THE UNIVERSITY OF MICHIGAN

SYSTEMS ENGINEERING LABORATORY

Department of Electrical Engineering
College of Engineering

Technical Report SEL-68-33

STRUCTURE OF SENSITIVITY REDUCTION

by

Charles L. Zahm

November 1968

This research was supported by NSF Grant GK-1925.

This report was also a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan.

ACKNOWLEDGEMENTS

I wish to acknowledge my debt of gratitude to Professor W. A. Porter for his guidance and enlightening discussions during the course of this research. I am also indebted to the other members of my committee for their constructive criticisms.

For the preparation of this manuscript, I especially want to thank Mrs. Joanne Aichler who worked so carefully and diligently at this task, as well as Miss Sharon Bauerle.

This work was supported by National Science Foundation Grant GK 1925.

Finally, I shall be forever thankful to my wife whose patience and encouragement made this effort possible.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF ILLUSTRATIONS

LIST OF APPENDICES

ABSTRACT

For several years control theorists have been examining the problem of reducing the sensitivity of a control system containing a plant, a precompensator and a feedback compensator. The purpose of this study is to ascertain the underlying *structure* of a feedback control system in order to reduce its sensitivity relative to a *nominally equivalent* open-loop system.

This problem is initially formulated in Banach spaces in order to emphasize the operator theoretic point of view. Attention is then restricted to Hilbert spaces, to obtain more detailed results including interpretation, from a physical point of view. Finally, the problem is formulated in terms of plants described by linear time-varying differential equations of the form

$$x(t) = A(t)x(t) + B(t)u(t), \; x(t_o) = x_o, \; t \in \Omega = [t_o, t_f]$$

$$y(t) = C(t)x(t).$$

Here the tuplet $x = (x_1, \ldots, x_n)$ represents the state variables, the tuplet $u = (u_1, \ldots, u_p)$ represents the independent plant inputs, and the tuplet $y = (y_1, \ldots, y_m)$ represents the independent plant outputs. The matrices $A(t)$, $B(t)$, and $C(t)$ are real and continuous on the set Ω and are of compatible dimensions. The pertinent question then answered in this thesis is: "Given a precompensator G and a feedback compensator M, what are the relationships between these compensators and the plant to ensure sensitivity improvement?"

To facilitate this study a concept of positive realness for time-varying kernels is introduced. This concept is demonstrated to indeed be a generalization of the standard definition given in the frequency domain for stationary systems. The concept of positive realness is then brought to bear on the pertinent issues, illustrating the natural relationship between positive realness and sensitivity improvement.

Chapter 1

INTRODUCTION

1.1 Basic Concept of Sensitivity

There are basically two notions of sensitivity both of which have the distinction of being considered classical. The first notion considers a system as represented by a set of vector differential equations whose solutions are dependent upon the parameters of the equations. The second notion, which is perhaps better known to control engineers, was introduced by Bode [9] and is almost as old as the theory of feedback.

The first method (see Kokotovic and Rutman [29]) considers the perturbations in the state vector due to disturbances in a parameter vector for systems, which can be described by

$$\dot{x} = f(x, u, \mu, t): \quad x(t_o) = x_o \tag{1.1}$$

where x is the $n \times 1$ state vector, u is the $m \times 1$ control vector, μ is the $p \times 1$ time-invariant parameter vector and t is the independent variable. If the vector μ is perturbed such that

$$\mu \rightarrow \tilde{\mu} = \mu + \Delta\mu,$$

then the corresponding state vector becomes

$$\dot{\tilde{x}} = f(\tilde{x}, u, \tilde{\mu}, t); \quad \tilde{x}(t_o) = x_o. \tag{1.2}$$

The perturbation in the state is then defined to be

$$\Delta x = \tilde{x} - x. \tag{1.3}$$

Forming a Taylor series expansion of equation 1.3 in powers of μ the perturbations in the state due to perturbations in μ is straightforward. First define the <u>trajectory sensitivity function</u>

$$\sigma_{ij} = \left(\frac{\partial x_i}{\partial \mu_j}\right)_{\Delta\mu = 0}.$$

To a first order approximation the perturbations in the state are then given by

$$\Delta x_i = \sum_{j=1}^{p} \sigma_{ij} \Delta\mu_j. \tag{1.4}$$

If f is continuously differentiable and uniformly Lipschitz, the solution of 1.2 exists and is given by equations 1.3 and 1.4 where σ_{ij} satisfies the sensitivity equation

$$\dot{\sigma}_{kj} = \sum_{i=1}^{n} \left(\frac{\partial f_k}{\partial x_i}\right) \sigma_{ij} + \frac{\partial f_k}{\partial \mu_j}$$

with the constraints

$$\sigma_{kj}(t_o) = 0, \ (k = 1, \ldots, n; \ j = 1, \ldots, p).$$

The second method, as devised by Bode, measures the system sensitivity by the ratio of the percentage change of the transfer function (transmission) $T(\mu)$ to the percentage change of the parameter μ. For differential variations, the sensitivity measure becomes

$$\$_{\mu}^{T} = \frac{\partial T(\mu)/T(\mu)}{\partial \mu/\mu} = \frac{\mu}{T(\mu)} \left[\frac{\partial T(\mu)}{\partial \mu}\right].$$

To illustrate an application of the sensitivity measure, consider the stationary single-variate feedback system of Figure 1.1.

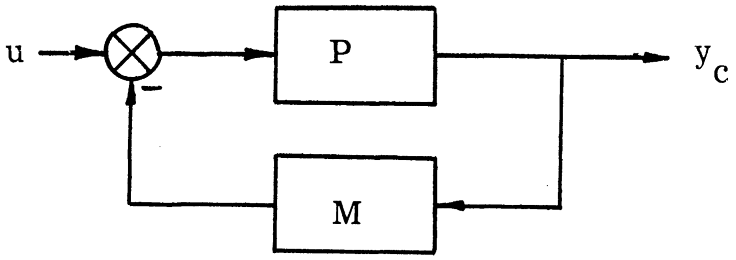

Figure 1.1 A Closed-Loop System

The plant and feedback compensator transfer functions are represented by P and M where y_c and u respectively represent the output and input of the feedback system. The transfer function of the system is given by

$$T(\mu) = \frac{P(s,\mu)}{1 + P(s,\mu)M(s)} ,$$

where it is tacitly agreed that P and M are Laplace transformable. Applying the definition of sensitivity reduction, and interpreting the parameter μ to be a parameter of the plant, the classical sensitivity measure becomes

$$\$_P^T(s,\mu_o) = [1 + P(s,\mu_o)M(s)]^{-1} \tag{1.5}$$

where μ_o is the nominal value of μ. Clearly, a good feedback design should insure that $|\$_P^T(j\omega,\mu_o)| \leq 1$ over the frequency band of interest.

Two primary reasons for introducing feedback are to reduce the effects of parameters variations upon the system behavior, and to improve the rejection of disturbance signals. If the feedback design does not include sensitivity considerations, then it is certainly possible that the effects of parameter variations could make the closed-loop system worse than the open-loop system.

Consider, for example, the stationary single-input, single-output feedback system of Figure 1.1 and the corresponding open-loop system of Figure 1.2

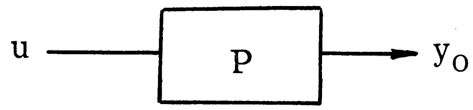

Figure 1.2 An Open-Loop System

It is easy to verify that the nominal system equations for the open-loop and closed-loop systems are given by

$$Y_o(s) = P(s, \mu_o)U(s) \tag{1.6}$$

and

$$Y_c(s) = P(s, \mu_o)[U(s) - M(s)Y_c(s)] \tag{1.7}$$

If the parameter changes from μ_o to $\Delta\mu + \mu_o$, the deviations from nominal for the open and closed-loop system are given by

$$\delta Y_o(s) = \Delta P(s, \mu)U(s) \tag{1.8}$$

$$[1 + P(s, \mu_o)M(s)]\,\delta Y_c(s) = \Delta P(s, \mu)[U(s) - M(s)(Y_c(s) + \delta Y_c(s))] \tag{1.9}$$

where $\Delta P(s, \mu) = \left(\dfrac{\delta P(s, \mu)}{\delta \mu}\right)_{\mu = \mu_o} \Delta\mu.$

From equations 1.8 and 1.9, it follows that the percentage change in the output of the closed and open-loop systems are related by

$$\frac{\delta Y_c(s)}{Y_c(s)} = [1 + P(s,\mu)M(s)]^{-1} \frac{\delta Y_o(s)}{Y_o(s)} \triangleq \$ \frac{\delta Y_o(s)}{Y_o(s)} .$$

Hence, if $|1 + P(j\omega, \mu)M(j\omega)|^{-1} \leq 1$ for all μ and $j\omega$, then the percentage change of the output of the closed-loop system will be less than that of the open-loop system. Note that for small plant variations such that $P(s, \mu)$ is approximately equal to $P(s, \mu_o)$, $\$$ is equal to the classical sensitivity given by equation 1.5. The above discussion forms the beginning of the modern concepts of sensitivity reduction of multivariate control systems. That is, it serves as a foundation from which the modern concepts are constructed. The following section gives a review of some of the more important developments in the modern theory of sensitivity reduction, and illustrates how these results reduce to the classical concepts described previously for single-variate stationary systems.

1.2 Review of the Literature

Recently efforts have been directed to extend the concept of sensitivity as presented by Bode to linear multivariate systems (references [11], [39] and [43] serve as a introduction into this subject). In particular, the works of Perkins and Cruz [11], [39] were the first successful attempts to establish a sensitivity reduction criteria for multivariate systems. Although the purpose of these papers were not to establish all the elegant structure pertinent to the sensitivity operator, they did serve to illustrate a methodology by which this objective could be achieved. The approach used by Perkins and Cruz established a criteria under which a closed-loop system would be less sensitive to disturbances than a nominally equivalent open-loop system. The criteria being that the closed loop system is said to be less sensitive than the open-loop system if[1] for all $t_f \epsilon (0, \infty)$,

$$\int_0^{t_f} \delta y_c'(t) Q \delta y_c(t) dt \leq \int_0^{t_f} \delta y_o'(t) Q \delta y_o(t) dt, \quad Q \geq 0, \qquad (1.10)$$

providing, of course, both integrals exist.

These authors considered the two system configurations illustrated in Figure 1.3.

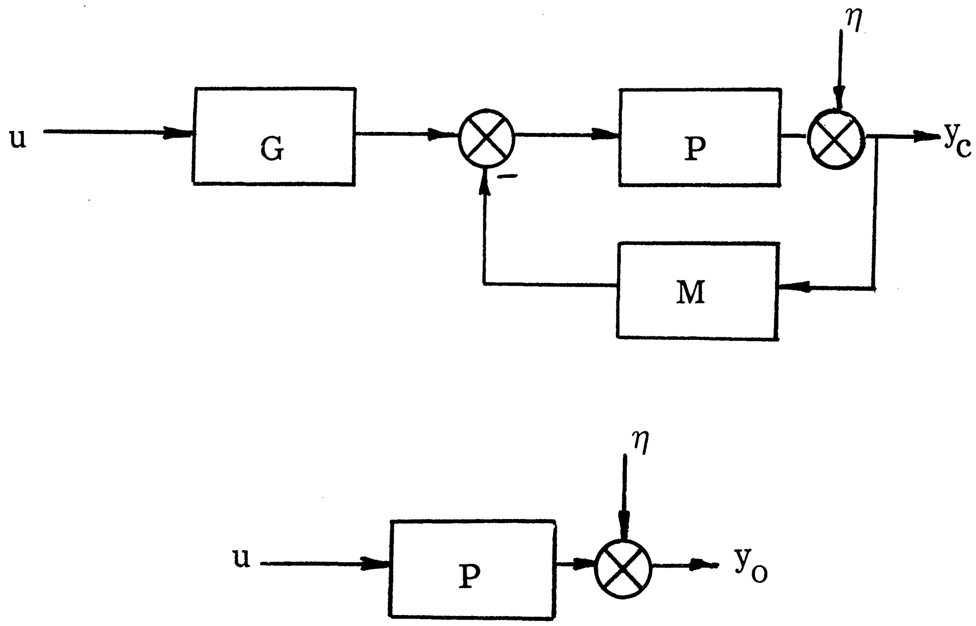

Figure 1.3 Two Nominally Equivalent Systems

---

[1] Throughout the thesis, prime is used to denote matrix transpose and the notation $A \geq 0$ is used to emply that A is a positive operator.

The plant, P, is linear, causal[1], and stationary and is represented by the matrix $P(s,\mu)$ and the feedback compensator is a constant matrix M. The compensator G is chosen so that the two systems have equivalent transfer functions with no disturbances in the form of plant perturbations or additive noise η.

If the plant is perturbed, then the relationship between the output of both systems is given by

$$\delta Y_c(s) = [1 + P(s,\mu)M]^{-1} \delta Y_o(s) \tag{1.11}$$

or

$$\delta Y_c(s) = \$(s,\mu)\,\delta Y_o(s). \tag{1.12}$$

Using Parseval's theorem and equation 1.12, it follows that the sensitivity criteria can be given a frequency domain interpretation via

$$\int_{-\infty}^{\infty} \delta Y_o^*(j\omega)[\$^*(j\omega)Q\$(j\omega) - Q]\,\delta Y_o(j\omega)d\omega \leq 0 \tag{1.13}$$

where * denotes the complex conjugate transpose. If inequality 1.10 exists and

$$Q - \$^*(j\omega)Q\$(j\omega) \geq 0 \text{ for all } \omega \epsilon (-\infty,\infty), \tag{1.14}$$

the closed-loop system is said to be less sensitive then the open-loop system. If Q is replaced by the identity operator, we see that for single-variate systems inequality, 1.14 reduces to

$$|1 + P(j\omega,\mu)M|^{-1} \leq 1, \text{ for all } \omega \,\epsilon\, (-\infty,\infty).$$

---

[1]For a definition of causality, see Appendix C.

Consequently, these results reduce to those discussed previously for single-variate systems. We now see that sensitivity reduction for multi-variate systems is given a new meaning via the performance index of inequality 1.10, which is a generalization of the classical definition for the single-variate case.

Having established this concept of sensitivity reduction for stationary multi-variate systems, Cruz and Perkins extended this concept to linear time-varying multi-variate systems [38]. This was accomplished by recognizing that $\delta y_c(t)$ and $\delta y_o(t)$ are related by a linear time-varying operator \$, such that

$$\delta y_c(t) = \$\delta y_o(t); \quad \$ = [I + PM]^{-1}. \qquad (1.15)$$

whenever I + PM is nonsingular. The sensitivity reduction criteria was then formulated by a natural generalization of inequality 1.10. This criteria states that the sensitivity of the closed-loop system is less than the open-loop system if the norm of δy_c is less than the norm of δy_o. It is evident this criteria is equivalent to the norm of \$ less than unity. This fact led the above authors to establish the relationship between the mapping \$, and contraction[1] mappings. These results although interesting for their own sake, were not sufficient to completely characterize the fundamental structure of \$.

Porter demonstrated, (see [41] through [45]), that the entire question of sensitivity reduction by means of feedback could be form-

---

[1] For a definition of contraction mappings, see [34].

ulated in terms of Banach spaces. This was accomplished by considering the two nominally equivalent systems of Figure 1.4 with η equal to zero. Letting B_1 and B_2 denote Banach spaces, the plant P is defined to be linear such that P: $B_1 \to B_2$. The compensators G and M are also linear and G: $B_1 \to B_1$ as well as M: $B_2 \to B_1$.

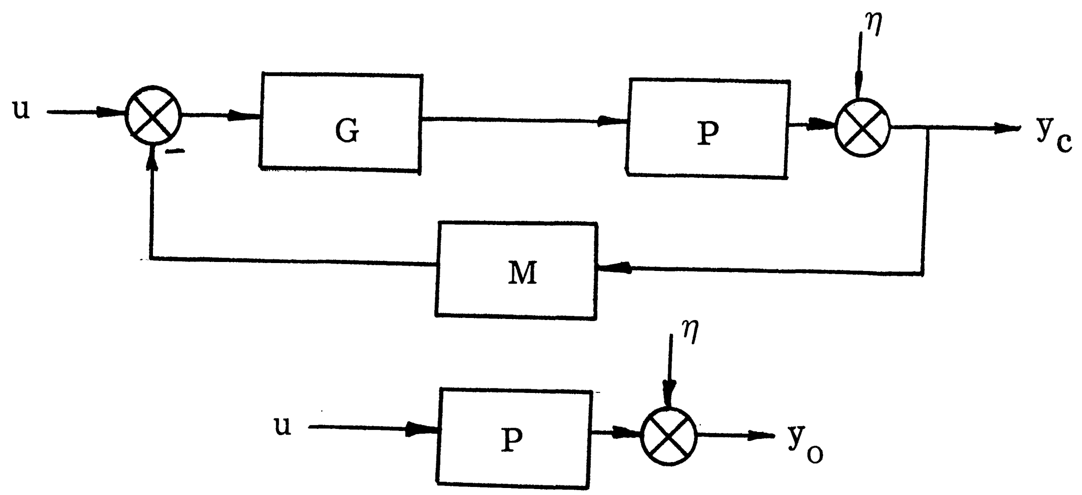

Figure 1.4 Two Nominally Equivalent Systems

As before when P is perturbed, the perturbations in the outputs of the two systems are related by a linear operator $\hat{\$}$, such that

$$\delta y_c = \hat{\$}\,\delta y_o; \quad \hat{\$} = [I + PGM]^{-1}. \qquad (1.16)$$

The sensitivity was then said to be improved if and only if the norm of $\hat{\$}$ is less than one. Porter, then focusing attention on time-varying systems whose operators map between finite cartesian products[1] of the Hilbert space, $L_2(-\infty,\infty)$, demonstrated that the

[1] See Appendix A for a definition of the Hilbert space $L_2(-\infty,\infty)$.

perturbations of the open and closed-loop systems can be related by

$$||\delta y_c||^2 = <\hat{\$}\delta y_o, \hat{\$}\delta y_o> = ||\delta y_o||^2 - <\delta y_o, (I - \hat{\$}^*\hat{\$})\delta y_o>.$$

Consequently a necessary but not sufficient condition for sensitivity reduction is that the operator $I-\hat{\$}^*\hat{\$}$ be positive. This condition is not sufficient in the sense that the inequality

$$\int_{-\infty}^{t_f} [\delta y_c(t), \delta y_c(t)]dt \leq \int_{-\infty}^{t_f} [\delta y_o(t), \delta y_o(t)]dt, \text{ for all } t_f \in (-\infty, \infty)$$

will not be satisfied for noncausal $\hat{\$}$. The above author then illustrated that if P, G, M, and $\hat{\$}$ are bounded and stationary;

$$I - \hat{\$}^*\hat{\$} \geq 0 \Longleftrightarrow I - \hat{\$}^*(j\omega)\hat{\$}(j\omega) \geq 0 \text{ for all } \omega \in (-\infty, \infty),$$

where $\hat{\$}(j\omega)$ is the frequency matrix representation of $\hat{\$}$. This result is identical to that given by Cruz and Perkins [11] with Q = I. It is emphasized that the results given by Porter are not restricted to the stationary case. In particular Porter extends the concepts of sensitivity reduction, in Hilbert spaces, to distributive systems (see [41] and [42]) as well as to time-varying systems [44].

Anderson and Newcomb [5], following the ideas of Cruz and Perkins, were the first to formalize the necessary as well as sufficient conditions for sensitivity reduction when the Hilbert space is $L_2^n(-\infty, \infty)$. By means of distribution theory, these authors demonstrated that the relation between the perturbations of the open and closed loop response due to plant parameter variations is linear and describable by a distributional[1] kernel.

For the systems of Figure 1.3, the kernel of the sensitivity operator \$ is given by

$$\$ = [\delta I_n + p_\mu \circledast m]^{-1}$$

where p_μ is the impulse response matrix of the perturbed plant $P(\mu)$, and consequently a distributional kernel dependent on μ, m is the impulse response matrix of the feedback compensator and δI_n and $\circledast$ denote the unit impulse matrix and convolution respectively.
The criteria for sensitivity reduction is not given by the usual norm on $L_2^n(-\infty,\infty)$, but instead by

$$\int_{-\infty}^{t_f} [\delta y_c(t), \delta y_c(t)]dt \leq \int_{-\infty}^{t_f} [\delta y_o(t), \delta y_o(t)]dt, \text{ for all } t_f \in (-\infty,\infty).$$

Employing this criteria, the necessary and sufficient conditions for sensitivity improvement are then demonstrated to be

1. $\$: L_2^n(-\infty,\infty) \to L_2^n(-\infty,\infty)$
2. \$ causal (A1)
3. $||\$|| \leq 1$.

For stationary systems the authors point out but do not prove that conditions (A1) are equivalent to

1. $\$(s)$ analytic for Re $s > 0$
2. $\bar{\$}(s) = \$(\bar{s})$ for Re $s > 0$ (A2)
3. $I - \$^*(s)\$(s) \geq 0$ for Re $s > 0$

---

[1] See Reference [50] for a comprehensive treatment of distribution theory.

where the overbar denotes complex conjugate and $\$(s)$ denotes the Laplace transform of the sensitivity kernel.

1.3 Sensitivity Reduction of Linear Optimal Control Systems[1]

Since the primary reason for introducing feedback is to reduce the effects of parameter variations upon the system behavior, the question of whether linear optimal regulator systems provide a closed-loop sensitivity reduction becomes pertinent. Kalman [23] was the first to answer this question affirmatively on the basis of the analogy of his results with the classical return difference of Bode. The result can be stated as follows: consider the completely controllable plant,

$$\dot{x}(t) = Ax(t) + bu(t); \quad x(0) = x_0, \tag{1.17}$$

with linear state feedback

$$u(t) = -m'x(t),$$

(where A, b, and m are $n\times n$, $n\times 1$, and $n\times 1$ matrices respectively), and a performance index of the form

$$J(u) = \int_0^\infty [x'(t)L'Lx(t) + u^2(t)]dt; \quad L'L > 0,$$

where $\{A, L\}$ is completely observable.[2] Then a necessary and sufficient condition for m to be an optimal control law is that m is a stable control law and

$$|1 + m'\Phi(j\omega)b|^2 = 1 + ||L\Phi(j\omega)b||^2 \tag{1.18}$$

hold for all ω, where $\Phi(j\omega) = (j\omega I - A)^{-1}$.

[1] The optimal regulator problem as well as associated concepts of modern control theory are discussed in Appendix B.

[2] Observability is discussed in Appendix B.

Cruz and Perkins [39] demonstrated that equation 1.18 was equivalent to the classical percentage-change sensitivity replacement

$$|\$(j\omega)| \leq 1 \quad \text{for all } \omega \epsilon (-\infty, \infty),$$

provided $\Phi(j\omega)b \neq 0$. Hence, Kalman's result demonstrated that for the single-input, multi-variate output, a necessary and sufficient condition for optimality is that the feedback control law reduce the sensitivity to disturbances.

Anderson, [6] following arguments similar to Kalman, obtained analogous results for the multi-variate feedback system. In particular, Anderson considered the controllable plant described by the equations,

$$\dot{x}(t) = Ax(t) + Bu(t);\ x(0) = x_o, \tag{1.19}$$

where

$$u(t) = -M'x(t)$$

(such that A, B, and M are $n \times n$, $n \times p$, and $n \times p$ matrices respectively), is the optimal control law for a performance index of the form

$$J(u) = \int_0^{\infty} [x'(t)L'(t)Lx(t) + u'(t)u(t)]\, dt. \tag{1.20}$$

The following theorems follow from Anderson [6]:

<u>Theorem 1.1</u> Let M be an optimal control law for the completely controllable plant 1.19 with the performance 1.20. If $\{A, L\}$ is completely observable, then

$$[I + B'\Phi^*(j\omega)M][I + M'\Phi(j\omega)B] > I, \text{ for all } \omega\epsilon(-\infty, \infty). \tag{1.21}$$

Theorem 1.2 Let M be a stable control law for the completely controllable plant 1.19 such that $\{A, M\}$ is completely observable and

$$[I + B'\Phi^*(j\omega)M][I + M'\Phi(j\omega)B] > I \text{ for all } \omega \in (-\infty, \infty),$$

then, there exists a matrix L such that $\{A, L\}$ is completely controllable such that M is an optimal control law for the performance index 1.20.

Kreindler [32], who was also working in this area, demonstrated that inequality 1.21 was equivalent to

$$\$^*(j\omega)Q\$(j\omega) \leq Q,$$

where

$$Q = MM^* \geq 0.$$

Consequently, Anderson's result illustrated that for the multi-variate feedback system, a necessary and sufficient condition for optimality is that the feedback reduce the weighted system sensitivity. Generalizing these results to multi-variate time-varying systems [3], Anderson was again able to demonstrate that over the infinite interval, the optimally derived controller reduced the sensitivity to a particular weighting of the output errors.

Kreindler, utilizing the notion of a trajectory sensitivity function, σ, defined by

$$\sigma(t) = \frac{\partial x(t)}{\partial \mu}$$

established a criteria under which the closed-loop system would be less sensitive to disturbances than a nominally equivalent open-loop system. The criteria being that the closed-loop system is less sensitive than the open-loop system if there exists a positive self-adjoint matrix Q such that

$$\int_0^{t_f} \sigma_c'(t) Q \sigma_c(t) dt < \int_0^{t_f} \sigma_o'(t) Q \sigma_o(t) dt \quad \text{for all } t_f \in (0, \infty),$$

where σ_o and σ_c represent the open and closed-loop trajectory sensitivity. For the multi-variate control system described by 1.19 where

$$u(t) = -M'x(t)$$

is the optimum control law for the performance index

$$J(u) = \int_0^{\infty} [x'(t)L'Lx(t) + u'(t)Ru(t)]dt; \quad R > 0,$$

then there exists a matrix Q namely, M'RM such that the sensitivity is reduced. Kreindler [32] also points out that in general it is not possible to reduce the sensitivity for arbitrary weighting matrices Q. To clarify the condition under which the optimal regulator reduces sensitivity, a necessary condition relating the matrices M and B is given in Chapter 5.

1.4 Dissertation Objectives

In the past, considerable effort has been devoted to identifying the structure of the sensitivity operator $. In the brief review given in section 1.2, it was seen that some of the tools used to tackle this problem owe their origin to functional analysis and distribution theory. However, made obvious from its scarcity, is the application of these tools to identify the structure of the compensators to achieve the aforementioned goal.

The purpose of this thesis is to apply Banach and Hilbert space techniques, as employed by Porter and Newcomb, to determine the relationships between the plant and its associated compensators to ensure sensitivity improvement. That is, necessary conditions and sufficient conditions relating the plant matrices $\{A(t), B(t), C(t)\}$ and the compensators G and M are given which guarantee sensitivity reduction to noise and plant parameter perturbations. It is also shown that the concept of positive realness plays an important role in developing these relationships.

Chapter 2

POSITIVE REAL OPERATORS

2.1 Introduction

For several years, network theorists have been employing a concept of positive real functions in the analysis of time-invariant networks (references [17], [18] and [33] are recommended as entries to this literature). More recently this concept has become an important tool for the control systems analyst. Popov, for example, successfully employed this concept in his development of a stability criterion for nonlinear feedback systems [40]. Anderson, employing the matrix analog of positive real function [4] explored the relationships between linear feedback control laws chosen to reduce system sensitivity, and linear feedback control laws derived on an optimal control basis from a quadratic loss function [6]. These and other applications provide in part the motivation of the present effort to formulate a notion of positive realness that can be fruitfully employed in the study of time-varying systems.

2.2 Integral Operators

We begin the discussion with the notion of an integral operator. Let $k(t,\tau)$ be a complex valued $m \times p$ matrix of functions continuous in both variables[1] on the set $\Omega \times \Omega$ where $\Omega \subset R$ denotes a closed

[1] Henceforth, if a kernel is said to be continuous it is assumed to be continuous in both variables.

interval of the real line. Denote by K the multivariate integral operator

$$(Ku)(t) = \int_{\Omega} k(t, \tau)u(\tau)d\tau, \quad t \in \Omega. \tag{2.1}$$

Let z = Ku, then in component form, each component of the vector z is given by

$$z_i(t) = \sum_{j=1}^{n} \int_{\Omega} k_{ij}(t, \tau)u_j(\tau)d\tau, \quad i = 1, \ldots, m.$$

The matrix $k(t, \tau)$ is called the kernel of the integral operator.

Throughout this chapter, we will focus our attention on bounded operators mapping between finite cartesian products of the Hilbert space[1] $L_2(\Omega)$. The scalar product between any two vectors $x, y \in L_2^n(\Omega)$ is given by

$$< x,y > = \sum_{i=1}^{n} \int_{\Omega} x_i(t)\bar{y}_i(t)dt$$

consequently the Hilbert space norm is computed by

$$||x||^2 = < x,x > = \sum_{i=1}^{n} \int_{\Omega} |x_i(t)|^2 dt = \int_{\Omega} [x(t), x(t)]dt,$$

where [,] denotes the inner product in the complex Euclidean space E^n.

The Hilbert space adjoint, K^*, of the operator K, can be determined in the following manner:

---

[1] Appendix A is recommended for a brief introduction into Hilbert spaces.

$$< y, Kx > = \int_\Omega [y(t), \int_\Omega k(t,\tau)x(\tau)d\tau]\,dt$$

$$= \int_\Omega \int_\Omega [y(t), k(t,\tau)x(\tau)]\,d\tau\,dt.$$

By Fubini's theorem[1], we can then write

$$< y, Kx > = \int_\Omega [\int_\Omega \overline{k'(t,\tau)}y(t), x(\tau)]\,dt\,d\tau = < K^*y, x >.$$

Hence, the adjoint operator is computed by

$$(K^*y)(t) = \int_\Omega \overline{k'(\tau,t)}y(\tau)d\tau, \quad t \in \Omega.$$

This serves to identify the kernel, k^*, of the adjoint operator namely

$$k^*(t,\tau) = \overline{k'(\tau,t)}, \quad (t,\tau) \in \Omega\times\Omega.$$

Let $w(t,\tau)$ be a continuous complex-valued $n\times m$ matrix defined on the set $\Omega\times\Omega$, and let W denote the associated integral operator. The application of W to z of equation 2.1 serves to identify the kernel of the composition WK

$$(Wz)(t) = \int_\Omega w(t,\tau) \int_\Omega k(\tau,s)u(s)ds\,d\tau.$$

Employing Fubini's theorem it follows that

$$(Wz)(t) = \int_\Omega \{\int_\Omega w(t,\tau)k(\tau,s)d\tau\}u(s)ds; \quad t \in \Omega$$

and thus the kernel of the composition N = WK is given by

$$n(t,s) = \int_\Omega w(t,\tau)k(\tau,s)d\tau; \quad (t,s) \in \Omega\times\Omega$$

---

[1] See Halmos [20], p. 148.

Remark 2.1 The composition of two operators may be used to introduce the idea of an inverse operator. That is, W is said to be the left inverse of K when

$$x(t) = (WKx)(t) \quad t \in \Omega,$$

for all x in the domain of K. Using the unit matrix delta function, the identity can be viewed as an integral operator; namely

$$Ix(t) = \int_\Omega I\,\delta(t-\tau)x(\tau)d\tau = x(t).$$

Therefore the kernel of the inverse system, $k^{-1}(t,\tau)$, may be viewed formally as the solution of the integral equation

$$I\,\delta(t-\tau) = \int_\Omega k^{-1}(t,\lambda)k(\lambda,\tau)d\lambda.$$

The following three lemmas (which are available in single variate form in the literature[1]) are helpful in characterizing kernels of integral operators which are bounded on the function space $L_2^n(\Omega)$ and in introducing the mathematical terminology. Because these kernels play an important role in the sequel, the proofs of these lemmas are included here.

Lemma 2.1 If the matrix kernel $f(t,\tau)$ satisfies the condition[2]

$$\sum_{ij}^{n} \int_{\Omega\times\Omega} |f_{ij}(t,\tau)|^2 d\tau dt < \infty$$

---

[1] See for example [28].

[2] The notation $\sum_{ij}^{n}$ is equivalent to $\sum_{i=1}^{n} \sum_{j=1}^{n}$.

it is called a Hilbert-Schmidt[1] kernel, and its associated integral operator F is bounded on $L_2^n(\Omega)$.

Proof: Let $y = Fx$ such that $x \epsilon L_2^n(\Omega)$ then in component form, each element of y is computed by

$$y_i(t) = \sum_j^n \int_\Omega f_{ij}(t, \tau)x_j(\tau)d\tau.$$

Employing the obvious inequality for $|y_i(t)|$ we obtain

$$|y_i(t)| \leq \sum_j^n \int_\Omega |f_{ij}(t, \tau)| \, |x_j(\tau)| d\tau.$$

By Schwartz's inequality for integrals

$$\begin{aligned} |y_i(t)| &\leq \sum_j^n [\int_\Omega |f_{ij}(t, \tau)|^2 d\tau]^{\frac{1}{2}} [\int_\Omega |x_j(\tau)|^2 d\tau]^{\frac{1}{2}} \\ &\leq [\sum_j^n \int_\Omega |f_{ij}(t, \tau)|^2 d\tau]^{\frac{1}{2}} [\sum_j^n \int_\Omega |x_j(\tau)|^2 d\tau]^{\frac{1}{2}} \\ &\leq [\sum_j^n \int_\Omega |f_{ij}(t, \tau)|^2 d\tau]^{\frac{1}{2}} ||x||. \end{aligned}$$

Using this inequality we obtain for $||y||$

$$||y|| = [\sum_i^n \int_\Omega |y_i(t)|^2 dt]^{\frac{1}{2}} \leq [\sum_{ij}^n \int_{\Omega \times \Omega} |f_{ij}(t, \tau)|^2 d\tau dt]^{\frac{1}{2}} ||x||.$$

Since $||x|| < \infty$, the lemma is proved.

Lemma 2.2 If the matrix kernel $f(t, \tau)$ satisfies the conditions

$$\int_\Omega |f_{ij}(t, \tau)| d\tau \leq M_1^2, \quad \int_\Omega |f_{ij}(t,\tau)| dt \leq M_2^2 \text{ for } i,j = 1, \ldots, n$$

---

[1] For further properties of Hilbert Schmidt operators see [12].

then, the associated operator F is bounded on $L_2^n(\Omega)$.

Proof: As in the previous lemma, for $y = Fx$, y_i is computed by

$$y_i(t) = \sum_j^n \int_\Omega f_{ij}(t, \tau)x_j(\tau)d\tau.$$

Taking absolute values

$$|y_i(t)| \leq \sum_j^n \int_\Omega |f_{ij}(t, \tau)x_j(\tau)|d\tau,$$

and using Schwartz's inequality

$$|y_i(t)| \leq \sum_j^n \int_\Omega [\,|f_{ij}(t, \tau)|\,|x_j(\tau)|^2]^{\frac{1}{2}}\,|f_{ij}(t, \tau)|^{\frac{1}{2}}d\tau$$

$$\leq \sum_j^n \{\int_\Omega |f_{ij}(t, \tau)|\,|x_j(\tau)|^2 d\tau\}^{\frac{1}{2}}\{\int_\Omega |f_{ij}(t, \tau)|d\tau\}^{\frac{1}{2}}$$

$$\leq M_1\{\sum_j^n \int_\Omega |f_{ij}(t, \tau)|\,|x_j(\tau)|^2 d\tau\}^{\frac{1}{2}}.$$

We obtain for $||y||$

$$||y|| = \{\sum_i^n \int_\Omega |y_i(t)|^2 dt\}^{\frac{1}{2}} \leq M_1\{\sum_{ij}^n \int_{\Omega\times\Omega} |f_{ij}(t, \tau)|\,|x_j(\tau)|^2 d\tau dt\}^{\frac{1}{2}}$$

$$\leq M_1\,\{\sum_j^n \int_\Omega |x_j(\tau)|^2 \sum_i^n \int_\Omega |f_{ij}(t, \tau)|dt\,d\tau\}^{\frac{1}{2}}$$

Hence,

$$||y|| \leq n^{\frac{1}{2}} M_1 M_2\,||x||,$$

and since $||x||$ is finite, the lemma is established.

If the matrix kernel $f(t, \tau)$ is of the stationary type, that is $f(t, \tau) = f(t - \tau)$, then the following lemma is pertinent.

<u>Lemma 2.3</u> If $f_{ij} \in L_1(\Omega)$ that is, $\int_\Omega |f_{ij}(\tau)| d\tau \leq M$ for $i, j = 1, \ldots, n$, then the associated integral operator is bounded on $L_2^n(\Omega)$ where $\Omega = (-\infty, \infty)$.

Proof: Let $Fx = y$ such that $x \in L_2^n(\Omega)$. y_i is computed by

$$y_i(t) = \sum_j^n \int_\Omega f_{ij}(t-\tau)x_j(\tau)d\tau = \sum_j^n \int_\Omega f_{ij}(\tau)x_j(t-\tau)d\tau.$$

Taking absolute values

$$|y_i(t)| \leq \sum_j^n \int_\Omega |f_{ij}(\tau)|\ |x_j(t-\tau)| d\tau,$$

and using Schwartz's inequality

$$|y_i(t)| \leq \sum_j^n \int_\Omega [f_{ij}(\tau)|\ |x_j(t-\tau)|^2]^{\frac{1}{2}} |f_{ij}(\tau)|^{\frac{1}{2}} d\tau$$

$$\leq \sum_j^n \{\int_\Omega |f_{ij}(\tau)|\ |x_j(t-\tau)|^2 d\tau\}^{\frac{1}{2}} \{\int_\Omega |f_{ij}(\tau)| d\tau\}^{\frac{1}{2}}$$

$$\leq M \{\sum_j^n \int_\Omega |f_{ij}(\tau)|\ |x_j(t-\tau)|^2 d\tau\}^{\frac{1}{2}}$$

Computing $||y||$

$$||y|| = \{\sum_i^n \int_\Omega |y_i(t)|^2 dt\}^{\frac{1}{2}} \leq M \{\sum_{ij}^n \int_{\Omega\times\Omega} |f_{ij}(\tau)|\ |x_j(t-\tau)|^2 d\tau\, dt\}^{\frac{1}{2}}$$

$$\leq M \{\sum_j^n \int_\Omega |x_j(t-\tau)|^2 \sum_i^n \int_\Omega |f_{ij}(\tau)| d\tau\, dt\}^{\frac{1}{2}}$$

$$\leq n^{\frac{1}{2}} M^2 \{\sum_j^n \int_\Omega |x_j(t-\tau)|^2 dt\}^{\frac{1}{2}}$$

$$\leq n^{\frac{1}{2}} M^2 ||x||.$$

Hence, the lemma is proved.

Definition 2.1 A continuous self adjoint matrix kernel $k(t,\tau)$, defined on $\Omega \times \Omega$, is called a positive kernel if[1]

$$< x, Kx > = \int_{\Omega\times\Omega} [x(t), k(t,\tau)x(\tau)]\, d\tau\, dt \geq 0 \tag{2.2}$$

for all $x \in L_2^n(\Omega)$.

To see that this class is nonvacuous, we consider, for example, the kernel defined by

$$k(t,\tau) = \int_{\Omega} \overline{f'(\lambda, t)} f(\lambda, \tau) d\lambda, \quad (t,\tau) \in \Omega \times \Omega \tag{2.3}$$

where $f(t,\tau)$ is a continuous complex valued $m \times m$ matrix.

Since

$$\begin{aligned} < x, Kx > &= \int_{\Omega\times\Omega\times\Omega} [f(\lambda,t)x(t), f(\lambda,\tau)x(\tau)]\, d\lambda\, d\tau\, dt \\ &= \int_{\Omega} [\int_{\Omega} f(\lambda,t)x(t)dt, \int_{\Omega} f(\lambda,\tau)x(\tau)d\tau]\, d\lambda \\ &= ||Fx||^2 \geq 0, \end{aligned}$$

it is clear the $k(t,\tau)$ is a positive kernel.

The following two lemmas are helpful in further characterizing positive kernels. Let Ω be the set $[t_o, t_f]$, then

Lemma 2.4 If the kernel $k(t,\tau)$ is continuous in both variables on the set $\Omega \times \Omega$, then a necessary and sufficient condition for $k(t,\tau)$ to be a positive kernel, is that for every finite uniform sequence $\{t_\alpha\}_1^m$ of distinct points belonging to Ω and arbitrary complex vectors

---

[1] In short a kernel is positive if its associated operator is positive. See also Zaanen [58].

$y^1, y^2, \ldots, y^m \epsilon E^n$, the following relation holds

$$\sum_{\alpha,\beta}^{m} [y^\alpha, k(t_\alpha, t_\beta) y^\beta] \geq 0. \qquad (2.4)$$

Proof: This proof is a straightforward generalization of the single variate case and follows that given in Widder [55]. Assume equation 2.4 is valid, and let

$$\Delta t = \frac{t_f - t_o}{m+1}.$$

Set

$$t_\alpha = t_o + \alpha \Delta t, \quad \alpha = 1, \ldots, m.$$

Letting $y^\alpha = x(t_\alpha)\left(\frac{t_f - t_o}{m+1}\right)$ and substituting into 2.4 yields

$$\sum_{\alpha,\beta}^{m} [x(t_\alpha), k(t_\alpha, t_\beta) x(t_\beta)] \left(\frac{t_f - t_o}{m+1}\right)^2 \geq 0. \qquad (2.5)$$

Equation 2.2 then follows from 2.5 by taking the limit $m \to \infty$. [56]

To prove necessity we proceed as follows: suppose that there exist $\{y^\alpha\}$ and $\{t_\alpha\}$ for which equation 2.4 is not positive. We shall then show that there exist continuous complex vectors x(t), such that equation 2.2 is not satisfied. Define a continuous scalar function $\theta_{\epsilon,\gamma}(t, \hat{t})$, such that $\hat{t}$ and t belong to Ω. Let ϵ and γ be small enough so that $t_o < \hat{t} - \epsilon - \gamma < \hat{t} + \epsilon + \gamma < t_f$. $\theta_{\epsilon,\gamma}(t, \hat{t})$ is then defined to be zero in the interval $(t_o \leq t \leq \hat{t} - \epsilon - \gamma)$ and $(\hat{t} + \epsilon + \gamma \leq t \leq t_f)$ and unity in $(\hat{t} - \gamma \leq t \leq \hat{t} + \gamma)$. In the rest of the interval $\theta_{\epsilon,\gamma}(t, \hat{t})$ is defined to be linear in t. Now choose $\epsilon > 0$, and $\gamma > 0$ so small that no two intervals $(-\epsilon - \gamma + t_\alpha, t_\alpha + \epsilon + \gamma)$ have a non-zero intersection.

Set

$$\theta(t) = \sum_{\alpha}^{m} y^\alpha \theta_{\epsilon,\gamma}(t, t_\alpha)$$

and let

$$K_m(t,\tau) = \sum_{\alpha,\beta}^{m} [y^\alpha, k(t+t_\alpha, \tau+t_\beta)y^\beta].$$

Hence

$$< \theta, K\theta > = \int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} K_m(t,\tau)\,dt\,d\tau + \sum_{\alpha,\beta}^{m} \iint_{\Delta_{\alpha\beta}} [\theta(t), k(t,\tau)\theta(\tau)]\,d\tau\,dt$$

where $\Delta_{\alpha\beta}$ is the region between the square

$$t_\alpha - \epsilon - \gamma \leq t \leq t_\alpha + \epsilon + \gamma, \quad t_\beta - \epsilon - \gamma \leq \tau \leq t_\beta + \epsilon + \gamma$$

and the square

$$t_\alpha - \gamma \leq t \leq t_\alpha + \gamma, \quad t_\beta - \gamma \leq \tau \leq t_\beta + \gamma.$$

For any point (t, τ) in the region $\Delta_{\alpha\beta}$

$$|[\theta(t), k(t,\tau)\theta(\tau)]| \leq M \sum_{i,j}^{n} |y_i^\alpha y_j^\beta|.$$

where

$$M = \sup_{(t,\tau)} |k_{ij}(t,\tau)| \text{ for } i, j = 1, \ldots, n.$$

Hence,

$$\left|\sum_{\alpha,\beta}^{m} \iint_{\Delta_{\alpha\beta}} [\theta(t), k(t,\tau)\theta(\tau)]\,d\tau\,dt\right| \leq 4\epsilon M(2\gamma+\epsilon) \sum_{\alpha,\beta}^{m} \sum_{i,j}^{n} |y_i^\alpha y_j^\beta|.$$

Since we assumed that equation 2.4 was non-positive, we will let

$$K_m(0,0) = -\delta$$

and choose an γ so small that

$$K_m(t,\tau) < -\delta/2; \quad |t| < \gamma, \quad |\tau| < \gamma.$$

Then, for such a γ,

$$\int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} K_m(t,\tau)d\tau\, dt < -2\delta\gamma^2$$

and therefore,

$$< \theta, K\theta >< -2\,\delta\gamma^2 + 4\,\epsilon\, M(2\gamma+\epsilon)\sum_{\alpha,\beta}\sum_{i,j} |y_i^\alpha y_j^\beta|.$$

By choosing ϵ small enough, the right-hand side of this inequality is negative and since $\theta(t)$ is a continuous vector function, we have a contradiction which completes the proof.

<u>Lemma 2.5</u> If a continuous matrix kernel $k(t,\tau)$, is positive, then $k(t,t)$ is a continuous positive matrix.

Proof: By lemma 2.4, we have, for any point t in (t_o, t_f),

$$[y, k(t,t)y] \geq 0$$

Hence, by definition, $k(t,t)$ is a positive matrix, and since $k(t,\tau)$ is continuous in $\Omega\times\Omega$, $k(t,t)$ is also continuous for $t \in \Omega$.

2.3 Linear Time-varying Systems

Turning now to more physical considerations we will define a concept of L_2-positive realness for integral operators representing time-varying dynamical systems as well as demonstrate their necessary structure to ensure L_2-positive realness. For time-invariant systems it will be shown that these concepts are generalizations of the standard definition.[1]

<u>Definition 2.2</u> An integral operator W: $L_2^m(\Omega) \rightarrow L_2^m(\Omega)$, whose kernel, $w(t, \tau)$ is continuous in both real variables on $\Omega \times \Omega$, is said to be <u>L_2-positive real</u> if

1. $w(t, \tau)$ is real
2. W is causal
3. W is bounded
4. $W + W^* \geq 0$.

Consider now the linear causal dynamical system with the mathematical model of the form

(S1)
$$x(t) = A(t)x(t) + B(t)u(t), x(t_o) = 0, \ t \in \Omega = [t_o, t_f] \qquad (2.6)$$
$$y(t) = C(t)x(t). \qquad (2.7)$$

Here the tuplet $u = (u_1, \ldots, u_m)$ represents the independent system inputs, the tuplet $x = (x_1, \ldots, x_n)$ represents the state variables, and $y = (y_1, \ldots, y_m)$ represents the output. The matrices $A(t), B(t)$, and $C(t)$ are real, and continuous on Ω, (that is, each element of each matrix is real and continuous on Ω), and are of compatible dimensions. The output of the linear system is related to the input by the integral operator

$$(Wu)(t) = \int_\Omega w(t,\tau)u(\tau)d\tau \qquad (2.8)$$

where w is the impulse response matrix[2] of the system described by S1 such that

---

[1]See definition B.8 of Appendix B.

[2]See Zadeh [59].

$$w(t, \tau) = C(t)\Phi(t,\tau)B(\tau) \qquad t \geq \tau$$

$$w(t, \tau) = 0 \qquad t < \tau$$

and where Φ is the transition matrix of the system described by

$$\dot{x}(t) = A(t)x(t) + u(t), \; x(t_o) = 0, \; t \in \Omega.$$

If the integral operator associated with S1, is L_2-positive real, then S1 is said to be an <u>L_2- positive real system</u>.

The next theorem establishes a sufficient condition for L_2 - positive realness in the context of S1.

<u>Theorem 2.1</u> Let the kernel of the causal integral operator W, be bounded on $L_2^m(\Omega)$, and given by $C(t)\Phi(t, \tau)B(\tau)$, If there exists a continuous, positive self-adjoint $n \times n$ matrix Q, and a continuous $r \times n$ matrix L such that

1. $-\dot{Q}(t) = Q(t)A(t) + A'(t)Q(t) + L'(t)L(t)$ (2.9)
2. $Q(t)B(t) = C'(t)$
3. $(Fx)(t) \triangleq \int_\Omega L(t)\Phi(t, \tau)B(\tau)x(\tau)d\tau$: $\|F\|$ is finite

where $Q(t_o)$ is chosen to satisfy condition (2) at $t = t_o$, then W is L_2- positive real.

Proof: We shall make use of the identities

$$\frac{d}{d\tau}\{\Phi'(\tau, t)Q(\tau)\Phi(\tau, s)\} = \Phi'(\tau, t)\{A'(\tau)Q(\tau) + \dot{Q}(\tau) + Q(\tau)A(\tau)\}\Phi(\tau, s)$$

$$= -\Phi'(\tau, t)L'(\tau)L(\tau)\Phi(\tau, s) \qquad (2.10)$$

$$\int_t^{t_f}\int_{t_o}^{\tau} f(t, \tau, s)ds\,d\tau = \int_{t_o}^{t}\int_t^{t_f} f(t, \tau, s)d\tau ds + \int_t^{t_f}\int_s^{t_f} f(t, \tau, s)d\tau ds.$$

Employing the above condition, F^*F can be written as

$$(F^*Fx)(t) = \int_{t_o}^{t}\int_{t}^{t_f} B'(t)\Phi'(\tau,t)L'(\tau)L(\tau)\Phi(\tau,s)B(s)x(s)d\tau ds$$
$$+ \int_{t}^{t_f}\int_{s}^{t_f} B'(t)\Phi'(\tau,t)L'(\tau)L(\tau)\Phi(\tau,s)B(s)x(s)d\tau ds. \qquad (2.11)$$

Now proceeding with the proof, it follows by substituting equation 2.10 into equation 2.11,

$$(F^*Fx)(t) = -\int_{t_o}^{t}\int_{t}^{t_f} B'(t)\frac{d}{d\tau}\{\Phi'(\tau,t)Q(\tau)\Phi(\tau,s)\}B(s)x(s)d\tau ds$$
$$-\int_{t}^{t_f}\int_{s}^{t_f} B'(t)\frac{d}{d\tau}\{\Phi'(\tau,t)Q(\tau)\Phi(\tau,s)\}B(s)x(s)d\tau ds.$$

Employing the fundamental theorem of integral calculus, it is clear that

$$(F^*Fx)(t) = \int_{t_o}^{t} B'(t)[Q(t)\Phi(t,s)-\Phi'(t_f,t)Q(t_f)\Phi(t_f,s)]B(s)x(s)ds$$
$$+\int_{t}^{t_f} B'(t)[\Phi'(s,t)Q(s)-\Phi'(t_f,t)Q(t_f)\Phi(t_f,s)]B(s)x(s)ds.$$

Since Φ is the transition matrix of a causal system, it follows from the preceding equation that

$$(F^*Fx)(t) = \int_{\Omega} B'(t)[Q(t)\Phi(t,s)+\Phi'(s,t)Q(s)]B(s)x(s)ds$$
$$-\int_{\Omega} B'(t)\Phi'(t_f,t)Q(t_f)\Phi(t_f,s)B(s)x(s)ds.$$

By employing condition (2) it is seen that

$$(F^*Fx)(t) = ([W+W^*]x(t) - \int_{\Omega} B'(t)\Phi'(t_f,t)Q(t_f)\Phi(t_f,s)B(s)x(s)ds$$

and since $F^{*}F$ is positive and bounded and since the integral operator associated the last term of this equation is positive and bounded, the theorem follows.

The following theorem gives a necessary condition for the L_2-positive realness of W in the context of the system described by S1.

Theorem 2.2 Let the kernel of the integral operator W be given by $C(t)\Phi(t,\tau)B(\tau)$ then a necessary condition for W to be L_2-positive real is

$$C(t)B(t) + B'(t)C'(t) \geq 0, \text{ for all } t \in \Omega.$$

Proof: From definition 2.2 we see that a necessary condition for positive realness is that the kernel

$$C(t)\Phi(t,\tau)B(\tau) + B'(t)\Phi'(\tau,t)C'(\tau)$$

be positive. Employing lemma 2.5 it follows

$$C(t)\Phi(t,t)B(t) + B'(t)\Phi'(t,t)C'(t) \geq 0.$$

Since $\Phi(t,t) = I$, the theorem follows.

Before continuing the analysis of L_2-positive realness it is pertinent to examine some of the characteristics of the matrix differential equation given by condition (1) of theorem 2.1

Remark 2.2 The solution of the differential equation

$$-\dot{Q}(t) = A'(t)Q(t) + Q(t)A(t) + L'(t)L(t)$$

exists and is unique (see Porter [46]) and is given by

$$Q(t) = \Phi'(t_o, t)\{Q(t_o) - \int_{t_o}^{t} \Phi'(\tau, t_o)L'(\tau)L(\tau)\Phi(\tau, t_o)d\tau\}\Phi(t_o, t),\ t \in \Omega.$$

Moreover, if $Q(t_o)$ is self-adjoint and chosen so that

$$Q(t_o) \geq \int_{t_o}^{t_f} \Phi'(\tau, t_o)L'(\tau)L(\tau)\Phi(\tau, t_o)d\tau$$

then $Q(t)$ is positive and self-adjoint for all $t \in \Omega$. It is also noted that if $L\Phi$ is the impulse response matrix of an exponentially asymptotically stable system[1], then $\lim_{t \to \infty} ||\Phi'(t, t_o)Q(t)\Phi(t, t_o)||$ is finite.

Remark 2.3 Assume that; $A(t) = A_1 + A_2(t)$ and $Q(t) = Q_1 + Q_2(t)$ where A_1 and Q_1 are stationary and $\text{Re}\{\sigma(A_1)\} \leq \epsilon < 0$ where $\sigma(A)$ denotes the spectrum[2] of A. Choose Q_1 to be the unique, positive, self-adjoint solution of

$$Q_1A_1 + A'_1Q_1 = -L'L.$$

Then sufficient conditions (see Porter [44]) for the existence of a matrix $L(t)$ and a bounded, uniformly positive matrix $Q(t)$ satisfying equation 2.9 are given by

1. for some scalars $k, \lambda > 0$, $||\Phi(t, t_o)|| \leq k e^{-\lambda(t - t_o)}, t \in (t_o, \infty)$
2. for some scalars $\delta, \beta > 0$, $||A_2(t)|| \leq \beta e^{-\delta t}, t \in (t_o, \infty)$
3. for some scalars $k', \lambda' > 0$, $||\Phi(t_o, t)|| \leq k' e^{\lambda'(t - t_o)}, t \in (t_o, \infty)$
4. $\dfrac{\beta(kk')^2||Q_1||}{\sqrt{2\lambda\delta}} < \gamma$ where $Q_1 > \gamma I > 0$.

---

[1] See Appendix B.

[2] For discussion of the spectrum of an operator, see Appendix A.

<u>Remark 2.4</u> If the matrices B and C are stationary then equation 2.9 becomes algebraic, namely;

$$QA(t) + A'(t)Q = -L'(t)L(t) \qquad (2.12)$$

Porter has shown (see [44]) that for the matrix A(t) such that

$$A(t) = A_1 + \gamma A_2(t); \quad \text{Re}\,[\sigma(A_1)] \leq \epsilon < 0,$$

it may still be possible to satisfy equation 2.12 with a constant Q. Following Porters development it follows that if $||A_2(t)|| \leq 1$ for $t \in [t_o, \infty)$ then equation 2.12 can be expressed by

$$QA(t) + A'(t)Q = A_1'Q + QA_1 + \gamma[A_2'(t)Q + QA_2(t)] = L'(t)L(t).$$

Hence it is sufficient that Q satisfies

$$A_1'Q + QA_1 = -\bar{L}'\bar{L}$$

where $\bar{L}'\bar{L} > 2\gamma\, ||Q||I$.

The following three examples serve to illustrate the previous theorems.

<u>Example 2.1</u> To illustrate the use of theorem 2.1, consider a single variate system. In equations 2.6, 7 let x, y and u be scalar valued functions. The matrices A, B, and C are replaced by the scalar functions -a(t), b, and c respectively.

Equation 2.6 reduces to

$$\dot{x}(t) + a(t)x(t) = bu(t), \quad x(0) = 0, \quad t \in \Omega = [0, \infty], \qquad (2.13.1)$$

$$y(t) = c\,x(t) \qquad (2.13.2)$$

where it is assumed that a(t), b, c are real and

$$a(t) = t + \delta, \text{ and } \delta, b, c > 0.$$

This system defines an integral operator given by

$$(Wu)(t) = \int_0^t c\,\Phi(t,\tau) b\, u(\tau) d\tau,$$

where

$$\Phi(t,\tau) = \exp\left\{ - \int_\tau^t a(s) ds \right\} \quad t \geq \tau$$

$$= 0 \qquad t < \tau.$$

Clearly (1) and (2) of definition 2.2 are satisfied. It is evident that W is bounded on $L_2(\Omega)$, since

$$\Phi(t,\tau) \leq e^{-\delta(t-\tau)} = e^{-\delta\lambda}, \lambda = t - \tau;$$

therefore

$$\int_0^t |c\,\Phi(\lambda) b| d\lambda < \infty \text{ for } t \in (0, \infty).$$

Hence, from lemma 2.3, Φ is an L_1 kernel and the boundedness of W follows. The operator W + W* is positive since there exists an $\ell(t)$ and a q(t), namely

$$\sqrt{2a(t)}q = \ell(t), \; q = {}^c/_b,$$

such that theorem 2.1 is satisfied. Consequently, the system described by equations 2.13 is L_2-positive real.

Example 2.2 The purpose of this example is to illustrate a kernel which doesn't satisfy theorem 2.2 and demonstrate that it is not L_2-positive real.

Consider the single-variate system represented by

$$\dot{x}(t) + x(t) = \text{Sin } t\, u(t), \quad x(0) = 0, \quad t \in [o, \infty]$$

$$y(t) = t x(t).$$

This system defines the integral operator given by

$$(Wu)(t) = \int_o^t t e^{-(t-\tau)} \text{Sin } \tau u(\tau) d\tau.$$

Let u be given by

$$u(t) = 1 \quad \pi \leq t \leq 2\pi$$

$$u(t) = 0 \quad \text{otherwise},$$

hence, $u \in L_2(o, \infty)$. It will now be demonstrated that $< u, (W + W^*)u >$ is non-positive, hence the single-variate system is not L_2-positive real.

It is clear that

$$< u, (W+W^*)u > = \int_\pi^{2\pi} \int_\pi^{2\pi} t e^{-(t-\tau)} \text{Sin } \tau d\tau dt + \int_\pi^{2\pi} \int_t^{2\pi} (\text{Sin } t) e^{-(\tau-t)} \tau d\tau dt$$

$$= -4\pi - 3 + e^{-\pi}(1+2\pi) < 0.$$

Example 2.3 The purpose of this example is to illustrate an operator that satisfies theorem 2.2 and is not L_2-positive real. The kernel under consideration is given by

$$w(t, \tau) = -\frac{1}{3} e^{-4(t-\tau)} + \frac{1}{3} e^{-(t-\tau)}, t \geq \tau,$$

$$w(t, \tau) = 0 \quad t < \tau$$

and Ω is the set $(-\infty, 1)$. Theorem 2.2 is satisfied since the matrix BC is the zero matrix. Let u be given by

$$u(t) = e^{(1+j4)t}, \quad -\infty \le t \le 1$$

$$u(t) = 0 \quad \text{otherwise},$$

hence $u \in L_2(-\infty, 1)$. Substituting u into the following inner product, it follows that

$$< u, (W + W^*)u > = \int_{-\infty}^{1} \int_{-\infty}^{t} \frac{1}{3} (e^{-(t-\tau)} - e^{-4(t-\tau)}) e^{(1+j4)\tau} e^{(1-j4)t} d\tau dt$$

$$+ \int_{-\infty}^{1} \int_{t}^{1} \frac{1}{3} (e^{-(\tau-t)} - e^{-4(\tau-t)}) e^{(1+j4)\tau} e^{(1-j4)t} d\tau dt$$

$$= -1.04.$$

consequently W is not L_2 - positive real.

2.4 Stationary Systems

In this section, we will focus our attention on the dynamical system S1 when A, B, and C are constant real matrices, and demonstrate that definition 2.2 is a generalization of the usual concept of positive realness given by definition B.8, when the integral operator representing S1 is defined in $L_2^m(-\infty, \infty)$.

The first order of business is that of establishing the relationship between L_2 - positive-realness in the time and frequency domains.

Lemma 2.6 If the integral operator W, defined on $L_2^m(-\infty, \infty)$, is L_2 - positive real and its kernel, $w(t-\tau)$ is continuous and stationary, then for Re $s > 0$

1. W(s) is analytic

2. $\overline{W(s)} = W(\bar{s})$

3. $W(s) + W^*(s) \geq 0$

where W(s) is the Laplace transform of $w(t-\tau)$.

Proof: 1. Since the kernel $w(t, \tau)$ is stationary and causal and since its associated integral operator is bounded on $L_2^m(-\infty, \infty)$, then it follows from theorem C.5 of Appendix C, that W(s) exists and is analytic for Re $s > 0$.

2. Condition (2) is equivalent to $w(t-\tau)$ being real.

3. Since $W + W^*$ is positive, it is also real and consequently we need only concern ourselves with Re $< x, Wx >$. Let

$$x(t) = \begin{cases} e^{st}\hat{x}, \text{ for } t < t_f \text{ where } s = \sigma + j\omega \ (\sigma > 0) \\ 0 \text{ for } t > t_f, \end{cases}$$

such that $\hat{x} \in E^m$, hence $x \in L_2^m(-\infty, \infty)$. Since $x(t) = 0$ for $t \geq t_f$,

$$\text{Re} < x, Wx > = \text{Re} \int_{-\infty}^{t_f} [x(t), \int_{-\infty}^{t} w(t-\tau)x(\tau)d\tau] \, dt$$

$$= \text{Re} \int_{-\infty}^{t_f} [x(t), \int_{0}^{\infty} w(\tau)x(t-\tau)d\tau] \, dt.$$

Consequently,

$$\text{Re} < x, Wx > = \text{Re} \int_{-\infty}^{t_f} [e^{st}\hat{x}, \int_{0}^{\infty} w(\tau)e^{-s\tau} d\tau e^{st}\hat{x}]dt$$

$$= \text{Re} [\hat{x}, W(s)\hat{x}] \int_{-\infty}^{t_f} e^{2\sigma t} dt.$$

Since

$$\int_{-\infty}^{t_f} e^{2\sigma t}\, dt > 0$$

and since Re < x, Wx >is positive, we obtain the desired result; namely

$$\mathrm{Re}\,[\hat{x}, W(s)\hat{x}] \geq 0, \ \mathrm{Re}\ s > 0. \tag{2.14}$$

Hence the definition for L_2 - positive realness of time-varying system is a natural one, since it is a generalization of the well established definition for time-invariant systems, when restricted to operators defined on $L_2^m(-\infty, \infty)$. It is for precisely this reason that the definition is given by "L_2 -positive real" instead of "positive real." If W(s) is represented by rational polynominals in s and is analytic for Re s $\geq$ 0, then the two definitions are equivalent.

Corollary 2.1 If, in addition to the three conditions describing W(s) in lemma 2.6, W(s) is absolutely convergent for Re s>0 and W(jω) is essentially bounded for $\omega \in (-\infty, \infty)$, then the converse of lemma 2.6 is easily established.

The proof follows from theorem C.6 and lemma 2.6.

Consider now the stationary representation of the system described by S1, which is given by

(S2)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0$$

$$y(t) = Cx(t)$$

where it is assumed that x, $u \in L_2^m(o, \infty)$. Taking Laplace transforms of the above system, it is easily seen that

$$Y(s) = C(sI - A)^{-1}BU(s) = W(s)U(s).$$

Since y is given by the convolution integral

$$y(t) = \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau,$$

it follows easily that the transform of the kernel $C\Phi B$, is given by the matrix $C(sI - A)^{-1}B$.

A pertinent question to ask at this point is "What relationships must exist between the matrices A, B, and C to ensure that W(s) is L_2-positive real?" The following two theorems are useful in resolving this question.

Theorem 2.3 A necessary condition for $C(sI - A)^{-1}B$ to be L_2-positive real[1] is that

$$CB = B'C' \geq 0$$

Proof: Since we are concerned with a real-rational matrix, equation 2.14 is valid for Re $s \geq 0$, and we can write the following chain of identities:

$$0 \leq \text{Re}\,[x, W(s)x] \Longleftrightarrow W(s) + W^*(s) \geq 0$$

$$\Longleftrightarrow C(sI - A)^{-1}B + B'(sI - A')^{-1}C' \geq 0, \text{ for Re } s \geq 0. \quad (2.15)$$

For large s inequality 2.15 reduces to

$$\frac{CB}{s} + \frac{B'C'}{\bar{s}} \geq 0, \text{ Re } s \geq 0. \quad (2.16)$$

[1] This theorem also applies for positive realness.

If $s = re^{j\theta}$ for $-\pi/2 \leq \theta \leq \pi/2$ then 2.16 becomes

$$[CB + B'C'] \frac{\cos\theta}{r} + j[B'C' - CB] \frac{\sin\theta}{r} \geq 0. \qquad (2.17)$$

Evaluating inequality 2.17 at $\theta = 0$; $\pi/2$; and $-\pi/2$ we obtain

$$\left.\begin{aligned} \theta = 0:&\quad CB + B'C' \geq 0 \\ \theta = \pi/2:&\quad j(B'C' - CB) \geq 0 \\ \theta = -\pi/2:&\quad -j(B'C' - CB) \geq 0 \end{aligned}\right\} \Longrightarrow B'C' = CB$$

Hence the theorem is proved. It is to be noted that these conditions are analogous to those of theorem 2.2.

Theorem 2.4 If the matrix A is nonsingular, a necessary condition for $C(sI-A)^{-1}B$ to be positive real is:

$$CA^{-1}B + B'(A')^{-1}C' \leq 0.$$

This is easily seen by evaluating the inequality 2.15 as $s \to 0$.

Returning now to the matrix differential equation

$$\dot{Q} + A'(t)Q(t) + Q(t)A(t) + L'(t)L(t) = 0. \qquad (2.18)$$

it is apparent that for stationary systems, equation 2.18 reduces to

$$A'Q + QA = -L'L.$$

Hence from theorem 2.1, sufficient conditions for positive realness can be stated by the following:

Corollary 2.2 If there exists a positive self adjoint matrix Q and a matrix L such that

1. $QA + A'Q = -L'L$, $\text{Re}\{\sigma(A)\} < 0$,

2. $QB = C'$

then the real time-invariant dynamical system S2 is L_2 - positive real. As pointed out in lemma B.5 of Appendix B, Anderson was able to sharpen this result for positive realness by showing that it was necessary as well as sufficient for $Q > 0$. His result, however, relied heavily on concepts developed in the complex frequency domain. At the present time, this Author has been unable to obtain the analogous necessary conditions for L_2 - positive realness in the time domain.

2.5 L_2 - Positive Realness and Stability

As a direct application of the preceding section, we shall demonstrate the utility of the concept of L_2 - positive realness as applied to stability in time-varying systems. The application of positive realness is not new to stationary plants (see for example [1] and [10]); however it is refreshing to see that a generalization of these concepts can be extended to time-varying systems.

Theorem 2.5 If the kernel of the composition of the operators P and M is given by $C(t)\Phi(t, \tau) B(\tau)M(\tau)$ and is L_2-positive real such that theorem 2.1 is satisfied with $Q(t) > 0$, then the closed-loop system illustrated in Figure 1.1 is stable.

Proof: Since theorem 2.1 is satisfied we have

1. $$-\dot{Q}(t) = A'(t)Q(t) + Q(t)A(t) + L'(t)L(t); \quad Q(t) > 0 \qquad (2.19)$$

2. $Q(t)B(t)M(t) = C'(t)$ (2.20)

The tentative Lyapunov function[1] V is defined by

$$V(x,t) = [x(t), Q(t)x(t)], \tag{2.21}$$

where

$$\dot{x}(t) = [A(t) - B(t)M(t)C(t)]x(t). \tag{2.22}$$

Differentiating equation 2.21

$$\dot{V}(x,t) = [x(t), \{[A(t)-B(t)M(t)C(t)]'Q(t)+\dot{Q}(t)+Q(t)[A(t)-B(t)M(t)C(t)]\}x(t)] \tag{2.23}$$

and writing equation 2.19 in the form

$$-\dot{Q}(t)-Q(t)B(t)M(t)C(t)-C'(t)M'(t)B'(t)Q(t) = Q(t)[A(t)-B(t)M(t)C(t)] + [A(t)-B(t)M(t)C(t)]'Q(t) + L'(t)L(t), \tag{2.24}$$

and by employing (2) it is clear that equation 2.23 becomes

$$\dot{V}(x,t) = [x(t), (-2C'(t)C(t)-L'(t)L(t))x(t)] \tag{2.25}$$

$$= -[L(t)\Psi(t,t_o)x(t_o), L(t)\Psi(t,t_o)x(t_o)] - 2[C(t)\Psi(t,t_o)x(t_o), C(t)\Psi(t,t_o)x(t_o)]$$

$$= -||L(t)\Psi(t,t_o)x(t_o)||^2 - 2||C(t)\Psi(t,t_o)x(t_o)||^2 \leq 0$$

where $\Psi(t,t_o)$ satisfies the equation

$$\dot{\Psi}(t,t_o) = [A(t) - B(t)M(t)C(t)]\Psi(t,t_o), \Psi(t_o,t_o) = I.$$

Therefore the closed-loop system is stable but not necessarily asymptotically stable. The following corollary gives a sufficient condition for the closed-loop system to be asymptotically stable.

---

[1] See Appendix B for a short review of Lyapunov functions.

Corollary 2.2 If either of the matrix pairs $\{A(t)-B(t)M(t)C(t), L(t)\}$ or $\{A(t)-B(t)M(t)C(t), C(t)\}$ are observable[1], and $||Q(t)||$ is finite for all $t \in (t_0, \infty)$, then the closed-loop system is asymptotically stable.

Another version of the stability criteria is given by the following theorem.

Theorem 2.6 If the kernel $C(t)\Phi(t,\tau)B(\tau)$ is L_2-positive real such that theorem 2.1 is satisfied and $Q(t) > 0$, $M(t) \geq 0$, then the closed-loop system illustrated in Figure 1 is stable.

Proof: Since theorem 2.1 is satisfied it follows that

1. $-\dot{Q}(t) = A'(t)Q(t) + Q(t)A(t) + L'(t)L(t) \quad Q(t) > 0$
2. $Q(t)B(t) = C'(t)$

The body of the proof is identical to the preceding theorem with the exception that

$$\dot{V}(x,t) = [x(t), (-2C'(t)M(t)C(t) - L'(t)L(t))x(t)] \tag{2.26}$$

$$= -||L(t)\Psi(t,t_0)x(t_0)||^2 - 2||C(t)M(t)^{\frac{1}{2}}\Psi(t,t_0)x(t_0)||^2 \leq 0,$$

where $M(t)^{\frac{1}{2}}$ denotes the unique positive square root[2] of M. For completeness we shall add the following corollary which is analogous to corollary 2.1.

---

[1] See Appendix B for the definition of observability.

[2] This terminology is discussed in Appendix A.

Corollary 2.3 If either of the matrix pairs $\{A(t)-B(t)M(t)C(t), L(t)\}$ or $\{A(t)-B(t)M(t)C(t), C(t)M^{\frac{1}{2}}(t)\}$ is observable and $||Q(t)||$ is finite for all $t \in (t_0, \infty)$, then the closed-loop system is asymptotically stable.

2.6 Summary

This chapter has served to define a generalization of the concept of positive realness to time-varying linear systems. A natural consequence of this generalization has led directly to, what network theorists call, passivity. That is, if the integral operator, W, defined in definition 2.2, is viewed as representing an n-port such that $W: v(t) \rightarrow i(t)$ where $\Omega = (-\infty, \infty)$, then it is easily seen that conditions 2 and 4 of this definition are equivalent to $\mathrm{Re} \int_{-\infty}^{\tau} v^*(t)i(t)dt \geq 0$ for all $\tau \in (-\infty, \infty)$. Hence, the material set forth in this chapter have their obvious implications in the area of linear passive network theory.

Starting with the definition of L_2-positive realness, necessary conditions and sufficient conditions were then formulated in the context of time-varying linear systems described by S1 and time-invariant systems described by S2 to illustrate the relationships between the matrices A(t), B(t), and C(t). Finally, an application of the concept of L_2-positive realness was given to illustrate its usefulness in the study of stability.

Chapter 3

SENSITIVITY REDUCTION IN LINEAR TIME-VARYING DYNAMICAL SYSTEMS

3.1 Introduction

One of the classical problems of control theory is to reduce, by means of feedback, the sensitivity of single variate systems to parameter variations of the plant and other disturbances. A lot of effort has been devoted to the pertinent aspects of this problem resulting in a considerable wealth of literature (see [9] and [22]). More recently efforts have been directed toward generalizing these concepts to include multivariate systems (references [11], [39], and [45] serve as a good introduction to this literature). This chapter and the following chapters establish the fundamental relationships between the plant matrices $\{A(t), B(t), C(t)\}$ and the compensators G and M to ensure sensitivity reduction of multivariate systems. This is clearly the first step which must be dealt with before an effective design procedure can be established.

3.2 Statement of the Problem

To begin the discussion of sensitivity reduction of linear systems, the problem is formulated in Banach spaces in order to emphasize the operator theoretic point of view. Attention is then restricted to Hilbert spaces, in order to obtain more detailed results including interpretations, from a physical point of view.

The problem of sensitivity reduction is based on the comparison of the perturbations of the outputs of two nominally equivalent systems under the influence of a disturbance. The system whose perturbation, in some sense, is the smallest, is said to be less sensitive. The disturbances may take the form of parameter variations or noise. For this analysis, attention is focused on the two nominally equivalent closed-loop and open-loop systems of Figure 1.3. The compensators G and M are to be chosen so that the closed-loop system is less sensitive, in some sense, than the open-loop system.

Let B_1 and B_2 denote Banach spaces. P is a linear plant with parameters $\{\alpha_1, \ldots, \alpha_n\}$ such that $P: B_1 \to B_2$. The compensators, G and M are such that $G: B_1 \to B_1$ and $M: B_2 \to B_1$. The symbols u and y denote the system input and output respectively while η denotes a system disturbance. The two systems are said to be nominally equivalent when the terminal mapping $u \to y$ is identical for both systems with no disturbances.

The equations describing the systems of Figure 1.3 are given by

$$y_c = P[Gu - My_c] \tag{3.1}$$

and

$$y_o = Pu. \tag{3.2}$$

It is easy to verify that for linear transformations such that (I + PM) is invertible, the above equations reduce to

$$y_c = [I + PM]^{-1}PGu \tag{3.3}$$

and

$$y_o = Pu, \tag{3.4}$$

which are defined to be the nominal system equations. To establish a terminal equivalence between them, it is necessary that

$$(I + PM)^{-1}PG = P$$

or equivalently

$$PG = P(I + MP).$$

Therefore, if G is chosen so that

$$G = I + MP, \tag{3.5}$$

the requirement is satisfied. Equation 3. 5 gives the explicit relationship for G in terms of the plant and the feedback compensator M, which is chosen to reduce the relative sensitivity of the closed-loop system.

If there are plant parameter perturbations, that is $\{\alpha_1, \alpha_2, \ldots\} \to \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\}$ such that the perturbations in P are bounded additive and linear, then

$$P \to \tilde{P}$$

where

$$\tilde{P} = P + \delta P.$$

The system equations are then given by

$$y_o + \delta y_o = (P + \delta P)u$$

and

$$y_c + \delta y_c = (P + \delta P)[(I + MP)u - M(y_c + \delta y_c)],$$

where δy represents the perturbation in the output. The perturbations in the open and closed-loop can then be shown to be related by the linear operator $\tilde{\$}$:

$$\delta y_c = \tilde{\$}\, \delta y_o \tag{3.6}$$

where

$$\tilde{\$} = [I + \tilde{P}M]^{-1}. \tag{3.7}$$

If the closed-loop system is to be less sensitive than the open-loop system with respect to the norm of the perturbations, then M must be chosen so that:

$$\|\delta y_c\| \leq \|\delta y_o\| \tag{3.8}$$

where the norm is taken on B_2. Hence, it is necessary and sufficient that $\tilde{\$}$ satisfy

$$\|\tilde{\$}\| \leq 1. \tag{3.9}$$

Clearly, M cannot be chosen a priori to reduce the sensitivity for all perturbations. However, it is possible to choose M to reduce the sensitivity for perturbations belonging to a specified class. This topic will be pursued in the sequel.

The aspect of sensitivity reduction which we shall discuss first, is that of reducing the perturbations of the output of the closed-loop system to additive noise η. The output of the closed-loop and open-loop systems due to the presence of noise, η, is given by

$$\tilde{y}_c = y_c + (I + PM)^{-1}\eta$$

and

$$\tilde{y}_o = y_o + \eta.$$

where $\tilde{y}$ represents the perturbed output. The perturbations in the closed-loop and open-loop systems are related by the linear operator $\$$.

$$\delta y_c = \$\, \delta y_o,$$

where $\delta y = \tilde{y} - y$, and evidently

$$\$ = (I + PM)^{-1}. \tag{3.10}$$

Therefore, if the closed-loop system is to be less sensitive than the open-loop system, $\$$ must also satisfy equation 3.9.

From equation 3.9, it is seen that the formulation of the sensitivity criterion in Banach spaces is not very rewarding, since it gives only a condition on the norm of the operator $\$$ and consequently doesn't lend itself to easy evaluation. If the problem is formulated on Hilbert spaces, additional structure on $\$$ can be obtained.

Since the perturbations are related by $\delta y_c = \$\delta y_o$, the norm $||\delta y_c||$ can be written as

$$||\delta y_c||^2 = <\delta y_c, \delta y_c> = <\$\, \delta y_o, \$\delta y_o> = <\delta y_o, \$^* \$\delta y_o>$$

$$= ||\delta y_o||^2 - <\delta y_o, (I - \$^*\$)\delta y_o>. \tag{3.11}$$

Hence, equation 3.8 is satisfied if

$$I - \$^*\$ \geq 0 \Longleftrightarrow ||\$|| \leq 1.$$

Therefore, in Hilbert spaces the necessary and sufficient condition for sensitivity reduction is

$$I - \$^*\$ \geq 0. \tag{3.12}$$

This equation, although abstract in nature, has proved very fruitful in understanding the nature of sensitivity reducing compensators. (See for example the papers of Porter [43] through [45].)

Throughout this discussion the sensitivity problem has been discussed in an abstract function space setting. A framework is now established which is helpful in bringing these abstract ideas to a concrete form. The analysis of linear time-varying systems will be the method by which this will be accomplished. Attention is focused on those transformations which act between finite cartesian products of $L_2(t_o, t_f)$ equipped with the usual inner product defined by

$$< x, y > = \int_\Omega [x(t), y(t)]dt, \quad \Omega = (t_o, t_f).$$

For time-varying systems, the closed-loop system is then said to be less sensitive than the open-loop system if

$$\int_{t_o}^{\tau} [\delta y_o(t), \delta y_o(t)]dt \geq \int_{t_o}^{\tau} [\delta y_c(t), \delta y_c(t)]dt, \text{ for all } \tau \in \Omega, \tag{3.13}$$

where it is tactitly agreed that δy_o and δy_c belong to complex E^m. Inequality 3.13, is a stronger condition than 3.12 since it not only implies that the norm of \$ must be less than unity but moreover, \$ must be causal on the interval Ω. Causality is easily established by selecting

$$\delta y_o(t) = 0 \text{ for } t < \tau \text{ and for all } \tau \in \Omega$$

consequently, from inequality 3.13 it must be the case that

$$\delta y_c(t) = 0 \text{ for } t < \tau \text{ and for all } \tau \in \Omega.$$

Therefore, for time-varying systems, the generalization of the necessary and sufficient condition for sensitivity improvement given by 3.12 can be formulated by the following

1. $\$: L_2^m(\Omega) \to L_2^m(\Omega), \quad \Omega = (t_o, t_f)$ (3.14)
2. $\$$ causal (3.15)
3. $I - \$^*\$ \geq 0.$ (3.16)

Now that the criteria for the sensitivity operator has been established, we shall investigate the implicit constraints that it imposes upon a plant which is described by S1, (where A(t), B(t), and C(t) are real continuous $n \times n$, $n \times p$, and $m \times n$ matrices respectively), and a feedback compensator, M(t), which is a real continuous $p \times m$ time-varying matrix.

3.3 Structure of the Sensitivity Operator

To understand the physical implications of conditions 3.14, 15, 16, it is necessary to identify the kernel of the operator \$. From equation 3.10, recall that

$$(\$\gamma)(t) = ([I + PM]^{-1}\gamma)(t), \tag{3.17}$$

and

$$(Pz)(t) = \int_{\Omega} C(t)\Phi(t,\tau)B(\tau)z(\tau)d\tau, \quad t \in \Omega \qquad (3.18)$$

To manipulate equation 3.17 to a more meaningful form, consider the following differential system

$$\dot{\alpha}(t) = A(t)\alpha(t) + B(t)M(t)\beta(t): \ \beta(t) = (\$\gamma)(t) \qquad (3.19)$$

$$\gamma(t) = C(t)\alpha(t) + \beta(t) \qquad (3.20)$$

The solution of this system is given by:

$$\gamma(t) = \int_{\Omega} \{I\,\delta(t-\tau) + C(t)\Phi(t,\tau)B(\tau)M(\tau)\}\beta(\tau)d\tau, \quad t \in \Omega,$$

which is an explicit form of $\$^{-1}$. The mapping \$, is identified as follows:

$$\beta(t) = \gamma(t) - C(t)\alpha(t) = \gamma(t) - \int_{\Omega} C(t)\Psi(t,\tau)B(\tau)M(\tau)\gamma(\tau)d\tau$$

$$= \int_{\Omega} \{I\,\delta(t-\tau) - C(t)\Psi(t,\tau)B(\tau)M(\tau)\}\gamma(\tau)\,d\tau, \qquad (3.21)$$

where $\Psi(t,\tau)$ satisfies:

$$\frac{d\Psi(t,\tau)}{dt} = [A(t) - B(t)M(t)C(t)]\Psi(t,\tau), \Psi(t_o, t_o) = I.$$

The mapping $\$: \gamma \rightarrow \beta$, given by equation 3.21, identifies the kernel of the sensitivity operator as:

$$\$(t,\tau) = I\,\delta(t-\tau) - C(t)\Psi(t,\tau)B(\tau)M(\tau). \qquad (3.22)$$

$$= I\,\delta(t-\tau) - h(t,\tau) \qquad (3.23)$$

where the m × m matrix kernel $h(t, \tau)$ is defined in the obvious way. It is therefore evident that the sensitivity operator is equivalent to

$$\$ = I - H, \tag{3.24}$$

where H is the integral operator associated with the kernel $h(t, \tau)$.

The following theorem is helpful for establishing the sensitivity conditions 3.14, 15, and 16, in terms of the integral operator H.

Theorem 3.1 A necessary and sufficient condition for sensitivity reduction is that the operator H be L_2 - positive real and $H + H^* - H^*H \geq 0$.

Proof: The proof is made obvious from the following four conditions and definition 2.2:

1. $\$: L_2^m(\Omega) \to L_2^m(\Omega) \Longleftrightarrow H: L_2^m(\Omega) \to L_2^m(\Omega), \; \Omega = (t_o, t_f)$
2. $\$$ causal $\Longleftrightarrow$ H causal
3. $I - \$^*\$ \geq 0 \Longleftrightarrow H + H^* - H^*H \geq 0$, hence $H + H^* \geq 0$.
4. Since A(t), B(t), C(t), and M(t) are real time varying matrices $h(t,\tau)$ is real.

In the theorems that follow, the concept of L_2 - positive realness is brought to bear on theorem 3.1 thus establishing the relationships between the parameters of the plant described by S1 and the feedback compensator M(t). For example, theorems 2.1 and 2.2 supply sufficient conditions and necessary conditions for L_2 - positive realness of H, by replacing the kernel $C(t)\Phi(t, \tau)B(\tau)$ with $C(t)\Psi(t, \tau)B(\tau)M(\tau)$.

The following three theorems help to identify the conditions under which sensitivity is improved in the context of S1.

Theorem 3.2 Let W be a integral operator bounded on $L_2^m(\Omega)$ whose kernel is given by $[C(t)-M'(t)B'(t)K(t)]\Psi(t,\tau)B(\tau)M(\tau)$ where K(t) is the unique self-adjoint solution of

$$-\dot{K}(t) = K(t)[A(t)-B(t)M(t)C(t)] + [A(t)-B(t)M(t)C(t)]'K(t)+C'(t)C(t), t \in \Omega,$$

and where $K(t_o)$ is given by

$$K(t_o) = \int_{t_o}^{t_f} \Psi'(\tau,t_o)C'(\tau)C(\tau)\Psi(\tau,t_o)d\tau.$$

If H is a bounded, causal and real operator defined on $L_2^m(\Omega)$, then the sensitivity is reduced if $W + W^*$ is positive.

Proof: The first step of the proof will be that of identifying the kernel of H^*H. Following the arguments of theorem 2.1, it is seen that

$$(H^*Hx)(t) = \int_{t_o}^{t}\int_{t}^{t_f} M'(t)B'(t)\Psi'(\tau,t)C'(\tau)C(\tau)\Psi(\tau,s)B(s)M(s)x(s)d\tau ds$$
$$+ \int_{t}^{t_f}\int_{s}^{t_f} M'(t)B'(t)\Psi'(\tau,t)C'(\tau)C(\tau)\Psi(\tau,s)B(s)M(s)x(s)d\tau ds.$$

Hence, substituting the differential equation for K into $(H^*Hx)(t)$

$$(H^*Hx)(t) = -\int_{t_o}^{t}\int_{t}^{t_f} M'(t)B'(t)\frac{d}{d\tau}\{\Psi'(\tau,t)K(\tau)\Psi(\tau,s)\}B(s)M(s)x(s)d\tau ds$$
$$-\int_{t}^{t_f}\int_{s}^{t_f} M'(t)B'(t)\frac{d}{d\tau}\{\Psi'(\tau,t)K(\tau)\Psi(\tau,s)\}B(s)M(s)x(s)d\tau ds.$$

Since Ψ is the transition matrix of a causal system, it follows that

$$(H^*Hx)(t) = \int_{t_o}^{t_f} M'(t)B'(t)[K(t)\Psi(t,s)+\Psi'(s,t)K(s)]B(s)M(s)x(s)ds$$
$$-\int_{t_o}^{t_f} M'(t)B'(t)\Psi'(t_f,t)K(t_f)\Psi(t_f,s)B(s)M(s)x(s)ds$$

Therefore, the kernel of H^*H is given by

$$M'(t)B'(t)[K(t)\Psi'(t,\tau) + \Psi'(\tau,t)K(\tau)]B(\tau)M(\tau)$$
$$- M'(t)B'(t)\Psi'(t_f,t)K(t_f)\Psi(t_f,\tau)B(\tau)M(\tau). \quad (3.25)$$

The kernel of $H + H^*$ is clearly given by

$$C(t)\Psi(t,\tau)B(\tau)M(\tau) + M'(t)B'(t)\Psi'(\tau,t)C(\tau), \quad (3.26)$$

and by subtracting equation 3.25 from 3.26 it follows that the kernel of $H + H^* - H^*H$ is

$$[C(t)-M'(t)B'(t)K(t)]\Psi(t,\tau)B(\tau)M(\tau)+M'(t)B'(t)\Psi'(\tau,t)[C'(\tau)-K(\tau)B(\tau)M(\tau)]$$
$$+ M'(t)B'(t)\Psi'(t_f,t)K(t_f)\Psi(t_f,\tau)B(\tau)M(\tau).$$

The theorem follows by recognizing that the first two terms of the preceding expression correspond to the kernel of $W + W^*$, and the last term corresponds to the kernel of a positive operator.

If Ω is the set (t_o, ∞), and $C\Psi$ is the impulse response matrix of an exponentially asymptotically stable system, then $\Psi'(\infty,t)K(\infty)\Psi(\infty,\tau)$ is the zero matrix and theorem 3.2 is necessary as well as sufficient.

Theorem 3.3 A necessary condition for $W + W^*$ to be positive is that

$$C(t)B(t)M(t) + M'(t)B'(t)C'(t) \geq 2M'(t)B'(t)K(t)B(t)M(t)$$

where K is defined in theorem 3.2.

The proof follows directly from theorem 2.2.

A sufficient condition for positivity of $W + W^*$ can be obtained via theorem 2.1, thus ensuring sensitivity reduction. This is established by the following

Theorem 3.4 A sufficient condition for positivity of $W + W^*$ is the existence of a continuous, positive, self-adjoint $n \times n$ matrix Q and a continuous $r \times n$ matrix L such that

1. $$-\dot{Q}(t) = Q(t)[A(t)-B(t)M(t)C(t)] + [A(t)-B(t)M(t)C(t)]'Q(t)+L'(t)L(t), \quad t \in \Omega$$
2. $$[Q(t)+K(t)]B(t)M(t) = C'(t)$$
3. $$(Fx)(t) \triangleq \int_{\Omega} L(t)\Psi(t,\tau)B(\tau)M(\tau)x(\tau)d\tau: \ \|F\| \text{ is finite,}$$

where $Q(t_0)$ is chosen to satisfy condition (2) at $t = t_0$.

The proof follows from theorem 3.2 and theorem 2.1.

Example 3.1 To illustrate the use of previous theorems we will consider a single-variate plant. In equations 2.6, 7, let x, y and u be scalar valued functions. The plant matrices A(t), B(t), C(t) and the feedback compensator M(t) are replaced by the scalar functions: -a(t), b, c and m respectively.

The plant equations reduce to

$$\dot{x}(t) + a(t)x(t) = bu(t), \quad x(t_o) = 0, \quad t \in (t_o, \infty)$$

$$y(t) = cx(t)$$

where it is assumed that a(t), b, c, and m are real and $\beta \geq a(t) \geq \delta \geq 0$ and $b, c, m > 0$. The sensitivity kernel is then given by

$$\$(t, \tau) = \delta(t-\tau) - c\Psi(t, \tau)bm$$

and Ψ is seen to be an L_1 kernel since

$$\Psi(t, \tau) = \exp\left\{-\int_{\tau}^{t} [a(s) + bmc]ds\right\} \quad t \geq \tau$$

$$\Psi(t, \tau) = 0 \qquad t < \tau.$$

The operator k(t) is the solution of the differential equation

$$-\dot{k}(t) = -2\,[a(t) + bmc]\,k(t) + c^2$$

which is given by

$$k(t) = \Psi^2(t_o, t) \int_t^{\infty} c^2 \Psi^2(\tau, t_o)d\tau,$$

and is positive and finite for all t and is bounded by

$$0 \leq k(t) \leq \frac{c^2}{2(\delta + bmc)} < \frac{c}{bm}.$$

Define the scalar q(t) by

$$q(t) = \frac{c}{bm} - k(t)$$

then substituting this expression into the differential equation for k(t) gives

$$q(t) = -2[a(t)+bmc]\left[\frac{c}{bm} - q(t)\right] + c^2,$$

hence,

$$-\dot{q}(t) = -2\,[a(t) + bmc]\,q(t) + c^2 + \frac{2a(t)c}{bm}$$

Consequently, if $\ell(t)$ is defined by

$$\ell(t) = [c^2 + \frac{2a(t)c}{bm}]^{\frac{1}{2}},$$

then $\ell(t)\,\Psi(t,\tau)bm$ is an L_1 kernel and theorem 3.4 is satisfied.

Throughout this section, structure of the sensitivity operator in the context of S1 has been identified and the necessary and sufficient conditions for sensitivity improvement have been interpreted in terms of the matrices A(t), B(t), C(t) and M(t). The relationship between positive realness and sensitivity reduction was also established. The following section serves to supplement the preceding conditions relating the structure of P and the compensator M to insure sensitivity improvement.

3.4 Sensitivity Analysis Via the Inverse Sensitivity Operator

The purpose of this section is to demonstrate that insight into sensitivity reduction can be obtained by employing the inverse of the sensitivity operator. As noted previously, the inverse operator for the system under discussion exists and is given by

$$\$^{-1} = I + PM, \tag{3.27}$$

where P is a real causal plant described by S1, and M is a continuous time-varying matrix. The necessary condition for sensitivity reduction given by inequality 3.16 can be reformulated via $\$^{-1}$ by the following:

$$0 \leq I - \$^*\$ = \$^* [(\$^*)^{-1}\$^{-1} - I]\,\$ \Longleftrightarrow (\$^*)^{-1}\$^{-1} - I \geq 0.$$

The necessary and sufficient conditions for sensitivity reduction can then be written in the form

1. $\$: L_2^m(\Omega) \to L_2^m(\Omega),\ \Omega = (t_o, t_f)$
2. $\$$ causal
3. $(\$^*)^{-1}\$^{-1} - I \geq 0.$ (3.28)

Since the implications of (1) and (2) have previously been examined, the pertinent question at this point is: "What is the sufficient structure of PM to insure that inequality 3.28 is satisfied?". The answer to this question is readily obtained by substituting equation 3.27 into 3.28 which results in

$$0 \leq (\$^*)^{-1}\$^{-1} - I \Longleftrightarrow PM + (PM)^* + (PM)^*PM \geq 0. \quad (3.29)$$

It is clear that if the operator $PM + (PM)^*$ is positive, then inequality 3.29 is satisfied. Translating this result into the context of the system described by S1, the following theorem is formally stated.

Theorem 3.5 A sufficient condition for $PM + (PM)^* \geq 0$ is the existence of a continuous positive self-adjoint $n \times n$ matrix Q a continuous $r \times n$ matrix L such that

1. $-\dot{Q}(t) = Q(t)A(t) + A'(t)Q(t) + L'(t)L(t)$
2. $Q(t)B(t)M(t) = C'(t)$
3. $(Fx)(t) \triangleq \int_\Omega L(t)\Phi(t,\tau)B(\tau)M(\tau)x(\tau)d\tau$: $\|F\|$ is finite,

where $Q(t_o)$ is chosen to satisfy condition (2) at $t = t_o$.
The proof follows that of theorem 2.1.

Remark 3.1 If the matrices B(t) and C(t) are equal to the identity and M(t) is invertible, then the above theorem for sensitivity reduction is equivalent to a condition established by Porter [44].

Theorem 3.5 is overly strong since the operator $(PM)^* PM$ was not considered to aid in the positiveness of 3.29. The following theorem provides a split of $(PM)^* PM$ and utilizes this to give a weaker sufficient condition for $(\$^*)^{-1}\$^{-1} - I \geq 0$.

Theorem 3.6 Let V be a integral operator bounded on $L_2^m(\Omega)$ whose kernel is given by $[C(t)+M'(t)B'(t)K(t)]\Phi(t,\tau)B(\tau)M(\tau)$ where K(t) is the unique self-adjoint solution of

$$-\dot{K}(t) = K(t)A(t) + A'(t)K(t) + C'(t)C(t), \quad t \in \Omega,$$

and where $K(t_o)$ is given by

$$K(t_o) = \int_{t_o}^{t_f} \Phi'(\tau, t_o)C'(\tau)C(\tau)\Phi(\tau, t_o)d\tau.$$

If PM is a bounded operator on $L_2^m(\Omega)$, and $V + V^* \geq 0$, then $PM + (PM)^* + (PM)^*PM \geq 0$.

The proof is straight forward and follows theorem 3.2.

It is clear that if Ω is the set (t_o, ∞) and if $C\Phi$ is the impulse response matrix of an exponentially asymptotically stable system, then $PM + (PM)^* + (PM)^*PM$ is positive if and only if $V + V^*$ is positive.

Moreover, it is also noted that by employing the inverse sensitivity operator, the sufficient conditions for sensitivity reduction impose a more severe restriction on the plant matrix A(t) then previously. This is easily seen by noting that the operators associated with the kernels $C(t)\Psi(t,\tau)B(\tau)M(\tau)$ and $C(t)\Phi(t,\tau)B(\tau)M(\tau)$ must both be bounded on $L_2^m(\Omega)$.

3.5 Sensitivity Reduction with Plant Perturbations

In the previous sections the structure of the plant and it's associated feedback compensator has been discussed from the point of view of reducing the sensitivity to additive noise η. In this section attention is directed to the question of sensitivity reduction when the parameters of the plant matrix A(t) are perturbed, such that

$$A(t) \rightarrow A(t) + \delta A(t) = \tilde{A}(t),$$

correspondingly,

$$H \rightarrow \tilde{H}.$$

For such disturbances, the necessary and sufficient conditions for sensitivity reduction are given by theorem 3.1 by replacing H with $\tilde{H}$.

The starting point for this discussion is the assumption that the closed-loop system does indeed have a reduced sensitivity to noise compared with the nominally equivalent open-loop system. It is also assumed that theorem 3.4 is satisfied. In view of these assumptions, the following theorem is presented.

Theorem 3.7 If $\tilde{H}$ is a real, causal, bounded integral operator on $L_2^m(\Omega)$ and if there exists positive, self-adjoint $n \times n$ matrices Q and K, a continuous $r \times n$ matrix L, such that

1. $$-\dot{Q}(t) = Q(t)[A(t) - B(t)M(t)C(t)] + [A(t)-B(t)M(t)C(t)]'Q(t) + L'(t)L(t)$$

2. $$-\dot{K}(t) = K(t)[A(t)-B(t)M(t)C(t)] + [A(t)-B(t)M(t)C(t)]'K(t) + C'(t)C(t)$$

3. $$[Q(t) + K(t)]B(t)M(t) = C'(t)$$

4. $(\tilde{F}x)(t) \triangleq \int_\Omega L(t)\tilde{\Psi}(t,\tau)B(\tau)M(\tau)x(\tau)d\tau$: $||\tilde{F}||$ is finite

where $Q(t_o)$ is chosen to satisfy condition (3) at $t = t_o$ and where $K(t_o)$ is given in theorem 3.2, then a sufficient condition for sensitivity reduction is

$$[Q(t) + K(t)]\delta A(t) + \delta A'(t)[Q(t) + K(t)] \leq 0.$$

Proof: Following the proof of theorem 2.1, we shall make use of the integral operator $\tilde{F}$:

$$(\tilde{F}x)(t) = \int_{t_o}^{t} L(t)\tilde{\Psi}(t,\tau)B(\tau)M(\tau)x(\tau)d\tau \quad t \in \Omega$$

and its adjoint, $\tilde{F}^*$, given by

$$(\tilde{F}^* z)(t) = \int_{t}^{t_f} M'(t)B'(t)\tilde{\Psi}'(\tau,t)L(\tau)z(\tau)d\tau, \quad t \in \Omega.$$

We shall also make use of the identity.

$$\frac{d}{d\tau}\{\tilde{\Psi}'(\tau,t)Q(\tau)\tilde{\Psi}(\tau,s)\} = \tilde{\Psi}'(\tau,t)\{\delta A'(\tau)Q(\tau) + Q(\tau)\delta A(\tau) - L'(\tau)L(\tau)\}\tilde{\Psi}(\tau,s).$$

Hence $\tilde{F}^*\tilde{F}$ can be written as

$$(\tilde{F}^*\tilde{F}x)(t) = -\int_{t_o}^{t}\int_{t}^{t_f} M'(t)B'(t)\frac{d}{d\tau}\{\tilde{\Psi}'(\tau,t)Q(\tau)\tilde{\Psi}(\tau,s)\}B(s)M(s)x(s)d\tau ds$$

$$-\int_{t}^{t_f}\int_{s}^{t_f} M'(t)B'(t)\frac{d}{d\tau}\{\tilde{\Psi}'(\tau,t)Q(\tau)\tilde{\Psi}(\tau,s)\}B(s)M(s)x(s)d\tau ds$$

$$+\int_{t}^{t_f}\int_{t_o}^{\tau}\tilde{\Psi}'(\tau,t)[\delta A'(\tau)Q(\tau)+Q(\tau)\delta A(\tau)]\tilde{\Psi}(\tau,s)x(s)ds\, d\tau.$$

Since $\tilde{\Psi}$ is the transition matrix of a causal system, it is clear that

$$(\tilde{F}^*\tilde{F}x)(t) = \int_{t_o}^{t_f} M'(t)B'(t)[Q(t)\tilde{\Psi}(t,s)+\tilde{\Psi}'(s,t)Q(s)]B(s)M(s)x(s)ds$$

$$-\int_{t_o}^{t_f} M'(t)B'(t)\tilde{\Psi}'(t_f,t)Q(t_f)\tilde{\Psi}(t_f,s)B(s)M(s)x(s)ds$$

$$+\int_{t}^{t_f}\int_{t_o}^{\tau}\tilde{\Psi}'(\tau,t)[\delta A'(\tau)Q(\tau)+Q(\tau)\delta A(\tau)]\tilde{\Psi}(\tau,s)x(s)ds\, d\tau.$$

Leaving this expression for the moment, and computing $\tilde{H}+\tilde{H}^*-\tilde{H}^*\tilde{H}$, it follows that

$$([\tilde{H}+\tilde{H}^*-\tilde{H}^*\tilde{H}]x)(t) = \int_{t_o}^{t_f}[C(t)-M'(t)B'(t)K(t)]\tilde{\Psi}(t,s)B(s)M(s)x(s)ds$$

$$+\int_{t_o}^{t_f} M'(t)B'(t)\tilde{\Psi}'(s,t)[C'(s)-K(s)B(s)M(s)]x(s)ds$$

$$+\int_{t_o}^{t_f} M'(t)B'(t)\tilde{\Psi}'(t_f,t)K(t_f)\tilde{\Psi}(t_f,s)B(s)M(s)x(s)ds$$

$$-\int_{t}^{t_f}\int_{t_o}^{\tau}\tilde{\Psi}'(\tau,t)[\delta A'(\tau)K(\tau)+K(\tau)\delta A(\tau)]\tilde{\Psi}(\tau,s)x(s)ds\, d\tau.$$

Using condition (3), it is seen that

$$([\tilde{H} + \tilde{H}^* - \tilde{H}^*\tilde{H}]x)(t) = (\tilde{F}^*\tilde{F}x)(t)$$
$$+ \int_{t_o}^{t_f} M'(t)B'(t)\tilde{\Psi}'(t_f, t)[Q(t_f)+K(t_f)]\tilde{\Psi}(t_f, s)B(s)M(s)x(s)ds$$
$$- \int_{t}^{t_f}\int_{t_o}^{\tau} \tilde{\Psi}'(\tau, t)\{[Q(\tau)+K(\tau)]\delta A(\tau)+\delta A'(\tau)[Q(\tau)+K(\tau)]\}\tilde{\Psi}(\tau, s)x(s)\,ds.$$

The theorem follows since $\tilde{H}+\tilde{H}^* - \tilde{H}^*\tilde{H}$ is positive.

It is remarked that in this last theorem, the assumption was made that $||\$|| \leq 1$ for all $\delta y_o \in L_2^m(\Omega)$. This assumption is not necessary for sensitivity reduction when the plant is perturbed but it is necessary that $||\tilde{\$}|| \leq 1$ when its domain is restricted to the range of δP. For elements belonging to the complement of the range of δP, $||\tilde{\$}||$ may be greater than one.

3.6 Summary

In this chapter, the basic problem of sensitivity reduction was formulated for multivariate systems in terms of the sensitivity operator \$. The structure of its kernel was given for the class of systems described by S1. The concept of L_2-positive realness then was brought to bear on the problem to establish fundamental relationships between the plant matrices $\{A(t), B(t), C(t)\}$ and the compensators G and M to ensure sensitivity improvement for disturbances in the form of noise and plant parameter perturbations.

Chapter 4

SENSITIVITY REDUCTION IN LINEAR STATIONARY SYSTEMS

4.1 Introduction

The last chapter discussed the problem of sensitivity reduction in the time domain and established the relationships between the time-varying plant matrices $\{A(t), B(t), C(t)\}$ and the feedback compensators G and M. Our present objective is to examine these previous results with the additional assumption of stationarity, and employ frequency domain techniques[1] as an added tool with which to work. It will become evident that many of the previous theorems, which proved to be only sufficient in the time domain, are necessary as well in the frequency domain. This is to be expected since as the definition of the system is sharpened, it is possible to obtain more detailed results about its structure. In short, the relationships between the constant plant matrices $\{A, B, C\}$ and the compensators G and M which ensure sensitivity reduction are established.

4.2 Conditions for Sensitivity Reduction in Stationary Systems

To begin the analysis of sensitivity reduction in stationary systems, it will first be helpful to translate the necessary and sufficient conditions for sensitivity improvement given by 3.14, 15, 16, to the complex-frequency domain when Ω is given by $[-\infty, \infty]$. The following lemma formalizes these conditions.

[1] See Appendix C.

Lemma 4.1 Let $\$(t)$ be the kernel of a stationary sensitivity operator, satisfying 3.14, 15, 16, then its Laplace transform, $\$(s)$, is such that for Re $s > 0$

1. $\$(s)$ is analytic
2. $I - \$^*(s)\$(s) \geq 0$.

Proof:

1. Employing the conditions given by statements 3.14 and 3.15 and using theorem C.5 of Appendix C, it follows that S(s) exists and is analytic for Re $s > 0$.
2. Inequality 3.16 implies condition (2) of this lemma by the following arguments:

$$I - \$^*\$ \geq 0 \Longleftrightarrow \langle x, (I - \$^*\$)x \rangle \geq 0 \text{ for all } x \in L_2^m(-\infty, \infty) \tag{4.1}$$

where

$$(\$x)(t) = \int_{-\infty}^{t} \$(t,\tau)x(\tau)d\tau = \int_{0}^{\infty} \$(\tau)x(t-\tau)d\tau.$$

Let

$$x(t) = e^{st}\hat{x}, \quad s = \sigma + j\omega, \ \sigma > 0, \ t \leq t_f$$

$$x(t) = 0, \quad t > t_f$$

where $\hat{x} \in E^m$ hence $x \in L_2^m(-\infty, \infty)$. Therefore,

$$(\$x)(t) = \int_{0}^{\infty} e^{s(t-\tau)}\$(\tau)\hat{x}\,d\tau$$
$$= \$(s)\,e^{st}\hat{x}.$$

Inequality 4.1 can now be written as

$$
\begin{aligned}
<x, (I - \$^*\$)x> &= < e^{st}\hat{x}, e^{st}\hat{x}> - < e^{st}\$(s)\hat{x}, e^{st}\$(s)\hat{x} > \\
&= <\hat{x}, e^{2\sigma t}\hat{x}> - <\hat{x}, e^{2\sigma t}\$^*(s)\$(s)\hat{x}> \quad \sigma > 0 \\
&= \int_{-\infty}^{t_f} e^{2\sigma t}[\hat{x}, (I - \$^*(s)\$(s))\hat{x}]\, dt \\
&= [\hat{x}, (I - \$^*(s)\$(s))\hat{x}] \int_{-\infty}^{t_f} e^{2\sigma t} dt, \quad \sigma > 0.
\end{aligned}
$$

Since $\int_{-\infty}^{t_f} e^{2\sigma t} dt$ is positive, the lemma follows.

The converse of this lemma is easily established from theorem C.6 if $\$(s)$ is absolutely convergent for Re $s > 0$ and $\$(j\omega)$ is essentially bounded for $\omega \epsilon (-\infty, \infty)$. It is pointed out that Newcomb, concerned with stationary sensitivity operators mapping between real Hilbert spaces, stated that conditions (1) and (2) of lemma 4.1 as well as the condition $\overline{\$}(s) = \$(\overline{s})$ (for Re $s > 0$) were equivalent to conditions 3.14, 15, and 16. These results were made possible by the additional structure employed in the use of distribution theory. Techniques in this direction are not within the intended scope of this thesis.

It is convenient to translate conditions 3.14, 15, 16 to an equivalent form along the $j\omega$ axis.

Lemma 4.2 Let s(t) be the kernel of a stationary sensitivity operator satisfying 3.14, 15, 16 such that $\$(s)$ is absolutely convergent for Re $s > 0$ and is described by a rational matrix then, the sensitivity is reduced if and only if

1. $\$(s)$ analytic for Re $s \geq 0$
2. $I - \$^*(j\omega)\$(j\omega) \geq 0$ for all ω

The proof of this lemma is well known and can be found in most network synthesis textbooks (for example Newcomb [36]).

Having now established the criteria that the stationary sensitivity operator must satisfy, the constraints that it imposes upon plants described by S2 and feedback compensators described by stationary matrices will be investigated. It will be noted that as we proceed through the development of sensitivity reduction for stationary systems, we shall heavily rely upon the results developed in chapter 3.

4.3 Structure of the Stationary Sensitivity Operator

For stationary systems, the sensitivity kernel given by equation 3.22 becomes

$$\$(t-\tau) = I\,\delta\,(t-\tau) - C\Psi(t-\tau)BM,$$

where $\Psi\,(t-\tau)$ satisfies

$$\frac{d\Psi(t-\tau)}{dt} = [A - BMC]\Psi(t-\tau),\ \Psi(t_0, t_0) = I. \quad (4.2)$$

Since the sensitivity kernel is time invariant, its Laplace transform is given by

$$\$(s) = I - C\,[sI - (A - BMC)]^{-1}BM \triangleq I - C\Psi(s)BM. \qquad (4.3)$$

It follows that $\$(s)$ is analytic and converges absolutely for Re $s \geq 0$ if and only if the irreducible realization[1] of

$$C[sI - (A - BMC)]^{-1}BM$$

is asymptotically stable, or if

$$\text{Re}\,\{\sigma(A - BMC)\} \leq \epsilon < 0,$$

[1] For a definition of irreducible realization, see Appendix B.

hence closed-loop stability.

As demonstrated in the preceding chapter, condition (2) of lemma 4.1 is equivalent to

$$0 \leq I - \$^*(j\omega)\$(j\omega) \Longleftrightarrow H(j\omega)+H^*(j\omega)-H^*(j\omega)H(j\omega) \geq 0 \quad (4.4)$$

where $H(j\omega)$ is identified by

$$H(j\omega) = C[j\omega I - (A - BMC)]^{-1}BM,$$

and is L_2-positive real.[1] The following theorem consolidates these results.

<u>Theorem 4.1</u> A necessary and sufficient condition for sensitivity reduction is that

1. The irreducible realization of $H(s)$ is asymptotically stable
2. $H(j\omega) + H^*(j\omega) - H^*(j\omega)H(j\omega) \geq 0 \quad \omega \in (-\infty, \infty)$

The proof follows directly from the preceding discussion, theorem 3.1 and lemma 4.2.

Hence, as expected from the preceding chapter, positive realness plays a dominant role in the basic structure of sensitivity reduction. Because of this, we are able to obtain additional structure on the matrices B, C, and M via theorem 2.3.

<u>Theorem 4.2</u> A necessary condition for sensitivity reduction is

$$CBM = M'B'C' \geq 0$$

---

[1] Since $H(s)$ is analytic in Re $s \geq 0$, L_2-positive realness is equivalent to positive realness.

Remark 4.1 Since it is necessary that H(s) be positive real, theorem 4.2 can be supplemented by utilizing the concepts developed by Falb and Wolovich [13]. It is demonstrated in section B.6 of Appendix B that a necessary condition for H(s) to be positive real is that $C_iBM \neq 0$ for $i = 1,2,\ldots,m$, where C_i denotes the ith row of the matrix C.

The following theorem serves to further clarify theorem 4.1 in the context of S2.

Theorem 4.3 If Re $\{\sigma(A - BMC)\} \leq \epsilon < 0$ then the necessary and sufficient condition for sensitivity reduction is that W(s) be positive real where

$$W(s) = (C - M'B'K)\Psi(s)BM, \tag{4.5}$$

and where K is the unique positive self-adjoint solution of

$$K(A - BMC) + (A - BMC)'K = -C'C$$

Proof: The proof follows from the definition of positive realness and by equation 4.4:

$$0 \leq C\Psi(j\omega)BM + M'B'\Psi^*(j\omega)C' - M'B'\Psi^*(j\omega)C'C\Psi(j\omega)BM =$$

$$C\Psi(j\omega)BM + M'B'\Psi^*(j\omega)C' - M'B'\Psi^*(j\omega)K[j\omega I - A + BMC]\Psi(j\omega)BM$$
$$- M'B'\Psi^*(j\omega)[j\omega I - A + BMC]^*\Psi(j\omega)BM =$$

$$C\Psi(j\omega)BM + M'B'\Psi^*(j\omega)C' - M'B'\Psi^*(j\omega)KBM - M'B'K\Psi(j\omega)BM =$$

$$(C - M'B'K')\Psi(j\omega)BM + [(C - M'B'K)\Psi(j\omega)BM]^* \geq 0.$$

The following two theorems give necessary conditions and sufficient conditions for $(C - M'B'K)\Psi(s)BM$ to be positive real.

Theorem 4.4 If $\text{Re}\{\sigma(A - BMC)\} \leq \epsilon < 0$ then a sufficient condition for $(C - M'B'K)\Psi(s)BM$ to be positive real is the existence of a positive self-adjoint matrix Q and a matrix L such that

1. $Q(A - BMC) + (A - BMC)'Q = -L'L$
2. $(Q + K)BM = C'$

The proof follows from theorem 3.4.

Theorem 4.5 Necessary conditions for $(C - M'B'K)\Psi(s)BM$ to be positive real are

1. $CBM = M'B'C'$
2. $CBM \geq M'B'KBM \geq 0.$

This sharpens the results of theorem 4.2 and is proved by application of theorem 2.3.

Example 4.1 To illustrate the previous theorems consider the plant described by

$$\dot{x} = \begin{bmatrix} .52 & 1.89 \\ -.74 & -1.94 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x$$

with the feedback matrix

$$M = \begin{bmatrix} .89 & .26 \\ -1.51 & .26 \end{bmatrix}.$$

A necessary condition for sensitivity reduction given by theorem 4.2, is satisfied since

$$CBM = M'B'C' = \frac{6}{115} \begin{bmatrix} 17 & 5 \\ 5 & 15 \end{bmatrix} \geq 0.$$

It is also seen that the eigenvalues of A - BMC are given by $\lambda_1 = -1$, $\lambda_2 = -2$, hence the closed-loop system is asymptotically stable and $(s) is analytic for Re $s \geq 0$. For sensitivity reduction, it is then necessary and sufficient that

$$W(s) = (C - M'B'K)[sI - (A - BMC)]^{-1}BM$$

be positive real where K is the unique positive, self-adjoint solution of

$$K(A - BMC) + (A - BMC)'K = -C'C.$$

Solving the preceding equation, K is given by

$$K = \begin{bmatrix} 7/12 & 1/2 \\ 1/2 & 2/3 \end{bmatrix}$$

hence W(s) can be written as

$$W(s) = \frac{3}{(115)^2(s+1)(s+2)} \begin{bmatrix} -18 & 166 \\ 130 & 130 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \begin{bmatrix} -12 & 10 \\ 17 & 5 \end{bmatrix}.$$

Since

$$W(j\omega)+W^*(j\omega) = \frac{(6/115)^2}{(2-\omega^2)^2+9\omega^2}\begin{bmatrix} 436+1171\omega^2 & -180+415\omega^2+920j\omega \\ -180+415\omega^2-920j\omega & 1300+325\omega^2 \end{bmatrix}$$

is a positive matrix for all ω, the sensitivity is reduced. This fact could also have been established from theorem 4. 4 by establishing the existence of positive self-adjoint matrix Q and a matrix L, namely;

$$Q = \frac{1}{6}\begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

such that theorem 4. 4 is satisfied.

By using the concept of irreducibility introduced in Appendix B, it is possible to obtain additional structure on the matrices B, C, and M.

<u>Theorem 4. 6</u> If $\text{Re}\{\sigma(A-BMC)\} \leq \epsilon < 0$ and $C\Psi(s)BM$ is irreducible then a necessary condition for sensitivity reduction is the existence of a positive definite self adjoint matrix Q and a matrix L such that

1. $Q(A-BMC) + (A-BMC)'Q = -L'L$
2. $QBM = C'$ which implies Rank (BM) = Rank (C)

The proof follows theorem 4. 1 and lemma B.5.

Theorem 4.7 If Re $\{\sigma(A-BMC)\} \leq \epsilon < 0$ and $(C - M'B'K)\Psi(s)BM$ is irreducable then a necessary and sufficient condition for sensitivity reduction is the existence of a positive definite self-adjoint matrix $\overline{Q}$ and a matrix $\overline{L}$ such that

1. $\overline{Q}(A - BMC) + (A - BMC)'\overline{Q} = \overline{L}'\overline{L}$
2. $(\overline{Q} + K)BM = C'$.

The proof follows from theorem 4.3 and lemma B.5.

Throughout this section we have developed the necessary and sufficient conditions, in the complex frequency domain, that the sensitivity operator must satisfy to insure sensitivity reduction. These conditions illustrate the strong ties that exist between the relatively old concept of positive realness and the relatively new concept of sensitivity reduction. Also exhibited were the necessary conditions relating the matrices B, C, and M, to ensure sensitivity improvement. The following section will employ the inverse of the sensitivity operator to supplement the conditions already developed.

4.4 Sensitivity Analysis Via the Inverse of the Stationary Sensitivity Operator

To begin this discussion we will borrow freely from lemma 4.2 and inequality 3.28 to formulate the sensitivity conditions as

Lemma 4.3 Let $\$(s)$ be described by equation 4.3, then the necessary and sufficient conditions for sensitivity reduction are

1. $\$(s)$ analytic in Re $s \geq 0$

2. $\$^*(j\omega)^{-1}\$(j\omega)^{-1} - I \geq 0$ for all $\omega \epsilon (-\infty, \infty)$. (4.6)

Since $\$^{-1}(j\omega) = I + P(j\omega)M$, condition (2) is equivalent to

$$0 \leq \$^*(j\omega)^{-1}\$^{-1}(j\omega) - I \Longleftrightarrow P(j\omega)M + [P(j\omega)M]^* + [P(j\omega)M]^* P(j\omega)M \geq 0. \tag{4.7}$$

To gain understanding of the physical implications of inequality 4.7, consider a single-variate time invariant system. Let P and M be represented by

$$P(s) = k\pi(s)/\psi(s), \quad M = m,$$

where $\pi(s)$ and $\psi(s)$ denote two polynominals in the complex variable s with real coefficients. The constants k and m are given by positive scalars. Inequality 4.7 then takes the form

$$0 \leq P(j\omega)m + P^*(j\omega)m + mP^*(j\omega)P(j\omega)m = \tag{4.8}$$

$$mk\left[\frac{\pi}{\psi}(j\omega) + \frac{\pi^*}{\psi^*}(j\omega)\right] + m^2k^2\left|\frac{\pi}{\psi}(j\omega)\right|^2 \geq 0. \tag{4.9}$$

By denoting

$$\left(\frac{\pi}{\psi}\right)(j\omega) + \left(\frac{\pi^*}{\psi^*}\right)(j\omega) = 2\,\mathrm{Re}\left[\left(\frac{\pi}{\psi}\right)(j\omega)\right],$$

a sufficient condition for $I - \$^*(j\omega)\$(j\omega)$ to be positive is for $2mk\,\mathrm{Re}\left[\left(\frac{\pi}{\psi}\right)(j\omega)\right] \geq 0$. This condition can be substantially relaxed by rearranging inequality 4.8 to the following

$$P(j\omega)m + mP^*(j\omega) + mP^*(j\omega)P(j\omega)m =$$

$$\frac{\pi^*}{\psi^*}(j\omega)\left\{mk\left[\left(\frac{\psi}{\pi}\right)(j\omega) + \frac{\psi^*}{\pi^*}(j\omega)\right] + m^2k^2\right\}\frac{\pi}{\psi}(j\omega).$$

It follows that

$$0 \leq \{mP^*+Pm+mP^*Pm\}(j\omega) \Longleftrightarrow \{mk(\frac{\psi}{\pi}+\frac{\psi^*}{\pi^*})+m^2k^2\}(j\omega) \geq 0.$$

Hence the necessary and sufficient conditions for $\$^*(j\omega)^{-1}\$(j\omega)^{-1}-I \geq 0$ for all ω is equivalent to

$$0 \leq \{mk(\frac{\psi}{\pi}+\frac{\psi^*}{\pi^*})+ m^2k^2\}(j\omega) =$$

$$2\,mk\,\mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)] + m^2k^2 \geq 0 \text{ for all } \omega \in (-\infty, \infty). \quad (4.10)$$

It can be seen from 4.10 that a positive scalar m can be found to satisfy this inequality provided that $-\infty < \inf_{\omega} \mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)]$. This is formalized by the following theorem.

<u>Theorem 4.8</u> If P and M are represented by $k\pi(s)/\psi(s)$ and m respectively where $\pi(s)$ and $\psi(s)$ denote two polynomials in the complex s with real coefficients, and if $-\infty < \inf_{\omega} \mathrm{Re}[(\frac{\psi}{\pi})(j\omega)]$ then there exists a positive scalar m such that $I-S^*(j\omega)S(j\omega) \geq 0$.

Proof: Writing 4.10 in the form

$$0 \leq \{mk(\frac{\psi}{\pi}+\frac{\psi^*}{\pi^*})+m^2k^2\}(j\omega) \Longleftrightarrow \begin{cases} mk \geq 0 \text{ and } \mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)] \geq \frac{-mk}{2} \\ mk \leq 0 \text{ and } \mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)] \leq \frac{-mk}{2} \end{cases} \quad (4.11)$$

and since the two conditions on the right side of 4.11 differ only by a change in equality sign, we will without loss of generality, consider only the case $mk \geq 0$. Let $\alpha = \inf_{\omega} \mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)]$, then for $m \in (\max\,[0, \frac{-2\alpha}{k}], \infty)$ the conditions: $mk \geq 0$ and $\mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)] \geq \frac{-mk}{2}$ hold.

Note, that the difference between the orders of the polynominals π and ψ play an important role in the existence of an m to satisfy theorem 4.8. This point is exemplified by the following:

Example 4.2 Consider a second order plant represented by

$$P(s) = \frac{k}{s^2 + a s + b} = k(\pi/\psi)(s); \; k, a, b > 0.$$

Equation 4.10 becomes

$$2mk(b - \omega^2) + m^2 k^2 \geq 0,$$

which can't be satisfied for any finite m as $\omega \to \infty$. From this example, it is clear that not all types of plants are suitable for satisfying the condition $\$^*(j\omega)^{-1}\$(j\omega)^{-1} - I \geq 0$. This point is well borne out in the multidimensional case, as we shall demonstrate in the following theorem.

Theorem 4.9 If $\$(s)$ is analytic in Re $s \geq 0$, and $\sigma(A) \leq \epsilon < 0$, then the necessary and sufficient condition for sensitivity reduction is that the matrix V(s) be positive real where

$$V(s) = (C + M'B'K)\Phi(s)BM$$

and where K is the positive self adjoint solution of

$$KA + A'K = -C'C.$$

Proof: The following chain of conditions follow from inequality 4.7:

$$C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'+M'B'\Phi^*(j\omega)C'C\Phi(j\omega)BM =$$

$$C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'+M'B'\Phi^*(j\omega)[K(j\omega I-A)+(j\omega I-A)^*K]\Phi(j\omega)BM =$$

$$C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'+M'B'\Phi^*(j\omega)KBM+M'B'K\Phi(j\omega)BM =$$

$$[(C+M'B'K)\Phi(j\omega)BM] + [(C+M'B'K)\Phi(j\omega)BM]^* \geq 0,$$

and the theorem follows from the definition of positive realness.

From the above theorem it is seen that if it is necessary to ascertain if the closed-loop system is less sensitive than the nominally equivalent open-loop system, (given that both are asymptotically stable), all that has to be done is to verify the positive realness of $(C + M'B'K)\Phi(s)BM$.

<u>Example 4.3</u> This example serves to illustrate theorem 4.9. Consider the multivariate plant given by

$$\dot{x} = \begin{bmatrix} -2/3 & 0 \\ 0 & -2/3 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x$$

with the feedback matrix

$$M = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} .$$

A necessary condition for sensitivity reduction is

$$CBM = M'B'C' \geq 0$$

which is satisfied since

$$CBM = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0.$$

To establish sensitivity reduction, it is seen that $\{\sigma(A\text{-}BMC)\} = \{-1\} < 0$ hence it is necessary and sufficient to establish the positive realness of

$$V(s) = (C + M'B'K)\Phi(s)BM.$$

Performing the indicated matrix multiplications it is seen that

$$V(j\omega) + V^*(j\omega) = \frac{5/12}{\omega^2 + 4/9} I \geq 0 \text{ for all } \omega \in (-\infty, \infty)$$

hence, sensitivity reduction. This result is expected from corollary 2.2, since there exists a positive self adjoint matrix Q and a matrix L, namely;

$$Q = \begin{bmatrix} 3.75 & 3.75 \\ 3.75 & 7.5 \end{bmatrix}, \text{ and } \quad L = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

such that

$$QA + A'Q = -L'L$$

and

$$(Q - K)BM = C'.$$

4.5 Sensitivity Reduction with Plant Perturbations

The previous sections have been discussing the structure of the plant and the control law to insure that the sensitivity of the closed-loop system to additive noise, η, is less than the nominally equivalent open-loop system. In this section attention is directed to the question of sensitivity reduction when the parameters of the plant matrix A are perturbed.

As a logical starting point for this discussion the assumption is made that the closed-loop system does indeed have a reduced sensitivity to noise η compared with the open-loop system. Hence, it is assumed that the operator H(s) is positive real, and, without loss of generality, it is also assumed that it is irreducible.

For notational convenience the perturbation in A is defined to be the mapping

$$A \to A + \delta A = \tilde{A}$$

consequently,

$$H \to \tilde{H}, \quad W \to \tilde{W}.$$

The necessary and sufficient conditions for sensitivity reduction with plant parameter variations are interpreted from lemma 4.2 and are formalized by

Lemma 4.4 The necessary and sufficient conditions for sensitivity reduction with perturbation in the matrix A are given by

1. $\tilde{\$}(s)$ analytic in Re $s \geq 0$ (4.12)
2. $I - \tilde{\$}^*(j\omega)\tilde{\$}(j\omega) \geq 0$ for all $\omega \epsilon (-\infty, \infty)$ (4.13)

where

$$\tilde{\$}(j\omega) = I - C\,[\,j\omega I - (\tilde{A} - BMC)]^{-1}BM.$$

Because of the perturbations, theorem 4.1 is written in the following form

<u>Lemma 4.5</u> A necessary and sufficient condition for sensitivity reduction with plant perturbations is

1. $\text{Re}\{\sigma(\tilde{A} - BMC)\} \leq \epsilon < 0$
2. $\tilde{H}(j\omega) + \tilde{H}^*(j\omega) - \tilde{H}^*(j\omega)\tilde{H}(j\omega) \geq 0$ for all $\omega \in (-\infty, \infty)$. (4.14)

Employing lemmas B.5, 4.4 and 4.5, sufficient conditions can be obtained on the perturbations, δA, to ensure sensitivity reduction.

<u>Theorem 4.10</u> If $\text{Re}\{\sigma(\tilde{A} - BMC)\} \leq \epsilon < 0$ and if $-\bar{Q}\delta A - \delta A'\bar{Q} + \bar{L}'\bar{L} - C'C \geq 0$, where $\bar{Q}$ and $\bar{L}$ are given by lemma B.5, then the sensitivity of the closed-loop system is reduced.

Proof: Since $H(s)$ is positive real, then by lemma B.5 there exists a positive definite self adjoint matrix Q and a matrix L such that

1. $\bar{Q}(A - BMC) + (A - BMC)'\bar{Q} = -\bar{L}'\bar{L}$. (4.15)
2. $\bar{Q}BM = C'$. (4.16)

Writing inequality 4.14 in the form

$$C\tilde{\Psi}(j\omega)BM + M'B'\tilde{\Psi}^*(j\omega)C' - M'B'\tilde{\Psi}^*(j\omega)C'C\tilde{\Psi}(j\omega)BM \geq 0$$

and employing equation 4.16, we obtain

$$M'B'\bar{Q}\tilde{\Psi}(j\omega)BM + M'B'\tilde{\Psi}^*(j\omega)\bar{Q}BM - M'B'\tilde{\Psi}^*(j\omega)C'C\tilde{\Psi}(j\omega)BM =$$

$$M'B'\tilde{\Psi}^*(j\omega)[j\omega I - (A - BMC + \delta A)]^*\bar{Q}\tilde{\Psi}(j\omega)BM$$

$$+ M'B'\tilde{\Psi}^*(j\omega)\{\bar{Q}[j\omega I - (A - BMC + \delta A)] - C'C\}\tilde{\Psi}(j\omega)BM =$$

$$M'B'\tilde{\Psi}^*(j\omega)\{-(A - BMC)'\bar{Q} - \bar{Q}(A - BMC) - C'C - \delta A'\bar{Q} - \bar{Q}\delta A\}\tilde{\Psi}(j\omega)BM =$$

$$M'B'\tilde{\Psi}^*(j\omega)\{\bar{L}'\bar{L} - C'C - \delta A'\bar{Q} - \bar{Q}\delta A\}\tilde{\Psi}(j\omega)BM \geq 0.$$

Hence, the theorem is proved.

Example 4.4 The purpose of this example is to illustrate theorem 4.10. We will again consider the system given by example 4.1 which is described by

$$\dot{x} = \begin{bmatrix} .52 & 1.89 \\ -.74 & -1.94 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x$$

with the feedback matrix

$$M = \begin{bmatrix} .89 & .26 \\ -1.51 & .26 \end{bmatrix} .$$

The matrices $\bar{Q}$ and $\bar{L}$, computed from equations 4.15, 16, are:

$$\bar{Q} = \begin{bmatrix} 1.41 & 1 \\ 1 & 1.83 \end{bmatrix}, \qquad \bar{L} = \begin{bmatrix} 1.41 & 1.41 \\ 0 & 2.23 \end{bmatrix}$$

Let the entries of the matrix A be given a 10% perturbation, so that

$$\tilde{A} = \begin{bmatrix} .52 & 1.89 \\ -.74 & -1.94 \end{bmatrix} + \begin{bmatrix} -.05 & -.19 \\ .07 & .19 \end{bmatrix}$$

The perturbed matrix $\tilde{W}$ is then given by

$$\tilde{W}(s) = \frac{3}{(115)^2(s^2+2.95s+1.82)} \begin{bmatrix} -18 & 166 \\ 130 & 130 \end{bmatrix} \begin{bmatrix} s+2.9 & 1.8 \\ -.93 & s+.05 \end{bmatrix} \begin{bmatrix} -12 & 10 \\ 17 & 15 \end{bmatrix}$$

Since

$$\tilde{W}(j\omega) + \tilde{W}^*(j\omega) =$$

$$\frac{6}{(115)^2[(1.82-\omega^2)^2 + 2.95^2]} \begin{bmatrix} 7520 + 13798\omega^2 & -2120 + 5000\omega^2 - j9440\omega \\ -2120 + 5000\omega^2 + j9440\omega & 13720 + 3960\omega^2 \end{bmatrix}$$

is a positive matrix for all ω, the sensitivity is still reduced. Employing theorem 4.10, it is seen that

$$L'L - C'C - Q\delta A - \delta A'Q = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 4.68 \end{bmatrix}$$

is also positive hence, this result also ensures sensitivity improvement.

4.6 Summary

In this chapter the problem of sensitivity reduction was formulated for stationary multivariate systems in terms of the sensitivity operator \$. The results of the previous chapter were then translated via frequency domain techniques to establish the relationships between the constant plant matrices $\{A, B, C\}$ and the compensators G and M to ensure sensitivity improvement for disturbances in the form of additive noise and plant parameter perturbations. It was also demonstrated that the concept of positive realness plays an important role for sensitivity operators of the form \$(s) = I-H(s). This concept is a very important one since it brings to bear on H(s), all the structure associated with positive real matrices. For example, for plants described by S2 and feedback compensators given by constant matrices, it can be shown by using the concepts developed in section B. 5 that every entry of H(s) must be a proper rational fraction of s and moreover, the order of the numerator and denominator can not differ by more than one. This is clearly a very severe limitation which was well illustrated by example 4. 2. This restriction could be reduced if it were desired to improve the sensitivity reduction only over some finite frequency band and not the entire frequency band. This is indeed a fruitful area for future research.

Chapter 5

A COLLECTION OF RESULTS IN SENSITIVITY REDUCTION

5.1 Sensitivity and Optimal Control

In Chapter 1, it was illustrated that there is a very definite relationship between sensitivity reduction and optimal control of regulator systems. In particular, Kalman [23] has shown that for single-input, single-output systems, the optimal control law does indeed reduce the sensitivity. More recently, Anderson [6] and Kreindler [32] extended the results of Kalman to include multi-variate systems. They have shown that the optimal control law does reduce sensitivity if a particular weighting is employed on the output errors. This weighting is not arbitrary, but follows from the optimization. Kreindler, points out that for an arbitrary weighting matrix, the optimum control law does not always reduce the sensitivity. Therefore, from previous discussions, it is germain to answer the question: "Under what conditions does the optimum control law reduce system sensitivity for a unity weighting matrix?" This question is not completely answered here, but a necessary condition is given to ensure reduction.

In this section, the optimal linear regulator problem, described in Appendix B is analyzed from the point of view of sensitivity reduction, and the salient features are presented.

To review, the linear dynamical system is characterized by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t);\ y(t) = x(t),$$

and the optimal control law is given by

$$u(t) = -M(t)x(t) = -R^{-1}B'(t)E(t)x(t),$$

where R(t) is a positive, definite self adjoint matrix, and E(t) is the unique positive self adjoint solution of the matrix Riccati equation discussed in section B. 3. To discuss the behavior of the optimal regulator system from the point of view of sensitivity reduction, it is pertinent to establish the sensitivity kernel $s(t,\tau)$. From equation 3. 22, it is clear that

$$\mathcal{S}(t, \tau) = I\,\delta(t-\tau) - \Psi(t, \tau)B(\tau)R^{-1}(\tau)B'(\tau)E(\tau), \quad (5.1)$$

where $\Psi(t, \tau)$ satisfies

$$\frac{d}{dt}\Psi(t, \tau) = [A(t) - B(t)R^{-1}(t)B'(t)E(t)]\Psi(t, \tau),\ \Psi(t_o, t_o) = I.$$

It is evident from equation 5. 1 that the sensitivity kernel can be written as

$$\mathcal{S}(t, \tau) = I\delta(t-\tau) - h(t, \tau), \quad (5.2)$$

where $h(t, \tau)$ is defined in the obvious way. Employing theorem 3. 1 we see that it is necessary for $h(t, \tau)$ to be a positive real kernel. Hence, from theorem 2. 2, a necessary condition for the optimal control law to reduce sensitivity is

$$B(t)R^{-1}(t)B'(t)E(t) + E(t)B(t)R^{-1}(t)B'(t) \geq 0. \quad (5.3)$$

This is clearly a pertinent result since it illustrates the relationship which must exist between B, R, and E for sensitivity reduction. For time invariant systems with regulation on the infinite interval, the necessary conditions for sensitivity reduction become more clearly defined.

Theorem 5.1 Necessary conditions for the optimal control law on the infinite interval to reduce sensitivity are

1. $EBR^{-1}B' = BR^{-1}B'E$
2. $BR^{-1}B'E \geq EBR^{-1}B'KBR^{-1}B'E \geq 0$

where K is the positive definite self adjoint solution

$$K(A - BR^{-1}B'E) + (A - BR^{-1}B'E)'K = -I.$$

Proof: Since, from theorem B.6, Re $\{\sigma(A-BR^{-1}B'E)\} \leq \epsilon < 0$, the theorem follows from theorem 4.5.

Hence, the solution of the algebraic form of the matrix Riccati equation must commute with $BR^{-1}B'$. A stronger condition relating E, B, and R can be obtained via theorem 4.3 if

$$(I - EBR^{-1}B'K)\Psi(s)BR^{-1}B'E \qquad (5.4)$$

is irreducible.

Theorem 5.2 If 5.4 is positive real and irreducible, then a necessary condition for sensitivity reduction is

$$BR^{-1}B' > 0.$$

Proof: From theorem 4.7 there exists a positive definite self adjoint matrix $\bar{Q}$, such that

$$(\bar{Q} + K)BR^{-1}B'E = I$$

or

$$\bar{Q} = -K + (BR^{-1}B'E)^{-1}$$

Therefore, it is necessary that $BR^{-1}B'$ be nonsingular, consequently positive definite. It is also to be noted that B' must be one to one.

This section has demonstrated the necessary conditions which must exist for the optimal regulator problem to have improved sensitivity as defined in Chapter 3. The next section shall discuss linear discrete systems and give sufficient conditions for sensitivity reduction.

5.2 Sensitivity Reduction in Linear Discrete Systems

The discussion of Chapter 3 and 4 illustrated the natural relationship between positive realness and sensitivity reduction of linear systems. In this section, we shall study the question of sensitivity reduction for discrete systems but, as we shall see, there is no relationship similar to positive realness to facilitate this study.

Consider the two nominally equivalent systems of Figure 1.3. The plant is described by the set of linear vector difference equations

$$x[(k+1)\tau] = Ax(k\tau) + Bu(k\tau) \tag{5.5}$$

$$y(k\tau) = Cx(k\tau), \tag{5.6}$$

where the mapping $u(k\tau) \rightarrow y(k\tau)$ acts between finite cartesian products of $\ell_2(\Omega)$ equipped with the usual inner product. The feedback operator M is given by a constant matrix. If the plant P is bounded linear and stationary, then P has a multiplicative form and the mapping $y = Pu$ can be written as

$$Y(z) = P(z)U(z), \tag{5.7}$$

where z is the complex variable associated with the z-transform. Taking z-transform of equations 5.5, 6, P(z) can be identified by the following:

$$Y(z) = C\Phi(z)BU(z),$$

where

$$\Phi(z) = (z\,I-A)^{-1}.$$

The discrete-time analog to the necessary and sufficient conditions for sensitivity reduction given by lemma 4.2 can be written as[1]

1. $\$(z)$ analytic $|z| \geq 1$ (5.8)
2. $I-\$^*(z)\$(z) \geq 0$ for all z such that $|z| = 1$, (5.9)

where

$$\$(z) = [I+P(z)M]^{-1}. \tag{5.10}$$

Rearranging equation (5.10) the sensitivity operator takes the now familiar form

$$\$(z) = I-C\Psi(z)BM,$$

---

[1] See also Perkins and Cruz [39].

where $\Psi(z)$ is given by

$$\Psi(z) = [zI - (A-BMC)]^{-1}.$$

Previously it was shown that the analysis of the single-variate case was fruitful for establishing insight into the multivariate case. With this in mind, consider the operator H(z)

$$H(z) = C\Psi(z)BM = k\ \frac{\pi(z)}{\psi(z)}\ m$$

where $\pi(z)$ and $\psi(z)$ are polynominals in z, with real coefficients. The gain k and the feedback operator m are real scalars. Condition (2) can then be written as

$$k\frac{\pi(z)}{\psi(z)}\, m + k\,\frac{\pi^*}{\psi^*}\, m - k^2m^2 \left|\frac{\pi}{\psi}(z)\right|^2 \geq 0, \text{ for all } z \ni |z| = 1 \quad (5.11)$$

Clearly a necessary condition for inequality 5.8 to be satisfied is for

$$mk\left[\frac{\pi}{\psi} + \frac{\pi^*}{\psi}\right](z) \geq 0; \text{ for all } z \ni |z| = 1.$$

Under general pole zero configuration, the above inequality cannot be satisfied. For example, consider the operator H(z) described by

$$H(z) = \frac{cm}{z-a} = mk\left(\frac{\pi}{\psi}\right)(z):\ c, m > 0,\quad |a| < 1.$$

Then,

$$mk\left[\frac{\pi}{\psi} + \frac{\pi^*}{\psi^*}\right](z) = \frac{2mc(\alpha - a)}{(\alpha - a)^2 + \beta^2}\ ;\ z = \alpha + j\beta. \qquad (5.12)$$

It is clear that equation 5.12 is negative for all α less than a. Hence, the positive realness condition which was used so fruitfully in Chapter

4 does not seem to have an analogous counterpart in the z-plane.

Even though positive realness doesn't carry over, it is still possible to reduce the sensitivity. Representing P(z) by

$$P(z) = \frac{k\gamma(z)}{\alpha(z)} ,$$

it is clear that

$$0 \leq \$^*(z)^{-1}\$^{-1}(z) - I \Longleftrightarrow \frac{\gamma^*(z)}{\alpha^*(z)} \{mk[(\frac{\alpha}{\gamma})(z) + (\frac{\alpha^*}{\gamma^*})(z)] + m^2k^2\} \frac{\gamma(z)}{\alpha(z)} \geq 0.$$

Hence we can formulate the following theorem.

Theorem 5.2 If H(z) is analytic for $|z| \geq 1$, and $\frac{\gamma(z)}{\alpha(z)}$ has no poles on the unit circle, then an m can be found so that $I - \$^*(z)\$(z)$ is positive.

The proof is trivial, and follows that given in theorem 4.8.

Turning now to the multivariate case we will again consider the question of sensitivity reduction when perturbations take the form of additive noise η and variations in the plant matrix A. If A, B, C and M are real matrices then sufficient conditions for sensitivity reduction in the presence of additive noise is given by

Theorem 5.3 If the absolute value of the eigenvalues of A-BMC are less then unity, then a sufficient condition for sensitivity reduction is the existence of a self adjoint matrix Q and a matrix W such that

1. $QBM = C'$
2. $-Q(A-BMC)-(A-BMC)'Q + 2\delta Q - C'C = W'W$, for all $\delta \in [-1,1]$.

Proof: $I-\$^*(z)\$(z) = C\Psi(z)BM+M'B'\Psi^*(z)C'-M'B'\Psi^*(z)C'C\Psi(z)BM$

$$= M'B'Q\Psi(z)BM+M'B'\Psi^*(z)QBM-M'B'\Psi^*(z)C'C\Psi(z)BM$$

$$= M'B'\Psi^*(z)[-Q(A-BMC)-(A-BMC)'Q+2\mathrm{Re}\, zQ-C'C]\,\Psi(z)BM$$

$$= M'B'\Psi^*(z)W'W\Psi(z)BM.$$

which is clearly positive.

As seen in Chapter 4, the problem of insuring sensitivity reduction in the presence of perturbations of the plant parameters is very difficult to solve satisfactorily. This problem is compounded since the concept of positive reality doesn't carry over in discrete systems. We can however obtain sufficient conditions on δA which guarantee sensitivity reduction.

Theorem 5.4 If the absolute value of the eigenvalues of $\tilde{A}-BMC$ are less than unity and the nominal closed-loop system satisfies conditions (1) and (2) of theorem 5.3, then the sensitivity of the perturbed closed-loop system is less than the open-loop system if $-Q\delta A-\delta A'Q+W'W$ is positive.

The proof follows directly from the previous theorem by replacing A with $\tilde{A}$.

From this section it is seen that the results, consistent with the preceding chapter, are not very conclusive and are left as an area of future research. In the next section we will study a different closed-loop configuration and discuss its structure to ensure sensitivity reduction.

5.3 Sensitivity Reduction of a Linear System

This section will discuss the question of sensitivity reduction for the nominally equivalent systems of Figure 1.4. It is to be noted that the only difference between the closed-loop system shown here and the one discussed in Chapters 3 and 4, is that the compensator G is inside the loop. We will see, however, that this makes a substantial difference in the structure of the plant and the associated compensators. As before: $P: B_1 \to B_2$ represents the response of the plant, $G: B_1 \to B_1$ and $M: B_2 \to B_1$ are compensation transformations. The system equations for the closed-loop and open-loop systems are given by

$$y_c = [I + PGM]^{-1} PGu$$

and

$$y_o = Pu$$

which are defined to be the nominal system equations. To insure terminal equivalence between the open and closed-loop systems, G is given by

$$G = (I - MP)^{-1}$$

provided that I - MP is nonsingular. It can easily be shown that the perturbations in the closed-loop and open-loop system in the presence of additive noise are related by

$$\delta y_c = (I + PGM)^{-1} \delta y_o.$$

Hence by the following sequence of identities

$$(I+PGM)^{-1} = [I+P(I-MP)^{-1}M]^{-1}$$
$$= [I+P(I-MP)^{-1}M(I-PM)(I-PM)^{-1}]^{-1}$$
$$= [I+PM(I-PM)^{-1}]^{-1}$$
$$= [(I-PM+PM)(I-PM)]^{-1}$$
$$= I - PM,$$

the sensitivity operator \$, is given by

$$\$ = (I - PM). \tag{5.13}$$

Now focusing attention on plants which are described by S1 and feedback compensators M which are real and continuous, it is clear from the discussion in Chapter 3 the kernel of \$ is given by

$$\$(t,\tau) = I\delta(t-\tau)-C(t)\Phi(t,\tau)B(\tau)M(\tau).$$

The necessary and sufficient conditions for sensitivity improvement given by 3.14, 15, 16 can be formulated in terms of the composition PM by the following

1. $S: L_2^m(\Omega) \to L_2^m(\Omega) \Longleftrightarrow PM: L_2^m(\Omega) \to L_2^m(\Omega), \quad \Omega = (t_o, t_f)$
2. $\$$ causal $\Longleftrightarrow$ PM causal
3 $I - \$^*\$ \geq 0 \Longleftrightarrow PM + (PM)^* - (PM)^*PM \geq 0.$

Since the system described by S1 is real, the following theorem is given;

Theorem 5.4 A necessary condition for sensitivity improvement is that the kernel associated with PM be L_2 - positive real. The proof follows from the above conditions and definition 2.2.

From the preceding theorem, it is clear that the necessary conditions for sensitivity reduction are similar to those of Chapter 3 The primary difference being the L_2 - positive realness of PM instead of H.

Theorem 5.5 Let G be an integral operator, bounded on $L_2^m(\Omega)$ whose kernel is given by

$$[C(t)-M'(t)B'(t)K(t)]\Phi(t,\tau)B(\tau)M(\tau)$$

where K(t) is the unique, positive, self adjoint solution of

$$-\dot{K}(t) = K(t)A(t) + A'(t)K(t) + C'(t)C(t)$$

such that $K(t_o)$ is given by

$$K(t_o) = \int_{t_o}^{t_f} \Phi'(\tau,t_o)C'(\tau)C(\tau)\Phi(\tau,t_o)d\tau.$$

If PM is a bounded, causal and real operator defined on $L_2^m(\Omega)$, then the sensitivity is reduced if $G + G^*$ is positive. The proof is straight forward and follows that of theorem 3.2.

Several theorems, similar to those following theorem 3.2, could be given here but because of their similarity, they will be omitted. Instead, attention is focused to the analysis of sensitivity improvement of stationary plants described by S2 with constant feedback matrices. The following theorem helps in establishing the necessary and sufficient conditions for sensitivity improvement for this system configuration.

Theorem 5.6 If $\sigma(A) \leq \epsilon < 0$, then the necessary and sufficient condition for sensitivity improvement is that $(C-M'B'K)\Phi(s)BM$ be positive real, where K is the positive, self adjoint solution of $KA+A'K = -C'C$.

Proof: The proof follows from definition B.8, and theorem 5.4

$$0 \leq I-S^*(j\omega)S(j\omega)=C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'-M'B'\Phi^*(j\omega)C'C\Phi(j\omega)BM$$

$$= C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'-M'B'\Phi^*(j\omega)[K(j\omega I-A)+(j\omega I-A)^*K]\Phi(j\omega)BM$$

$$= C\Phi(j\omega)BM+M'B'\Phi^*(j\omega)C'-M'B'\Phi^*(j\omega)KBM-M'B'K\Phi(j\omega)BM$$

$$= (C-M'B'K)\Phi BM+[(C-M'B'K)\Phi(j\omega)BM]^* \geq 0 \text{ for all } \omega \in (-\infty,\infty).$$

To gain a better understanding of the restrictions placed in the plant, consider a single variate system with the plant represented by

$$P(s) = \frac{k\pi(s)}{\psi(s)}.$$

The function π and ψ denote two polynominals of a complex variable s with real coefficients such that all the zeros of ψ are in the left half plane. The operators k and m are also real scalars. Hence

$$0 \leq \{m'P^*+Pm-m'P^*Pm\}(j\omega) \Longleftrightarrow mk\left(\frac{\psi}{\pi}+\frac{\psi}{\pi^*}\right)(j\omega)-m^2k^2 \geq 0.$$

Noting that $(\psi/\pi + \psi^*/\pi^*(j\omega) = 2\,\mathrm{Re}\,[(\frac{\psi}{\pi})(j\omega)]$, the preceding inequality, can be written in the form

$$0 \leq mk\,(\psi/\pi + \psi^*/\pi^*)(j\omega) - m^2k^2 \Longleftrightarrow \begin{cases} mk \geq 0 \text{ and } \mathrm{Re}\,[(\psi/\pi)(j\omega)] \geq mk/2 \\ \\ mk \leq 0 \text{ and } \mathrm{Re}\,[(\psi/\pi)(j\omega)] \leq mk/2 \end{cases} \tag{5.14}$$

Without loss of generality we will consider only the case; $mk \geq 0$.

<u>Theorem 5.7</u> A scalar m can be found which satisfies 5.14 if and only if the frequency plot of $(\psi/\pi)(j\omega)$ lies in either the left half or the right half of the complex plane.

Proof: (Sufficiency) Suppose that $0 \leq \alpha = \inf_{\omega} \{\mathrm{Re}[(\psi/\pi)]\}$. Then for $m \in [0, 2\alpha/k]$ the conditions; $mk \geq 0$ and $\mathrm{Re}[(\psi/\pi)(j\omega)] \geq mk/2$ hold. Similarly if $\sup_{\omega} \{\mathrm{Re}\,[(\psi/\pi)(j\omega)]\} = \beta \leq 0$ then for $m \in [2\beta/k, 0]$ the condition; $mk \leq 0$ and $\mathrm{Re}\,[(\psi/\pi)(j\omega)] \leq mk/2$ hold. Necessity is obvious.

<u>Corollary 5.1</u> The value of m which maximizes I-\$*(jω)\$(jω) is given by

$$m = \frac{1}{k} \inf_{\omega} \mathrm{Re}[(\psi/\pi)(j\omega)]$$

if the frequency plot of $(\psi/\pi)(j\omega)$ lies in the right half plane.

<u>Corollary 5.2</u> The value of m which maximizes I-\$*(jω)\$(jω) is given by

$$m = \frac{1}{k} \sup_{\omega} \mathrm{Re}\,[(\psi/\pi)(j\omega)]$$

if the frequency plot of $(\psi/\pi)(j\omega)$ lies in the left hand plane.

Hence, by the preceding corollaries it is possible to select the feedback compensator M, to minimize the sensitivity. This feasibility was demonstrated for single-variate systems, however, it is felt that there may be a generalization of this result for the multivariate case. This problem is left open as an area of future research.

5.4 Summary

In summary, this chapter has discussed three problems pertinent to sensitivity reduction. First, a necessary condition relating the solution of the matrix Riccati equation and the plant matrix B was given to ensure that the optimal regulator does indeed reduce sensitivity. Second, in the study of discrete systems, a sufficient condition for sensitivity improvement was established relating the plant matrices $\{A, B, C\}$ and the feedback compensators G and M. Finally, a variation of the closed-loop system was studied resulting in necessary and sufficient conditions relating the plant matrices and its associated compensators.

Appendix A

A REVIEW OF FUNCTIONAL ANALYSIS CONCEPTS AND TERMINOLOGY

The books by Porter [48], Liusternik and Sobolev [34], Kolmogorov and Fomin [30], Simmons [52], Taylor [53], Riesz and Nagy [49], and Bachman and Narici [7] are the principal sources of definitions.

A.1 Linear Space

A set X of elements $\{x, y, \ldots, \ldots\}$ is called a linear space if the following conditions are satisfied.

A. For any two elements, $x, y \in X$ (read "x and y belong to the set X"), there is a uniquely defined third element $z = x + y, z \in X$, called their sum, such that

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. There exists an element 0 having the property that $x + 0 = x$ for all $x \in X$
4. For every $x \in X$ there exists an element $-x$ such that $x + (-x) = 0$.

B. For arbitrary scalars $\alpha, \beta \in F$ and any element $x \in X$, there is defined an element $\alpha x \in X$ such that

1. $\alpha(\beta x) = (\alpha\beta)x$
2. $1 \cdot x = x$.

A linear space is said to be <u>normed</u> if to each element $x \in X$ there exists a mapping $|| \quad ||$ called a <u>norm</u> such that

$$|| \quad || : X \to R$$

$$x \to ||x||$$

satisfying the conditions

1. $||x|| \geq 0$
2. $||x|| = 0$ if and only if $x = 0$
3. $||\alpha x|| = |\alpha| \cdot ||x||$
4. $||x+y|| \leq ||x|| + ||y||$. $x, y \in X$

where R is the space of real numbers.

When applied to the difference between two elements $x, y \in X$, the norm $||x - y||$ has the geometric interpretation of being the distance between x and y in the space X ($||x - y||$ defines a metric for the space X).

A sequence of elements $\{x_n\}$ of a normed linear space X is called a <u>fundamental sequence</u> or <u>Cauchy sequence</u> if for every number $\epsilon > 0$ there exists an index number N such that $||x_m - x_n|| < \epsilon$ for all $m, n \geq N$. An element x of a normed linear space X is called the limit of a sequence $\{x_n\}$ of X (written $x_n \to x$ or $\lim x_n = x$) if $||x_n - x|| \to 0$ and $n \to \infty$. A normed linear space X is called a <u>complete space</u> if every fundamental sequence of this space has a limit in X. Complete normed linear spaces are by convention called <u>Banach spaces</u>. We now describe some of the main examples of

Banach spaces.

<u>Example 1</u> Let X be an n-dimensional real Euclidean space with the norm defined by $||x|| = |x_1|^2 + \ldots + |x_n|^2$ for all $x \in X$.

<u>Example 2</u> Let M be the set of all measurable functions on the interval Ω, then the linear space defined by

$$X = L_2(\Omega) = \{x \in M: \int_\Omega |x(t)|^2 dt < \infty\} \text{ where}$$
$$||x|| = \{\int_\Omega |x(t)|^2 dt\}^{\frac{1}{2}} \text{ is a Banach space.}$$

<u>Example 3</u> Let $X = L_1(\Omega) = \{x \in M: \int_\Omega |x(t)|\, dt < \infty\}$ where $||x|| = \int_\Omega |x(t)|dt$, then X is a Banach space.

<u>Example 4</u> Let $X = L_\infty = \{x \in M: \text{ess sup } |x(t)| < \infty\}$ where $||x|| = \text{ess sup } |x(t)|$, then X is a Banach space.

It is to be noticed that all the definitions given up to now do not say anything about the 'angle' between two elements. The concept of the 'angle' helps to determine when two elements are orthogonal (or perpendicular). Precisely to take care of this need the definition of an <u>inner product space</u> is introduced.

A linear space X is said to be an inner product space if there exists a function (denoted by $< , >$) defined on X which maps $X \times X$ the cartesian product space, into the scalar field F such that

$$X \times X \rightarrow F$$
$$(x, y) \rightarrow < x, y >$$

(where (,) represents an order pair in $X \times X$) with the following properties:

1. $< x,y > = < \overline{y,x} >$, $x,y \in X$ (where bar denotes complex conjugate)
2. $< \alpha x + \beta y, z > = \alpha < x,z > + \beta < y,z >$, $\alpha, \beta \in F$ and $x,y,z \in X$
3. $< x,x > \; > 0$ if $x = 0$
4. $< x,x > = 0$ if and only if x is the zero vector.

Any two elements $x,y \in X$ are said to be orthogonal if $< x,y > = 0$. Because of conditions (3) and (4) we can define $||x|| = < x,x >^{\frac{1}{2}}$ and it is a simple exercise to check that this defines a norm on X. In this case it is said that the norm is induced by the inner product. A Hilbert space is an inner product space which is complete with respect to the norm induced by the inner product. The following examples illustrate two of the more important Hilbert spaces.

Example 1   An n-dimensional real Euclidean space is a real Hilbert space.

Example 2   The space $L_2(\Omega)$ is a complex Hilbert space where $< x,y > = \int_\Omega x(t)\overline{y(t)}dt$.

Many Banach spaces are not Hilbert spaces since an inner product does not exists which generates a particular norm. An example of two such spaces are

1. $L_1(\Omega)$
2. $L_\infty(\Omega)$.

A. 2 Functions

If there exists a correspondence between the elements of one space X and the elements of another space Y, then the mechanism by which the relationship is established is called a transformation or mapping. The relation between an element $x \in X$ and its image element $y \in Y$ may be denoted by

$$y = F(x)$$

or

$$y = Fx.$$

The space X is called the domain of the operator F and Y the range. It is assumed here that both X and Y are normed linear spaces. If the domain and range of F are both in X then F is said to be an operator on X.

At times it so happens that Y is a subset of the real line, i.e., the image elements y are simply real numbers. Then the operator is called a functional.

A transformation is said to be bounded if there exists a constant c such that

$$||Fx|| \leq c \; ||x||$$

for all $x \in X$.

To site two examples of bounded operators consider the following:

Example 1 Let $x \in L_1(\Omega)$ and let $y \in L_\infty(\Omega)$ be a fixed element, then

$$Fx = \int_\Omega x(t)\,\overline{y(t)}\,dt$$

is a bounded linear operator and

$$||Fx|| = |Fx| \leq \int_\Omega |x(t)| \cdot |y(t)| dt \leq ||y||_\infty \int_\Omega |x(t)| dt$$
$$= ||y||_\infty \cdot ||x||_1.$$

<u>Example 2</u> Let $x \in L_2(\Omega)$ and let $y \in L_2(\Omega)$ be fixed then

$$Fx = \int_\Omega x(t) \overline{y(t)} \, dt$$

is a bounded linear operator and by the Schwartz inequality

$$||Fx||^2 \leq [\int_\Omega |x(t)| \cdot |y(t)| dt]^2 \leq \int_\Omega |x(t)|^2 dt \cdot \int_\Omega |y(t)|^2 dt$$
$$= ||x||_2 \cdot ||y||_2.$$

Unbounded operators often arise in control problems and the following examples illustrate two such operators.

<u>Example 1</u> Let $X = \{x \in L_2(o, \infty): x$ is absolutely continuous and $\dot{x} \in L_2(o, \infty)\}$. If $Fx = \dot{x}$, then F is an unbounded operator.

<u>Example 2</u> Let $X = \{x \in L_2(o, \infty): t \cdot x(t) \in L_2(o, \infty)\}$. If $(Fx)(t) = t \cdot x(t)$, then F is an unbounded operator.

A transformation F is said to be <u>continuous</u> if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$||Fx_1 - Fx_2|| < \epsilon$$

whenever

$$||x_1 - x_2|| < \delta$$

for all $x_1, x_2 \in X$. Note that the norms in these expressions can be interpreted according to whether the domain X is involved or the image space Y. That is $||Fx||$ refers to the norm in the range space while $||x||$ refers to the norm in the domain space.

A. 3 Linear Operators

An operator F is said to be linear if it satisfies the condition:

$$F(\alpha x_1 + \beta x_2) = \alpha F(x_1) + \beta F(x_2)$$

for any two elements $x_1, x_2 \in X$ and arbitrary scalrs α and β. If a linear operator is continuous, then it is also bounded. The reverse is also true.

The norm of a linear operator F, (denoted by $||F||$), is defined as the greatest lower bound of the numbers c which satisfy the boundedness condition:

$$||F|| = \sup_{||x||=0} \frac{||Fx||}{||x||} = \sup_{||x|| \leq 1} ||Fx|| = \sup_{||x||=1} ||Fx||.$$

If F_1 and F_2 are two linear operators on a normed linear space X, then the inequality

$$||F|| \leq ||F_1|| + ||F_2||$$

holds for their sum $F_1 + F_2 = F$.

If F is a linear transformation from the space X into space Y and G is a linear transformation from Y to Z, then the composite operator H = GF defined by

$$Z = H(x) = G(F(x)), \; x \in X, \; z \in Z$$

is called the product of the transformations F and G. The norm $||H||$ satisfies the inequality

$$||H|| \leq ||G|| \cdot ||F||.$$

Some Additional Definitions

In dealing with operators on a Hilbert space several additional concepts occur with sufficient frequency to merit specific mention in this Appendix.

A. 4 Adjoint Operators

Before defining an adjoint operator, it is worthwhile to introduce the concept of a conjugate space.

It is known that the set of all bounded linear functionals defined on a normed linear space X forms itself a normed linear space called the conjugate space of X and denoted by X^*.

It is true that for any arbitrary bounded functional J defined on the Hilbert space H there is a unique element $y \in H$ such that

$$J(z) = \langle z, y \rangle$$

for every $z \in H$. Letting $z = Tx$ where T is a linear bounded operator on H, the definition of the adjoint of T (denoted by T^*) is arrived at by letting

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for every $x, y \in H$. T^* maps into H, is linear, bounded and the equality

$$||T^*|| = ||T||$$

holds. A linear, bounded operator T on H is said to be self-adjoint if $T = T^*$.

Four basic properties of adjoint operators are the following: If T_1 and T_2 are two linear operators on H, then

1. The adjoint of their sum is equal to the sum of their adjoints,

$$(T_1 + T_2)^* = T_1^* + T_2^*$$

2. The adjoint of their product is equal to the product of their adjoint in reverse order,

$$(T_1 T_2)^* = T_2^* T_1^*$$

3. The identity operator is self-adjoint

$$I^* = I$$

4. $(T^*)^* = T^{**} = T.$

A.5 Positive Operators on a Hilbert Space

A linear, bounded operator T different from zero defined on a real Hilbert space H is said to be positive if for every $f \in H$ the condition

$$< Tf, f > \geq 0$$

holds. The operator T is said to be positive definite if 0 in this equation is replaced by $\delta < f, f >$ $(\delta > 0)$. It is customary to normalize f in which case the inequality becomes

$$< Tf, f > \geq \delta$$

for all $f \in H$ with $||f|| = 1$. Symbolically the notation $T \geq \delta I$ is used.

It is true that a positive definite, self-adjoint operator T on H possesses a unique, self-adjoint, positive definite operator $T^{\frac{1}{2}}$ well-defined on H called the square root of T such that

$$T^{\frac{1}{2}}T^{\frac{1}{2}} = T.$$

Five basic properties of positive and self-adjoint operators are the following: If T_1 and T_2 are operators mapping a complex Hilbert space H into itself then

1. T_1 is self adjoint if and only if $< T_1 x, x >$ is real for all x. Hence every positive operator on H is self adjoint.
2. If T_1 and T_2 are bounded and if T_1 and T_2 commute, then $T_1 \geq 0$ and $T_2 \geq 0$ implies that $T_1 T_2 \geq 0$.
3. If T_1 and T_2 are bounded and if $T_1 \geq 0$, $T_2 \geq 0$, and $T_1 T_2 \geq 0$ then T_1 and T_2 commute.
4. If T_1 is positive definite, then T_1 is non-singular.
5. If T_1 is one to one then $T_1^* T_1 > 0$ otherwise $T_1^* T_1 \geq 0$.

A.6 Unitary Transformations

The linear transformation U of one Hilbert space H_1 into another Hilbert space H_2 is said to be isometric if it leaves the inner products invariant, i.e., for every $f, g \in H_1$ the equality

$$< Uf, Ug > = < f, g >$$

holds. It is to be noted that the inner products are on different spaces and have to be interpreted accordingly. If the image of the space H_1 under the transformation coincides with H_2, in other words, the mapping under U is 'onto', then the transformation U is said to be unitary. Equivalently, the relation

$$U^* = U^{-1}$$

holds on H_1.

A. 7 Eigenvalues and Eigenvectors

If T is an operator on a complex Hilbert space, and if $Tx = \lambda x$ for some non-zero x and for some scalar λ, then λ is called an eigenvalue of T and x is called an eigenvector of T. The set of all eigenvalues of T is called the spectrum of T and is denoted by $\sigma(T)$. If Re $\{\sigma(T)\} < 0$ (≤ 0) then T is asymptotically stable (stable).

Appendix B

CONCEPTS IN MODERN CONTROL THEORY

B. 1 Stability Theory

A fundamental aspect of the study of dynamic systems is the determination of their stability. There are several powerful methods for the study of stability but perhaps the most general method is that of Lyapunov which was developed 70 years ago in Russia [35] and which has become an invaluable tool in optimal control theory.

Intuitively the problem of stability is that of determining the behavior of a physical system in the neighborhood of an equilibrium state. If the system returns to this state after being perturbed it is called stable; if not it is unstable. Unfortunately, this intuitive concept excludes many forms of motion which are also recognized as being stable. It is for this reason that terms like "asymptotically stable", "uniformly stable", "stable in the large", etc. have arisen.

For the purposes of this thesis, the stability concepts introduced are related to linear systems which are characterized by

$$\dot{x}(t) = A(t)x(t); \quad x(t_o) = x_o \tag{B. 1}$$

where A(t) is an $n \times n$ matrix whose elements are continuous functions of time on $(-\infty, \infty)$.

Definition B. 1 The state $x = 0$ is said to be Lyapunov stable, if for any t_o and any $\epsilon > 0$, there is a $\delta > 0$ depending on ϵ and t_o such that $||x_o|| < \delta$ implies $||x(t:x_o, t_o)|| < \epsilon$ for all $t \geq t_o$, where $x(t;x_o, t_o)$ is the response of time t to the perturbations $x_o = x(t_o)$, and $||x||$ is the Euclidean norm defined by $||x||^2 = |x_1|^2 + \ldots + |x_n|^2$.

Definition B. 2 The state $x = 0$ is said to be asymptotically stable if (1) it is Lyapunov stable and (2) for any x_o sufficiently close to 0, $x(t;x_o, t_o) \to 0$ as $t \to \infty$.

Theorem[1] B. 1 The system described by equation B. 1 is Lyapunov stable if and only if there exists a constant M, which may depend on t_o, such that $||\Phi(t, t_o)|| \leq M$ for all $t \geq t_o$.

Theorem[1] B. 2 The system described by equation B.1 is asymptotically stable if and only if there is a constant M such that

$$||\Phi(t, t_o)|| \leq M \text{ for } t \geq t_o; \quad \lim_{t \to \infty} ||\Phi(t, t_o)|| = 0 \text{ for all } t_o,$$

hence $||x(t;x_o, t_o)|| \to 0$ as $t \to \infty$.

Definition B. 3 The system described by B. 1 is said to be exponentially asymptotically stable if there exists scalars λ, $k > 0$ such that

$$||\Phi(t, \tau)|| \leq k e^{-\lambda(t-\tau)} \text{ for all } t \geq \tau \epsilon [t_o, \infty).$$

Lyapunov devised a method of applying quadratic functionals to discuss the questions of asymptotic behavior of solutions of linear differential equations. Before proceeding with the method of Lyapunov, the following terminology is introduced.[2]

---

[1]See Zadeh [59].

[2]See Hahn [19].

Function of Class K: A function $\phi(r)$ which is a continuous real function defined on the closed interval $[0, h]$ such that $\phi(r) = 0$ when $r = 0$ and increasing strictly monotonically with r is called a function of class K.

Decrescent function: A function $V(x, t)$ is said to be a decrescent function if $|V(x,t)| \leq \phi(||x||)$ in the region

$$||x|| \leq h \text{ for all } t \geq t_o$$

where $\phi(||x||)$ is a function of class K.

Radially Unbounded function: A scalar function $V(x, t)$ is said to be radially unbounded if $V(x, t) \geq \psi(||x||)$ in the region

$$||x|| \leq h \text{ for all } t \geq t_o$$

for arbitrarily large h where $\psi(||x||)$ is a function of class K.

Positive (Negative) Definite function: A scalar function $V(x, t)$ is said to be positive (negative) definite if $V(0, t) = 0$ and $V(x, t) \geq \psi(||x||)$ $(\leq -\psi(||x||))$ in the region

$$||x|| \leq h \text{ for all } t \geq t_o$$

where $\psi(||x||)$ is a function of class K.

In studying the stability of the system characterized by equation B.1, the following lemmas are very important.

Lemma[1] B.1 The origin is stable in the sense of Lyapunov if there exists a positive definite function $V(x, t)$ such that $\frac{d}{dt} V(x, t)$ evaluated along the solution of equation B.1 is not positive.

---

[1] Ibid.

Lemma[1] B. 2 The origin is asymptotically stable if there exists a function $V(x, t)$ which is everywhere positive definite, radially unbounded, descrescent, and $\frac{d}{dt} V(x, t)$ evaluated along the solution of equation B.1 is negative definite.

Definition B. 4 A function V which satisfies lemmas 3.1 and 3.2 is called a Lyapunov function for the differential equation B.1.

One of the important uses of Lyapunov functions other than for purposes of stability, is to establish the nature of the solution of the algebraic equation

$$A'E + EA = Q \tag{B.2}$$

Lemma B. 3 Consider equation B. 2 where E, A and Q are $n \times n$ matrices and $Q < 0$, then there exists a unique solution $E > 0$ if and only if $\text{Re}\,\{\sigma(A)\} \leq \epsilon < 0$.

Proof: Define the solution of equation B. 2 by the integral

$$E = - \int_0^\infty e^{A't} Q e^{At} dt. \tag{B.3}$$

E is clearly positive definite since e^{At} is never singular and

$$\begin{aligned} [X, Ex] &= - \int_0^\infty [x, e^{A't} Q e^{At} x] \, dt \\ &= - \int_0^\infty [e^{At} x, Q e^{At}] \, dt. \end{aligned}$$

Integrating B. 3, premultiplied by A, by parts it follows that

[1] Ibid.

$$AE = -\int_0^{\infty} A'e^{A't}Qe^{At}dt = -[e^{A't}Qe^{At}]_0^{\infty} + \int_0^{\infty} E^{A't}Qe^{At}A\,dt$$

$$= Q - EA.$$

Conversely assume $E > 0$ and consider the differential equation

$$\dot{x} = Ax.$$

A Lyapunov function $V(x)$ is defined by

$$V(x) = [x, Ex]$$

then

$$\dot{V}(x) = [\dot{x}, Ex] + [x, E\dot{x}]$$

$$= [Ax, Ex] + [x, EAx]$$

$$= [x, (A'E + EA)x].$$

Hence $V(x) = [x, Qx] < 0$ and $x(t) \to 0$ which implies $\text{Re}\{\sigma(A)\} \leq \epsilon \leq 0$.

To prove uniqueness consider the existence of a solution E_1 such that

$$A'E_1 + E_1A = Q. \tag{B. 4}$$

Subtracting B. 4 from B. 2 we obtain

$$A'(E - E_1) = -(E - E_1)A.$$

Since[1] $\sigma(A) \cap \sigma(-A') = \phi$, $E = E_1$ and the lemma is proved.

B. 2 Controllability and Observability

The fundamental concepts of controllability and observability of linear systems were first introduced by Kalman and play an

[1] See Gantmacher Ref [14] page 220.

important role in modern control theory. In this section we will restrict our attention to continuous and discrete, linear multi-variate control systems. However, since most of the results for discrete-time systems can be obtained from the continuous-time case by replacing the variable t with discrete points t_k, and by replacing integrals with summations, it will be sufficient to study only the continuous time case.

Consider the multi-variate linear system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{B.4.1}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{B.4.2}$$

where: x is an n-vector, the state; u(t) is an m-vector, the input; y(t) is a p-vector, the output; A(t), B(t), C(t), D(t) are $n\times n$, $n\times m$, $p\times n$, $p\times m$ matrices, respectively.

<u>Definition B. 5</u> A plant is said to be <u>completely state-controllable</u> if for each pair of points x_o and x_f in R^n, there exists a bounded measurable controller $u(t) \in R^m$ on some finite interval $t_o < t \leq t_f$ which steers x_o to x_f.

<u>Definition B. 6</u> A plant is said to be <u>completely output-controllable</u> if for each pair of points y_o and y_f in R^p, there exists a bounded measurable controller $u(t) \in R^m$ on some finite interval $t_o < t < t_f$ which steers y_o to y_f.

It has been shown [31] that the plant is completely output-controllable on $[t_o, t_f]$, if and only if, the Gramian matrix

$$P(t_o, t_f) + D(t_f)D^*(t_f)$$

where

$$P(t_o, t_f) = \int_{t_o}^{t_f} C(t_f)\Phi(t_f, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, \tau)C^*(t_f)\, d\tau,$$

is nonsingular. The necessary and sufficient conditions for the doublet $\{A, B\}$ identified with equation B. 4.1 to be completely state-controllable is that the Gramian matrix

$$W(t_o, t_f) = \int_{t_o}^{t_f} \Phi(t_o, \tau)B(\tau)B^*(\tau)\Phi^*(t_o, \tau)\, d\tau,$$

be nonsingular.

From the above equation it is clear that the necessary and sufficient conditions for controllability depend explicitly on the state transition matrix $\Phi(t_o, t)$. Consequently, it is necessary to obtain a solution of the time-varying differential equations. To overcome this difficulty Silverman and Meadows [51] define state controllability matrices characterized in terms of $A(t), B(t)$, and $C(t)$ and their derivatives. The state controllability matrix of the system described by equation B. 4 is

$$Q_c(t) = [P_o(t), P_1(t), \ldots, P_{n-1}(t)] \qquad \text{(B. 5. 1)}$$

where

$$P_{k+1}(t) = -A(t)P_k(t) + P_k(t),\ P_o(t) = B(t). \qquad \text{(B. 5. 2)}$$

It can be shown that the system is completely state controllable if and only if $Q_c(t)$ has rank n on the interval $[t_o, t_f]$.

For stationary systems, the preceding conditions for state controllability reduce the fact that

$$Q_c = [B, AB, \ldots, A^{n-1}B]$$

has rank n.

A necessary and sufficient condition for complete output-controllability is that the $p \times (n+1)\,m$ matrix

$$\Gamma_1 = [CB, CAB, \ldots, CA^{n-1}B, D]$$

have rank p. Note that although output and state-controllability are conceptually similar, they do not imply each other. This is easily seen by setting $D = 0$. The rank of Γ is less than or equal to the rank of C or Q_c. Thus if the rank of $\Gamma_1 < p$ the system would not be completely output-controllable. Conversely, the rank of $\Gamma_1 = p$ does not imply rank $Q_c = n$.

<u>Definition B. 7</u> An unforced plant is said to be <u>completely observable</u> on $[t_o, t_f]$ if for a given t_o and t_f every $x(t_o)$ in R^n can be determined from the knowledge of $y(t)$ on $[t_o, t_f]$.

It is well known [48] that the necessary and sufficient conditions for the doublet $\{A, C\}$ identified with the plant, to be completely observable is that

$$M(t_o, t_f) = \int_{t_o}^{t_f} \Phi^*(t, t_o)C^*(t)C(t)\Phi(t, t_o)dt$$

be non-singular. The corresponding observability matrix due to Silverman and Meadows, is given by

$$Q_o(t) = [R_o(t), R_1(t), \ldots, R_{n-1}(t)]$$

where

$$R_{k+1}(t) = A'(t)R_k(t) + \dot{R}_k(t), \quad R_o(t) = C'(t).$$

Hence the system described by equations B.4 is completely observable on the interval $[t_o, t_f]$ if and only $Q_o(t)$ has rank n on the interval. For stationary systems this condition is equivalent to the $n \times np$ matrix

$$[C', A'C', \ldots, A'^{(n-1)}C']$$

having rank n.

Further simplification of these conditions is possible for special forms of the constant matrix A. For example, if A has distinct eigenvalues and if A is a diagonal matrix then

1. The stationary system is completely controllable if and only if B has no all-zero rows.
2. The stationary system is completely observable if and only if C has no all-zero columns.

B.3 Optimal Regulator Problem

In this section the optimal regulator problem is formulated and the salient features of the optimal solution are presented.

Consider the linear dynamical system characterized by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \; x(t_o) = x_o.$$

where x(t) is the n dimensional state vector, u(t) is the m-dimensional control input and the matrices A(t) and B(t) are locally measurable and of compatable dimensions. The optimal linear regulator problem is to determine the control u on the interval (t_o, t_f) which minimizes the cost functional:

$$J(u) = x'(t_f)Fx(t_f) + \int_{t_o}^{t_f} [x'(t)L'(t)L(t)x(t) + u'(t)Ru(t)]dt$$

where the terminal state $x(t_f)$ is unconstrained, F is a positive matrix and R(t) is a continuous positive definite matrix.

The optimal control $\hat{u}$ is given by the linear feedback control law

$$\hat{u}(t) = -R^{-1}(t)B'(t)E(t)x(t) = -M(t)x(t)$$

where E(t) is the unique positive solution of the matrix Riccati equation

$$-\dot{E}(t) = -A'(t)E(t) + E(t)A(t) - E(t)B(t)R^{-1}(t)B'(t)E(t) + L'(t)L(t)$$

satisfying the boundry condition

$$E(t_f) = F.$$

The matrix E(t) has the property that for arbitrary $t \epsilon [t_o, t_f]$

$$J(\hat{u}) = \min_u J(u) = [x(t), E(t)x(t)].$$

Restricting our attention to the case when the matrices A, B, R, and W are stationary and $t_f \rightarrow \infty$, the cost functional takes the form

$$J(u) = \lim_{t_f \rightarrow \infty} \int_{o}^{t_f} \{[x(t), L'Lx(t)] + [u(t), Ru(t)]\} dt$$

where an admissible controller u is measurable on $(0 \leq t < \infty)$ and is such that the cost functional converges to a finite value.

By a suitable normalization of the input vector, it is possible to take R = I, the unit matrix; thus the cost function takes the form

$$J(u) = \lim_{t_f \to \infty} \int_0^{t_f} \{[x(t), L'Lx(t)] + [\hat{u}(t), \hat{u}(t)]\}dt \qquad (B.6)$$

where $\hat{u} = R^{\frac{1}{2}}u$.

Now stating the results due to Kalman [23], the problem may be solved as follows:

Let $\pi(t: t_f, 0) = E(t)$ be the unique self adjoint solution of

$$-\dot{E}(t) = E(t)A + A'E(t) - E'(t)BB'E(t) + L'L \qquad (B.7)$$

such that

$$\pi(t_f; t_f, 0) = E(t_f) = 0.$$

If the plant is completely controllable, then

$$\lim_{t_f \to \infty} \pi(t; t_f, 0) = E(t)$$

exists for all t and

$$\lim_{t_f \to \infty} \pi(0; t_f, 0) = E_\infty \stackrel{\Delta}{=} E \qquad (B.8.1)$$

is the steady state solution of equation B.7.

Let the control law be given by

$$M = B'E \qquad (B.8.2)$$

then as given by Kalman:

Theorem B.3 If $\{A, B\}$ is completely controllable and the cost functional is given by B.6 then the optimal control law is given by B.8.2 where E is obtained by evaluating the limit of B.8.1.

Theorem B.4 If $\{A, B\}$ is completely controllable and $\{A, L\}$ is completely observable, then E is positive definite and self adjoint and the optimal control law is asymptotically stable.

Theorem B.5 If $\{A, B\}$ is completely controllable, a necessary and sufficient condition for the stability of the optimal control law is that all the eigenvalues of A restricted to uncontrollable states have negative real parts.

Theorem B.6 Let $\{A, B\}$ be completely controllable and $\{A, L\}$ be completely observable, then a necessary and sufficient condition for M to be a stable optimal control law is that there exists a matrix E such that

$$E > 0 \text{ and } E = E'$$

$$M = B'E$$

$$-E(A - BM) - (A - BM)'E = L'L + M'M$$

B.4 Irreducible Dynamic Systems

Consider the linear, time-invariant, finite dimensional plant, described by the following equations

$$\dot{x} = Ax + Bu \tag{B.9.1}$$

$$y = Cx \tag{B.9.2}$$

where u, x, and y are m, n, and p-vectors respectively. The number n is defined to be the dimension of the system. This description is isomorphic with the triple of matrices $\{A, B, C\}$ which is called a realization of the transfer function matrix

$$T(s) = C(sI - A)^{-1}B. \tag{B. 10}$$

Any matrix triple $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ satisfying B. 10 is a realization of T(s) but this realization is not unique. A realization of T(s) is called irreducible if the dimension of A is as small as possible. Kalman [26] has shown that every rational, proper matrix T(s) possesses an irreducible realization. Moreover if $\{A_T, B_T, C_T\}$ and $\{\tilde{A}_T, \tilde{B}_T, \tilde{C}_T\}$ are two irreducible realizations of T(s) then there exists a constant nonsingular matrix Γ such that

$$\tilde{A}_T = \Gamma A_T \Gamma^{-1}$$
$$\tilde{B}_T = \Gamma B_T$$
$$\tilde{C}_T = C_T \Gamma^{-1}$$

Hence the two realizations represent the same system but with respect to two different basis. This is easily seen since

$$T(s) = C_T(sI - A_T)^{-1}B_T = C_T\Gamma^{-1}\Gamma(sI - A_T)^{-1}\Gamma^{-1}\Gamma B_T$$
$$= \tilde{C}_T(sI - \Gamma A_T\Gamma^{-1})^{-1}\tilde{B}_T = \tilde{C}_T(sI - \tilde{A}_T)^{-1}\tilde{B}_T.$$

Kalman [26] has shown that if T(s) is a transfer function matrix satisfying

$$T(s) = C(sI - A)^{-1}B$$

then,

1. T(s) is irreducible if and only if it is completely state-controllable and completely observable.

2. If $\{A, B, C\}$ is a non-minimal realization, then there exists a nonsingular matrix Γ such that the triple $\{A, B, C\}$ is brought into the canonical form $\{A_c, B_c, C_c\}$ with

$$A_c = \Gamma A \Gamma^{-1} = \begin{bmatrix} A^{AA} & A^{AB} & A^{AU} \\ 0 & A^{BB} & A^{BU} \\ 0 & 0 & A^{UU} \end{bmatrix}$$

$$B_c = \Gamma B = \begin{bmatrix} B^{A} \\ B^{B} \\ 0 \end{bmatrix}$$

$$C_c = C\Gamma^{-1} = [0, C^B, C^U].$$

The system of differential equations for this realization is identical with that of equations B.9 since x_c is related to x via the nonsingular operator Γ such that

$$x_c = \Gamma x = (x^A, x^B, x^U)',$$

The superscript letters have the following meanings

A: controllable and unobservable

B: controllable and observable

U: uncontrollable.

The triple $\{A^{BB}, B^{B}, C^{B}\}$ is irreducible and its realization

$$T(s) = C^{B}(sI - A^{BB})^{-1}B^{B}$$

is equivalent to T(s).

B.5 Positive Realness

For several years, network theorists have been employing the concept of positive realness. More recently this concept has become an important tool for control systems analyst. This section reviews the concept of positive real matrices and gives an important lemma due to Anderson [4].

The definition of a positive real matrix can be found in several books on network synthesis (for example Newcomb [36]) and is given by

Definition B. 8 An $n \times n$ matrix $\Gamma(s)$ is called positive real if for Re $s > 0$

1. T(s) is analytic
2. $\overline{T(s)} = T(\overline{s})$
3. $T^{*}(s) + T(s) \geq 0$

where the overbar denotes complex conjugate and * denotes complex conjugate transpose. If T(s) is composed of rational polynominals in s then the above statements are equivalent to

1. T(s) is real-rational
2. T(s) analytic in Re $s > 0$

3. Poles of $T(s)$ on $\text{Re } s = 0$ are simple
4. For each pole on $\text{Re } s = 0$, the residue matrix is positive and self-adjoint
5. $T^*(j\omega) + T(j\omega) \geq 0$ for all $\omega \in (-\infty, \infty)$.

It is of interest to see what necessary conditions are placed on the entries of a matrix described by equation B. 10 to ensure positive realness. To facilitate this, it is helpful to write equation B. 10 in the form

$$T(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \cdots & a_{1p}(s) \\ a_{21}(s) & & & \\ \vdots & & & \\ a_{p1}(s) & & \cdots & a_{pp}(s) \end{bmatrix}.$$

Consider an arbitrary entry $a_{ij}(s)$ of $T(s)$, given by

$$a_{ij}(s) = \pi_{ij}(s)/\psi_{ij}(s).$$

Let π_{ij} and ψ_{ij} be the nth and mth order polynominals in s respectively. It is well known that for the given system of differential equations, described by equations B. 9, $n < m$ for all i, j and since the poles of $T(s)$ must be simple on $\text{Re } s = 0$, it must be the case that $m = n + 1$ for all i, j. A little reflection also shows that all of the entries along the main diagonal must be positive real functions.

Lemma B.5 (Anderson [4]) Let T(s) be a matrix of proper rational functions such that T(s) has poles which lie in Re s < 0 and are simple on Re s = 0. Let {A, B, C} be an irreducible realization of T(s). Then T(s) is positive real if and only if there exists a positive definite self adjoint matrix Q and a matrix L such that

1. $QA + A'Q = -L'L$
2. $QB = C'$.

B.6 Decoupling and the Inverse System

In the recent months there have been several studies (see [13], [15] and [47]) concerned with the problem of decoupling multi-variate systems by state feedback. In particular the paper by Falb and Wolovich [13] give the necessary and sufficient for this characterization for stationary systems, as well as conditions relating to the inverse system.

Consider the time-invariant linear feedback system shown in Figure C.1.

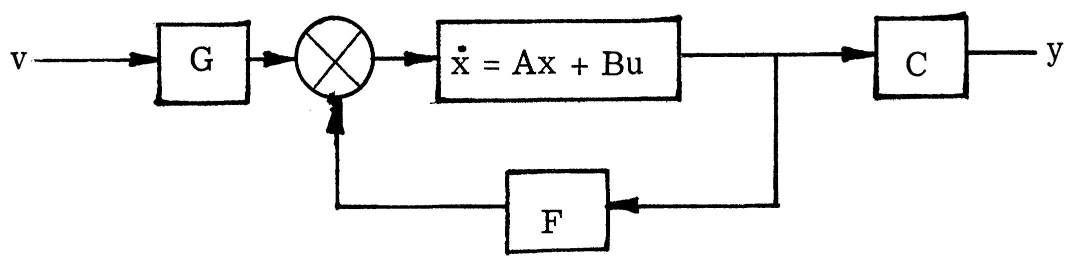

Figure C. 1 A Time-invariant Linear Feedback System

The plant is described by equations B. 9 with the control vector u, given by

$$u = Fx + Gv; \quad G \text{ is nonsingular.}$$

The closed-loop form is then given by

$$\dot{x} = (A + BF)x + BGv; \quad y = Cx.$$

Definition B. 9 The system is said to be decoupled if the i^{th} input affects only the i^{th} output.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be given by

$$\alpha_i = \min\{j: C_i A^j B \neq 0, \ j = 0,1,\ldots,n-1\} \text{ or}$$

$$\alpha_i = n-1 \text{ if } C_i A^j B = 0 \text{ for all } j,$$

where C_i is the i^{th} row of the matrix C. Let $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ denote the matrices

$$\tilde{A} = \begin{bmatrix} -C_1 A^{\alpha_1} - \\ \vdots \\ -C_m A^{\alpha_m} - \end{bmatrix}; \quad \tilde{B} = \tilde{A}B = \begin{bmatrix} -C_1 A^{\alpha_1} B - \\ \vdots \\ -C_m A^{\alpha_m} B - \end{bmatrix}; \quad D = \begin{bmatrix} \frac{d^{\alpha_1+1}}{dt^{\alpha_1+1}} & & 0 \\ & \ddots & \\ 0 & & \frac{d^{\alpha_m+1}}{dt^{\alpha_m+1}} \end{bmatrix}$$

The next two lemmas follow from [13].

Lemma B. 5 There exist a pair of matrices F and G which decouple the system if and only if $\tilde{B}$ is nonsingular. In particular the matrix pair

$$F = \tilde{B}^{-1}\tilde{A}A, \quad G = \tilde{B}^{-1}$$

satisfies the requirement of decoupling the system.

<u>Lemma B.6</u> If the matrix $\tilde{B}$ is nonsingular then an inverse system exists which can be represented by

$$\text{(S3)} \qquad \begin{aligned} v &= (\tilde{B}G)^{-1}\{Dy - \tilde{A}(A+BF)z\} \\ \dot{z} &= (I - B\tilde{B}^{-1}\tilde{A})A + B\tilde{B}^{-1}Dy. \end{aligned}$$

A question of interest is "What are the restrictions on the plant described by B.9,1 , 2 in the context of the previous discussion, to be positive real?" To help answer this question, the representation of the plant in the context of (S3) with $G = I$ and $F = 0$ (in the transformed domain) is given by

$$\begin{aligned} D(s)Y(s) &= \tilde{A}AX(s) + BU(s) \\ X(s) &= (sI - A)^{-1}BU(s), \end{aligned}$$

which combine to the form

$$D(s)Y(s) = \{\tilde{A}A(sI - A)^{-1}B + \tilde{B}\}U(s).$$

With some manipulation the plant transfer function simplifies to the form

$$P(s) = D^{-1}(s)\tilde{A}(I - A/s)^{-1}B$$

for all values of s for which the inverse exists. Since $D^{-1}(s)$ is diagonal it follows that

$$D^{-1}(s) = \sum_{i=1}^{m} s^{-(\alpha_i+1)} E_i$$

where E_i is the orthogonal projection on the ith coordinate subspace. Hence, we have the expression

$$P(s) = \sum_{i=1}^{m} s^{-(\alpha_i+1)} E_i\tilde{A}(I - A/s)^{-1}B. \qquad (B.11)$$

For large s, equation B.11 becomes

$$\lim_{s \to \infty} P(s) = \sum_{i=1}^{m} s^{-(\alpha_i+1)} E_i\tilde{A}B.$$

For P(s) to be positive real, it is necessary that all the poles on Re s = 0 be simple, hence $\alpha_i = 0$ for all i. Therefore, $C_iB \neq 0$ for any i. We can formulate the following:

Theorem B.7 A necessary condition for the plant described by equations B.9 to be positive real is that $C_iB \neq 0$ for all i.

Appendix C

A SUMMARY OF FOURIER ANALYSIS

C.1 Introduction

Throughout this Appendix L_p will denote the usual Lebesque space on the infinite interval $(-\infty, \infty)$. That is L_p consists of all measurable complex valued functions, f, of a real variable such that

$$||f|| = [\int_{-\infty}^{\infty} |f(t)|^p dt]^{1/p} < \infty$$

holds (with integration in the Lebesque sense). The cases $p = 1, 2$ will be of primary interest. The Hilbert space L_2 is equipped with the innerproduct (,) where

$$(x, y) = \int_{-\infty}^{\infty} x(s)\overline{y}(s)\, ds, \quad x, y \in L_2.$$

The following discussion extends easily to finite products of L_2. The notation L_2^n will denote the space of all tuplets

$$f = (f_1, \ldots, f_n), \quad f_i \in L_2$$

being finite with respect to the norm

$$||f|| = [\sum_{i=1}^{n} \int_{-\infty}^{\infty} |f_i(t)|^2 dt]^{\frac{1}{2}}.$$

Again L_2^n is a Hilbert space with the innerproduct < , > where

$$< f, g > = \sum_{i=1}^{n} \int_{-\infty}^{\infty} f_i(t)\overline{g}_i(t)dt, \quad f, g \in L_2^n.$$

L_2^n is equipped with the usual algebraic operations.

In dealing with physical systems an important notion is that of stationarity. Let D(T) denote the domain of T and K_a, for real a, denote the time translation operator

$$(K_a x)(t) = x(t-a), \quad x \in L_2^n.$$

Then T is <u>stationary</u> whenever

1. $D(T) = K_a D(T)$ all real a
2. $TK_a = K_a T$ all real a

In other words if $x \in D(T)$ and if $y = Tx$ then $K_a x \in D(T)$ and $K_a y = TK_a x$.

In dealing with stationary systems Fourier transforms play an important role. The Fourier transform $\mathcal{F}$ of a function $x \in L_2$ is to be defined by the expression

$$(\mathcal{F}x)(\omega) = \underset{N \to \infty}{\text{l.i.m.}} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-N}^{N} e^{j\omega t} x(t)dt. \qquad \text{(C.1)}$$

Here l.i.m. denotes limit in the mean. Some salient properties of F are summarized in the following theorem of Plancherel (See [54], page 51).

<u>Theorem C.1</u> As an operator on L_2, $\mathcal{F}$ is one-to-one, onto, and norm preserving. The inverse of $\mathcal{F}$ is determined by

$$(\mathcal{F}^{-1}y)(t) = \underset{N \to \infty}{\text{l.i.m.}} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-N}^{N} e^{-j\omega t} y(\omega)d\omega. \qquad \text{(C.2)}$$

In other words $\mathcal{F}$ is an isometric isomorphism of L_2 onto itself. A consequence of this is that for $x, y \in L_2$

$$(\mathcal{F}x, \mathcal{F}y) = (x, y)$$

holds, moreover $\mathfrak{F}^* = \mathfrak{F}^{-1}$. The Fourier transform is also extended to the space L_2^n in the natural way; the Fourier transform of a tuplet being the tuplet of component Fourier transforms. We remark only that the innerproduct relationship takes the form

$$< \mathfrak{F}x, \mathfrak{F}y > = \sum_{i=1}^{n} \int_{-\infty}^{\infty} (\mathfrak{F}x)_i(\omega)(\mathfrak{F}y)_i(\omega)\, d\omega$$
$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} x_i(t)\bar{y}_i(t)dt = < x, y >$$

The next result which is apparently due to Bochner (see [21]) is a cornerstone in the study of stationary systems. In this theorem $y = Tx$ is mapping from L_2^m into L_2^n while $\hat{y}$ and $\hat{x}$ denote the L_2 Fourier transform of y and x respectively. $\hat{T}$ denotes an $n \times m$ matrix of measurable functions.

<u>Theorem C.2</u> A necessary and sufficient condition for T to be linear bounded and stationary is that

$$\hat{y}(\omega) = \hat{T}(\omega)\hat{x}(\omega), \quad \omega \epsilon (-\infty, \infty).$$

where each component of the matrix $\hat{T}$ is a uniformly bounded measurable function.

This theorem may be paraphrased as stating that the transformation T from L_2^m into L_2^n is linear bounded and stationary, if and only if, T has a multiplicative form. Since $y = Tx \Longleftrightarrow \hat{y} = \hat{T}\hat{x}$ the matrix $\hat{T}$, which represents T in its multiplicative form can, as we see from the equality chain

$$\hat{y} = \mathcal{F}y + \mathcal{F}Tx = \mathcal{F}T\mathcal{F}^{-1}\mathcal{F}x$$

be represented as $\hat{T} = \mathcal{F}T\mathcal{F}^{-1}$.

C.2 Convolutions

The examples to be presented later come from the class of systems which may be identified with convolution operators. The convolution operator "$f\circledast$" is defined by

$$(f\circledast g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds, \quad t\in(-\infty,\infty) \tag{C.3}$$

The well known operational properties: $f\circledast g = g\circledast k$ and $f\circledast(g\circledast k) = (f\circledast g)\circledast k$ are easily verified. The convolution of a $n\times m$ matrix and a $m\times k$ matrix is defined in the obvious manner. In particular the case $m\times 1$ is of interest for, as we shall see, such convolutions can define the type of linear transformation under discussion.

The next theorem (see [12], pg. 951) isolates several important facts concerning convolutions on L_1 and L_2.

Theorem C.3 For f, $x\in L_1$ the convolution $f\circledast x$ is well defined and satisfies

$$\|f\circledast x\|_1 \leq \|f\|_1 \cdot \|x\|_1.$$

For $f\in L_1$, $x\in L_2$ the convolution $f\circledast x$ exists in L_2 and satisfies

$$\|f\circledast x\|_2 \leq \|f\|_1 \cdot \|x\|_2.$$

If f, $x\in L_2$ the convolution $f\circledast x$ defines a continuous function with norm (sup norm) at most $\|f\|_2 \cdot \|x\|_2$.

In view of equation C. 3 and this theorem it is clear that for $f \in L_1$ the convolution $f\circledast$ defines a bounded linear transformation on both L_1 and L_2. With the domain of $f\circledast$ being the entire space L_2, it is easily verified that this operator is stationary. The theorem also generalizes easily to the multivariate setting. For instance if W is a $n \times m$ matrix of functions $W_{ij} \in L_1$ such that $|W| \in L_1$ where $|W|(t) = |W(t)|$ denotes the norm of W(t) as a mapping from $\ell_1(m)$ into $\ell_1(n)$ then $W\circledast$ is a bounded linear stationary transformation sending L_2^m into L_2^n with norm satisfying

$$||W\circledast|| \leq \int_{-\infty}^{\infty} |W(t)|dt$$

Finally it is noted that $f\circledast$, as an operator on L_2, has a Hilbert space adjoint $(f\circledast)^*$. This adjoint is itself a convolution namely $(f\circledast)^* = \tilde{f}\circledast$ where $\tilde{f}(t) = f(-t)$, $t \in (-\infty, \infty)$. More generally for any bounded linear stationary system T, acting between finite products of L_2 with $\hat{T}$ the matrix multiplicative representation T, the identity chain

$$< T^*z, x > = < z, Tx > = < \hat{z}, \hat{T}\hat{x} > = < (\hat{T})^*\hat{z}, \hat{x} >$$

shows that $(\hat{T})^*$, the conjugate transpose of $\hat{T}$, is the multiplicative matrix representation of T^*.

C. 3 Causal Systems

Heuristically a noncausal system is one in which present values of the output are not influenced by future values of the input. To sharpen this somewhat let P_τ, for real τ, denote the projection operator defined by

$$(P_\tau x)(t) = \begin{cases} x(t) & -\infty < t \leq \tau \\ 0 & \tau < t \end{cases}$$

In other words P_τ is computed by multiplication with the characteristic function of the interval $(-\infty, \tau]$. A function T is said to be causal if for every $x_1, x_2 \in D(T)$ such that $P_\tau x_1 = P_\tau x_2$, for any real τ, then $P_\tau Tx_1 = P_\tau Tx_2$.

A convolution $f\circledast$ causal if and only if $f(t) = 0$ for $t \leq 0$. In this case with $y = f\circledast x$ we have

$$y(t) = \int_{-\infty}^{t} f(t-s)x(s)\,ds = \int_{0}^{\infty} f(s)x(t-s)\,ds, \quad t \in (-\infty, \infty)$$

and the output y at any $t \in (-\infty, \infty)$ clearly depends only on past values of the input. Similarly a convolution $f\circledast$, where $f(t) = 0$ for $t \geq 0$, is called purely anticausal. Evidently

$$(f\circledast x)(t) = \int_{t}^{\infty} f(t-s)x(s)ds, \quad t \in (-\infty, \infty)$$

and hence present output values depend only on future values of the input.

The Fourier transform representation of a causal convolution has a certain familiar and important property. To state this result we introduce the complex Fourier transform on L_2 by means of definition

$$\hat{x}(s) = (\mathfrak{F}x)(s) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega.$$

From the definition it is not difficult to show that if $x(t) = 0,\ t \leq 0$ then $\hat{x}(s)$ exists for $\sigma \geq 0$ and is analytic in the right half (i.e. $\sigma > 0$) of the complex plane.[1] Furthermore for $\sigma > 0$

$$\int_{-\infty}^{\infty} |x(\sigma + j\omega)|^2 d\omega = \int_{0}^{\infty} |x(t)|^2 e^{-2\sigma t} dt \leq ||x||^2 < \infty.$$

The converse of this result is also true as is stated in the next theorem (see [37] section 1.4).

<u>Theorem C.4</u> The two subsets of $L_2(-\infty, \infty)$

$$E = \{ x: x(t) = 0,\ t \leq 0\}$$

$$\hat{E} = \{x: \hat{x}(s) \text{ is analytic in } \sigma > 0,\ ||\hat{x}(\sigma)|| \leq ||x||.\ \sigma > 0\}$$

are identical. Moreover for $\hat{x} \in E$

$$\hat{x}(\omega) = \underset{\sigma \to 0^+}{\mathrm{l.i.m.}}\ \hat{x}(s).$$

where the limit exists for almost all $\omega \in (-\infty, \infty)$.

Theorem C.4 extends to L_2^n with norms replacing absolute values where necessary.

Suppose now that $x \in L_2$ satisfies $x(t) = 0$ for $t \geq 0$ and that y is the function defined by $y(t) = x(-t)$. Then $y \in L_2$ and $y(t) = 0$ for $t < 0$. The equality chain

$$\hat{x}(s) = \int_{0}^{\infty} e^{-st} x(t)dt = - \int_{0}^{\infty} e^{-st} x(-t)dt$$
$$= \int_{-\infty}^{0} e^{-st} y(t)dt = \hat{y}(-s)$$

---

[1] See Widder [55] pp. 80-81.

then shows that the properties set forth in Theorem C.4 for y in the half space $\sigma > 0$ hold also for $\hat{x}$ in the half space $\sigma < 0$. This observation establishes the corollary: the conjugate of a causal convolution is a purely anticausal convolution. The proof of this statement follows from the above remarks and the fact that if $\Phi(s) \leftrightarrow \phi\circledast$ then $\Phi(s) \leftrightarrow (\phi\circledast).^*$

Employing the preceding theorems it follows that stationary, causal convolutions associated with operators bounded on $L_2(-\infty, \infty)$ have complex Fourier transforms that are analytic for $\sigma > 0$. This is formalized by the following.

<u>Theorem C.5</u> Let g(t) be a continuous function for $t \epsilon (-\infty, \infty)$, such that $g\circledast$ is a causal stationary convolution bounded on $L_2(-\infty, \infty)$, then $\hat{g}(s)$ exists and is analytic for $\sigma > 0$.

Proof: Let $y = g\circledast x$ where x is chosen so that

$$x(t) = e^{st} \text{ for } t \leq t_f: \; s = \sigma + j\omega, \; \sigma > 0$$

$$x(t) = 0 \text{ for } t > t_f.$$

Then, since $g\circledast$ is causal and bounded on $L_2(-\infty, \infty)$,

$$y(t) = e^{st} \int_{-\infty}^{t} g(t-\tau) e^{-s(t-\tau)} d\tau$$

exists for almost all t and belongs to $L_2(-\infty, \infty)$.

Making the change in variable; $\lambda = t - \tau$,

$$y(t) = e^{st} \int_{0}^{\infty} g(\lambda) e^{-s\lambda} d\lambda, \; \sigma > 0.$$

Since y(t) exists for almost all t, it follows that $\hat{g}(s)$ exists, and converges for $\sigma > 0$. Moreover, $\hat{g}(s)$ is analytic[1] for $\sigma > 0$.

With a few additional assumptions placed on $\hat{g}(s)$, the converse of theorem C.5 is demonstrated.

<u>Theorem C.6</u> If $\hat{g}(s)$ is absolutely convergent for $\sigma > 0$ such that $\hat{g}(\omega)$ is essentially bounded for almost all $\omega \epsilon (-\infty, \infty)$, then $g\circledast$ is a causal stationary convolution bounded on $L_2(-\infty, \infty)$.

Proof: Let $x \epsilon L_2(o, \infty)$, then $\hat{x}(s)$ exists and converges absolutely for $\sigma > 0$. Hence,[2] $\hat{g}(s)\hat{x}(s) = \{\widehat{g\circledast x}\}(s) = \hat{y}(s)$. Since the l.i.m. as $\sigma \to 0$ of $\hat{g}$, $\hat{x}$, and $\hat{y}$ exist, then by theorems C.2 and C.4, $g\circledast$ is linear, bounded, causal, and stationary on $L_2(-\infty, \infty)$.

<u>Theorem C.7</u> If $g\circledast$ is a causal stationary convolution bounded on $L_2(-\infty, \infty)$ such that g(t) is continuous on $t \epsilon (0, \infty)$ and if $\hat{g}(s)$ consists of rational polynomials in s with real coefficients such that $\hat{g}(\infty) = \alpha$, then $\hat{g}(s)$ is analytic for $\sigma \geq 0$.

Proof: By hypothesis, the inverse complex Fourier transform of $\hat{g}(s)$ consists of finite linear combinations of terms of the form

$$g(t) = \alpha\,\delta(t) + t^n e^{-\gamma t}, \quad t \geq 0$$
$$= 0 \qquad\qquad\qquad\quad , \quad t < 0$$

where n is a nonnegative interger and γ is a complex constant. If there

[1] See Widder [55], pp. 57.

[2] See Widder [56], pp. 453.

exist any poles on the $j\omega$ axis, then g(t) can be written in the form

$$g(t) = g_1(t) + g_2(t)$$

where $g_1(t)$ consists of finite linear combinations of terms of the form

$$g_1(t) = t^n e^{j(\omega_m t + \gamma_m)}$$

Let $\tilde{y} = \tilde{g} * x$ be the response when $\tilde{g}(t) = t^n e^{j(\omega_o t + \gamma_o)}$ where $x(t) = 1$ for $t \in [0, 1]$ and zero otherwise. Then

$$\tilde{y}(t) = \int_0^1 \{t^n + nt^{n-1}\tau + \ldots + \tau^n\} e^{j(\omega_o t + \gamma_o)} e^{-j\omega_o \tau} d\tau, \; t \geq 1$$

$$= [a_n t^n + \ldots + a_o] e^{j(\omega_o t + \gamma_o)}.$$

where $a_n, \ldots, a_o$ are constants resulting from the integration.

Consequently, the total response at $\omega_m = \omega_o$ is the form

$$y(t) = [A_n t^n + \ldots A_o] e^{j\omega_o t}$$

where $A_n, \ldots, A_o$ are complex constants. Since y(t) must belong to $L_2(1, \infty)$, it is the case that $A_n, \ldots, A_o = 0$ and the theorem follows.

<u>Corollary C. 1</u> If $g*$ is a causal stationary convolution bounded on $L_2(-\infty, \infty)$ such that g(t) is continuous on $t \in (0, \infty)$ and if $\hat{g}(s)$ consists of rational polynomials in s with real coefficients such that $\hat{g}(\infty) = 0$, then $\hat{g}(s)$ is analytic for $\sigma \geq 0$ and $g(t) \in L_1(0, \infty) \cap L_2(0, \infty)$.

REFERENCES

1. Anderson, B. D. O., "Stability of Control Systems with Multiple Nonlinearities", Journal of the Franklin Institute, Vol. 282, No. 3, September 1966, p. 155.

2. Anderson, B. D. O., Newcomb, R. W., and Kalman, R. E., "Equivalence of Linear Time-Invariant Dynamical Systems", Journal of the Franklin Institute, Vol. 281, No. 5, May 1966.

3. Anderson, B. D. O., "Sensitivity Improvement Using Optimal Design", Proceedings I.E.E., Vol. 113, No. 6, June 1966.

4. Anderson, B. D. O., "A Control Theory Viewpoint of Positive Reality", Standford Electronics Laboratory, Technical Report No. 6558-4, February 1966.

5. Anderson, B. D. O., and Newcomb, R. W., "An Approach to the Time-Varying Sensitivity Problem", Standford Electronics Laboratory, Technical Report No. 6560-1, June 1966.

6. Anderson, B. D. O., "The Inverse Problem of Optimal Control", Standford Electronics Laboratory, Technical Report No. 6560-3, April 1966.

7. Bachman, G., and Narici, L., Functional Analysis, Academic Press, New York, 1966.

8. Bochner, S., and Chankrasekharan, K., Fourier Transforms, Annals of Mathematics Studies, No. 6.19, Princeton, 1949.

9. Bode, H. W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand Company Inc., Princeton, New Jersey, 1945.

10. Brockett, R. W. and Willems, J. L., "Frequency Domain Stability Criteria, Part I," I.E.E.E. Transactions, A.C., Vol. Ac-10, No. 3, July 1965, p. 225-261.

11. Cruz, Jr., J. B. and Perkins, W. R., "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design", I.E.E.E. Transactions on A.C., Vol. AC-9, No. 3, July 1964.

12. Dunford, N., and Schwartz, J. T., Linear Operators, Part II, Wiley, New York, 1963.

13. Falb, P. L., and Wolovich, W. A., "On the Decoupling of Multivariable Systems," 1967 J.A.C.C., Philadelphia, Pennsylvania.

14. Gantmacher, F. R., The Theory of Matrices, Volume 1, Chelsea Publishing Co., 1960.

15. Gilbert, E., "The Decoupling of Multivariable Systems by State Feedback," to be published in SIAM, 1969.

16. Goldberg, R. R., Fourier Transforms, Cambridge at the University Press, 1962.

17. Guillemin, E. A., Synthesis of Passive Networks, New York, John Wiley and Sons, Inc., 1959.

18. Guillemin, E. A., The Mathematics of Circuit Analysis, John Wiley and Sons, Inc., New York, 1959.

19. Hahn, W., Theory and Application of Lyapunov's Direct Method, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963.

20. Halmos, P. R., Measure Theory, D. Van Nostrand Company, Inc., Princeton, New Jersey.

21. Hille, E., Functional Analysis and Semi-Groups, American Math. Soc. Coll. Publ. Vol. 31, New York, 1948.

22. Horowitz, I. M., Synthesis of Feedback Systems, Academic Press, 1963.

23. Kalman, R. E., "When is a Linear Control System Optimal", Trans. Amer. Soc. Mech. Engrs., March 1964, 86, Part D, p. 1.

24. Kalman, R. E., "Contributions to the Theory of Optimal Control", Boln. Soc. Mat. Mex., 1960, p. 102.

25. Kalman, R. E. and Narendra, K. S., "Controllability of Linear Dynamical Systems", Contributions to Differential Equations, Vol. 1, No. 2, 1962.

26. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems", J. S.I.A.M. Control, Series A, Vol. 1, No. 2, 1963.

27. Kalman, R. E. and Bertram, J. E., "Control System Analysis and Design Via the "Second Method" of Lyapunov", Journal of Basic Engineering, June 1960, p. 371.

28. Kantorovich, L. V., Functional Analysis in Normed Spaces, Macmillan, New York, 1964.

29. Kokotovic, P. V. and Rutman, R. S., "Sensitivity of Automatic Control Systems", (Survey) Automation and Remote Control, Vol. 26, No. 4, April 1965.

30. Kolmogorov, A. N. and Fomin, S. V., Elements of the Theory of Functions and Functional Analysis, Vol. 1 and 2, Graylock Press, Rochester, N. Y., 1957.

31. Kreindler, E. and Sarachik, P. E. "On the Concepts of Controllability and Observability of Linear Systems", Trans. I.E.E.E. on A.C., AC-9, 1964.

32. Kreindler, E., "Closed-Loop Sensitivity Reduction of Linear Optimal Control Systems", Research Dept. Grummand Aircraft Engineering Corporation, Bethpage, New York.

33. Kuh, E. S. and Rohrer, R. A., Theory of Linear Active Networks, Holden-Day, San Francisco, 1967.

34. Liusternik, L. A. and Sobolev, V. J., Elements of Functional Analysis, Frederick Unger Publ. Co., New York, 1961.

35. Lyapunov, A., "Probléme général de la stabilite du mouvement", Ann. Math. Studies, No. 12, 1947.

36. Newcomb, R. W., Linear Multiport Synthesis, McGraw-Hill, New York, 1966.

37. Paley, R.E.A.C., and Wiener, N., Fourier Transforms in the Complex Domain, American Mathematical Soc. Colloquium Publ., Vol. 19, 1934.

38. Perkins, W. R. and Cruz, Jr., J. B., "Sensitivity Operators for Linear Time-Varying Systems", Proc. IFAC Symp. on Sensitivity Analysis, Proceedings of an International Symposium, Dubrovnik, 1964, Pergamon Press, 1966.

39. Perkins, W. R. and Cruz, Jr., J. B., "The Parameter Variation Problem in State Feedback Control Systems", Trans. ASME, Series D, Vol. 87, March 1965.

40. Popov, V. M., "Absolute Stability of Nonlinear Systems of Automatic Control", Automation and Remote Control, Vol. 22, No. 8, 1961, p. 961-979.

41. Porter, W. A., Parameter Sensitivity in Distributive Feedback Systems," Internal Journal of Control, Vol. 5, No. 5, 1967.

42. Porter, W. A., "Sensitivity Problems in Distributive Systems," Internal Nournal of Control, Vol. 5, No. 5, 1967.

43. Porter, W. A., "On the Reduction of Sensitivity in Multivariate Systems, Intl. J. Control, Vol. 5, No. 1, 1967, pp. 1-9.

44. Porter, W. A., "On Sensitivity in Multivariate Nonstationary Systems," International Journal of Control, Vol. 7, No. 5, 1968.

45. Porter, W. A., "Sensitivity Problems in Linear Systems," I.E.E.E. Trans. on A.C., Vol. AC-10, No. 3, July 1965.

46. Porter, W. A., "On the Matrix Riccati Equation," I.E.E.E. Trans. on A.C., Vol. AC-12, No. 6, December 1967.

47. Porter, W. A., "Some Structure Problems in Systems Analysis, Dept. of Electrical Engineering, Systems Engineering Laboratory, The University of Michigan, Ann Arbor, 1968.

48. Porter, W. A., Modern Foundations of System Engineering, Macmillan Publ. Co., 1966.

49. Riesz, F. and Sz. Nagy, B., Functional Analysis, F. Ungar Publ. Co., New York, 1955.

50. Schwartz, L., Théorie des distributions, Hermann, Paris, 1966.

51. Silverman, L. M. and Meadows, H. E., "Controllability and Observability in Time-Variable Linear Systems, SIAM Journal of Control, Vol. 5, No. 1, 1967.

52. Simmons, G. F., Introduction to Topology and Modern Analysis, McGraw-Hill, 1963.

53. Taylor, A. E., Introduction to Funcational Analysis, John Wiley and Sons, Inc., New York, 1964.

54. Titchmarsh, E. C., Theory of Fourier Integrals, Oxford at the Clarendon Press, 1937.

55. Widder, D. V., The Laplace Transform, Princeton University Press, 1946.

56. Widder, D. V., Advanced Calculus, Prentice-Hall, Englewood Cliffs, N. J.

57. Youla, D. C., Castriota, L. J., and Carlin, H. J., "Bounded Real Scattering Matrices and the Foundations of Linear Passive Network Theory," I. R. E. Trans. on Circuit Theory, March 1959.

58. Zaanen, A. C., Linear Analysis, Interscience Publishers, Inc., 1953.

59. Zadeh, L. A. and Desoer, C. A., Linear System Theory, McGraw-Hill Book Company, Inc., New York, 1963.