also investigated and the limits are defined.

This study has been divided into two major phases. The first phase includes the formulation of the transient flow problem including the inertial multiplier by following the characteristics-method procedures. The second phase concerns the investigation of errors in transient solutions due to the inertial multiplier and the numerical approximation.

The basic gas dynamic equations including the inertial multiplier are developed in Chapter II. The method-of-characteristics solution of transient flow equations is examined in Chapter III. The fundamental property of the characteristic equations is presented in a broadened sense so that the computational procedures permit not only the analysis of transients to be carried forward in time, but also the control solution to be formulated by utilizing the techniques of valve stroking. The concept of valve stroking was proposed by Streeter(21,22) to calculate a desirable valve motion procedure in liquid systems so that the surge pressure during the transient period is maintained within the allowable maximum or minimum value and, furthermore, the transient in the system ceases at the end of valve motion. The same concept of stroking computation is also useful in a gas system,(17) particularly in scheduling the operation of the system in order to meet the customer load demand and the contract minimum pressure specifications.

Chapter IV proposes a new approach to obtain a semi-analytical solution to the non-linear finite difference equations for steady oscillatory flow in natural gas systems. The solution to the non-linear equations is obtained by a process of successive approximation. The result of the second-order approximation is verified by the numerical solution of the method of characteristics. One of the most useful aspects of the proposed solution is the capability of yielding explicitly an error bound due to the discretization, and to the employment of the inertial multiplier for a particular set of system parameters. By pursuing the numerical experiments for a wide range of system parameters, the correlation of the error bound with the system parameters may be found. The results of the error investigations are presented in diagrams included in Chapter V. The application of the error diagrams to transient analysis and control computations are illustrated in Chapter VI. The transient solution by using the computed Δx and α is checked by an accurate solution to justify the reliability of the result in the error diagrams.

Along with this study, a field experiment was conducted in a natural gas transmission line owned by the Consumers Power Company, Jackson, Michigan. The result of the experiment is included in Chapter VII to demonstrate the validity of the concept of the inertial multiplier in the simulation of transient flow in natural gas systems.

II. BASIC EQUATIONS INCLUDING THE INERTIAL MULTIPLIER

The partial differential equations describing transient flow in a natural gas pipeline have been developed in a number of references (2,17,18,23). The derivation of these equations from the considerations of mass conservation, equation of motion, and equation of state is reviewed and presented herein. A discussion is also included on the relative significance of the terms in the equation of motion and the validity of introducing an inertial multiplier into the equation when the transient condition in the system is rather mild or of long duration.

In the development of the basic equations for natural gas transient flows, the following assumptions are normally made:

1. The pipe is inelastic and the flow in the pipe is considered to be one-dimensional.
2. The slope and the cross-sectional area of the pipe over any particular reach are constant.
3. The steady state friction factor may be used to describe the transient friction loss. In particular, the Darcy-Weisbach friction factor is used in this thesis.
4. The flow in natural gas pipelines is isothermal and the gas temperature is assumed constant and maintained at the average system temperature.
5. The compressibility factor in a specific system is constant.

2.1 Equation of State

The equation of state for an isothermal system commonly used in the natural gas industry is

$$\frac{p}{\rho} = \frac{zgRT}{M_w} \qquad (2.1)$$

where

p = absolute pressure in lb/ft^2

ρ = gas mass density in slugs/ft^3

z = compressibility factor

g = acceleration of gravity in ft/sec^2

R = gas constant being 1545 ft-lb/°R - mole

T = absolute temperature in °R

M_w = gas molecular weight in lb/mole

The compressibility factor z is a correction factor for the deviation of the real gas from the ideal gas law. It is dependent upon temperature, pressure and the composition of gas. However, in natural gas transmission systems, the change in z due to pressure variation is only about one percent per 100 psi pressure fluctuation at normal pressure level. The gas composition and temperature in normal operating conditions are substantially constant. Thus the use of a constant z in a particular system is sufficient for most transient analyses. The entire term on the right-hand side of Equation (2.1) is therefore a constant and the equation of state becomes

$$\frac{p}{\rho} = \frac{zgRT}{M_w} = B^2 \qquad (2.2)$$

where B is the isothermal wave speed in ft/sec.

2.2 Continuity Equation

The continuity equation describing the conservation of mass in a control volume states that the net influx of mass must equal the time rate of increase of mass in the control volume. With reference to Figure 1, this statement expressed in terms of variables becomes

$$M - [M + \frac{\partial M}{\partial x} \delta x] = \frac{\partial}{\partial t}(\rho A \delta x) \tag{2.3}$$

in which

M = the mass flow rate in slugs/sec

A = cross-sectional area of the pipe in ft^2

δx = length of control volume

and x, t are distance and time, respectively.

After combining Equation (2.3) with Equation (2.2),

$$\frac{\partial p}{\partial t} + \frac{B^2}{A} \frac{\partial M}{\partial x} = 0 \tag{2.4}$$

2.3 Equation of Motion

Newton's Second Law of Motion as written for the control volume in Figure 2 yields

$$pA - [pA + \frac{\partial}{\partial x}(pA)\delta x] - \tau_o \pi D \, \delta x - \rho g A \, \delta x \sin\theta = (\rho A \delta x) \frac{DV}{Dt} \tag{2.5}$$

where V is the fluid transport velocity in ft/sec; θ is the angle of the inclination of pipe measured upward from the horizontal; τ_o, lb/ft^2, is the wall shear stress, which is generally taken to be the steady state shear stress for the same velocity and can be replaced by the following relationship.(20)

$$\tau_o = \frac{\rho f V |V|}{8} \tag{2.6}$$

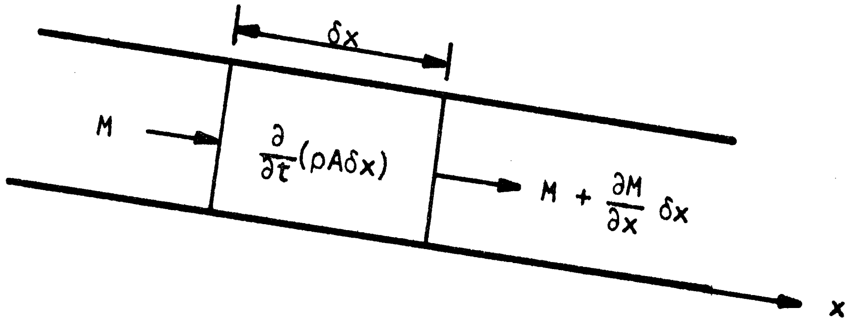


Figure 1. Control Volume for Mass Conservation.

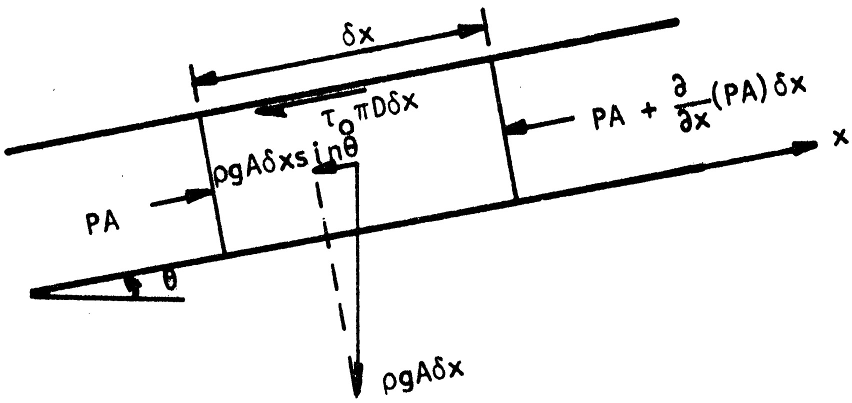


Figure 2. Control Volume for Equation of Motion.

with f the Darcy-Weisbach friction factor. The absolute value of V is introduced to maintain the correct shear stress direction for negative velocities. The substantial derivative, $\frac{DV}{Dt}$, in Equation (2.5) includes two parts: $V\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial t}$. If the mass flow rate

$$M = \rho AV = \frac{pAV}{B^2} \tag{2.7}$$

is differentiated with respect to x and rearranged, $V\frac{\partial V}{\partial x}$ can be expressed as

$$V\frac{\partial V}{\partial x} = \frac{MB^4}{A^2p^2}\left(\frac{\partial M}{\partial x} - \frac{M}{P}\frac{\partial p}{\partial x}\right) \tag{2.8}$$

Similarly, if Equation (2.7) is differentiated with respect to t and Equation (2.4) is used to replace $\partial p/\partial t$, then

$$\frac{\partial V}{\partial t} = \frac{B^2}{Ap}\left(\frac{\partial M}{\partial t} + \frac{MB^2}{pA}\frac{\partial M}{\partial x}\right) \tag{2.9}$$

After combining Equation (2.5) with Equations (2.6) to (2.9),

$$\left(1 - \frac{B^2M^2}{A^2p^2}\right)\frac{\partial P}{\partial x} + \frac{fB^2M|M|}{2DA^2p} + \frac{pg\sin\theta}{B^2} + \frac{1}{A}\left(\frac{\partial M}{\partial t} + \frac{2B^2M}{pA}\frac{\partial M}{\partial x}\right) = 0 \tag{2.10}$$

In normal conditions, $\frac{B^2M^2}{A^2p^2}\left(= \frac{V^2}{B^2}\right)$ in the first term of Equation (2.10) is very small compared with unity and may be neglected. The entire last term, which is formed mainly from inertia, is small compared with other terms in the equation, and the second part of this term, being on the order of $\frac{2V}{B}\frac{\partial M}{\partial t}$, is even smaller than the first part. In this treatment, the term containing $\partial M/\partial x$ is dropped and the term containing $\partial M/\partial t$ is not only retained but enlarged to some extent through multiplication by a constant α^2, where α is the inertial multiplier. Thus the equation of motion is simplified to

$$\frac{\partial p}{\partial x} + \frac{fB^2M|M|}{2DA^2p} + \frac{pg\sin\theta}{B^2} + \frac{\alpha^2}{A}\frac{\partial M}{\partial t} = 0 \tag{2.11}$$

The concept of employing the inertial multiplier is quite similar to that of neglecting the entire inertial term, provided a judicious treatment regarding the limitation on the magnitude of the multiplier can be achieved. The advantages of employing the inertial multiplier can be visualized when the method of characteristics is used for problem solution. The inertial multiplier not only enlarges the time increment for standard characteristics-method analysis of transient flows, but also provides the capability of obtaining a more feasible control solution when the method of characteristics is used to perform the valve stroking computations.[17,18]

The selection of an appropriate magnitude of the inertial multiplier is primarily based upon the severity of transient in the particular system. For a long duration transient, a big α may be used to permit a large time step for a stable numerical solution. On the other hand, if the problem involves a rapid transient, a small α should be used to produce an accurate result. The manner in which α may be used to suit the particular problem provides a great deal of flexibility in the application of the method of characteristics to analyze general natural gas transient flows. The investigation of the limitation on the allowable magnitude of α will be deferred until Chapter V where the errors in the solutions due to the inertial multiplier and the numerical approximation are considered.

Equations (2.4) and (2.11) are the basic dynamic equations governing the solution of natural gas transient flow problems. Consistent units have been used in the derivation. The variables M and p are expressed in terms of slugs/sec and lb/ft^2, respectively. However, in the natural gas industry, the mass flow rate is normally described in terms of millions of cubic feet per day (mmcfd) at the base pressure (P_b) and the base temperature (T_b), and the pressure is normally described by pounds per square inch (psi). For direct engineering applications, the units of M and p used by the gas industry are incorporated into Equations (2.4) and (2.11). Thus

$$\frac{\partial P}{\partial t} + \frac{KB^2}{A}\frac{\partial Q}{\partial x} = 0 \tag{2.12}$$

$$\frac{\partial P}{\partial x} + \frac{K\alpha^2}{A}\frac{\partial Q}{\partial t} + \frac{Pg\sin\theta}{B^2} + \frac{K^2 f B^2 Q|Q|}{2DA^2 P} = 0 \tag{2.13}$$

where Q and P are, respectively, the mass flow rate in mmcfd and the pressure in psi, and K is the constant

$$K = \frac{10^6}{86400}\frac{M_w P_b}{gRT_b} \tag{2.14}$$

The units of other variables and the constant parameters in Equations (2.12) and (2.13) remain as previously defined.

III. TRANSIENT SOLUTIONS BY THE METHOD OF CHARACTERISTICS

Equations (2.12) and (2.13) in the previous chapter constitute a set of hyperbolic partial differential equations with two dependent variables P and Q and two independent variables x and t. Due to the nonlinearities occuring in Equation (2.13), an exact closed form solution to these equations has not been available. Numerical approximation must be applied for the solution. Without employing the inertial multiplier in the system equations, the numerical solution to these equations has been accomplished by one of the three most widely accepted current numerical procedures, namely, the method of characteristics and the implicit and other explicit finite difference methods. The first method converts the partial differential equations into four particular ordinary differential equations, which are integrated numerically.(17,18,23) The last two methods involve the placement of the partial differential equations into finite difference form, which leads to a solution of a set of non-linear simultaneous equations,(7,24,25) or to an explicit solution.(2,6,15) There are advantages and disadvantages inherent in each method. The implicit procedure offers the advantage of guaranteed stability for a large time increment, but has the disadvantage of requiring the solution of a set of non-linear simultaneous equations at each time step. The computer storage for a complicated network is therefore very large. The explicit procedure requires relatively little storage requirement but the time increment is limited by certain stability criteria so that the computational time becomes excessive for long duration transients in a complex system. The method of characteristics provides a stable accurate solution of the equations as long as the time increment is restricted to the reach length divided by the wave speed. Consequently, the computation of slow transients with this method is again very time-consuming and uneconomical.

The inertial multiplier introduced in the previous chapter provides the capability of relaxing the restriction on the time increment with the method-of-characteristics procedure. Herein the method of characteristics is used to derive the natural gas transient flow solution.

3.1 Characteristic Equations

Let Equation (2.12) be denoted by J_1 and Equation (2.13) by J_2. These two equations can be combined linearly by employing an unknown multiplier λ. Thus

$$J_2 + \lambda J_1 = 0 \tag{3.1}$$

Equation substitutions and rearrangement are made to yield

$$\lambda\left(\frac{\partial P}{\partial t} + \frac{1}{\lambda}\frac{\partial P}{\partial x}\right) + \frac{K\alpha^2}{A}\left(\frac{\partial Q}{\partial t} + \frac{\lambda B^2}{\alpha^2}\frac{\partial Q}{\partial x}\right) + \frac{Pg\sin\theta}{B^2} + \frac{K^2 f B^2 Q|Q|}{2DA^2P} = 0 \tag{3.2}$$

Any two real, distinct non-zero values of λ in Equation (3.2) will produce two independent equations whose solution satisfies the original system of Equations (2.12) and (2.13). In particular, from the theory of characteristics, Equation (3.2) may be transformed into the total differential equation

$$\lambda\frac{dP}{dt} + \frac{K\alpha^2}{A}\frac{dQ}{dt} + \frac{Pg\sin\theta}{B^2} + \frac{K^2 f B^2 Q|Q|}{2DA^2P} = 0 \tag{3.3}$$

provided the following relation exists

$$\frac{dx}{dt} = \frac{1}{\lambda} = \frac{\lambda B^2}{\alpha^2} \tag{3.4}$$

The two particular values of λ to make this transformation are obtained from the right part of Equations (3.4). Thus

$$\lambda = \pm \frac{\alpha}{B} \tag{3.5}$$

By substituting these values of λ back into Equations (3.4),

$$\frac{dx}{dt} = \pm \frac{B}{\alpha} \tag{3.6}$$

As can be seen in Equation (3.6), the previous independent variable, x, is now dependent upon t. Equation (3.6) represents two characteristic lines, denoted by C^+ and C^-, in the x-t plane. After combining Equation (3.5) with Equation (3.3) and grouping the results with the corresponding Equation (3.6), one obtains the following two pairs of ordinary differential equations, identified as C^+ and C^-

$$C^+ \begin{cases} \dfrac{\alpha}{B}\dfrac{dP^2}{dt} + \dfrac{2K\alpha^2}{A} P\dfrac{dQ}{dt} + \dfrac{2P^2 g\sin\theta}{B^2} + \dfrac{K^2 f B^2 Q|Q|}{DA^2} = 0 & (3.7) \\[2ex] \dfrac{dx}{dt} = \dfrac{B}{\alpha} & (3.8) \end{cases}$$

$$C^- \begin{cases} \dfrac{\alpha}{B}\dfrac{dP^2}{dt} - \dfrac{2K\alpha^2}{A} P\dfrac{dQ}{dt} - \dfrac{2P^2 g\sin\theta}{B^2} - \dfrac{K^2 f B^2 Q|Q|}{DA^2} = 0 & (3.9) \\[2ex] \dfrac{dx}{dt} = -\dfrac{B}{\alpha} & (3.10) \end{cases}$$

Equations (3.7) and (3.9) are the compatibility relations which describe the variations in the dependent variables along the respective characteristic lines. Equation (3.7) can only be applied when Equation (3.8) is satisfied, likewise, Equation (3.9) is valid only when Equation (3.10) is satisfied. Equations (3.7) to (3.10) are of a particularly simple form, inasmuch as each equation contains only total differentials of the variables with respect to time. It must be emphasized that according to the derivation, every solution of the original partial differential equations satisfies this set of total differential equations and the converse is also true.[4]

Before proceeding further with the possible numerical techniques that may be constructed to obtain a solution, we shall discuss briefly the basic concept of the application of the characteristic equations for the calculations of analysis and control of transient flow problems.

3.2 Concept of Analysis and Control Computations

The characteristic equations, Equations (3.7) to (3.10), are used as the working equations for the simulation of transient flows in natural gas pipeline systems. The application of these equations can easily be visualized by reference to Figure 3. Equations (3.8) and (3.10) are represented by the C^+ and C^- characteristic lines in the x-t plane. At the intersection point, P, all four equations are valid and a solution for the variables, x, t, P, Q is possible. Thus if all conditions are known at the points R and S, the conditions at point P can be obtained from Equations (3.7) to (3.10). It can also be recognized that if conditions are known at some point along the continuation of C^+ characteristic, say at R' in place of R, an equally valid solution can be obtained for conditions at P. Similarly, a solution at P may also be determined if conditions at R and S' are known.

Along with this interpretation in the x-t plane, it can be observed that if the initial conditions along a pipe AB are completely known, a solution can be obtained at any point within ABC in Figure 4. The area within ABC is therefore the domain of dependence of initial conditions AB since the conditions within this area are uniquely determined by the given conditions. By specifying boundary conditions at x=0 and x=L, a complete solution can be carried

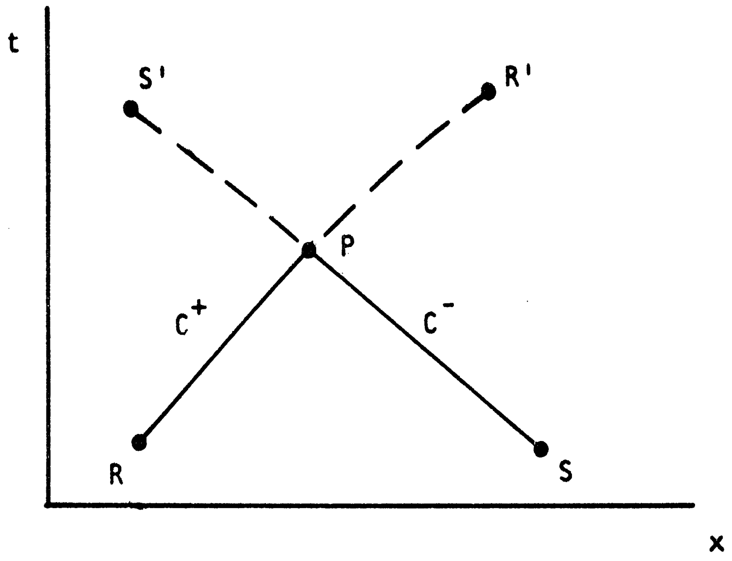


Figure 3. Characteristic Lines in the x-t Plane.

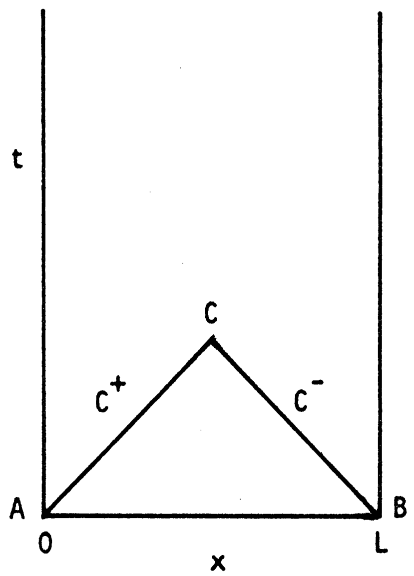


Figure 4. Domain of Dependence, Initial Conditions Given.

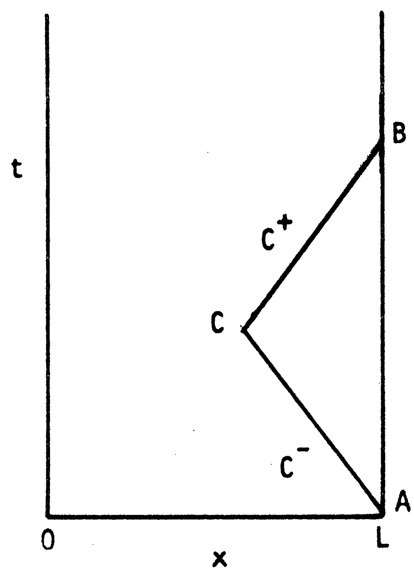


Figure 5. Domain of Dependence, Conditions Given at $x = L$.

forward in time. This is the standard procedure of analysis of a transient flow problem when the method of characteristics is used.[22]

The same concept of analysis of a transient may be extended into an idea of controlling transient conditions in natural gas pipelines, particularly in scheduling operations for gas distribution systems in order to fulfill the variable load demand and contract pressure specifications. To demonstrate the capability of making such a control calculation, it is assumed that the pressure and the flow rate at the downstream end of the pipe are specified for a short duration. A complete solution is possible in the region ABC of Figure 5 because the region within ABC is the domain of dependence of the given conditions at AB. By extending the duration of time over which conditions are specified at the downstream boundary, the entire x-t plane can be made a domain of dependence except the portions above the upper C^+ and below the lower C^- characteristics in Figure 6. Thus the conditions at the upstream boundary are completely determined over the duration CD, Figure 6. In order to adequately define the operation of the control mechanisms at the upstream boundary, it is necessary to complete the control computations over the entire transient duration. The conditions below AC and above BD can be calculated from the initial and final conditions, respectively. It is not necessary to have steady initial or final conditions in a natural gas pipeline, however, the initial data and the desired final conditions must be defined for any specific problem.

The above concept of obtaining a transient control solution has been referred to as valve stroking[14,17,21,22] and has theoretical justification. However, it has been noted that without employing the

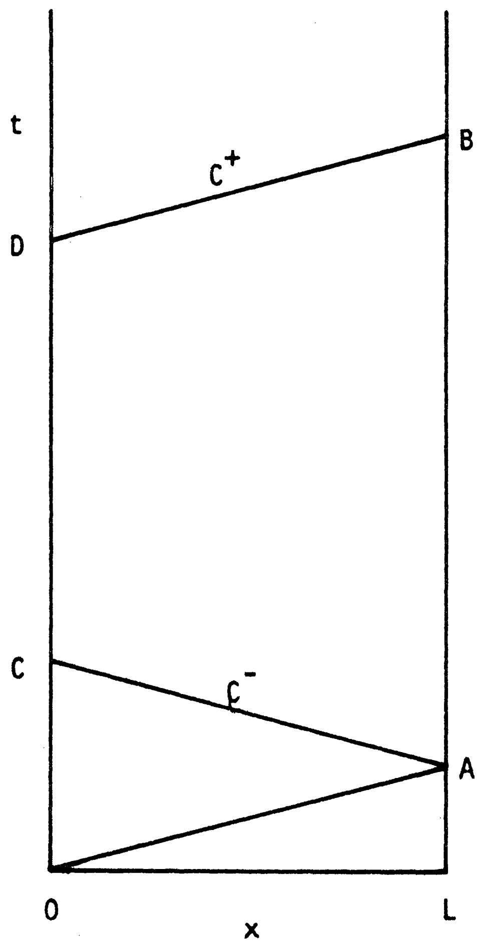


Figure 6. The x-t Plane Used to Illustrate the Control Computational Scheme.

inertial multiplier, it is possible to obtain a feasible control solution in a natural gas system only when it is subjected to some specially assigned boundary conditions, particularly when the pressure specification is substantially smooth. In cases when severe change in pressure specification at the downstream boundary is required, it is physically necessary that a tremendous amount of flow rate be fed into the system at the upstream boundary to afford this pressure change, inasmuch as the inertia in the gas system is negligible. Thus the control solution so obtained may not be feasible in practical operations. However, the inertial multiplier α incorporated into the dynamic equation not only increases the inertial effect in the system but also alters the characteristic directions as can be seen in Equations (3.7) to (3.10). The change in the characteristic directions permits an earlier control action to be taken, which, consequently, yields a more practical solution. An example illustrating the computation of the control solution in a compressor station will be shown in Chapter VI.

3.3 Finite Difference Approximation

The characteristic equations, Equations (3.7) to (3.10), must be placed in finite difference forms before they can be used for a problem solution. The second-order finite difference approximation may be constructed by employing the trapezoidal rule formula

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{1}{2}\left[f(x_0) + f(x_1)\right](x_1 - x_0) \qquad (3.11)$$

Equations (3.8) and (3.10) are used to determine the mesh size ratio in a grid system if the method of specified-time intervals[11] is employed. Thus the time increment Δt is related to the reach length Δx by

$$\Delta t = \frac{\alpha}{B} \Delta x \tag{3.12}$$

With reference to Figure 7, Equations (3.7) and (3.9) are now multiplied by dt and integrated along respective characteristic lines utilizing the formula in Equation (3.11). The results are

$$C^+ \quad P_P^2 - P_R^2 + \frac{KB\alpha}{A}(P_P + P_R)(Q_P - Q_R) + \frac{g\Delta x \sin\theta}{B^2}(P_P^2 + P_R^2)$$

$$+ \frac{K^2 f B^2 \Delta x}{2DA^2}(Q_P|Q_P| + Q_R|Q_R|) = 0 \tag{3.13}$$

$$C^- \quad P_P^2 - P_S^2 - \frac{KB\alpha}{A}(P_P + P_S)(Q_P - Q_S) - \frac{g\Delta x \sin\theta}{B^2}(P_P^2 + P_S^2)$$

$$- \frac{K^2 f B^2 \Delta x}{2DA^2}(Q_P|Q_P| + Q_S|Q_S|) = 0 \tag{3.14}$$

in which $B\Delta t/\alpha$ has been replaced by Δx from Equation (3.12) and the subscripts are used to define the location of the variables in the x-t plane.

It can be observed from Equation (3.12) that with the use of the inertial multiplier the time increment is proportional to this multiplier α. The value of α allowable in a given system is dependent primarily upon the severity of transient. For a very slow transient, an appreciable α may be used which allows a significant increase in Δt for numerical computation. The determination of both α and Δx for a given problem will be discussed in Chapter V.

3.4 Steady State Equations

Transient analysis calculations usually start from an initial steady state condition. The development of an equation to calculate

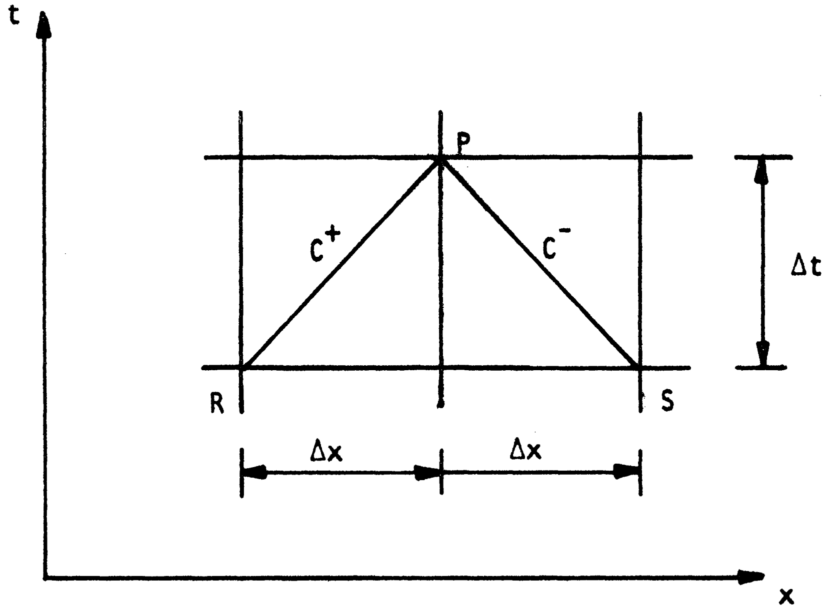


Figure 7. The x-t Diagram with the Method of Specified-Time Intervals.

the steady state solution which confirms the one obtained from the characteristic equations is of primary interest in modelling a transient flow analysis program.

If the term of time variation in the equation of motion, Equation (2.11), is dropped, the differential steady state equation becomes

$$\frac{dP}{dx} + \frac{Pg\sin\theta}{B^2} + \frac{K^2fB^2Q^2}{2DA^2P} = 0 \tag{3.15}$$

This equation is then multiplied by $2Pdx$ and integrated over the length L from P_1 upstream to P_2 downstream. The result is

$$\frac{P_1^2 + K^2fB^4Q^2/(2gDA^2\sin\theta)}{P_2^2 + K^2fB^4Q^2/(2gDA^2\sin\theta)} = e^{2gL\sin\theta/B^2} \tag{3.16}$$

After rearranging,

$$P_1^2 - e^sP_2^2 = \frac{K^2fB^2L}{DA^2}\,\frac{e^s-1}{s}\,Q^2 \tag{3.17}$$

in which $s = 2gL\sin\theta/B^2$ (3.18)

Equation (3.17) is applicable to obtain steady state solution in any pipeline at uniform slope. For a horizontal pipeline, $(e^s-1)/s = 1$, $e^s = 1$, and Equation (3.17) is reduced to

$$P_1^2 - P_2^2 = \frac{K^2fB^2L}{DA^2}\,Q^2 \tag{3.19}$$

In the natural gas industry, empirical equations for horizontal steady state flow in the form of

$$P_1^2 - P_2^2 = C\,\frac{Q^m}{D^n}\,L \tag{3.20}$$

are used. The constants C, m, n are dependent upon the pipeline characteristics and fluid properties. Equation (3.17) may also be used to calculate the steady vertical flow in the wells. In this application, θ is taken to be 90°, and P_1 is considered to be the sand face pressure at the depth L, P_2 is the well head pressure.[10]

In natural gas transmission systems, the pipelines are normally horizontal or of very small slope. Thus, if Equation (3.16) is rearranged as

$$\left[P_1^2 + \frac{K^2 f B^4 Q^2}{2gDA^2 \sin\theta}\right] e^{-gL\sin\theta/B^2} = \left[P_2^2 + \frac{K^2 f B^4 Q^2}{2gDA^2 \sin\theta}\right] e^{gL\sin\theta/B^2}$$

and the substitution of

$$e^{\pm gL\sin\theta/B^2} \approx 1 \pm gL\sin\theta/B^2$$

is made, the following results

$$P_2^2 - P_1^2 + \frac{gL\sin\theta}{B^2}(P_1^2 + P_2^2) + \frac{K^2 f B^2 L}{DA^2} Q^2 = 0 \qquad (3.21)$$

This equation matches with Equations (3.13) and (3.14), when $Q_P = Q_R = Q_S$. Equation (3.21) will be used to set the initial steady state condition for a transient calculation when slightly inclined pipelines exist in the system.

3.5 Basic Analysis Computational Procedures

The concept of transient analysis computation by the method of characteristics has been explained briefly in Section 3.2. The computational scheme using Equations (3.13) and (3.14) can further be illustrated by considering an example of a single pipeline. By reference to Figure 8, consider a pipe to be divided into N equal reaches. The spacings of a mesh in the x-t plane are determined as $\Delta x = L/N$ and $\Delta t = \alpha L/BN$ in which N and α are found a priori by the criteria presented in Chapter V. It is assumed that the transient calculation starts at some time, t_0, when conditions P and Q are completely known for the sections 0, 1, ...N-1, N. The initially known

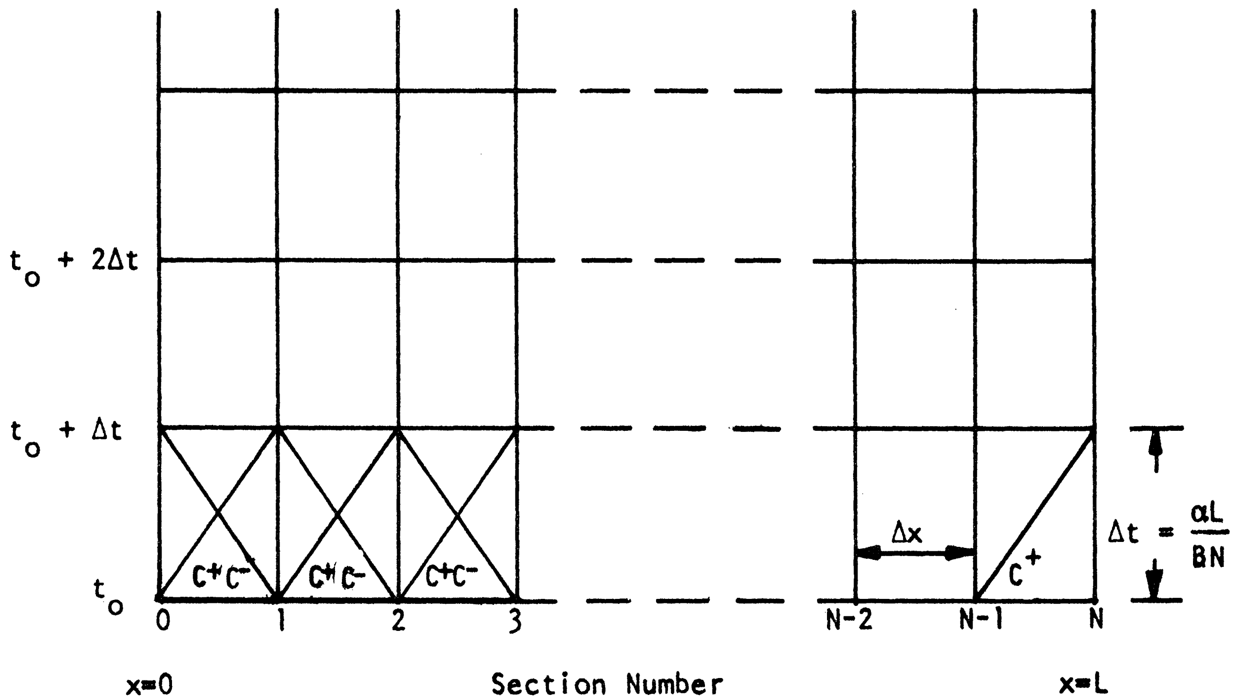


Figure 8. The Characteristic Grid of a Single Pipeline.

conditions may be the steady state solution or a transient solution obtained from previous calculation. The conditions P and Q for the internal sections 1, 2, ...N-1 at time $t_0 + \Delta t$ may be found one section at a time by solving Equations (3.13) and (3.14) simultaneously. The subscripts R and S in these two equations refer to the appropriate points on the line of known information, t_0, and the subscript P refers to the point on line $t_0 + \Delta t$ where conditions are to be determined. The solution of Equations (3.13) and (3.14) requires an iterative procedure. The Newton-Raphson iterative method is used for this work.

At the boundary section 0 or N, one condition at each boundary must be specified. The unknown condition at each boundary may be found by using an appropriate characteristic equation. Equation (3.14) is used with the upstream boundary condition and Equation (3.13) is used with the downstream boundary condition. The solution may be obtained directly or by an iterative scheme dependent upon the complexity of the boundary condition. Now all the conditions along the pipe at time $t_0 + \Delta t$ are completely known. The solution may then be advanced one time step to $t_0 + 2\Delta t$ by the same procedures.

Although a single pipeline is considered in the above procedures, the same computational scheme can also be applied to a complicated network system, where only the conditions at the boundaries need special treatment. The following section describes some of the boundary conditions normally encountered in a natural gas system.

3.6 Boundary Conditions

Boundary conditions in a natural gas network may have various configurations but in almost every case the mass conservation at the boundaries must be observed. Also, the inertial effects of the gas at the boundaries are neglected.

The most frequently encountered boundary in a network is the junction where more than two pipes interconnect. The boundary conditions at a junction require that the net mass inflow to the junction be zero and the pressure be common to all pipes leaving or entering the junction. Equations describing the junction boundary conditions can be easily formulated and incorporated into a transient analysis model.

An orifice or a regulator installed in a line causes an abrupt drop in pressure. The amount of pressure-drop is dependent upon the area of the valve opening, the inlet pressure and the flow rate. The regulator can be modelled by the following equations.(19)

When $P_u \leq 1.82\ P_d$ (Subsonic flow condition)

$$Q = C\sqrt{(P_u - P_d)P_d} \tag{3.22}$$

When $P_u > 1.82\ P_d$ (Sonic flow condition)

$$Q = 0.5\ C\ P_u \tag{3.23}$$

in which P_u and P_d are respectively the upstream and downstream pressures in psia, Q is the flow through the valve in mmcfd, C is the constant dependent upon the valve flow area and the loss coefficient. The regulators are normally controlled by discharge pressure or flow rate. In either case, the valve opening is adjusted automatically so that the control variable is maintained at its set-point value.

A compressor also has a very small gas storage capacity. It responds so rapidly that it affects the transient response by changing the magnitude of pressure or flow rather than the timing or phase of these variables. The equation modelling a compressor is given by[19,24]

$$HP = Q\left[k_1 \left(\frac{P_d}{P_s}\right)^{k_3} - k_2\right] \qquad (3.24)$$

in which HP is the horsepower used by the compressor, Q is the flow rate in mmcfd, P_s and P_d are respectively the suction and discharge pressures in psia. k_1, k_2 and k_3 are constant for a given compressor unit. Compressor operations are normally controlled by discharge pressure, suction pressure or flow rate. In each case the controlled variable is maintained at its set-point value until the power calculated from Equation (3.24) exceeds the maximum power limit at the station; then the unit operates at the maximum available power.

In each of the above boundary conditions, the equation describing the appropriate boundary condition combined with one characteristic equation from each connecting pipe yields the solution of the unknown conditions at the boundary.

IV. SEMI-ANALYTICAL SOLUTION OF NONLINEAR DIFFERENCE EQUATIONS

In the method of characteristics presented in Chapter III, the characteristic equations in total differential form are replaced by the corresponding finite difference equations from which the numerical solution is carried forward in time. With the aid of high-speed computing equipment, this method of solution is applicable for any complicated system. In fact, due to the existence of non-linearities in the equations, the numerical procedure provides the most effective means of accomplishing a solution to the transient flow problem.

However, as mentioned before, the numerical error produced with the finite difference approach is appreciable in natural gas systems if too large a reach length is used in the computation. The error bound in the numerical solution corresponding to the magnitude of reach length must be determined before the method can be used with confidence for problem solutions. In particular, with the model developed in this thesis, the error bound due to both the numerical approximation and the use of the inertial multiplier shall be investigated.

In order to pursue this investigation, a semi-analytical method to obtain a solution of the non-linear finite difference equations including the inertial multiplier, Equations (3.13) and (3.14), is developed below. The term semi-analytical is used, since the procedure to obtain the solution involves the theory of impedance for the variables with steady oscillations with a numerical technique employed to obtain the transfer matrix extended over the entire length of pipe. The error bound determined by the application of the semi-analytical solution is studied in the next chapter.

The problem of steady oscillatory flow in a single pipeline is considered herein. Steady oscillations occur in a system if a periodically changing boundary condition exists long enough to establish flow conditions which are periodic at every point of the system. The problems of steady oscillatory flow in the liquid systems are generally handled by the impedance method which yields an analytical solution to the linearized partial differential Equations.[22] In natural gas systems, the linearized equations are solved by the technique of transfer functions.[24] However, the results are not quite satisfactory for high friction systems, because the linearized friction term does not adequately represent the high friction effect in a natural gas pipeline. In view of this fact, a second order representation of the friction must be included to improve the result. Although the second order friction term again constitutes the non-linearity in the system equations, the order of magnitude of this non-linear term is much smaller than that of the other ones. The solution of these non-linear equations may be obtained by a process of successive approximations. This process of successive approximation is performed as follows. The first approximation to the solution is obtained from the linearized differential equations by the impedance method. By employing the solution of first approximation to estimate the non-linear terms which form the non-homogeneous part of the equations, the non-linear differential equations are transformed into the non-homogeneous linear differential equations. The solution of the second approximation can be achieved by solving the non-homogeneous linear differential equations. Higher approximations may be accomplished by this process of repeated substitutions. However, the solution of the second approximation is sufficient for engineering purposes.

Much of the theory developed for the solution of differential equations may be applicable for the solution of difference equations. (References 5, 8 and 9.) Although the above-mentioned procedure is applied to the differential equations, the same approach is also suitable for a system of difference equations.(5,9,12) In the following sections, the first and second approximations to the solution of finite difference equations are developed.

4.1 Steady Oscillatory Flow

The system to be considered is a single pipeline with fixed inlet pressure P_o and variable flow demand at the outlet end expressed as

$$Q(L,t) = Q_o + \Delta Q \sin\omega t \tag{4.1}$$

where Q_o is the steady state flow rate in mmcfd, ΔQ is the amplitude of the flow variation in mmcfd, ω is the frequency in rad/sec, t is time in seconds. Equations (3.13) and (3.14) in the previous chapter together with these boundary condition specifications are used to obtain a solution.

It is convenient to convert all the variables, P, Q, x, t, into dimensionless form by using the following substitutions

$$P^* = \frac{P}{P_o} \;,\; Q^* = \frac{Q}{Q_o} \;,\; x^* = \frac{x}{L} \;,\; t^* = \frac{tB}{L} \tag{4.2}$$

where L and B are, respectively, the length of pipe and the wave speed. The finite difference equations, Equations (3.13) and (3.14), expressed in dimensionless form become

$$P^*(I,J)^2 - P^*(I-1,J-1)^2 + m\alpha\,[P^*(I,J)+P^*(I-1,J-1)].$$
$$[Q^*(I,J)-Q^*(I-1,J-1)] + \sigma m^2 h\,[Q^*(I,J)^2+Q^*(I-1,J-1)^2] = 0 \tag{4.3}$$

$$P^*(I-1,J+1)^2-P^*(I,J)^2 - m\alpha\,[P^*(I-1,J+1)+P^*(I,J)].$$
$$[Q^*(I-1,J+1)-Q^*(I,J)] - \sigma m^2 h\,[Q^*(I-1,J+1)^2+Q^*(I,J)^2] = 0 \quad (4.4)$$

in which the slope of the pipe is omitted and the absolute value sign on the flow rate in friction terms is removed assuming that no flow reversal occurs in the system. The subscripts defining the locations in the x-t plane are changed to the double indices according to the notations shown in Figure 9, where the indices I, J are the discrete variables in space and time respectively. The parameters m, σ, and h are defined as

$$m = \frac{KBQ_O}{AP_O}\,, \quad \sigma = \frac{fL}{2D}\,, \quad h = \frac{\Delta x}{L} \quad (4.5)$$

The boundary conditions are

$$P^*(0,J) = 1 \quad (4.6a)$$

$$Q^*(N,J) = 1 + \Delta q\,\sin(J\omega^*k) \quad (4.6b)$$

in which $\Delta q = \Delta Q/Q_O, \quad \omega^* = \omega L/B$ (4.7)

and k is the time increment in dimensionless form. According to Equation (3.12)

$$k = \alpha h$$

For simplicity, the superscript asterisk is dropped hereafter, yet the quantity is understood to be dimensionless unless otherwise indicated.

The instantaneous pressure $P(I,J)$ is divided into two parts, the mean or average pressure $\overline{P}(I)$ and oscillatory pressure $p(I,J)$. Thus

$$P(I,J) = \overline{P}(I) + p(I,J) \quad (4.9)$$

where $\overline{P}(I)$ is a discrete function of space only. Similarly, the flow rate can be expressed

$$Q(I,J) = \overline{Q} + q(I,J) = 1 + q(I,J) \quad (4.10)$$

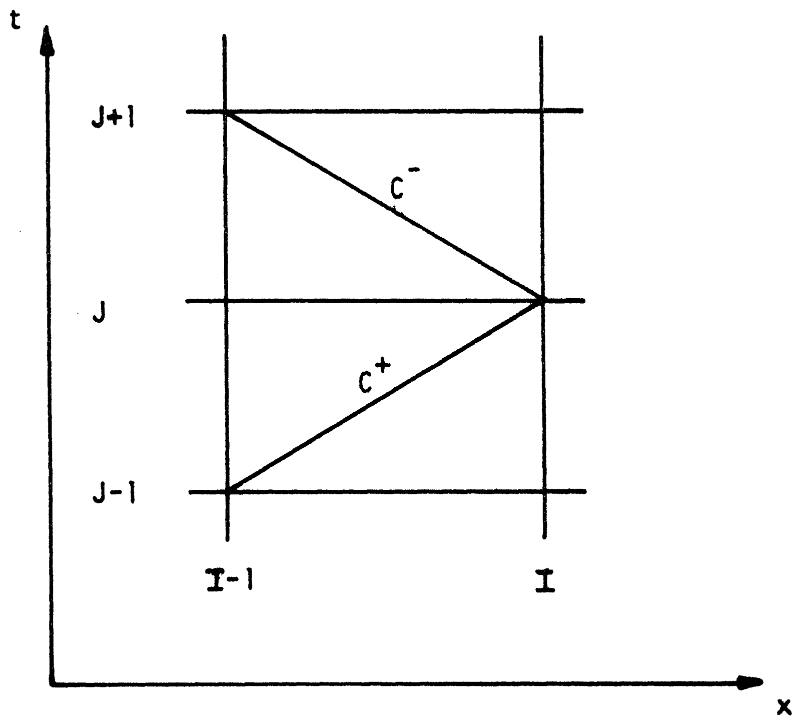


Figure 9. The x-t Plane Used for the Semi-Analytical Solution.

where $\overline{Q}$ is the mean or steady state flow rate in dimensionless form and is unity. $q(I,J)$ is the fluctuation from the mean. By substituting Equations (4.9) and (4.10) into Equations (4.3) and (4.4), and rearranging

$$\begin{aligned}&[\overline{P}(I)^2-\overline{P}(I-1)^2+2\sigma m^2 h] + 2\{\overline{P}(I)p(I,J)+[m\alpha\tilde{P}(I)+\sigma m^2 h]q(I,J)\\ &\quad -\overline{P}(I-1)p(I-1,J-1) - [m\alpha\tilde{P}(I)-\sigma m^2 h]q(I-1,J-1)\}\\ &\quad = -\{p(I,J)^2-p(I-1,J-1)^2 + m\alpha[p(I,J)+p(I-1,J-1)].\\ &\quad [q(I,J)-q(I-1,J-1)] + \sigma m^2 h[q(I,J)^2+q(I-1,J-1)^2]\}\end{aligned} \tag{4.11}$$

$$\begin{aligned}&[\overline{P}(I-1)^2-\overline{P}(I)^2-2\sigma m^2 h] + 2\{-\overline{P}(I)p(I,J)+[m\alpha\tilde{P}(I)-\sigma m^2 h]q(I,J)\\ &\quad +\overline{P}(I-1)p(I-1,J+1) - [m\alpha\tilde{P}(I)+\sigma m^2 h]q(I-1,J+1)\}\\ &\quad = -\{p(I-1,J+1)^2-p(I,J)^2 - m\alpha[p(I-1,J+1)+p(I,J)].\\ &\quad [q(I-1,J+1)-q(I,J)] - \sigma m^2 h[q(I-1,J+1)^2+q(I,J)^2]\}\end{aligned} \tag{4.12}$$

in which

$$\tilde{P}(I) = \frac{1}{2}[\overline{P}(I) + \overline{P}(I-1)]$$

The terms in the right-hand side of both Equations (4.11) and (4.12) are non-linear oscillatory variables which may be neglected in the first approximation, inasmuch as the magnitude of these terms is much smaller than those on the left-hand side. The solution of the first approximation is then used to estimate the non-linear terms for the second approximation.

In the following derivations, the subscripts 1 and 2 to the variables $\overline{P}$, p, and q are used to indicate the solutions of the first and second approximations, respectively. To simplify the writing, we shall adopt two linear difference operators

$$L_1[p_n,q_n] = \overline{P}_n(I)p_n(I,J) + [m\alpha\tilde{P}_n(I)+\sigma m^2h]\ q_n(I,J) - \overline{P}_n(I-1)p_n(I-1,J-1) - [m\alpha\tilde{P}_n(I) - \sigma m^2h]\ q_n(I-1,J-1) \tag{4.13}$$

$$L_2[p_n,q_n] = - \overline{P}_n(I)p_n(I,J) + [m\alpha\tilde{P}_n(I)- \sigma m^2h]\ q(I,J) + \overline{P}_n(I-1)p_n(I-1,J+1) - [m\alpha\tilde{P}_n(I) + \sigma m^2h]\ q_n(I-1,J+1) \tag{4.14}$$

to represent the sequences of linear oscillation in Equations (4.11) and (4.12). The arguments p_n,q_n in either operator represent the dependent variables in the sequences. The subscript n ranges from 1 to 2. It is convenient to employ the mathematical short-hand of using the complex variable to express a sinusoidal variation. For instance,

$$C_1 \cos\omega t = Re[C_1 e^{i\omega t}]$$

where "Re" stands for the "real part of". The letters Re are dropped but are understood to exist. The real part of the final result represents the solutions.

4.2 First Approximation

By neglecting the non-linear orcillating terms, one may reduce Equations (4.11) and (4.12) to

$$[\overline{P}_1(I)^2 - \overline{P}_1(I-1)^2 + 2\ \sigma m^2h] + 2L_1[p_1,q_1] = 0 \tag{4.15}$$

$$[\overline{P}_1(I-1)^2 - \overline{P}_1(I)^2 - 2\ \sigma m^2h] + 2L_2[p_1,q_1] = 0 \tag{4.16}$$

In either of these two equations, the terms in the first square bracket are independent of time while those denoted by the difference operator are linear oscillations with respect to time. Each equation may be transformed into uncoupled equations as follows

$$P_1(I)^2 = P_1(I-1)^2 - 2\ \sigma m^2h \tag{4.17}$$

$$L_1\ [p_1,q_1] = 0 \tag{4.18}$$

$$L_2\ [p_1,q_1] = 0 \tag{4.19}$$

where Equation (4.17) results from both Equations (4.15) and (4.16).

Equation (4.17) together with the given condition, $\overline{P}_1(0) = 1$, may be used as a recursive relation to yield the solution of the mean pressure. Equations (4.18) and (4.19) with the boundary conditions

$$p_1(0,J) = 0 \tag{4.20a}$$

$$q_1(N,J) = -\, i\, \Delta q\; e^{iJ\omega k} \tag{4.20b}$$

are used to achieve the solution of the oscillatory variables.

In a steady oscillatory flow problem, the solution is oscillating with respect to time at any location of the pipe. Thus, by using the separation-of-variable technique, one may assume

$$p_1(I,J) = \xi_1(I)\; e^{iJ\omega k} \tag{4.21}$$

$$q_1(I,J) = \eta_1(I)\; e^{iJ\omega k} \tag{4.22}$$

where $i = \sqrt{-1}$, and $\xi_1(I)$ and $\eta_1(I)$ are unknown complex variables and are discrete functions of space only. After substituting Equations (4.21) and (4.22) into Equations (4.18) and (4.19), and dropping the common factor $e^{iJ\omega k}$

$$\overline{P}_1(I)\xi_1(I) + [m\alpha\tilde{P}_1(I) + \sigma m^2 h]\,\eta_1(I) - E^{-1}\overline{P}_1(I-1)\xi_1(I-1)$$
$$- E^{-1}[m\alpha\tilde{P}_1(I) - \sigma m^2 h]\,\eta_1(I-1) = 0 \tag{4.23}$$

$$- \overline{P}_1(I)\,\xi_1(I) + [m\alpha\tilde{P}_1(I) - \sigma m^2 h]\eta_1(I) + E\,\overline{P}_1(I-1)\,\xi_1(I-1)$$
$$- E\,[m\alpha\tilde{P}_1(I) + \sigma m^2 h]\,\eta_1(I-1) = 0 \tag{4.24}$$

in which

$$E = e^{i\omega k} = e^{i\omega\alpha h} \tag{4.25}$$

The boundary conditions, Equations (4.20), become

$$\xi_1(0) = 0 \tag{4.26a}$$

$$\eta_1(N) = -\, i\, \Delta q \tag{4.26b}$$

Equations (4.23) and (4.24) are a set of homogeneous linear difference equations with variable coefficients. The solution may be obtained by the technique of transfer matrix.[3] First of all, Equations (4.23) and (4.24) are combined and rearranged to yield

$$\xi_1(I) = C11(I)\ \xi_1(I-1) + C12(I)\ \eta_1(I-1) \tag{4.27}$$

$$\eta_1(I) = C21(I)\ \xi_1(I-1) + C22(I)\ \eta_1(I-1) \tag{4.28}$$

in which

$$C11(I) = \frac{\bar{P}_1(I-1)}{\bar{P}_1(I)}\left[\cos\omega\alpha h + \frac{i\sigma mh\ \sin\omega\alpha h}{\alpha\tilde{P}_1(I)}\right] \tag{4.29a}$$

$$C12(I) = -\left[im\alpha\tilde{P}_1(I)\sin\omega\alpha h + 2\sigma m^2h\ \cos\omega\alpha h + \frac{i\sigma^2m^3h^2\sin\omega\alpha h}{\alpha\tilde{P}(I)}\right]\Big/\bar{P}_1(I) \tag{4.29b}$$

$$C21(I) = -\frac{i\bar{P}_1(I-1)\sin\omega\alpha h}{m\alpha\tilde{P}_1(I)} \tag{4.29c}$$

$$C22(I) = \cos\omega\alpha h + \frac{i\sigma mh\ \sin\omega\alpha h}{\alpha\tilde{P}(I)} \tag{4.29d}$$

and the following substitutions have been made

$$\frac{1}{2}(E + E^{-1}) = \cos\omega\alpha h$$

$$\frac{1}{2}(E - E^{-1}) = i\ \sin\omega\alpha h$$

To facilitate the mathematical derivations, matrix notations will be employed. Let the state vector $\{Z_1(I)\}$ at Ith section be defined

$$\{Z_1(I)\} = \begin{Bmatrix} \xi_1(I) \\ \eta_1(I) \end{Bmatrix} \tag{4.30}$$

If the matrix

$$[TN(I)] = \begin{bmatrix} C11(I) & C12(I) \\ C21(I) & C22(I) \end{bmatrix} \tag{4.31}$$

is also defined, then Equations (4.27) and (4.28) can be written as

$$\{Z_1(I)\} = [TN(I)] \ \{Z_1(I-1)\} \tag{4.32}$$

As can be seen in Figure 10, Equation (4.32) is applied to the Ith reach of a pipeline. $[TN(I)]$, which relates the state vectors at both ends of the Ith reach, is the field transfer matrix of this reach. Extension of Equation (4.32) over the entire N reaches of the pipe may be made to yield

$$\{Z_1(N)\} = [SN] \ \{Z_1(0)\} \tag{4.33}$$

where

$$[SN] = [TN(N)] \ [TN(N-1)] \ \dots \ [TN(1)] \tag{4.34}$$

$[SN]$ is the over-all transfer matrix which relates the state vector at one end of a pipe to that at the other end. Although the elements of a field transfer matrix involve complex arithmetic, the multiplication of matrices shown in Equation (4.34) may be performed numerically by the use of the computer.

In view of the relation in Equation (4.33) and the boundary conditions in Equations (4.26), one may immediately obtain the solution for the unknown variables $\xi_1(N)$ and $\eta_1(0)$ at the boundaries. The recursive relationship in Equation (4.32) together with the initial value $\{Z_1(0)\}$ may be used for the solution of the intermediate sections. The solution is thus completely determined at the entire N+1 sections. According to Equations (4.21) and (4.22), the oscillatory pressure $p_1(I,J)$ and the flow rate $q_1(I,J)$ may also be determined by multiplying $e^{iJ\omega k}$ to the solution of $\xi_1(I)$ and $\eta_1(I)$. By taking the real part of the final result, one may express the solution as

$$p_1(I,J) = PC(I) \cos [J\omega k + PS(I)] \tag{4.35}$$

$$q_1(I,J) = QC(I) \cos [J\omega k + QS(I)] \tag{4.36}$$

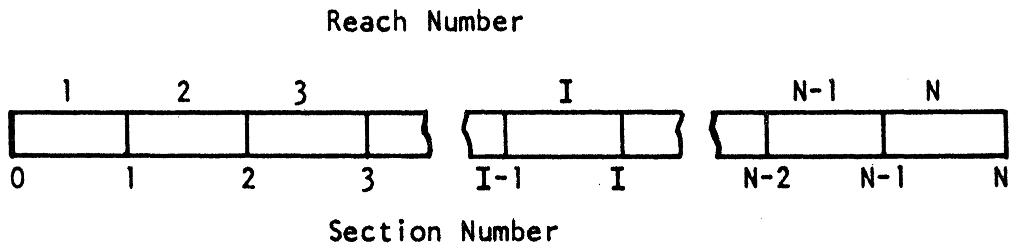


Figure 10. Portion of a Single Pipeline.

where PC and PS are, respectively, the modulus and the argument of the complex variable ξ_1. QC and QS are the corresponding relations of η_1.

4.3 Second Approximation

The solution of the second approximation is to satisfy

$$[\overline{P}_2(I)^2 - \overline{P}_2(I-1)^2 + 2\sigma m^2 h] + 2\ L_1[p_2, q_2] = G(I,J) \qquad (4.37)$$

$$[\overline{P}_2(I-1)^2 - \overline{P}_2(I)^2 - 2\sigma m^2 h] + 2\ L_2[p_2, q_2] = H(I,J) \qquad (4.38)$$

with the boundary conditions

$$p_2(0,J) = 0 \qquad (4.39a)$$

$$q_2(N,J) = -i\ \Delta q\ e^{iJ\omega k} \qquad (4.39b)$$

in which

$$G(I,J) = -\ \{p_1(I,J)^2 - p_1(I-1,J-1)^2 + m\alpha[p_1(I,J) + p_1(I-1,J-1)]. \\ [q_1(I,J) - q_1(I-1,J-1)] + \sigma m^2 h[q_1(I,J)^2 + q_1(I-1,J-1)^2]\} \qquad (4.40)$$

$$H(I,J) = -\{p_1(I-1,J+1)^2 - p_1(I,J)^2 - m\alpha[p_1(I-1,J+1) + p_1(I,J)]. \\ [q_1(I-1,J+1) - q_1(I,J)] - \sigma m^2 h[q_1(I-1,J+1)^2 + q_1(I,J)^2]\} \qquad (4.41)$$

G and H represent the sequences of non-linear oscillating terms, which are evaluated by the solution of the first approximation. If the solution of the first approximation shown in Equations (4.35) and (4.36) is substituted into Equations (4.40) and (4.41) and the complex variable notation to express a sinusoidal function is employed, one obtains, after simplification,

$$G(I,J) = A0(I) + 2[R1(I) + E^{-1}R2(I) + E^{-2}R3(I)]\ e^{i2J\omega k} \qquad (4.42)$$

$$H(I,J) = -A0(I) + 2[S1(I) + E\ S2(I) + E^2 S3(I)]\ e^{i2J\omega k} \qquad (4.43)$$

where E is evaluated as previously. A0 is a real variable and R1..., S1... etc. are complex variables. A0(I), R1(I)... and S1(I)... are terms representing the combinations of the known quantities of PC, PS and QC, QS at the sections I-1 and I. They are not shown herein because of the tediousness, however, they may be seen in the computer program listing in Appendix B.

In view of Equations (4.37) and (4.38), together with Equations (4.42) and (4.43), one may again uncouple each of Equations (4.37) and (4.38) to yield

$$\overline{P}_2(I)^2 = \overline{P}_2(I-1)^2 - 2\sigma m^2 h + AO(I) \qquad (4.44)$$

and

$$L_1[p_2, q_2] = [R1(I) + E^{-1}R2(I) + E^{-2}R3(I)]\, e^{i2J\omega k} \qquad (4.45)$$

$$L_2[p_2, q_2] = [S1(I) + E\ S2(I) + E^2 S3(I)]\, e^{i2J\omega k} \qquad (4.46)$$

Equation (4.44) with the specified condition, $\overline{P}_2(0) = 1$, yields the solution of the mean pressure recursively. The oscillatory solution must satisfy a set of non-homogeneous linear difference equations, Equations (4.45) and (4.46), together with the boundary conditions

$$p_2(0,J) = 0 \qquad (4.47a)$$

$$q_2(N,J) = -\,i\ \Delta q\ e^{iJ\omega k} \qquad (4.47b)$$

The system of Equations (4.45), (4.46) and (4.47) includes the oscillation due to two different harmonic frequencies. The non-homogeneous parts of Equations (4.45) and (4.46) contain known terms of the second harmonic oscillation, while the bounary condition, Equation (4.47b), is of fundamental excitation. In a linear system, the solution due to the excitation of two different harmonic frequencies may be treated independently and the results superposed to yield a complete oscillatory solution. Thus, Equations (4.45) and (4.46) together with Equations (4.47) are reduced to the following two systems.

The first system due to the fundamental frequency excitation is to satisfy

$$L_1[p_2', q_2'] = 0 \tag{4.48}$$

$$L_2[p_2', q_2'] = 0 \tag{4.49}$$

with the boundary conditions

$$p_2'(0,J) = 0 \tag{4.50a}$$

$$q_2'(N,J) = -i\ \Delta q\ e^{iJ\omega k} \tag{4.50b}$$

The second one corresponding to the second harmonic frequency must satisfy

$$L_1[p_2'', q_2''] = [R1(I) + E^{-1}R2(I) + E^{-2}R3(I)]\ e^{i2J\omega k} \tag{4.51}$$

$$L_2[p_2'', q_2''] = [S1(I) + E\ S2(I) + E^2 S3(I)]\ e^{i2J\omega k} \tag{4.52}$$

with the boundary conditions

$$p_2''(0,J) = 0 \tag{4.53a}$$

$$q_2''(N,J) = 0 \tag{4.53b}$$

The single and double primes on the variables p_2 and q_2 are used to denote the solution of each system.

The procedures to obtain a solution to the system of Equations (4.48), (4.49) and (4.50) are identical to those of the first approximation in the previous section except now the mean pressures which appear as the variable coefficients in the difference operators are calculated from Equation (4.44). The details will not be repeated herein.

We shall consider the solution of Equations (4.51), (4.52) and (4.53) in which the forcing functions appear in the non-homogeneous parts of the equations rather than at the boundary. We may again assume

$$p_2''(I,J) = \xi_2(I)\ e^{i2J\omega k} \tag{4.54}$$

$$q_2''(I,J) = \eta_2(I)\ e^{i2J\omega k} \tag{4.55}$$

where $\xi_2(I)$ and $\eta_2(I)$ are complex variables to be determined. After substituting Equations (4.54) and (4.55) into Equations (4.51) and

(4.52), and dropping the factor $e^{i2J\omega k}$

$$\bar{P}_2(I)\xi_2(I) + [m\alpha\tilde{P}_2(I)+\sigma m^2h]\,\eta_2(I) - E^{-2}\bar{P}_2(I-1)\xi_2(I-1)$$
$$-E^{-2}[m\alpha\tilde{P}_2(I)-\sigma m^2h]\eta_2(I-1) = R1(I)+E^{-1}R2(I)+E^{-2}R3(I) \tag{4.56}$$

$$-\bar{P}_2(I)\xi_2(I) + [m\alpha\tilde{P}_2(I)-\sigma m^2h]\eta_2(I) + E^2\bar{P}_2(I-1)\xi_2(I-1)$$
$$-E^2[m\alpha\tilde{P}_2(I)+\sigma m^2h]\eta_2(I-1) = S1(I)+E\ S2(I)+E^2S3(I) \tag{4.57}$$

These two equations may be combined and rearranged to yield

$$\xi_2(I) = C11(I)\ \xi_2(I-1) + C12(I)\ \eta_2(I-1) + ZP(I) \tag{4.58}$$

$$\eta_2(I) = C21(I)\ \xi_2(I-1) + C22(I)\ \eta_2(I-1) + ZQ(I) \tag{4.59}$$

in which

$$ZP(I) = [R1(I)+E^{-1}R2(I)+E^{-2}R3(I)-S1(I)-E\ S2(I)-E^2S3(I)]\ /\ [2\bar{P}_2(I)]$$
$$-\sigma mh[R1(I)+E^{-1}R2(I)+E^{-2}R3(I)+S1(I)+E\ S2(I)+E^2S3(I)]\ /\ [2\alpha\tilde{P}_2(I)\ \bar{P}_2(I)] \tag{4.60a}$$

$$ZQ(I) = [R1(I)+E^{-1}R2(I)+E^{-2}R3(I)+S1(I)+E\ S2(I)+E^2S3(I)]\ /\ [2m\alpha\tilde{P}_2(I)] \tag{4.60b}$$

and

$$C11(I) = \frac{\bar{P}_2(I-1)}{\bar{P}_2(I)}\cos 2\omega\alpha h + \frac{i\sigma mh\ \sin 2\omega\alpha h}{\alpha\tilde{P}_2(I)} \tag{4.61a}$$

$$C12(I) = -[im\alpha\tilde{P}_2(I)\ \sin 2\omega\alpha h + 2\sigma m^2h\ \cos 2\omega\alpha h + \frac{i\sigma^2m^3h^2\sin 2\omega\alpha h}{\alpha\tilde{P}_2(I)}]\ /\ \bar{P}_2(I) \tag{4.61b}$$

$$C21(I) = -\frac{i\bar{P}_2(I-1)\ \sin 2\omega\alpha h}{m\alpha\tilde{P}_2(I)} \tag{4.61c}$$

$$C22(I) = \cos 2\omega\alpha h + \frac{i\sigma mh\ \sin 2\omega\alpha h}{\alpha\tilde{P}_2(I)} \tag{4.61d}$$

According to Equations (4.54) and (4.55), the boundary conditions become

$$\xi_2(0) = 0 \tag{4.62a}$$

$$\eta_2(N) = 0 \tag{4.62b}$$

Equations (4.58) and (4.59) may also be expressed in matrix notation as

$$\{Z_2(I)\} = [TN(I)]\,\{Z_2(I-1)\} + \{F(I)\} \tag{4.63}$$

where the state vector $\{Z_2(I)\}$ and the transfer matrix $[TN(I)]$ are defined as previously. The forcing vector $\{F(I)\}$ at the Ith reach is defined

$$\{F(I)\} = \begin{Bmatrix} ZP(I) \\ ZQ(I) \end{Bmatrix} \tag{4.64}$$

Equation (4.63) subjected to the boundary conditions, Equations (4.62), is a non-homogeneous linear equation. From the theory of corresponding linear differential equations, the general solution of Equation (4.63) is obtained by summing the particular and the complementary solutions.[5,8,9] If the initial value of $\{Z_2(0)\}$ is assumed, say

$$\{Z_2(0)\} = \begin{Bmatrix} a \\ b \end{Bmatrix} \tag{4.65}$$

where a and b are any arbitrary constants, the particular solution may be obtained numerically by the recursive relationship of Equation (4.63). The particular solution of η at $I = N$ so obtained may not satisfy the boundary condition shown in Equation (4.62b). Let us denote the result of $\eta_2(N)$ by, say, c. The complementary solution must therefore satisfy the corresponding homogeneous equation

$$\{Z_2(I)\} = [TN(I)]\,\{Z_2(I-1)\} \tag{4.66}$$

subjected to the boundary conditions

$$\xi_2(0) = -a \tag{4.67a}$$

$$\eta_2(N) = -c \tag{4.67b}$$

The procedures to obtain the solution of Equations (4.66) and (4.67) by the technique of transfer matrix have already been shown in the previous section, thus the particular and the complementary solutions to the system of Equations (4.62) and (4.63) are completely determined.

After summing these particular and complementary solutions and multiplying the result by $e^{i2J\omega K}$, according to Equations (4.54) and (4.55), one obtains a general solution of the oscillatory variables, p_2'' and q_2'', corresponding to the excitation of second harmonic frequency. Superimposing this result to that of the fundamental frequency, p_2' and q_2', one obtains the complete oscillatory solution with second-order approximation.

4.4 Verification by Numerical Solution

The semi-analytical solution of the non-linear finite difference equations with second-order approximation developed previously is compared with the numerical solution by the method of characteristics. There are two reasons for making this comparison: (1) the accuracy of the method-of-characteristics solution has been verified experimentally. (References 17 and 18.) A good comparison ensures the accuracy of the second-order solution, too. (2) The solution of the second-order approximation will be used in the next chapter to investigate the error bound in the numerical solution with the method of characteristics. Thus, it is necessary to verify that the second-order solution is able to represent the numerical one for any magnitudes of reach length and the inertial multiplier.

The example considered is a single horizontal pipe of 12 miles, 1.1 ft diameter with a fixed inlet pressure of 446 psia and a variable flow demand at outlet end of $[100 + 25 \sin(\frac{2\pi t}{30})]$ in mmcfd, where t is in minutes.

The solution for the pressure at the outlet end of the pipeline determined both numerically and semi-analytically by using various reach lengths are presented. Figure 11 shows an accurate solution both with numerical method and second-order approximation by using $\Delta x = 1$ mile, $\alpha = 1$ and correspondingly $\Delta t = 4.4$ seconds. The solution of the first-order approximation is also shown in this figure to illustrate the fact that the linearized solution is not sufficient to describe transient phenomena in natural gas systems. The numerical and the second-order solutions of the same problem by using only two reaches $\Delta x = 6$ miles and one reach $\Delta x = 12$ miles are shown in Figures 12 and 13, respectively. An appropriate time increment and $\alpha = 1$ are used for each case. It can be seen from these Figures that the second-order solution is in good agreement with the numerical one when the same Δx and α are used. The results in Figures 12 and 13 are incorrect due to large Δx. However, they are presented to illustrate the fact that the second-order solution is able to represent the numerical one even though the length of reach used is unreasonably large.

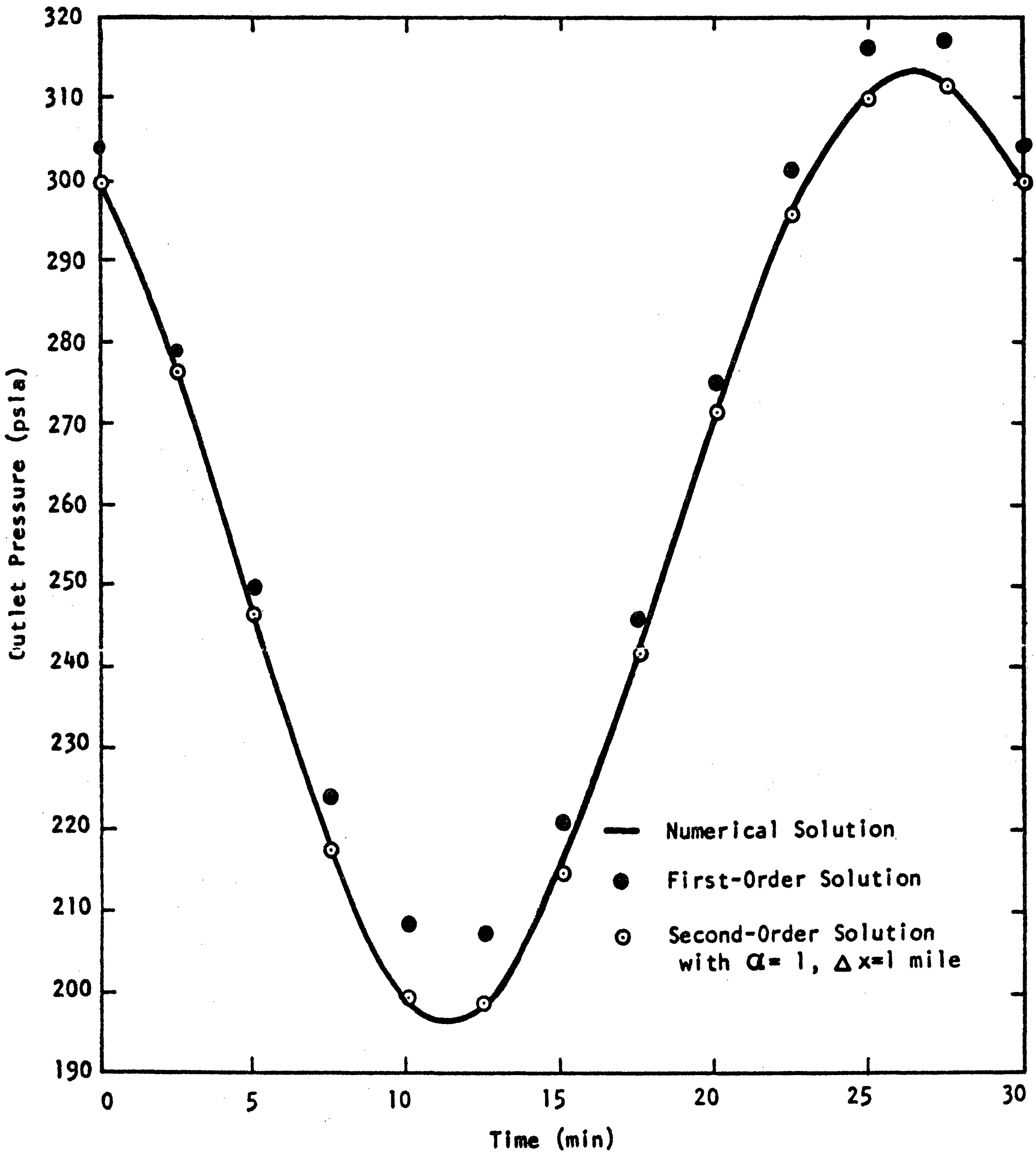


Figure 11. Comparison of the Analytical and the Numerical Solutions of the Outlet Pressure in a Simple Pipeline (1).

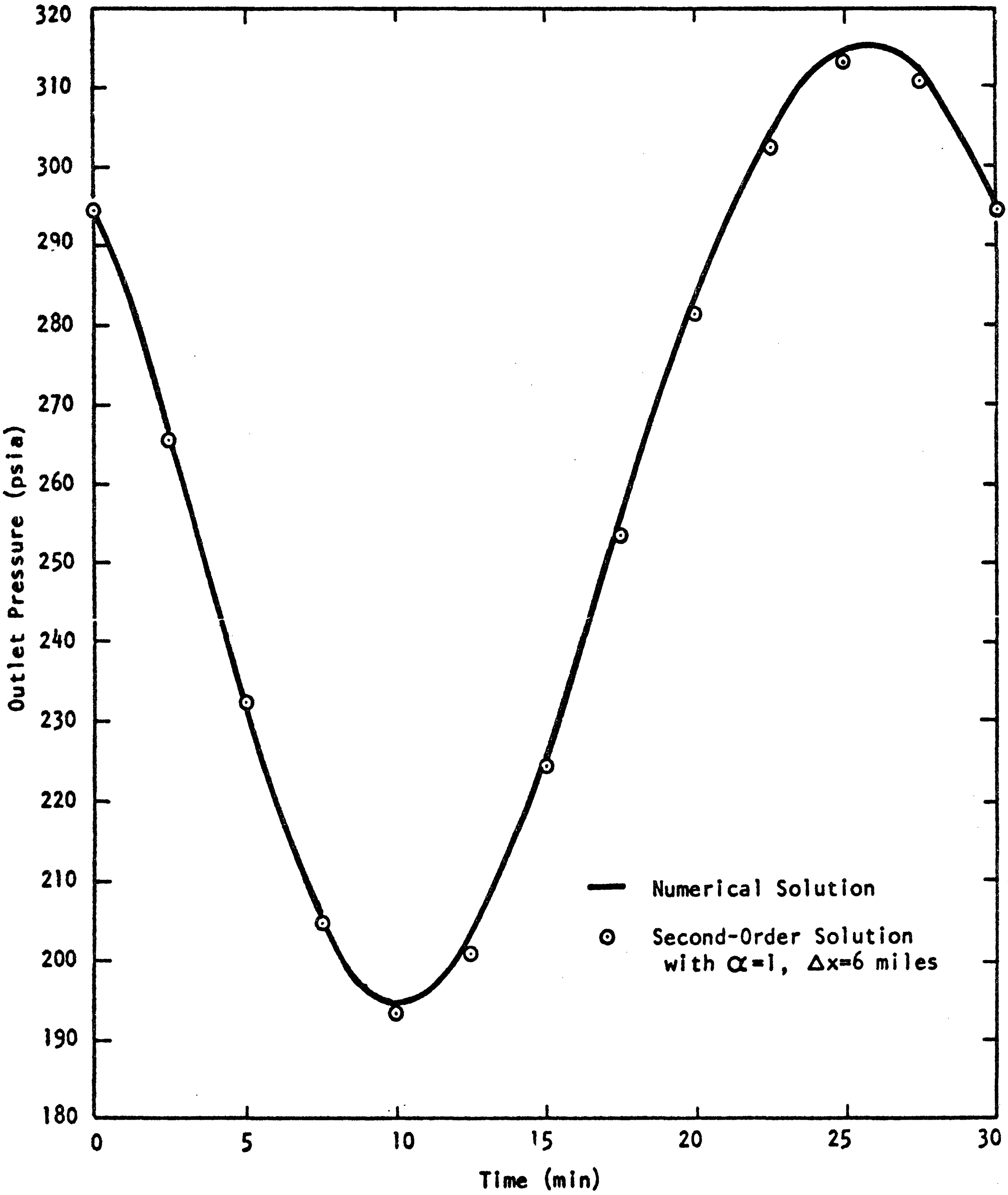


Figure 12. Comparison of the Analytical and the Numerical Solutions of the Outlet Pressure in a Simple Pipeline (II).

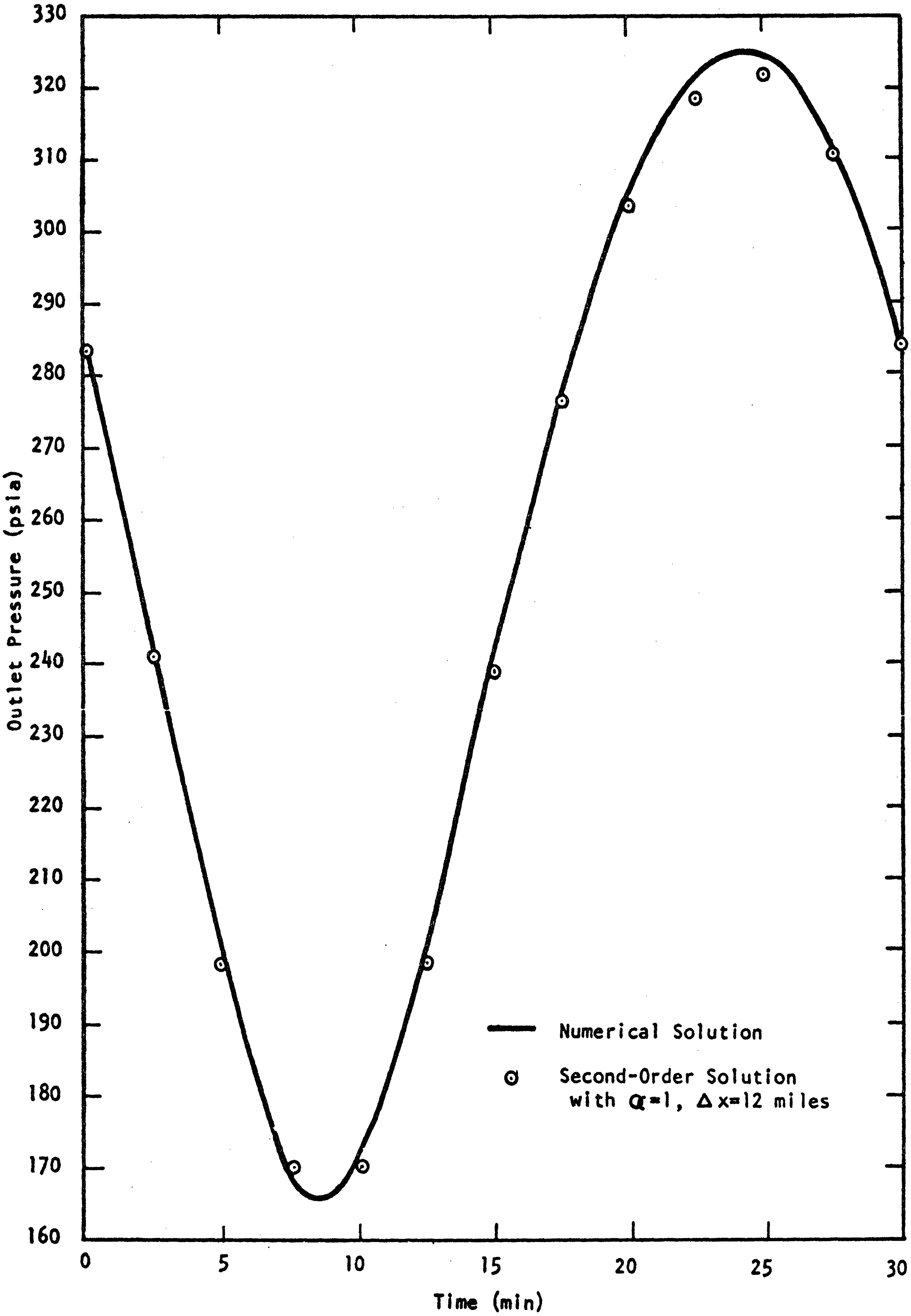


Figure 13. Comparison of the Analytical and the Numerical Solutions of the Outlet Pressure in a Simple Pipeline (III).

V. ERRORS DUE TO THE DISCRETIZATION AND THE INERTIAL MULTIPLIER

The existence of error in the solution of natural gas transient problems with the method of characteristics has been well recognized.[23] This error being a result of finite difference approximation is commonly referred to as the discretization or truncation error. Users of this method for transient solutions may eliminate this error by limiting the reach to a small length Δx . Although the solution so obtained is accurate, the computational time for the analysis of a moderately slow transient in a complicated system is very lengthy and uneconomical.

The theory developed in Chapter II has employed an inertial multiplier α in the dynamic equation. This multiplier in the model of the characteristics method significantly increases the magnitude of time increment for numerical computation. However, the question arises as to how large an inertial multiplier may be used for a system without introducing appreciable error. Consequently, it is most desirable to investigate the error existing in the characteristics-method solution thereby imposing limitations on the allowable Δx and α in a given problem. The work in this chapter is devoted to such an investigation.

5.1 Investigating Errors by Numerical Experiments

The second-order solution developed in the previous chapter is used specifically as an alternative to the numerical result when the steady oscillatory problem is concerned. The reliability of the second-order solution has also been demonstrated by the comparison with the numerical solution (see Figures 11 to 13). The advantage of adopting this solution to represent the numerical one is obvious as it expresses the result by a transfer matrix in terms of the system parameters such as Δx, α and the pipeline characteristics. Once the system parameters are given, the calculated transfer matrix combined with the given boundary conditions will express the solution at any location along the pipe explicitly as a function of time. The most useful aspect of this feature is to provide a direct means of evaluating the system error. The error in the solution may be obtained when the solution of a problem with $\Delta x \to 0$ and $\alpha \to 1$ is compared to the one with certain Δx and α. Although the solution involves complex arithmetic, the numerical experiments can be pursued by the use of the computer which permits the error to be investigated for a wide spectrum of system parameters.

It must be realized that the above procedures can determine only the numerical value of the error for a given set of parameters. For a different set of parameters, a different value of error will be obtained. Consequently, the problem of correlating the error with the corresponding system parameters arises.

From the similarity analysis, the relevant system parameters can be found as

$$\psi, \phi = F(m, \sigma, \omega, \Delta q, h, \alpha) \tag{5.1}$$

in which ψ and ϕ are, respectively, the error bounds of pressure and flow rate in dimensionless forms. m, σ, ω, etc. are dimensionless quantities defined as in the previous chapter. It has been shown from the experience during this investigation that each parameter in the argument of F in Equation (5.1) influences the errors at approximately the same degree so that the number of the governing parameters may not be reduced. Thus it is extremely difficult to present the experimental results for such a wide scope of governing parameters.

However, by examining the solution of the first-order approximation in the previous chapter and neglecting terms of small effect, one may derive the solution in algebraic form from which the error functions may also be determined algebraically (see Appendix A). The algebraic equations of the error functions so obtained may be used effectively to determine the controlling parameters that govern the error bounds from the model of the second-order approximation. The results are shown in functionals as follows

$$\frac{\psi \sqrt{P_d(1 + \sigma^2 m^2 \omega^2)}}{m \cdot \omega \cdot \Delta q} = f_1\left(\alpha, \frac{\sigma m h}{\sqrt{P_d}}\right) \qquad (5.2)$$

$$\frac{\phi \, P_d^{4/3}(1 + \sigma^2 m^2 \omega^2)}{\omega^2 \, \Delta q} = f_2\left(\alpha, \sigma m h \sqrt{\frac{m \, P_d^{1/3}}{\omega}}\right) \qquad (5.3)$$

where P_d is the steady state outlet pressure in dimensionless notation. Equations (5.2) and (5.3) are of the form which was conjectured by Equation (5.1). Furthermore, the grouping of the parameters in these two equations demonstrates the feasibility of presenting the experimental results.

Although Equations (5.2) and (5.3) are derived on the basis of the first-order approximation, the functional relations f_1 and f_2 in these two equations will be obtained by pursuing the numerical experiments with the use of the computer on the model of the second-order approximation. The results from the numerical experiments are shown in Figures 14 and 15. It can be observed from these two diagrams that in the case of $\alpha = 1$, where only the discretization error

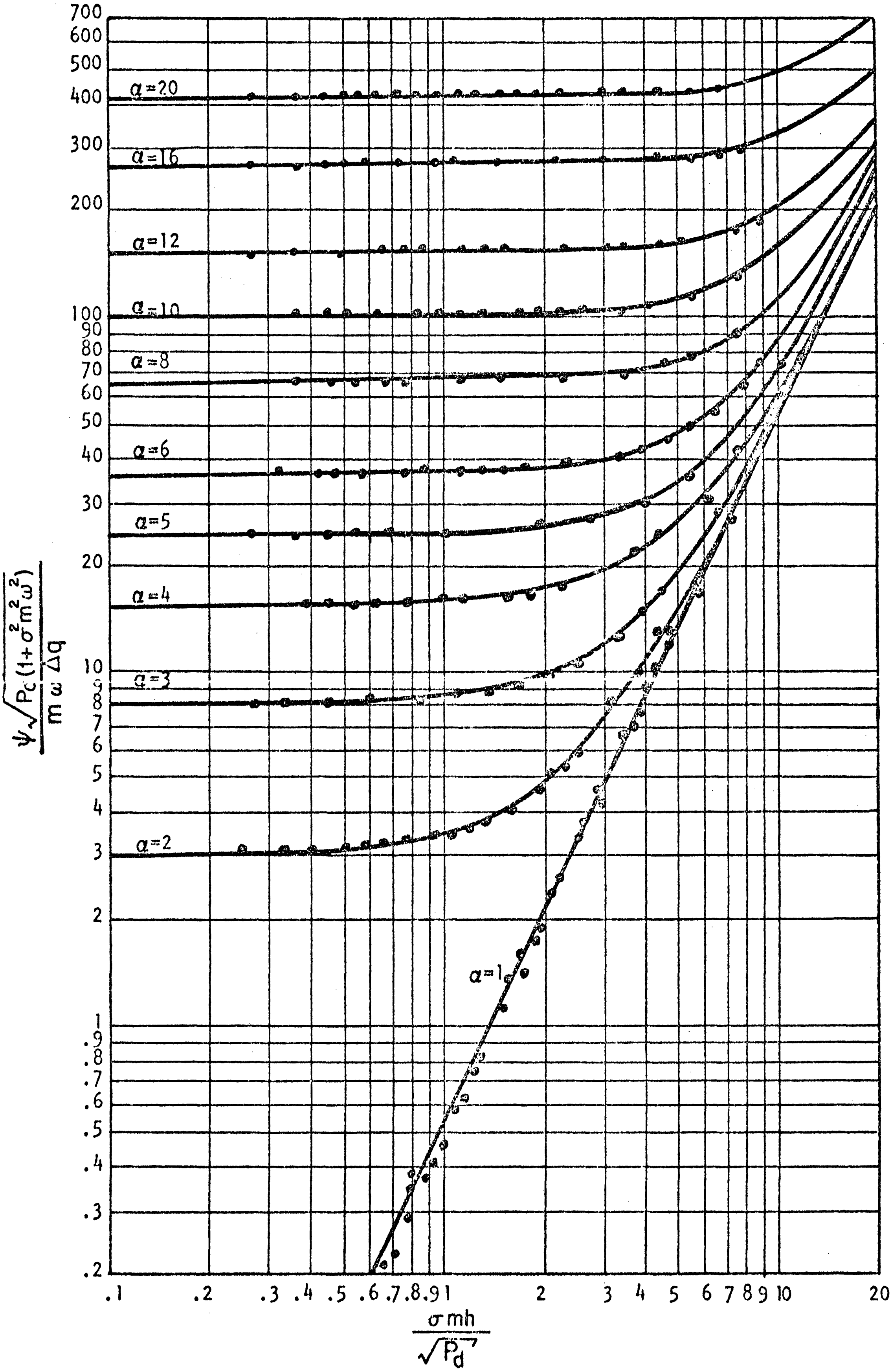


Figure 14. Error Diagram (I).

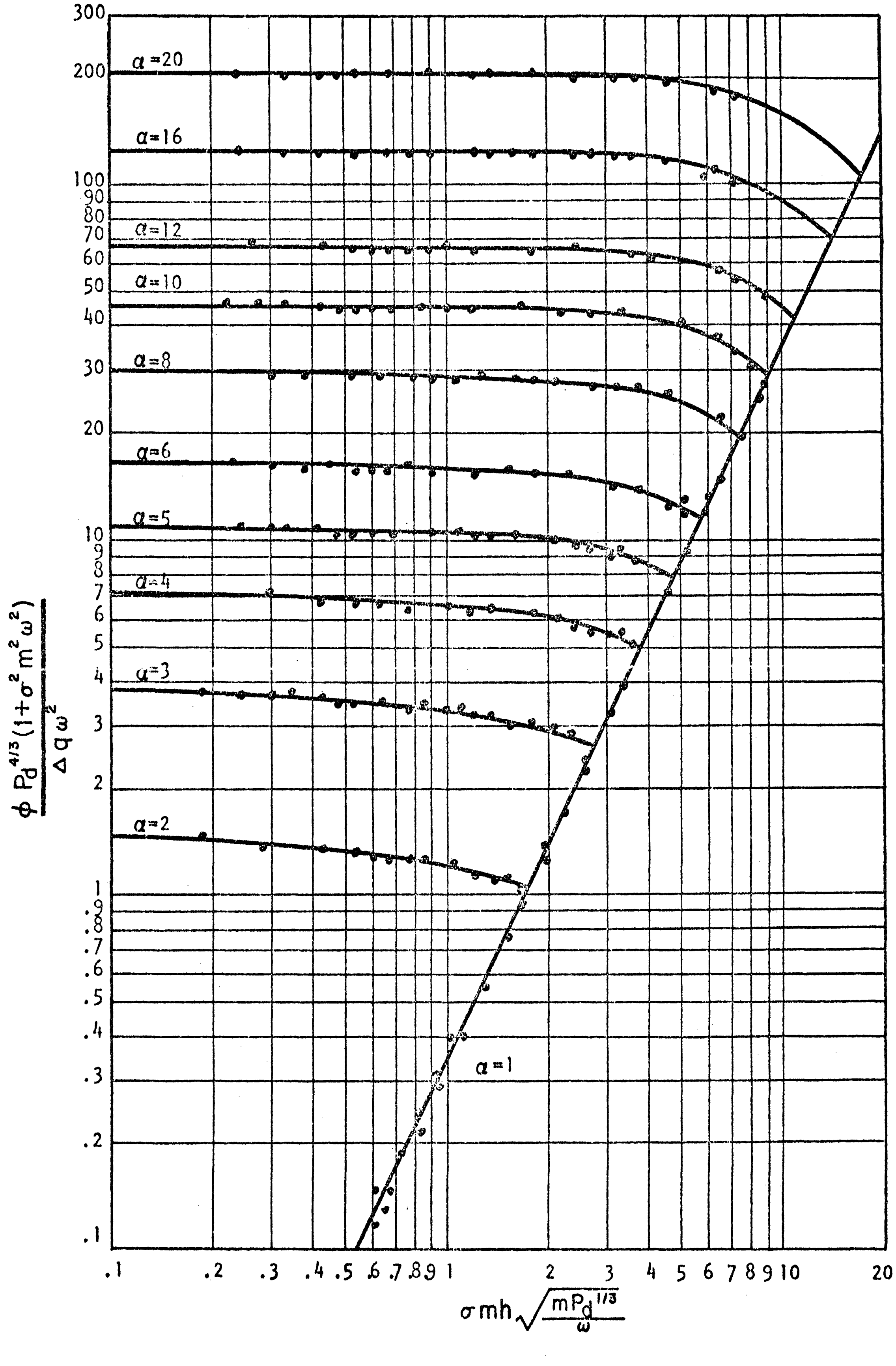


Figure 15. Error Diagram (II).

dominates, the error appears as a straight line with the slope of two. This confirms most of the mathematical derivations that the discretization error in the trapezoidal rule approximation is of second order of the grid size,[5,8] i.e.

$$\psi, \phi = O(h^2) \tag{5.4}$$

In the cases of α other than one, the results of a wide range of system parameters may well be represented by curves, which justify further that the grouping of the parameters in Equations (5.2) and (5.3) is valid.

5.2 Use of the Error Diagrams

The diagrams shown in Figures 14 and 15 can be used to select the optimal h and α so that the largest time increment in the transient analysis program may be used. It is recognized that in the natural gas industry, the measurement of flow rate is rather inaccurate. The requirement of an accurate solution in flow rate is of less significance. It is therefore suggested that the values of h and α be determined from the specification of the allowable pressure error with the use of Figure 14, while Figure 15 is used only for the purpose of checking the accuracy in the flow solution.

It must be remembered that these two diagrams are constructed on the basis of a single line system where the existence of steady oscillation in the solution is assumed. With slight modification in interpreting the parameters, they may also be applied to the analysis of a network transient with non-sinusoidal boundary condition specifications. In the application to the network transient analysis, the parameters used in these two diagrams are determined as follows. σ of each line is found from the pipeline characteristics (f, L, D), while $m \; (= \frac{KBQ_o}{AP_o})$ and $P_d \; (= \sqrt{1 - 2\sigma m^2})$ are calculated directly from the initial steady state condition. The determination of Δq and ω of each line needs some approximations. For the transient analysis of a distribution or transmission system, the flow demand at each distributing node is normally given as a function of time. If it is assumed that the amount of the amplitude in the demand curve of each node is distributed to each adjoining line during the transient condition according to the proportion of the initial steady state flow rate through the line, then, Δq of each line may be determined as the ratio of the amplitude of the flow demand curve with respect to the mean flow rate at the downstream node of the line. ω of this line may also be determined from the frequency of the same curve. Whenever the flow demand specification is not a sine function, ω may be obtained by assuming that the frequency of the demand curve is equivalent to that of a fictitious sine curve whose steepest slope matches with that of the real demand curve. Thus

$$\omega = \frac{Q_t \, L}{\Delta Q \, B} \tag{5.5}$$

in which Q_t is the steepest slope in the flow demand curve in mmcfd/sec. ΔQ is the amplitude of the demand curve in mmcfd. L and B are, respectively, the length in feet and the wave speed in ft/sec. The idea for this assumption is based upon the realization that it is the rate of change of the flow demand that governs the severity of a transient rather than the duration of the transient itself. In case the demand is altered instantaneously by a substantial amount, such as in the occasion when a valve is suddenly shut off, this assumption cannot yield a realistic frequency value. However, in that case, one understands that $\alpha = 1$ and a small Δx should be used for an accurate solution.

Although the assumptions set forth to determine Δq and ω in a network transient are rather conjectural, its applicability can be illustrated by practical examples. It is apparent that more work could be done to determine with justification the parameters in the error diagrams for the examples of complex network systems with general boundary condition specifications. The examples that illustrate the use of the error diagrams described previously together with the comparison of the solution so obtained with an accurate solution are presented in the next chapter.

VI. TRANSIENT ANALYSIS AND CONTROL APPLICATIONS

In the preceding chapters, the development of a model for the solution of natural gas transient flow problems with the method of characteristics including the inertial multiplier has been presented. The error existing in the solution due to the numerical approximation and the use of the inertial multiplier has also been investigated and the results have been presented in diagrams so that the users of the model may properly select the magnitudes of Δx and α for an optimum time increment in the numerical computations. In this chapter, the application of the transient analysis model is illustrated by examples. An example is also included to demonstrate the capability of the use of a valve-stroking computation to obtain a feasible control solution taken in a compressor station when an unexpected swing of load demand occurs in the system and the minimum delivery pressure must be maintained.

6.1 Applications to Transient Analysis Computation

Two examples are presented herein. The first is a single pipeline with sinusoidal boundary condition specification. The second is a simple network system subjected to arbitrarily specified boundary conditions. In both examples, the procedure to determine Δx and α is illustrated and the solution obtained by using the computed Δx and α with the standard characteristics-method analysis procedure is compared to the accurate solution.

Example 1 - Consider a single pipeline with a fixed inlet pressure of 500 psia, and a variable flow at the demand end of $[80+20\sin(\frac{2\pi t}{60})]$ in mmcfd, where t is in minutes. The parameters in the problem are L = 12 miles, D = 1.2 ft, f = 0.012, B = 1190 ft/sec.

According to the definitions set forth in the previous chapters, the dimensionless parameters are calculated as

$$\sigma = 316.8,\ m = 0.019,\ \Delta q = 0.25,\ \omega = 0.092,\ P_d = 0.874$$

If the allowable error in pressure solution is five psi, which corresponds to one percent of the inlet pressure, one calculates the values of the grouped parameters

$$\frac{\psi \sqrt{P_d(1 + \sigma^2 m^2 \omega^2)}}{m\omega\Delta q} = 25$$

$$\frac{\sigma m}{\sqrt{P_d}} = 6.54$$

With these numbers and the use of the diagram in Figure 14 and also bearing in mind that the value of h (being $\Delta x/L$) must be the inverse of an integer number of reaches, one may determine the most desirable values of h and α for the maximum time increment through a trial-and-error procedure. In this example, $h = 1$ and $\alpha = 3$ are determined for the maximum time increment. The corresponding Δx and Δt are respectively 12 miles and 160 seconds. Now, by reference to Figure 15, the error of inlet flow rate, ϕ, with the values of h and α determined is found to be 0.77 percent of the inlet flow which corresponds to a discrepancy of 0.6 mmcfd.

The transient analysis solution using the computed Δx and Δt is shown in Figure 16. An accurate solution by using a small Δx of one mile, $\alpha = 1$ and correspondingly Δt of 4.4 seconds is also shown in Figure 16. It can be seen that the maximum discrepancy in pressure solution is about five psi and the discrepancy in flow solution is less than one mmcfd.

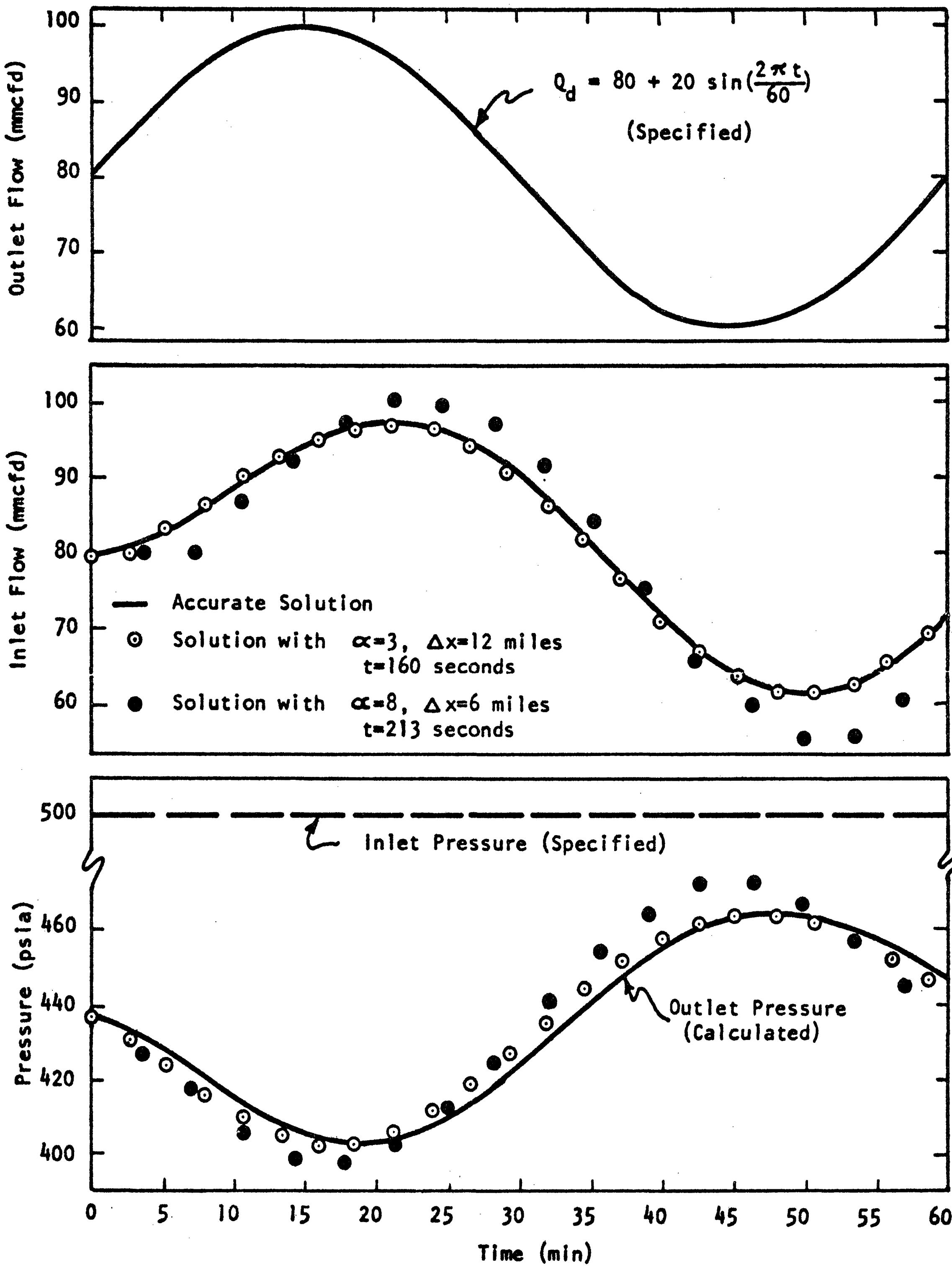
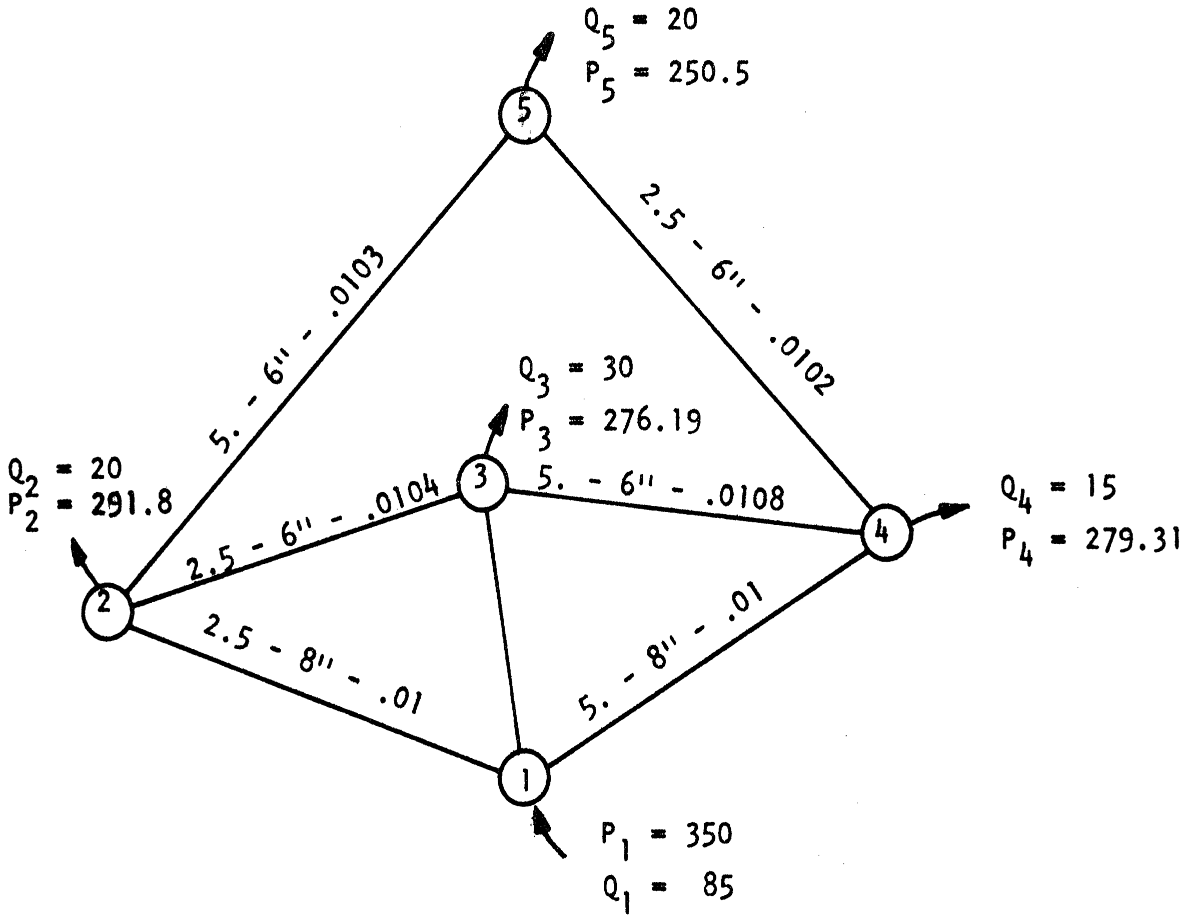


Figure 16. Example 1 -- Transient Pressure and Flow Variation in a Single Pipeline.

If it were assumed one can tolerate an error of three percent (15 psi discrepancy) in the pressure solution of the same problem, the values of h and α determined by following the same procedures of calculations are $h = 0.5$ and $\alpha = 8$. The corresponding Δx and Δt are six miles and 213 seconds, respectively. The error in the flow solution determined by using the computed h and α and with reference to Figure 15 is 8.25 percent of the initial flow which corresponds to a discrepancy of 6.6 mmcfd. The transient solution by using the computed Δx and Δt is also shown in Figure 16. The maximum deviation of the solution from the accurate one falls within the range being specified or determined by the use of the error diagrams as can be noted in this figure.

Example 2 - Figure 17 shows the geometric configuration of a network with initial steady state flows and nodal pressures indicated. The pressure at Node 1 is held at 350 psig while the flow demands at the other nodes are specified as shown in Figure 18. The error in pressure solution of 3.5 psi (i.e. $\psi = 1$ percent) is allowable.

If the Δx of 2.5 miles is used throughout the entire system, the pipeline characteristics and the calculated dimensionless parameters are listed in Table I. By using the data in the last two columns of Table I and with reference to Figure 14, one can see that the pipeline from Node 1 to Node 4 is most critical to yield the smallest value of α based upon the same degree of error. One should use the data from this line to determine the time increment Δt for the whole system. The value of α obtained from this line is 10.5 and the corresponding Δt is two minutes. The values of α at the other lines shall be calculated on the basis of this Δt. It can be



The Sequences of the Line Data are L(mi)-D-f

Figure 17. Schematic Diagram of the Network of Example 2.

TABLE I

PROPERTIES OF THE PIPELINES IN THE NETWORK OF EXAMPLE 2

From Node	To Node	D (inch)	L (mile)	Q_o (mmcfd)	f	σ	m	Δq	ω	h*	P_d	$\frac{\sigma mh}{\sqrt{P_d}}$	$\psi \frac{\sqrt{P_d(1+\sigma^2 m^2 \omega^2)}}{m\omega\Delta q}$ **
1	2	8	2.5	36.75	.0099	98	.0404	.25	.0031	1.	.834	4.34	292
1	3	6	2.5	19.95	.01	132	.0386	.17	.0031	1.	.789	5.74	437
1	4	8	5.0	28.3	.01	198	.031	.33	.0062	.5	.798	3.44	141
2	3	6	2.5	8.03	.0104	137.3	.02	.17	.0031	1.	.947	2.82	923
4	3	6	5.0	2.02	.0108	285	.0064	.17	.0062	.5	.989	0.92	1474
2	5	6	5.0	8.72	.0103	272	.0225	.25	.0062	.5	.858	3.30	266
4	5	6	2.5	11.28	.0102	134	.0275	.33	.0031	1.	.897	3.89	336

* Use Δx of 2.5 miles in each pipeline.

** ψ = 1 percent is used in each line.

observed that by the use of the inertial multiplier in a transient analysis program, it does not require any interpolation or adjustment of the length or wave speed for some odd lengths of the pipeline in a network in order to have the same Δt for each line in the system and also satisfy the characteristic equations.

The solutions by using Δx of 2.5 miles and the calculated Δt of two minutes are presented in Figure 18. The execution time for this three-hour transient in the IBM 360/67 system is 11 seconds. The accurate results by using $\alpha = 1$, $\Delta x = 2.5$ miles and correspondingly $\Delta t = 11$ seconds are also shown in Figure 18. The computer execution time is now 52.7 seconds. And, it can be noted that the approximate results deviate from the accurate ones by less than one percent as being desired.

These two examples have demonstrated the validity and the usefulness of the error diagrams presented in the previous chapter. The allowable error in the solution of transient simulation may be specified a priori and the desirable length of reach and the inertial multiplier can be determined by the use of these error diagrams.

6.2 Application to Transient Control Computation

The concept of control computation by use of the valve stroking principles applied to the characteristic equations for natural gas systems has been described briefly in Chapter III. An illustrative example is presented below to demonstrate the capability of obtaining a feasible stroking solution when the inertial multiplier is employed into the model.

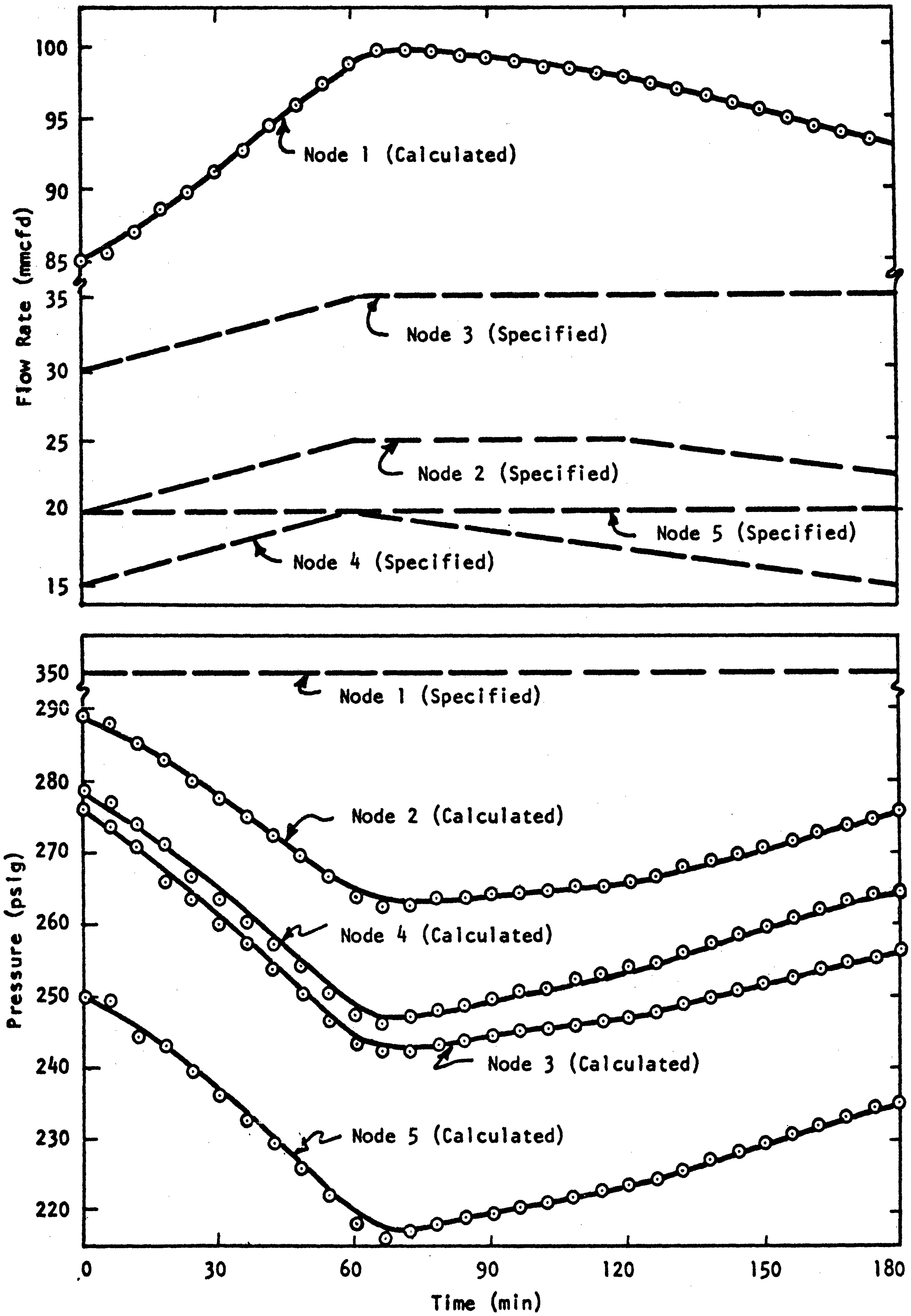


Figure 18. Example 2 -- Transient Pressure and Flow Variation in a Network System.

Example 3 - A single pipeline with a compressor station at the inlet end and the delivery point to the customer at the outlet end is hypothesized. The compressor station is assumed to be located at a storage field so that the suction pressure of the compressor is substantially constant. The compressor is assumed to be operating normally at the discharge pressure control with the set-point value of 500 psia. The system is initially in steady state condition with flow rate being 30 mmcfd until at time $t = 0$; then the customer load demand is specified as shown in Figure 20. It is desired that the compressor be kept operating at the same discharge pressure unless the delivery pressure at downstream cannot be maintained at the contract minimum value of 450 psia.

The computational scheme can be visualized by reference to an x-t plane shown in Figure 19. In the first phase of the computations, the analysis procedures are carried forward in time by specifying the fixed compressor discharge pressure of 500 psia at the inlet end and the variable load demand at the outlet end until at time t_1 when the delivery pressure drops below the contract minimum value. The second phase of the computations is to determine the controlling variables in the compressor station so that the pressure at downstream is maintained at the contract value while the customer load demand is also satisfied. The solution for this phase of operation is obtained by performing the valve-stroking computations. By specifying the desired boundary conditions at the downstream end over the duration DH, Figure 19, one obtains a particular solution for the corresponding boundary variations at the upstream end over the period CG. The computational procedures may be carried out by re-calculating the

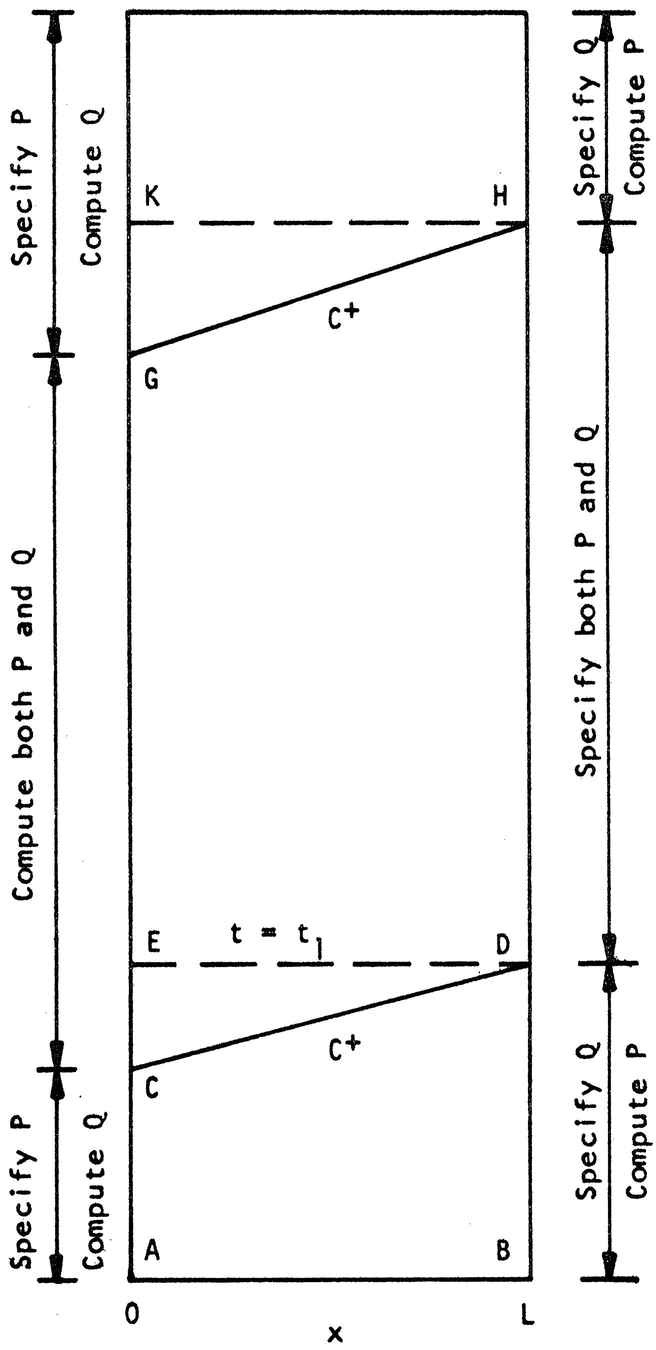


Figure 19. The x-t Diagram Used for Example 3.

conditions on the C^+ characteristic CD with the specified conditions at D and the known information below CD. The computations advance in time by obtaining the solution on the diagonal lines parallel to CD until at GH when the pressure at the upstream end G falls back again to the compressor discharge pressure setting of 500 psia. In the last stage of the computations, the analysis procedures proceed by specifying the upstream pressure fixed at 500 psia and the prescribed load demand at the downstream boundary. The solution within GHK is obtained first, then the computation is advanced in time until the transient is over.

According to the criteria in the previous chapter, the computation can be handled by dividing the pipe into five equal reaches and using α of 3.5 in order to achieve an accuracy of 0.5 percent error in pressure solution. The specified and calculated conditions at both upstream and downstream boundaries are shown in Figure 20.

The actual operation of the control mechanisms in the compressor station versus time can be computed according to the preceding analysis if the calibration of the horsepower versus the driving speed of the compressor unit is known, because the horsepower requirement may be found by using Equation (3.24).

As a check on the accuracy of the results shown in Figure 20, the computed upstream pressure and the prescribed load demand at the downstream end are placed as boundary conditions on a standard characteristic method analysis. The same initial conditions are also used. The objective is to see if the action of the control devices actually permits the delivery pressure to be maintained at the specified minimum value even when the load demand keeps increasing. This analysis

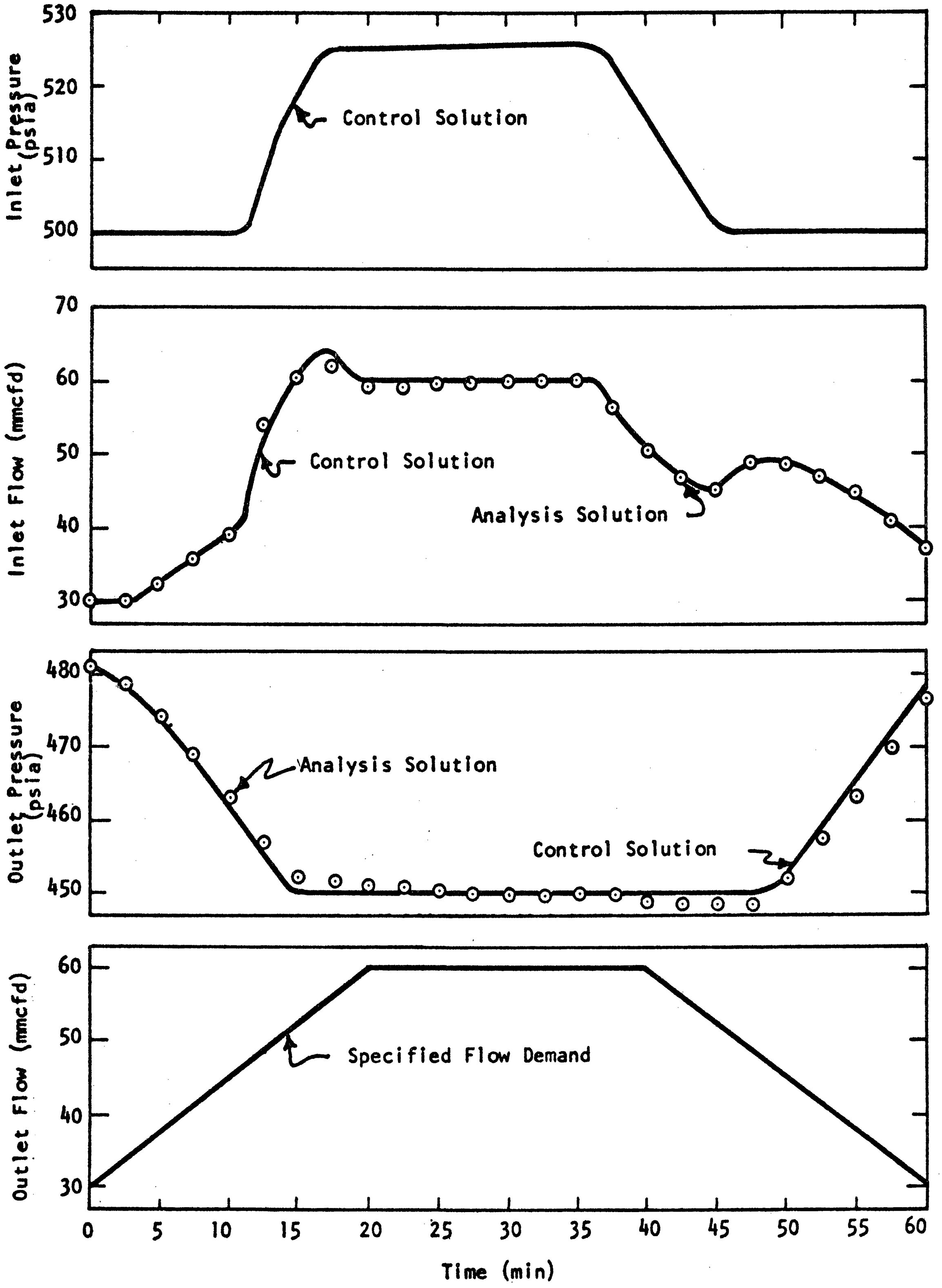


Figure 20. Results of Valve-Stroking Computations in Example 3.

confirms with the control computations within 0.5 percent as can be seen in Figure 20. Also, the unsteady flow conditions are identical during the transient period.

VII. NATURAL GAS FIELD EXPERIMENT

In order to verify the theory developed in the previous chapters, a field experiment in a natural gas transmission line was planned along with this investigation. The experiment was originally designed to show the validity of the solution of the control computation as illustrated by example 3 in Chapter VI except that the pressure, flow rate and the pipeline characteristics in the experiment were of different values. The system was initially maintained at steady state condition. The valves at both ends of the line were operated in such a way as to produce the calculated flow rate at the upstream end and the specified flow demand at the downstream end. The pressures at both ends were measured to compare with the calculated solutions. It is apparent that if the measured pressures at both ends of the line confirm the computed ones, one can be convinced that the theory of the valve stroking is applicable as well to the controlled operation of a natural gas system.

Unfortunately, the friction factor of the test line during the experiment was much higher than what had been used in making the computation. The pressure-drop across the line was so substantial as to cause the objective of this experiment to be unfulfillable. However, the result of the experiment is still valuable for the verification of the transient flow analysis solution, particularly, for the illustration of the validity of the concept of the inertial multiplier in the simulation of natural gas transient flows. The result of the experiment will be shown below following the description of the experiment.

7.1 Description of the Experiment

The experiment was conducted on a section of the gas transmission system belonging to the Consumers Power Company, Jackson, Michigan. The line chosen is a 12 inch pipe carrying gas from the St. Clair Compressor Station to the town of Mt. Clemens. Figure 21 shows the schematic of the test line.

One Sanborn strip chart recorder was used at each end of the line for measuring pressures. Statham strain gage type pressure transducers were used as the pressure sensing device. Each unit of the recorder and the transducer was calibrated before the test by the use of the Heise Pressure gage in the G. G. Brown Laboratory, The University of Michigan, and was checked during the test in the field by the use of a dead weight gage. During the test the amount of gas to feed into the line at St. Clair end was controlled by a computerized device located at the control room in the compressor station. A flow recording station at the St. Clair end of the line was used to check the actual flow into the line. A control valve at the Mt. Clemens end of the line was operated manually to produce the desired flow rate. A flow recording station located upstream of the control valve was used to monitor the valve operation versus time. The measurement of gas temperature was available at Mt. Clemens. The set-up of the measuring instruments at each end of the line is shown in Plates I through III.

Before the test, the flow in the line was supposed to be in steady state condition. The line was originally in such a condition with a pressure-drop of 22 psi at a flow rate of 15 mmcfd. However, there existed a slight transient in the line when the test

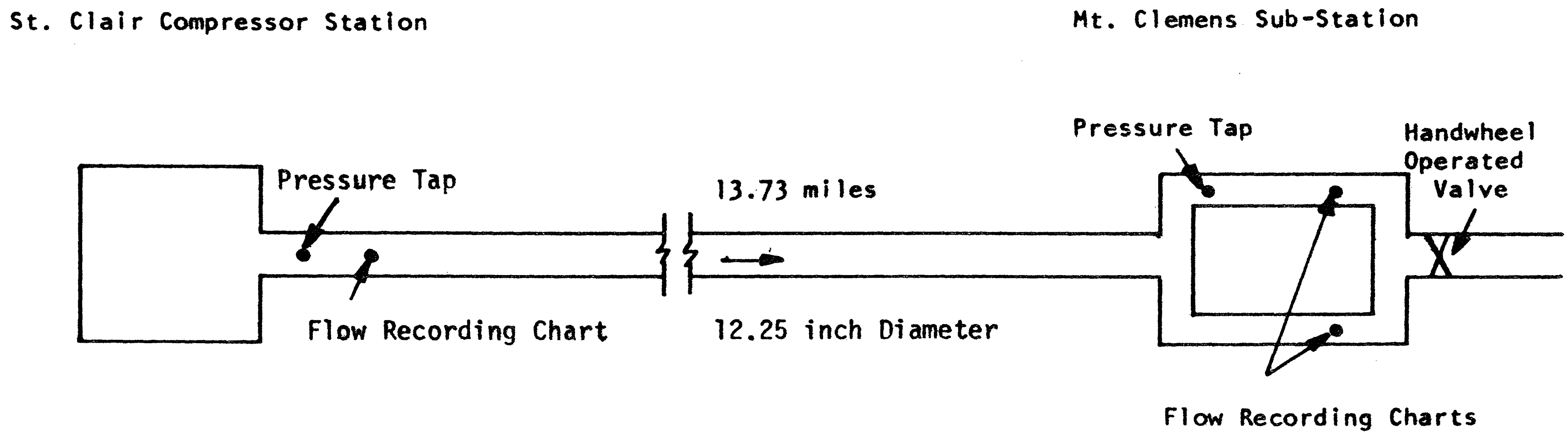


Figure 21. Schematic of the Consumers Power Company Transmission Line from St. Clair, Michigan to Mt. Clemens, Michigan.

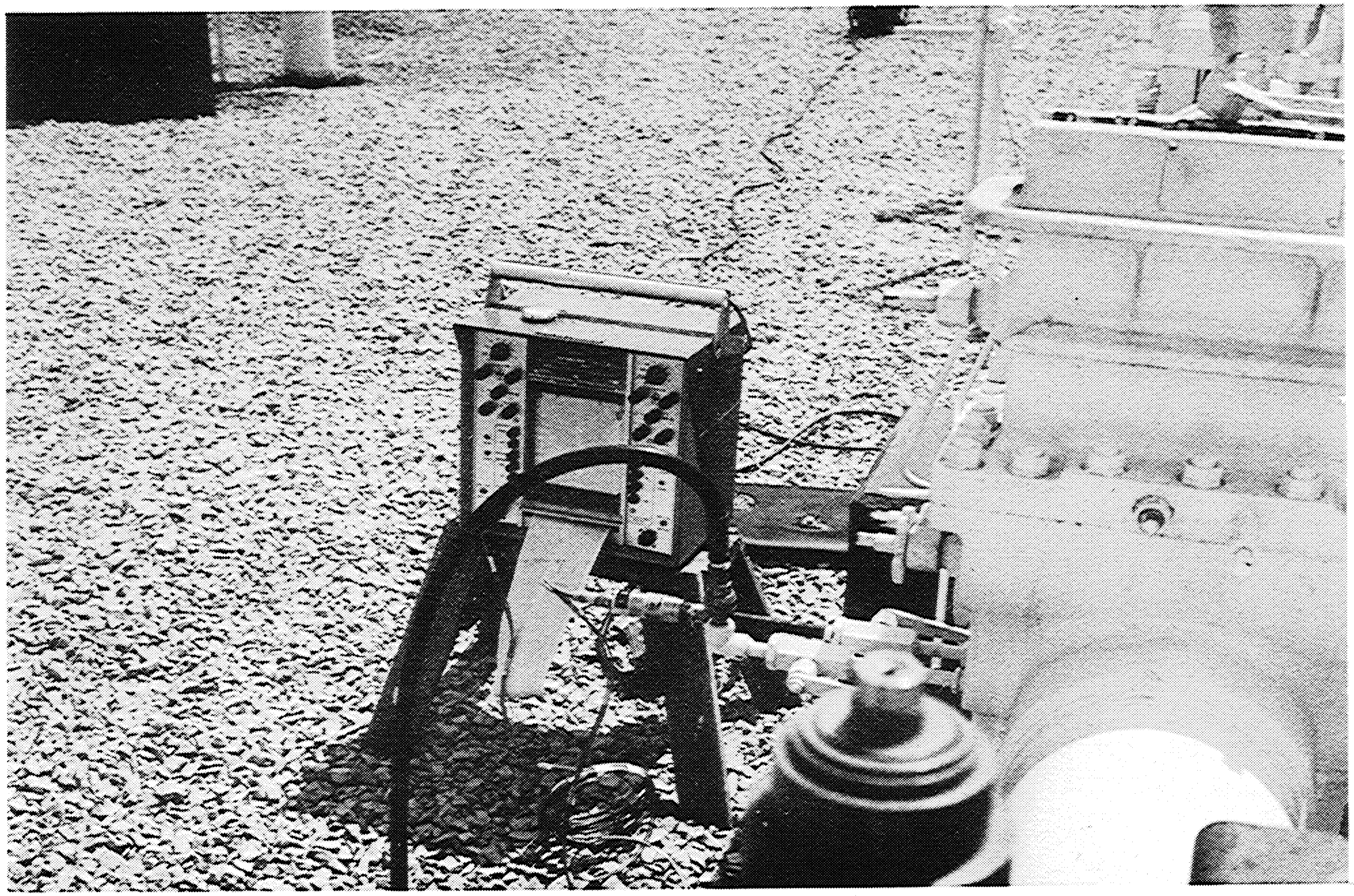


Plate 1. The Strip Chart Recorder and the Pressure Transducer at the St. Clair End of the Line.

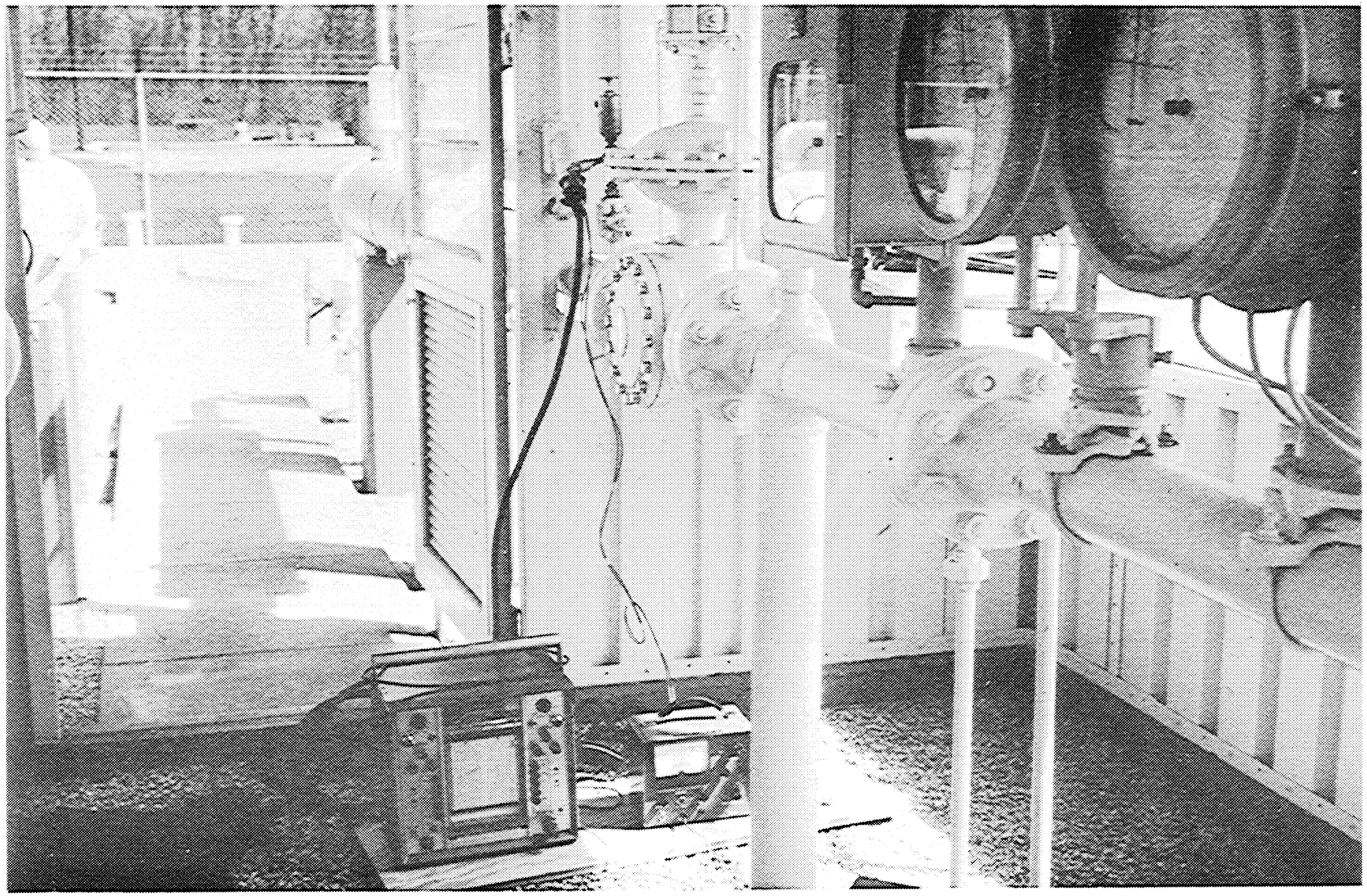


Plate 2. The Measuring Instruments at the Mt. Clemens End of the Line.

Plate 3. The Pressure Transducer and the Dead Weight Gage Used at the Mt. Clemens End.

started. Radio communication was used to synchronize the valve operations and the strip chart recorders at both ends of the line. Gas was fed into the line at the upstream end and was taken out of the line by the operation of the control valve at the downstream end according to the curves shown in Figure 22. The adjustment of the operations was made at two-minute intervals. Unfortunately, owing to the excessive pressure-drop across the line during the test, the control valve at Mt. Clemens was fully open at approximately ten minutes after the test started, yet the maximum flow rate had not been reached. The valve at Mt. Clemens was then left wide open and the actual flow through the valve was measured until the peak of the downstream flow demand was over so that the control valve could be closed gradually to yield the specified flow rate. The flow rate at the upstream end of the line was still controlled as planned. The entire period of the measurements in transient flow conditions lasted eighty minutes.

7.2 Result of the Experiment

Since the original intention of the experiment could not be fulfilled due to the unexpected high friction in the test line which caused the control valve at Mt. Clemens to be unworkable during the test, the result measured could only be used to check the solution of the ordinary transient analysis computations. The measured pressures at both ends of the line were placed as the specified boundary conditions in a transient analysis program, and the computed solutions of the flow rate at both ends from this analysis program were compared with measured flows.

The isothermal wave speed in a natural gas system may be calculated by Equation (2.2), i.e.

$$B^2 = \frac{gzRT}{M_W}$$

The molecular weight of the gas was 17.37 lb/mole. The compressibility factor was 0.96. The system temperature was determined to be 500°R. Thus, the wave speed was obtained as 1170 ft/sec. A friction factor was determined from the pre-transient steady flow condition by the use of Equation (3.19). The friction factor so obtained was 0.026 which was about 2.5 times larger than the value of the friction factor previously available. Nevertheless, this high friction factor is used during the transient analysis calculations.

The method-of-characteristics solution by using $\alpha = 1$ and ten reaches $\Delta x = 1.37$ miles is shown in Figure 22. The time increment used is 6.18 seconds. The execution time in the computer system of IBM 360/67 is 16.0 seconds. The result, in general, is agreeable with the experimental data, although slight discrepancy in the solution existed as the sensitivity of the flow measuring device used in this experiment is doubtful. Another computer solution by using $\alpha = 4$ and five reaches $\Delta x = 2.75$ miles and correspondingly $\Delta t = 49.5$ seconds is also shown in Figure 22. The result confirms the previously computed solution yet the computer execution time is now only 5.0 seconds.

The natural gas transient flow solution with the method of characteristics has been verified experimentally by a previous investigator.[17,18] The above comparison demonstrates the usefulness of the concept of the inertial multiplier incorporated into the

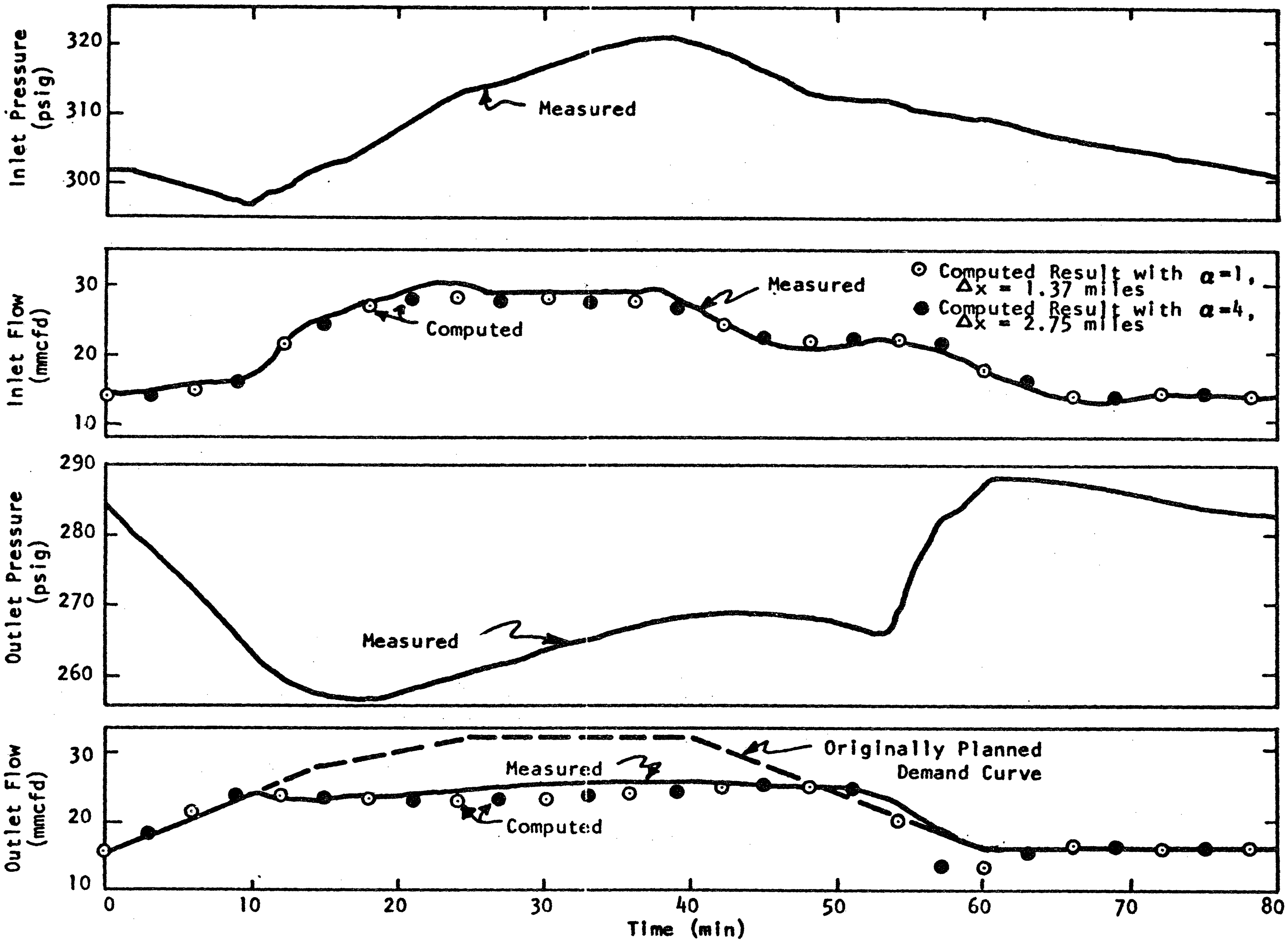


Figure 22. Comparison of the Measured and the Computed Results.

method-of-characteristics procedures. The use of the inertial multiplier improves the effectiveness of this method of solution so far as the computational cost and the accuracy of solution are concerned.

VIII. SUMMARY AND CONCLUSIONS

In the foregoing chapters, the concept of the inertial multiplier incorporated with the characteristics procedures as a method of solution for natural gas transient flow problems was developed. The use of the inertial multiplier permits not only an optimum time increment for numerical computation to be used under various transient flow conditions, but also a feasible control solution to be calculated directly by using the valve-stroking principles applied to the characteristic equations for natural gas systems.

A solution with the second-order approximation to the non-linear finite difference characteristic equations was developed in a steady oscillatory flow system. By employing the solution of the first approximation to estimate the non-linear terms which form the non-homogeneous part of the linearized equations, the solution of the second-order approximation was achieved. This was accomplished by solving the non-homogeneous linear difference equations. Although a higher approximation may be accomplished by the same process of repeated substitutions, the solution of the second approximation is sufficiently accurate for engineering purposes. The results of the second-order approximation were shown to confirm the numerical solution with the method of characteristics.

By the application of the second-order solution, the validity and the range of usefulness of the inertial multiplier and the criterion on the allowable length of reach for a specific transient flow analysis problem have been investigated. With the use of the computer, the error bound in the solution corresponding to the discretization and the use of the inertial multiplier was obtained quantitatively by pursuing the numerical experiments for a wide spectrum of system parameters. The results were presented in diagrams for engineering uses. Examples illustrating the application of the error diagrams to specific problems were included. The transient solution by using the computed Δx and α was checked with the accurate solution, which showed the reliability of the error diagrams. Although the diagrams were constructed on the basis of a steady oscillatory flow model, the applicability of the results to general transient flow conditions was demonstrated by practical examples. However, it is recommended that further study in this area be conducted to ensure that the error diagrams are applicable to any complex network systems with arbitrary boundary condition specifications.

The computations in connection with the field experiment which was performed in a natural gas transmission line demonstrated the usefulness of the concept of the inertial multiplier. By permitting a large value of the inertial multiplier to be used in a slow transient condition, the simulation of such a transient flow problem in natural gas systems can be accomplished with significant reduction in the computational time, while the accuracy of solution is still maintained.

APPENDIX A

ALGEBRAIC SOLUTION OF THE FIRST-ORDER APPROXIMATION

The solution of the first-order approximation in algebraic form is derived herein to reveal the controlling parameters that govern the error functions. The first-order solution by the use of the transfer-matrix technique can be expressed as

$$\begin{Bmatrix} p_1(N,t) \\ q_1(N,t) \end{Bmatrix} = [SN] \begin{Bmatrix} p_1(0,t) \\ q_1(0,t) \end{Bmatrix} \tag{A.1}$$

where [SN] is the over-all transfer matrix of the pipeline and is determined by Equation (4.34) which is reproduced below.

$$[SN] = [TN(N)]\,[TN(N-1)]\,\ldots\,[TN(1)] \tag{A.2}$$

The key factor that dominates the difficulty in obtaining the solution expressed in the algebraic equations lies in the manipulation of the matrix multiplication shown in Equation (A.2). It is most effective to perform this multiplication by the use of the computer as has been done in pursuing the error investigations. However, in order to determine the controlling parameters of error functions, a slight approximation can be made to permit the manipulation of the process of Equation (A.2).

By using the following substitutions

$$\sin\omega\alpha h \approx \omega\alpha h - \frac{\omega^3\alpha^3h^3}{6}$$

$$\cos\omega\alpha h \approx 1 - \frac{1}{2}\,\omega^2\alpha^2h^2$$

one may approximate the elements of the field transfer matrix [TN(I)], which were shown in Equations (4.29a) to (4.29d).

$$C11(I) = \frac{\bar{P}_1(I-1)}{\bar{P}_1(I)}\left[1 - \frac{\omega^2\alpha^2h^2}{2} + \frac{i\sigma m\omega h^2}{\tilde{P}_1(I)}\right] \tag{A.3a}$$

$$C12(I) = \frac{-1}{\tilde{P}_1(I)} \left[2\sigma m^2 h + im\omega\alpha^2 h \tilde{P}_1(I) + \frac{i\omega\sigma^2 m^3 h^3}{\tilde{P}_1(I)} - \sigma\omega^2 m^2 \alpha^2 h^3 - \frac{im\omega^3 \alpha^4 h^3 \tilde{P}_1(I)}{6} \right] \tag{A.3b}$$

$$C21(I) = - \frac{i\bar{P}_1(I-1)}{m\alpha\tilde{P}_1(I)} \left[\omega\alpha h - \frac{\omega^3\alpha^3 h^3}{6} \right] \tag{A.3c}$$

$$C22(I) = 1 - \frac{\omega^2\alpha^2 h^2}{2} + \frac{i\sigma m\omega h^2}{\tilde{P}_1(I)} \tag{A.3d}$$

in which, and in the following derivations as well, the terms containing h^4 or higher order are neglected. With these expressions and the following approximation

$$\bar{P}_1^{(I)} = \sqrt{1 - 2\sigma m^2 I h} \approx 1 - \sigma m^2 I h$$

the multiplication of the field transfer matrix shown in Equation (A.2) can be accomplished by performing a simple algebraic manipulation. The result is

$$R11 = \frac{1}{\bar{\bar{P}}_1(N)} \left[1 - \frac{\omega^2\alpha^2}{2} \quad \frac{\sigma m^2 \omega^2 \alpha^2}{6} (1 - h^2) + i\sigma m\omega + \frac{i\sigma^2 m^3 \omega (2+h^2)}{6} \right] \tag{A.4a}$$

$$R12 = \frac{-1}{\bar{P}_1(N)} \left[im\omega\alpha^2 \left(1 - \frac{\omega^2\alpha^2}{6} \right) \bar{P}_1(N)^{\frac{1}{2}} + 2\sigma m^2 - \frac{1}{3} \sigma m^2 \omega^2 \alpha^2 (2 + h^2) + \frac{i\omega\sigma^2 m^3 (2+h^2)}{3\bar{P}_1(N)^{\frac{1}{2}}} \right] \tag{A.4b}$$

$$R21 = - \frac{i\omega}{m} \bar{P}_1(N)^{\frac{1}{2}} + \frac{i\omega^3\alpha^2}{m} \left[\frac{\sigma m^2}{12} + \left(\frac{h^2}{3} - \frac{h}{2} + \frac{1}{3} \right) \right] + \frac{\sigma\omega^2 (1-h^2)}{3\bar{P}_1(N)} \tag{A.4c}$$

$$R22 = 1 - \frac{1}{2}\omega^2\alpha^2\left[1 + \frac{\sigma m^2(1-h^2)}{3}\right] + i\sigma m\omega\left[1 - \frac{\sigma m^2}{6}(4 - h^2)\right] \tag{A.4d}$$

where R11, R12 et al are the elements of the over-all transfer matrix, [SN]. That is

$$[SN] = \begin{bmatrix} R11 & R12 \\ R21 & R22 \end{bmatrix} \tag{A.5}$$

By combining Equation (A.1) with Equation (A.5) and using the following boundary conditions

$$\begin{aligned} p_1(0,t) &= 0 \\ q_1(N,t) &= -i\Delta q e^{i\omega t} \end{aligned} \tag{A.6}$$

one obtains the unkonwn conditions at the boundaries as

$$p_1(N,t) = -\, i\Delta q e^{i\omega t} R12/R22 \tag{A.7}$$

$$q_1(0,t) = -\, i\Delta q e^{i\omega t}/R22 \tag{A.8}$$

where R12 and R22 are determined from Equations (A.4b) and (A.4d), respectively.

Equations (A.7) and (A.8) express the solution of the first approximation in algebraic equations, in which the factors of the discretization h and the inertial multiplier α are included. These two equations may be used to determine the error functions explicitly. The error functions are determined by subtracting the solution with $h \to 0$ and $\alpha \to 1$ from the one with certain h and α. The error functions so obtained are, after neglecting the terms of small effect

$$\tilde{\psi} = \frac{m\omega\Delta q e^{i\omega t}}{\bar{P}_1(N)^{\frac{1}{2}}(1 + i\sigma m\omega)}\left[(\alpha^2 - 1) + \frac{\sigma^2 m^2 h^2}{3\bar{P}_1(N)}\right] \tag{A.9}$$

$$\tilde{\phi} = \frac{-i\omega^2\Delta q e^{i\omega t}}{2\bar{P}_1(N)^{\frac{4}{3}}(1 + i\sigma m\omega)^2}\left[(\alpha^2 - 1) - \frac{i\sigma^2 m^3 h^2 \bar{P}_1(N)^{\frac{1}{3}}}{3\omega}\right] \tag{A.10}$$

in which $\tilde{\psi}$ and $\tilde{\phi}$ are error functions in the solution of the outlet pressure and inlet flow, respectively. The absolute values of these functions determine the error bounds. Thus

$$\psi = |\tilde{\psi}| = \frac{m\omega\Delta q}{\sqrt{P_d(1 + \sigma^2 m^2 \omega^2)}} \left[(\alpha^2 - 1) + \frac{1}{3}\left(\frac{\sigma^2 m^2 h^2}{P_d}\right) \right] \tag{A.11}$$

$$\phi = |\tilde{\phi}| = \frac{\omega^2 \Delta q}{2P_d^{\frac{4}{3}}(1 + \sigma^2 m^2 \omega^2)} \left[(\alpha^2 - 1)^2 + \left(\frac{\sigma^2 m^3 h^2 P_d^{\frac{1}{3}}}{3\omega}\right)^2 \right]^{\frac{1}{2}} \tag{A.12}$$

where ψ and ϕ are, respectively, the error bounds in pressure and flow, and P_d is used to replace $\overline{P}_1(N)$, the steady state outlet pressure.

Equations (A.11) and (A.12) are derived from the solution of the first-order approximation. The grouped parameters which appear in these two equations may be valid as well in the model of the second-order approximation, since the difference in the magnitudes of the solution of these two approximations is small. Thus, for the model of the second-order approximation, the following functionals are used.

$$\frac{\psi \sqrt{P_d(1 + \sigma^2 m^2 \omega^2)}}{m\omega\Delta q} = f_1(\alpha, \frac{\sigma m h}{\sqrt{P_d}}) \tag{A.13}$$

$$\frac{\phi\, P_d^{\frac{4}{3}}(1 + \sigma^2 m^2 \omega^2)}{\omega^2 \Delta q} = f_2(\alpha, \sigma m h \sqrt{\frac{m\, P_d^{\frac{1}{3}}}{\omega}}) \tag{A.14}$$

In view of these two equations, it is possible to present the results of the error investigations pursued by numerical experiments. The functional relationships f_1 and f_2 are shown graphically in Figures 14 and 15, respectively.

APPENDIX B

A FORTRAN IV LANGUAGE PROGRAM FOR THE SOLUTION OF SECOND-ORDER APPROXIMATION

```
C     TRANSIENT ANALYSIS OF A NATURAL GAS PIPELINE
C     SECOND-ORDER SOLUTION OF THE CHARACTERISTIC FINITE DIFFERENCE
C     EQUATIONS
C     FMOLWT = GAS MOLECULAR WEIGHT -- LB/MOLE
C     TEMP = AVERAGE SYSTEM TEMPERATURE -- DEGREES RANKINE
C     RHO = MASS DENSITY AT THE BASE CONDITIONS -- SLUGS/CU. FOOT
C     CZ = COMPRESSIBILITY FACTOR
C     XL = LENGTH OF THE PIPE -- MILES
C     D = DIAMETER -- FEET
C     AL = THE INERTIA MULTIPLIER
C     F = DARCY-WEISBACH FRICTION FACTOR
C     PER = PERIOD -- SECONDS
C     PO = INLET PRESSURE -- PSIA
C     QO = STEADY-STATE FLOW RATE -- MMCFD
C     DQ = AMPLITUDE OF FLOW VARIATION -- MMCFD
C     TMAX = DURATION OF TRANSIENT -- SECONDS
C     GIV = AN ARBITRARY CONSTANT
C     B = WAVE SPEED -- FEET/SECOND
C     B2 = B**2
C     N = NUMBER OF REACHES
C     DX = LENGTH OF A REACH -- FEET
C     DT = TIME INCREMENT -- SECONDS
C     OM = FREQUENCY -- RADIAN/SECOND
C     PBAR = MEAN PRESSURE -- PSIA
C     DTT = TIME STEP FOR PRINT-OUT -- SECONDS
C
      IMPLICIT COMPLEX(Z)
      COMMON/FDE/ZR11,ZR12,ZR21,ZR22,Z11(20),Z12(20),Z21(20),Z22(20),N,
     1CA,F2,DT,PBAR(20)
      DIMENSION ZPP(20),ZQQ(20),PC(20),PS(20),QC(20),QS(20)
      NAMELIST/DIN/N,FMOLWT,TEMP,CZ,XL,D,AL,F,PER,PO,QO,DQ,TMAX,DTT,GIV
     1/OUT/RHO,B2,B,CK,CA,A,DX,DT,FF,OM,PN,DQO,PHQO,DQO1,PHQO1,DPN,
     2PHPN,DPN1,PHPN1
    1 READ(5,DIN,END=999)
      WRITE(6,DIN)
      RHO=FMOLWT*14.73*144./(520.*1545.*32.2)
      B2=CZ*1545.*TEMP*32.2/FMOLWT
      B=SQRT(B2)
      CK=RHO*1.E6/(86400.*144.)
      XL=XL*5280.
      A=.7854*D*D
      DX=XL/FLOAT(N)
      DT=AL*DX/B
      FF=F*B2*DX*CK*CK/(D*A*A)*QO*QO
      CA=CK*B*AL/A
      OM=6.2832/PER
      OM2=2.*OM
      F1=FF/(QO*QO)
      F2=.5*FF/QO
      NS=N+1
      DO 5 I=1,NS
    5 PBAR(I)=SQRT(PO*PO-FLOAT(I-1)*FF)
```

```
      PN=PBAR(NS)
C    THE FIRST-ORDER APPROXIMATION
C    THE FIRST-ORDER SOLUTION IS EXPRESSED AS
C         P(I)=PC(I)*COS(OM*T)+PS(I)*SIN(OM*T)
C         Q(I)=QC(I)*COS(OM*T)+QS(I)*SIN(OM*T)
      CALL FDETM(AL,OM)
      ZQN=DQ*CMPLX(0.,-1.)
      ZQO=ZQN/ZR22
      QC(1)=REAL(ZQO)
      QS(1)=-AIMAG(ZQO)
      PC(1)=0.
      PS(1)=0.
      ZPP(1)=CMPLX(PC(1),-PS(1))
      ZQQ(1)=CMPLX(QC(1),-QS(1))
      DO 6 J=1,N
      I=J+1
      ZPP(I)=Z11(J)*ZPP(J)+Z12(J)*ZQQ(J)
      ZQQ(I)=Z21(J)*ZPP(J)+Z22(J)*ZQQ(J)
      PC(I)=REAL(ZPP(I))
      PS(I)=-AIMAG(ZPP(I))
      QC(I)=REAL(ZQQ(I))
      QS(I)=-AIMAG(ZQQ(I))
    6 CONTINUE
C    THE SECOND-ORDER APPROXIMATION
C    CALCULATE THE MEAN PRESSURE
      DO 8 I=2,NS
      J=I-1
      AO=.5*(PC(I)**2+PS(I)**2-PC(J)**2-PS(J)**2+.5*F1*(QC(I)**2+QS(I)
     1**2+QC(J)**2+QS(J)**2)+CA*(PC(I)*QC(I)+PS(I)*QS(I)-PS(J)*QS(J)-
     2PC(J)*QC(J)+COS(OM*DT)*(QC(I)*PC(J)+QS(I)*PS(J)-PC(I)*QC(J)-PS(I)*
     3QS(J))+SIN(OM*DT)*(PC(J)*QS(I)+PC(I)*QS(J)-PS(I)*QC(J)-PS(J)*QC(I)
     4)))
    8 PBAR(I)=SQRT(PBAR(J)**2-FF-AO)
      PN=PBAR(NS)
      WRITE(6,110) (PBAR(I),I=1,NS)
  110 FORMAT(//'OMEAN PRESSURE = ',13F8.2/(17X,13F8.2))
      ZE1=CMPLX(COS(OM*DT),SIN(OM*DT))
      ZE2=ZE1*ZE1
C    THE SOLUTION CORRESPONDING TO THE SECOND HARMONIC FREQUENCY
      CALL FDETM(AL,OM2)
      ZPP(1)=GIV*CMPLX(1.,-1.)
      ZQQ(1)=ZPP(1)
      DO 10 I=1,N
      J=I+1
      R1C=.5*(PC(J)**2-PS(J)**2+CA*(PC(J)*QC(J)-PS(J)*QS(J))+.5*F1*(QC(J
     1)**2-QS(J)**2))
      R1S=PC(J)*PS(J)+.5*CA*(PC(J)*QS(J)+PS(J)*QC(J))+.5*F1*QC(J)*QS(J)
      ZR1=-.5*CMPLX(R1C,-R1S)
      R2C=.5*CA*(QC(J)*PC(I)-PC(J)*QC(I)+PS(J)*QS(I)-QS(J)*PS(I))
      R2S=.5*CA*(PS(I)*QC(J)-PC(J)*QS(I)+PC(I)*QS(J)-PS(J)*QC(I))
      ZR2=-.5*CMPLX(R2C,-R2S)
      R3C=.5*(PS(I)**2-PC(I)**2+CA*(PS(I)*QS(I)-PC(I)*QC(I))+.5*F1*(QC(I
     1)**2-QS(I)**2))
```

```
      R3S=-PC(I)*PS(I)-.5*CA*(PC(I)*QS(I)+PS(I)*QC(I))+.5*F1*QC(I)*QS(I)
      ZR3=-.5*CMPLX(R3C,-R3S)
      S1C=-.5*(PC(J)**2-PS(J)**2+CA*(PS(J)*QS(J)-PC(J)*QC(J))+.5*F1*(QC(
     1J)**2-QS(J)**2))
      S1S=-PC(J)*PS(J)+.5*CA*(PC(J)*QS(J)+PS(J)*QC(J))-.5*F1*QC(J)*QS(J)
      ZS1=-.5*CMPLX(S1C,-S1S)
      S2C=-.5*CA*(PC(J)*QC(I)+PS(I)*QS(J)-PC(I)*QC(J)-PS(J)*QS(I))
      S2S=-.5*CA*(QC(I)*PS(J)+QS(I)*PC(J)-PC(I)*QS(J)-PS(I)*QC(J))
      ZS2=-.5*CMPLX(S2C,-S2S)
      S3C=.5*(PC(I)**2-PS(I)**2-CA*(PC(I)*QC(I)-PS(I)*QS(I))-.5*F1*(QC(I
     1)**2-QS(I)**2))
      S3S=PC(I)*PS(I)-.5*CA*(PC(I)*QS(I)+QC(I)*PS(I))-.5*F1*QC(I)*QS(I)
      ZS3=-.5*CMPLX(S3C,-S3S)
      ZP=(ZR1+ZR2/ZE1+ZR3/ZE2-ZS1-ZS2*ZE1-ZS3*ZE2)/(2.*PBAR(J))-(ZR1+ZR2
     1/ZE1+ZR3/ZE2+ZS1+ZS2*ZE1+ZS3*ZE2)*F1/(2.*CA*PBAR(J)*(PBAR(I)+PBAR
     2(J)))
      ZQ=(ZR1+ZR2/ZE1+ZR3/ZE2+ZS1+ZS2*ZE1+ZS3*ZE2)/(CA*(PBAR(I)+PBAR(J))
     1)
      ZPP(J)=Z11(I)*ZPP(I)+Z12(I)*ZQQ(I)+ZP
      ZQQ(J)=Z21(I)*ZPP(I)+Z22(I)*ZQQ(I)+ZQ
   10 CONTINUE
      ZQ20=ZPP(1)*ZR21/ZR22-ZQQ(NS)/ZR22
      ZP2N=-ZR11*ZPP(1)+ZR12*ZQ20
C    THE SOLUTION CORRESPONDING TO THE FUNDAMENTAL FREQUENCY
      CALL FDETM(AL,OM)
      WRITE(6,101)
  101 FORMAT('1    FIELD TRANSFER MATRICES'/18X,'C11',24X,'C12',24X,'C21'
     1,24X,'C22')
      DO 15 J=1,N
   15 WRITE(6,102) J,Z11(J),Z12(J),Z21(J),Z22(J)
  102 FORMAT('0   J=',I3,4(2E12.4,3X))
      WRITE(6,103) ZR11,ZR12,ZR21,ZR22
  103 FORMAT(//'0    THE OVER-ALL TRANSFER MATRIX'/8X,4(2E12.4,3X)////)
      ZQ0=ZQN/ZR22
      ZPN=ZR12*ZQ0
      DQ0=CABS(ZQ0)
      PHQ0=ATAN2(AIMAG(ZQ0),REAL(ZQ0))
      DPN=CABS(ZPN)
      PHPN=ATAN2(AIMAG(ZPN),REAL(ZPN))
      DQ01=CABS(ZQQ(1)+ZQ20)
      PHQ01=ATAN2(AIMAG(ZQQ(1)+ZQ20),REAL(ZQQ(1)+ZQ20))
      DPN1=CABS(ZPP(NS)+ZP2N)
      PHPN1=ATAN2(AIMAG(ZPP(NS)+ZP2N),REAL(ZPP(NS)+ZP2N))
      WRITE(6,OUT)
      WRITE(6,104)
  104 FORMAT('1    TRANSIENT SOLUTION BY THE SECOND-ORDER APPROXIMATION'/
     117X,'TIME',10X,'PU',13X,'QU',18X,'PD',13X,'QD')
      T=0.
   20 QU=Q0+DQ0*COS(OM*T+PHQ0)+DQ01*COS(2.*OM*T+PHQ01)
      QD=Q0+DQ*SIN(OM*T)
      PU=P0
```

```
      PD=PN+DPN*COS(OM*T+PHPN)+DPN1*COS(2.*OM*T+PHPN1)
      WRITE(6,105) T,PU,QU,PD,QD
 105  FORMAT(1H0,5X,3F15.3,5X,2F15.3)
      T=T+DTT
      IF(T.GT.TMAX) GO TO 1
      GO TO 20
 999  STOP
      END


      SUBROUTINE FDETM(AL,OM)
C   FINITE DIFFERENCE EQUATION TRANSFER MATRIX SUBROUTINE
C   CALCULATE THE FIELD TRANSFER MATRICES AND THE OVER-ALL TRANSFER
C   MATRIX
C   Z11, Z12 ETC. ARE THE ELEMENTS OF A FIELD TRANSFER MATRIX
C   ZR11, ZR12 ETC. ARE THE ELEMENTS OF THE OVER-ALL TRANSFER MATRIX
C
      IMPLICIT COMPLEX(Z)
      COMMON/FDE/ZR11,ZR12,ZR21,ZR22,Z11(20),Z12(20),Z21(20),Z22(20),N,
     1CA,F2,DT,PBAR(20)
      ZN=CMPLX(0.,OM*DT)
      ZE=CEXP(ZN)
      DO 10 J=1,N
      I=N-J+1
      PO=.5*(PBAR(I)+PBAR(I+1))
      ZDEL=2.*CA*PO*ZE*PBAR(I+1)
      Z11(I)=PBAR(I)/ZDEL*(CA*PO*(1.+ZE*ZE)-F2*(1.-ZE*ZE))
      Z12(I)=((CA*PO-F2)**2-(ZE*(CA*PO+F2))**2)/ZDEL
      Z21(I)=PBAR(I)*PBAR(I+1)*(1.-ZE*ZE)/ZDEL
      Z22(I)=PBAR(I+1)/ZDEL*(CA*PO*(1.+ZE*ZE)-F2*(1.-ZE*ZE))
      IF(J.EQ.1) GO TO 8
      ZM11=ZR11*Z11(I)+ZR12*Z21(I)
      ZM12=ZR11*Z12(I)+ZR12*Z22(I)
      ZM21=ZR21*Z11(I)+ZR22*Z21(I)
      ZM22=ZR21*Z12(I)+ZR22*Z22(I)
      ZR11=ZM11
      ZR12=ZM12
      ZR21=ZM21
      ZR22=ZM22
      GO TO 10
   8  ZR11=Z11(I)
      ZR12=Z12(I)
      ZR21=Z21(I)
      ZR22=Z22(I)
  10  CONTINUE
      RETURN
      END
```

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