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STRUCTURAL INFERENCE AND IDENTIFICATION OF DISCRETE TIME SYSTEMS

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ABSTRACT

STRUCTURAL INFERENCE AND IDENTIFICATION
OF DISCRETE TIME SYSTEMS

by
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This dissertation develops a general theory of coordinatized sets and structured functions. This theory is then applied to structure inference and identification of discrete time systems with coordinatized state spaces.

The aforementioned inference and identification are based on partial data produced by the system. To every set of experiments there corresponds a family of structured partial models of the system. As experimentation progresses, a sequence of families of partial models is obtained. Those models and their interrelations are studied.

Several measures of model performance such as structural confidence, predictive range and confidence are introduced. Properties of these measures and their dependence on parameters are discussed. We show that the structural confidence measure for a sequence of partial models never decreases as the partial data set grows.

We show how the system model can be identified on special subsets

of the state space, given certain complexity bounds on system structure. The construction of a parameterized family of such subsets with desirable properties is described and their computational properties investigated.

Several experimentation strategies are suggested and their advantages and disadvantages discussed. A novel feature of these strategies is that they employ a methodology for predicting not-yet-observed state transitions which can be formally justified.

Finally, we point out that the theory developed provides a basis for computer aided methodology of model structure inference.

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In Memory of My Father.

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LIST OF SYMBOLS

Symbol	Meaning
D	index set
P_α	projection onto α-coordinate
$P_{D'}$	projection onto D' subset of D
$\Pi^S_{D'}$	basic partition induced by D' on S
$\leq$	refinement relation or inequality symbol
$\subseteq$	set inclusion
e.p.	extension property
s.e.p.	strong extension property
G(S)	graph of S
$L(\Pi)$	set of locations of Π
$f\|X$	restriction of a function f to X
Π_f	partition of kernel equivalence of f
$L(f)$	set of locations of f
L(f)	unique location of f
$F_{(S,R)}$	set of all functions from S to R
$E^{\bar{f}(X)}_{(S,R)}$	set of all extensions f of $\bar{f}$ from X to S, $f\varepsilon F_{(S,R)}$
$E^{\bar{f}(X)}_{(S,R)}(L,\cdot)$	set of all extensions f of $\bar{f}$ from X to S s.t. $f\varepsilon F_{(S,R)}$ and $L\varepsilon L(f)$
$E^{\bar{f}(X)}_{(S,R)}(\cdot,ub)$	set of all f s.t. $f\varepsilon E^{\bar{f}(X)}_{(S,R)}$ and there is an $\tilde{L}\varepsilon L(f)$ s.t. $\|\tilde{L}\| \leq ub$

$COMP_S^L(X)$	completion of X w.r.t. L and S
$F_{(X,R)}(L,\cdot)$	set of all functions $\bar{f}$ in $F_{(X,R)}$ s.t. $L\varepsilon\mathcal{L}(\bar{f})$
$F_{(X,R)}(\cdot,ub)$	set of all $\bar{f}$ in $F_{(X,R)}$ s.t. $\exists$ an $\bar{L}\varepsilon\mathcal{L}(\bar{f})$ with $\lvert\bar{L}\rvert \leq ub$
$F_{(X,R)}(L,ub)$	$F_{(X,R)}(L,\cdot) \cap F_{(X,R)}(\cdot,ub)$
X_L^y	$\{s \mid s\varepsilon S \text{ and } P_{D-L}(s) = P_{D-L}(y)\}$
Y_k^y	$\bigcup_{L\varepsilon\mathcal{L}_k} X_L^y$, where $\mathcal{L}_k = \{L \mid L \subseteq D \text{ and } \lvert L\rvert = k\}$
n	cardinality of D
$CONF_{(S,R)}^{\bar{f}(X)}(L,ub)$	confidence that L is a location of actual f under ub condition, where $f\varepsilon E_{(S,R)}^{\bar{f}(X)}$
$CONF_{(S,R)}^{\bar{f}(X)}(L)$	$CONF_{(S,R)}^{\bar{f}(X)}(L,ub)$, when $ub = n = \lvert D\rvert$
$ACONF_{(S,R)}^X(ub)$	average confidence on X relative to ub
$EXPL_{(S,R)}^{\bar{f}(X)}(\bar{L})$	explanatory range of $\bar{f}$ relative to $\bar{L}$
$PRED_{(S,R)}^{\bar{f}(X)}(\bar{L})$	predictive range of $\bar{f}$ relative to $\bar{L}$
$PCONF_{(S,R)}^{\bar{f}(X)}(\bar{L})$	predictive confidence of $\bar{f}$ relative to $\bar{L}$
$VAL_{(S,R)}^{\bar{f}(X)}(L)$	validity range of $\bar{f}$ w.r.t. L
P(A)	probability of an event A
P(A|B)	probability of an event A under condition B
E(X)	expected value of a random variable X
$\mathbb{R}$	reals
Bnd(S)	boundary of a set S
Int(S)	interior of a set S

CHAPTER I

INTRODUCTION

1.1 Review and Motivation

In many real life situations, we deal with systems for which only partial data is available to construct a system model. We often restrict our attention to a subclass of all possible models, constrained by considerations of elegancy and the sheer necessity to substantially cut down the search space to be explored.

In this context we consider the problem of modelling autonomous discrete time systems, with special emphasis on structure inference and identification.

It is assumed that the state space of the system is structured (coordinatized). Each system state thus is a finite tuple, where the coordinates represent the chosen attributes of the system.

Examples of such a system are: a bacterial cell, specified by the set of vectors giving the concentration levels of chosen molecule types and their transitions ([Z1]), digital networks ([HS1]) and tessellation automata ([YA1]). Systems with coordinatized state spaces have been discussed in [K1] and [ZW1].

Klir ([K1]) talks about the need for classifying appearances of observed attributes as an important aspect of any empirical investigation. He discusses the level of refinement (resolution level) used in observing system attributes. Zeigler and Weinberg ([ZW1]) use the above system description in simulation of a living bacterial cell (E. coli).

In this work we shall not be concerned with the way in which system attributes come about, but simply assume that they have been chosen and can be measured.

In a typical modelling situation, we seek structured models, whose state spaces are identical with the one assumed for the system and whose transition functions have been structured. Essentially, this means that the global transition function is thought of as composed of local (component) transition functions. With each coordinate transition function there is associated a subset of coordinates (its "influence set") such that the knowledge of the present states of "influencers" is sufficient to determine the next component state. A given transition function can be structured in many ways. We will refer to a family of influence sets (one influence set for every component) as a model structure. In general we must explore a multitude of structured models for a given system.

It is here that the choice of coordinatization can have important ramifications for modelling. Of special importance are the irredundant coordinatizations, which guarantee the uniqueness of a structured model with minimal interaction (i.e. minimal influencer sets).

We discuss various kinds of coordinatizations and their relationships. We also introduce a hierarchy of coordinatizations and discuss methods of irredundant set generation.

The framework used follows that of Zeigler ([Z1],[Z2]), where a formalism for handling structured automata is developed. Zeigler discusses coordinatizations of abstract sets and introduces the concept of structured functions. He points out potential implications for modelling of a state set coordinatization used.

After every set of experiments (or every data generation phase) we can construct a family of structured partial models--partial in the sense that in general they are defined only on a proper subset of the system's "operating range". This subset always includes the set of states visited in previous experiments and all partial models match the real system behavior on the latter set.

Partial models and various proposals for their performance evaluation and comparison of rival models have been discussed in [M1], [H1] and [G1]. Maciejowski ([M1]) suggests that the problem of choosing between rival models is the same as that of assessing confidence in the models--the model, in which one has higher confidence is the one to be preferred. Hanna ([H1]) suggests evaluation of learning models based on the information content of a model. Gaines ([G1]) proposes model evaluation based on complexity, with preference for less complex models. He also points out a trade-off between complexity of a model and the degree to which it approximates a given behavior.

We are interested in experimentation and construction strategies, which efficiently generate "credible" models. To this end, we define a probability measure reflecting, for every partial model, the confidence we have, that the total model of the system (i.e. defined on the whole state set) is identical to it in structure. All the measures are defined under the assumption that "everything we have not seen yet" is equally likely to happen. The degree of confidence is also used by Klir ([K2]) to evaluate correctness of structure candidates representing a given data system.

Alternate rounds of experimentation and model construction result in a sequence of families of partial models. In case that each of the experimental sets in the sequence is irredundant, every family consists of just one partial model. We show that in this case, there exists a total ordering (by inclusion) of partial model structures. Moreover, as long as the number of coordinates is finite, even if the state space is infinite, there is a point in the sequence beyond which each partial model has the same structure as the total system model. However, unless an upper bound on structure complexity is known and attained, we do not know at what point the structure has been identified.

This result is analogous to results in language and grammar identification ([G3], [F1]). Gold ([G3]) discusses concepts of language identification in the limit and finite language identification. Feldman ([F1]) discusses grammar identification in the limit and grammar approachability. In case of language identification in the limit, the learner guesses a language at each time. After finite time all the guesses are same and correct, but the learner does not necessarily know when his guess is correct and so must go on processing the information. Finite identification is analogous to our structure identification, when a bound on structure complexity is given and known to be attained.

In general, we show that the structural confidence measure for a sequence of partial models never decreases as the partial data grows. We define and discuss several other measures of model performance, in particular predictive range and confidence. Predictions are made for the predictive range of a partial model and are constrained by so-far-acquired data. They are made in belief that regularity detected in the data will continue to be present in the behavior of the system on the predictive range (i.e. that the structure will not change).

This is analogous in spirit to a methodology of predictions proposed by Klir ([Kl]). Klir suggests identification of time-invariant properties representing the data, by processing the empirical activity matrix (acquired data). Those properties are then used for generation of activity matrices (further data points). A rule of generation of further data is thus based on the same properties as the empirical activity matrix.

We show that the more a partial model is able to predict beyond the experimental set, on which it is constructed (the predictive range), the smaller our confidence must be that what it predicts is correct. But every misprediction on the predictive range is informative--it invalidates the hypothesis that the actual function has same structure as its observed portion, and so forces us to extend at least one of the influence sets. Thus if structural information is our main goal, the larger the predictive range the better.

Based on the theory developed, we offer a modeller several experimentation strategies based on various trade-offs between expected confidence, expected predictive range and computational complexity. Some of the strategies proposed use special domain-subset construction methods. Under certain upper bound conditions on system structure, those allow a modeller not only to determine model structure but also system transition function.

The theory developed here provides a basis for computer-aided methodology of model structure identification. Feasibility of a software package, aiding a modeller in constructing a system model, is largely due to the simplicity of evaluation of the proposed measures.

The problem of designing computer aids to help the modeller in dealing with plurality of partial models has been raised and discussed by Zeigler ([Z3]). Zeigler points out the requirements that need to be met by such a software system.

1.2 Organization

This dissertation is organized in two parts.

The first part consists of Chapters 2-4 and is concerned with the theory of coordinatizations and functions from structured domains. Chapter 2 develops a theory of coordinatizations. It introduces a hierarchy of coordinatizations. In particular irredundant coordinatizations are analyzed and ways of generating irredundant sets proposed.

Chapter 3 discusses properties of functions from structured domains. We introduce the concept of a location of a function and investigate properties of locations of a function, when restricted to a family of nested subsets. Based on those results we proceed to discuss determination of locations for a finite family of functions on the basis of a proper domain subset. Finally, methods for construction of special domain-subsets of a Cartesian domain with desirable properties are set forth.

Chapter 4 addresses itself to the problem of location inference. Methods of location inference for a function known on a proper domain subset only are discussed. Notions of structural confidence, average confidence on a subset, predictive range and predictive confidence are introduced. Computational methods for their evaluation are provided and their properties and dependence on parameters analyzed.

The second part consists of Chapter 5 and is devoted to the application of the theory developed in the first part to discrete time systems. It discusses structured transition functions, i.e. structured functions on coordinatized state space of a system. A concept of a partial system model is formalized and structural confidence, predictive range and predictive confidence for a partial model are discussed. A methodology for predicting state-transitions not yet observed is proposed. Ways of comparing rival partial models are suggested. Finally, several experimentation strategies are proposed and their advantages and disadvantages discussed.

The above parts are followed by Chapter 6, which summarizes the results obtained in them and suggests a number of further research topics. Finally, Appendix A discusses irredundance of open convex subsets of $\mathbb{R}^n$, with potential applications to identification of stochastic automata.

1.3 Some Notational Conventions

Each chapter in this dissertation is divided into sections. Section m of chapter n is numbered according to the scheme n.m. Theorems, lemmas, corollaries, etc. within each section n.m are numbered according to the scheme n.m.1 and delimited by the symbol $\square$.

Lines are tagged by numbers or lower case letters. References to a line tag made within the scope of a theorem, lemma, corollary, etc., are always local, unless otherwise specified. A referenced acknowledgement is provided whenever a theorem, definition, etc. is reproduced from another source; all other theorems, definitions, etc. are original to this dissertation.

The reader is referred to page x for a detailed list of symbols.

CHAPTER II

THEORY OF COORDINATIZATIONS

2.1 Introduction

In this chapter we develop the theory of coordinatizations of a single set. This development is largely based on work of B. P. Zeigler (see [Z1]). His formalism and definitions are used here as the starting point. The importance of this theory stems from the fact that the type of coordinatization of a state space of a discrete time system has important implications for modelling enterprise.

We explore here several types of coordinatizations and their interrelations. The spectrum of coordinatizations, which fall in between independent ones at one end and Cartesian ones at the other, is introduced. Irredundant coordinatizations will be particularly emphasized. Their special importance results from the fact, that when a state space of an autonomous discrete time system is irredundant, there is a unique structured model of this system with transition behaviour identical to that of the system.

We will demonstrate ways of constructing irredundant sets, for example using as constructing elements the sets already known to be irredundant, like the Cartesian ones.

It is often easier to determine whether a given set is irredundant or not, by looking at the way it is built from basic elements, rather than by using other criteria (like the one following directly from the definition of irredundance).

2.2 Types of Coordinatizations

The next few definitions and theorems follow those of [Z1], with only minor deviations.

We start with a concept of a structured set.

A set S is said to be structured if it is a subset of a cross product of an indexed family of sets, that is $S \subseteq \bigtimes_{\alpha\in D} S_\alpha$. With a structured set S, we associate a family of coordinate projections $\{P_\alpha | \alpha\in D\}$, where $P_\alpha : S \to S_\alpha$ is defined in the natural manner.

With index set D totally ordered, we extend the projections to project on all nonempty subsets of coordinates. Thus for any $D' \subseteq D$, $P_{D'} : S \to \bigtimes_{\alpha\in D'} S_\alpha$ and $P_{D'} = \bigtimes_{\alpha\in D'} P_\alpha$, where the order of coordinates is the one induced by the order of D. P_ϕ is defined to be any constant function with domain S.

From now on we assume that a set S we are dealing with is structured over a finite index set D, where cardinality of D is at least 2.

Definition 2.2.1 ([Z1])

A partition Π on S is said to be induced by a subset D' of D, if for every pair $x,y\in S$, $x\Pi y$ iff $P_{D'}(x) = P_{D'}(y)$. □

We will denote Π as above by $\Pi^S_{D'}$, and refer to it as a <u>basic partition</u> on S induced by D'. In case D' is a singleton, e.g. $D' = \{\alpha\}$, we will often write Π^S_α rather than $\Pi^S_{\{\alpha\}}$.

When it is clear what structured set we have in mind, we will sometimes write $\Pi_{D'}$, when $\Pi^S_{D'}$ is actually meant.

In general there might be more than one subset of D inducing the same partition on S. In other words the map $f: 2^D \to P^S$, defined by $f: D' \mapsto \Pi_{D'}$, is not one-to-one for arbitrary coordinatizations. As we shall see later however, this condition will turn out to be a necessary and sufficient one for S to be independent.

We remind the reader that P^S denotes the set of all partitions on S and P_D^S the set of all basic partitions on S.

When S is a proper subset of $\bigtimes_{\alpha \in D} P_\alpha(S)$, intervariable dependence may arise. This leads us to the following definition.

Definition 2.2.2 ([Z1])

Coordinate $\alpha \in D$ is dependent on S, whenever $\Pi_{D-\alpha}^S \leq \Pi_\alpha^S$. □

Coordinate $\alpha \in D$ is independent, if it is not dependent. We will say that S is coordinatized independently (or is independent) if all coordinates of D are independent.

It follows directly from the definition, that coordinate α is dependent on S, if for arbitrary tuple in S, knowledge of its projection onto $D-\alpha$ suffices to determine α-coordinate value of the tuple.

A concept of a location of a partition or function we are just about to introduce will play a central role in this thesis. With every partition Π on S, we associate a family of subsets of D with special properties-locations of Π.

Definition 2.2.3 ([Z1])

Let Π be a partition on S. $D' \subseteq D$ is a location of Π on S, if $\Pi_{D'}^S \leq \Pi$ and for any $D'' \subseteq D'$, if $\Pi_{D''}^S \leq \Pi$, then $D'' = D'$. □

We see that a location of Π is a minimal subset D' of D, such that the basic partition associated with D' refines Π.

In general a partition may have many locations. For special types of coordinatizations though, every partition on S has a unique location.

We will call such coordinatizations irredundant.

Definition 2.2.4 ([Z1])

A coordinatization of S is irredundant if every partition on S has a unique location. □

The next theorem states necessary and sufficient conditions for a set S to be irredundant.

Theorem 2.2.1 ([Z1])

A coordinatization of S is irredundant

a) iff for all $D_1, D_2 \subseteq D$, $\Pi^S_{D_1} \cup \Pi^S_{D_2} = \Pi^S_{D_1 \cap D_2}$

b) iff for all $D_1, D_2 \subseteq D$, $\Pi^S_{D_1 \cap D_2} \leq \Pi^S_{D_1} \cup \Pi^S_{D_2}$,

where $\Pi^S_{D_1} \cup \Pi^S_{D_2}$ is the transitive closure of the set union of $\Pi^S_{D_1}$ and $\Pi^S_{D_2}$.

Proof

Can be found in [Z1]. □

We remind the reader that if Π_1 and Π_2 are any relations on S, then the transitive closure of Π_1 and Π_2, $\Pi_1 \cup \Pi_2$, is defined by $s(\Pi_1 \cup \Pi_2)s' \Leftrightarrow \exists\, s_1, s_2, \ldots, s_n = s'$ such that $s_i \Pi_1 s_{i+1}$ for all odd i, and $s_i \Pi_2 s_{i+1}$ for all even i.

Definition 2.2.5

A coordinatization of S is Cartesian if $S = \bigtimes_{\alpha \in D} P_\alpha(S)$. □

Cartesian coordinatizations are always irredundant. This is proven in the next proposition.

Proposition 2.2.1

If S is Cartesian, then it is irredundant.

Proof

We need to show that for any $D_1, D_2 \subseteq D$, $\Pi_{D_1 \cap D_2} \leq \Pi_{D_1} \cup \Pi_{D_2}$. If $|S| = 1$, S is clearly irredundant. So we assume $|S| \geq 2$. Let $s, s' \varepsilon S$, where $s\Pi_{D_1 \cap D_2} s'$. We define z by $z(\alpha) = \begin{cases} s(\alpha) & \text{for } \alpha\varepsilon D_1 \\ s'(\alpha) & \text{for } \alpha\varepsilon D-D_1 \end{cases}$.

z is well defined and $z\Pi_{D_1} s$, $z\Pi_{D_2} s'$. This follows, since $D_2 = (D_2 \cap D_1) \cup (D_2 \cap (D-D_1))$ and $s\Pi_{D_1 \cap D_2} s'$. Thus $s\Pi_{D_1} z$ and $z\Pi_{D_2} s'$ implies that $s(\Pi_{D_1} \cup \Pi_{D_2})s'$, which was to be proved. □

We will now show that for any finite family of partitions on S with unique locations, the location of their intersection is also unique and equal to the union of the locations of all partitions in the family.

Lemma 2.2.1

Let $\Pi = \{\Pi_1, \Pi_2, \ldots, \Pi_n\}$ be a finite family of partitions on S, with the property that Π_i has a unique location L_i, for all $i = 1,\ldots,n$. Then the location of $\bigcap_{i=1}^{n} \Pi_i$ is unique and equal to $\bigcup_{i=1}^{n} L_i$.

Proof

We begin by showing that for arbitrary location L of $\bigcap_{i=1}^{n} \Pi_i$, $L \supseteq \bigcup_{i=1}^{n} L_i$. Since $\Pi_L \leq \bigcap_{i=1}^{n} \Pi_i \leq \Pi_i$, for $i = 1,\ldots,n$, L contains a location of Π_i, for all i. But Π_i's have unique locations and thus $L \supseteq L_i$, $\forall_i$, which implies that $L \supseteq \bigcup_{i=1}^{n} L_i$.

Since $\Pi_{L_i} \leq \Pi_i$, $\forall_i$, $\bigcap_{i=1}^{n} \Pi_{L_i} \leq \bigcap_{i=1}^{n} \Pi_i$. But $\bigcap_{i=1}^{n} \Pi_{L_i} = \Pi_{\bigcup_{i=1}^{n} L_i}$. Hence $\bigcup_{i=1}^{n} L_i$ contains a location of $\bigcap_{i=1}^{n} \Pi_i$, say $\tilde{L}$. Then $L \supseteq \tilde{L}$ and since both L and $\tilde{L}$ are locations of $\bigcap_{i=1}^{n} \Pi_i$, $L = \tilde{L}$. We showed then that for an arbitrary location L of $\bigcap_{i=1}^{n} \Pi_i$, $L \supseteq \bigcup_{i=1}^{n} L_i \supseteq L$ holds. Thus $L = \bigcup_{i=1}^{n} L_i$ is the unique location of the intersection. □

The next theorem gives several characterizations of an independent coordinatization.

We shall denote by P_D^S the set of all basic partitions on S. We note in passing, that for any $D'' \subseteq D' \subseteq D$, $\Pi_{D'} \leq \Pi_{D''}$.

<u>Theorem 2.2.2</u>

A coordinatization of S is independent

a) iff for all $\alpha \varepsilon D$, α is the unique location of Π_α.

b) iff for all $D' \subseteq D$, D' is the unique location of $\Pi_{D'}$.

c) iff for all $D', D'' \subseteq D$, if $D'' \neq D'$ then $\Pi_{D'} \neq \Pi_{D''}$.

<u>Proof</u>

a) First we show, that if for $\forall \alpha \varepsilon D$, α is the unique location of Π_α, then S is independent.

Suppose S is not independent. Then $\exists \alpha \varepsilon D$ s.t. $\Pi_{D-\alpha} \leq \Pi_\alpha$. This implies that $D-\alpha$ contains a location for Π_α, which is distinct from α. But this leads to a contradiction.

We now prove that if S is independent, then α is a unique location of Π_α, for all $\alpha \varepsilon D$.

Suppose this is not the case. Then $\exists$ an $\alpha \varepsilon D$ s.t. α is not a unique location of Π_α. Thus either α is not a location of Π_α or α is a location of Π_α, but not a unique one. In either case $\exists$ L s.t. L is a location of Π_α and $\alpha \notin L$. This implies that $L \subseteq D-\alpha$, implies that $\Pi_{D-\alpha} \leq \Pi_L \leq \Pi_\alpha$, which in turn implies that α is dependent. This however contradicts our assumption.

b) Clearly if for all $D' \subseteq D$, $\Pi_{D'}$ has a unique location D', then for all $\alpha \varepsilon D$, Π_α has a unique location α. So by part a) S is independent.

We need to show that if S is independent, then D' is the unique location of $\Pi_{D'}$, for all $D' \subseteq D$.

By part a) we know that independence of S implies that for every $\alpha \varepsilon D'$, Π_α has a unique location α. We note that $\Pi_{D'} = \bigcap_{\alpha \varepsilon D'} \Pi_\alpha$. Using finite induction and Lemma 2.2.1, we show that $\Pi_{D'}$ has a unique location D'.

c) We show first that if S is independent, then $D' \neq D''$ implies $\Pi_{D'} \neq \Pi_{D''}$. Since S is independent $\Pi_{D'}$ has a unique location D' and $\Pi_{D''}$ has a unique location D''. This follows from part b) of the theorem. Since $D' \neq D''$, $\Pi_{D'}$ and $\Pi_{D''}$ have distinct locations and thus cannot be equal.

We now prove that if for $\forall$ D',D'' s.t. $D' \neq D''$, $\Pi_{D'} \neq \Pi_{D''}$, then S is independent. Suppose this is not the case. Then $\exists$ $\alpha \varepsilon D$ s.t. α is dependent, i.e. $\Pi_{D-\alpha} \leq \Pi_\alpha$. This implies that $\Pi_{D-\alpha} \cap \Pi_\alpha = \Pi_{(D-\alpha)\cup\{\alpha\}} = \Pi_D = \Pi_{D-\alpha}$, which contradicts our assumption. □

There are other necessary and sufficient conditions for irredundance besides those of Theorem 2.2.1. Which condition is used as an irredundance criterion depends of course on a particular situation involved.

The next theorem states those conditions.

Theorem 2.2.3

A coordinatization of S is irredundant

a) iff for $\forall\ \Pi \varepsilon P^S$ with $\Pi \neq I$, $\bigcap_{\Pi_{D'} \leq \Pi} D' = L \neq \phi$ and $\Pi_L \leq \Pi$.

b) iff for $\forall \Pi \varepsilon P^S$, $\forall\ D_1, D_2 \subseteq D$, if $\Pi_{D_1} \leq \Pi$ and $\Pi_{D_2} \leq \Pi$, then $\Pi_{D_1 \cap D_2} \leq \Pi$.

Proof

a) If coordinatization is irredundant, then any D' s.t. $\Pi_{D'} \leq \Pi$ contains the unique location of Π, say L, for any $\Pi \neq I$. Since $\Pi \neq I$, $L \neq \phi$. Also clearly $\Pi_L \leq \Pi$ and thus $\bigcap_{\Pi_{D'} \leq \Pi} D' = L \neq \phi$.

We now need to show that if for every $\Pi \neq I$, $\bigcap_{\Pi_{D'} \leq \Pi} D' = L \neq \phi$ and $\Pi_L \leq \Pi$, then S is irredundant. It suffices to show that L is the unique location of Π. Suppose $\tilde{L}$ is an arbitrary location of Π. Then $\Pi_{\tilde{L}} \leq \Pi \Rightarrow \tilde{L} \supseteq L$. Since $\Pi_L \leq \Pi$, L contains a location of Π, say $\hat{L}$. But then $\tilde{L} \supseteq \hat{L}$, and since $\tilde{L}$ and $\hat{L}$ are both locations, $\tilde{L} = \hat{L}$. Since $\tilde{L} \supseteq L \supseteq \hat{L}$ holds, this implies $\tilde{L} = L$. Thus L is the unique location of Π.

b) We first assume that S is irredundant and show that for any Π on S, any $D_1, D_2 \subseteq D$, if $\Pi_{D_1} \leq \Pi$ and $\Pi_{D_2} \leq \Pi$, then $\Pi_{D_1 \cap D_2} \leq \Pi$. This follows immediately from Theorem 2.2.1, since $\Pi_{D_1} \leq \Pi$, $\Pi_{D_2} \leq \Pi$ $\Longrightarrow \Pi_{D_1} \cup \Pi_{D_2} \leq \Pi$ and by Theorem 2.2.1 $\Pi_{D_1} \cup \Pi_{D_2} = \Pi_{D_1 \cap D_2}$.

We now show that if for every $\Pi \varepsilon P^S$, $\forall\ D_1, D_2 \subseteq D$ $\Pi_{D_1} \leq \Pi$ & $\Pi_{D_2} \leq \Pi$ implies $\Pi_{D_1 \cap D_2} \leq \Pi$, then S is irredundant.

Suppose S is not irredundant. Then $\exists$ a Π on S, $\Pi \neq I$, with at least two different locations, say L_1 and L_2. Then $\Pi_{L_1} \leq \Pi$, $\Pi_{L_2} \leq \Pi$. By our assumption $\Pi_{L_1 \cap L_2} \leq \Pi$ and thus $L_1 \cap L_2$ contains a location of Π. Since $L_1 \supseteq L_1 \cap L_2$, $L_2 \supseteq L_1 \cap L_2$ and both L_1, L_2 are locations of Π, $L_1 = L_1 \cap L_2$ and $L_2 = L_1 \cap L_2$. This implies $L_1 = L_2$, which contradicts our hypothesis. □

It turns out that for any coordinatization there exists a relation between the number of indpendent coordinates and the cardinality of the coordinatized set. This is expressed in the next proposition.

Proposition 2.2.2 ([Z1])

Let $\{S_\alpha | \alpha \varepsilon D\}$ be a coordinatization of a finite set S. Let $D' \subseteq D$ be any independent subset of D. Then $|D'| \leq |S| - 1$.

Proof

See [Z1]. □

The implication of the above proposition is that if S is independent, then the cardinality of S is at least one greater than the cardinality of the index set, S is coordinatized over.

We will now illustrate the concepts introduced by a few examples.

Example 2.2.1

Consider $S \subseteq \{a,d\} \times \{b,e\} \times \{c,f\}$, where $D = \{1,2,3\}$ and $S = \{(a,b,c),(a,e,c),(d,b,f),(d,e,f),(a,e,f)\}$. First we list all the basic partitions on S.

$$\Pi_1^S = \{\overline{(a,b,c),(a,e,c),(a,e,f)},\ \overline{(d,b,f),(d,e,f)}\}$$

$$\Pi_2^S = \{\overline{(a,b,c),(d,b,f)},\ \overline{(a,e,c),(d,e,f),(a,e,f)}\}$$

$$\Pi_3^S = \left\{\overline{(a,b,c),(a,e,c)},\ \overline{(d,b,f),(d,e,f),(a,e,f)}\right\}$$

$$\Pi_{\{1,2\}}^S = \left\{\overline{(a,e,c),(a,e,f)},\ \overline{(a,b,c)},\ \overline{(d,b,f)},\ \overline{(d,e,f)}\right\}$$

$$\Pi_{\{1,3\}}^S = \left\{\overline{(a,b,c),(a,e,c)},\ \overline{(d,b,f),(d,e,f)},\ \overline{(a,e,f)}\right\}$$

$$\Pi_{\{2,3\}}^S = \left\{\overline{(a,b,c)},\ \overline{(a,e,c)},\ \overline{(d,b,f)},\ \overline{(d,e,f),(a,e,f)}\right\}$$

$$\Pi_\phi = I = \left\{\overline{(a,b,c),(a,e,c),(d,b,f),(d,e,f),(a,e,f)}\right\}$$

$$\Pi_{\{1,2,3\}} = 0 = \left\{\overline{(a,b,c)},\ \overline{(a,e,c)},\ \overline{(d,b,f)},\ \overline{(d,e,f)},\ \overline{(a,e,f)}\right\}$$

It can be easily verified that S is independent. The easiest way to do it is to use part c) of Theorem 2.2.2.

We will now demonstrate that although S is independent, it is not irredundant. To do so, it suffices to exhibit one partition on S with more than one location.

Take $\Pi = \left\{\overline{(a,b,c)},\ \overline{(a,e,c),(d,b,f),(d,e,f),(a,e,f)}\right\}$.

We check that $\Pi_{\{1,2\}}^S \leq \Pi$ and $\Pi_{\{2,3\}}^S \leq \Pi$, while Π_1^S, Π_2^S and Π_3^S do not refine Π. Thus $\{1,2\}$ and $\{2,3\}$ are locations of Π and S is not irredundant.

□

Example 2.2.2

Consider $S = \{(a,b,a),(b,b,b),(b,b,a),(a,b,b)\}$, $D = \{1,2,3\}$. We note that $P_1(S) = \{a,b\}$, $P_2(S) = \{b\}$ and $P_3(S) = \{a,b\}$. Clearly, $S = P_1(S) \times P_2(S) \times P_3(S)$ and thus by Definition 2.2.5 is Cartesian. □

Example 2.2.3

Consider any open circle C in $\mathbb{R}^2$. C is irredundant. To show that we need to prove that for any $x = (x_{\alpha_1}, x_{\alpha_2})$ and any $y = (y_{\alpha_1}, y_{\alpha_2})$, where $x,y \in C$ $x \Pi_{\alpha_1} \cup \Pi_{\alpha_2} y$.

From the definition of a circle it follows that $x_{\alpha_1}^2 + x_{\alpha_2}^2 < r^2$ and $y_{\alpha_1}^2 + y_{\alpha_2}^2 < r^2$ both hold. (r is the radius of the circle C.) Hence clearly $(x_{\alpha_1}^2 + y_{\alpha_2}^2) + (y_{\alpha_1}^2 + x_{\alpha_2}^2) < 2r^2$ is true. But then either $x_{\alpha_1}^2 + y_{\alpha_2}^2 < r^2$ or $y_{\alpha_1}^2 + x_{\alpha_2}^2 < r^2$ must hold (or both). W.l.o.g. assume that $x_{\alpha_1}^2 + y_{\alpha_2}^2 < r^2$. Let $p = (x_{\alpha_1}, y_{\alpha_2})$. $p \varepsilon C$ and $p \Pi_{\alpha_1} x$, $p \Pi_{\alpha_2} y$ hold. Thus $x \Pi_{\alpha_1} \cup \Pi_{\alpha_2} y$, which proves irredundance of C. □

For the illustration of Example 2.2.3 refer to Figure 2.2.1.

We remark that the above result can be generalized for any open ball in $\mathbb{R}^n$.

<u>Example 2.2.4</u>

Consider any open convex subset S of the plane, $\mathbb{R}^2$. S is thought of as coordinatized in a natural manner. We will denote the coordinates of D by α_1 and α_2. We will prove that such an S is irredundant.

For let p_1 and p_2 be any two points of S. Since S is convex the line segment joining p_1 and p_2, L_{p_1,p_2} is in S. Since S is open, for every x on L_{p_1,p_2} there exists at least one open ball with center x contained in S. We will denote it by $B(x)$.

Let $\Psi = \{B(x) | x \varepsilon L_{p_1,p_2}\}$. Then clearly Ψ forms an open cover of L_{p_1,p_2}. L_{p_1,p_2} is compact and thus Ψ contains a finite subcover of L_{p_1,p_2}. We will denote this finite subcover by $C = \{B_1(x_1), B_2(x_2), \ldots, B_n(x_n)\}$, where $x_i \varepsilon L_{p_1,p_2}$ for $i = 1,\ldots,n$, for some integer n. Without loss of generality we will assume that the enumeration of x_i's is such that $d(x_{i+1}, p_2) < d(x_i, p_2)$, where d is the usual Euclidean distance in $\mathbb{R}^2$. Also we remove from our finite cover all open balls properly contained in other balls. Then any two neighboring balls must intersect, i.e. $B(x_i) \cap B(x_{i+1}) \neq \phi$. If this were not the case the balls would not form an open cover. Choose any $z_1 \varepsilon B(x_1) \cap B(x_2)$, $z_2 \varepsilon B(x_2) \cap B(x_3), \ldots, z_{n-1} \varepsilon B(x_{n-1}) \cap B(x_n)$. Since as was shown in Example 2.2.3 any open ball in $\mathbb{R}^2$ is irredundant, the following holds: $p_1 \Pi_{\alpha_1} \cup \Pi_{\alpha_2} z_1 \Pi_{\alpha_1} \cup \Pi_{\alpha_2} z_2 \ldots z_{n-2} \Pi_{\alpha_1} \cup \Pi_{\alpha_2} z_{n-1} \Pi_{\alpha_1} \cup \Pi_{\alpha_2} p_2$. This in turn implies that $p_1 \Pi_{\alpha_1} \cup \Pi_{\alpha_2} p_2$. Thus S is irredundant (refer to Figure 2.2.2). □

Remark

We note in passing that the above result can be quite easily extended to $\mathbb{R}^n$. Namely, every open convex subset of $\mathbb{R}^n$ is irredundant, for an arbitrary integer n. Also it can be shown that $S_n = \{(p_1, \ldots, p_n) | p_i \geq 0, \forall_i, \text{ and } \sum_{i=1}^{n} p_i \leq \text{constant}\}$ is irredundant. For the proof the reader is referred to Appendix A. This fact has implications for identification of probabilistic automata and other types of systems dealing with concentrations, populations, etc. □

We will now turn to other types of coordinatizations.

For lack of a better name we will refer to the first three of them as coordinatizations of type 1, type 2 and type 3.

Definition 2.2.6

A coordinatization of S is of

a) type 1 if for all $\alpha \varepsilon D$, $\Pi^S_{D-\alpha} \cup \Pi^S_{\alpha} = I$

b) type 2 if for all $D_1, D_2 \subseteq D$ s.t. $D_1 \cap D_2 = \phi$, $\Pi^S_{D_1} \cup \Pi^S_{D_2} = I$

c) type 3 if for all $\Pi \varepsilon P^S, \Pi \neq I, \bigcap_{L \varepsilon \mathcal{L}} L \neq \phi$, where $\mathcal{L}$ is the family of all locations of Π. □

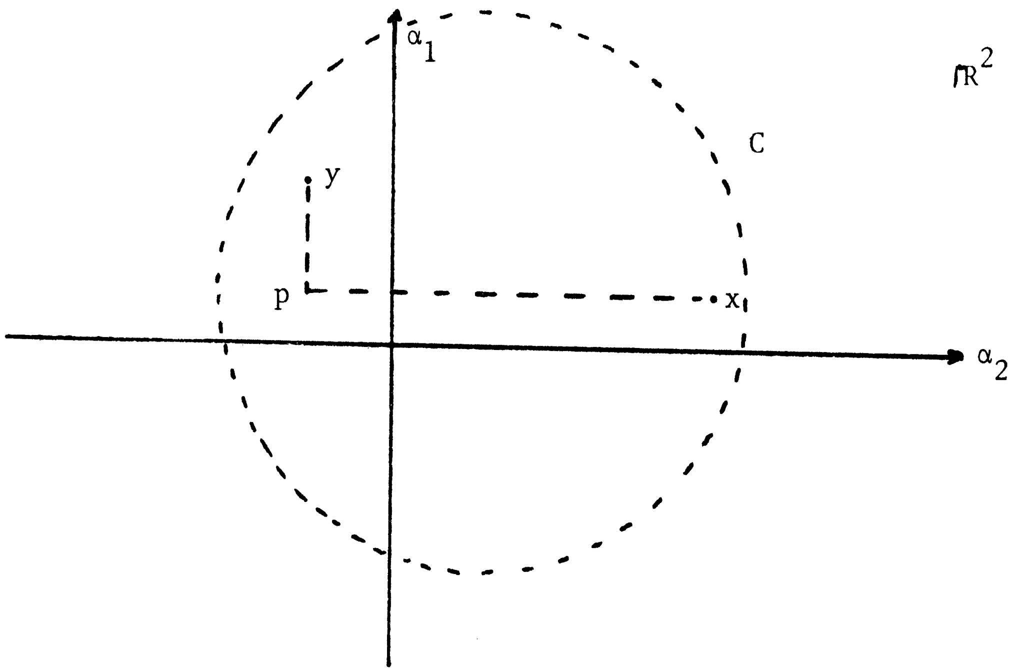


Figure 2.2.1: An Open Circle in $\mathbb{R}^2$ Is Irredundant.

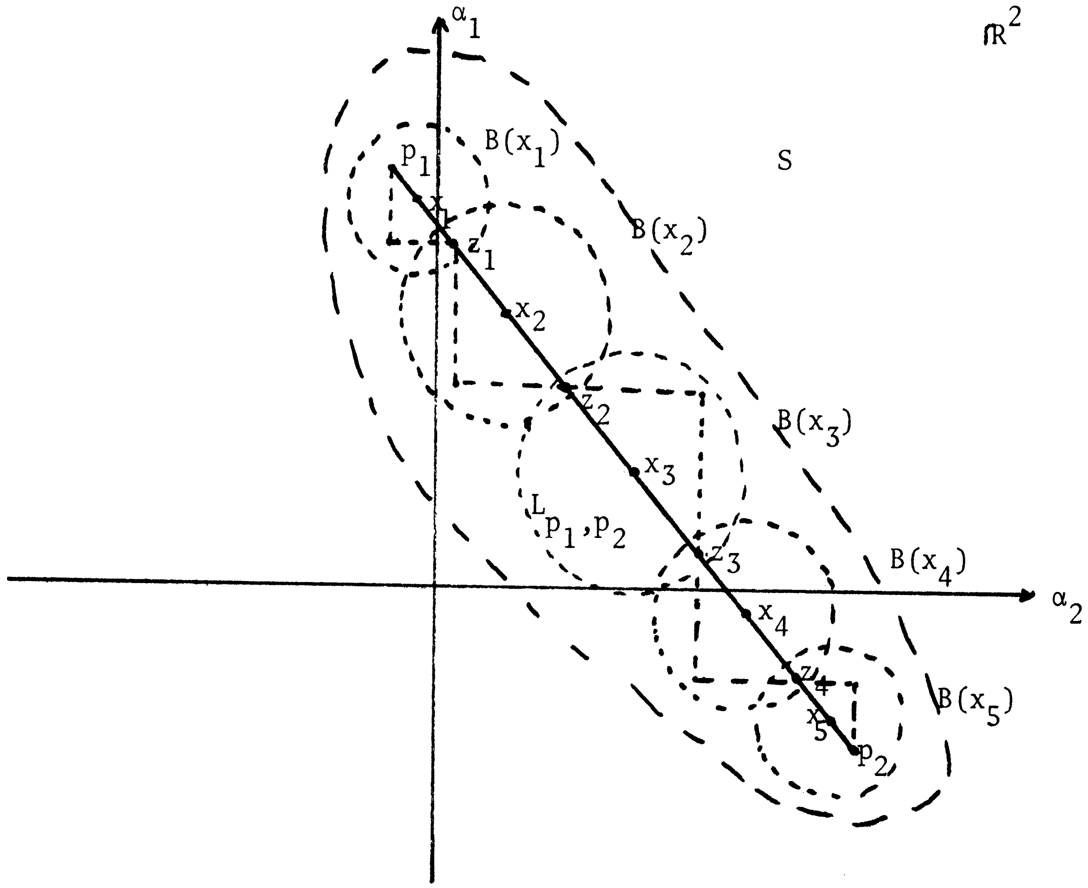


Figure 2.2.2: Every Open Convex Set S in $\mathbb{R}^2$ Is Irredundant.

We will now elaborate some more on the properties of the coordinatizations just introduced.

Proposition 2.2.3

A coordinatization of S is of type 1 iff for every $\Pi \varepsilon P^S$ if α is a location of Π, for some $\alpha \varepsilon D$, then α is the only location of Π.

Proof

We first show that if S is of type 1 (i.e. $\Pi_\alpha \cup \Pi_{D-\alpha} = I, \forall \alpha \varepsilon D$) and α is a location of Π, then α is the only location of Π. Clearly $\Pi \neq I$, for I has the unique location ϕ. Suppose $L \neq \alpha$ is a location of Π. Since $\alpha \notin L$, $L \subseteq D-\alpha$ and $\Pi_L \leq \Pi$. $L \subseteq D-\alpha \Rightarrow \Pi_{D-\alpha} \leq \Pi_L$ and $\Pi_\alpha \cup \Pi_{D-\alpha} = I \Rightarrow \Pi_\alpha \cup \Pi_L = I$. But $\Pi_\alpha \leq \Pi$, $\Pi_L \leq \Pi \Rightarrow \Pi_\alpha \cup \Pi_L \leq \Pi$. This in turn implies that $\Pi = I$ which is a contradiction to our assumption.

We now show that if for every Π on S, s.t. α is a location of Π, for some $\alpha \varepsilon D$, α is its unique location, S is of type 1, i.e. that $\Pi_\alpha \cup \Pi_{D-\alpha} = I$, for all $\alpha \varepsilon D$. Suppose this is not so. Then $\exists\, \alpha \varepsilon D$, s.t. $\Pi_\alpha \cup \Pi_{D-\alpha} \neq I$. $\Pi_\alpha \leq \Pi_\alpha \cup \Pi_{D-\alpha}$ and $\Pi_{D-\alpha} \leq \Pi_\alpha \cup \Pi_{D-\alpha}$. This clearly implies that α contains a location of $\Pi_\alpha \cup \Pi_{D-\alpha}$ and so does $D-\alpha$. Since $\Pi_\alpha \cup \Pi_{D-\alpha} \neq I$, α is a location itself. Thus α and L are distinct locations of $\Pi_\alpha \cup \Pi_{D-\alpha}$, where L is some subset of $D-\alpha$. This contradicts our assumption. Hence S is of type 1. □

Proposition 2.2.3 proves that S is of type 1 if and only if all partitions with singleton locations have unique locations.

We now proceed to show that S is of type 2 if and only if for any partition on S, no two of its locations are disjoint.

Proposition 2.2.4

A coordinatization of S is of type 2 iff for every $\Pi\varepsilon P^S$, if L_1 and L_2 are any distinct locations of Π, then $L_1 \cap L_2 \neq \phi$.

Proof

We first prove that if S is of type 2 (i.e. for any $D_1, D_2 \subseteq D$ with $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$) and if Π is any partition on S with more than one location, then for any two distinct locations L_1 and L_2 of Π, $L_1 \cap L_2 \neq \phi$. Clearly $\Pi \neq I$, since I has the unique location ϕ. Suppose $L_1 \cap L_2 = \phi$. Since L_1, L_2 are locations of Π, $\Pi_{L_1} \cup \Pi_{L_2} \leq \Pi$. But by our assumption $\Pi_{L_1} \cup \Pi_{L_2} = I$, which implies $\Pi = I$. This however is a contradiction.

Secondly we show that if for any Π with more than one location, intersection of any two of its locations is nonempty, then for all D_1, D_2 s.t. $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$, i.e. that S is of type 2. Suppose not. Then $\exists\, D_1, D_2$ s.t. $D_1 \cap D_2 = \phi$, but $\Pi_{D_1} \cup \Pi_{D_2} \neq I$. Let $\Pi = \Pi_{D_1} \cup \Pi_{D_2}$. Then D_1 contains a location of Π, say L_1, and D_2 contains a location of Π, say L_2. L_1 and L_2 are different from ϕ, since $\Pi \neq I$. $D_1 \cap D_2 = \phi \Rightarrow L_1 \cap L_2 = \phi$. This however is a contradiction. Thus S is of type 2, which was to be proved. □

The next two types of coordinatizations are motivated graphically. These are coordinatizations with extension property (e.p.) and strong extension property (s.e.p.) defined below.

Definition 2.2.7

Let $\{S_{\alpha_i} | \alpha_i \varepsilon D\}$ be a coordinatization of S, where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the total ordering of D. S is said to have

a) e.p. if for all $x,y \in S$ and any $i \in \{1,\ldots,n\}$ if $x \Pi_{\alpha_i} y$, then $w,z \in S$, where $w(\alpha_j) = \begin{cases} x(\alpha_j) & \text{for } j \leq i \\ y(\alpha_j) & \text{for } j > i \end{cases}$ and

$$z(\alpha_j) = \begin{cases} y(\alpha_j) & \text{for } j \leq i \\ x(\alpha_j) & \text{for } j > i \end{cases}$$

b) s.e.p. if S has e.p. for all permutations of D. □

Every set S with e.p. has a graphical representation G(S) by a multi-level graph. G(S) is obtained in the following manner. The set of vertices V equals $\bigcup_{\alpha_i \in D} S_{\alpha_i}$, where all S_{α_i} are treated as distinct. Every edge of G(S) represents a subset of $S_{\alpha_i} \times S_{\alpha_{i+1}}$, for some $1 \leq i \leq n-1$, where n is the cardinality of the index set D. There is an edge joining $(s_{\alpha_i}, s_{\alpha_{i+1}})$ in G(S) just in case there is a point in S with the projection onto $\{\alpha_i,\alpha_{i+1}\}$ equal to $(s_{\alpha_i},s_{\alpha_{i+1}})$. Every point s of S is thus represented by a path $(s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_n})$ of the graph. It is clear from its description that this representation is one-to-one. Thus given G(S) of the type as above we can determine S uniquely.

To make the representation more readable we will align the vertices along vertical lines. This is illustrated by Figure 2.2.3.

<u>Remark</u>

Every set S coordinatized over an index set D with $|D| = 2$, has a one-to-one graphical representation G(S). This is so, because every such S has e.p.

We note that the above type of graphical representation does not provide a one-to-one map of sets without e.p. Looking at G(S) we cannot "retrieve" S, but only its closure with respect to e.p. By the closure of S w.r.t. e.p. we mean the smallest subset X of $\bigtimes_{\alpha \in D} S_\alpha$, such that $X \supseteq S$ and X has e.p. It can be easily verified that the above closure is unique. □

We will say that a coordinate $\alpha \varepsilon D$ is independent of the coordinates $D' \subseteq D-\alpha$ on S if $\Pi_{D'} \not\leq \Pi_\alpha$. α is independent on S then, if it is independent of coordinates $(D-\alpha)$ on S.

For sets with e.p. "local independence" implies independence. This is stated more formally below.

Proposition 2.2.5

Let S have e.p. and let $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ be the ordering of D. Then S is independent iff α_i is independent of its neighbors, for $i = 1,\ldots,n$, where the neighbors of α_i are α_{i-1} and α_{i+1} if $2 \leq i \leq n-1$, and the neighbors of α_1 are α_2 and α_n and of α_n, α_{n-1} and α_1.

Proof

If α_i is independent on S, then it is clearly independent of its neighbors.

We need to prove the implication the other way. Let $i\varepsilon(2,\ldots,n-1)$. α_i independent of its neighbors $\Rightarrow \exists\, x,y \varepsilon S$ with $x_{\alpha_{i-1}} = y_{\alpha_{i-1}}$ and $x_{\alpha_{i+1}} = y_{\alpha_{i+1}}$ but $x_{\alpha_i} \neq y_{\alpha_i}$. But then $\exists\, z,w \varepsilon S$ s.t. $z_{\alpha_k} = w_{\alpha_k}$ for $k \neq i$, where $z_{\alpha_{i-1}} = w_{\alpha_{i-1}} = x_{\alpha_{i-1}} = y_{\alpha_{i-1}}$ and $z_{\alpha_{i+1}} = w_{\alpha_{i+1}} = x_{\alpha_{i+1}} = y_{\alpha_{i+1}}$ and $z_{\alpha_i} = x_{\alpha_i}$, $w_{\alpha_i} = y_{\alpha_i}$. This follows from e.p. property of S. Thus $z \Pi_{D-\alpha_i} w$ but $z \not\!\Pi_{\alpha_i} w$, which implies independence of α_i on S. The proof for α_1 and α_n follows in a similar way. □

We will now take a closer look at coordinatizations with s.e.p. First, we will establish some auxiliary results to be used later.

Lemma 2.2.2

Let $S \subseteq \bigtimes_{i=1}^{n} S_{\alpha_i}$, where S has e.p. Then S has the property, that for any $x,y\varepsilon S$, any α_i, if $x\Pi_{\alpha_i} y$, then for any α_j

a) $(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{j-1}}, x_{\alpha_j}, y_{\alpha_{j+1}}, \ldots, y_{\alpha_n}) \varepsilon S$

and

b) $(y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{j-1}}, y_{\alpha_j}, x_{\alpha_{j+1}}, \ldots, x_{\alpha_n}) \varepsilon S$ hold

$\Longleftrightarrow$ for any two disjoint subsets of $D = \{\alpha_1, \ldots, \alpha_n\}$, D' and D'' s.t.

$D' \cup D'' = D$ and any point z s.t. $z(\alpha_i) = \begin{cases} x_{\alpha_i} & \text{if } \alpha_i \varepsilon D' \\ y_{\alpha_i} & \text{if } \alpha_i \varepsilon D'' \end{cases}$, $z\varepsilon S$.

Proof

Clearly if any z as above is in S, conditions a) and b) are met.

We have to prove that if S has e.p. and a) and b) hold then any z as above is in S. Let x,y,α_i be such that $x\Pi_{\alpha_i} y$.

Let $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}$ denote the coordinates of D at which z changes from x to y or y to x, ordered by order of D. W.l.o.g. assume $z_{\alpha_i} = x_{\alpha_i}$ for all $\alpha_i < \alpha_{i_1}$, $z_{\alpha_i} = y_{\alpha_i}$ for $\alpha_{i_1} \leq \alpha_i < \alpha_{i_2}$, $z_{\alpha_i} = x_{\alpha_i}$ for $\alpha_{i_2} \leq \alpha_i < \alpha_{i_3}$, etc. Then if a) and b) hold, $x\Pi_{\alpha_i} y \Rightarrow z_1 =$ $(x_{\alpha_1}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_n}) \varepsilon S$. But $z_1 \Pi_{\alpha_i} x \Rightarrow z_2 =$ $(x_{\alpha_1}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, \ldots, y_{\alpha_{i_2-1}}, x_{\alpha_{i_2}}, \ldots, x_{\alpha_n}) \varepsilon S$. Again $z_2 \Pi_{\alpha_i} y \Rightarrow z_3 =$ $(x_{\alpha_1}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, \ldots, y_{\alpha_{i_2-1}}, x_{\alpha_{i_2}}, \ldots, x_{\alpha_{i_3-1}}, y_{\alpha_{i_3}}, \ldots, y_{\alpha_n}) \varepsilon S$, etc. Finally $z_k = z\varepsilon S$, which was to be proved. □

<u>Lemma 2.2.3</u>

$S \subseteq \mathop{\times}\limits_{i=1}^{n} S_{\alpha_i}$ has s.e.p. iff for any x,yεS, any $\alpha_i \varepsilon D$, if $x \Pi_{\alpha_i} y$, then for any $\alpha_j \varepsilon D$

a) $(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{j-1}}, x_{\alpha_j}, y_{\alpha_{j+1}}, \ldots, y_{\alpha_n}) \varepsilon S$

and

b) $(y_{\alpha_1}, y_{\alpha_2}, \ldots y_{\alpha_{j-1}}, y_{\alpha_j}, x_{\alpha_{j+1}}, \ldots, x_{\alpha_n}) \varepsilon S$ hold.

<u>Proof</u>

I. If S has s.e.p. then S has e.p. for all permutations of D.

If $\alpha_j = \alpha_{i-1}$ (i.e. j=i-1) both a) and b) hold since S has e.p. and $x_{\alpha_{j+1}} = x_{\alpha_i} = y_{\alpha_i} = y_{\alpha_{j+1}}$, same is true for $\alpha_j = \alpha_i$.

1) We assume that $j < i - 1$.

We permute the α_i and α_{j+1} coordinates, denoting so permuted S by $S^{(\alpha_{j+1}, \alpha_i)}$.

Then
$$\begin{cases} (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{j-1}}, x_{\alpha_j}, \overline{x_{\alpha_{j+1}}}, x_{\alpha_{j+2}}, \ldots, \overline{x_{\alpha_i}}, x_{\alpha_{i+1}}, \ldots, x_{\alpha_n}) \varepsilon S \\ (y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{j-1}}, y_{\alpha_j}, y_{\alpha_{j+1}}, y_{\alpha_{j+2}}, \ldots, y_{\alpha_i}, y_{\alpha_{i+1}}, \ldots, y_{\alpha_n}) \varepsilon S \end{cases}$$

$$\Rightarrow \begin{cases} (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{j-1}}, x_{\alpha_j}, \overline{x_{\alpha_i}}, x_{\alpha_{j+2}}, \ldots, \overline{x_{\alpha_{j+1}}}, x_{\alpha_{i+1}}, \ldots, x_{\alpha_n}) \varepsilon S^{(\alpha_{j+1}, \alpha_i)} \\ (y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{j-1}}, y_{\alpha_j}, y_{\alpha_i}, y_{\alpha_{j+2}}, \ldots, y_{\alpha_{j+1}}, y_{\alpha_{i+1}}, \ldots, y_{\alpha_n}) \varepsilon S^{(\alpha_{j+1}, \alpha_i)} \end{cases}$$

But since S has s.e.p. $S^{(\alpha_{j+1}, \alpha_i)}$ has e.p. and thus

$$\begin{cases} (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{j-1}}, x_{\alpha_j}, \overline{y_{\alpha_i}}, y_{\alpha_{j+2}}, \ldots, \overline{y_{\alpha_{j+1}}}, y_{\alpha_{i+1}}, \ldots, y_{\alpha_n}) \varepsilon S^{(\alpha_{j+1}, \alpha_i)} \\ (y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{j-1}}, y_{\alpha_j}, x_{\alpha_i}, x_{\alpha_{j+2}}, \ldots, x_{\alpha_{j+1}}, x_{\alpha_{i+1}}, \ldots, x_{\alpha_n}) \varepsilon S^{(\alpha_{j+1}, \alpha_i)} \end{cases}$$

which $\Rightarrow \begin{cases} (x_{\alpha_1},x_{\alpha_2},\ldots,x_{\alpha_{j-1}},x_{\alpha_j},y_{\alpha_{j+1}},y_{\alpha_{j+2}},\ldots,y_{\alpha_i},y_{\alpha_{i+1}},\ldots,y_{\alpha_n})\varepsilon S \\ (y_{\alpha_1},y_{\alpha_2},\ldots,y_{\alpha_{j-1}},y_{\alpha_j},x_{\alpha_{j+1}},x_{\alpha_{j+2}},\ldots,x_{\alpha_i},x_{\alpha_{i+1}},\ldots,x_{\alpha_n})\varepsilon S \end{cases}$

Thus a) and b) hold for any $j \leq i - 1$, if S has s.e.p.

2) for $j > i - 1$, we proceed in an analogous way, except that we permute coordinates (α_i,α_j) rather than (α_{j+1},α_i).

II. We now prove that if for any $x,y\varepsilon S$ and any $\alpha_i \varepsilon D$ with $x\Pi_{\alpha_i} y$, a) and b) of Lemma 2.2.3 hold for any $\alpha_j \neq \alpha_i$, then S has s.e.p. Suppose this is not the case. Then $\exists$ a permutation of D, p, such that S^p does not have e.p. That is $\exists\ x^p,y^p \varepsilon S^p$ and an $\alpha_i \varepsilon D$, s.t. $x^p \Pi_{\alpha_i} y^p$, i.e.

$(x_{p(\alpha_1)},\ldots,x_{p(\alpha_i)},\ldots,x_{p(\alpha_n)})\Pi_{p(\alpha_i)}(y_{p(\alpha_1)},\ldots,y_{p(\alpha_i)},\ldots,y_{p(\alpha_n)})$ but

either $z^p = (x_{p(\alpha_1)},\ldots,x_{p(\alpha_i)},y_{p(\alpha_{i+1})},\ldots,y_{p(\alpha_n)})\notin S^p$

or $w^p = (y_{p(\alpha_1)},\ldots,y_{p(\alpha_i)},x_{p(\alpha_{i+1})},\ldots,x_{p(\alpha_n)})\notin S^p$.

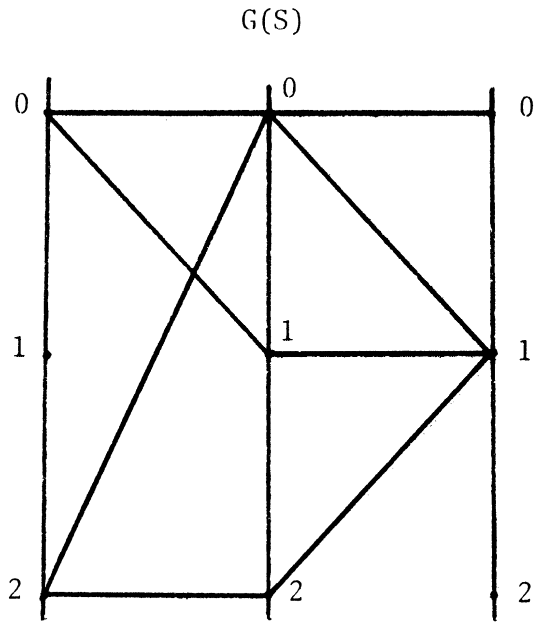
W.l.o.g. we assume that $z^p \notin S^p$. We note that $z^p \notin S^p \Rightarrow z \notin S$, where

$$z = (z_{\alpha_i}) \text{ and } z_{\alpha_i} = \begin{cases} x_{\alpha_i} & \text{if } z^p_{p^{-1}(\alpha_i)} = x_i \\ y_{\alpha_i} & \text{if } z^p_{p^{-1}(\alpha_i)} = y_i \end{cases}$$

$x^p \Pi^{S^p}_{\alpha_i} y^p \Rightarrow x\Pi^S_{\alpha_i} y$. $z \notin S \Rightarrow$ a) or b) do not hold for some α_j, by Lemma 2.2.2. This however contradicts our assumption.

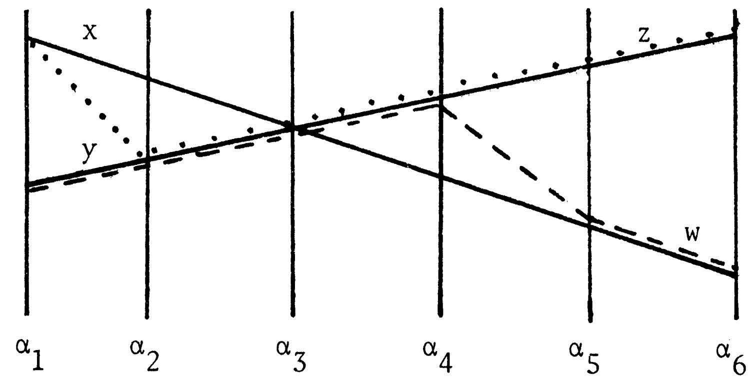
For the illustration of the proof we refer the reader to Figure 2.2.4. □

In the next proposition we prove that a set S with e.p. has s.e.p. if and only if it has e.p. under all cycle permutations of D, or, alternatively, if and only if it has e.p. under all transpositions of D.



$S = \{(0,0,0),(0,0,1),(2,0,0),(2,0,1),(0,1,1),(2,2,1)\}$

Figure 2.2.3: A Graphical Representation of S.



$x \Pi_{\alpha_3} y$ $\quad$ $x,y \varepsilon S$ imply $z,w \varepsilon S$

Figure 2.2.4: Properties of S with S.E.P.

Proposition 2.2.6

Let S have e.p., where $S \subseteq \bigtimes_{i=1}^{n} S_{\alpha_i}$ and $D = \{\alpha_1,\dots,\alpha_n\}$. Then S has s.e.p.

a) iff S has e.p. for every transposition of D

b) iff S has e.p. for every cycle permutation of D.

Proof

a) Clearly if S has s.e.p. then for all transpositions t of D, S^t has e.p. We want to show then that if for all transpositions t of D, S^t has e.p., then S has s.e.p.

By Lemma 2.2.3 to show that S has s.e.p. we just need to show that if $x\Pi_{\alpha_i} y$, then for any $\alpha_j \neq \alpha_i$, $(x_{\alpha_1},\dots,x_{\alpha_j},y_{\alpha_{j+1}},\dots,y_{\alpha_n})$ and $(y_{\alpha_1},\dots,y_{\alpha_j},x_{\alpha_{j+1}},\dots,x_{\alpha_n})$ are in S.

The proof of this however is exactly the same as the proof of I in Lemma 2.2.3, since only transposition type of permutations were used in the proof.

b) We just need to show that if S has e.p. and S^c has e.p., for every cycle c of D, then S has s.e.p. Equivalently, we need to show that for every $x,y \varepsilon S$, any $\alpha_i \varepsilon D$, where $x\Pi_{\alpha_i} y$, $(x_{\alpha_1},\dots,x_{\alpha_j},y_{\alpha_{j+1}},\dots,y_{\alpha_n})\varepsilon S$ and $(y_{\alpha_1},\dots,y_{\alpha_j},x_{\alpha_{j+1}},\dots,x_{\alpha_n})\varepsilon S$, for any $\alpha_j \varepsilon D$.

1) We first show that for any $x,y\varepsilon S$ with $x\Pi_{\alpha_1} y$, any point z s.t. $z(\alpha_i) = y(\alpha_i)$ or $z(\alpha_i) = x(\alpha_i)$, z is in S. Let c_k denote the cycle of D with α_1 in k'th position, i.e. $c_1 = \{\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n\}$ and for $k \geq 2$, $c_k = \{\alpha_{n-k+2},\alpha_{n-k+3},\dots,\alpha_n,\alpha_1,\alpha_2,\dots,\alpha_{n-k+1}\}$. There are n cycles of $D = \{\alpha_1,\dots,\alpha_n\}$. Let $\{\alpha_{i_1},\alpha_{i_2},\dots,\alpha_{i_k}\}$ be ordered by the order of D,

where those are the points of change of z from x to y or y to x. W.l.o.g. take $z = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_i}, \ldots, y_{\alpha_{i_2-1}}, x_{\alpha_{i_2}}, \ldots, x_{\alpha_{i_3-1}}, y_{\alpha_{i_3}}, \ldots)$.
Consider c_{ℓ_m}, where $\ell_m = (n+1)-(i_m-1) = n+2-i_m$, i.e. $\ell_1 = n+2-i_1$, $\ell_2 = n+2-i_2, \ldots, \ell_k = n+2-i_k$. By our assumption $S^{c_{i_m}}$ has e.p. for $m=1,\ldots,k$. Now $x\varepsilon S, y\varepsilon S \Rightarrow x^{c_{\ell_1}} \varepsilon S^{c_{\ell_1}}$ and $y^{c_{\ell_1}} \varepsilon S^{c_{\ell_1}}$, i.e.

$$(\overbrace{x_{\alpha_{i_1}}, x_{\alpha_{i_1+1}}, \ldots, x_{\alpha_n}, x_{\alpha_1}}^{\ell_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}) \varepsilon S^{c_{\ell_1}} \text{ and}$$

$(y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_n}, y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{i_1-1}}) \varepsilon S^{c_{\ell_1}}$. Since $S^{c_{\ell_1}}$ has e.p. and $x^{c_{\ell_1}} \Pi_{\alpha_1} y^{c_{\ell_1}}$, $(y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_n}, x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}) \varepsilon S^{c_{\ell_1}}$. This clearly implies that $z_1 = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_n}) \varepsilon S$, where $z_{\alpha_j} = y_{\alpha_j}$ for all $\alpha_j \geq \alpha_{i_1}$.

$z_1^{c_{\ell_2}} \varepsilon S^{c_{\ell_2}}$ and $z_1^{c_{\ell_2}} = (z_{\alpha_{i_1}}, z_{\alpha_{i_2+1}}, \ldots, z_{\alpha_n}, z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_{i_2-1}})$, i.e.

$$z_1^{c_{\ell_2}} = (y_{\alpha_{i_2}}, y_{\alpha_{i_2+1}}, \ldots, y_{\alpha_n}, x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_{i_2-1}}) \varepsilon S^{c_{\ell_2}}$$

$$x^{c_{\ell_2}} = (x_{\alpha_{i_2}}, x_{\alpha_{i_2+1}}, \ldots, x_{\alpha_n}, x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, \ldots\ldots\ldots\ldots\ldots, x_{\alpha_{i_2-1}}) \varepsilon S^{c_{\ell_2}}$$

Since $z_1^{c_{\ell_2}} \Pi_{\alpha_1} x^{c_{\ell_2}}$ and $S^{c_{\ell_2}}$ has e.p.,

$$(x_{\alpha_{i_2}}, x_{\alpha_{i_2+1}}, \ldots, x_{\alpha_n}, x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_{i_2-1}}) \varepsilon S^{c_{\ell_2}}$$

$$\Rightarrow z_2 = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{i_1-1}}, y_{\alpha_{i_1}}, y_{\alpha_{i_1+1}}, \ldots, y_{\alpha_{i_2-1}}, x_{\alpha_{i_2}}, x_{\alpha_{i_2+1}}, \ldots, x_{\alpha_{i_n}}) \varepsilon S$$

Proceeding this way we show that $z\varepsilon S$.

2) We now want to show that for any $x,y\varepsilon S$ any $\alpha_i \varepsilon D$ if $x \Pi_{\alpha_i} y$ and S^c has e.p. for all cycles c, then for any $\alpha_j \varepsilon D$

$w = (x_{\alpha_1},\ldots,x_{\alpha_j},y_{\alpha_{j+1}},\ldots,y_{\alpha_n})$ and $p = (y_{\alpha_1},y_{\alpha_2},\ldots,y_{\alpha_j},x_{\alpha_{j+1}},\ldots,x_{\alpha_n})$ are in S. It suffices to show that $w \varepsilon S$, since the proof for p is analogous. Let c be a ((n+2)-i)'th cycle, i.e. $c = \{\alpha_i,\alpha_{i+1},\ldots,\alpha_n,\ldots,\alpha_{i-1}\}$ ($i \geq 2$, since for i = 1 the assertion was proven).

To show that $w \varepsilon S$ it suffices to show that $w^c \varepsilon S^c$. We note that x^c and y^c are related by first coordinate in S^c. Also S^c has e.p. and so do all cycle type of permutations of c. Thus by part 1) $w^c \varepsilon S^c$, which implies w S.

The proof of 2) is illustrated by Figure 2.2.5. □

2.3 Relations among Coordinatizations

In this section we will analyze the spectrum of coordinatizations. The levels of the spectrum will be numbered I through VI and we will prove that if coordinatization is of level j, then it is also of level i, for any i smaller than j, provided coordinatization has no constant coordinates. The case of coordinatization with constant coordinates will be discussed separately.

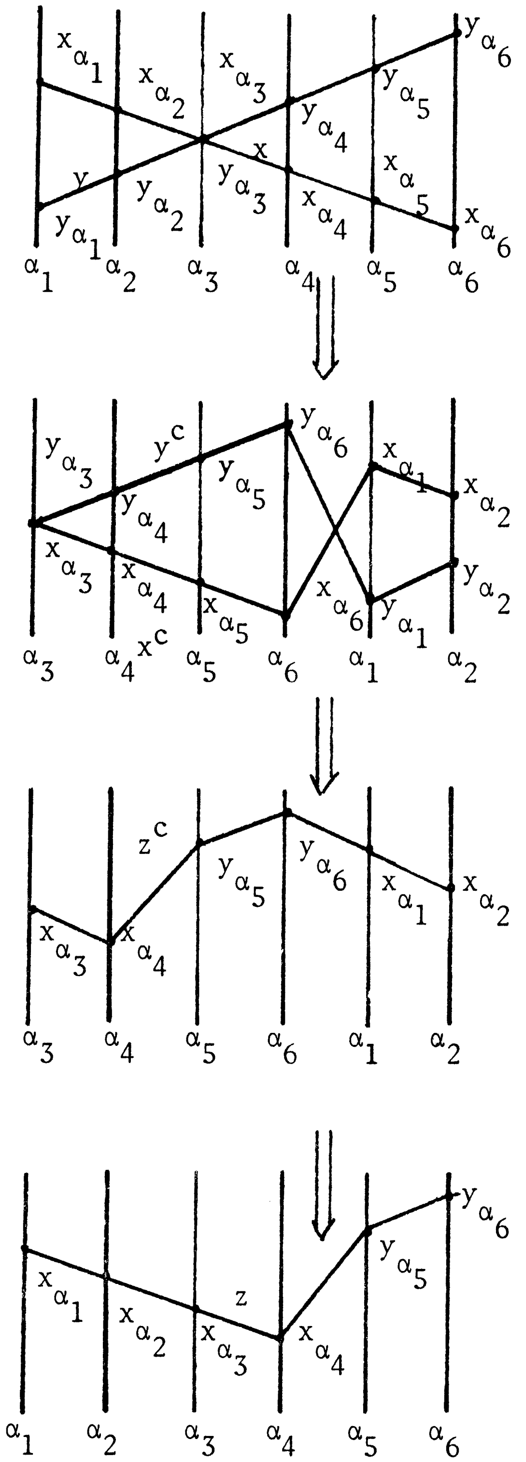
We will also provide and prove some sufficient conditions, under which the implication goes from lower to higher levels.

The spectrum of coordinatizations is shown in Figure 2.3.1.

Theorem 2.3.1

Let $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ be a set with no constant coordinates. Then if coordinatization of S is of level j in the spectrum of coordinatizations, for any $2 \leq j \leq 6$, it is also of level j - 1.

$x,y \varepsilon S, x\Pi_{\alpha_3} y$ implies $(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}, y_{\alpha_5}, y_{\alpha_6}) \varepsilon S$



$x,y \varepsilon S$

$x\Pi_{\alpha_3} y$

$x^c, y^c \varepsilon S^c$

$c = (\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_1, \alpha_2)$

$z^c = (x_{\alpha_3}, x_{\alpha_4}, y_{\alpha_5}, y_{\alpha_6}, x_{\alpha_1}, x_{\alpha_2}) \varepsilon S^c$

$z = (x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}, y_{\alpha_5}, y_{\alpha_6}) \varepsilon S$

Figure 2.2.5: S Has E.P. and S^c Has E.P. for Any Cycle c.

Proof

a) $j = 6$. By Proposition 2.2.1 S is irredundant and hence of level $j - 1 = 5$.

b) $j = 5$. S is irredundant, which implies that every $\Pi \varepsilon P^S$, $\Pi \neq I$, has a unique location $L_\Pi \neq \phi$. Thus $\bigcap_{L_\Pi \varepsilon L} L_\Pi = L_\Pi \neq \phi$ and S is of level $j - 1 = 4$ in the spectrum (i.e. of type 3).

c) $j = 4$, i.e. S is of type 3. Let Π be any partition on S with more than one location. Let $L_1 \neq L_2$ be any two locations of Π. Then clearly since intersection of all locations of Π is nonempty, so is $L_1 \cap L_2$. By Proposition 2.2.4 this implies that S is of type 2, i.e. level III in the spectrum.

d) $j = 3$. Clearly if for all $D_1, D_2 \subseteq D$ with $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$ then $\Pi_\alpha \cup \Pi_{D-\alpha} = I$ for $\forall \alpha \varepsilon D$. This is true because $\alpha \cap (D-\alpha) = \phi$ (we note that $|D|$ was assumed to be ≥ 2). Thus if S is of level III in the spectrum it is also of level II.

e) $j = 2$, i.e. S has the property that $\Pi_\alpha \cup \Pi_{D-\alpha} = I$, for all $\alpha \varepsilon D$. Since S has no constant coordinates, $\Pi_\alpha \neq I$ and thus α is a location of Π_α. By Proposition 2.2.3 this implies that α is the unique location of Π_α, for all $\alpha \varepsilon D$. Finally, by part a) of Theorem 2.2.2 this implies that S is independent, and thus of level I in the spectrum of coordinatizations. □

We will now show that all levels of the spectrum are distinct (in general). For every level we will exhibit a coordinatization, which is of this level, but not of level one higher.

Theorem 2.3.2

For every j, $1 \leq j \leq 5$, there exists a coordinatized S s.t. S is

$S \subseteq \bigtimes_{\alpha \in D} S_\alpha$

I Independent , for $\forall\ \alpha \in D,\ \Pi_{D-\alpha} \not\leq \Pi_\alpha$

$\Uparrow$

II Type 1 , for $\forall\ \alpha \in D,\ \Pi_\alpha \cup \Pi_{D-\alpha} = I$

$\Uparrow$

III Type 2 , for $\forall\ D_1, D_2 \subseteq D$, with $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$

$\Uparrow$

IV Type 3 , for $\forall\ \Pi \in P^S,\ \Pi \neq I, \bigcap_{L \in \mathcal{L}} L \neq \phi$, where $\mathcal{L}$ is the family of all locations of Π.

$\Uparrow$

V Irredundant , for $\forall\ \Pi \in P^S$, Π has a unique location

$\Uparrow$

VI Cartesian , $S = \bigtimes_{\alpha \in D} P_\alpha(S)$

Figure 2.3.1: Spectrum of Coordinatizations (No Constant Coordinates).

of level j, but not of level j + 1.

Proof

a) I $\not\Rightarrow$ II

Consider $S = \{(a_1,b_1),(a_2,b_1),(a_3,b_2),(a_3,b_3)\}$. Clearly S is independent. With $D = \{\alpha_1,\alpha_2\}$, we note that $\Pi_{D-\alpha_2} \cup \Pi_{\alpha_2} = \Pi_{\alpha_1} \cup \Pi_{\alpha_2} =$

$$\{\overline{(a_1,b_1)}, \overline{(a_2,b_1)}, \overline{(a_3,b_2),(a_3,b_3)}\} \cup \{\overline{(a_1,b_1),(a_2,b_1)}, \overline{(a_3b_2)}, \overline{(a_3,b_3)}\} =$$

$$\{\overline{(a_1,b_1),(a_2,b_1)}, \overline{(a_3,b_2),(a_3,b_3)}\} \neq I$$

b) II $\not\Rightarrow$ III

Consider $S = \{(a_1,b_1,c_1,d_1), (a_2,b_2,c_2,d_2), (a_2,b_1,c_1,d_1), (a_1,b_2,c_1,d_1), (a_1,b_1,c_2,d_1), (a_1,b_1,c_1,d_2)\}$.

It is easy to show that

$\Pi_{\alpha_1} \cup \Pi_{D-\alpha_1} = I$

$\Pi_{\alpha_2} \cup \Pi_{D-\alpha_2} = I$

$\Pi_{\alpha_3} \cup \Pi_{D-\alpha_3} = I$

$\Pi_{\alpha_4} \cup \Pi_{D-\alpha_4} = I$.

However the partition $\Pi =$

$$\{\overline{(a_1,b_1,c_1,d_1),(a_1,b_1,c_2,d_1),(a_1,b_1,c_1,d_2),(a_2,b_1,c_1,d_1),(a_1,b_2,c_1,d_1)}, \overline{(a_2,b_2,c_2,d_2)}\}$$

has locations $\{\alpha_1,\alpha_2\}$ and $\{\alpha_3,\alpha_4\}$, which are disjoint. Thus by Proposition 2.2.4 S is not of type 2 and hence not of level III in the spectrum.

c) III $\not\Rightarrow$ IV

Consider $S = \{(a,b,c),(a,e,c),(d,b,f),(d,e,c),(a,e,f)\}$, $D = \{\alpha_1,\alpha_2,\alpha_3\}$.

We first show that S is of level III, i.e. that for all $D_1, D_2 \subseteq D$ with $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$. Actually to show that it suffices to show that for all $D_1, D_2 \subseteq D$ with $D_1 \cap D_2 = \phi$ and $D_1 \cup D_2 = D$, $\Pi_{D_1} \cup \Pi_{D_2} = I$. Thus we need to show that

$\Pi_{\alpha_1} \cup \Pi_{\alpha_2 \cup \alpha_3} = I$, $\Pi_{\alpha_2} \cup \Pi_{\alpha_1 \cup \alpha_3} = I$ and $\Pi_{\alpha_3} \cup \Pi_{\alpha_1 \cup \alpha_2} = I$. This however follows easily by inspection.

Let $\Pi = \left\{ \overline{(d,b,f)}, \overline{(a,b,c),(a,e,c),(d,e,c),(a,e,f)} \right\}$. It can be verified that $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$ and $\{\alpha_2, \alpha_3\}$ are locations of Π. Obviously $\{\alpha_1, \alpha_2\} \cap \{\alpha_1, \alpha_3\} \cap \{\alpha_2, \alpha_3\} = \phi$. Thus S is not of level IV.

d) IV $\not\Rightarrow$ V

Let $S = \{(a,b,c),(a,e,c),(d,b,f),(d,e,f),(a,e,f)\}$. We first prove that S is of level IV, i.e. for any partition Π on S with more than one location, intersection of all its locations is nonempty. Since $|D| = 3$ however, to prove that, it suffices to show that for any $D_1, D_2 \subseteq D$ s.t. $D_1 \cap D_2 = \phi$, $\Pi_{D_1} \cup \Pi_{D_2} = I$, i.e. that no two distinct locations of Π are disjoint. (For then no Π can have more than two locations.) To prove that, it suffices to show that

$\Pi_{\alpha_1} \cup \Pi_{\{\alpha_2, \alpha_3\}} = I$, $\Pi_{\alpha_2} \cup \Pi_{\{\alpha_1, \alpha_3\}} = I$ and $\Pi_{\alpha_3} \cup \Pi_{\{\alpha_1, \alpha_2\}} = I$, which clearly holds.

S is not irredundant, for example $\Pi = \{\overline{(a,b,c)}, \overline{(a,e,c),(d,b,f),(d,e,f),(a,e,f)}\}$ has two locations $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_3\}$.

e) V $\not\Rightarrow$ VI

Consider $S = \{(0,0),(1,0),(1,1)\}$. S is irredundant but not Cartesian, since $(0,1) \notin S$. □

Remark

For S with constant coordinates and any $3 \leq j \leq 6$, if S is of level j then it is of level (j-1). If coordinatization of S is of level II however, then every $\alpha \varepsilon D$ is either independent or constant. Thus II $\not\Rightarrow$ I. The above can be easily verified by tracing through the proof of Theorem 2.3.1.

We shall now give some conditions under which the implication in the spectrum goes from a lower to a higher level.

We note that if cardinality of an index set is two, levels II, III, IV and V are all equivalent.

Also as was pointed out in part d) of the proof of Theorem 2.3.2, if $|D| = 3$, then levels III and IV are equivalent.

Other conditions are stated and proved below.

Proposition 2.3.1

Let S be independent. Then S is irredundant if either a) or b) holds, where

a) $\Pi = \{\Pi | \Pi \varepsilon P^S$ and Π has a unique location$\}$ is a sublattice of P^S.

b) $P^S_D = \{\Pi^S_{D'} | D' \subseteq D\}$ is a sublattice of P^S.

Proof

1) We prove that if a) holds, S is irredundant. We need to show that for any $D_1, D_2 \subseteq D$, $\Pi^S_{D_1 \cap D_2} \leq \Pi^S_{D_1} \cup \Pi^S_{D_2}$.

Since S is independent, by part b) of Theorem 2.2.2 Π_{D_1} has a unique location D_1 and Π_{D_2} a unique location D_2. Since a) holds $\Pi = \Pi_{D_1} \cup \Pi_{D_2}$ has a unique location, say L. Since $\Pi_{D_1} \leq \Pi$ and $\Pi_{D_2} \leq \Pi$, $D_1 \supseteq L$ and $D_2 \supseteq L$. Thus $D_1 \cap D_2 \supseteq L$. This implies that $\Pi_{D_1 \cap D_2} \leq \Pi_L$. But L being the location of Π implies $\Pi_L \leq \Pi$. Thus $\Pi_{D_1 \cap D_2} \leq \Pi_{D_1} \cup \Pi_{D_2}$.

2) We prove that if b) holds S is irredundant. Again we need to show that $\Pi_{D_1 \cap D_2} \leq \Pi_{D_1} \cup \Pi_{D_2}$. Since P_D^S is a sublattice of P^S, $\Pi_{D_1} \cup \Pi_{D_2} = \Pi_{D_3}$, for some $D_3 \subseteq D$. Now $\Pi_{D_1} \leq \Pi_{D_1} \cup \Pi_{D_2} = \Pi_{D_3}$ and $\Pi_{D_2} \leq \Pi_{D_3}$. Since S is independent, D_3 is the unique location of Π_{D_3}. Thus $D_1 \supseteq D_3$ and $D_2 \supseteq D_3$, which $\Rightarrow D_1 \cap D_2 \supseteq D_3$. But this in turn implies that $\Pi_{D_1 \cap D_2} \leq \Pi_{D_3} = \Pi_{D_1} \cup \Pi_{D_2}$. □

As was stated in Proposition 2.2.2 cardinality of an independent set S is at least one larger than the cardinality of its index set. If those two are equal, i.e. if $|S| = |D| + 1$ we will refer to S as a minimal independent set. We will prove in the next proposition that every minimal independent set is also irredundant. First we will state and prove an auxiliary lemma.

Lemma 2.3.1

Let S be an independent subset of $\bigtimes_{\alpha \in D} S_\alpha$, where $|D| = n$ and $|S| = n+1$. Then $\exists$ an enumeration of D, $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\}$ and an enumeration of S, $\{s_0, s_1, \ldots, s_n\}$ with the following property. For Y_i, i=1,...,n, defined recursively by $Y_1 = \{s_0, s_1\}$, and $Y_i = Y_{i-1} \cup \{s_i\}$ for i=2,...,n, $s_1 \Pi_{D-\alpha_{i_1}} s_0$ and for every s_ℓ, where $2 \leq \ell \leq n$, $\exists$ an $s \in Y_{\ell-1}$ s.t. $s_\ell \Pi_{D-\alpha_{i_\ell}} s$.

Proof

Pick any $\alpha \in D$ and let $\alpha_{i_1} = \alpha$. Then since coordinate α_{i_1} is independent on S, $\exists$ $s_0, s_1 \in S$ s.t. $s_0 \Pi_{D-\alpha_{i_1}} s_1$. (Of course $s \not\Pi_{\alpha_{i_1}} s_1$.) Let $Y_1 = \{s_0, s_1\}$. We now show that $\exists$ a $\beta \in D-\alpha_{i_1}$ and an $s_\beta \in S-Y_1$ s.t.

$s_\beta \Pi_{D-\beta} s_0$ or $s_\beta \Pi_{D-\beta} s_1$. This is true for the following reason. All coordinates of $D-\alpha_{i_1}$ are independent. So for every $\gamma \varepsilon D-\alpha_{i_1}$ $\exists$ $s_\gamma, z_\gamma \varepsilon S$ s.t. $s_\gamma \Pi_{D-\gamma} z_\gamma$ (while $s_\gamma \not\Pi_\gamma z_\gamma$). Now $s_\gamma = s_0$ and $z_\gamma = s_1$ cannot hold for any $\gamma \varepsilon D-\alpha_{i_1}$, since $\alpha_{i_1} \varepsilon D-\gamma$ and $s_0 \Pi_{\alpha_{i_1}} s_1$. If for no $\gamma \varepsilon D-\alpha_{i_1}$ $\exists$ a $z_\gamma \varepsilon S-Y_1$ s.t. $z_\gamma \Pi_{D-\gamma} s_0$ or $z_\gamma \Pi_{D-\gamma} s$, all coordinates of $(D-\alpha_{i_1})$ are independent on $S-Y_1$. Since $|D-\alpha_{i_1}|=n-1$, this implies by Proposition 2.2.2 that $|S-Y_1| \geq (n-1)+1=n$, which implies that $|S| = |S-Y_1|+|Y_1| \geq n+2$. This however is a contradiction to $|S| = n + 1$. So $\exists$ a $\beta \varepsilon D-\alpha_{i_1}$ and $s_\beta \varepsilon S-Y_1$ s.t. $s_\beta \Pi_{D-\beta} s_0$ or $s_\beta \Pi_{D-\beta} s_1$. Let $\alpha_{i_2} = \beta$, $s_2 = s_\beta$ and $Y_2 = Y_1 \cup \{s_2\}$.

We then show that $\exists$ a $\gamma \varepsilon D-\{\alpha_{i_1}, \alpha_{i_2}\}$ and an $s_\gamma \varepsilon S-Y_2$ s.t. $s_\gamma \Pi_{D-\gamma} s_0$ or $s_\gamma \Pi_{D-\gamma} s_1$ or $s_\gamma \Pi_{D-\gamma} s_2$ holds. In an analogous manner we prove at the k'th stage that $\exists$ a $\gamma \varepsilon D-\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ and an $s_\gamma \varepsilon S-Y_k$ s.t. $s_\gamma \Pi_{D-\gamma} z$, for some $z \varepsilon Y_k$. □

We will now illustrate Lemma 2.3.1 by an example.

Example 2.3.1

Consider $S = \{(0,0,0,0), (0,0,0,1), (0,1,0,1), (0,0,1,1), (1,1,0,1)\}$, where $D = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. S is clearly independent and $|S| = |D|+1 = 5$. We demonstrate one enumeration of D and S, as of Lemma 2.3.1. Let $s_0 = (0,0,0,1)$, $s_1 = (0,1,0,1)$, $s_2 = (0,0,1,1)$, $s_3 = (1,1,0,1)$, $s_4 = (0,0,0,0)$. $\alpha_{i_1} = \alpha_2$, $\alpha_{i_2} = \alpha_3$, $\alpha_{i_3} = \alpha_1$ and $\alpha_{i_4} = \alpha_4$. Thus $Y_1 = \{(0,0,0,1), (0,1,0,1)\}$, $Y_2 = \{(0,0,0,1)\ (0,1,0,1), (0,0,1,1)\}$, $Y_3 = \{(0,0,0,1), (0,1,0,1), (0,0,1,1), (1,1,0,1)\}$. We verify that $s_0 \Pi_{D-\alpha_2} s_1$, $s_2 \Pi_{D-\alpha_3} s_0$, $s_3 \Pi_{D-\alpha_1} s_1$ and $s_4 \Pi_{D-\alpha_4} s_0$. Hence the enumeration is as desired.

Proposition 2.3.2

Let S be an independent subset of $\bigtimes_{\alpha \varepsilon D} S_\alpha$, where $|D| = n (n \geq 2)$ and

$|S| = n + 1$. Then S is irredundant.

Proof

We proceed by induction on n.

a) induction base

We show that for $n = 2 = |\{\alpha_1,\alpha_2\}|$ and $S = \{s_0,s_1,s_2\}$ if S is independent then it is irredundant. For S independent either 1) $s_0\Pi_{\alpha_1}s_1$, $s_1\Pi_{\alpha_2}s_2$ or 2) $s_0\Pi_{\alpha_1}s_1$, $s_0\Pi_{\alpha_2}s_2$ or 3) $s_0\Pi_{\alpha_1}s_2$, $s_0\Pi_{\alpha_2}s_1$ or 4) $s_0\Pi_{\alpha_1}s_2$, $s_1\Pi_{\alpha_2}s_2$ or 5) $s_1\Pi_{\alpha_1}s_2$, $s_0\Pi_{\alpha_2}s_1$ or 6) $s_1\Pi_{\alpha_1}s_2$, $s_2\Pi_{\alpha_2}s_0$ holds. In all those cases S is irredundant.

b) induction step

(*) We assume that if $|D| = k$ and S is any independent set coordinatized over D, with $|S| = k + 1$, then S is irredundant. We want to prove that if $|D| = k + 1$ and S is any independent set coordinatized over D, with $|S| = k + 2$, (*) $\Rightarrow$ that S is irredundant.

Let $S = \{s_0,s_1,\ldots,s_k,s_{k+1}\}$ be an enumeration of S and $\{\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_k},\alpha_{i_{k+1}}\}$ an enumeration of D as in Lemma 2.3.1. $S = Y_k \cup \{s_{k+1}\}$, where $Y_k = \{s_0,\ldots,s_k\}$. The coordinates $\{\alpha_{i_1},\ldots,\alpha_{i_k}\}$ are all independent on Y_k, since for every S_ℓ, $1 \le \ell \le k$, $\exists$ an $s\epsilon Y_k$ s.t. $s_\ell\Pi_{D-\alpha_i}s$ and $s_\ell \neq s$. We also note that coordinate $\alpha_{i_{k+1}}$ is constant on Y_k. This follows, since $s_1\Pi_{D-\alpha_{i_1}}s_0 \Rightarrow P_{\alpha_{i_{k+1}}}(s_1) = P_{\alpha_{i_{k+1}}}(s_0)$, $s_2\Pi_{D-\alpha_{i_2}}s_1$ or $s_2\Pi_{D-\alpha_{i_2}}s_0 \Rightarrow P_{\alpha_{i_{k+1}}}(s_2) = P_{\alpha_{i_{k+1}}}(s_1)$ (or $P_{\alpha_{i_{k+1}}}(s_0)$), etc. Let $\bar{Y}_k$ denote the projection of Y_k onto $\bar{D} = D-\alpha_{i_{k+1}}$, i.e. $\bar{Y}_k = P_{D-\alpha_{i_{k+1}}}(Y_k)$. Then since $\bar{Y}_k$ is independent, $|\bar{Y}_k| = k + 1$ and $|\bar{D}| = k$, by (*) $\bar{Y}_k$ is irredundant. This implies (trivially) irredundance of Y_k.

We now need to prove irredundance of S. We have to show that for any $z \varepsilon Y_k$ any $D_1, D_2 \subseteq D$, if $s_{k+1} \Pi^S_{D_1 \cap D_2} z$, then $s_{k+1} \Pi^S_{D_1} \cup \Pi^S_{D_2} z$. Let z, D_1, D_2 be as above. Since $z \not\Pi^S_{\alpha_{i_{k+1}}} s_{k+1}$, $\alpha_{i_{k+1}} \notin D_1 \cap D_2$. W.l.o.g. assume that $\alpha_{i_{k+1}} \notin D_1$, i.e. $D_1 \subseteq D-\alpha_{i_{k+1}}$. Now $s_{k+1} \Pi^S_{D-\alpha_{i_{k+1}}} s$, for some $s \varepsilon Y_k$ (by construction of Y_k's). Since $D_1 \subseteq D-\alpha_{i_{k+1}}$, this implies that $s_{k+1} \Pi^S_{D_1} s$. Since $s_{k+1} \Pi^S_{D_1 \cap D_2} z$ and $s \Pi^S_{D_1 \cap D_2} s_{k+1}$ (for $D_1 \cap D_2 \subseteq D_1$), $s \Pi^S_{D_1 \cap D_2} z$ holds. Hence clearly $s \Pi^{Y_k}_{D_1 \cap D_2} z$. Since Y_k is irredundant, $s(\Pi^{Y_k}_{D_1} \cup \Pi^{Y_k}_{D_2}) z$ and so $s(\Pi^S_{D_1} \cup \Pi^S_{D_2}) z$. Since $s_{k+1} \Pi^S_{D_1} s$, $s_{k+1}(\Pi^S_{D_1} \cup \Pi^S_{D_2}) s$. So $s(\Pi^S_{D_1} \cup \Pi^S_{D_2}) z$, which completes the proof. (We note that this proof works for $D_1 \cap D_2 = \phi$.) □

<u>Example 2.3.2</u>

Let $S = \bigtimes_{\alpha \varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$ and $|D| = n$. We construct a subset X of S in the following way. We pick any $s_1 \varepsilon S$. Let s_2 be any point of S s.t. $P_{\alpha_1}(s_2) \neq P_{\alpha_1}(s_1)$ and $P_{D-\alpha_1}(s_2) = P_{D-\alpha_1}(s_1)$, let s_3 be any point of S s.t. $P_{\alpha_2}(s_3) \neq P_{\alpha_2}(s_2)$ and $P_{D-\alpha_2}(s_3) = P_{D-\alpha_2}(s_2)$, etc. That is s_i is any point of S s.t. $P_{\alpha_{i-1}}(s_i) = P_{\alpha_{i-1}}(s_{i-1})$ and $P_{D-\alpha_{i-1}}(s_i) = P_{D-\alpha_{i-1}}(s_{i-1})$, for $2 \leq i \leq n+1$. $|X| = n+1$ and X is clearly independent by construction. Hence it is irredundant by Proposition 2.3.2.

For $S = \{0,1\}^5$ the following set X (constructed as above) is irredundant.

$$X = \begin{array}{l} \{(0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ )\} \\ \cup \\ \{(\textcircled{1},\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ )\} \\ \cup \\ \{(\ 1\ ,\ \textcircled{1},\ 0\ ,\ 0\ ,\ 0\ )\} \\ \cup \\ \{(\ 1\ ,\ 1\ ,\ \textcircled{1},\ 0\ ,\ 0\ )\} \\ \cup \\ \{(\ 1\ ,\ 1\ ,\ 1\ ,\ \textcircled{1},\ 0\ )\} \\ \cup \\ \{(\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ \textcircled{1})\}. \end{array}$$

□

We will next show that if S is of level II and has strong extension property then it is Cartesian, in case the cardinality of an index set is greater than or equal to three.

Proposition 2.3.3

Let $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$, where $|D| \geq 3$ and S has s.e.p. Then if S is of level II (type 1) it is Cartesian.

Proof

First we note that if S is of type 1, $\Pi_\alpha \cup \Pi_{D-\alpha} = I$ for all $\alpha\varepsilon D$. This implies that for any $\alpha,\beta\varepsilon D$, $\alpha \neq \beta$, $\Pi_\alpha \cup \Pi_\beta = I$ on S. We will show that for any $x_i \varepsilon P_i(S)$, $i=1,2,\ldots,n$, where $|D| = n$, $(x_1,x_2,\ldots,x_n)\varepsilon S$, provided $\Pi_\alpha \cup \Pi_\beta = I$, for all $\alpha,\beta\varepsilon D(\alpha\neq\beta)$. So actually we will prove a somewhat stronger result than the one stated. To show that $(x_1,x_2,\ldots,x_n)\varepsilon S$ it suffices to show that for every i, $1 \leq i \leq n-1$, $\exists$ some point s_i in S s.t. $P_i(s_i) = x_i$ and $P_{i+1}(s_i) = x_{i+1}$.
This follows since S has s.e.p. and $s^i \Pi_{i+1} s^{i+1}$, where s^i's are as above. $x_i \varepsilon P_i(S)$, $x_{i+1}\varepsilon P_{i+1}(S) \Rightarrow \exists\ z^i$, $w^{i+1} \varepsilon S$ s.t. $P_i(z^i) = x_i$ and $P_{i+1}(w^{i+1}) = x_{i+1}$. Since $\Pi_i \cup \Pi_{i+1} = I$, $\exists\ p_1,\ldots,p_\ell \varepsilon S$ s.t. $z^i \Pi_i p_1 \Pi_{i+1} p_2 \ldots \Pi_{i+1} w^{i+1}$.
This implies that $\exists$ a broken line from the point x_i of G(S) (the graph

of S) to the point x_{i+1} of G(S). This line is composed of segments (p_i, p_{i+1}), where $p_i \varepsilon P_i(S)$ and $p_{i+1} \varepsilon P_{i+1}(S)$. By the length of this line we will mean the number of segments it is composed of. We want to show that there is always a broken line of length 1 connecting x_i and x_{i+1} (for any i, i+1, any x_i, x_{i+1}). This will clearly imply that s^i's as defined before are in S.

We note the length of the broken line is always odd, i.e. 1, 3, 5 etc. This is illustrated by Figure 2.3.2.

We will proceed by induction on ℓ, the length of the broken line connecting x_i and x_{i+1}. We show that for any odd k, $k \geq 1$, if there is a broken line of length k from x_i to x_{i+1}, then $\exists$ a broken line of length 1 from x_i to x_{i+1}.

a) induction base, for k = 1 this is obvious

b) induction hypothesis. Let k be any odd number ≥ 1.

(*) We assume that if there is a broken line of length k connecting any two points $p_i \varepsilon P_i(S)$ and $p_{i+1} \varepsilon P_{i+1}(S)$, then $\exists$ a broken line of $\ell = 1$ connecting p_i and p_{i+1}.

We will show that (*) implies that for any $x_i \varepsilon P_i(S)$, any $x_{i+1} \varepsilon P_{i+1}(S)$, if $\exists$ a broken line of $\ell = k + 2$, then there is a broken line of $\ell = 1$ from x_i to x_{i+1}. Let L be such a line, $\ell(L) = k + 2$. Then let y_{i+1} denote a point on L s.t. $y_{i+1} \varepsilon P_{i+1}(S)$ and $\ell(L(y_{i+1}, x_{i+1})) = 2$. $L(y_{i+1}, x_{i+1})$ denotes the segment of L between y_{i+1} and x_{i+1} and $\ell(L(y_{i+1}, x_{i+1}))$ its length.

Then $\ell(L(x_i, y_{i+1})) = k$ and thus by (*) $\exists$ a line segment $\bar{L}(x_i, y_{i+1})$ s.t. $\ell(\bar{L}(x_i, y_{i+1})) = 1$. Let $\hat{L}$ be a broken line between x_i and x_{i+1} defined by $\hat{L}(x_i, y_{i+1}) = \bar{L}$ and $\hat{L}(y_{i+1}, x_{i+1}) = L(y_{i+1}, x_{i+1})$. Clearly $\ell(\hat{L}) = 1 + 2 = 3$. Let $y_i \varepsilon P_i(S)$ denote the point of $\hat{L}$, s.t. $\ell(\hat{L}(y_i, x_{i+1})) = 1$,

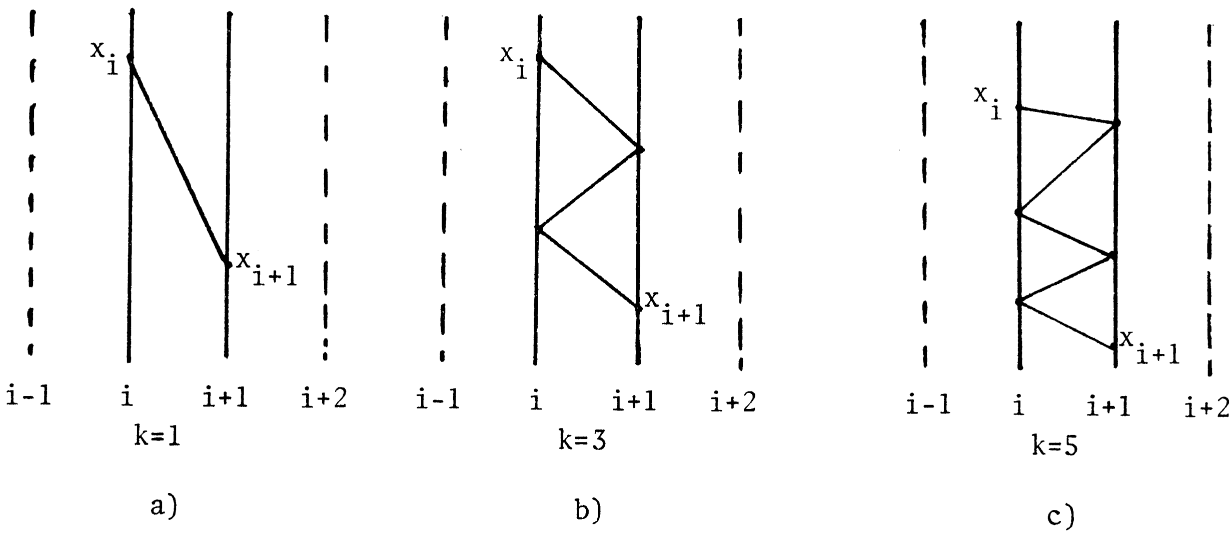


Figure 2.3.2: Broken Lines of Length k Connecting x_i and x_{i+1}.

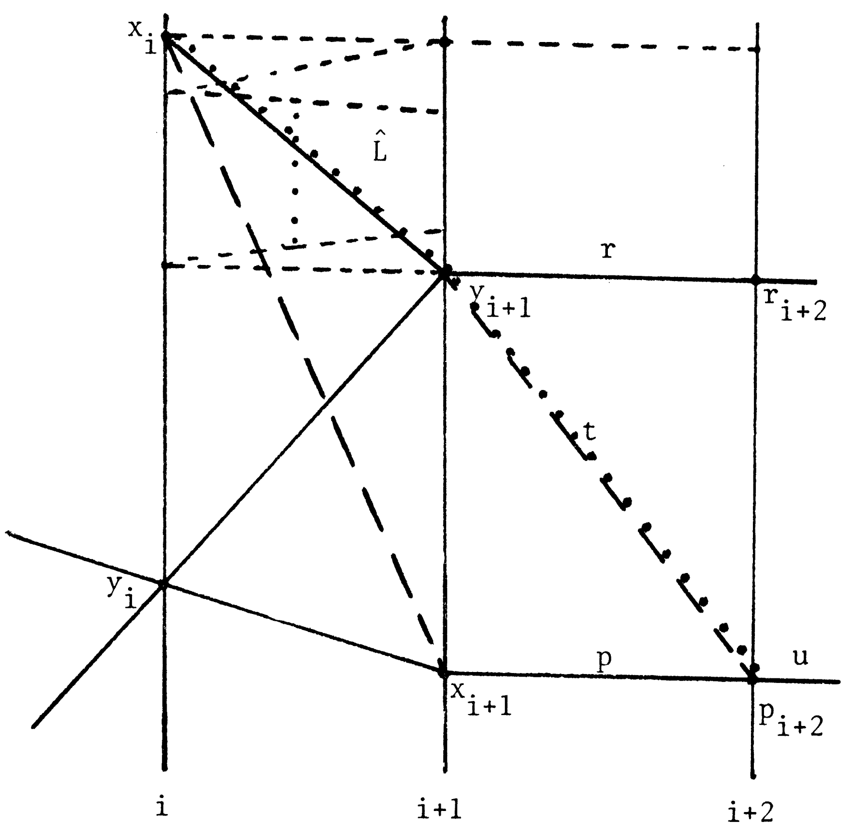


Figure 2.3.3: Illustration of the Proof of Proposition 2.3.2.

i.e. (y_i, x_{i+1}) is the last segment of $\hat{L}$. We assume that $i + 1 \leq n - 1$. Then $\exists$ a $p \varepsilon S$ s.t. $p_i = y_i$, $p_{i+1} = x_{i+1}$, $p_{i+2} = p_{i+2}$. Also $\exists$ $r \varepsilon S$ s.t. $r_i = y_i$ and $r_{i+1} = y_{i+1}$. Since $p \Pi_i r$ and S has s.e.p. $\exists$ a $t \varepsilon S$ s.t. $t_i = y_i$, $t_{i+1} = y_{i+1}$ and $t_{i+2} = p_{i+2}$. Also since (x_i, y_{i+1}) is a segment of $\hat{L}$, $\exists$ an $h \varepsilon S$ s.t. $h_i = x_i$ and $h_{i+1} = y_{i+1}$. $h \Pi_{i+1} t$ holds, which implies that $\exists$ a $u \varepsilon S$ s.t. $u_i = x_i$, $u_{i+1} = y_{i+1}$ and $u_{i+2} = p_{i+2}$. Now $u \Pi_{i+2} p \Rightarrow \exists$ a $s^i \varepsilon S$ s.t. $s^i_i = x_i$ and $S^i_{i+1} = x_{i+1}$. Thus $\exists$ a length-1 segment connecting x_i and x_{i+1}.

In case $i + 1 = n$, we use (i-1)'th coordinate as an auxiliary one (rather than (i+2)'th coordinate) and proceed in an analogous way.

The proof is illustrated by Figure 2.3.3. □

2.4 Theory of Irredundance

In this section we will discuss ways of obtaining irredundant sets from other sets, for instance those which are known to be irredundant.

We will start by pointing out a graphical interpretation of irredundance for sets coordinatized over an index set with cardinality two.

Proposition 2.4.1

Let $S \subseteq S_{\alpha_1} \times S_{\alpha_2}$. Then S is irredundant iff for any two points $p_1, p_2 \varepsilon G(S)$, $\exists$ a path connecting p_1 and p_2.

Proof

We first point out that any S as above has e.p. and so G(S) is well defined. S is irredundant $\Leftrightarrow \Pi^S_{\alpha_1} \cup \Pi^S_{\alpha_2} = I \Leftrightarrow$ for any two points $x, y \varepsilon S$, $x \Pi_{\alpha_1} \cup \Pi_{\alpha_2} y \Leftrightarrow \exists\ z_1, \ldots, z_k \varepsilon S$ s.t. $x \Pi_{\alpha_1} z_1 \Pi_{\alpha_2} z_2, \ldots, \Pi_{\alpha_2} z_k \Pi_{\alpha_1} y$.
$\Leftrightarrow$ for any p_1, p_2 G(S), $\exists$ a path connecting p_1 and p_2. □

Example 2.4.1

Consider $S_1, S_2 \subseteq S_{\alpha_1} \times S_{\alpha_2}$, where $S_{\alpha_1} = \{0,1,2\} = S_{\alpha_2}$ and $S_1 = \{(0,0),(0,2),(1,0),(2,2)\}$, $S_2 = \{(0,0),(1,0),(1,1),(2,2)\}$. Then (refer to Figure 2.4.1) S_1 is irredundant but S_2 is not. □

We will now prove that if S is irredundant then its projection onto any subset of an index set is also irredundant.

Proposition 2.4.2

Let $S \subseteq \bigtimes_{\alpha \in D} S_\alpha$ be irredundant and let $\bar{D}$ be any subset of D. Then $\bar{S} = P_{\bar{D}}(S)$ is an irredundant subset of $\bigtimes_{\alpha \in \bar{D}} S_\alpha$.

Proof

Suppose $\bar{S} = P_{\bar{D}}(S)$ is not irredundant. Then $\exists$ a partition $\bar{\Pi}$ on $\bar{S}$ with more than one location. Let L_1, L_2 be two distinct locations of $\bar{\Pi}$. We will show that then $\exists$ a Π on S s.t. L_1, L_2 are locations of Π. Let Π on S be defined by $x\Pi y \Leftrightarrow P_{\bar{D}}(x)\bar{\Pi}P_{\bar{D}}(y)$. Π is well defined and we show that $\Pi_{L_1} \leq \Pi$, $\Pi_{L_2} \leq \Pi$.

Let $x\Pi^S_{L_1} y$, where $x,y \in S$. Then $P_{\bar{D}}(x)\Pi^{\bar{S}}_{L_1} P_{\bar{D}}(y)$, since $L_1 \subseteq \bar{D}$. But $\Pi^{\bar{S}}_{L_1} \leq \bar{\Pi} \Rightarrow P_{\bar{D}}(x)\bar{\Pi}P_{\bar{D}}(y) \Rightarrow x\Pi y$. Similarly $\Pi_{L_2} \leq \Pi$. We now show that for any $L \subseteq \bar{D}$, if $\Pi^S_L \leq \Pi$ then $\Pi^{\bar{S}}_L \leq \bar{\Pi}$. Let $\bar{x}\Pi^{\bar{S}}_L\bar{y}$, where $\bar{x},\bar{y} \in \bar{S}$. $\bar{x} = P_{\bar{D}}(x)$, $\bar{y} = P_{\bar{D}}(y)$ for some $x,y \in S$. Clearly $x\Pi^S_L y$ holds. Since $\Pi^S_L \leq \Pi$, $x\Pi y$. This $\Rightarrow P_{\bar{D}}(x)\bar{\Pi}P_{\bar{D}}(y)$, i.e. $\bar{x}\bar{\Pi}\bar{y}$. So L_1 and L_2 are locations of Π. (L_1, L_2 both contain locations of Π, but by the above argument this containment cannot be proper.) Thus S is not irredundant, which contradicts our assumption. □

We will now show that a Cartesian product of a finite family of sets is irredundant iff all the sets in the family are irredundant.

Proposition 2.4.3

Let $S^i \subseteq \bigtimes_{\alpha\varepsilon D_i} S^i_\alpha$, for i=1,...,n, where $D_i \cap D_j = \phi$ for $i \neq j$.

Then $S = \bigtimes_{i=1}^{n} S^i$ is an irredundant subset of $\bigtimes_{i=1}^{n}\left(\bigtimes_{\alpha\varepsilon D_i} S^i_\alpha\right)$ iff S^i is an irredundant subset of $\bigtimes_{\alpha\varepsilon D_i} S^i_\alpha$, for i=1,...,n.

Proof

a) Since $S^i = P_{D_i}(S)$, clearly if S is irredundant so are all S^i (by Proposition 2.4.2).

b) We now prove that if all S^i are irredundant, then so is S.

We will prove it for n = 2 case, since this can be easily generalized by induction. Let S^1, S^2 be given, where S^1, S^2 are irredundant, $S = S^1 \times S^2$. We need to show that $\Pi^S_{\bar D\cap\tilde D} \underline{\angle} \Pi^S_{\bar D} \cap \Pi^S_{\tilde D}$, for every $\bar D, \tilde D \subseteq D$, where $D = D_1 \cup D_2$. Let $\bar D_1 = \bar D \cap D_1$ and $\bar D_2 = \bar D \cap D_2$. Similarly let $\tilde D_1 = \tilde D \cap D_1$ and $\tilde D_2 = \tilde D \cap D_2$. Clearly $\bar D = \bar D_1 \cup \bar D_2$, $\tilde D = \tilde D_1 \cup \tilde D_2$, where $\bar D_1 \cap \bar D_2 = \phi$, $\tilde D_1 \cap \tilde D_2 = \phi$, $\bar D_1 \cap \tilde D_2 = \phi$ and $\bar D_2 \cap \tilde D_1 = \phi$. Thus $\bar D \cap \tilde D = (\bar D_1 \cap \tilde D_1) \cup (\bar D_2 \cap \tilde D_2)$, and $\Pi^S_{\bar D \cup \tilde D} = \Pi^S_{(\bar D_1 \cap \tilde D_1)\cup(\bar D_2\cap\tilde D_2)} = \Pi^S_{(\bar D_1\cap\tilde D_1)} \cap \Pi^S_{\bar D_2\cap\tilde D_2}$. We will denote any point $x\varepsilon S$ by (x_1, x_2), where $x_1 \varepsilon S^1$ and $x_2 \varepsilon S^2$ are such that $x_1 = P_{D_1}(x)$ and $x_2 = P_{D_2}(x)$. Let $x \Pi^S_{\bar D\cap\tilde D} y$. This implies $x_1 \Pi^{S^1}_{\bar D_1\cap\tilde D_1} y_1$ and $x_2 \Pi^{S^2}_{\bar D_2\cap\tilde D_2} y_2$. Since S^1, S^2 are irredundant $\exists\ z_1^1, z_1^2, \ldots, z_1^k \varepsilon S^1$ and $w_2^1, w_2^2, \ldots, w_2^\ell$ s.t. $x_1 \Pi^{S^1}_{\bar D_1} z_1^1 \Pi^{S^1}_{\tilde D_1} z_1^2, \ldots, \Pi^{S^1}_{\bar D_1} z_1 \Pi^{S^1}_{\tilde D_1} y_1$ and $x_2 \Pi^{S_2}_{\bar D_2} w_2^1 \Pi^{S_2}_{\tilde D_2} w_2^2, \ldots, \Pi^{S_2}_{\bar D_2} w_2^\ell \Pi^{S_2}_{\tilde D_2} y_2$. Let $m = \max(k,\ell)$. W.l.o.g. we assume $\ell < k$ and set $w_2^{\ell+1} = w_2^{\ell+2} = \ldots = w_2^k = y_2$. $x_1 \Pi^{S^1}_{\bar D_1} z_1^1$ and $x_2 \Pi^{S^2}_{\bar D_2} w_2^1 \Rightarrow (x_1, x_2) \Pi^S_{\bar D_1 \cup \bar D_2} (z_1^1, w_2^1)$. By similar argument it is clear that

$$(x_1,x_2)\Pi^S_{\bar D_1\cup\bar D_2}(z_1^1,w_2^1)\Pi^S_{\tilde D_1\cup\tilde D_2}(z_1^2,w_2^2)\ldots\Pi^S_{\bar D_1\cup\bar D_2}(z_1^k,w_2^k)\Pi^S_{\tilde D_1\cup\tilde D_2}(y_1,y_2),$$ i.e.

$x\Pi^S_D p_1 \Pi^S_{\bar{D}} p_2 \ldots \Pi^S_D p_k \Pi^S_{\bar{D}}\, y$, where $p_i = (z^i_1, w^i_2)$. So $x(\Pi^S_D \cup \Pi^S_{\bar{D}})y$, which was to be proved. □

Proposition 2.4.3 is illustrated in Figure 2.4.2.

In general, union of pairwise nondisjoint family of irredundant sets is not irredundant. This is shown in the example below. We will state the conditions under which such a union is irredundant.

Example 2.4.2

Let $S_1 = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$. S_1 is of the form of Example 2.3.2 and so is irredundant. Let $S_2 = \{(0,1,1),(1,1,1)\}$. S_2 is obviously irredundant. $S_1 \cap S_2 = \{(1,1,1)\} \neq \phi$. $S = S_1 \cup S_2 = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1),(0,1,1)\}$ however is not irredundant. Consider a partition on S with two equivalence classes, $\Pi = \{\overline{(0,1,1)}, \overline{S-\{(0,1,1)\}}\}$. Then $L_1 = \{\alpha_1, \alpha_2\}$ and $L_2 = \{\alpha_1, \alpha_3\}$ are both locations of Π. □

Proposition 2.4.4

Let F be a family of irredundant subsets of $\bigtimes_{\alpha \in D} S_\alpha$ s.t. $S \cap \bar{S} \neq \phi$, for any $S, \bar{S} \in F$. Then $X = \bigcup_{S \in F} S$ is irredundant if

a) for any $S, \bar{S} \in F$ any $s \in S$, any $\bar{s} \in \bar{S}$ and any $L \subseteq D$, if $s\Pi_L \bar{s}$, then $\exists$ a point $p \in S \cap \bar{S}$ s.t. $p\Pi_L s$.

Proof

We need to show that for any $D_1, D_2 \subseteq D$, $\Pi^X_{D_1 \cap D_2} \leq \Pi^X_{D_1} \cup \Pi^X_{D_2}$. Let D_1, D_2 be given and let x,y be arbitrary points of X s.t. $x\Pi_{D_1 \cap D_2} y$. Then $x \in S, y \in \bar{S}$ for some $S, \bar{S} \in F$. If $S = \bar{S}$ the above clearly holds, since S is irredundant. So assume $S \neq \bar{S}$. Then by condition a)

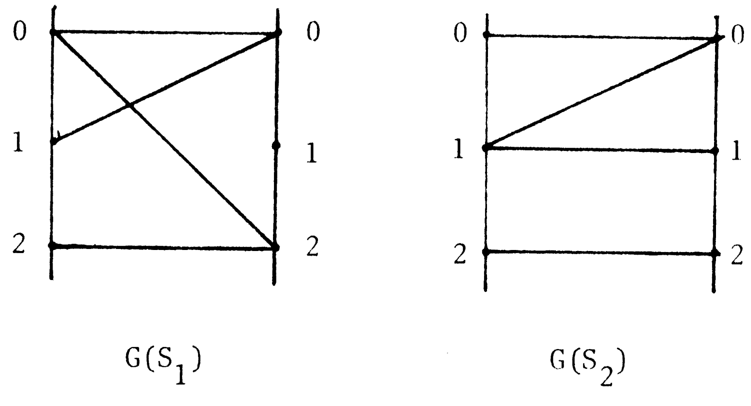
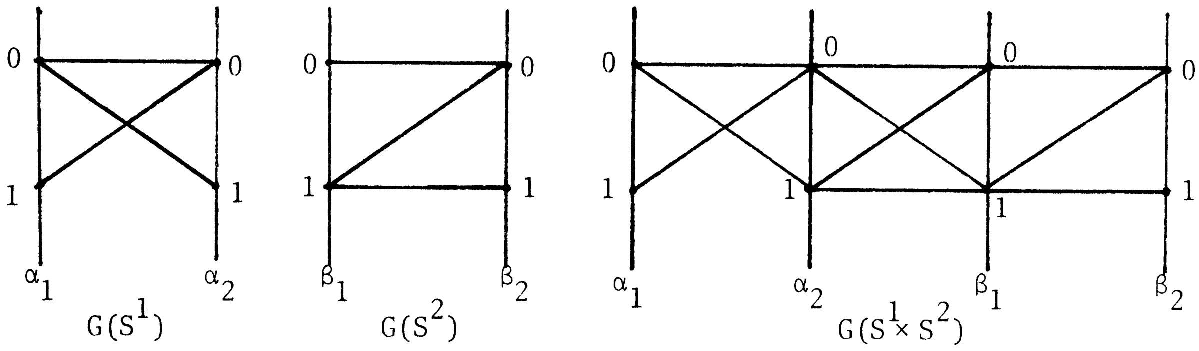


Figure 2.4.1: Graphical Interpretation of Irredundance.



$S^1 = \{(0,0),(0,1),(1,0)\}$ $S^2 = \{(0,0),(1,0),(1,1)\}$

$S^1 \times S^2 = \{(0,0,0,0),(0,0,1,0),(0,0,1,1),(0,1,0,0),(0,1,1,0),(0,1,1,1),$
$(1,0,0,0),(1,0,1,0),(1,0,1,1)\}$

Figure 2.4.2: Irredundance of S^1, S^2 Implies Irredundance of $S^1 \times S^2$.

$\exists$ a $z \varepsilon S \cap \bar{S}$ s.t. $x \Pi^S_{D_1 \cap D_2} z$ and $y \Pi^{\bar{S}}_{D_1 \cap D_2} z$. (If $D_1 \cap D_2 = \phi$, then this says simply $x, z \varepsilon S$ and $y, z \varepsilon \bar{S}$.) Now by irredundance of S and $\bar{S}$ this implies $x(\Pi^S_{D_1} \cup \Pi^S_{D_2})z$ and $y(\Pi^{\bar{S}}_{D_1} \cup \Pi^{\bar{S}}_{D_2})z$. This clearly implies that $x(\Pi^X_{D_1} \cup \Pi^X_{D_2})y$, which was to be proved. □

Remark

The sufficient condition of Proposition 2.4.4 is not a necessary one. This is demonstrated by the example below. □

Example 2.4.3

Let $X_1 = \{(a_1,b_1),(a_2,b_1),(a_2,b_2)\}$ and $X_2 = \{(a_1,b_2),(a_2,b_2),(a_3,b_2)\}$ be subsets of $\{a_1,a_2,a_3\} \times \{b_1,b_2\}$. X_1 and X_2 are both irredundant and so is $X = X_1 \cup X_2$ (refer to Figure 2.4.3). $X_1 \cap X_2 = \{(a_2,b_2)\}$ and although for $(a_1,b_1) \varepsilon X_1$, $(a_1,b_2) \varepsilon X_2$ $(a_1,b_1)\Pi_{\alpha_1}(a_1,b_2)$, there is no point z in $X_1 \cap X_2$ s.t. $(a_1,b_1)\Pi_{\alpha_1} z$.

As a corollary to Proposition 2.4.4 we will now show that a union of a family of pairwise nondisjoint Cartesian sets is always irredundant.

Corollary 2.4.1

Let Ψ be a family of Cartesian subsets of $\bigtimes_{\alpha \varepsilon D} S_\alpha$ with the property that for any C, $\bar{C}$ in Ψ, $C \cap \bar{C} \neq \phi$.

Then $X = \bigcup_{C \varepsilon \Psi} C$ is an irredundant subset of $\bigtimes_{\alpha \varepsilon D} S_\alpha$.

Proof

We just need to show that condition a) of Proposition 2.4.4 is met. Let $L \subseteq D$ and let $x \varepsilon C, y \varepsilon \bar{C}$ be such that $x \Pi^X_L y$. We need to show that $\exists$ a point $z \varepsilon C \cap \bar{C}$ s.t. $P_L(z) = P_L(x) = P_L(y)$. If $L = \phi$ this is clear for $C \cap \bar{C} \neq \phi$. If $L = D$, $x = y \in C \cap \bar{C}$. So assume $\phi \neq L \subsetneq D$. Since $C \cap \bar{C} \neq \phi$, $\exists$ a point $w \in C \cap \bar{C}$. Let z be a point defined by

$$z(\alpha) = \begin{cases} x(\alpha) & \text{for } \alpha \in L \\ w(\alpha) & \text{for } \alpha \in D-L \end{cases}$$

. Since C is Cartesian and $x,w \in C$, $z \in C$. Since $x(\alpha) = y(\alpha)$ for $\forall \alpha \in L$, and $y,w \in \bar{C}$, $z \in \bar{C}$. Thus $z \in C \cap \bar{C}$ and z is as required. □

In general, complements of irredundant sets are not necessarily irredundant, as is illustrated below. Complements of some irredundant sets however are always irredundant. The next few propositions will demonstrate types of irredundant sets, for which this is the case.

Example 2.4.4

Let $S = \{a_1,a_2,a_3\}^2$ and let $X = \{(a_1,a_1),(a_1,a_2),(a_2,a_3),(a_3,a_1),(a_3,a_2),(a_3,a_3)\}$. Then $X^c = S - C = \{(a_1,a_3),(a_2,a_1),(a_2,a_2)\}$. X is irredundant while its complement is not. For the graphical interpretation refer to Figure 2.4.4. □

We will now prove that a complement of a Cartesian set is always irredundant (in general not Cartesian).

Proposition 2.4.5

Let $S = \bigtimes_{\alpha \in D} S_\alpha$ and let Y be a Cartesian subset of S. Then S-Y is irredundant.

Proof

Since Y is Cartesian, $Y = \bigtimes_{\alpha \in D} Y_\alpha$. $S-Y = \bigtimes_{\alpha \in D} S_\alpha - \bigtimes_{\alpha \in D} Y_\alpha = \bigcup_{\alpha \in D} \left(\bigtimes_{\beta \in D} \bar{S}^\alpha_\beta \right)$,

where $\bar{S}^\alpha_\beta = \begin{cases} S_\beta - Y_\beta & \text{, for } \beta = \alpha \\ S_\beta & \text{, for } \beta \neq \alpha \end{cases}$. This can be easily shown by induction on $|D|$. Let $\bar{D} = \{\alpha | \alpha \in D \text{ \& } Y_\alpha = S_\alpha\}$. If $\bar{D} = D$, then $S = Y$ and

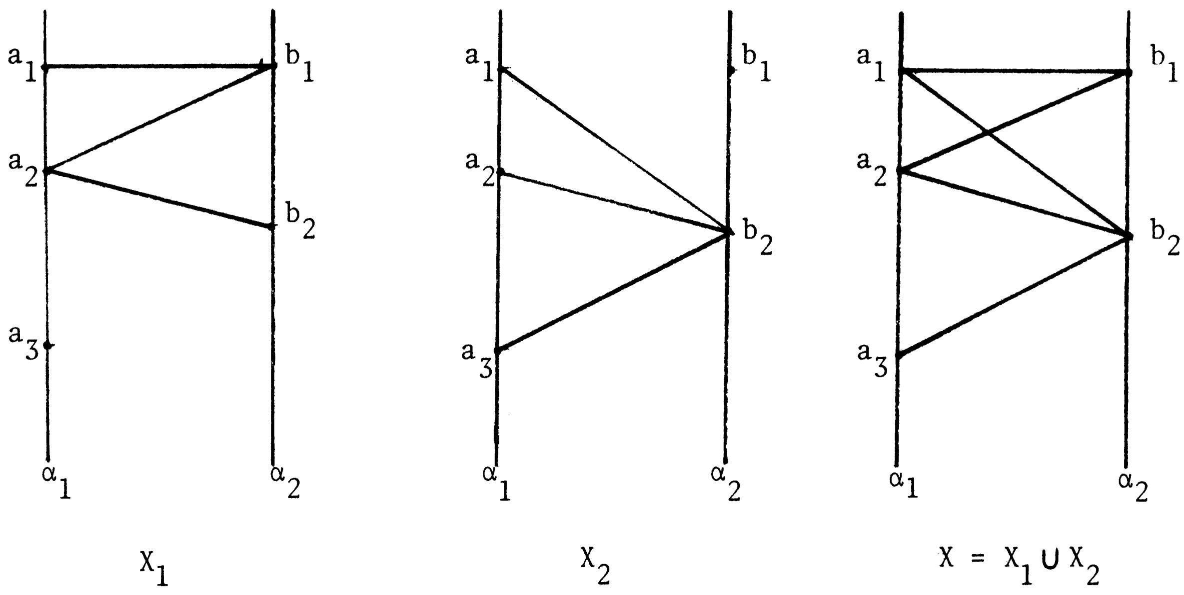


Figure 2.4.3: Irredundance of Union of Sets.

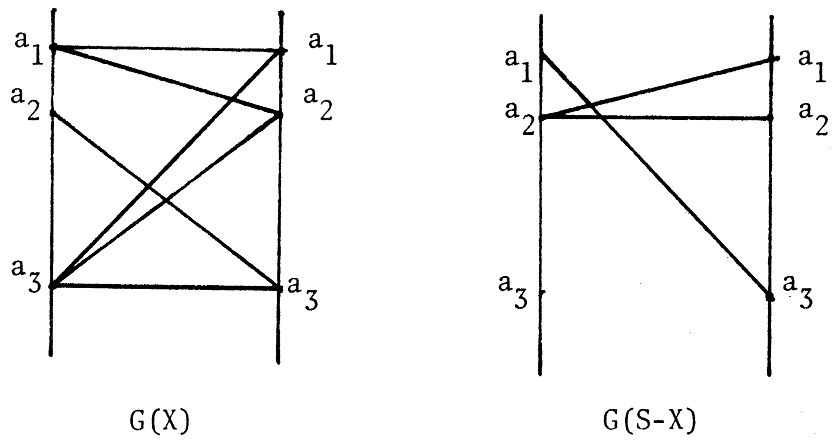


Figure 2.4.4: Irredundance of X Does Not Imply Irredundance of its Complement.

$S-Y = \phi$ is vacuously irredundant. So assume $\bar{D} \subsetneq D$. Then for every $\alpha \in \bar{D}$, $\bigtimes_{\beta \in D} \bar{S}^{\alpha}_{\beta} = \phi$. Hence $S-Y = \bigcup_{\alpha \in D-\bar{D}} (\bigtimes_{\beta \in D} \bar{S}^{\alpha}_{\beta})$. Now $\bigcap_{\alpha \in D-\bar{D}} (\bigtimes_{\beta \in D} \bar{S}^{\alpha}_{\beta}) = \bigtimes_{\alpha \in D} \hat{S}_{\alpha}$,

where $\hat{S}_{\alpha} = \begin{cases} S_{\alpha} & \text{for } \alpha \in \bar{D} \\ S_{\alpha} - Y_{\alpha} & \text{for } \alpha \in D-\bar{D} \end{cases}$. $\hat{S}_{\alpha} \neq \phi$, for $\forall \alpha$. Thus S-Y is a union of a family of Cartesian sets with nonempty intersection and is irredundant by Corollary 2.4.1. □

Remark

Since every singleton is a Cartesian set, its complement is always irredundant. In other words removal of any point from a Cartesian set does not change its irredundance. □

As was shown in Proposition 2.3.2 every minimal independent set is irredundant. We shall now prove that its complement is also irredundant.

Proposition 2.4.6

Let S be Cartesian, $S = \bigtimes_{\alpha \in D} S_{\alpha}$, where $|D| = n$. Let X be an independent subset of S with $|X| = n+1$.

Then (S-X) is irredundant.

Proof

We will use induction on $n (n \geq 2)$.

1) Induction base. $n = 2$ case. Let $D = \{\alpha_1, \alpha_2\}$ and $X = \{x^1, x^2, x^3\}$. Then $\exists\, x \in X$ s.t. α_1 is constant on $X - \{x\}$ and its value on $X - \{x\}$ is distinct from $P_{\alpha_1}(x)$. (This follows from the proof of Lemma 2.3.1.) W.l.o.g. we assume that $x = x^1$. Thus $P_{\alpha_1}(x^2) = P_{\alpha_1}(x^3) \neq P_{\alpha_1}(x^1)$. Also $P_{\alpha_2}(x^1) = P_{\alpha_2}(x^2)$ or $P_{\alpha_2}(x^1) = P_{\alpha_2}(x^3)$ holds. W.l.o.g. assume

$P_{\alpha_2}(x^1) = P_{\alpha_2}(x^2)$. $X = \{x^1\} \cup \{x^2,x^3\} = \{(x^1_{\alpha_1},x^2_{\alpha_2})\} \cup \{x^2_{\alpha_1}\} \times \{x^2_{\alpha_2},x^3_{\alpha_2}\}$.
Thus $S_{\alpha_1} \times S_{\alpha_2} - X = \{x^2_{\alpha_1}\} \times (S_{\alpha_2} - \{x^2_{\alpha_2},x^3_{\alpha_2}\}) \cup (S_{\alpha_1} - \{x^2_{\alpha_1}\}) \times S_2 - \{(x^1_{\alpha_1},x^2_{\alpha_2})\} = \{x^2_{\alpha_1}\} \times (S_2 - \{x^2_{\alpha_2}\} - \{x^3_{\alpha_2}\}) \cup (S_{\alpha_1} - \{x^2_{\alpha_1}\} - \{x^1_{\alpha_1}\}) \times S_2 \cup \{x^1_{\alpha_1}\} \times (S_2 - \{x^2_{\alpha_2}\})$. We note that $S_2 - \{x^2_{\alpha_2}\} - \{x^3_{\alpha_2}\} \subseteq S_2 - \{x^2_{\alpha_2}\} \subseteq S_2$ and also that each of the three sets in the union is irredundant. Thus $\Pi_{\alpha_1} \cup \Pi_{\alpha_2} = I$ holds on S-X and S-X is irredundant.

2) (*)We assume that for $n = \ell(\geq 2)$ the proposition holds. We need to prove it holds then for $n = \ell+1$.

W.l.o.g. let $D = \{\alpha_1,\alpha_2,\ldots,\alpha_\ell,\alpha_{\ell+1}\}$ and $X = \{x^1,x^2,\ldots,x^\ell,x^{\ell+1},x^{\ell+2}\}$ be enumerations of D and X s.t. $P_{\alpha_1}(x^i)$ is constant for all $i > 1$ and $P_{\alpha_1}(x^1) \neq P_{\alpha_1}(x^i)$ for all $i \neq 1$. (We know a γ and an x^γ with this property do exist.) We also note that $\exists$ an $x^j \in X - \{x^1\} = \{x^2,x^3,\ldots,x^{\ell+2}\}$ s.t. $P_{D-\alpha_1}(x^1) = P_{D-\alpha_1}(x^j)$, i.e. $P_{D-\alpha_1}(x^1) \subseteq P_{D-\alpha_1}(\{x^2,x^3,\ldots,x^j,\ldots,x^{\ell+2}\})$. $S_1 \times S_2 \times \ldots \times S_\ell \times S_{\ell+1} - \{x^1,x^2,\ldots,x^\ell,x^{\ell+1}\} = \{x^1_{\alpha_1}\} \times S_2 \times \ldots \times S_\ell \times S_{\ell+1} - \{x^1\} \cup$

$(S_1 - \{x^1_{\alpha_1}\}) \times S_2 \times \ldots \times S_\ell \times S_{\ell+1} - \{x^2,x^3,\ldots,x^\ell,x^{\ell+1}\} =$

$\{x^1_{\alpha_1}\} \times (S_2 \times \ldots \times S_\ell \times S_{\ell+1} - P_{D-\alpha_1}(x^1)) \cup$

$(S_1 - \{x^1_{\alpha_1}\} - \{x^2_{\alpha_1}\} \cup \{x^2_{\alpha_1}\}) \times S_2 \times \ldots \times S_\ell \times S_{\ell+1} -$

$\{x^2_{\alpha_1}\} \times P_{D-\{\alpha_1\}}(\{x^2,x^3,\ldots,x^{\ell+1}\}) = \{x^1_{\alpha_1}\} \times (S_2 \times \ldots \times S_{\ell+1} - P_{D-\alpha_1}(x^1)) \cup$

$(S_1 - \{x^1_{\alpha_1}\} - \{x^2_{\alpha_1}\}) \times S_2 \times \ldots \times S_\ell \times S_{\ell+1} \cup$

$\{x^2_{\alpha_1}\} \times (S_2 \times \ldots \times S_\ell \times S_{\ell+1} - P_{D-\{\alpha_1\}}(\{x^2,\ldots,x^{\ell+1}\})) = Y_1 \cup Y_2 \cup Y_3$.

We note that Y_1 is irredundant as a product of irredundant sets (by Proposition 2.4.5 and 2.4.3). Y_2 is irredundant since it is Cartesian.

Finally since the set $\hat{X} = P_{D-\{\alpha_1\}}(\{x^2,\ldots,x^{\ell+1}\})$ is a minimal independent subset of $S_2 \times \ldots \times S_{\ell+1}$, by the assumption (*), $(S_2 \times \ldots \times S_{\ell+1} - \hat{X})$ is irredundant. Thus by Proposition 2.4.3 Y is irredundant.

Hence Y_1, Y_2, Y_3 are all irredundant. We need to prove that so is their union. We need to show that for any $x,y \in S-X$, any $D_1,D_2 \subseteq D$ s.t. $x\Pi_{D_1 \cap D_2}y$, $x(\Pi_{D_1} \cup \Pi_{D_2})y$. If both x and y are in Y_i, for some i, then $x(\Pi_{D_1} \cup \Pi_{D_2})y$ clearly holds, since $\forall\, Y_i$ are irredundant. Now if $x \in Y_i$, $y \in Y_j$, $i \neq j$, then $\alpha_1 \notin D_1 \cap D_2$. This is so, because $P_{\alpha_1}(Y_i) \cap P_{\alpha_2}(Y_j) = \phi$ for $i \neq j$. W.l.o.g. we assume that $\alpha_1 \notin D_2$, i.e. that $D_2 \subseteq D-\alpha_1$.

We note that $P_{D-\alpha_1}(Y_3) \subseteq P_{D-\alpha_1}(Y_1) \subseteq P_{D-\alpha_1}(Y_2)$ holds, so for any pair i,j, $i \neq j$ either $P_{D-\alpha_1}(Y_i) \subseteq P_{D-\alpha_1}(Y_j)$ or $P_{D-\alpha_1}(Y_j) \subseteq P_{D-\alpha_1}(Y_i)$. Say, $P_{D-\alpha_1}(Y_i) \subseteq P_{D-\alpha_1}(Y_j)$ holds. Then for $x \in Y_i$ $\exists$ a $z \in Y_j$ s.t. $P_{D-\alpha_1}(x) = P_{D-\alpha_1}(z)$. But $D_2 \subseteq D-\alpha_1 \Rightarrow x\Pi_{D_2}z \Rightarrow x(\Pi_{D_1} \cup \Pi_{D_2})z$. Now $z \in Y_j$, $y \in Y_j$ and $z\Pi_{D_1 \cap D_2}x \Rightarrow z\Pi_{D_1 \cap D_2}y$. But Y_j is irredundant and so $z(\Pi_{D_1} \cup \Pi_{D_2})y$. This implies that $x(\Pi_{D_1} \cup \Pi_{D_2})y$, which was to be proved. □

We will next prove that a complement of an arbitrary subset of a Cartesian set, all of whose coordinate projections are properly contained in corresponding coordinate projections of a Cartesian set, is irredundant. Further we will show that every superset of such a complement is irredundant. First we will prove an auxiliary lemma.

Lemma 2.4.1

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$ for all $\alpha\varepsilon D$. Let $C \subset S$, where $C = \bigtimes_{\alpha\varepsilon D} C_\alpha$ and $C_\alpha \subsetneq S_\alpha$.

Then any set $Y \supseteq S-C$ is irredundant.

Proof

Let $X = S-C$. Then by Proposition 2.4.5 X is irredundant. We will assume then that $Y \supsetneq S-C$. We will now show that for every $z\varepsilon C$, every $\alpha\varepsilon D$, $\exists$ an $x^\alpha \varepsilon X$ s.t. $z\Pi_{D-\alpha}x^\alpha$. Let $a_\alpha \; \varepsilon \; S_\alpha - C_\alpha$. Then $x^\alpha = (z_1,\ldots,z_{\alpha-1},a_\alpha,z_{\alpha+1},\ldots,z_n)\varepsilon X$, for $a_\alpha \notin C_\alpha$. Obviously $x^\alpha \Pi_{D-\alpha} z$.

We need to show that for any $x,y\varepsilon Y$ any $D_1,D_2 \subseteq D$, if $x\Pi^Y_{D_1\cap D_2}y$, then $x(\Pi^Y_{D_1} \cup \Pi^Y_{D_2})y$.

Now if $D_1 = D$, $D_1 = \phi$, $D_2 = D$ or $D_2 = \phi$ the above clearly holds. So assume $\phi \neq D_1 \neq D$, $\phi \neq D_2 \neq D$. Let $x,y\varepsilon Y$ be as above. Then $\exists$ $z,w\varepsilon X$ s.t. $x\Pi^Y_{D_1} z$, $y\Pi^Y_{D_1} w$. If $x\varepsilon X$ take $z = x$, if $y\varepsilon X$, take $w = y$. Since $X = S-C$, if $x\varepsilon Y-X$, then $x\varepsilon C$. But we showed before that then $\exists$ a point $z\varepsilon X$ s.t. $z\Pi_{D_1} x$. (Note that $D_1 \subseteq D-\alpha$ for some $\alpha\varepsilon D$.) The same holds for $y\varepsilon Y-X$. Since X is irredundant and $w\Pi^X_{D_1\cap D_2}z$, $w(\Pi^X_{D_1} \cup \Pi^X_{D_2})z$. Thus $x(\Pi^Y_{D_1} \cup \Pi^Y_{D_2})y$, which we set to prove.

For an illustration of this proof refer to Figure 2.4.5. □

Corollary 2.4.2

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$ for $\forall$ $\alpha\varepsilon D$. Let Y be an arbitrary subset of S with the property, that $P_\alpha(Y) \subsetneq S_\alpha$, for all $\alpha\varepsilon D$. Then every set X s.t. $X \supseteq S-Y$ is irredundant.

Proof

Clearly $Y \subseteq \bigtimes_{\alpha \in D} P_\alpha(Y)$ and so $S-Y \supseteq S - \bigtimes_{\alpha \in D} P_\alpha(Y)$. Every superset of $S-Y$ is clearly a superset of $S - \bigtimes_{\alpha \in D} P_\alpha(Y)$ and so is irredundant by Lemma 2.4.1. □

We will illustrate the above corollary by an example.

Example 2.4.5

1) Let $S = \{0,1,2\}^2$. Then $X = S - \{1,2\} \times \{1,2\} = S-C$ is irredundant and so are all supersets of X (refer to Figure 2.4.6).

2) Let $S = \{0,1,2\}^3$. Then $X = S - \{(0,1,0),(1,2,1),(0,2,0)\}$ is irredundant and so are all its supersets. □

2.5 Conclusions

The results of sections 2.3 and 2.4 have important implications for automatic construction of irredundant sets. We will summarize the more important ones among them.

Suppose that a Cartesian set S is given and we want to automatically generate irredundant subsets of S.

Clearly any proper Cartesian subset C of S can be generated easily. Then so can the set difference S-C. Both of those were proved to be irredundant.

If C is such that $C_\alpha \subsetneq S_\alpha$, for all $\alpha \varepsilon D$, we proved that every superset of S-C is irredundant. All such supersets can again be very easily generated.

Finally, we can easily generate minimal independent subsets of S (especially those of Example 2.3.2), which were also proved to be irredundant.

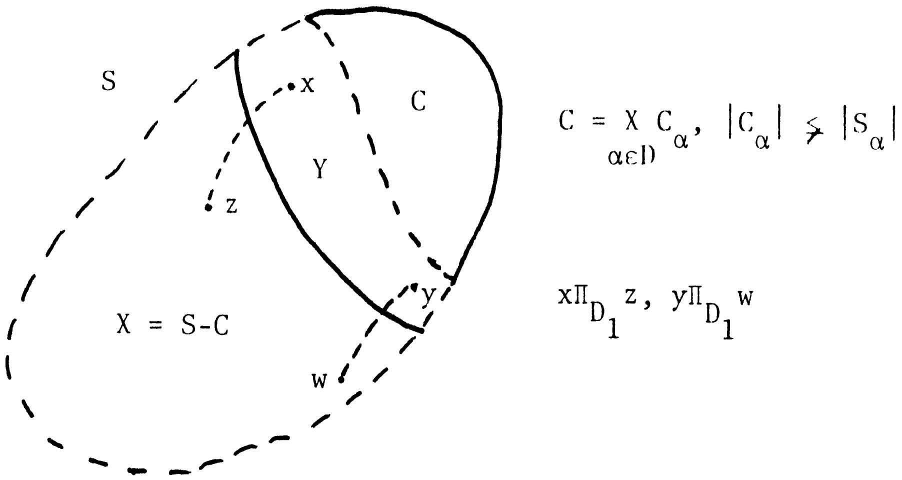


Figure 2.4.5: Illustration of the Proof of Lemma 2.4.1.

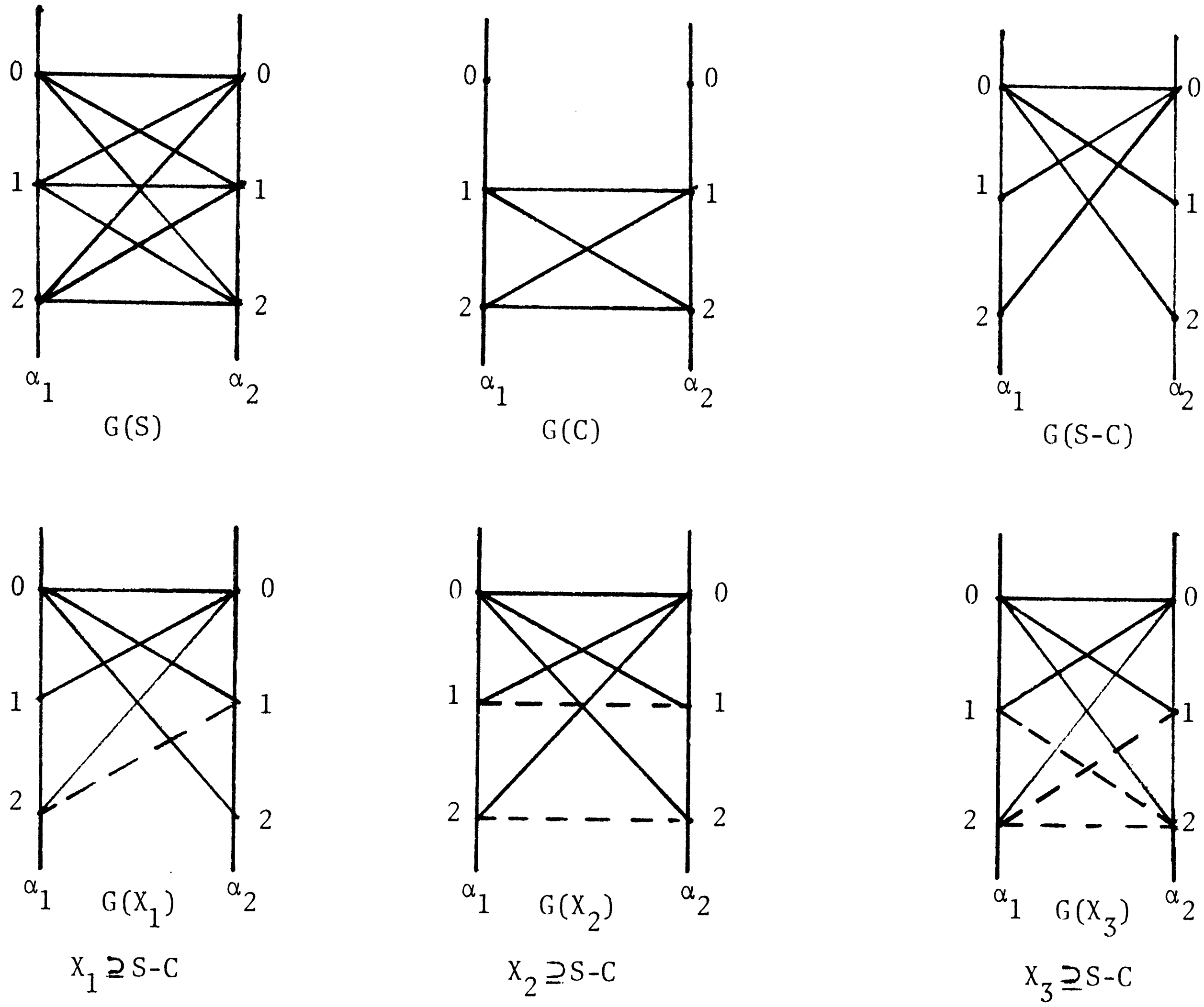


Figure 2.4.6: All Supersets of $(\underset{\alpha\varepsilon D}{X} S_\alpha - \underset{\alpha\varepsilon D}{X} C_\alpha)$ Are Irredundant, Where $C_\alpha \subsetneq S_\alpha$.

Thus there is a subclass of all irredundant subsets of S, which can be easily automatically generated on the computer.

The results of this chapter also often facilitate the verification of irredundance of sets under consideration. For example once we know that a set is a union of a family of nondisjoint Cartesian sets, we know it is irredundant without any need for further verification. Same goes for other types of sets, for example a projection of an irredundant set onto any subset of its index set.

If a set is known to have s.e.p. property it is irredundant if and only if it is Cartesian. Verifying whether a set is Cartesian is clearly much easier than verifying its irredundance directly from the definition.

Checking if S has s.e.p. is again facilitated by Proposition 2.2.6. We do not need to check whether every permutation of S has e.p., but just whether all cycle type of permutations of S have e.p.

The problem, which still remains open, is that of finding a minimal irredundant superset of a given set X. It is hoped that the closure of a set X with respect to extension property will turn out to be useful in generating a minimal irredundant superset of X. The solution of this problem will have implications for the design of experimentation strategies. Why this is so will become clear in later chapters of the thesis.

CHAPTER III

FUNCTIONS WITH STRUCTURED DOMAINS

3.1 Introduction

In this chapter we will discuss properties of functions from structured domains, whose codomains are arbitrary abstract sets (not structured). The results obtained here will be later used, when functions from structured sets to structured sets will be considered. In particular the results are going to be applied to finite autonomous discrete time systems, where the functions involved will be the transition functions on a structured state space.

We will start by introducing the concept of a location of a function. This concept refers to a minimal subset L of an index set, such that Π_L refines the partition of kernel equivalence of a function. We next explore the relations between locations of functional restrictions to a sequence of nested subsets of a function domain. Based on those results, we prove that for any finite family of functions with a common, infinitely countable domain, there exists a finite domain-subset X with the following property: for every function f in the family, $\mathcal{L}(f) = \mathcal{L}(f|X)$, where $\mathcal{L}(f)$ and $\mathcal{L}(f|X)$ are the families of locations of f and $f|X$ respectively.

We will then discuss extensions of functions from proper domain subsets with given locations.

Finally, for finite Cartesian domains, we show how to construct proper domain subsets with special properties. Namely, given an upper bound on a location size of the function, we construct a minimal proper domain subset such that the restriction of the function to the subset determines uniquely the function itself.

3.2 Properties of Function Restrictions to a Family of Nested Domain-Subsets

The notion of a location of a function is intimately related to that of a location of a partition. Actually it will be defined to be a location of a partition of kernel equivalence of a function.

We recall that for a function f from S, a kernel partition of f denoted by Π_f, is a partition on S defined by: $s\Pi_f s'$ iff $f(s) = f(s')$, for all s and s' in S.

Definition 3.2.1 ([Z1])

Let f be an arbitrary function from S, where $S \subseteq \bigtimes_{\alpha\in D} S_\alpha$. $D' \subseteq D$ is a location of f iff D' is a location of Π_f. □

Definition 3.2.2 ([Z1])

A function f from S, where $S \subseteq \bigtimes_{\alpha\in D} S_\alpha$, is in reduced form if D is a location of f. □

In the sequel we will denote the family of all locations of f by $\mathcal{L}(f)$. In case the location of f is unique we will often denote it by L(f).

The restriction of f to a subset X of S will be denoted by $f|X$.

We will now show that for any two subsets X_1 and X_2 of S, where $X_1 \subseteq X_2$, the following relation holds. For an arbitrary function f from S, every location of $f|X_2$ contains at least one location of $f|X_1$. We also prove that the converse of this statement is not true. That is, there might exist a location of $(f|X_1)$, which is not contained in any of the locations of $(f|X_2)$.

Proposition 3.2.1

Let f be a function from $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ and let X_1, X_2 be any subsets of S, where $X_1 \subseteq X_2$. Then

a) for every $L \varepsilon \mathcal{L}(f|X_2)$ $\exists$ an $\tilde{L} \varepsilon \mathcal{L}(f|X_1)$ s.t. $L \supseteq \tilde{L}$

but

b) the statement

"for every $\tilde{L} \varepsilon \mathcal{L}(f|X_1)$ $\exists$ an $L \varepsilon \mathcal{L}(f|X_2)$ s.t. $\tilde{L} \subseteq L$ "

is not generally true.

Proof

a) Let $L \varepsilon \mathcal{L}(f|X_2)$. Then $\Pi_L^{X_2} \leq \Pi_{f|X_2}^{X_2}$. We note that $\Pi_L^{X_1} = (\Pi_L^{X_2}|X_1)$ and $\Pi_{f|X_1}^{X_1} = (\Pi_{f|X_2}^{X_2}|X_1)$. Hence $(\Pi_L^{X_2}|X_1) \leq (\Pi_{f|X_2}^{X_2}|X_1)$, i.e. $\Pi_L^{X_1} \leq \Pi_{f|X_1}^{X_1}$. But this implies that L contains a location of $f|X_1$.

b) It suffices to give a counterexample. Consider $S = \{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2)\} = \{a_1,a_2\} \times \{b_1,b_2\}$, where $D = \{\alpha,\beta\}$. Let $X_1 = \{(a_1,b_1),(a_2,b_2)\}$ and let $X_2 = S$. Clearly $X_1 \subsetneq X_2$. Let f be an arbitrary function from S with a kernel partition $\Pi_f^S = \{\overline{(a_1,b_1),(a_1,b_2)}, \overline{(a_2,b_1),(a_2,b_2)}\}$. X_2 is irredundant and thus f has a unique location. $L(f|X_2) = L(f) = L(\Pi_f^S) = \{\alpha\}$.

$$\Pi_{f|X_1}^{X_1} = \{\overline{(a_1,b_1)}, \overline{(a_2,b_2)}\} \text{ and } \mathcal{L}(f|X_1) = \mathcal{L}(\Pi_{f|X_1}^{X_1}) = \{\{\alpha\},\{\beta\}\}.$$

Obviously $\beta \not\subseteq \alpha$, which completes the proof. □

Remarks

1) If $|L(f|X_1)| = 1$, that is if $f|X_1$ has a unique location, then this location is contained in every location of $f|X_2$, where $X_1 \subseteq X_2$.

2) For any sequence of nested irredundant subsets of S, the corresponding sequence of locations of function restrictions to the subsets is totally ordered by set inclusion. □

We show next that for an irredundant set S and any irredundant nonempty subset X of S the following holds. If f_1 and f_2 are arbitrary functions from S with equal locations, such that the kernel partition of $f_1|X$ refines the kernel partition of $f_2|X$, then the equality of the location of f_2 and the location of $f_2|X$ implies the equality of the location of f_1 and the location of $f_1|X$.

Proposition 3.2.2

Let S be irredundant and let f_1, f_2 be arbitrary functions from S s.t. $L(f_1) = L(f_2)$. Then for an arbitrary nonempty irredundant subset X of S, if $(\Pi_{f_1}|X) \leq (\Pi_{f_2}|X)$ and if $L(f_2) = L(f_2|X)$, then $L(f_1) = L(f_1|X)$.

Proof

$\Pi_{f_1}|X \leq \Pi_{f_2}|X \Rightarrow L(\Pi_{f_1}|X) \supseteq L(\Pi_{f_2}|X)$, i.e. $L(f_1|X) \supseteq L(f_2|X)$. But $L(f_2|X) = L(f_2) = L(f_1)$ and so $L(f_1|X) \supseteq L(f_1)$. By Proposition 3.2.1 $L(f_1) \supseteq L(f_1|X)$ and hence $L(f_1) = L(f_1|X)$. □

Suppose that $X_1 \subsetneq X_2 \subsetneq X_3 \subseteq S$ and that $L(f|X_1) = L(f|X_2)$. One might then attempt a guess that from this point on the equality of families of locations of f restricted to supersets of X_2 will obtain, in particular that $L(f|X_3) = L(f|X_2)$. As we illustrate below, this turns out to be a misleading guess.

Example 3.2.1

Consider $S = \{0,1\}^3$ and $X_1 = \{(0,0,0),(1,0,0)\}$, $X_2 = X_1 \cup \{(0,1,0),(1,1,0)\}$, $X_3 = S$. Let $D = \{\alpha_1,\alpha_2,\alpha_3\}$ and let f be any function from S with a kernel partition

$\Pi_f = \{\overline{(0,0,0),(0,1,0),(1,0,1)},\ \overline{(1,0,0),(1,1,0),(0,0,1),(0,1,1),(1,1,1)}\}$.

We note that X_1,X_2,X_3 are Cartesian and so irredundant.

$\Pi_{f|X_1} = \{\overline{(0,0,0)},\ \overline{(1,0,0)}\}$ and $\Pi_{f|X_2} = \{\overline{(0,0,0),(0,1,0)},\ \overline{(1,0,0),(1,1,0)}\}$

$L(f|X_1) = \{\alpha_1\}$, $L(f|X_2) = \{\alpha_1\}$ and $L(f) = \{\alpha_1,\alpha_2,\alpha_3\}$. So $L(f|X_1) = L(f|X_2) \neq L(f|X_3)$. This illustrates the above assertion. □

The example is illustrated by Figure 3.2.1.

We next show that if a set is a union of a pairwise nondisjoint family of Cartesian sets, then for any function from this set there exists a decomposition of its location along the "constructing components". Namely, the location of the function is equal to the union of the locations of its restrictions to all the sets of the family.

Proposition 3.2.3

Let $S = \bigtimes_{\alpha\in D} S_\alpha$ and let Ψ be a family of Cartesian subsets of S s.t. for any $C,\tilde{C}\in\Psi$, $C \cap \tilde{C} \neq \phi$. Further let $X = \bigcup_{C\in\Psi} C$.

Then for any function f from X

$$L(f) = \bigcup_{C\in\Psi} L(f|C).$$

Proof

Let $L_C = L(f|C)$, for all $C\in\Psi$. Since $C \subseteq X$, by Proposition 3.2.1 $L = L(f) \supseteq L_C$, for all $C\in\Psi$. So $L \supseteq \bigcup_{C\in\Psi} L_C$ holds. We need to show that $\bigcup_{C\in\Psi} L_C \supseteq L$, or equivalently that for any $x,y\in X$ if $x\Pi^X_{\cup_{C\in\psi} L_C}\, y$ then

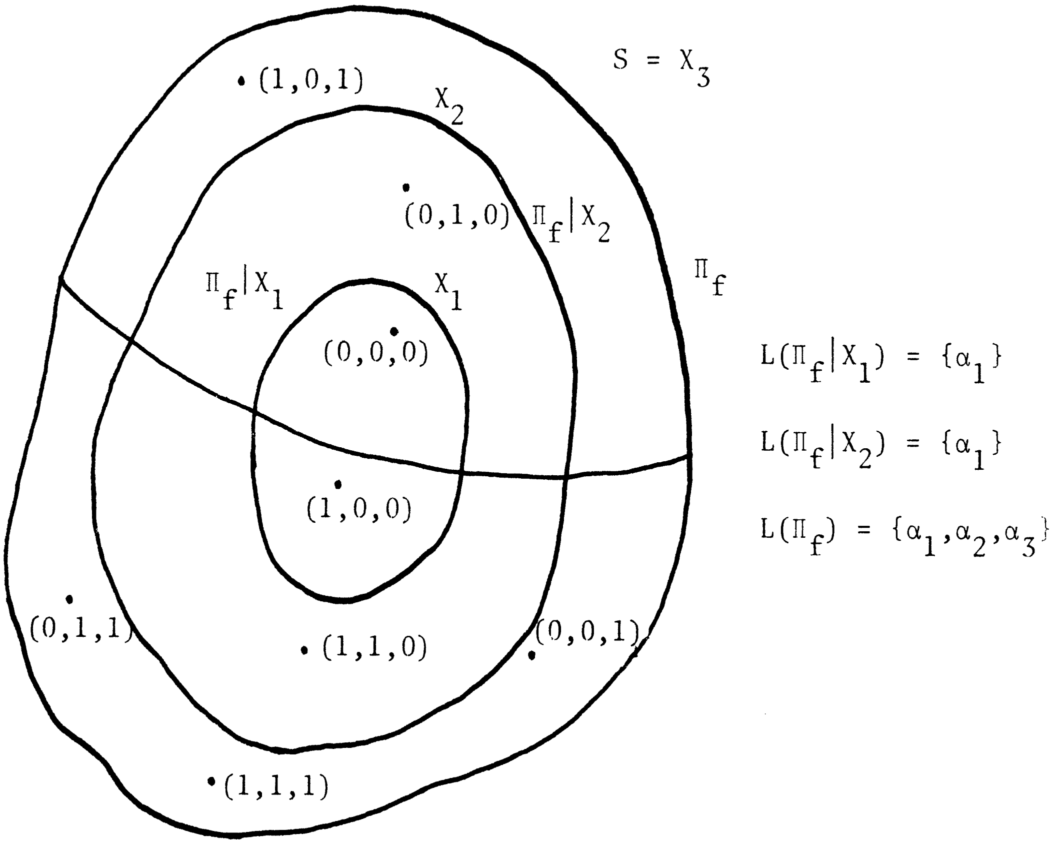


Figure 3.2.1: Illustration of Example 3.2.1.

$f(x) = f(y)$. Let x,y be such points.

Then $x \in C, y \in \tilde{C}$, for some $C, \tilde{C}$ in Ψ. Also $x \Pi_{L_C \cup L_{\tilde{C}}} y$ and since $C \cap \tilde{C} \neq \phi$, $\exists$ a $z \in C \cap \tilde{C}$ s.t. $x \Pi_{L_C \cup L_{\tilde{C}}} z \Pi_{L_C \cup L_{\tilde{C}}} y$. (See the proof of Corollary 2.4.1.) So $x \Pi_{L_C} z$ and $y \Pi_{L_{\tilde{C}}} z$. Since $x,z \in C$, $y,z \in \tilde{C}$ and L_C and $L_{\tilde{C}}$ are the locations of $f|C$ and $f|\tilde{C}$ respectively, $f(x) = f(z)$ and $f(y) = f(z)$ hold. Thus $f(x) = f(y)$, which was to be proved. □

<u>Remark</u>

The equality of Proposition 3.2.3 does not necessarily hold for a pairwise nondisjoint family Ψ of irredundant sets, which are not Cartesian, even if their union is irredundant. The location of every function f from $\bigcup_{X \in \Psi} X$ always contains $\bigcup_{X \in \Psi} L(f|X)$, but this containment may be proper. □

<u>Example 3.2.2</u>

a) Consider $X = \{(0,0),(1,0),(1,1)\} \cup \{(1,1),(2,1),(2,2),(0,2)\} = X_1 \cup X_2$. All X_1, X_2 and X are irredundant and $X_1 \cap X_2 \neq \phi$.

Let $\Pi_f = \{\overline{(0,0),(1,0),(1,1)}, \overline{(2,1),(2,2),(0,2)}\}$. Then $L(\Pi_f) = \{\alpha_1, \alpha_2\}$.

$\Pi_f|X_1 = \{\overline{(0,0),(1,0),(1,1)}\} = I$ and $L(\Pi|X_1) = \phi$.

$\Pi_f|X_2 = \{\overline{(1,1)}, \overline{(2,1),(2,2),(0,2)}\}$ and $L(\Pi|X_2) = \{\alpha_1\}$.

$L(\Pi_f) \supsetneq L(\Pi_f|X_1) \cup L(\Pi_f|X_2)$, for $\{\alpha_1, \alpha_2\} \supsetneq \{\alpha_1\}$.

(For illustration refer to Figure 3.2.2.)

b) Consider $X = \{(0,0),(1,0)\} \cup \{(1,0),(1,1)\} = C_1 \cup C_2$.

$C_1 \cap C_2 = \{(1,0)\} \neq \phi$ and C_1, C_2 are both Cartesian. Let f be any function from X with a kernel partition $\Pi_f = \{\overline{(0,0)}, \overline{(1,0),(1,1)}\}$.

Then $L(f) = \{\alpha_1\}$ and $L(f|C_1) = \{\alpha_1\}$, $L(f|C_2) = \phi$. Clearly $L(f) = L(f|C_1) \cup L(f|C_2)$. (For illustration refer to Figure 3.2.3.) □

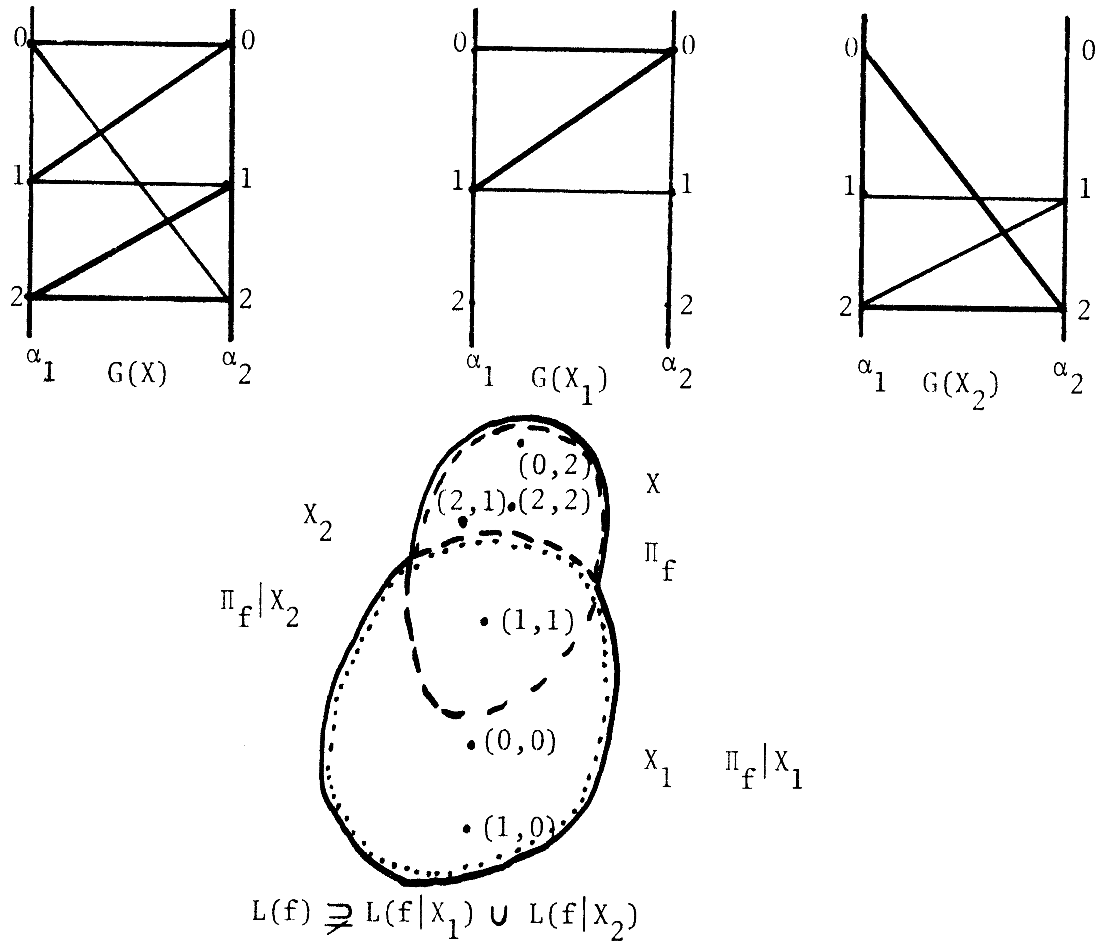


Figure 3.2.2: Illustration of Example 3.2.2 a).

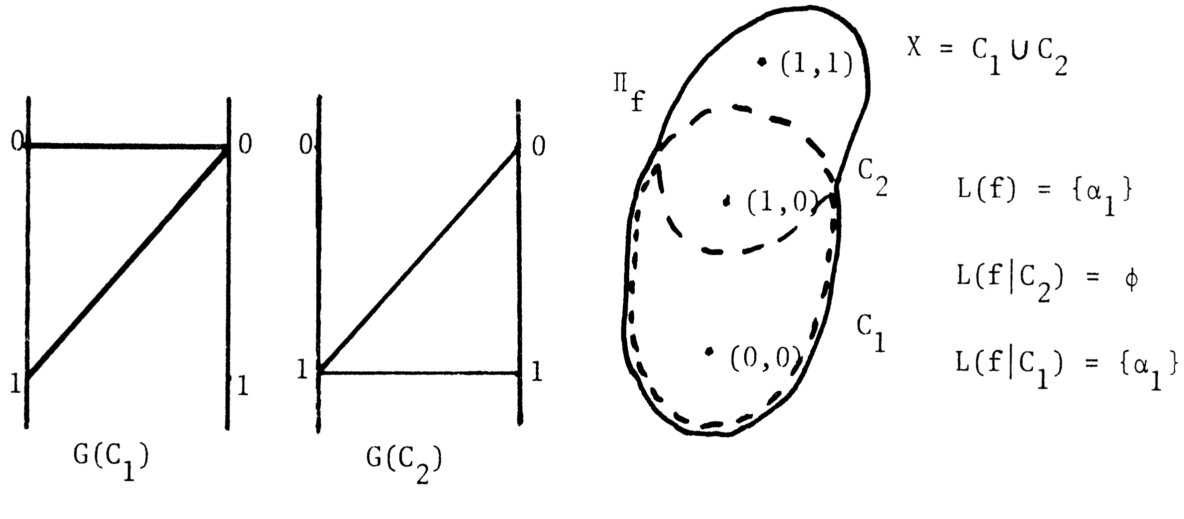


Figure 3.2.3: Illustration of Example 3.2.2 b).

Next we will consider an infinite set S structured over a finite index set D and an arbitrary function from S. We will show that for any sequence $\{X_i\}_{i=1}^{\infty}$ of nested subsets of S, which constitute its cover, and any function f from S, there exists a point N in the sequence, such that for every $i \geq N$ $L(f|X_i) = L(f)$.

As a corollary we will then prove that same holds for every finite family of functions from S.

Before we do that however, we need to establish some auxiliary results.

We will now prove that for S and a sequence $\{X_i\}_{i=1}^{\infty}$ as above the following is true. Given any f from S and any sequence $\{L_i\}$ of locations, such that $L_i \varepsilon L(f|X_i)$ and $\{L_i\}$ is totally ordered by set inclusion, the sequence $\{L_i\}$ becomes constant and equal to one of the locations of f.

Lemma 3.2.1

Let $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$, where $|D| = n$. Let f be an arbitrary function from S and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of subsets of S s.t. $X_{i+1} \supseteq X_i$ for all i and $\bigcup_{i=1}^{\infty} X_i = S$. Then if $\{L_i\}_{i=1}^{\infty}$ is a sequence of subsets of D, s.t. $L_i \subseteq L_{i+1}$ and $L_i \varepsilon L(f|X_i)$, $\exists$ N s.t. for $\forall i \geq N$, $L_i = L_N$ and $L_N \varepsilon L(f)$.

Proof

Our proof will consist of two parts. First we will show that $\exists$ N s.t. $L_i = L_N$ for all $i \geq N$. Then we will prove that $L_N \varepsilon L(f)$.

a) Since all L_i are finite (this follows from finiteness of D) and totally ordered the sequence becomes constant after a finite number of elements. So N as above exists.

b) We first show that L_N as above contains a location of f. To do so it suffices to show that $\Pi^S_{L_N} \leq \Pi_f$, i.e. that for any $x,y\varepsilon S$ s.t. $x\Pi_{L_N} y$, $f(x) = f(y)$. Since $\cup X_i = S$ and $X_{i+1} \supseteq X_i$, $\exists\, N_{x,y}$ s.t. both x and y are in $X_{N_{x,y}}$. Let $\bar{N} = \max(N,N_{x,y})$. Then $x,y\varepsilon X_{\bar{N}}$ and $L_N \varepsilon L(f|X_{\bar{N}})$. Clearly $x\Pi^{X_{\bar{N}}}_{L_N} y$, and thus $(f|X_{\bar{N}})(x) = (f|X_{\bar{N}})(y)$, which implies that $f(x) = f(y)$. Hence L_N contains a location of f, say $L, L_N \supseteq L$. We will show that $L_N = L$. By Proposition 3.2.1, L contains a location of $f|X_{\bar{N}}$, say $\tilde{L}, L \supseteq \tilde{L}$. Hence $L_N \supseteq \tilde{L}$, where $L_N, \tilde{L}$ are both locations of $f|X_{\bar{N}}$. This implies that $L_N = \tilde{L}$ and so $L_N \supseteq L \supseteq \tilde{L}$ implies $L = L_N$. Thus for all $i \geq N$, $L_i = L_N$ and $L_N \varepsilon L(f)$, which was to be proved. □

From now on, unless mentioned otherwise, we assume that $S \subseteq \bigtimes_{\alpha\varepsilon D} S_\alpha$, where cardinality of D is finite and equal n.

In the next lemma we prove the following result. We assume that a sequence $\{X_i\}_{i=1}^{\infty}$ of nested subsets of S is given and that f is a function from S. For every set X_i in the sequence we choose a nonempty subset of $L(f|X_i)$, $\hat{L}_i$.

This way we obtain a sequence $\{\hat{L}_i\}_{i=1}^{\infty}$. Provided that $\hat{L}_i$ were chosen so that for every i and every location in $\hat{L}_{i+1}$ there is some location in $\hat{L}_i$ contained in it, there exists an infinite sequence of locations $\{L_i\}_{i=1}^{\infty}$, where $L_i \varepsilon \hat{L}_i$ and $L_{i+1} \supseteq L_i$.

The existence of such a sequence is not obvious at all and to prove it we employ a graph theoretic result known as The Infinity

Lemma. (For the statement and proof of generalized Infinity Lemma the reader is referred to [BW1].)

Lemma 3.2.2

Let $\left\{X_i\right\}_{i=1}^{\infty}$ be a sequence of subsets of S s.t. $X_{i+1} \supseteq X_i$, for all i, and let f be a function from S, where $\mathcal{L}_i = \mathcal{L}(f|X_i)$. For every i, let $\hat{\mathcal{L}}_i$ be a nonempty subset of $\mathcal{L}_i$.

If $\left\{\hat{\mathcal{L}}_i\right\}_{i=1}^{\infty}$ has the property that for every i, every $L \varepsilon \hat{\mathcal{L}}_{i+1}$, $\exists$ an $\bar{L} \varepsilon \hat{\mathcal{L}}_i$ s.t. $L \supseteq \bar{L}$, then $\exists$ an infinite sequence $\left\{L_i\right\}_{i=1}^{\infty}$ s.t. $L_i \varepsilon \hat{\mathcal{L}}_i$ and $L_{i+1} \supseteq L_i$.

Proof

Since D is finite so is clearly each $\hat{\mathcal{L}}_i$. Let Q_i be a subset of $\hat{\mathcal{L}}_i \times \hat{\mathcal{L}}_{i+1}$, $i = 1,2,\ldots$, s.t. for any $L_1 \varepsilon \hat{\mathcal{L}}_i$, any $L_2 \varepsilon \hat{\mathcal{L}}_{i+1}$, $(L_1,L_2)\varepsilon Q_i \Leftrightarrow L_1 \subseteq L_2$. Then each Q_i is finite and nonempty, since every $L \varepsilon \hat{\mathcal{L}}_{i+1}$ contains at least one $\tilde{L} \varepsilon \hat{\mathcal{L}}_i$.

Also the first point of every pair in $Q_{i+1}(i=1,2,\ldots)$ is the same as the second point of some pair in Q_i. All conditions of the Infinity Lemma are satisfied and thus $\exists$ an infinite sequence $\left\{L_i\right\}_{i=1}^{\infty}$ s.t. $(L_i,L_{i+1})\varepsilon Q_i$, $\forall$ i.

We note that this sequence is totally ordered by set inclusion, since by our definition of Q_i, $(L_i,L_{i+1})\varepsilon Q_i \Leftrightarrow L_i \subseteq L_{i+1}$. Also $L_i \varepsilon \hat{\mathcal{L}}_i$. This is a desired sequence.

The proof is illustrated by Figure 3.2.4. □

Theorem 3.2.1

Let $\left\{X_i\right\}_{i=1}^{\infty}$ be a sequence of subsets of S s.t. $X_{i+1} \supseteq X_i$ and

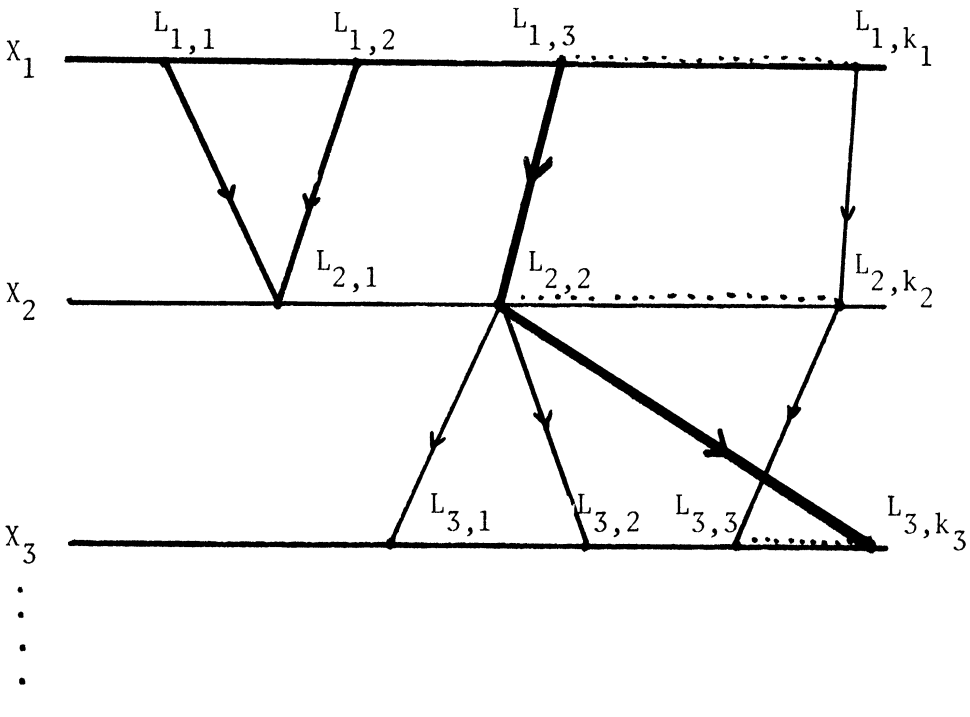


Figure 3.2.4: Graphical Illustration of Lemma 3.2.2.

$\bigcup_{i=1}^{\infty} X_i = S$. Then for any function f from S, $\exists$ N s.t. for all $i \geq N$, $L(f|X_i) = L(f)$.

Proof

1) First we show that $\exists$ N s.t. for all $i \geq N$, $L(f) \subseteq L(f|X_i)$. We will denote $L(f|X_i)$ by L_i. Let $|L(f)| = \ell \geq 1$. We will proceed by finite induction on the number of locations of f contained in L_i.

a) Induction base.

We show that at least one of the locations of f is contained in L_i, for $\forall\, i \geq N$, some N. By Proposition 3.2.1 for every $\tilde{L} \varepsilon L_{i+1}$ $\exists$ an $\hat{L} \varepsilon L_i$ s.t. $\tilde{L} \supseteq \hat{L}$. With $\hat{L}_i = L_i$ Lemma 3.2.2 can be applied. Thus $\exists$ a sequence $\left\{L_i\right\}_{i=1}^{\infty}$ s.t. $L_i \varepsilon L_i$ and $L_i \subseteq L_{i+1}$. By Lemma 3.2.1 then $\exists$ N s.t. $L_i = L_N$ for $\forall\, i \geq N$ and $L_N \varepsilon L(f)$. Thus $\exists$ N s.t. at least one of the locations of $f(L_N)$ is contained in L_i, for $\forall\, i \geq N$.

b) Induction step.

We show that if $\exists\, \hat{N}$ s.t. k locations of f are locations of $(f|X_i)$ for $\forall\, i \geq \hat{N}$, where $k < \ell$, then $\exists$ an $N \geq \hat{N}$ s.t. (k+1) locations of f are locations of $(f|X_i)$ for all $i \geq N$.

Let $\{L_{i_1}, L_{i_2}, \ldots, L_{i_k}\}$ be locations of f s.t. $L_{i_j} \varepsilon L_i$, for $j = 1,\ldots,k$, $i \geq \hat{N}$. Then L_{i_j} are locations of $(f|X_i)$ for all $i \geq \hat{N}$ and thus it suffices to show that $\exists\, N \geq \hat{N}$ and $\exists\, L_{i_{k+1}} \varepsilon L(f) - \{L_{i_1}, L_{i_2}, \ldots, L_{i_k}\}$ s.t. $L_{i_{k+1}} \varepsilon L_i$ for all $i \geq N$.

Let $\hat{L}_i = L_i - \{L_{i_1}, \ldots, L_{i_k}\}$ for all $i \geq \hat{N}$. We note that $\hat{L}_i \neq \phi$, $\forall\, i \geq \hat{N}$. For suppose $\hat{L}_{\bar{i}} = \phi$ for some $\bar{i} \geq \hat{N}$. Since $|L(f)| > k$, $\hat{L}(f) = L(f) - \{L_{i_1}, \ldots, L_{i_k}\} \neq \phi$. So let $\tilde{L} \varepsilon \hat{L}(f)$. Then by

Proposition 3.2.1 $\exists\, L \varepsilon \mathcal{L}_{\bar{i}}$ s.t. $\tilde{L} \supseteq L$, but then $L = L_{\bar{i}_j}$ for some $j\varepsilon(1,\ldots,k)$. This implies $\tilde{L} \supseteq L_{\bar{i}_j}$. Since $L_{\bar{i}_j} \varepsilon \mathcal{L}(f)$, this in turn implies $\tilde{L} = L_{\bar{i}_j}$, in contradiction to our assumption that $\tilde{L}\varepsilon\hat{\mathcal{L}}(f)$.

We now want to prove that for all $i \geq \hat{N}$, if $L\varepsilon\hat{\mathcal{L}}_{i+1}$, then $\exists$ an $\hat{L}\varepsilon\hat{\mathcal{L}}_i$ s.t. $L \supseteq \hat{L}$. We know that this holds for $\mathcal{L}_{i+1}$ and $\mathcal{L}_i$, i.e. that for every $L\varepsilon\hat{\mathcal{L}}_{i+1}$ $\exists$ an $\hat{L}\varepsilon\mathcal{L}_i$ s.t. $L \supseteq \hat{L}$. We need to show that $\hat{L}\varepsilon\hat{\mathcal{L}}_i$ or equivalently that $\hat{L} \neq L_{i_j}$ for $j = 1,\ldots,k$. Suppose $\hat{L} = L_{i_j}$ for some j.

L_{i_j} is a location of $f|X_{i+1}$ and so $L \supseteq L_{i_j} \Rightarrow L = L_{i_j}$. This contradicts $L\varepsilon\hat{\mathcal{L}}_{i+1}$. Thus $\hat{L}\varepsilon\hat{\mathcal{L}}_i$.

Now all the conditions of Lemma 3.2.2 are satisfied and thus $\exists$ a sequence $\left\{L_i\right\}_{i=\hat{N}}^{\infty}$ s.t. $L_i\varepsilon\hat{\mathcal{L}}_i$ and $L_i \subseteq L_{i+1}$. By Lemma 3.2.1 then $\exists\, N \geq \hat{N}$ s.t. $L_i = L_{i+1} = \tilde{L}$ for $i \geq N$ and $\tilde{L}\varepsilon\mathcal{L}(f)$. Since $L_i\varepsilon\hat{\mathcal{L}}_i$, $\forall\, i$, $\tilde{L}\varepsilon\hat{\mathcal{L}}(f)$ holds.

We showed then that $\exists\, N$ s.t. $\mathcal{L}(f) \subseteq \mathcal{L}(f|X_i)$, $\forall\, i \geq N$.

2) We still have to show that $\exists$ an $\bar{N}$ s.t. for all $i \geq \bar{N}$, $\mathcal{L}_i = \mathcal{L}(f|X_i) = \mathcal{L}(f)$.

Let N be s.t. $\mathcal{L}(f) \subseteq \mathcal{L}(f|X_i)$, $\forall\, i \geq N$. Let $\hat{\mathcal{L}}_i = \mathcal{L}_i - \mathcal{L}(f)$, $\forall\, i \geq N$. If $\exists$ an $\bar{N} \geq N$ s.t. $\hat{\mathcal{L}}_i = \phi$ for all $i \geq \bar{N}$, then obviously $\mathcal{L}_i = \mathcal{L}(f)$ for $i \geq \bar{N}$ and we are done.

Suppose that such an $\bar{N}$ does not exist. Then $\exists$ a sequence of natural numbers $\left\{N_i\right\}_{i=1}^{\infty}$ s.t. $N_{i+1} \geq N_i$, $N_i \geq N$ and $\hat{\mathcal{L}}_{N_i} \neq \phi$. As before we show that $\hat{\mathcal{L}}_{N_i}$ satisfy the conditions of Lemma 3.2.2 and so $\exists$ a sequence of locations $\left\{L_i\right\}_{i=1}^{\infty}$ s.t. $L_i \subseteq L_{i+1}$, $L_i\varepsilon\hat{\mathcal{L}}_{N_i}$. By Lemma 3.2.1 then $\exists\, \bar{N}$ s.t. $L_i = \hat{L}$, for all $i \geq \hat{N}$ and $\hat{L}\varepsilon\mathcal{L}(f)$. But $L_i\varepsilon\hat{\mathcal{L}}_{N_i} = \mathcal{L}_{N_i} - \mathcal{L}(f)$, $\forall\, i$, which leads to a contradiction.

So $\hat{N}$ as above exists. $\square$

Corollary 3.2.1

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of subsets of S s.t. $X_{i+1} \supseteq X_i$ and $\bigcup_{i=1}^{\infty} X_i = S$. Then for any finite family F of functions from S $\exists$ an N s.t. for $\forall$ $i \geq N$ and any $f \in F$, $L(f|X_i) = L(f)$.

Proof

By Theorem 3.2.1 for every $f \in F$ $\exists$ N_f s.t. for all $i \geq N_f$, $L(f|X_i) = L(f)$. Simply take $N = \max_{f \in F} N_f$. □

As a corollary to Theorem 3.2.1 we will show that for any finite family F of functions from an infinitely countable set S, there exists a finite subset X of S with the property that for every f in F, $L(f) = L(f|X)$.

This is to say that to find a set of locations of any function in the family we just need to know the function on a subset X as above.

It is important to mention that although the existence of such a set is guaranteed, this set is not known a priori. For any nested sequence of finite sets, which cover S, from some point on every set in the sequence has the desired property.

Suppose that S is irredundant and without loss of generality that we consider a single function from S, cardinality of whose location is a priori known.

Then "experimenting" we obtain the function values on a sequence of nested subsets and after a finite number of steps we find the location of the function.

It is the knowledge of location cardinality which tells us when to stop. We stop when the cardinalities of the location of the restriction and of the location of the function are equal.

Without this knowledge however, we never know whether we have already reached the location or whether we still need to go further, unless of course one of the locations is equal to the index set D.

Corollary 3.2.2

Let $|S| = \aleph_0$. Then for any finite family F of functions from S $\exists$ a finite subset X of S s.t. $L(f) = L(f|X)$ for all $f \in F$.

Proof

Let $\{s_i\}_{i=1}^{\infty}$ be an enumeration of S. Let $X_1 = \{s_1\}$ and $X_{i+1} = X_i \cup \{s_{i+1}\}$, for $\forall\ i \geq 1$. Then $X_{i+1} \supseteq X_i$, $\bigcup_{i=1}^{\infty} X_i = S$ and $\forall\ X_i$'s are finite. Thus be Corollary 3.2.1 $\exists$ N s.t. for $\forall\ i \geq N$, $L(f|X_i) = L(f)$ for all $f \in F$. Take $X = X_N$. □

Remark

We note that for arbitrary S there is no proper subset X of S such that for every function f from S to some codomain R (with $|R| \geq 2$), $L(f) = L(f|X)$.

(Given X let f be a function with the kernel partition $\Pi_f = \{\overline{X}, \overline{S-X}\}$. Then $L(f|X) = \{\phi\}$ and since $\Pi_f \neq I$, $L(f) \neq L(f|X)$). □

Corollary 3.2.3

Let S be a countably infinite set, s.t. $S = \times_{\alpha \in D} S_\alpha - \times_{\alpha \in D} C_\alpha$, where $C_\alpha \subseteq S_\alpha$, for all $\alpha \in D$. Then for every finite family of functions from S $\exists$ a finite irredundant subset X of S of the same form as S s.t. for every $f \in F$, $L(f) = L(f|X)$.

Proof

Let $\left\{s_{\alpha}^{i}\right\}_{i=0}^{\infty}$ be an enumeration of S_{α}. With $|D| = n$ we construct a sequence $\left\{X_i\right\}_{i=1}^{\infty}$ of subsets of $\bigtimes_{\alpha\varepsilon D} S_{\alpha}$, where $X_0 = \{(s_{\alpha_1}^0, s_{\alpha_2}^0, \ldots, s_{\alpha_n}^0)\}$ $X_1 = X_0 \cup \{(s_{\alpha_1}^1, s_{\alpha_2}^0, s_{\alpha_3}^0, \ldots, s_{\alpha_n}^0)\}$, $X_2 = X_1 \cup \{(s_{\alpha_1}^0, s_{\alpha_2}^1, s_{\alpha_3}^0, \ldots, s_{\alpha_n}^0),$ $(s_{\alpha_1}^1, s_{\alpha_2}^1, s_{\alpha_3}^0, \ldots, s_{\alpha_n}^0)\}$, etc. That is at the i'th stage we add to X_{i-1} all points in $\bigtimes_{\alpha\varepsilon D} S_{\alpha}$, which are in

$$P_{\alpha_1}(X_{i-1}) \times P_{\alpha_2}(X_{i-1}) \times \cdots \times P_{\alpha_{k_i-1}}(X_{i-1}) \times \left\{s_{\alpha_{k_i}}^{j_i}\right\} \times P_{\alpha_{k_i+1}}(X_{i-1}) \times \cdots \times P_{\alpha_n}(X_{i-1}),$$

$$\text{where } j_i = \begin{cases} \left[\frac{i}{n}\right] + 1 & \text{if } \left[\frac{i}{n}\right] < \frac{i}{n} \\ \frac{i}{n} & \text{if } \left[\frac{i}{n}\right] = \frac{i}{n} \end{cases} \quad \text{and} \quad k_i = \begin{cases} i_{\bmod n} & \text{if } i_{\bmod n} \neq 0 \\ n & \text{if } i_{\bmod n} = 0 \end{cases}$$

For any number r [r] denotes the largest integer not greater than r. Clearly all X_i's are Cartesian by construction and $\bigcup_{i=0}^{\infty} X_i = \bigtimes_{\alpha\varepsilon D} S_{\alpha}$, $X_{i+1} \supseteq X_i$ hold for all i.

For every i, let $Y_i = X_i - \bigtimes_{\alpha\varepsilon D} C_{\alpha} = X_i - X_i \cap C$, where X_i's are as constructed above and $C = \bigtimes_{\alpha\varepsilon D} C_{\alpha}$. $X_i \cap C$ are Cartesian, since both X_i and C are. Thus Y_i is a set of the form of S (and of Proposition 2.4.5) and thus irredundant. Clearly $Y_{i+1} \supseteq Y_i$ for all i and $\bigcup_{i=0}^{\infty} Y_i = \bigcup_{i=1}^{\infty} X_i - C = S-C$. So by Corollary 3.2.1 $\exists$ N s.t. for $\forall$ $i \geq N$, $\forall$ $f\varepsilon F$ $L(f) = L(f|Y_i)$. Take $X = Y_N$. □

Example 3.2.3

Consider $S = S_1 \times S_2 \times S_3$, where $S_i = \left\{s_i^j\right\}_{j=0}^{\infty}$. Then the first few sets in the sequence $\{X_i\}$ of Cartesian subsets of S, such that $X_{i+1} \supseteq X_i$ and $\bigcup_i X_i = S$ are given below.

$$X_0 = \{(s_1^0, s_2^0, s_3^0)\}$$

$$X_1 = X_0 \cup \{(s_1^1, s_2^0, s_3^0)\}$$

$$X_2 = X_1 \cup \left\{\begin{matrix}(s_1^0, s_2^1, s_3^0), \\ (s_1^1, s_2^1, s_3^0)\end{matrix}\right\}$$

$$X_3 = X_2 \cup \left\{\begin{matrix}(s_1^0, s_2^0, s_3^1), \\ (s_1^1, s_2^0, s_3^1), \\ (s_1^0, s_2^1, s_3^1), \\ (s_1^1, s_2^1, s_3^1)\end{matrix}\right\}$$

$$X_4 = X_3 \cup \left\{\begin{matrix}(s_1^2, s_2^0, s_3^0), \\ (s_1^2, s_2^1, s_3^0), \\ (s_1^2, s_2^1, s_3^1), \\ (s_1^2, s_2^0, s_3^1)\end{matrix}\right\}$$

$$X_5 = X_4 \cup \left\{\begin{matrix}(s_1^0, s_2^2, s_3^0), \\ (s_1^0, s_2^2, s_3^1), \\ (s_1^1, s_2^2, s_3^0), \\ (s_1^1, s_2^2, s_3^1), \\ (s_1^2, s_2^2, s_3^0), \\ (s_1^2, s_2^2, s_3^1)\end{matrix}\right\}$$

□

3.3 Properties of Extensions of Functions Defined on Proper Domain-Subsets

Throughout this section we deal with functions from a structured domain S to a codomain R. It is assumed that the cardinality of R is at least two, thus allowing nonconstant functions.

R is treated as an abstract nonstructured set.

When a function $\bar{f}$ is known on a proper subset X of S only, several questions arise.

For example, for every location $\bar{L}$ of $\bar{f}$ we want to be able to count all extensions f of $\bar{f}$ to S, with the property that $\bar{L}\varepsilon L(f)$. Also for every $\bar{L}\varepsilon L(\bar{f})$ and an arbitrary subset L of the index set D containing $\bar{L}$, we want to count all the extensions f of $\bar{f}$ to S with the property, that there exists a location of f equal to L.

The importance of answering those questions will be clear in later chapters, where the above results are going to be used to compute confidence in a given partial model, average confidence, predictive confidence and other parameters of interest.

We first establish some notation.

With $\bar{f}$ a function from a proper subset X of S to R and F the family of all functions from S to R

$E^{\bar{f}(X)}_{(S,R)}$	denotes the set of all extensions of $\bar{f}$ to S, in F.
$E^{\bar{f}(X)}_{(S,R)}(L,\cdot)$	denotes the set of all extensions f of $\bar{f}$ s.t. $f\varepsilon F$ and $L\varepsilon L(f)$.
$E^{\bar{f}(X)}_{(S,R)}(\subseteq L,\cdot)$	denotes the set of all extensions f of $\bar{f}$, s.t. $f\varepsilon F$ and $\exists$ an $\tilde{L}\varepsilon L(f)$ s.t. $\tilde{L}\subseteq L$.
$E^{\bar{f}(X)}_{(S,R)}(\cdot,ub)$	denotes the set of all f s.t. $f\varepsilon E^{\bar{f}(X)}_{(S,R)}$ and $\exists$ an $\tilde{L}\varepsilon L(f)$ s.t. $\lvert\tilde{L}\rvert \leq ub$.

We will often write $E^{\bar{f}(X)}_{(S,R)}(L)$ instead of $E^{\bar{f}(X)}_{(S,R)}(L,\cdot)$ and $E^{\bar{f}(X)}_{(S,R)}(\subseteq L)$ in place of $E^{\bar{f}(X)}_{(S,R)}(\subseteq L,\cdot)$.

Also when the sets S and R remain constant, we will drop (S,R) subscripts, for example $E^{\bar{f}(X)}_{(S,R)}$ will be shortened to $E^{\bar{f}(X)}$. Same holds for superscripts, e.g., when X is constant we will often write $E^{\bar{f}}_{(S,R)}$ instead of $E^{\bar{f}(X)}_{(S,R)}$.

For an arbitrary subset X of S and an arbitrary subset L of D we introduce a set called completion of X w.r.t. L and S, denoted by $COMPL^L_S(X)$.

Definition 3.3.1

For an arbitrary $X \subseteq S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ and an arbitrary $L \subseteq D$,

$COMPL^L_S(X) = \{s \mid s\varepsilon S \text{ and } \exists\ x\varepsilon X \text{ s.t. } s\Pi_L x\}$. □

We remark that for any $L \subseteq D$, $COMPL^L_S(X) \supseteq X$. Also $COMPL^\phi_S(X) = S$, while $COMPL^D_S(X) = X$. It is possible that although $L \neq D$, $COMPL^L_S(X) = X$.

We will now show that for any chain of subsets of D the corresponding completions of X also form a chain, but reversely ordered.

Proposition 3.3.1

Let X be a subset of S and let $\{L_1,\dots,L_k\}$ be subsets of D, where $L_i \subseteq L_{i+1}, i=1,\dots,k-1$. Then $\{COMPL^{L_1}_S(X), COMPL^{L_2}_S(X),\dots,COMPL^{L_k}_S(X)\}$ form a chain, where $COMPL^{L_i}_S(X) \supseteq COMPL^{L_{i+1}}_S(X)$, for all $i = 1,\dots,k-1$.

Proof

The proof is trivial and follows directly from Definition 3.3.1. Fix an $i\varepsilon(1,\dots,k-1)$. Then $s\varepsilon COMPL^{L_{i+1}}_S(X)$ iff $\exists$ an $x\varepsilon X$ s.t. $s\Pi_{L_{i+1}} x$. But $L_{i+1} \supseteq L_i \Rightarrow s\Pi_{L_i} x, \Rightarrow s\varepsilon COMPL^{L_i}_S(X)$. □

Our next result is the following. Given a function $\bar{f}$ from a subset X of S to R, such that $\bar{L}\varepsilon \mathcal{L}(\bar{f})$, we show that for any $L \supseteq \bar{L}$, if there exists an extension f of $\bar{f}$ to S, such that $f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L)$, then the restriction $\hat{f}$ of f to $COMPL^L_S(X)$ is unique and $\bar{L}$ is a location of $\hat{f}$.

Further we give the definition of f on $COMPL^L_S(X)$.

Lemma 3.3.1

Let $X \subseteq S$ and let $\bar{f}$ be a function from X to R with $\bar{L} \varepsilon \mathcal{L}(\bar{f})$. Then for any L s.t. $\bar{L} \subseteq L \subseteq D$ and any $f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L)$ (provided $E^{\bar{f}(X)}_{(S,R)}(L) \neq \phi$)

a) $\hat{f} = f|COMPL^L_S(X)$ is unique

b) $\hat{f}$ is defined by

$\hat{f}(s) = \bar{f}(x)$, for all $s \varepsilon COMPL^L_S(X)$, where $x \varepsilon X$ is such that $s \Pi_L x$.

c) $\bar{L} \varepsilon \mathcal{L}(\hat{f})$

hold.

Proof

Let $f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L)$ be given.

1) We first show the uniqueness of the restriction $\hat{f}$ of f to $COMPL^L_S(X) = Y$.

$f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L) \Rightarrow \Pi^S_L \leq \Pi_f$. Let $s \varepsilon COMPL^L_S(X)$. Then by Definition 3.3.1 $\exists$ an $x \varepsilon X$ s.t. $s \Pi_L x$. This implies that $f(s) = f(x)$. Thus $\hat{f}(s) = \bar{f}(x)$. We need to show $\hat{f}$ is well defined. Let $y \neq x$ be a point in $COMPL^L_S(X)$ s.t. $s \Pi_L y$. Then $x \Pi_L y$ holds and so $x \Pi_{\bar{L}} y$, since $L \supseteq \bar{L}$. But $\bar{L} \varepsilon \mathcal{L}(\bar{f})$ implies then that $\bar{f}(x) = \bar{f}(y)$.

Hence a) and b) are proved.

2) We now prove part c).

To show that $\bar{L} \varepsilon \mathcal{L}(\hat{f})$ it suffices to show that $\Pi^Y_{\bar{L}} \leq \Pi_{\hat{f}}$. For this inequality implies that $\bar{L}$ contains some location of $\hat{f}$, say $\tilde{L}$. But then by Proposition 3.2.1 $\exists$ an $L' \varepsilon \mathcal{L}(\bar{f})$ s.t. $\tilde{L} \supseteq L'$. So $\bar{L} \supseteq \tilde{L} \supseteq L'$ holds. Since $\bar{L} \varepsilon \mathcal{L}(\bar{f})$ and $L' \varepsilon \mathcal{L}(\bar{f})$ however, $\bar{L} = L'$ and thus $\tilde{L} = \bar{L}$, which means that if $\Pi^{Y}_{\bar{L}} \leq \Pi_{\hat{f}}$, then $\bar{L}\varepsilon L(\hat{f})$. So let $s,s'\varepsilon Y = COMPL^{L}_{S}(X)$ be such that $s\Pi_{\bar{L}}s'$. $\exists$ $x,x'\varepsilon X$ s.t. $s\Pi_{L}x$ and $s'\Pi_{L}x'$. By part 1) of the proof $\hat{f}(s) = \bar{f}(x)$ and $\hat{f}(s') = \bar{f}(x')$. Since $L \supseteq \bar{L}$, $s\Pi_{\bar{L}}x$ and $s'\Pi_{\bar{L}}x'$ hold. Also $s\Pi_{\bar{L}}s' \Rightarrow x\Pi_{\bar{L}}x'$. But $\bar{L}\varepsilon L(\bar{f}) \Rightarrow \bar{f}(x) = \bar{f}(x')$. Thus $\hat{f}(s) = \hat{f}(s')$ and $\Pi^{Y}_{\bar{L}} \leq \Pi_{\hat{f}}$, which was to be proved. □

Corollary 3.3.1

Let $X \subseteq S$ and let $\bar{f}:X \to R$, where $\bar{L}$ is a location of $\bar{f}$, i.e. $\bar{L}\varepsilon L(\bar{f})$. Then for any L, s.t. $\bar{L} \subseteq L \subseteq D$

$$E^{\bar{f}(X)}_{(S,R)}(\subseteq L) = E^{\hat{f}(COMPL^{L}_{S}(X))}_{(S,R)}(\subseteq L) \text{ , where}$$

$\hat{f}$ is the unique extension of $\bar{f}$ to $COMPL^{L}_{S}(X)$ of part b) of Lemma 3.3.1.

Proof

Since $\hat{f}$ is an extension of $\bar{f}$, clearly $E^{\hat{f}(Y)}(\subseteq L) \subseteq E^{\bar{f}(X)}(\subseteq L)$, where $Y = COMPL^{L}_{S}(X)$.

We need to show that $E^{\bar{f}(X)}(\subseteq L) \subseteq E^{\hat{f}(Y)}(\subseteq L)$. Let $f\varepsilon E^{\bar{f}(X)}(\subseteq L)$. Then $\exists$ $\tilde{L},\hat{L} \subseteq \tilde{L} \subseteq L$ s.t. $\tilde{L}\varepsilon L(f)$ and $\hat{L}\varepsilon L(\bar{f})$. But by Lemma 3.3.1 then $f|COMPL^{\tilde{L}}_{S}(X) = \tilde{f}$, where $\tilde{f}(s) = \bar{f}(x)$, for $s\varepsilon COMPL^{\tilde{L}}_{S}(X)$, where $x\varepsilon X$ is such that $s\Pi_{\tilde{L}}x$. Since $\tilde{L} \subseteq L$, $COMPL^{L}_{S}(X) \subseteq COMPL^{\tilde{L}}_{S}(X)$ holds.

We show that $\tilde{f}|COMPL^{L}_{S}(X) = \hat{f}$, where $\hat{f}$ is as above.

For any $s\varepsilon COMPL^{L}_{S}(X)$, $\hat{f}(s) = \bar{f}(y)$, where $y\varepsilon X$ is such that $s\Pi_{L}y$. But $s\Pi_{L}y \Rightarrow s\Pi_{\tilde{L}}y \Rightarrow \tilde{f}(s) = \bar{f}(y) = \hat{f}(s)$ for $\forall$ $s\varepsilon COMPL^{L}_{S}(X)$. Clearly $f|COMPL^{L}_{S}(X) = \tilde{f}|COMPL^{L}_{S}(X) = \hat{f}$ and so $f\varepsilon E^{\hat{f}(Y)}(\subseteq L)$, which was to be proved. □

Given a function $\bar{f}$ from a subset X of S s.t. $\bar{L}\varepsilon L(\bar{f})$ and any

$L \supseteq \bar{L}$, we show how to construct all extensions f of $\bar{f}$ to S, such that at least one of the locations of f is contained in L.

Theorem 3.3.1

Let $X \subseteq S$ and let $\bar{f}:X \to R$, where $\bar{L}$ is one of the locations of $\bar{f}$. Then for any L, s.t. $\bar{L} \subseteq L \subseteq D$,

$$E^{\bar{f}(X)}_{(S,R)}(\subseteq L) = \{f \mid f \in E^{\bar{f}(X)}_{(S,R)},\ f \mid COMPL^L_S(X) = \hat{f} \text{ and } \Pi_f \geq \Pi\},$$

where $\hat{f}$ is the unique extension of $\bar{f}$ to $COMPL^L_S(X)$ of Lemma 3.3.1 and Π is a partition on S defined by

$$[s]_\Pi = \begin{cases} [s]_{\Pi_{\hat{f}}} & \text{for } s \in COMPL^L_S(X) \\ [s]_{\Pi_L} & \text{for } s \in S-COMPL^L_S(X) \end{cases}$$

Proof

We note that by Corollary 3.3.1 we need to "count" only $E^{\hat{f}(COMPL^L_S(X))}(\subseteq L)$. Thus clearly $f \in E^{\bar{f}(X)}_{(S,R)}(\subseteq L) \Rightarrow f \mid COMPL^L_S(X) = \hat{f}$.

We show next that Π as above is well defined. It suffices to show that $S-COMPL^L_S(X) = \bigcup_{z \in S-COMPL^L_S(X)} [z]_{\Pi^S_L}$. Equivalently we just need to show that for every $z \in S-COMPL^L_S(X)$, $[z]_{\Pi^S_L} \subseteq S-COMPL^L_S(X)$ or that for any $s \in S$ s.t. $z \Pi^S_L s$, $\nexists\, x \in X$ s.t. $s \Pi^S_L x$. This is obvious for given an $s \in S$ s.t. $s \Pi^S_L z$, if $\exists$ an $x \in X$ s.t. $x \Pi^S_L s$, then $x \Pi^S_L z$, # to $z \in S-COMPL^L_S(X)$.

So Π is well defined. Also since $\bar{L} \in \mathcal{L}(\hat{f})$, it is easy to show that $\Pi^S_L \leq \Pi$.

a) We show that if f is such that $f \mid COMPL^L_S(X) = \hat{f}$ and $\Pi_f \geq \Pi$, then $f \in E^{\hat{f}(Y)}(\subseteq L)$, where $Y = COMPL^L_S(X)$. Clearly f is an extension of $\hat{f}$ so we have to show that $\exists\, \tilde{L} \subseteq L$ s.t. $\tilde{L} \in \mathcal{L}(f)$. To do so we just need to show that $\Pi_L^S \leq \Pi_f$. But $\Pi_L^S \leq \Pi \leq \Pi_f \Rightarrow \Pi_L^S \leq \Pi_f$.

b) We now show that if $f \in E^{\hat{f}(Y)}(\subseteq L)$ then $f|Y = \hat{f}$ and $\Pi_f \geq \Pi$. That $f|Y = \hat{f}$ is obvious.

Now $\exists$ an $\tilde{L} \subseteq L$ s.t. $\tilde{L} \in L(f)$. This implies that $\Pi_{\tilde{L}}^S \leq \Pi_f$, implies that $\Pi_L^S \leq \Pi_f$. Let $x,y \in S$ be such that $x\Pi y$. Then by definition of Π either $x,y \in Y$ or $x,y \in S-Y$. If $x,y \in Y$, then $x\Pi y \Rightarrow \hat{f}(x) = \hat{f}(y) \Rightarrow f(x) = f(y)$. If $x,y \in S-Y$, $x\Pi y \Rightarrow x\Pi_L^S y \Rightarrow x\Pi_f y$, since we showed that $\Pi_L^S \leq \Pi_f$. This completes the proof. □

Theorem 3.3.1 will be later illustrated by an example.

Given $\bar{f}$ as in Theorem 3.3.1 and an arbitrary $L \supseteq \bar{L}$ we will now show how to find the set of all extensions f of $\bar{f}$, such that $L \in L(f)$.

Proposition 3.3.2

Let X be a subset of S and let $\bar{f}: X \to R$, where $\bar{L} \in L(\bar{f})$. Then for an arbitrary L s.t. $\bar{L} \subseteq L \subseteq D$,

$$E_{(S,R)}^{\bar{f}(X)}(L) = E_{(S,R)}^{\bar{f}(X)}(\subseteq L) - \bigcup_{\substack{\tilde{L} \\ \tilde{L} \subsetneq L}} E_{(S,R)}^{\bar{f}(X)}(\tilde{L}).$$

Proof

This is obvious, since from the definition of $E_{(S,R)}^{\bar{f}(X)}(\subseteq L)$ it follows that $E^{\bar{f}}(\subseteq L) = \bigcup_{\tilde{L} \subseteq L} E^{\bar{f}}(\tilde{L}) = \bigcup_{\tilde{L} \subsetneq L} E^{\bar{f}}(\tilde{L}) \cup E^{\bar{f}}(L)$. Also for any $\tilde{L} \subsetneq L$, $E^{\bar{f}}(L) \cap E^{\bar{f}}(\tilde{L}) = \phi$. □

Proposition 3.3.3

Let $X \subseteq S$ and let $\bar{f}: X \to R$, where $\bar{L} \in L(\bar{f})$. Then

a) $E_{(S,R)}^{\bar{f}(X)}(\subseteq \bar{L}) = E_{(S,R)}^{\bar{f}(X)}(\bar{L})$

and

b) $E^{\bar{f}(X)}_{(S,R)}(\bar{L}) = E^{\hat{f}(Y)}_{(S,R)}(\bar{L})$, where $Y = COMPL^{\bar{L}}_{S}(X)$ and $\hat{f}$ is the extension of $\bar{f}$ to $COMPL^{\bar{L}}_{S}(X)$ defined by Lemma 3.3.1.

Proof

a) Let $f \varepsilon E^{\bar{f}}(\subseteq \bar{L})$. Then $\exists$ an $\tilde{L} \subsetneq \bar{L}$ s.t. $\tilde{L} \varepsilon \mathcal{L}(f)$. Also $\exists$ an $\hat{L} \varepsilon \mathcal{L}(\bar{f})$ s.t. $\tilde{L} \supseteq \hat{L}$. This implies that $\bar{L} \supseteq \hat{L}$. But since both $\bar{L}$ and $\hat{L}$ are locations of $\bar{f}$, $\bar{L} = \hat{L}$ and so $\tilde{L} = \bar{L}$. So $f \varepsilon E^{\bar{f}}(\bar{L})$ and $E^{\bar{f}}(\subseteq \bar{L}) \subseteq E^{\bar{f}}(\bar{L})$.

Clearly $E^{\bar{f}}(\bar{L}) \subseteq E^{\bar{f}}(\subseteq \bar{L})$ and thus a) has been proved.

b) Part b) follows directly from Lemma 3.3.1. □

We will now illustrate by an example how to construct extensions with desired locations.

Example 3.3.1

Consider $S = \{0,1\}^3$ and $X = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$. Then S and X are irredundant (X by Proposition 2.3.2) and S is actually Cartesian.

Let $\bar{f}$ be a function from X to $R = \{0,1\}$ defined in Figure 3.3.1. Then $\bar{f}$ has the unique location $\{\alpha_1\}$.

We are seeking all extensions of $\bar{f}$ to S with locations contained in $\{\alpha_1,\alpha_2\}$. Since $L(\bar{f}) = \{\alpha_1\}$ we are thus seeking all extensions of $\bar{f}$ to S with locations $\{\alpha_1\}$ and $\{\alpha_1,\alpha_2\}$.

To construct those we first find the $COMPL^{\{\alpha_1,\alpha_2\}}_{S}(X)$ and then use Corollary 3.3.1.

$$COMPL^{\{\alpha_1,\alpha_2\}}_{S}(X) = X \cup \{(0,0,1),\ (1,0,1)\}.$$

$\hat{f}$ as of Lemma 3.3.1 is defined by $\hat{f}|X = \bar{f}$ and $\hat{f}((0,0,1)) = \bar{f}((0,0,0)) = 1$, while $\hat{f}((1,0,1)) = \bar{f}((1,0,0)) = 0$.

x	$\bar{f}(x)$
(0,0,0)	1
(1,0,0)	0
(1,1,0)	0
(1,1,1)	0

$L(\bar{f}) = \{\alpha_1\}$

X

$COMPL_S^{\{\alpha_1,\alpha_2\}}(X)$

x	$\hat{f}(x)$
(0,0,0)	1
(1,0,0)	0
(1,1,0)	0
(1,1,1)	0
(0,0,1)	1
(1,0,1)	0

$L(\hat{f}) = \{\alpha_1\}, \ \hat{f}|X = \bar{f}$

x	$f_1(x)$
(0,0,0)	1
(1,0,0)	0
(1,1,0)	0
(1,1,1)	0
(0,0,1)	1
(1,0,1)	0
(0,1,0)	0
(0,1,1)	0

S

$f_1|COMPL_S^{\{\alpha_1,\alpha_2\}}(X) = \hat{f}$

$L(f_1) = \{\alpha_1,\alpha_2\}$

S

x	$f_2(x)$
(0,0,0)	1
(1,0,0)	0
(1,1,0)	0
(1,1,1)	0
(0,0,1)	1
(1,0,1)	0
(0,1,0)	1
(0,1,1)	1

$f_2|COMPL_S^{\{\alpha_1,\alpha_2\}}(X) = \hat{f}$

$L(f_2) = \{\alpha_1\}$

Figure 3.3.1: Tables of **Functions** of Example 3.3.1.

We note that $S\text{-}COMPL_S^{\{\alpha_1,\alpha_2\}}(X) = \{(0,1,0),(0,1,1)\}$, that is, there is one $\Pi^S_{\{\alpha_1,\alpha_2\}}$ block in $(S\text{-}COMPL_S^{\{\alpha_1,\alpha_2\}}(X))$.

We will now use Theorem 3.3.1. Π of this theorem is defined by

$$\Pi = \{\overline{(0,0,0),(0,0,1)},\ \overline{(1,0,0),(1,1,0),(1,1,1),(1,0,1)},\ \overline{(0,1,0),(0,1,1)}\}$$

All the extensions f of $\bar{f}$, whose locations are contained in $\{\alpha_1,\alpha_2\}$ are those, for which $f|COMPL_S^{\{\alpha_1,\alpha_2\}}(X) = \hat{f}$ and $\Pi \le \Pi_f$. There are two such extensions. One of them, f_1, assigns value 0 to (0,1,0) and (0,1,1), and f_2 assigns value 1 to (0,1,0) and (0,1,1). f_1 and f_2 are as in Figure 3.3.1. We easily check that $L(f_1) = \{\alpha_1,\alpha_2\}$ and $L(f_2) = \{\alpha_1\}$. □

We will show in the next corollary that for an arbitrary subset X of S, if $\bar{f}$ is a function from X to R such that $\bar{L}\varepsilon\mathcal{L}(\bar{f})$, there always exists an extension f of $\bar{f}$ to S, such that $\bar{L}\varepsilon\mathcal{L}(f)$.

<u>Corollary 3.3.2</u>

Let $X \subsetneq S$ and let $\bar{f}:X \to R$. Then for any $\bar{L}\varepsilon\mathcal{L}(\bar{f})$ $\exists$ an f such that $f\varepsilon E^{\bar{f}(X)}_{(S,R)}(\bar{L})$. If $|\mathcal{P}_{\bar{L}}(S)|$ and $|R|$ are finite, then

$$|E^{\bar{f}(X)}_{(S,R)}(\bar{L})| = |R|^{|\mathcal{P}_{\bar{L}}(S)|-|\mathcal{P}_{\bar{L}}(X)|}.$$

<u>Proof</u>

By Proposition 3.3.3, $E^{\bar{f}(X)}(\bar{L}) = E^{\hat{f}(Y)}(\bar{L}) = E^{\hat{f}(Y)}(\subseteq\bar{L})$, where $Y = COMPL_S^{\bar{L}}(X)$ and $\hat{f}$ is as in Proposition 3.3.3. But then by Theorem 3.3.1

(*) $$E^{\bar{f}(X)}(\bar{L}) = \{f|f\varepsilon E^{\hat{f}(Y)} \text{ and } \Pi_{\bar{L}} \le \Pi_f\}.$$

But since $|R| \ne 0$, clearly f as above exists. For example assign any value of R to all points of $S\text{-}COMPL_S^{\bar{L}}(X)$. (If the latter is empty $S = COMPL_S^{\bar{L}}(X)$ and there is a unique extension of $\bar{f}$ to S s.t. $\bar{L}\varepsilon\mathcal{L}(f)$).

b) From equation (*) above it clearly follows that we get all the desired extensions by assigning all possible sets of values of R to $\Pi_{\bar{L}}$ blocks of $(S\text{-}COMPL_S^{\bar{L}}(X))$.

There are $(|P_{\bar{L}}(S)| - |P_{\bar{L}}(X)|)$ $\Pi_{\bar{L}}$ blocks of S in $(S\text{-}COMPL_S^{\bar{L}}(X))$. So there are $|R|^{|P_{\bar{L}}(S)|-|P_{\bar{L}}(X)|}$ extensions f of $\bar{f}$ s.t. $\bar{L}\varepsilon L(f)$, which was to be proved. □

As a corollary to Corollary 3.3.2 we will now show that if $\bar{f}$ is a function from X to R and $\bar{L}\varepsilon L(\bar{f})$, there exists a unique extension of $\bar{f}$ to any subset of $COMPL_S^{\bar{L}}(X)$ such that $\bar{L}$ is a location of this extension.

Corollary 3.3.3

Let $X \subsetneq S$ and let $\bar{f}:X \rightarrow R$. Then for any $\bar{L}\varepsilon L(\bar{f})$ and any subset Z of $COMPL_S^{\bar{L}}(X)$ s.t. $Z \supseteq X$, $\exists$ a unique extension $\hat{f}$ of $\bar{f}$ to Z s.t. $\bar{L}\varepsilon L(\hat{f})$.

If $\bar{L} = L(\bar{f})$, then $\bar{L} = L(\hat{f})$ holds.

Proof

1) We first show that $\exists$ an extension $\hat{f}$ of $\bar{f}$ to Z s.t. $\bar{L}\varepsilon L(\hat{f})$. By Corollary 3.3.2 $\exists$ an f, an extension of $\bar{f}$ to S s.t. $\bar{L}\varepsilon L(f)$. Let $\hat{f} = f|Z$. Then $\bar{L} \supseteq \tilde{L}$, where $\tilde{L}\varepsilon L(\hat{f})$ and $\tilde{L} \supseteq \bar{\bar{L}}$, where $\bar{\bar{L}}\varepsilon L(\bar{f})$. Thus $L \supseteq \bar{\bar{L}}$ and so $\bar{L} = \bar{\bar{L}}$. This implies $\bar{L} = \tilde{L}$. Thus $\bar{L}\varepsilon L(\hat{f})$ and the desired extension exists.

2) We next show the uniqueness of the extension. Suppose the extension is not unique. Let $\hat{f}_1$ and $\hat{f}_2$ be two distinct extensions of $\bar{f}$ to Z, s.t. $\bar{L}\varepsilon L(\hat{f}_1)$ and $\bar{L}\varepsilon L(\hat{f}_2)$. Then by Corollary 3.3.2 applied to Z and S, $\exists$ f_1 and f_2 s.t. $f_1\varepsilon E^{\hat{f}_1(Z)}(\bar{L})$ and $f_2\varepsilon E^{\hat{f}_2(Z)}(\bar{L})$. Since $X \subseteq Z \subseteq COMPL_S^{\bar{L}}(X)$, and $\hat{f}_1 = (f_1|Z) \neq \hat{f}_2 = (f_2|Z)$ holds, clearly $f_1|COMPL_S^{\bar{L}}(X) \neq f_2|COMPL_S^{\bar{L}}(X)$. This however contradicts part a) of

Lemma 3.3.1. Thus the uniqueness of the above extension follows.

If $\bar{L}$ is the unique location of $\bar{f}$, then for every $\tilde{L} \in \mathcal{L}(\hat{f})$, $\tilde{L} \supseteq \bar{L}$. But $\bar{L} \in \mathcal{L}(\hat{f}) \Rightarrow \bar{L} = L(\hat{f})$. □

Remark

1) For $\bar{f}: X \to R$ and $\bar{L} \in \mathcal{L}(\bar{f})$, if $L \supsetneq \bar{L}$ is such that $\mathrm{COMPL}_S^L(X) = S$, then $E^{\bar{f}}(L) = \phi$. This follows directly from Lemma 3.3.1.

2) It can be shown that with $\bar{f}$ and $\bar{L}$ as above, even if $S \supsetneq \mathrm{COMPL}_S^L(X)$ and both X and S are irredundant, given $L \supsetneq \bar{L}$, the set $E^{\bar{f}}(L)$ may be empty.

3) If S is Cartesian however and all the conditions of 2) are met, $E^{\bar{f}}(L) \neq \phi$. □

It is easy to show that given an irredundant subset X of a Cartesian S and any subset L of D, the $\mathrm{COMPL}_S^L(X)$ is irredundant. This need not be the case when S is irredundant but not Cartesian.

In either case though, if $\bar{f}: X \to R$ and $\bar{L} = L(\bar{f})$, every extension $\hat{f}$ of $\bar{f}$ to $\mathrm{COMPL}_S^{\bar{L}}(X)$ such that $\bar{L} \in \mathcal{L}(\hat{f})$ has the unique location $\bar{L}$. (This was proved in Corollary 3.3.3.)

3.4 Construction of Domain-Subsets with Special Properties

Throughout this section we will assume that S is Cartesian and that cardinality of every S_α is at least two.

We will start by constructing, for every subset L of the index set D, an irredundant subset X of S with the following property: for every function f from S, whose location is contained in L, $L(f) = L(f|X)$. Actually we will construct a parametrized family of such subsets.

All sets of the family will be alike--have same cardinality and essential structure. They will be shown to be minimal in cardinality among all the subsets with the described property.

Using as constructing elements the sets described above, for every integer k, ($0 \leq k \leq |D|$), we will construct a parametrized family of irredundant subsets of S, each of which has the following property: for every function f from S, such that $|L(f)| \leq k$, the locations of f and its restriction to the subset are the same.

In the sequel we will use the following notation.

For any subset X of S

$F_{(X,R)}$ denotes the set of all functions from X to R

$F_{(X,R)}(L,\cdot)$ denotes the set of all functions $\bar{f}$ in $F_{(X,R)}$, such that $L \varepsilon \mathcal{L}(\bar{f})$

$F_{(X,R)}(\subseteq L,\cdot)$ denotes the set of all $\bar{f} \varepsilon F_{(X,R)}$ such that $\exists$ an $\tilde{L} \varepsilon \mathcal{L}(\bar{f})$ such that $\tilde{L} \subseteq L$.

$F_{(X,R)}(\cdot,ub)$ denotes the set of all $\bar{f} \varepsilon F_{(X,R)}$ such that $\exists$ an $\tilde{L} \varepsilon \mathcal{L}(\bar{f})$ with $|\tilde{L}| \leq ub$.

$F_{(X,R)}(L,ub)$ denotes the set $F_{(X,R)}(L,\cdot) \cap F_{(X,R)}(\cdot,ub)$.

We will often shorten our notation in an analogous way to the one described in section 3.

Definition 3.4.1

Let $S = \bigtimes_{\alpha \varepsilon D} S_\alpha$ and let L be an arbitrary subset of D. Then for any $y \varepsilon S$, X_L^y is defined by

$$X_L^y = \{s \mid s \varepsilon S \text{ and } P_{D-L}(s) = P_{D-L}(y)\}. \qquad \square$$

It follows directly from the definition of X_L^y that for any two points y and $\bar{y}$ of S, such that $P_{D-L}(y) = P_{D-L}(\bar{y})$, X_L^y and $X_L^{\bar{y}}$ are equal.

As a matter of fact X_L^y and $X_L^{\bar{y}}$ are equal if and only if $P_{D-L}(y) = P_{D-L}(\bar{y})$.

We also note that $P_{\bar{L}}(S) = P_{\bar{L}}(X_L^y)$ for arbitrary $\bar{L} \subseteq L$ and any y in S. This obviously implies that for every $\bar{L} \subseteq L$, the $COMPL_S^{\bar{L}}(X_L^y)$ is equal to S.

We can think about X_L^y as defined above as a subset of S, for which all α coordinates of L are open, that is vary over entire S_α, and all coordinates of (D-L) are fixed and equal to those of y.

We will sometimes denote X_L^y schematically using □ to denote an open coordinate. Thus for example if D = {1,2,3,4}, then □□ $y_3 y_4$ denotes $X_{\{1,2\}}^y$, where $y = (y_1, y_2, y_3, y_4)$.

We note that $X_\phi^y = \{y\}$ and $X_D^y = S$.

We will summarize the important properties of X_L^y sets in the theorem to follow. First we will establish an auxiliary result to be used in its proof.

Lemma 3.4.1

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$ for all $\alpha\varepsilon D$, and let L be an arbitrary subset of D. If $Z_L \neq \phi$ is an arbitrary subset of S with the property that for every $f\varepsilon F_{(S,R)}(L)$, $L(f) = L(f|Z_L)$, then Z_L contains at least one point from every equivalence class of Π_L^S.

Proof

If $L = \phi$, then any Z_L s.t. $|Z_L| \geq 1$ works. (Since then Π_ϕ^S has 1 equivalence class.) So assume $L \neq \phi$. Suppose ∃ an equivalence class of Π_L^S, $[x]_{\Pi_L^S}$ s.t. $Z_L \cap [x]_{\Pi_L^S} = \phi$. Let $a,b\varepsilon R$, where $a \neq b$ and let f be a function defined by $f([x]_{\Pi_L^S}) = a$ and $f(S-[x]_{\Pi_L^S}) = b$.

Then $\Pi_f = \{\overline{[x]_{\Pi_L^S}}, \overline{S-[x]_{\Pi_L^S}}\}$. $L(f) = L$, since $\Pi_L^S \leq \Pi_f$ and for every $\alpha\varepsilon L$, $\Pi_{L-\alpha}^S \nleq \Pi_f$. (This follows, since for every $\alpha\varepsilon L$ $\exists$ $x_\alpha \varepsilon S-[x]_{\Pi_L^S}$ s.t. $x_\alpha \Pi_{L-\alpha}^S x$ but $x_\alpha \not\Pi_L^S x$.)

We note that $Z_L \subseteq S - [x]_{\Pi_L^S}$ and so $L(f|Z_L) = \phi \neq L$. Thus $\exists$ f, $f\varepsilon F_{(S,R)}(L)$ s.t. $L(f) \neq L(f|Z_L)$ and this contradicts our assumption. So indeed Z_L contains at least one point from every equivalence class of Π_L^S. □

Corollary 3.4.1

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$, $\forall$ $\alpha\varepsilon D$. Then the only subset Z_D of S with the property, that for every $f\varepsilon F_{(S,R)}(D)$, $L(f) = L(f|Z_D)$ is S itself.

Proof

We simply notice that $[x]_{\Pi_D^S} = \{x\}$, $\forall$ $x\varepsilon S$. We then apply the above lemma. □

We will now illustrate the construction of X_L^y sets by an example.

Example 3.4.1

Consider $S = \{0,1\}^3$, where $D = \{\alpha_1,\alpha_2,\alpha_3\}$.

Then

$$X_\phi^{(0,0,0)} = \{(0,0,0)\}$$

$$X_{\{\alpha_1\}}^{(0,0,0)} = \{(0,0,0),(1,0,0)\}$$

$$X_{\{\alpha_1,\alpha_2\}}^{(0,0,0)} = \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$$

$$X_{\{\alpha_1,\alpha_2,\alpha_3\}}^{(0,0,0)} = X_D^{(0,0,0)} = S .$$

We note that $X^{(0,0,0)}_{\{\alpha_1\}} = X^{(1,0,0)}_{\{\alpha_1\}}$.

Also $X^{(1,1,1)}_{\{\alpha_1\}} = \{(0,1,1),(1,1,1)\}$ and $|X^{(1,1,1)}_{\{\alpha_1\}}| = |X^{(0,0,0)}_{\{\alpha_1\}}|$.
We will show later that $|X^y_L| = |X^{\bar{y}}_L|$, for any $y,\bar{y}\varepsilon S$. For the illustration refer to Figure 3.4.1. □

Theorem 3.4.1

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| \geq 2$. Then the following hold.

a) for every $y\varepsilon S$, every $L \subseteq D$, X^y_L is irredundant.

b) $|X^y_L| = \prod_{\alpha\varepsilon L} m_\alpha$, where $m_\alpha = |S_\alpha|$, for $\forall\, L \neq \phi$ and $|X^y_\phi| = 1$

c) for any $y,\bar{y}\varepsilon S$ and any $L \subseteq D$, $|X^y_L| = |X^{\bar{y}}_L|$

d) for any $y\varepsilon S$, any $L_1 \subseteq L_2 \subseteq D$, $X^y_{L_1} \subseteq X^y_{L_2}$, and if $L_1 \subsetneq L_2$, then $X^y_{L_1} \subsetneq X^y_{L_2}$.

e) for any $y\varepsilon S$, any $L \subseteq D$, if $f\varepsilon F_{(S,R)}(\subseteq L)$, then $L(f) = L(f|X^y_L)$

f) for any $y\varepsilon S$, any $L \subseteq D$ and arbitrary $\hat{L} \subseteq L$, if $\bar{f}\varepsilon F_{(X^y_L,R)}(\hat{L})$ then $\exists$ a unique extension f of $\bar{f}$ to S, s.t. $\hat{L} = L(f)$.

g) for any $L \subseteq D$, if Z_L is any subset of S s.t. for $\forall\, f\varepsilon F_{(S,R)}(\subseteq L)$, $L(f) = L(f|Z_L)$, then $|Z_L| \geq |X^y_L|$ for every $y\varepsilon S$.

Proof

a) X^y_L is actually Cartesian, for $X^y_L = \bigtimes_{\alpha\varepsilon D} X_\alpha$,

$$\text{where } X_\alpha = \begin{cases} S_\alpha & \text{if } \alpha\varepsilon L \\ \{y_\alpha\} & \text{if } \alpha\varepsilon D-L \end{cases}$$

b) This follows from part a) of the proof. Since for $L = \phi$, $X^y_L = \{y\}$, $|X^y_\phi| = 1$. For $L \neq \phi$, since $X^y_L = \bigtimes_{\alpha\varepsilon D} X_\alpha$, where X_α's

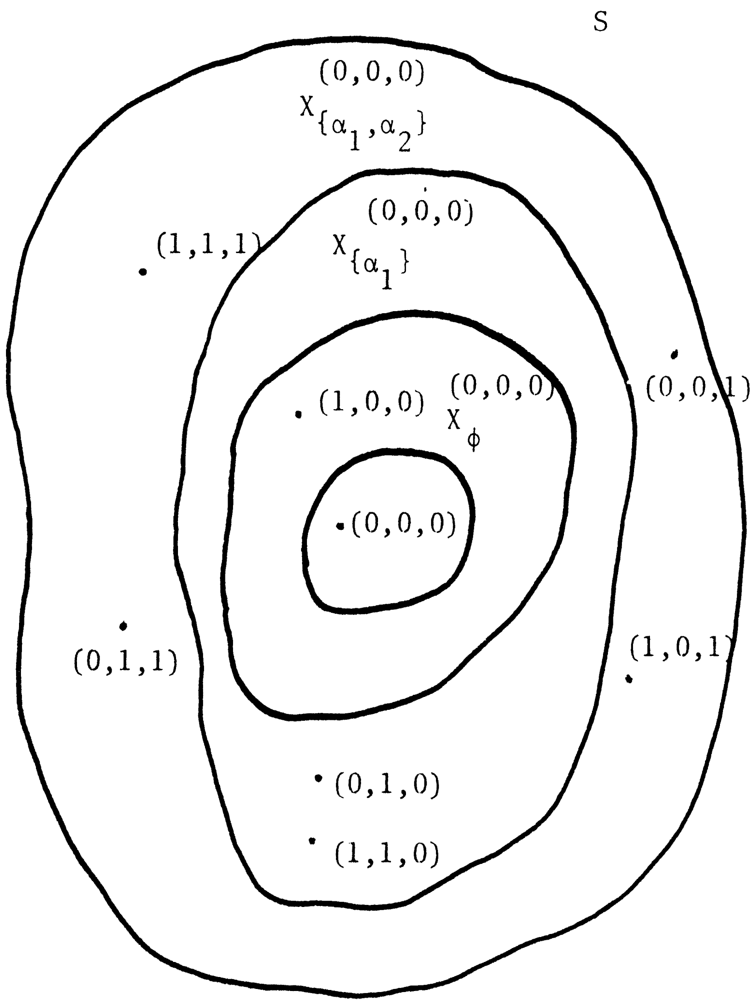


Figure 3.4.1: X_L^y Subsets of S.

are as above, $|X_L^y| = \prod_{\alpha\varepsilon D}|X_\alpha| = \prod_{\alpha\varepsilon L}|X_\alpha| = \prod_{\alpha\varepsilon L}|S_\alpha|$.

c) This follows directly from part b), since $|X_L^y|$ is independent of y and depends only on L.

d) Let $s\varepsilon X_{L_1}^y$. Then $s\Pi_{D-L_1}y$. But since $L_1 \subseteq L_2$, $D-L_1 \supseteq D-L_2$. Hence $s\Pi_{D-L_2}y$ holds and thus $s\varepsilon X_{L_2}^y$ by Definition 3.4.1.

For $L_1 \subsetneq L_2$, let s be the point of S defined by

$$s_\alpha = \begin{cases} y_\alpha & \text{for } \alpha\varepsilon D-L_2 \\ z_\alpha & \text{for } \alpha\varepsilon L_2 \end{cases}, \text{ where } y = (y_\alpha) \text{ and } z = (z_\alpha).$$

z is any point of S s.t. $z_\alpha \neq y_\alpha$ for all $\alpha\varepsilon D$. Then clearly $s\varepsilon X_{L_2}^y$, but $s\not\varepsilon X_{L_1}^y$, since $P_{D-L_1}(s) \neq P_{D-L_1}(y)$ ($z_\alpha \neq y_\alpha$ on (L_2-L_1)).

e) Let L,y be arbitrary but given and let $f\varepsilon F_{(S,R)}(\subseteq L)$. Then $\exists\ \hat{L} \subseteq L$ s.t. $L(f) = \hat{L}$. We denote $\bar{f} = f|X_L^y$. Let $\tilde{L} = L(\bar{f})$. Then clearly $\hat{L} \supseteq \tilde{L}$. Also $f\varepsilon E^{\bar{f}(X_L^y)}(\hat{L})$. As we noted before $COMPL_S^{\hat{L}}(X) = S$. We use Lemma 3.3.1, all of whose conditions are met. By this lemma $\tilde{L}\varepsilon \mathcal{L}(f)$. But S is irredundant, which implies $\tilde{L} = \hat{L}$. Thus $L(f) = L(f|X_L^y)$, which was to be shown.

f) Again we note that for every $\hat{L} \subseteq L$, $COMPL_S^{\hat{L}}(X_L^y) = S$ and we apply Corollary 3.3.3. (Remember that both X_L^y and S are irredundant.)

g) We notice that X_L^y has exactly one point from every equivalence class of Π_L^S. By Lemma 3.4.1 any Z_L as of part g) has to contain at least one point from every equivalence class of Π_L^S. So clearly $|Z_L| \geq |X_L^y|$. □

Although we do not intend to do it here, it can be shown that every set Z_L satisfying g) of Theorem 3.4.1, whose cardinality is minimal, is an X_L^y set, for some $y\varepsilon S$.

Using X_L^y sets as defined above we will construct Y_k^y sets, where k is an integer ranging from 0 through n.

Given an index set D with cardinality n, we will denote by L_k the set of all subsets of D with cardinality k. Of course $|L_k| = \binom{n}{k} = n_k$.

Definition 3.4.2

Let $S = \times_{\alpha\varepsilon D} S_\alpha$, where $|D| = n$, and let k be an arbitrary integer s.t. $0 \le k \le n$. Then for any $y\varepsilon S$

$$Y_k^y = \bigcup_{L\varepsilon L_k} X_L^y \, . \qquad \square$$

It follows from the above definition that Y_k^y is the set of all points in S, which differ from y on at most k coordinates.

In other words for every $s\varepsilon Y_k^y$ there is at least (n-k) coordinates α of D such that $s_\alpha = y_\alpha$.

If d is a Hamming distance on S (i.e. $d(x,y) = |\{\alpha | P_\alpha(x) \neq P_\alpha(y)\}|$), then $Y_k^y = \{s | s\varepsilon S \text{ and } d(s,y) \le k\}$.

Unlike for X_L^y sets, $Y_k^y = Y_k^{\bar{y}}$ if and only if $y = \bar{y}$, except when $k = n$.

We also note that for any $\bar{L}$, such that $|\bar{L}| \le k$, $P_{\bar{L}}(S) = P_{\bar{L}}(Y_k^y)$ and thus $COMPL_S^{\bar{L}}(Y_k^y) = S$. This follows directly from the fact that for any such $\bar{L}$ there is a subset L of D such that $\bar{L} \subseteq L$ and $|L| = k$. But then $X_L^y \subseteq Y_k^y$ holds, and as was noted before $COMPL_S^{\bar{L}}(X_L^y) = S$.

As is easy to see $Y_0^y = X_\phi^y = \{y\}$ and $Y_n^y = X_D^y = S$. Also for every $k < n$, Y_k^y is a proper subset of S.

We are now ready to summarize the most important properties of Y_k^y sets.

Theorem 3.4.2

Let $S = \bigtimes_{\alpha\in D} S_\alpha$, where $|S_\alpha| \geq 2$. Then the following hold.

a) for every k, $0 \leq k \leq n$, every $y\in S$ Y^y_k is irredundant.

b) $|Y^y_0| = 1$ for all $y\in S$, and for $m_\alpha = |S_\alpha|$

$$|Y^y_k| = 1 + \sum_{i=1}^{k} \sum_{L\in \mathcal{L}_i} \prod_{\alpha\in L} (m_\alpha - 1) \ , \ \forall\, k \geq 1$$

In case $m_\alpha = m$ for $\forall\ \alpha\in D$

$$|Y^y_k| = \sum_{i=0}^{k} \binom{n}{i} (m-1)^i$$

c) for any $y, \bar{y}\in S$ and any $1 \leq k \leq n$, $|Y^y_k| = |Y^{\bar{y}}_k|$

d) for any $y\in S$ and any $k_1 \leq k_2$, $Y^y_{k_1} \subseteq Y^y_{k_2}$. Further if $k_1 \lneq k_2$ then $Y^y_{k_1} \subsetneq Y^y_{k_2}$.

e) for any $\bar{L} \subseteq D$ s.t. $|\bar{L}| \leq k$,

$L(f) = L(f|Y^y_k)$ holds for every $f\in F_{(S,R)}(\bar{L})$ (for $\forall$ y, $\forall$ k).

f) for any k, any y if $\bar{f}: Y^y_k \to R$ and $|L(\bar{f})| \leq k$, then $\exists$ a unique extension f of $\bar{f}$ to S such that $L(\bar{f}) = L(f)$.

Proof

a) $Y^y_k = \bigcup_{L\in \mathcal{L}_k} X^y_L$. X^y_L are Cartesian and clearly $\{y\}\in \bigcap_{L\in \mathcal{L}_k} X^y_L$. Thus by Corollary 2.4.1 Y^y_k is irredundant.

b) $Y^y_k = \bigcup_{i=0}^{k} Z^y_i$, where $Z^y_i = \{s | s\in S \text{ and } d(s,y) = i\}$. Assume $k \geq 1$.

All Z^y_i sets are disjoint and so $|Y^y_k| = \sum_{i=0}^{k} |Z^y_i| = 1 + \sum_{i=1}^{k} |Z^y_i|$.

We can choose subsets of D with cardinality i in $\binom{n}{i}$ ways. $\mathcal{L}_i$ is the class of all such subsets. Clearly $Z^y_i = \bigcup_{L\in \mathcal{L}_i} W^y_L$, where W^y_L is the subset of Z^y_i, whose points are different from y on all coordinates of L and on those only. $|W_L^y| = \prod_{\alpha\varepsilon L}(m_\alpha-1)$. Again for $L \neq \tilde{L}$ $W_L^y \cap W_{\tilde{L}}^y = \phi$ and so $|Z_i^y| = \sum_{L\varepsilon\mathcal{L}_i}\prod_{\alpha\varepsilon L}(m_\alpha-1)$. Thus $|Y_k^y| = 1 + \sum_{i=1}^{k}\sum_{L\varepsilon\mathcal{L}_i}\prod_{\alpha\varepsilon L}(m_\alpha-1)$.

For $m = m_\alpha$, $\forall\ \alpha$, this becomes

$$|Y_k^y| = 1 + \sum_{i=1}^{k}\binom{n}{i}(m-1)^i = \binom{n}{0}(m-1)^0 + \sum_{i=1}^{k}\binom{n}{i}(m-1)^i = \sum_{i=0}^{k}\binom{n}{i}(m-1)^i$$

c) follows directly from part b)

d) $Y_{k_1}^y = \{s|d(s,y) \leq k_1\}$, $Y_{k_2}^y = \{s|d(s,y) \leq k_2\}$.

So clearly if $k_1 \leq k_2$ $Y_{k_1}^y \subseteq Y_{k_2}^y$ holds.

For $k_1 \lneq k_2$, $Y_{k_1}^y \subsetneq Y_{k_2}^y$, since the set of points with distance k_2 from y is nonempty.

e) Let $\bar{L}$ be given, $|\bar{L}| \leq k$. Then $\exists$ L s.t. $\bar{L} \subseteq L \subseteq D$, $|L| = k$ and $X_L^y \subseteq Y_k^y$. Let $f\varepsilon F_{(S,R)}(\bar{L})$ be given. Let $\tilde{L} = L(f|Y_k^y)$. By part e) of Theorem 3.4.1, $L(f|X_L^y) = L(f)$. But then since $S \supseteq Y_k^y \supseteq X_L^y$,

$L(f) \supseteq \tilde{L} \supseteq L(f)$ holds, which implies $\tilde{L} = L(f)$. This completes the proof.

f) Suppose there are two extensions of $\bar{f}$, f_1 and f_2 s.t. $f_1 \neq f_2$ and $L(\bar{f}) = L(f_1) = L(f_2) = \bar{L}$, where $|\bar{L}| \leq k$ $(\bar{f}:Y_k^y \to R)$. Clearly $X_{\bar{L}}^y \subseteq Y_k^y$ and by part e) of Theorem 3.4.1, $L(f_1|X_{\bar{L}}^y) = L(f_1) = L(f_2|X_{\bar{L}}^y) = \bar{L}$. Since both f_1 and f_2 are extensions of $\bar{f}$, $f_1|X_{\bar{L}}^y = f_2|X_{\bar{L}}^y = \hat{f}$.

But then $L(\hat{f}) = \bar{L}$ and f_1, f_2 are two different extensions of $\hat{f}$ to S with location $\bar{L}$. This however contradicts part f) of Theorem 3.4.1. □

<u>Corollary 3.4.2</u>

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|S_\alpha| = m$, $\forall\ \alpha$. Then the following hold.

a) for any $L \subseteq D$ with $|L| = \ell \leq k$

$$c_\ell = |P_L(S)| - |P_L(Y_k^y)| = 0$$

b) for any $\ell > k$ and any $L \subseteq D$ with $|L| = \ell$

$$c_\ell = |P_L(S)| - |P_L(Y_k^y)| = \sum_{i=0}^{\ell-k-1} \binom{\ell}{i} (m-1)^{\ell-i}$$

Proof

a) This is clear from the definition of Y_k^y.

b) the proof of this part is analogous to the proof of part b) of Theorem 3.4.2, except that projections are taken. □

As we shall see later c_ℓ-values will be required for confidence computations. It is then important that they be easily computable.

Remark

Every superset of Y_{n-1}^y is irredundant. We simply note that $S - Y_{n-1}^y = \{z | d(z,y)=n\} = \bigtimes_{\alpha\in D}(S_\alpha - \{y_\alpha\})$ and use Corollary 2.4.2 □

Suppose that given an integer k, $0 < k < n$ and some enumeration of L_k we are seeking a family $\{Y_i\}$ of irredundant subsets of S with the following property. For every $i \leq k$ and any subset $\bar{L}$ of L_i (where $L_1, L_2, \ldots$ is the enumeration of L_k), if $f \in F_{(S,R)}(\bar{L})$, then $L(f|Y_j) = L(f)$ holds for all $j \geq i$, and $|Y_{i+1} - Y_i|$ is minimal. In other words at i'th stage we add a minimal number of points to Y_{i-1} to achieve the desired property. $X_{L_1}^y, X_{L_1}^y \cup X_{L_2}^y, \ldots, \bigcup_{L_i \in L_k} X_{L_i}^y$ turns out to be such a family (for an arbitrary $y \in S$).

Proposition 3.4.1

Let $S = \bigtimes_{\alpha\in D} S_\alpha$, where $|D| = n$ and let k be any integer, $0 < k < n$. Let $\{L_i\}_{i=1}^{n_k}$ be an enumeration of L_k and let $\{Y_i\}_{i=1}^{n_k}$ be a family of subsets of S, where $Y_1 = X^y_{L_1}$ and $Y_{i+1} = Y_i \cup X^y_{L_{i+1}}$, for all i, $1 \leq i \leq n_k - 1$.

Then the following hold.

a) $\{Y_i\}_{i=1}^{n_k}$ has the property, that given any i, $1 \leq i \leq n_k$, and any $\bar{L} \subseteq L_i$, for every $f \varepsilon F_{(S,R)}(\bar{L})$, $L(f) = L(f|Y_j)$ holds for all $j \geq i$.

b) for every i, $1 \leq i \leq n_k - 1$, if Z_{i+1} is a subset of S such that $Z_{i+1} \supseteq Y_i$ and for every $\bar{L} \subseteq L_{i+1}$, any $f \varepsilon F_{(S,R)}(\bar{L})$, $L(f) = L(f|Z_{i+1})$ holds, then $|Z_{i+1} - Y_i| \geq |Y_{i+1} - Y_i|$.

<u>Proof</u>

a) The fact that the family $\{Y_i\}$ satisfies property a) follows directly from part e) of Theorem 3.4.1.

b) It follows from Lemma 3.4.1 that Z_{i+1} has to contain at least one point from every equivalence class of $\Pi^S_{L_{i+1}}$. To prove the minimality of $(Y_{i+1} - Y_i)$ then, it suffices to show that for every $z \varepsilon Y_{i+1} - Y_i$, $[z]_{\Pi^S_{L_{i+1}}} \cap Y_i = \phi$. Since $X^y_{L_{i+1}}$ contains exactly one point from every equivalence class of $\Pi^S_{L_{i+1}}$, this will show that every point of $(Y_{i+1} - Y_i)$ is the only point from its $\Pi^S_{L_{i+1}}$ equivalence class in Y_{i+1}, and thus at least as many as $|Y_{i+1} - Y_i|$ points have to be added to Y_i to achieve the desired property.

Let $z \varepsilon Y_{i+1} - Y_i = X^y_{L_{i+1}} - \bigcup_{\ell=1}^{i} X^y_{L_\ell}$

$z \varepsilon X^y_{L_{i+1}} \Rightarrow z_\alpha = y_\alpha$ for all $\alpha \varepsilon D - L_{i+1}$.

$z \not\varepsilon Y_i \Rightarrow z \not\varepsilon X^y_{L_\ell}$ for $\forall\, \ell = 1,\ldots,i$. From Definition 3.4.1 then it follows that for every $\ell \varepsilon (1,\ldots,i)$ $\exists$ an $\alpha_\ell \varepsilon D - L_\ell$ s.t. $z_{\alpha_\ell} \neq y_{\alpha_\ell}$. But since $z_\alpha = y_\alpha$ for all $\alpha \in D-L_{i+1}$, this implies that for every ℓ $\exists$ an $\alpha_\ell \in (D-L_\ell) \cap L_{i+1}$ s.t. $z_{\alpha_\ell} \neq y_{\alpha_\ell}$. (It can be checked easily that since $L_\ell \neq L_{i+1}$ for all ℓ, $(D-L_\ell) \cap L_{i+1}$ is nonempty.)

Suppose $\exists$ an $s \in Y_i$ s.t. $P_{L_{i+1}}(s) = P_{L_{i+1}}(z)$, i.e. $s \in [z]_{\Pi^S_{L_{i+1}}}$.

Then $P_\alpha(z) = P_\alpha(s)$ for all $\alpha \in L_{i+1}$. If $s \in Y_i$ then clearly $s \in X^y_{L_\ell}$, for some ℓ. This implies $s_\alpha = y_\alpha$ for all $\alpha \in D-L_\ell$. But $s_\alpha = z_\alpha$ for all $\alpha \in L_{i+1}$ and by the above argument $\exists$ an $\alpha_\ell \in (D-L_\ell) \cap L_{i+1}$ s.t. $s_{\alpha_\ell} = z_{\alpha_\ell} \neq y_{\alpha_\ell}$ #. So indeed for every $z \in Y_{i+1} - Y_i$, $[z]_{\Pi^S_{L_{i+1}}} \cap Y_i = \phi$, which completes the proof. □

It follows directly from Proposition 3.2.3, that for any function f from S to R, $L(f|Y_i) = \bigcup_{\ell=1}^{i} L(f|X^y_{L_\ell})$.

We remark that the $\{Y_i\}$ family of Proposition 3.4.1 has the following property: for any i and any $\bar{L} \subseteq L_i$, given any $j \geq i$ and any function $\bar{f} \in F_{(Y_j,R)}(\bar{L})$, there is a unique extension f of $\bar{f}$ to S with location $\bar{L}$.

We also note that given k, $0 < k < n$ constructing $\{Y_i\}$ family of subsets of S as of Proposition 3.4.1 is a way of building up Y^y_k set, since $Y_{n_k} = Y^y_k$.

Suppose that although the function f on S is not known, an upper bound on the size of its location is given and equal to k, where $0 < k < n$.

Then if $f|Y^y_k$ is known we find L(f) and f itself without needing any more information. That is guaranteed by parts e) and f) of Theorem 3.4.2. This is also the manner in which Y^y_k sets will be used for model building.

Another way to proceed would be to experiment on Y_1, then Y_2, etc., eventually finding the desired function and its location. Although in particular cases it might be possible to quit at some earlier stage, before $Y_{n_k} = Y_k^y$ is reached, this is not the case in general.

Essentially, this results from the fact that for $L_1 \cup L_2 \supseteq L_3$, and f any function from S s.t. $L(f) \subseteq L_3$, although $L(f|X^y_{L_1}) \cup L(f|X^y_{L_2}) = L(f|X^y_{L_1} \cup X^y_{L_2}) \subseteq L(f) = L(f|X^y_{L_3})$, the above containment may be proper. Thus it is not sufficient to choose enough elements of $\mathcal{L}_k$ to cover D, we might need them all.

We will now illustrate the construction of Y_k^y sets by an example.

<u>Example 3.4.2</u>

Consider $S = \{0,1\}^3$ and $y = (0,0,0)$. Then

$$Y_0^y = X_\phi^y = \{(0,0,0)\}$$

$$Y_1^y = X^y_{\alpha_1} \cup X^y_{\alpha_2} \cup X^y_{\alpha_3} = \{(0,0,0),(1,0,0)\} \cup \{(0,0,0),(0,1,0)\} \cup \{(0,0,0),(0,0,1)\} = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$$

$$Y_2^y = X^y_{\{\alpha_1,\alpha_2\}} \cup X^y_{\{\alpha_1,\alpha_3\}} \cup X^y_{\{\alpha_2,\alpha_3\}} = \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\} \cup \{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\} \cup \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\} = \{(0,0,0),(0,1,0)(1,0,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1)\}$$

$$Y_3^y = S$$

$Y_0^y \subsetneq Y_1^y \subsetneq Y_2^y \subsetneq Y_3^y$ holds. □

We would like to remark now on a relative size of Y_k^y sets.

It follows from part b) of Theorem 3.4.2 that if all m_α's are

same and equal to m, $|Y^y_{n-1}| = \sum_{i=0}^{n-1}\binom{n}{i}(m-1)^i =$
$\sum_{i=0}^{n}\binom{n}{i}(m-1)^i - \binom{n}{n}(m-1)^n = m^n - (m-1)^n.$

For large m the relative size of Y^y_{n-1}, $\frac{|Y^y_{n-1}|}{|S|} = \frac{m^n - (m-1)^n}{m^n} =$
$1 - (1-\frac{1}{m})^n$ becomes very small. Some example relative sizes of Y^y_{n-1} are given in the table of Figure 3.4.2.

In the next proposition we will show that every superset of Y^y_{n-1} is irredundant. This result will be used in later chapters.

<u>Proposition 3.4.2</u>

Let $S = \bigtimes_{\alpha\varepsilon D} S_\alpha$, where $|D| = n$. Then for any $y\varepsilon S$ and any $X \supseteq Y^y_{n-1}$, X is irredundant.

<u>Proof</u>

We recall that $Y^y_{n-1} = \{s|s\varepsilon S \text{ and } d(s,y) \leq n-1\}$. Since $|D| = n$, for every $s\varepsilon S$, $d(s,y) \leq n$ holds. Also since $|S_\alpha| \geq 2$,
$S - Y^y_{n-1} = \{s|s\varepsilon S \text{ and } d(s,y) = n\} = \bigtimes_{\alpha\varepsilon D}(S_\alpha - \{y_\alpha\}) = \bigtimes_{\alpha\varepsilon D} C_\alpha = C$, where $C_\alpha = S_\alpha - \{y_\alpha\}$, $\forall\ \alpha\varepsilon D$. Thus $S - Y^y_{n-1} = C$ is Cartesian, $C_\alpha \subsetneq S_\alpha$, and by Lemma 2.4.1 every $X \supseteq S-C = Y^y_{n-1}$ is irredundant. This completes the proof. □

<u>Remark</u>

In case S is not Cartesian but is a set difference of two Cartesian sets, $S = \bigtimes_{\alpha\varepsilon D} S_\alpha - \bigtimes_{\alpha\varepsilon D} C_\alpha$, where $|S_\alpha - C_\alpha| \geq 2$, for all α the following observations are in order.

1) For every $y\varepsilon \bigtimes_{\alpha\varepsilon D}(S_\alpha - C_\alpha)$ the following hold.

$S = \underset{\alpha\varepsilon D}{X} S_\alpha \qquad |S_\alpha| = m$, for all $\alpha\varepsilon D$. $\qquad |D| = n = 5$

m	3	5	6	7	8
$\lvert Y^y_{n-1}\rvert$	211	2101	4651	9031	15961
$\lvert S\rvert$	243	3125	7776	16807	32768
$\frac{\lvert Y^y_{n-1}\rvert}{\lvert S\rvert}$	.868	.672	.598	.537	.487

Figure 3.4.2: Relative Size of Y^y_{n-1} as a Function of m.

a) for every $L \subseteq D$, such that $|L| < n$, $X_L^y \subsetneq S$

b) for every $k < n$ $Y_k^y \subsetneq S$ (where X_L^y, Y_k^y are subsets of $\bigtimes_{\alpha \varepsilon D} S_\alpha$ as defined before).

2) Theorems 3.4.1, 3.4.2 and Proposition 3.4.1 essentially hold for S as above, except for obvious minor changes. □

CHAPTER IV

LOCATION INFERENCE

4.1 Introduction

In this chapter we will deal with the problem of location inference for functions from a structured domain S to a codomain R. It will be assumed throughout, that all sets involved are finite. Thus S, the index set D, which S is structured over, and R will be finite sets ($|R| \geq 2$).

It is often the case, that the function involved is known on some proper subset of its domain only. This happens for instance, when the functional values are obtained through experimentation and so practical limitations on the number of experiments conducted are present.

Given a function $\bar{f}$ from a subset X of S to R, the "actual" function f from S to R (not known) is an extension of $\bar{f}$ to S.

Relative to a probability space defined, for every location L of $\bar{f}$, we find the probability that L is a location of f. This probability reflects the confidence we have in L being a location of f.

We will discuss here the properties of confidence function and its dependence on parameters (location, upper bound on location sizes, a subset size, etc.).

We will show that for any two irredundant domain subsets X_1 and X_2, where $X_1 \subseteq X_2$, and any function f from S, confidence in the location of $f|X_2$ is at least as large as confidence in the location of

$f|X_1$. It is in this sense, that confidence is nondecreasing.

We will also talk here about an average or expected confidence on a subset X of S. This will be formally computed as an expected value of a properly defined random variable on X.

Finally we will describe how to make predictions of f to S or some proper subset of S, given knowledge of f on a subset X of S. We will then discuss the relation between the size of the predictive range and the confidence we have in predictions made.

4.2 Confidence in an Inferred Location

In this section we will define the confidence function. The probability measure chosen, will be the one making every function from S to R equally likely. This reflects our assumption that "everything we have not seen yet" is equally likely to happen.

We will then analyze the most important properties of confidence function introduced. We will use the notation established in previous chapters. Let $\langle F_{(S,R)}, 2^{F(S,R)}, P\rangle$ be a probability space, where $2^{F(S,R)}$ is the power set of $F_{(S,R)}$ and for every $A \varepsilon 2^{F(S,R)}$, $P(A) = \frac{|A|}{|F_{(S,R)}|}$.

Definition 4.2.1

Let $X \subseteq S$ and let $\bar{f}: X \to R$. Then for any $L \subseteq D$, any integer ub such that $0 \leq ub \leq |D|$

a) $CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L, ub) = P(\text{actual } f \varepsilon F_{(S,R)}(\subseteq L, \cdot) | \text{actual } f \varepsilon E^{\bar{f}(X)}_{(S,R)}(\cdot, ub))$

b) $CONF^{\bar{f}(X)}_{(S,R)}(L, ub) = P(\text{actual } f \varepsilon F_{(S,R)}(L, \cdot) | \text{actual } f \varepsilon E^{\bar{f}(X)}_{(S,R)}(\cdot, ub))$ □

When $ub = n = |D|$ we will denote $CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L, n)$ by $CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L)$ and similarly $CONF^{\bar{f}(X)}_{(S,R)}(L, ub)$ by $CONF^{\bar{f}(X)}_{(S,R)}(L)$.

We note that

$$CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L) = P(\text{actual } f\varepsilon F_{(S,R)}(\subseteq L)\,|\,\text{actual } f\varepsilon E^{\bar{f}(X)}_{(S,R)}) \text{ and}$$

$$CONF^{\bar{f}(X)}_{(S,R)}(L) = P(\text{actual } f\varepsilon F_{(S,R)}(L)\,|\,\text{actual } f\varepsilon E^{\bar{f}(X)}_{(S,R)}).$$

Proposition 4.2.1

Let $X \subseteq S$ and let $\bar{f}:X \rightarrow R$. Then for any $L \subseteq D$,

a) $$CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L,ub) = \begin{cases} \dfrac{|E^{\bar{f}}_{(S,R)}(\subseteq L,ub)|}{|E^{\bar{f}}_{(S,R)}(\cdot,ub)|} & \text{if } E^{\bar{f}}_{(S,R)}(\cdot,ub) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$CONF^{\bar{f}(X)}_{(S,R)}(L,ub) = \begin{cases} \dfrac{|E^{\bar{f}}_{(S,R)}(L,ub)|}{|E^{\bar{f}}_{(S,R)}(\cdot,ub)|} & \text{if } E^{\bar{f}(X)}_{(S,R)}(\cdot,ub) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

b) In case $ub = n$ and $\exists$ an $\bar{L}\varepsilon\mathcal{L}(\bar{f})$ such that $L \supseteq \bar{L}$

$$CONF^{\bar{f}(X)}_{(S,R)}(\subseteq L) = |R|^{-\{(|S|-|P_L(S)|) - (|X|-|P_L(X)|)\}}$$

In case $ub = n$ and $L\varepsilon\mathcal{L}(\bar{f})$

$$CONF^{\bar{f}(X)}_{(S,R)}(L) = |R|^{-\{(|S|-|P_L(S)|) - (|X|-|P_L(X)|)\}}$$

Proof

a) It follows directly from Definition 4.2.1 that

$$CONF^{\bar{f}}(\subseteq L,ub) = P(\text{actual } f\varepsilon F(\subseteq L)\,|\,\text{actual } f\varepsilon E^{\bar{f}}(\cdot,ub)).$$

In case $E^{\bar{f}}(\cdot,ub) = \phi$ the above is clearly 0. Otherwise we employ a well known formula for a conditional probability and

$$\mathrm{CONF}^{\bar{f}}(\subseteq L,ub) = \frac{P(\text{actual } f\varepsilon F\ (\subseteq L) \text{ and actual } f\varepsilon E^{\bar{f}}(\cdot,ub))}{P(\text{actual } f\varepsilon E^{\bar{f}}(\cdot,ub))} =$$

$$\frac{P(\text{actual } f\varepsilon E^{\bar{f}}(\subseteq L,ub))}{P(\text{actual } f\varepsilon E^{\bar{f}}(\cdot,ub))} = \frac{|E^{\bar{f}}(\subseteq L,ub)| / |F|}{|E^{\bar{f}}(\cdot,ub)| / |F|} .$$

The last equality follows directly from the definition of P.

Similarly we derive the formula for $\mathrm{CONF}^{\bar{f}(X)}_{(S,R)}(L,ub)$.

b) It follows directly from part a) that

$$\mathrm{CONF}^{\bar{f}(X)}(\subseteq L) = \frac{|E^{\bar{f}(X)}(\subseteq L)|}{|E^{\bar{f}(X)}|}$$

Obviously since there are $|S-X| = |S|-|X|$ points of S-X, there are $|R|^{|S|-|X|}$ possible extensions of $\bar{f}$ to S.

It follows from Theorem 3.3.1 that with L as assumed

$$|E^{\bar{f}(X)}_{(S,R)}(\subseteq L)| = |R|^{|P_L(S)|-|P_L(X)|}$$

There are $|P_L(S)|-|P_L(X)|$ blocks of Π^S_L in $\text{S-COMPL}^L_S(X)$ and $|R|$ values may be assigned to every such block.

Thus

$$\mathrm{CONF}^{\bar{f}(X)}(\subseteq L) = \frac{|R|^{|P_L(S)|-|P_L(X)|}}{|R|^{|S|-|X|}}$$ and the final expression for $\mathrm{CONF}^{\bar{f}(X)}(\subseteq L)$ readily follows.

To compute $\mathrm{CONF}^{\bar{f}(X)}_{(S,R)}(L)$, where $L\varepsilon L(\bar{f})$ we use part a) of Proposition 3.3.3 and just proven result. □

Remark

1) We note that for an irredundant subset X of S and any function $\bar{f}:X \to R$ such that $L = L(\bar{f})$ and $|L| = ub$ $\mathrm{CONF}^{\bar{f}(X)}_{(S,R)}(L,ub) = 1$.

2) For any two functions $\bar{f}_1$ and $\bar{f}_2$ from X to R, if $L\varepsilon L(\bar{f}_1)$ and $L\varepsilon L(\bar{f}_2)$, $\mathrm{CONF}^{\bar{f}_1(X)}_{(S,R)}(L) = \mathrm{CONF}^{\bar{f}_2(X)}_{(S,R)}(L)$ holds. □

Proposition 4.2.1 will be used essentially in the following way. Having observed $\bar{f}$ on X, for every $L \varepsilon L(\bar{f})$ we will compute $CONF^{\bar{f}}_{(S,R)}(L,ub)$, where ub is a given bound on location size, known or assumed a priori.

When $|L(\bar{f})| > 1$ we might choose the location with highest confidence as a location of f and the corresponding confidence gives us simply an idea as to how good our choice is. When $|L(\bar{f})| = 1$ we are dealing of course with the unique location of $\bar{f}$.

We note that if S is irredundant $E^{\bar{f}(X)}_{(S,R)}(L,ub) = 0$ when $|L| > ub$. Also $E^{\bar{f}(X)}_{(S,R)}(L,ub) = E^{\bar{f}(X)}_{(S,R)}(L)$ for L with $|L| \leq ub$ and arbitrary S.

We will next show that confidence is nondecreasing with the decreasing upper bound. This should be intuitively obvious, for as an upper bound decreases we decrease the number of possible candidates for our function.

Proposition 4.2.2

Let $X \subseteq S$ and let $\bar{f}:X \to R$. Then for any $\bar{L} \varepsilon L(\bar{f})$, any ub_1, ub_2 such that $|\bar{L}| \leq ub_2 \leq ub_1$

$$CONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub_1) \leq CONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub_2).$$

Proof

Since $|\bar{L}| \leq ub_2 \leq ub_1$ holds $E^{\bar{f}}(\bar{L},ub_1) = E^{\bar{f}}(\bar{L},ub_2) = E^{\bar{f}}(\bar{L})$. Thus to show the above inequality, we just need to show that $|E^{\bar{f}}(\cdot,ub_1)| \geq |E^{\bar{f}}(\cdot,ub_2)|$ (see Proposition 4.2.1). $ub_1 \geq ub_2$ implies that $E^{\bar{f}}(\cdot,ub_2) \subseteq E^{\bar{f}}(\cdot,ub_1)$ and the above inequality clearly holds. □

Our next result will be following. Given any two irredundant subsets of irredundant S, X_1 and X_2 such that $X_1 \subseteq X_2$ and any two functions $\bar{f}_1$ and $\bar{f}_2$, where $\bar{f}_1 \varepsilon F_{(X_1,R)}$, $\bar{f}_2 \varepsilon F_{(X_2,R)}$, and $\bar{f}_1 = \bar{f}_2 | X_1$ holds, the confidence in $L(\bar{f}_2)$ being a location of actual function is at least as large as confidence in $L(\bar{f}_1)$ being a location of actual function (provided upper bound condition is met in both cases). So as we go on experimenting the confidence will never decrease.

Before we state and prove it, we need to establish some auxiliary results reflecting the relations between cardinalities of projections on X_1 and X_2 sets.

Lemma 4.2.1

For any $X_1 \subseteq X_2 \subseteq S$ and any $\hat{L}, L \subseteq D$, where $\hat{L} \supseteq L$

$$|P_{\hat{L}}(X_2)| - |P_{\hat{L}}(X_1)| \geq |P_L(X_2)| - |P_L(X_1)|$$

Proof

$$|P_{\hat{L}}(X_2)| - |P_{\hat{L}}(X_1)| = |P_{\hat{L}}(X_2 - \{x | x \in X_2 \text{ and } \exists \text{ a } y \in X_1 \text{ s.t. } y \Pi_{\hat{L}} x\})| =$$

$$|P_{\hat{L}}(X_2 - \{\cup [x]_{\Pi_{\hat{L}}^{X_2}} \mid [x]_{\Pi_{\hat{L}}^{X_2}} \cap X_1 \neq \phi\})| = |P_{\hat{L}}(X_2 - \mathrm{COMPL}_{X_2}^{\hat{L}}(X_1))| = |P_{\hat{L}}(Y_{\hat{L}})|$$

Similarly, $|P_L(X_2)| - |P_L(X_1)| = |P_L(X_2 - \mathrm{COMPL}_{X_2}^{L}(X_1))| = |P_L(Y_L)|$.

Since $\hat{L} \supseteq L$, $Y_{\hat{L}} \supseteq Y_L$ holds and so $|P_L(Y_{\hat{L}})| \geq |P_L(Y_L)|$. But then $|P_{\hat{L}}(Y_{\hat{L}})| \geq |P_L(Y_{\hat{L}})|$ implies that $|P_{\hat{L}}(Y_{\hat{L}})| \geq |P_L(Y_L)|$, which was to be proved. □

Corollary 4.2.1

For any $X_1, X_2 \subseteq S$ and any $\hat{L}, L \subseteq D$, where $X_1 \subseteq X_2$ and $\hat{L} \supseteq L$, the following hold.

a) $|S - X_1| - |S - X_2| \geq (|P_L(S)| - |P_L(X_1)|) - (|P_{\hat{L}}(S)| - |P_{\hat{L}}(X_2)|)$

b) $(|P_{\hat{L}}(S)| - |P_{\hat{L}}(X_1)|) - (|P_{\hat{L}}(S)| - |P_{\hat{L}}(X_2)|) \geq$
$(|P_L(S)| - |P_L(X_1)|) - (|P_L(S)| - |P_L(X_2)|)$

Proof

a) Since $X_1 \subsetneq X_2 \subsetneq S$, $|S-X_1| - |S-X_2| = |X_2-X_1| = |X_2| - |X_1| = |P_D(X_2)| - |P_D(X_1)|$.

By Lemma 4.2.1

$|P_D(X_2)| - |P_D(X_1)| \geq |P_L(X_2)| - |P_L(X_1)|$ holds.

Also by the very same lemma

$|P_{\hat{L}}(S)| - |P_{\hat{L}}(X_2)| \geq |P_L(S)| - |P_L(X_2)|$.

Adding the above inequalities we obtain

$|X_2| - |X_1| + |P_{\hat{L}}(S)| - |P_{\hat{L}}(X_2)| \geq |P_L(S)| - |P_L(X_1)|$,

which implies

$|X_2| - |X_1| \geq (|P_L(S)| - |P_L(X_1)|) - (|P_{\hat{L}}(S)| - |P_{\hat{L}}(X_2)|)$,

which was to be shown.

b) We note that the left hand side of the inequality is simply equal to $|P_{\hat{L}}(X_2)| - |P_{\hat{L}}(X_1)|$ and similarly the right hand side of the inequality to $|P_L(X_2)| - |P_L(X_1)|$.

So the inequality of part b) is simply another form of the inequality of Lemma 4.2.1. □

We are now ready to state and prove next theorem.

Theorem 4.2.1

Let X_1, X_2, S be all irredundant, where $\phi \neq X_1 \subseteq X_2 \subseteq S$ holds. Let $\bar{f}_2 : X_2 \to R$ and let $\bar{f}_1 = \bar{f}_2 | X_1$. Then with $L_1 = L(\bar{f}_1)$ and $L_2 = L(\bar{f}_2)$

(1) $CONF^{\bar{f}_1(X_1)}_{(S,R)}(L_1, ub) \leq CONF^{\bar{f}_2(X_2)}_{(S,R)}(L_2, ub)$ holds for any integer ub, $0 \leq ub \leq n$, provided that if $|L_1| \leq ub$ holds then $|L_2| \leq ub$ also holds.

Proof

We first note that since X_1 and X_2 are both irredundant, $L_1 \subseteq L_2$.

If $|L_1| > ub$ then clearly $|L_2| > ub$ and both sides of (1) are equal to 0.

So assume $|L_1| \leq ub$ and $|L_2| \leq ub$ holds. Then $E^{\bar{f}_1(X_1)}_{(S,R)}(\cdot,ub) \neq \phi$ and $E^{\bar{f}_2(X_2)}_{(S,R)}(\cdot,ub) \neq \phi$. Hence we need to show that

$$(2) \quad \frac{|E^{\bar{f}_1}(L_1)|}{|E^{\bar{f}_1}(\cdot,ub)|} \leq \frac{|E^{\bar{f}_2}(L_2)|}{|E^{\bar{f}_2}(\cdot,ub)|} \quad \text{holds}$$

or equivalently that

$$(3) \quad \frac{|E^{\bar{f}_1}(L_1)|}{\left|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_1}} E^{\bar{f}_1}(\subseteq \tilde{L})\right|} \leq \frac{|E^{\bar{f}_2}(L_2)|}{\left|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{f}_2}(\subseteq \tilde{L})\right|} \quad \text{holds.}$$

Since $L_2 \supseteq L_1$, $\{\tilde{L} \mid |\tilde{L}| = ub \text{ and } \tilde{L}\supseteq L_1\} \supseteq \{\tilde{L} \mid |\tilde{L}| = ub \text{ and } \tilde{L}\supseteq L_2\}$ and so

$$\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_1}} E^{\bar{f}_1}(\subseteq \tilde{L}) \supseteq \bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{f}_1}(\subseteq \tilde{L}) .$$

To prove (3) then it suffices to prove that

$$(4) \quad \frac{|E^{\bar{f}_1}(L_1)|}{\left|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{f}_1}(\subseteq \tilde{L})\right|} \leq \frac{|E^{\bar{f}_2}(L_2)|}{\left|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{f}_2}(\subseteq \tilde{L})\right|}$$

From now on we will denote the set $\{\tilde{L} \mid |\tilde{L}| = ub \;\&\; \tilde{L}\supseteq L_2\}$ by $\mathcal{L}$.

We note that (4) holds if and only if

$$(5) \quad \frac{\left|\bigcup_{\tilde{L}\in\mathcal{L}} E^{\bar{f}_1}(\subseteq \tilde{L})\right|}{|E^{\bar{f}_1}(L_1)|} \geq \frac{\left|\bigcup_{\tilde{L}\in\mathcal{L}} E^{\bar{f}_2}(\subseteq \tilde{L})\right|}{|E^{\bar{f}_2}(L_2)|} \quad \text{holds.}$$

Since $E^{\bar{f}_1}(L_1) = E^{\bar{f}_1}(\subseteq L_1)$ and $L_1 \subseteq L_2$, clearly $E^{\bar{f}_1}(\subseteq L_1) \subseteq E^{\bar{f}_1}(\subseteq L_2)$ holds. Also since $L_2 = L(\bar{f}_2)$, $E^{\bar{f}_2}_{(S,R)}(L_2) = E^{\bar{f}_2}_{(S,R)}(\subseteq L_2)$.

To prove (5) then it suffices to show that

$$(6)\quad \frac{\left|\bigcup_{\tilde{L}\in L} E^{\bar{f}_1}(\subseteq \tilde{L})\right|}{|E^{\bar{f}_1}(\subseteq L_2)|} \geq \frac{\left|\bigcup_{\tilde{L}\in L} E^{\bar{f}_2}(\subseteq \tilde{L})\right|}{|E^{\bar{f}_2}(L_2)|} \text{ holds.}$$

We note that for any irredundant subset X of S (irredundant), any $\bar{f}:X \to R$ and any $L,\hat{L} \subseteq D$

$$E^{\bar{f}(X)}(\subseteq L) \cap E^{\bar{f}(X)}(\subseteq \hat{L}) = E^{\bar{f}(X)}(\subseteq (L \cap \hat{L})).$$

Also for any family of finite sets $A_1, A_2, \ldots, A_n$

$$|\cup A_i| = |A_1| + (|A_2| - |A_1 \cap A_2|) +$$
$$(|A_3| - |A_3 \cap A_1| - |A_3 \cap A_2| + |A_1 \cap A_2 \cap A_3|) + \ldots \text{ holds.}$$

Let $\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_k$ be an enumeration of L.

Then $\left|\bigcup_{\tilde{L}\in L} E^{\bar{f}_1}(\subseteq \tilde{L})\right| = |E^{\bar{f}_1}(\subseteq \tilde{L}_1)| + (|E^{\bar{f}_1}(\subseteq \tilde{L}_2)| - |E^{\bar{f}_1}(\subseteq (\tilde{L}_1 \cap \tilde{L}_2))|) +$
$(|E^{\bar{f}_1}(\subseteq \tilde{L}_3)| - |E^{\bar{f}_1}(\subseteq (\tilde{L}_3 \cap \tilde{L}_1))| - |E^{\bar{f}_1}(\subseteq (\tilde{L}_3 \cap \tilde{L}_2))| +$
$|E^{\bar{f}_1}(\subseteq (\tilde{L}_1 \cap \tilde{L}_2 \cap \tilde{L}_3))|) + \ldots = A_1 + A_2 + A_3 + \ldots$, where each A_i is of the form

$$A_i = |E^{\bar{f}_1}(\subseteq \tilde{L}_i)| - \sum_{p\in P_i} |E^{\bar{f}_1}(\subseteq L_p)| + \sum_{q\in Q_i} |E^{\bar{f}_1}(\subseteq L_q)|,$$

for some index sets P_i and Q_i. For every $p \in P_i, L_2 \subseteq L_p \subsetneq \tilde{L}_i$ and for every $q \in Q_i, L_2 \subseteq L_q \subsetneq \tilde{L}_i$ holds. $|\tilde{L}_i| = ub$ for all i and $\tilde{L}_i \supseteq L_2$, $\forall$ i.

To prove that (6) holds then, it suffices to show that for every i

$$(7)\quad \frac{|E^{\bar{f}_1}(\subseteq \tilde{L}_i)| - \sum_{p\in P_i} |E^{\bar{f}_1}(\subseteq L_p)| + \sum_{q\in Q_i} |E^{\bar{f}_1}(\subseteq L_q)|}{|E^{\bar{f}_1}(\subseteq L_2)|} \geq$$
$$\frac{|E^{\bar{f}_2}(\subseteq \tilde{L}_i)| - \sum_{p\in P_i} |E^{\bar{f}_2}(\subseteq L_p)| + \sum_{q\in Q_i} |E^{\bar{f}_2}(\subseteq L_q)|}{|E^{\bar{f}_2}(\subseteq L_2)|}$$

We will now show that if $Q_i \neq \phi$, then for every $q \in Q_i$

$$\frac{|E^{\bar{f}_1}(\subseteq L_q)|}{|E^{\bar{f}_1}(\subseteq L_2)|} \geq \frac{|E^{\bar{f}_2}(\subseteq L_q)|}{|E^{\bar{f}_2}(\subseteq L_2)|}$$

We need to show that

$$\frac{\dfrac{|\mathcal{P}_{L_q}(S)| - |\mathcal{P}_{L_q}(X_1)|}{|R|}}{\dfrac{|\mathcal{P}_{L_2}(S)| - |\mathcal{P}_{L_2}(X_1)|}{|R|}} \geq \frac{\dfrac{|\mathcal{P}_{L_q}(S)| - |\mathcal{P}_{L_q}(X_2)|}{|R|}}{\dfrac{|\mathcal{P}_{L_2}(S)| - |\mathcal{P}_{L_2}(X_2)|}{|R|}} .$$

Since $L_2 \subseteq L_q$, $\forall\, q \varepsilon Q_i$ and $X_1 \subseteq X_2$ this follows from part b) of Corollary 4.2.1.

So to prove (7) it suffices to show that for every i

$$(8) \quad \frac{|E^{\bar{f}_1}(\subseteq \tilde{L}_i)| - \sum_{p \varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{|E^{\bar{f}_1}(\subseteq L_2)|} \geq \frac{|E^{\bar{f}_2}(\subseteq \tilde{L}_i) - \sum_{p \varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|}{|E^{\bar{f}_2}(\subseteq L_2)|}$$

holds.

If $P_i = \phi$, (8) holds by part b) of Corollary 4.2.1. So assume $P_i \neq \phi$.

We note that since for every $p \varepsilon P_i$, $L_1 \subseteq L_2 \subseteq L_p$ holds, $E^{\bar{f}_1}(\subseteq L_p) \neq \phi$ and $E^{\bar{f}_2}(\subseteq L_p) \neq \phi$ for $\forall\, p \varepsilon P_i$. Thus $\sum_{p \varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|$ and $\sum_{p \varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|$ are both greater than 0. Also as can be easily seen both nominators of (8) are non-negative, while both denominators are strictly greater than 0.

If RHS of inequality (8) is equal to 0, then the inequality clearly holds. So assume it is greater than 0. Then to show that (8) holds it suffices to show that

$$(9) \quad \frac{|E^{\bar{f}_1}(\subseteq \tilde{L}_i)| - \sum_{p \varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{|E^{\bar{f}_2}(\subseteq \tilde{L}_i)| - \sum_{p \varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|} \geq \frac{|E^{\bar{f}_1}(\subseteq L_2)|}{|E^{\bar{f}_2}(\subseteq L_2)|} \quad \text{holds.}$$

We will now show that

$$(10)\quad \frac{\sum_{p\varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{\sum_{p\varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|} \geq \frac{|E^{\bar{f}_1}(\subseteq L_2)|}{|E^{\bar{f}_2}(\subseteq L_2)|} \quad \text{holds.}$$

To show that, it suffices to show that for every $p\varepsilon P_i$

$$(11)\quad \frac{|E^{\bar{f}_1}(\subseteq L_p)|}{|E^{\bar{f}_2}(\subseteq L_p)|} \geq \frac{|E^{\bar{f}_1}(\subseteq L_2)|}{|E^{\bar{f}_2}(\subseteq L_2)|} \quad \text{is true.}$$

But (11) follows immediately from Corollary 4.2.1, since $L_2 \subsetneq L_p$ for $\forall\, p\varepsilon P_i$ and $X_1 \subsetneq X_2$.

Thus (10) holds and to prove (9) it suffices to show that

$$(12)\quad \frac{|E^{\bar{f}_1}(\subseteq \tilde{L}_i)| - \sum_{p\varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{|E^{\bar{f}_2}(\subseteq \tilde{L}_i)| - \sum_{p\varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|} \geq \frac{\sum_{p\varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{\sum_{p\varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|} .$$

Inequality (12) holds if and only if

$$(13)\quad \frac{|E^{\bar{f}_1}(\subseteq \tilde{L}_i)|}{|E^{\bar{f}_2}(\subseteq \tilde{L}_i)|} \geq \frac{\sum_{p\varepsilon P_i} |E^{\bar{f}_1}(\subseteq L_p)|}{\sum_{p\varepsilon P_i} |E^{\bar{f}_2}(\subseteq L_p)|} \quad \text{holds.}$$

This follows from Corollary 4.2.1. (We show it for every $p\varepsilon P_i$.) So (1) of Theorem 4.2.1 holds, which was to be proved. □

We remark that for $X = \phi$, $CONF^{\bar{f}(X)}_{(S,R)}(L,ub)$ can be interpreted as an a priori confidence in L being a location of actual f, under given upper bound condition. We will denote it simply by $CONF_{(S,R)}(L,ub)$.

We will now show that for an irredundant S and any $L \subsetneq D$, any irredundant subset Y of S, $Y \neq \phi$, any function $\bar{f}: Y \rightarrow R$ such that

$L = L(\bar{f})$, $CONF_{(S,R)}(L,ub) \leq CONF^{\bar{f}(Y)}_{(S,R)}(L,ub)$. In other words having seen a function from Y with location L may increase our confidence in L, but will never decrease it.

<u>Proposition 4.2.3</u>

Let X be an irredundant subset of irredundant S. Let $\bar{f}:X \to R$ and let $L = L(\bar{f})$. Then for any ub

$$(1)\quad CONF_{(S,R)}(L,ub) \leq CONF^{\bar{f}(X)}_{(S,R)}(L,ub)$$

<u>Proof</u>

When $X = \phi$ (1) clearly holds. So we assume $X \neq \phi$.

If $|L| > ub$, then both sides of (1) are 0 and we are done. So assume $|L| \leq ub$.

$CONF(L,ub) = \dfrac{|F_{(S,R)}(L)|}{|F(\cdot,ub)|}$. Since $F_{(S,R)}(L) \subseteq F_{(S,R)}(\subseteq L)$ and

$F_{(S,R)}(\cdot,ub) \supseteq \bigcup_{\substack{|\tilde{L}|=ub \\ \tilde{L}\supseteq L}} F_{(S,R)}(\tilde{L})$, $CONF_{(S,R)}(L,ub) \leq \dfrac{|F_{(S,R)}(\subseteq L)|}{\left|\bigcup_{\substack{|\tilde{L}|=ub \\ \tilde{L}\supseteq L}} F_{(S,R)}(\tilde{L})\right|}$

We will denote $\{\tilde{L} \mid \tilde{L} \supseteq L \text{ and } |\tilde{L}| = ub\}$ by $\mathcal{L}$.

Since

$$CONF^{\bar{f}(X)}(L,ub) = \frac{|E^{\bar{f}}(L)|}{\left|\bigcup_{\tilde{L}\in\mathcal{L}} E^{\bar{f}}(\tilde{L})\right|}$$

to prove (1) of Proposition 4.2.3 it suffices to show that

$$(1)\quad \frac{|F(\subseteq L)|}{\left|\bigcup_{\tilde{L}\in\mathcal{L}} F(\tilde{L})\right|} \leq \frac{|E^{\bar{f}}(L)|}{\left|\bigcup_{\tilde{L}\in\mathcal{L}} E^{\bar{f}}(\tilde{L})\right|} \quad \text{holds.}$$

By analogous argument as in the proof of Theorem 4.2.1 it suffices to show that for any $\tilde{L}_i \varepsilon \mathcal{L}$ and any index sets P_i, Q_i where for every $p \varepsilon P_i$, $L \subseteq L_p \subseteq \tilde{L}_i$, for every $q \varepsilon Q_i$, $L \subsetneq L_q \subsetneq \tilde{L}_i$

$$(2)\quad \frac{|F(\tilde{L}_i)| - \sum_{p\varepsilon P_i} |F(\subseteq L_p)| + \sum_{q\varepsilon Q_i} |F(\subseteq L_q)|}{|F(\subseteq L)|} \geq \frac{|E^{\bar{f}}(\tilde{L}_i)| - \sum_{p\varepsilon P_i} |E^{\bar{f}}(\subseteq L_p)| + \sum_{q\varepsilon Q_i} |E^{\bar{f}}(\subseteq L_q)|}{|E^{\bar{f}}(\subseteq L)|} \quad \text{holds.}$$

We show that for every $q \varepsilon Q_i$ (if $Q_i \neq \phi$)

$$(3)\quad \frac{|F(\subseteq L_q)|}{|F(\subseteq L)|} \geq \frac{|E^{\bar{f}}(\subseteq L_q)|}{|E^{\bar{f}}(\subseteq L)|} \quad .$$

LHS of (3) equals $\dfrac{|R|^{|\mathcal{P}_{L_q}(S)|}}{|R|^{|\mathcal{P}_L(S)|}}$ and RHS of (3) equals $\dfrac{|R|^{|\mathcal{P}_{L_q}(S)| - |\mathcal{P}_{L_q}(X)|}}{|R|^{|\mathcal{P}_L(S)| - |\mathcal{P}_L(X)|}}$

Since $L_q \supseteq L$, $\forall\, q \varepsilon Q_i$, $|\mathcal{P}_{L_q}(X)| \geq |\mathcal{P}_L(X)|$ and hence (3) holds.

Thus to prove (2) it suffices to show that

$$(4)\quad \frac{|F(\subseteq \tilde{L}_i)| - \sum_{p\varepsilon P_i} |F(\subseteq L_p)|}{|F(\subseteq L)|} \geq \frac{|E^{\bar{f}}(\subseteq \tilde{L}_i)| - \sum_{p\varepsilon P_i} |E^{\bar{f}}(\subseteq L_p)|}{|E^{\bar{f}}(\subseteq L)|} \quad \text{holds.}$$

Again it is clear that both nominators in (4) are nonnegative while both denominators of (4) are positive.

Also $\sum_{p\varepsilon P_i} |F(\subseteq L_p)|$ and $\sum_{p\varepsilon P_i} |E^{\bar{f}}(\subseteq L_p)|$ are both positive.

We will now show that

$$(5)\quad \frac{\sum_{p\varepsilon P_i} |F(\subseteq L_p)|}{\sum_{p\varepsilon P_i} |E^{\bar{f}}(\subseteq L_p)|} \geq \frac{|F(\subseteq L)|}{|E^{\bar{f}}(\subseteq L)|} \quad \text{holds.}$$

It suffices to show that for every $p \in P_i$

$$(6)\quad \frac{|F(\subseteq L_p)|}{|E^{\bar{f}}(\subseteq L_p)|} \geq \frac{|F(\subseteq L)|}{|E^{\bar{f}}(\subseteq L)|}$$

$$\text{But}\quad \frac{|R|^{|P_{L_p}(S)|}}{|R|^{|P_{L_p}(S)| - |P_{L_p}(X)|}} \geq \frac{|R|^{|P_L(S)|}}{|R|^{|P_L(S)| - |P_L(X)|}} \quad \text{holds}$$

since $L_p \supseteq L$ implies $|P_{L_p}(X)| \geq |P_L(X)|$.

So (5) is true and to prove (4) it suffices to show, that

$$(7)\quad \frac{|F(\subseteq \tilde{L}_i)| - \sum_{p \in P_i} |F(\subseteq L_p)|}{\sum_{p \in P_i} |F(\subseteq L_p)|} \geq \frac{|E^{\bar{f}}(\subseteq \tilde{L}_i)| - \sum_{p \in P_i} |E^{\bar{f}}(\subseteq L_p)|}{\sum_{p \in P_i} |E^{\bar{f}}(\subseteq L_p)|} \quad \text{holds.}$$

(7) holds if and only if

$$(8)\quad \frac{|F(\subseteq \tilde{L}_i)|}{\sum_{p \in P_i} |F(\subseteq L_p)|} \geq \frac{|E^{\bar{f}}(\subseteq \tilde{L}_i)|}{\sum_{p \in P_i} |E^{\bar{f}}(\subseteq L_p)|} \quad \text{holds}$$

if and only if for every $p \in P_i$

$$(9)\quad \frac{|F(\subseteq \tilde{L}_i)|}{|F(\subseteq L_p)|} \geq \frac{|E^{\bar{f}}(\subseteq \tilde{L}_i)|}{|E^{\bar{f}}(\subseteq L_p)|} \quad \text{holds}$$

if and only if

$$(10)\quad \frac{|R|^{|P_{\tilde{L}_i}(S)|}}{|R|^{|P_{L_p}(S)|}} \geq \frac{|R|^{|P_{\tilde{L}_i}(S)| - |P_{\tilde{L}_i}(X)|}}{|R|^{|P_{L_p}(S)| - |P_{L_p}(X)|}} \quad \text{holds.}$$

But since $\tilde{L}_i \supseteq L_p$, $|P_{\tilde{L}_i}(X)| \geq |P_{L_p}(X)|$ and so (10) holds. This completes the proof. □

Remark

We remark that Theorem 4.2.1 holds for arbitrary X_1, X_2 and S (rather than irredundant) in case ub = n and $L_1 \varepsilon \mathcal{L}(\bar{f}_1)$, $L_2 \varepsilon \mathcal{L}(\bar{f}_2)$ are such that $L_1 \subseteq L_2$.

This follows easily from Corollary 4.2.1 (part a)), since

$$\mathrm{CONF}^{\bar{f}_1}_{(S,R)}(L_1) = \frac{|R|^{|P_{L_1}(S)| - |P_{L_1}(X_1)|}}{|R|^{|S|} - |X_1|} \quad \text{and}$$

$$\mathrm{CONF}^{\bar{f}_2}_{(S,R)}(L_2) = \frac{|R|^{|P_{L_2}(S)| - |P_{L_2}(X_2)|}}{|R|^{|S|} - |X_2|} .$$ □

We will next show that for any two functions $\bar{f}$ and $\bar{g}$ from X to R, if $L(\bar{f}) \subseteq L(\bar{g})$ holds, then confidence in $L(\bar{f})$ being a location of actual function is not greater than confidence in $L(\bar{g})$ being its location. Thus as the complexity of the function increases, so does the confidence. By complexity we mean here simply the size of the location of the function.

The above is made precise in the following.

Proposition 4.2.4

Let X be an irredundant subset of irredundant S. Let $\bar{f}$ and $\bar{g}$ be functions from X to R, where $L(\bar{f}) \subseteq L(\bar{g})$ holds. Then

(1) $\mathrm{CONF}^{\bar{f}(X)}_{(S,R)}(L(\bar{f}),ub) \leq \mathrm{CONF}^{\bar{g}(X)}_{(S,R)}(L(\bar{g}),ub)$ holds for any integer ub, $0 \leq ub \leq n$, provided that if $|L(\bar{f})| \leq ub$ holds then so does $|L(\bar{g})| \leq ub$.

Proof

Let $L(\bar{f}) = L_1$ and $L(\bar{g}) = L_2$. If $|L_1| > ub$, then $|L_2| > ub$ and both sides of inequality (1) are equal to 0. Thus (1) holds.

So we assume that $|L_1| \le ub$ and $|L_2| \le ub$ hold. Then

$$CONF^{\bar{f}(X)}_{(S,R)}(L_1,ub) = \frac{|E^{\bar{f}(X)}(\subseteq L_1)|}{|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_1}} E^{\bar{f}(X)}(\subseteq\tilde{L})|} \quad \text{and}$$

$$CONF^{\bar{g}(X)}_{(S,R)}(L_2,ub) = \frac{|E^{\bar{g}(X)}(\subseteq L_2)|}{|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{g}(X)}(\subseteq\tilde{L})|} \quad .$$

Since $L_2 \supseteq L_1$, $|P_{L_2}(S)| - |P_{L_2}(X)| \ge |P_{L_1}(S)| - |P_{L_1}(X)|$. This follows from part a) of Corollary 4.2.1, where $X_1 = X_2$. Thus by Corollary 3.3.2 $|E^{\bar{f}(X)}(L_1)| \le |E^{\bar{g}(X)}(L_2)|$.

Since $L_2 \supseteq L_1$, $\{\tilde{L} | \tilde{L} \supseteq L_1 \text{ and } |\tilde{L}|=ub\} \supseteq \{\tilde{L} | \tilde{L} \supseteq L_2 \text{ and } |\tilde{L}|=ub\}$. Thus

$$|\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_1}} E^{\bar{f}(X)}(\subseteq\tilde{L})| \ge |\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{f}(X)}(\subseteq\tilde{L})| = |\bigcup_{\substack{\tilde{L} \\ |\tilde{L}|=ub \\ \tilde{L}\supseteq L_2}} E^{\bar{g}(X)}(\subseteq\tilde{L})|.$$

So clearly (1) holds, which we were to prove. □

We note that if X and S are arbitrary but ub = n, Proposition 4.2.4 still applies.

<u>Remark 1</u>

In Lemma 4.2.1 and Corollary 4.2.1, when X_1 and X_2 are $Y^y_{k_1}$ and $Y^y_{k_2}$ sets respectively, ($|S_\alpha| = m$ for $\forall\, \alpha\varepsilon D$) the containment of the locations is not a necessary condition. Rather than assume that $\hat{L} \supseteq L$ holds it suffices to assume that $|\hat{L}| \ge |L|$.

This follows from the fact that for any Y_k^y and any L_1, L_2 with $|L_1| = |L_2|, |P_{L_1}(Y_k^y)| = |P_{L_2}(Y_k^y)|$.

Similarly, in Proposition 4.2.4 if X is a Y_k^y set, it suffices to assume that $|L(\bar{f})| \leq |L(\bar{g})|$ holds. □

Remark 2

1) We note that for any subset X of S (S not necessarily irredundant) and any function $\bar{f}$ from X to R, if D is a location of $\bar{f}$, then $CONF_{(S,R)}^{\bar{f}(X)}(D) = 1$.

First we notice that if $D \in \mathcal{L}(\bar{f})$, then D is the unique location of $\bar{f}$. But then this implies that every extension of $\bar{f}$ to S has location D and the above statement follows.

2) Given an upper bound ub, ub $\leq n - 1$, and any function $\bar{f}$ from Y_{ub}^y to R, such that $|L(\bar{f})| \leq$ ub, $CONF_{(S,R)}^{\bar{f}(Y_{ub}^y)}(L(\bar{f}), ub) = 1$. This follows from the fact that for any $\tilde{L} \supsetneq L(\bar{f})$ such that $|\tilde{L}| \leq$ ub, $E_{(S,R)}^{\bar{f}(Y_{ub}^y)}(\tilde{L}) = \phi$ (see part e) of Theorem 3.4.2).

3) For any $X \subseteq S$ and any $\bar{f}: X \to R$, any $L \in \mathcal{L}(\bar{f})$,

$$CONF_{(S,R)}^{\bar{f}(X)}(L, ub) \geq \frac{1}{|R|^{|S-X|}}, \text{ provided } |L| \leq ub.$$

This follows from Corollary 3.3.2 and the fact that $E^{\bar{f}(X)}(\cdot, ub) \subseteq E^{\bar{f}(X)}$. Clearly $|E^{\bar{f}(X)}| = |R|^{|S-X|}$. □

We will illustrate the above results by an example.

Example 4.2.1

Let $S = \{0,1,2\}^3$ and let X_1, X_2 be following subsets of S.
$X_1 = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$ and $X_2 = X_1 \cup \{(2,1,1)\} = X_1 \cup \{(2,1,1),(1,1,1)\} = X_1 \cup C$.

X_1 is clearly irredundant as a minimal independent set. X_2 is irredundant by Proposition 2.4.4.

a) Consider $\bar{f}_1\colon X_1 \to R$, where $R = \{0,1,2\}$ and $\bar{f}_1$ is defined by a table of Figure 4.2.1. Then $L(\bar{f}_1) = \{\alpha_2\}$. With $ub_1 = 3$ and $ub_2 = 2$

$$|E^{\bar{f}_1(X_1)}(\{\alpha_2\},ub_1)| = |E^{\bar{f}_1(X_1)}(\{\alpha_2\})| = |E^{\bar{f}_1(X_1)}(\{\alpha_2\},ub_2)| =$$

$$|R|^{|P_{\alpha_2}(S)| - |P_{\alpha_2}(X_1)|} = 3^{3-2} = 3^1 = 3$$

$$|E^{\bar{f}_1(X_1)}(\cdot,3)| = |E^{\bar{f}(X_1)}| = |R|^{|S-X_1|} = 3^{27-4} = 3^{23}$$

$$|E^{\bar{f}_1(X_1)}(\cdot,2)| = |E^{\bar{f}_1(X_1)}(\subseteq \{\alpha_1,\alpha_2\}) \cup E^{\bar{f}_1(X_1)}(\subseteq \{\alpha_2,\alpha_3\})| =$$

$$|E^{\bar{f}_1(X_1)}(\subseteq \{\alpha_1,\alpha_2\})| + |E^{\bar{f}_1(X_1)}(\subseteq \{\alpha_2,\alpha_3\})| - |E^{\bar{f}_1(X_1)}(\subseteq \{\alpha_2\})| =$$

$3^{9-3} + 3^{9-3} - 3 = 3(2\cdot 3^5-1)$. Thus $\mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub_1) = \dfrac{3}{3^{23}} = \dfrac{1}{3^{22}}$

and $\mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub_2) = \dfrac{3}{3(2\cdot 3^5-1)} = \dfrac{1}{2\cdot 3^5-1}$. Clearly

$\mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub_2) < \mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub_1)$.

The above illustrates Proposition 4.2.2.

b) We shall now illustrate Theorem 4.2.1.

Let $ub = 2$ and let X_1,X_2 be as above. Further, let $\bar{f}_2$ be an extension of $\bar{f}_1$ to X_2 given by Figure 4.2.1.

Then $L(\bar{f}_2) = \{\alpha_2\}$. We compute $\mathrm{CONF}_{(S,R)}^{\bar{f}_2(X_2)}(\{\alpha_2\},2)$.

$$|E^{\bar{f}_2(X_2)}(\{\alpha_2\})| = |R|^{|P_{\alpha_2}(S)| - |P_{\alpha_2}(X_2)|} = 3^{3-2} = 3^1 = 3$$

$$|E^{\bar{f}_2(X_2)}(\cdot,2)| = |E^{\bar{f}_2(X_2)}(\subseteq \{\alpha_1,\alpha_2\})| + |E^{\bar{f}_2(X_2)}(\subseteq \{\alpha_2,\alpha_3\})| -$$

$$|E^{\bar{f}_2(X_2)}(\{\alpha_2\})| = 3^{9-4} + 3^{9-3} - 3 = 3(3^4+3^5-1).$$

x	$\overline{f}_1(x)$
(0,0,0)	0
(1,0,0)	0
(1,1,0)	1
(1,1,1)	1

$\overline{f}_1: X_1 \to R$

x	$\overline{f}_2(x)$
(0,0,0)	0
(1,0,0)	0
(1,1,0)	1
(1,1,1)	1
(2,1,1)	1

$\overline{f}_2: X_2 \to R$

x	$\overline{g}_1(x)$
(0,0,0)	0
(1,0,0)	1
(1,1,0)	2
(1,1,1)	2

$\overline{g}_1: X_1 \to R$

Figure 4.2.1: Function Tables of Example 4.2.1.

Thus $CONF_{(S,R)}^{\bar{f}_2(X_2)}(\{\alpha_2\},2) = \dfrac{3}{3(3^4+3^5-1)} = \dfrac{1}{3^4+3^5-1}$, while

$CONF_{(S,R)}^{\bar{f}_1(X_1)}(\{\alpha_2\},2) = \dfrac{1}{2\cdot 3^5-1}$. Clearly, $CONF_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub) <$

$CONF_{(S,R)}^{\bar{f}_2(X_2)}(L(\bar{f}_2),ub)$ holds.

c) We will now illustrate Proposition 4.2.3. We compute $CONF_{(S,R)}(\{\alpha_2\},2)$, an a priori confidence that actual function has location $\{\alpha_2\}$, given that the cardinality of the location of all possible functions is at most 2.

$$CONF_{(S,R)}(\{\alpha_2\},2) = \frac{|F_{(S,R)}(\{\alpha_2\})|}{|F_{(S,R)}(\cdot,2)|} \cdot |F_{(S,R)}(\{\alpha_2\})| =$$

$$|F_{(S,R)}(\subseteq\{\alpha_2\})| - |F_{(S,R)}(\phi)| = |R|^{|P_{\alpha_2}(S)|} - 1 = 3^3 - 1$$

$$|F_{(S,R)}(\cdot,2)| = |F_{(S,R)}(\subseteq\{\alpha_1,\alpha_2\})| + (|F_{(S,R)}(\subseteq\{\alpha_1,\alpha_3\})| -$$

$$|F_{(S,R)}(\subseteq\{\alpha_1\})|) + (|F_{(S,R)}(\subseteq\{\alpha_2,\alpha_3\})| - |F_{(S,R)}(\subseteq\{\alpha_2\})| -$$

$$|F_{(S,R)}(\subseteq\{\alpha_3\})| + |F_{(S,R)}(\subseteq\phi)|) =$$

$$3^9 + (3^9-3^3) + (3^9-3^3-3^3+1) = 3\cdot 3^9 - 3\cdot 3^3 + 1$$

So $CONF_{(S,R)}(\{\alpha_2\},2) = \dfrac{3^3-1}{3\cdot 3^9-3\cdot 3^3+1} < \dfrac{3^3}{3^{10}-3^4+1} = \dfrac{1}{3^7-3+3^{-3}} <$

$\dfrac{1}{2\cdot 3^5-1} = CONF_{(S,R)}^{\bar{f}_1(X_1)}(\{\alpha_2\},2)$.

d) Finally we will illustrate Proposition 4.2.4. Consider a function $\bar{g}_1$ from X_1 to R, as given in Figure 4.2.1.

$L(\bar{g}_1) = \{\alpha_1,\alpha_2\} \supset \{\alpha_2\} = L(\bar{f}_1)$.

With ub = 2 as was computed in part a) of this example

$$\mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1),ub) = \frac{1}{2\cdot 3^5-1}$$

$$\mathrm{CONF}_{(S,R)}^{\bar{g}_1(X_1)}(\{\alpha_1,\alpha_2\},2) = \frac{|E^{\bar{g}_1(X_1)}(\{\alpha_1,\alpha_2\})|}{|\bigcup_{\substack{\tilde{L}\supseteq\{\alpha_1,\alpha_2\}\\ |\tilde{L}|=2}} E^{\bar{g}_1(X_1)}(\subseteq\tilde{L})|} = \frac{|E^{\bar{g}_1(X_1)}(\{\alpha_1,\alpha_2\})|}{|E^{\bar{g}_1(X_1)}(\{\alpha_1,\alpha_2\})|} = 1.$$

So clearly $\mathrm{CONF}_{(S,R)}^{\bar{f}_1(X_1)}(L(\bar{f}_1)ub) < \mathrm{CONF}_{(S,R)}^{\bar{g}_1(X_1)}(L(\bar{g}_1),ub)$ holds. □

We will now discuss the notion of average confidence for a subset X of S. As will become clear later, this is essentially motivated by the desire of a modeller to choose a domain subset to experiment (or generate the data) on, based on expected confidence on this subset.

Relative to an upper bound ub, we define the average confidence on an irredundant subset X of irredundant S (denoted by $\mathrm{ACONF}_{(S,R)}^{X}(ub)$) in the following manner.

Given ub, let Z_{ub} be a random variable, $Z_{ub}:F_{(S,R)} \to \mathbb{R}$, defined by $Z_{ub}(f) = \mathrm{CONF}_{(S,R)}^{f|X}(L(f|X),ub)$.

Then

$$\mathrm{ACONF}_{(S,R)}^{X}(ub) = E(Z_{ub}|ub) = \sum_{f\in F} P(f|ub)\cdot Z_{ub}(f).$$

When ub = $|D|$, we will denote $\mathrm{ACONF}_{(S,R)}^{X}(ub)$ by $\mathrm{ACONF}_{(S,R)}^{X}$.

<u>Proposition 4.2.5</u>

Let $X \subseteq S$, where X and S are irredundant. Let ub be an integer $0 \le ub \le |D|$. Then

(1) $$\mathrm{ACONF}_{(S,R)}^{X}(ub) = \left(\sum_{\bar{f}\in F_{(X,R)}(\cdot,ub)} |E^{\bar{f}}(L(\bar{f}))|\right) \Big/ \left(\sum_{\bar{f}\in F_{(X,R)}(\cdot,ub)} |E^{\bar{f}}(\cdot,ub)|\right)$$

Proof

2) $\sum_{f\varepsilon F} P(f|ub)\cdot CONF^{f|X}(L(f|X),ub) =$

$$\frac{1}{|F_{(S,R)}(\cdot,ub)|}\cdot\sum_{f\varepsilon F_{(S,R)}(\cdot,ub)} CONF^{f|X}(L(f|X),ub).$$

We note that every $f\varepsilon F_{(S,R)}(\cdot,ub)$ is an extension of some $\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)$. Actually, $F_{(S,R)}(\cdot,ub) = \bigcup_{\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)} E^{\bar{f}}(\cdot,ub)$.

Thus $|F_{(S,R)}(\cdot,ub)| = \sum_{\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)} |E^{\bar{f}}(\cdot,ub)|$.

We also note that for every $\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)$ and any $f,g\varepsilon E^{\bar{f}}(\cdot,ub)$, $CONF^{f|X}(L(f|X),ub) = CONF^{g|X}(L(g|X),ub)$. So

$$\sum_{f\varepsilon F_{(S,R)}(\cdot,ub)} CONF^{f|X}(L(f|X),ub) = \sum_{\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)} CONF^{\bar{f}}(L(\bar{f}),ub)\cdot|E^{\bar{f}}(\cdot,ub)| =$$

$$\sum_{\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)} |E^{\bar{f}}(L(\bar{f}))|.$$

Equality (1) clearly follows. □

We next show that average confidence is nondecreasing with an increasing size of the subset of S.

Proposition 4.2.6

Let X_1,X_2 be irredundant subsets of irredundant S, where $X_1 \subseteq X_2$ holds. Then for any ub

$ACONF^{X_1}_{(S,R)}(ub) \leq ACONF^{X_2}_{(S,R)}(ub)$ holds.

Proof

The above follows immediately since for every $f\varepsilon F_{(S,R)}(\cdot,ub)$,

$CONF^{f|X_1}(L(f|X_1),ub) \leq CONF^{f|X_2}(L(f|X_2),ub)$ by Theorem 4.2.1. □

Next, we will give some lower bounds for average confidence. Before we do that however we prove the following.

Lemma 4.2.2

Let S be Cartesian, $S = \bigtimes_{\alpha\in D} S_\alpha$. Then for any $L_1, L_2 \subseteq D$ with $L_1 \subseteq L_2$,

$$|F_{(S,R)}(L_1)| \leq |F_{(S,R)}(L_2)|.$$

Proof

Let $|L_1| = k$. If $k = n$, then $L_2 = L_1$ and we are done. So assume $k < n$. Also we assume $L_1 \subsetneq L_2$.

It follows from Theorem 3.4.2, that there is a 1-1 correspondence h between $F_{(S,R)}(L_1)$ and $F_{(Y_k^y,R)}(L_1)$, where $h : F_{(S,R)}(L_1) \to F_{(Y_k^y,R)}(L_1)$ is defined by $h(f) = f|Y_k^y$.

Since $|L_2| > |L_1| = k$, $COMPL_S^{L_2}(Y_k^y) \subsetneq S$. It follows from Remark 3) after Corollary 3.3.3, that for $\forall \bar{f}\in F_{(Y_k^y,R)}(L_1)$ there exists an extension of $\bar{f}$ to S with location L_2. This implies that $|F_{(S,R)}(L_2)| \geq$ $|\bigcup_{\bar{f}\in F_{(Y_k^y,R)}(L_1)} E^{\bar{f}}_{(S,R)}(L_2)| \geq |F_{(Y_k^y,R)}(L_1)| = |F_{(S,R)}(L_1)|$, which was to be proved. □

Proposition 4.2.7

Let X be an irredundant subset of $S = \bigtimes_{\alpha\in D} S_\alpha$, where $|D| = n$.

Then

a) $ACONF^X_{(S,R)} \geq \frac{1}{2^n}$

b) for $n \geq 2$ ub and $|S_\alpha| = m$, $\forall\, \alpha\varepsilon D$

$$ACONF^X_{(S,R)}(ub) \geq \frac{\sum_{\substack{L \\ |L|=ub}} |F_{(X,R)}(L)| \cdot |E^{\bar{f}_L}_{(S,R)}(L)|}{(ub+1)\binom{n}{ub} |F_{(S,R)}(L_{ub})|},$$

where $\bar{f}_L$ is any function in $F_{(X,R)}(L)$ and L_{ub} is an arbitrary subset of D with $|L_{ub}| = ub$.

Proof

a) $F_{(S,R)} = \bigcup_{\bar{f}\varepsilon F_{(X,R)}} E^{\bar{f}}_{(S,R)}$ implies $|F_{(S,R)}| = |E^{\bar{f}}_{(S,R)}| \cdot |F_{(X,R)}|$, since for any $\bar{f},\bar{g}\varepsilon F_{(X,R)}$, $|E^{\bar{f}}_{(S,R)}| = |E^{\bar{g}}_{(S,R)}|$.

Further $F_{(X,R)} = \bigcup_{L\varepsilon 2^D} F_{(X,R)}(L)$. It follows then from Lemma 4.2.2, that $|F_{(X,R)}| \leq 2^n \cdot |F_{(X,R)}(D)|$ and so that $|F_{(S,R)}| \leq |E^{\bar{f}}_{(S,R)}| \cdot |F_{(X,R)}(D)| \cdot 2^n$.

Now $ACONF^X_{(S,R)} = \frac{1}{|F_{(S,R)}|} \sum_{f\varepsilon F_{(S,R)}} CONF^{f|X}(L(f|X)) =$

$$\frac{1}{|F_{(S,R)}|} \sum_{\bar{f}\varepsilon F_{(X,R)}} CONF^{\bar{f}}(L(\bar{f}))\cdot|E^{\bar{f}}| \geq \frac{1}{|F_{(S,R)}|} \sum_{\bar{f}\varepsilon F_{(X,R)}(D)} 1\cdot|E^{\bar{f}}| =$$

$$\frac{|E^{\bar{f}}| \cdot |F_{(X,R)}(D)|}{|F_{(S,R)}|}.$$

$$\text{So } ACONF^X_{(S,R)} \geq \frac{|E^{\bar{f}}| \cdot |F_{(X,R)}(D)|}{|E^{\bar{f}}| \cdot |F_{(X,R)}(D)| \cdot 2^n} = \frac{1}{2^n}$$

b) Since $|S_\alpha| = m$ for $\forall\, \alpha\varepsilon D$ and S is Cartesian, for any $L_1, L_2 \subseteq D$ with $|L_1| = |L_2|$, $|F_{(S,R)}(L_1)| = |F_{(S,R)}(L_2)|$.

For $n \geq 2ub$ $\binom{n}{i} \leq \binom{n}{ub}$ holds for every $i \leq ub$. It follows from just made remarks that

$$\mathrm{CONF}^{f|Y_2^y}_{(S,R)}(\{\alpha_1,\alpha_2\}) = \frac{1}{2^{8-7}} = \frac{1}{2} \quad \text{and}$$

$$\mathrm{CONF}^{f}_{(S,R)}(\{\alpha_1,\alpha_2\}) = 1.$$

For the confidence curve see Figure 4.2.2.

b) We will now compute $\mathrm{ACONF}^{Y_k^y}_{(S,R)}$ for each of the Y_k^y sets.

We recall that $\mathrm{ACONF}^{X}_{(S,R)} = \frac{1}{|F_{(S,R)}|} \sum_{f \in F} \mathrm{CONF}^{f|X}(L(f)) =$

$$\frac{1}{|F_{(S,R)}|} \sum_{L \subseteq D} |F_{(X,R)}(L)| \cdot |E^{\bar{f}_L}| \mathrm{CONF}^{\bar{f}_L(X)}(L) =$$

$$\frac{1}{|F_{(S,R)}|} \sum_{L \subseteq D} |F_{(X,R)}(L)| \cdot |E^{\bar{f}_L}(L)|,$$ where $\bar{f}_L$ is an arbitrary function from X to R with location L.

$$\mathrm{ACONF}^{Y_0^y}_{(S,R)} = \frac{1}{2^8}(|F_{(Y_0^y,R)}(\phi)| \cdot 1 = \frac{2}{2^8} = \frac{1}{2^7}$$

$$\mathrm{ACONF}^{Y_1^y}_{(S,R)} = \frac{1}{2^8}(|F_{(Y_1^y,R)}(\phi)| \cdot 1 + \binom{3}{1}|F_{(Y_1^y,R)}(\{\alpha_1\})| \cdot |E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| +$$

$$\binom{3}{2}|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\})| \cdot |E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| + |F_{(Y_1^y,R)}(D)| \cdot |E^{\bar{f}_D}|).$$

We compute

$$|F_{(Y_1^y,R)}(\{\alpha_1\})| = 2^2-2 = 2 \text{ and } |E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| = 1$$

$$|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\})| = 2^{|P_{\{\alpha_1,\alpha_2\}}(Y_1^y)|} - |F_{(Y_1^y,R)}(\phi)| - 2|F_{(Y_1^y,R)}(\{\alpha_1\})| =$$

$$2^3 - 2 - 2 \cdot 2 = 8 - 6 = 2 \text{ and } |E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| = 2^{4-3} = 2.$$

Finally,

$$|F_{(Y_1^y,R)}(D)| = 2^{|Y_1^y|} - |F_{(Y_1^y,R)}(\phi)| - 3|F_{(Y_1^y,R)}(\{\alpha_1\})| - 3|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\})| =$$

$$2^4 - 2 - 3 \cdot 2 - 3 \cdot 2 = 16 - 14 = 2 \text{ and } |E^{\bar{f}_D}| = 2^{8-4} = 16.$$

Thus

$$\mathrm{ACONF}^{Y_1^y}_{(S,R)} = \frac{1}{2^8}(2 + 3\cdot 2\cdot 1 + 3\cdot 2\cdot 2 + 2\cdot 16) = \frac{1}{2^8}\cdot 52 = \frac{26}{2^7}$$

Similarly we compute $\mathrm{ACONF}^{Y_2^y}_{(S,R)}$. To do so, we first compute all the necessary elements.

$$|F_{(Y_2^y,R)}(\{\alpha_1\})| = 2^2 - 2 = 2 \text{ and } |E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| = 1$$

$$|F_{(Y_2^y,R)}(\{\alpha_1,\alpha_2\})| = 2^4 - 2 - 2\cdot 2 = 16 - 6 = 10 \text{ and}$$

$$|E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| = 1$$

Finally,

$$|F_{(Y_2^y,R)}(D)| = 2^7 - 2 - 3\cdot 2 - 3\cdot 10 = 128 - 38 = 90 \text{ and } |E^{\bar{f}_D}| = 2. \text{ So}$$

$$\mathrm{ACONF}^{Y_2^y}_{(S,R)} = \frac{1}{2^8}(2\cdot 1 + 3\cdot 2\cdot 1 + 3\cdot 10\cdot 1 + 90\cdot 2) = \frac{218}{2^8} = \frac{109}{2^7}.$$

The values of average confidence are tabulated in Figure 4.2.2.

4.3 Prediction Range and Correctness

On the basis of knowledge of the function on a proper subset X of S, we would like to make further predictions of the function.

In this section we will discuss the manner in which such predictions are made, the range of predictions and their correctness.

Suppose then that a function $\bar{f}$ from X to R is given, where X is a subset of S. For every location $\bar{L}$ of $\bar{f}$, we will predict $\bar{f}$ to the set $\mathrm{COMPL}^{\bar{L}}_S(X)$. We shall denote this set in the prediction-making context by $\mathrm{EXPL}^{\bar{f}(X)}_{(S,R)}(\bar{L})$ and call it the explanatory range of $\bar{f}$ relative to $\bar{L}$. We will denote the set $(\mathrm{EXPL}^{\bar{f}(X)}_{(S,R)}(\bar{L})-X)$ by $\mathrm{PRED}^{\bar{f}(X)}_{(S,R)}(\bar{L})$ and call it the predictive range of $\bar{f}$ relative to $\bar{L}$.

$$\mathrm{CONF}^{f|Y_2^y}_{(S,R)}(\{\alpha_1,\alpha_2\}) = \frac{1}{2^{8-7}} = \frac{1}{2} \quad \text{and}$$

$$\mathrm{CONF}^{f}_{(S,R)}(\{\alpha_1,\alpha_2\}) = 1.$$

For the confidence curve see Figure 4.2.2.

b) We will now compute $\mathrm{ACONF}^{Y_k^y}_{(S,R)}$ for each of the Y_k^y sets.

We recall that $\mathrm{ACONF}^{X}_{(S,R)} = \frac{1}{|F_{(S,R)}|} \sum_{f\varepsilon F} \mathrm{CONF}^{f|X}(L(f)) =$

$$\frac{1}{|F_{(S,R)}|} \sum_{L\subseteq D} |F_{(X,R)}(L)|\cdot|E^{\bar{f}_L}|\,\mathrm{CONF}^{\bar{f}_L(X)}(L) =$$

$\frac{1}{|F_{(S,R)}|} \sum_{L\subseteq D} |F_{(X,R)}(L)|\cdot|E^{\bar{f}_L}(L)|$, where $\bar{f}_L$ is an arbitrary function from X to R with location L.

$$\mathrm{ACONF}^{Y_0^y}_{(S,R)} = \frac{1}{2^8}\,(|F_{(Y_0^y,R)}(\phi)|\cdot 1 = \frac{2}{2^8} = \frac{1}{2^7}$$

$$\mathrm{ACONF}^{Y_1^y}_{(S,R)} = \frac{1}{2^8}\,(|F_{(Y_1^y,R)}(\phi)|\cdot 1 + \binom{3}{1}|F_{(Y_1^y,R)}(\{\alpha_1\})|\cdot|E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| +$$

$$\binom{3}{2}|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\})|\cdot|E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| + |F_{(Y_1^y,R)}(D)|\cdot|E^{\bar{f}_D}|).$$

We compute

$$|F_{(Y_1^y,R)}(\{\alpha_1\})| = 2^2-2 = 2 \text{ and } |E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| = 1$$

$$|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\})| = 2^{|P_{\{\alpha_1,\alpha_2\}}(Y_1^y)|} - |F_{(Y_1^y,R)}(\phi)|-2|F_{(Y_1^y,R)}(\{\alpha_1\})| =$$

$$2^3 - 2 - 2\cdot 2 = 8 - 6 = 2 \text{ and } |E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| = 2^{4-3} = 2.$$

Finally,

$$|F_{(Y_1^y,R)}(D)| = 2^{|Y_1^y|} - |F_{(Y_1^y,R)}(\phi)|-3|F_{(Y_1^y,R)}(\{\alpha_1\})|-3|F_{(Y_1^y,R)}(\{\alpha_1,\alpha_2\}) =$$

$$2^4 - 2 - 3\cdot 2 - 3\cdot 2 = 16 - 14 = 2 \text{ and } |E^{\bar{f}_D}| = 2^{8-4} = 16.$$

Thus

$$ACONF^{Y_1^y}_{(S,R)} = \frac{1}{2^8}(2 + 3\cdot2\cdot1 + 3\cdot2\cdot2 + 2\cdot16) = \frac{1}{2^8}\cdot 52 = \frac{26}{2^7}$$

Similarly we compute $ACONF^{Y_2^y}_{(S,R)}$. To do so, we first compute all the necessary elements.

$$|F_{(Y_2^y,R)}(\{\alpha_1\})| = 2^2 - 2 = 2 \text{ and } |E^{\bar{f}_{\alpha_1}}(\{\alpha_1\})| = 1$$

$$|F_{(Y_2^y,R)}(\{\alpha_1,\alpha_2\})| = 2^4 - 2 - 2\cdot2 = 16 - 6 = 10 \text{ and}$$

$$|E^{\bar{f}_{\{\alpha_1,\alpha_2\}}}(\{\alpha_1,\alpha_2\})| = 1$$

Finally,

$|F_{(Y_2^y,R)}(D)| = 2^7 - 2 - 3\cdot2 - 3\cdot10 = 128 - 38 = 90$ and $|E^{\bar{f}_D}| = 2$. So

$$ACONF^{Y_2^y}_{(S,R)} = \frac{1}{2^8}(2\cdot1 + 3\cdot2\cdot1 + 3\cdot10\cdot1 + 90\cdot2) = \frac{218}{2^8} = \frac{109}{2^7}.$$

The values of average confidence are tabulated in Figure 4.2.2.

4.3 Prediction Range and Correctness

On the basis of knowledge of the function on a proper subset X of S, we would like to make further predictions of the function.

In this section we will discuss the manner in which such predictions are made, the range of predictions and their correctness.

Suppose then that a function $\bar{f}$ from X to R is given, where X is a subset of S. For every location $\bar{L}$ of $\bar{f}$, we will predict $\bar{f}$ to the set $COMPL_S^{\bar{L}}(X)$. We shall denote this set in the prediction-making context by $EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})$ and call it the explanatory range of $\bar{f}$ relative to $\bar{L}$. We will denote the set $(EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})-X)$ by $PRED^{\bar{f}(X)}_{(S,R)}(\bar{L})$ and call it the predictive range of $\bar{f}$ relative to $\bar{L}$.

s	f(s)
(0,0,0)	0
(1,0,0)	0
(0,1,0)	1
(1,1,0)	1
(0,0,1)	0
(1,0,1)	0
(0,1,1)	1
(1,1,1)	1

X	$ACONF^{X}_{(S,R)}$
Y^y_0	$\frac{1}{2^7}$
Y^y_1	$\frac{26}{2^7}$
Y^y_2	$\frac{109}{2^7}$
S	1

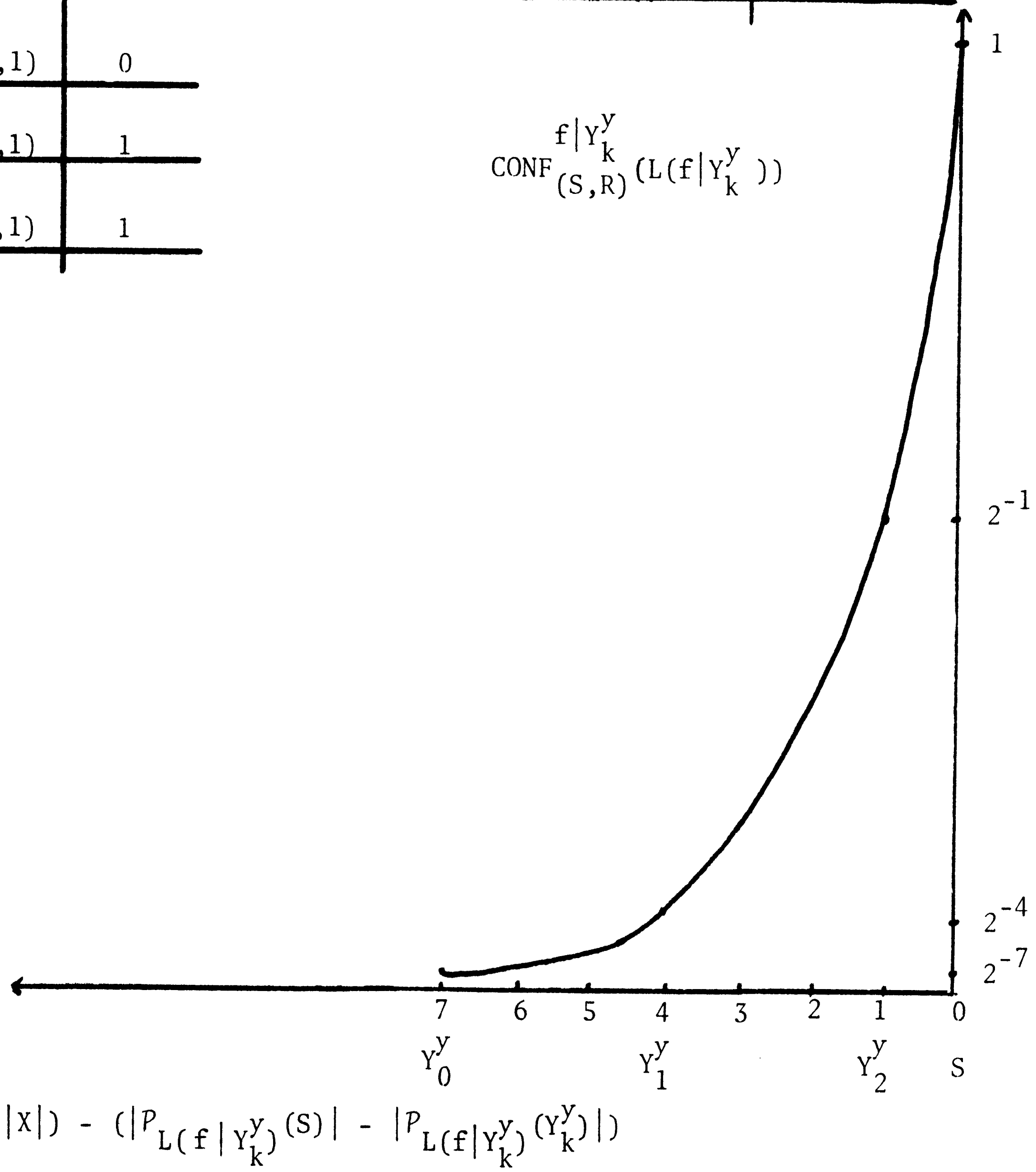


Figure 4.2.2: Tables and **Functions** of Example 4.2.2.

Let $\bar{f}$ and $\bar{L}$ be as described. Let $\hat{f}_{\bar{L}}$ denote the predicted function on $EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})$. Then we define $\hat{f}_{\bar{L}}$ to be the unique extension of $\bar{f}$ to $COMPL^{\bar{L}}_{S}(X)$, namely for every $y \varepsilon COMPL^{\bar{L}}_{S}(X)$, $\hat{f}_{\bar{L}}(y) = \bar{f}(x)$, where $x \varepsilon X$ is such that $x \Pi_{\bar{L}} y$. (See Lemma 3.3.1 and Corollary 3.3.3.) Clearly $\hat{f}_{\bar{L}}|X = \bar{f}$.

Remark

1) We note that if $COMPL^{\bar{L}}_{S}(X) = X$, $PRED^{\bar{f}(X)}_{(S,R)}(\bar{L}) = \phi$ and so we make no new predictions. This happens for example when $\bar{L} = D$.

2) For any two functions $\bar{f}$ and $\bar{g}$ from X to R and any $\bar{L}_1, \bar{L}_2$ such that $\bar{L}_1 \varepsilon L(\bar{f})$, $\bar{L}_2 \varepsilon L(\bar{g})$, where $\bar{L}_1 \subseteq \bar{L}_2$ holds, $PRED^{\bar{f}(X)}_{(S,R)}(\bar{L}_1) \supseteq PRED^{\bar{g}(X)}_{(S,R)}(\bar{L}_2)$. This follows directly from the fact that $COMPL^{\bar{L}_1}_{S}(X) \supseteq COMPL^{\bar{L}_2}_{S}(X)$.

3) We note that $COMPL^{\bar{L}}_{S}(X)$ is the largest subset Z of S with the property, that for all $f \varepsilon E^{\bar{f}}_{(S,R)}(\bar{L})$, $f|Z$ is unique.

This follows directly from Proposition 3.3.3 and Theorem 3.3.1, provided $|R| \geq 2$. □

In the sequel we will denote by $PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L})$ the probability that all predictions were made correctly (when $\bar{L}$ was "chosen" to make them) and call it the predictive confidence.

More precisely

$PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L}) = P((\text{actual function } f|EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})) = \hat{f}_{\bar{L}}|f \varepsilon E^{\bar{f}})$

Similarly, when an upper bound ub is given $PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub) = P((\text{actual function } f|EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})) = \hat{f}_{\bar{L}} \mid \text{actual } f \varepsilon E^{\bar{f}}_{(S,R)}(\cdot,ub))$.

In the next proposition we find an expression for $PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub)$.

Proposition 4.3.1

Let X be a subset of S and let $\bar{f}:X \to R$, where $\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)$. Then for any $\bar{L}\varepsilon L(\bar{f})$ and any integer ub, $0 \leq ub \leq n$,

a) $$PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub) = \frac{|E^{\hat{f}_{\bar{L}}(Y)}_{(S,R)}(\cdot,ub)|}{|E^{\bar{f}(X)}_{(S,R)}(\cdot,ub)|}$$

where $Y = EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})$.

b) If ub = n

$$PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L}) = \frac{1}{|R|^{|PRED^{\bar{f}(X)}_{(S,R)}(\bar{L})|}}$$

Proof

a) Since $\bar{f}\varepsilon F_{(X,R)}(\cdot,ub)$, $E^{\bar{f}(X)}(\cdot,ub) \neq \phi$ and we are not dividing by 0. (See Corollary 3.3.2.)

The expression for $PCONF^{\bar{f}(X)}_{(S,R)}(\bar{L},ub)$ follows directly from its definition, when the formula for conditional probability is employed.

b) It follows from part a) that

$$PCONF^{\bar{f}(X)}(\bar{L}) = \frac{|E^{\hat{f}_{\bar{L}}(Y)}|}{|E^{\bar{f}(X)}|} = \frac{|R|^{|S|-|EXPL^{\bar{f}(X)}(\bar{L})|}}{|R|^{|S|-|X|}} = \frac{1}{|R|^{|EXPL^{\bar{f}(X)}(\bar{L})|-|X|}} =$$

$$\frac{1}{|R|^{|EXPL^{\bar{f}(X)}(\bar{L})-X|}} = \frac{1}{|R|^{|PRED^{\bar{f}(X)}(\bar{L})|}}$$, which was to be proved. □

It can be easily seen from Proposition 4.3.1 that the relation between the number of predictions made and the confidence in the made predictions is reversely proportional. That is, the more we predict the smaller the certainty, that we are correct in our predictions.

We note that if actual function f from S to R has a location $\bar{L}$, then $\hat{f}_{\bar{L}}$ as described above is the correct prediction of f to $EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})$.

When actual f has a location L, where $L \supsetneq \bar{L}$ holds, however, we are assured of the correctness of our predictions to the set $COMPL^{L}_{S}(X)$, rather than $EXPL^{\bar{f}(X)}_{(S,R)}(\bar{L})$. That at least those among our predictions are correct, follows directly from Lemma 3.3.1.

We will call the set $COMPL^{L}_{S}(X)$ in this context the <u>validity range</u> of $\bar{f}$ with respect to L and denote it by $VAL^{\bar{f}(X)}_{(S,R)}(L)$.

We will show that for a Cartesian S, $VAL^{\bar{f}(X)}_{(S,R)}(L)$ is the largest subset Z of S with the property, that for every $f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L)$, $f|Z$ is equal to the prediction by $\bar{f}$.

<u>Proposition 4.3.2</u>

Let X be an irredundant subset of $S = \bigtimes_{\alpha \varepsilon D} S_{\alpha}$, where $|S_{\alpha}| \geq 2, \forall \alpha \varepsilon D$. Let $\bar{f}: X \to R$, where $|R| \geq 2$ and let $\bar{L} = L(\bar{f})$. Then for any $L \supseteq \bar{L}$, $COMPL^{L}_{S}(X)$ is the largest subset Z of $COMPL^{\bar{L}}_{S}(X)$ with the property, that for every $f \varepsilon E^{\bar{f}(X)}_{(S,R)}(L)$, $f|Z = \hat{f}_{\bar{L}}|Z$, where $\hat{f}_{\bar{L}}$ is the prediction of $\bar{f}$ to $COMPL^{L}_{S}(X)$.

<u>Proof</u>

If $L = \bar{L}$ or $COMPL^{\bar{L}}_{S}(X) = COMPL^{L}_{S}(X)$, the proposition is clearly true. So we assume that $COMPL^{\bar{L}}_{S}(X) \supsetneq COMPL^{L}_{S}(X)$ holds.

It follows directly from Lemma 3.3.1 that $Z \supseteq COMPL^{L}_{S}(X)$. We want to show that $Z = COMPL^{L}_{S}(X)$. Suppose that $Z \supsetneq COMPL^{L}_{S}(X)$. We show that then $\exists$ a function $g \varepsilon E^{\bar{f}(X)}(L)$, such that $g|Z \neq \hat{f}_{\bar{L}}|Z$, thus contradicting the properties of Z.

Let $p\varepsilon Z\text{-}COMPL_S^L(X)$. (Clearly $Z \subseteq COMPL_S^{\bar{L}}(X)$, since this is the set we make predictions to.)

We shall distinguish two cases: $\bar{L} = \phi$ and $\bar{L} \neq \phi$.

a) $\bar{L} = \phi$. So $COMPL_S^{\bar{L}}(X) = S$.

Then $\hat{f}_{\bar{L}}$ is constant on $COMPL_S^{\bar{L}}(X)$, say $\hat{f}_{\bar{L}}(x) = c$ for $\forall\, x\varepsilon COMPL_S^{\bar{L}}(X)$. We define g on S in the following way. For every $x\varepsilon[p]_{\Pi_L}$, $g(x) = d \neq c$. (Remember that $|R| \geq 2$.) For all other x in S, $g(x) = \hat{f}_{\bar{L}}(x) = c$. Clearly $g|Z \neq \hat{f}_L|Z$, since $g(p) = d \neq \hat{f}_L(p) = c$. (Since $p\varepsilon S\text{-}COMPL_S^L(X)$, $[p]_{\Pi_L} \subseteq S\text{-}COMPL_S^L(X)$ and so $g|COMPL_S^L(X) = \hat{f}_{\bar{L}}|COMPL_S^L(X)$.) We need to show that $g\varepsilon E^{\bar{f}(X)}(L)$. Clearly $\Pi_L^S \leq \Pi_g$. Since S is Cartesian and $|S_\alpha| \geq 2, \forall\, \alpha\varepsilon D$, for every $\alpha\varepsilon L$ $\exists$ an $x_\alpha\varepsilon S$ s.t. $x_\alpha \Pi_{L-\alpha}^S p$ but $x_\alpha \not\Pi_L^S p$.

Thus by our definition of g, for every $\alpha\varepsilon L, \exists\, x_\alpha$ s.t. $x_\alpha \Pi_{L-\alpha}^S p$ but $g(x_\alpha) \neq g(p)$. This implies that for every $\alpha\varepsilon L$, $\Pi_{L-\alpha}^S \not\leq \Pi_g$ and thus L is the location of g, which was to be proved.

b) $\bar{L} \neq \phi$. So $|\Pi_{\bar{f}}| \geq 2$.

Let p be as above, i.e. $p\varepsilon Z\text{-}COMPL_S^L(X)$. Again we will construct a g from S to R, such that $g\varepsilon E^{\bar{f}(X)}(L)$, but $g|Z \neq \hat{f}_{\bar{L}}|Z$.

Actually it suffices to construct an $h{:}COMPL_S^{\bar{L}}(X) \to R$ such that $L(h) = L$ but $h|Z \neq \hat{f}_{\bar{L}}|Z$. Then an existence of g as above follows from Corollary 3.3.2.

Since $p\varepsilon COMPL_S^{\bar{L}}(X)$, $\exists\, x\varepsilon X$ s.t. $x\Pi_{\bar{L}}p$. Since $|\Pi_{\bar{f}}| \geq 2$ $\exists$ an $w\varepsilon X$ s.t. $\bar{f}(w) \neq \bar{f}(x)$. This clearly implies that $w \not\Pi_{\bar{L}} x$, since $\bar{L} = L(\bar{f})$. So $w \not\Pi_{\bar{L}} p$.

Now since S is Cartesian and $|S_\alpha| \geq 2$, for every $\alpha\varepsilon L\text{-}\bar{L}$ $\exists$ an $x_\alpha\varepsilon S$ s.t. $x_\alpha \Pi_{L-\alpha} p$ but $x_\alpha \not\Pi_L p$. We note that all such x_α's are related to each other and to p by $\Pi_{\bar{L}}$. With x as above $(x\Pi_{\bar{L}}p)$, all x_α's are also

$\Pi_{\bar{L}}$ related to x, and none of the x_α's is $\Pi_{\bar{L}}$ related to w. It follows then that all x_α's as above are in $COMPL_S^{\bar{L}}(X)$.

We define h to be the following function.

For $\forall\, y \in COMPL_S^{\bar{L}}(X) - [p]_{\Pi_L}$, $h(y) = \hat{f}_{\bar{L}}(y)$ and for $\forall\, y \in [p]_{\Pi_L}$, $h(y) = \bar{f}(w)$.

We need to show that $h \in E^{\bar{f}(X)}_{(COMPL_S^{\bar{L}}(X),R)}$, in other words that $h|X = \bar{f}$. But since $p \notin COMPL_S^L(X)$, $[p]_{\Pi_L} \cap COMPL_S^L(X) = \phi$, $\Rightarrow [p]_{\Pi_L} \cap X = \phi$. Thus for $\forall\, y \in X$, $h(y) = \hat{f}_{\bar{L}}(y) = \bar{f}(y)$, which shows that $h|X = \bar{f}$.

Finally, we have to show that $L = L(h)$. Clearly $\Pi_L \leq \Pi_h$. (We recall that $\bar{L} = L(\hat{f}_{\bar{L}})$ and so $\bar{L} = L(h|COMPL_S^{\bar{L}}(X)-[p]_{\Pi_L})$.

$L \supseteq \bar{L} \Rightarrow \Pi_L \leq \Pi_{h|COMPL_S^{\bar{L}}(X)-[p]_{\Pi_L}})$.

Also since h is an extension of $\bar{f}$, $L(h) \supseteq \bar{L}$. To show $L = L(h)$ then, it suffices to show that for every $\alpha \in L-\bar{L}$, $\exists\, x_\alpha \in COMPL_S^{\bar{L}}(X)$ s.t. $x_\alpha \Pi_{L-\alpha}\, p$ but $h(x_\alpha) \neq h(p)$. Now $h(p) = \bar{f}(w)$. With x_α's as before $h(x_\alpha) = \hat{f}_{\bar{L}}(x_\alpha) = \bar{f}(x) \neq \bar{f}(w)$, which completes the proof.

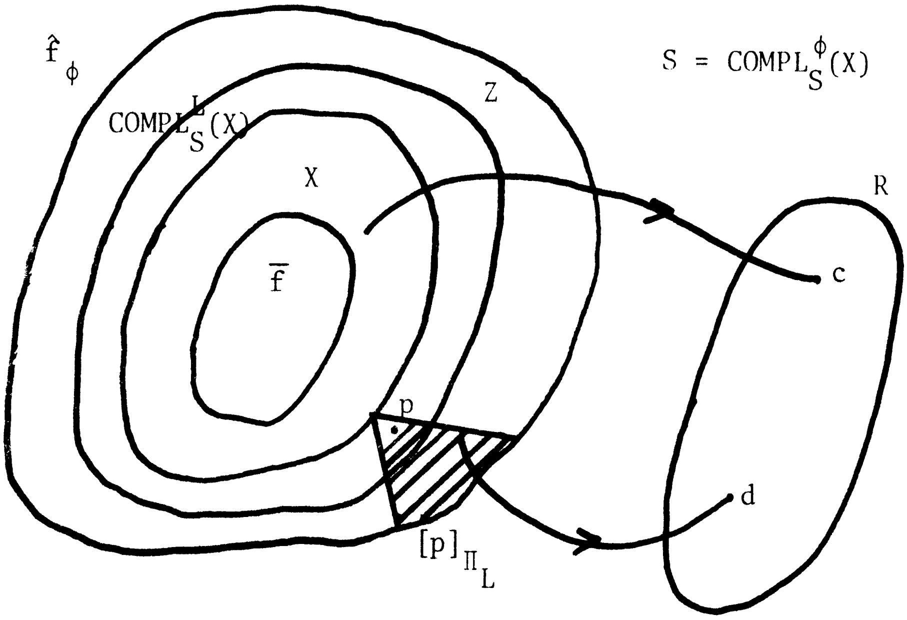
For the illustration of the above proof refer to Figure 4.3.1. □

In the following example we will illustrate that the number of predictions made is ambivalent with the size of X.

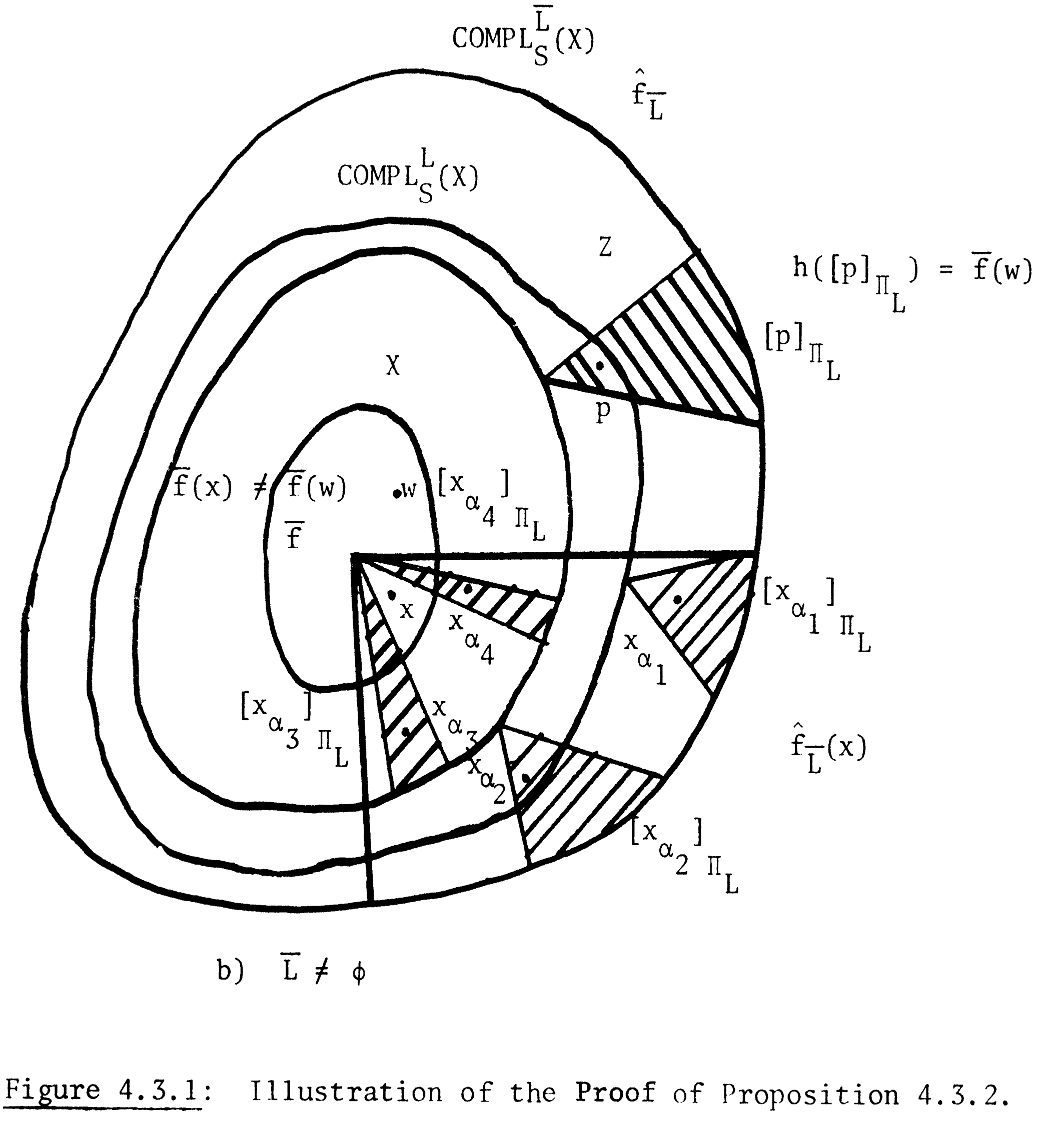
<u>Example 4.3.1</u>

a) Consider S Cartesian and $L_1 \subsetneq L_2 \subsetneq D$. Then $X^y_{L_1} \subsetneq X^y_{L_2}$. For any f from S, $L(f|X^y_{L_1}) \subseteq L_1$ and $L(f|X^y_{L_2}) \subseteq L_2$. Thus

$COMPL_S^{L(f|X^Y_{L_1})}(X^y_{L_1}) = COMPL_S^{L(f|X^y_{L_2})}(X^y_{L_2}) = S$. But this implies that



a) $\bar{L} = \phi$ $g([p]_{\Pi_L}) = d$, $g(S-[p]_{\Pi_L}) = c$



b) $\bar{L} \neq \phi$

Figure 4.3.1: Illustration of the Proof of Proposition 4.3.2.

$$|PRED^{f|X^y_{L_1}}(L(f|X^y_{L_1}))| \gtrless |PRED^{f|X^y_{L_2}}(L(f|X^y_{L_2}))|.$$

b) Consider $S = \{0,1,2\}^3$ and

$X_1 = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$, $X_2 = X_1 \cup \{(2,1,1)\}$.

Both X_1 and X_2 are irredundant.

Let f be a function from S to $R = \{0,1,2\}$ with $\Pi_f = \Pi^S_{\alpha_1}$. Let

$\bar{f}_1 = f|X_1$ and $\bar{f}_2 = f|X_2$. Then $L(\bar{f}_1) = L(\bar{f}_2) = \{\alpha_1\}$.

It can be easily verified that

$PRED^{\bar{f}_2}_{(S,R)}(\{\alpha_1\}) = PRED^{\bar{f}_1}_{(S,R)}(\{\alpha_1\}) \cup \{(2,0,0),(2,0,1),(2,0,2),(2,1,0),$
$(2,1,2),(2,2,0),(2,2,1),(2,2,2)\}$.

Thus

$$|PRED^{\bar{f}_2}(\{\alpha_1\})| \gtrless |PRED^{\bar{f}_1}(\{\alpha_1\})|.$$

The example then clearly illustrates the above assertion. □

CHAPTER V

APPLICATIONS TO DISCRETE TIME SYSTEMS

5.1 Introduction

In this chapter we will deal with functions from structured domains, whose codomains have been also structured. We will define a concept of structured model (partial model) of such a function (system).

We will be seeking reduced models only. Those models are minimal in a sense to be made precise later.

In general, there are many possible reduced structure assignments for a function. When a function domain is irredundant however, the uniqueness of such an assignment will result.

Similarly as in Chapter IV we will discuss here concepts of confidence, average confidence, predictive confidence, etc.

We will derive the expressions for the above, in case of a Cartesian codomain.

We will apply all the results to finite autonomous discrete time systems. In this case the domain and codomain sets are same and equal to the structured state space of the system. The functions involved will be the state transition functions.

For those systems we will discuss several strategies, which a modeller might follow during the experimental (modelling) process. The one chosen will depend on his objectives and a priori information available about the system being modelled.

5.2 Structured Functions

We shall first give the definition of a structured function in terms of locations of its components.

Definition 5.2.1 ([Z1])

Let S and S' be structured sets, where $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ and $S' \subseteq \bigtimes_{\beta \varepsilon D'} S'_\beta$. A function $f:S \to S'$ is structured by an indexed family of functions $\{f_\beta | f_\beta : P_{I_\beta}(S) \to S'_\beta,\ I_\beta \subseteq D, \beta \varepsilon D'\}$, if $f = \bigtimes_{\beta \varepsilon D'} f_\beta \cdot P_{I_\beta}$. □

We will refer to a family $\{I_\beta | \beta \varepsilon D'\}$ as above as a structure of f.

In general a function f can be structured in many ways leading to different structures of f. For example $\{I_\beta | \beta \varepsilon D'\}$, where $I_\beta = D$ for all $\beta \varepsilon D'$ is always a structure of f. We simply define for every $\beta \varepsilon D'$ $f_\beta : S \to S'_\beta$ to be the function $f_\beta(s) = P_\beta \cdot f(s)$.

A function f is structured in the sense of Definition 5.2.1 if for every $\beta \varepsilon D'$ and every $s \varepsilon S$ the diagram of Figure 5.2.1 commutes.

In structuring the function we will try to find those structures, which are minimal in the following sense. If $\{I_\beta | \beta \varepsilon D'\}$ is such a structure, then for any other structure $\{L_\beta | \beta \varepsilon D'\}$ of S, if $L_\beta \subseteq I_\beta$ then $L_\beta = I_\beta$, $\forall \beta \varepsilon D'$. We will call those structures reduced.

More formally,

Definition 5.2.2 ([Z1])

A function f with domain $S, S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$, is in reduced form if D is a location of f. □

Definition 5.2.3 ([Z1])

A structured function $f:S \to S'$, $f = \bigtimes_{\beta \varepsilon D'} f_\beta \cdot P_{I_\beta}$ is in reduced form if every f_β is in reduced form. □

We note that a structured function f is in reduced form if I_β is a location of f_β, where $\{I_\beta|\beta\varepsilon D'\}$ is a structure of f. We will then call $\{I_\beta|\beta\varepsilon D'\}$ a <u>reduced structure</u> of f.

When S is irredundant, every structured f has a unique reduced structure. This structure is given by $I_\beta = L(P_\beta \cdot f)$, for all β in D'.

In case S is not irredundant, but $\{I_\beta|\beta\varepsilon D'\}$ is reduced, $I_\beta \varepsilon L(P_\beta \cdot f)$, for every $\beta\varepsilon D'$.

We will illustrate the above concepts for a transition function of a discrete time system.

<u>Example 5.2.1</u>

Consider S = {(0,0,0,0),(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)} and δ on S to be an identity on S, i.e. $\delta(s) = s$ for all $s\varepsilon S$. S is clearly irredundant since it is a minimal independent set.

As can be easily verified the unique reduced structure of δ is given by $I_{\alpha_1} = \{\alpha_1\}$, $I_{\alpha_2} = \{\alpha_2\}$, $I_{\alpha_3} = \{\alpha_3\}$ and $I_{\alpha_4} = \{\alpha_4\}$.

We will demonstrate two nonreduced ways of structuring δ.

a) Consider δ_{α_i} as given in Figure 5.2.2 a).

Clearly $\delta = \bigtimes_i \delta_{\alpha_i} \cdot P_{I_{\alpha_i}}$, where

$I_{\alpha_1} = \{\alpha_1, \alpha_2\}$ and $I_{\alpha_i} = \{\alpha_i\}$, for i = 2,3,4. This structure is not reduced since $L(\delta_{\alpha_1}) = \{\alpha_1\} \subsetneq \{\alpha_1, \alpha_2\}$.

b) Consider δ_{α_i} as given in Figure 5.2.2 b).

Again it is easy to check that $\delta = \bigtimes_i \delta_{\alpha_i} \cdot P_{I_{\alpha_i}}$, where

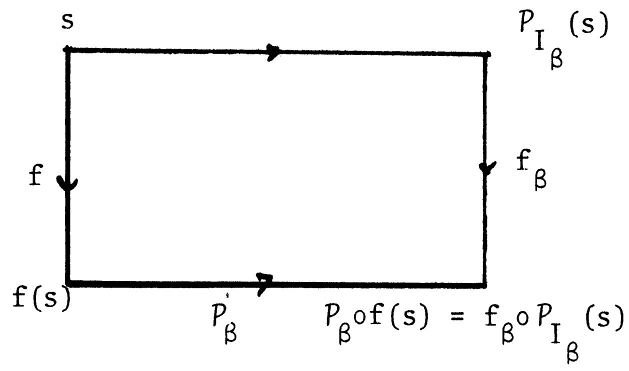


Figure 5.2.1: Diagram of a Structured Function.

x	$\delta_{\alpha_1}(x)$
(0,0)	0
(1,0)	1
(1,1)	1

$\delta_{\alpha_i}(P_{\alpha_i}(s)) = P_{\alpha_i}(s)$

for i = 2,3,4

a)

x	$\delta_{\alpha_4}(x)$
(0,0)	0
(1,0)	0
(1,1)	1

$\delta_{\alpha_i}(P_{\alpha_i}(s)) = P_{\alpha_i}(s)$

for i = 1,2,3

b)

$\langle S,\delta \rangle$

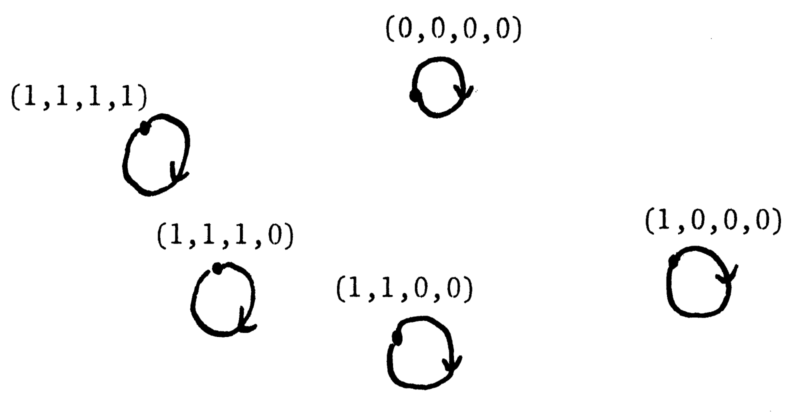


Figure 5.2.2: Illustration of Example 5.2.1.

$I_{\alpha_i} = \{\alpha_i\}$, for i = 1,2,3 and $I_{\alpha_4} = \{\alpha_1,\alpha_4\}$. This structure is not reduced since $L(\delta_{\alpha_4}) = \{\alpha_4\} \subsetneq \{\alpha_1,\alpha_4\}$.

5.3 Structured Partial Models

We consider here the family $F_{(S,S')}$ of all function from S to S', where S and S' are structured sets.

We will refer to $\langle S,S',f\rangle$ as a system, where $f \in F_{(S,S')}$. We shall now explain what is meant by a partial (structured) model of $\langle S,S',f\rangle$.

Definition 5.3.1

Given a system $\langle S,S',f\rangle$, $\langle S_1,S_1',\bar{f}\rangle$ is a partial model of $\langle S,S',f\rangle$ if

a) $S_1 \subseteq S$

b) $S_1' = S'$

c) $\bar{f}:S_1 \to S_1'$. □

We will refer to $\langle S_1,S_1',\bar{f}\rangle$ as a model of $\langle S,S',f\rangle$ just in case $S_1 = S$. We will seek partial models with $\bar{f} = f|S_1$.

Definition 5.3.2

$\langle S_1,S_1',\bar{f}\rangle$ is a structured partial model of $\langle S,S',f\rangle$ if $\langle S_1,S_1',\bar{f}\rangle$ is a partial model of $\langle S,S',f\rangle$ and if $\bar{f}$ is a structured function. □

We will then refer to a structure of $\bar{f}$ as a partial model structure (structure of a partial model).

We are going to seek reduced structured partial models, namely partial models of the form $\langle S_1,S',\bar{f}\rangle$, where $\bar{f}$ is in reduced form.

The notation we will use here is analogous to the one introduced in Chapters 3 and 4. Thus with S and S' as described before and $\bar{f}$ a function from some subset X of S to S'

$E^{\bar{f}(X)}_{(S,S')}$ denotes the set of all extensions f of $\bar{f}$ to S s.t. $f \varepsilon F_{(S,S')}$

$E^{\bar{f}(X)}_{(S,S')}(\{L_\beta\}_{\beta\varepsilon D'},\cdot)$ denotes the set of all extensions f of $\bar{f}$ to S, such that $L_\beta \varepsilon L(P_\beta \cdot f)$

$E^{\bar{f}(X)}_{(S,S')}(\subseteq \{L_\beta\},\cdot)$ denotes the set of all $f\varepsilon E^{\bar{f}(X)}_{(S,S')}$ s.t. for all $\beta\varepsilon D'$ $\exists$ an $\tilde{L}_\beta \subseteq L_\beta$ with $\tilde{L}_\beta \varepsilon L(P_\beta \cdot f)$.

$E^{\bar{f}(X)}_{(S,S')}(\cdot,\{ub_\beta\}_{\beta\varepsilon D'})$ denotes the set of all $f\varepsilon E^{\bar{f}(X)}_{(S,S')}$ s.t. for every β $\exists$ an $L_\beta \varepsilon L(P_\beta \cdot f)$ with $|L_\beta| \leq ub_\beta$.

Similarly, $F_{(X,S')}$, $F_{(X,S')}(\{L_\beta\}_{\beta\varepsilon D'},\cdot)$, $PRED^{\bar{f}(X)}_{(S,S')}(\{\bar{L}_\beta\}_{\beta\varepsilon D'},\cdot)$ etc. are defined. Again sometimes the notation will be shortened, when the intent is clear.

As in Chapter 4 we define a probability space $\langle F_{(S,S')}, 2^{F_{(S,S')}}, P\rangle$, where for every $A\varepsilon F_{(S,S')}$, $P(A) = \frac{|A|}{|F_{(S,S')}|}$.

The various concepts introduced previously can be readily extended to structured functions, relative to just introduced probability space. Thus we will define confidence in the following way.

<u>Definition 5.3.3</u>

Let $X \subseteq S$ and let $\bar{f}:X \to S'$. Then for any family $L = \{L_\beta\}_{\beta\varepsilon D'}$, where $L_\beta \subseteq D$, for all $\beta\varepsilon D'$, and any family of integers $UB = \{ub_\beta\}_{\beta\varepsilon D'}$, where $0 \leq ub_\beta \leq |D|$

$$CONF^{\bar{f}(X)}_{(S,S')}(L,UB) = P(\text{actual } f\varepsilon F_{(S,S')}(L,\cdot) \mid \text{actual } f\varepsilon E^{\bar{f}(X)}_{(S,S')}(\cdot,UB)). \quad \Box$$

When $ub_\beta = n = |D|$ holds for all β, we will denote $CONF^{\bar{f}(X)}_{(S,S')}(L,UB)$ by $CONF^{\bar{f}(X)}_{(S,S')}(L)$.

Proposition 5.3.1

Let $X \subseteq S$ and let $\bar{f}:X \to S'$. Then for any family $L = \{L_\beta\}_{\beta\epsilon D'}, L_\beta \subseteq D, \forall \beta\epsilon D'$, and any family $UB = \{ub_\beta\}_{\beta\epsilon D'}$ of integers, $0 \leq ub_\beta \leq |D|$,

$$CONF^{\bar{f}(X)}_{(S,S')}(L,UB) = \begin{cases} \dfrac{|E^{\bar{f}(X)}_{(S,S')}(L,UB)|}{|E^{\bar{f}(X)}_{(S,S')}(\cdot,UB)|} & \text{if } E^{\bar{f}(X)}_{(S,S')}(\cdot,UB) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

Proof

Almost identical to the proof of Proposition 4.2.1. □

Remark

1) We note that for an irredundant subset X of S and any function $\bar{f}:X \to S'$ such that $L_\beta = L(P_\beta \cdot \bar{f})$ and $|L_\beta| = ub_\beta$, $CONF^{\bar{f}(X)}_{(S,S')}(\{L_\beta\},\{ub_\beta\})$ is equal to either 0 or 1.

2) For arbitrary X and S if L is a structure of $\bar{f}$, $\bar{f}:X \to S'$ and UB is such that $|L_\beta| \leq ub_\beta$ for all $L_\beta \epsilon L$, then $E^{\bar{f}(X)}_{(S,S')}(L,UB) = E^{\bar{f}(X)}_{(S,S')}(L)$ holds. □

Roughly speaking, we showed in Chapter 4 that in case of non-structured codomain, confidence in structure of functional restrictions had an important property of being nondecreasing with an increasing domain size.

When we deal however with structured functions the above property is in general not necessarily preserved. As a matter of fact structure preserving extensions do not always exist in the structured case, as they do in the nonstructured one. Therefore as we go on experimenting and construct a sequence of partial models, our confidence in the structure of a "later" partial model may be smaller than in the structure of an earlier one. As we will prove later the mentioned property will hold however, when S' is Cartesian.

This is made more precise by the following example.

Example 5.3.1

Consider $S = \{0,1\}^3 - \{0,1,1\}$. S is irredundant but not Cartesian.

Let X_1 and X_2 be the following subsets of S:
$X_1 = \{(0,0,0),(1,0,0),(1,1,0)\}$ and $X_2 = X_1 \cup \{(1,1,1),(0,1,0),(1,0,1)\}$.
X_1 is irredundant, since $P_{\alpha_3}(X_1) = \{0\}$ is constant and $P_{\{\alpha_1,\alpha_2\}}(X_1)$ is irredundant. We will now prove irredundance of X_2. X_2 can be written as the following union:
$X_2 = \{(0,1,0),(1,0,1)\} \cup \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\} = Z_1 \cup Z_2$.
We first note that for every $\alpha \varepsilon D$ $\exists$ an $x_\alpha \varepsilon Z_2$ s.t. $x_\alpha \Pi_\alpha (0,1,0)$. Also for every $\alpha \varepsilon D$ $\exists$ an $x_\alpha \varepsilon Z_2$ s.t. $x_\alpha \Pi_\alpha (1,0,1)$. Since Z_2 is irredundant (as a minimal independent set), $\Pi_{D_1} \cup \Pi_{D_2} = I$ holds on $Z_1 \cup Z_2$ for all $D_1, D_2 \subseteq D$ with $D_1 \cap D_2 = \phi$.

To prove irredundance of X_2, we still have to show that for any $\alpha \varepsilon D$, any D_1, D_2 with $D_1 \cap D_2 = \{\alpha\}$, $\Pi^{X_2}_{\{\alpha\}} \leq \Pi^{X_2}_{D_1} \cup \Pi^{X_2}_{D_2}$. This follows easily by inspection, since Z_2 is irredundant. So X_2 is irredundant. We note that $S = X_2 \cup \{(0,0,1)\}$.

Let $S' = S$, $D' = D$ and let $\bar{f}_2$ be a function from X_2 to S given by the table of Figure 5.3.1. We will denote by $\bar{f}_1$ the restriction of $\bar{f}_2$ to X_1. Then $L(P_{\alpha_1} \cdot \bar{f}_1) = \{\alpha_1\}$, $L(P_{\alpha_2} \cdot \bar{f}_1) = \{\alpha_2\}$ and $L(P_{\alpha_3} \cdot \bar{f}_1) = \phi$. Thus $L_1 = \{\{\alpha_1\},\{\alpha_2\},\phi\}$ is the reduced structure of $\bar{f}_1$. One of the structure preserving extensions of $\bar{f}_1$ to S is given by g_1 of

X_2 X_1

x	$\bar{f}_2(x)$
(0,0,0)	(0,1,0)
(1,0,0)	(1,1,0)
(1,1,0)	(1,0,0)
(1,1,1)	(1,0,1)
(0,1,0)	(0,0,0)
(1,0,1)	(1,1,1)

a)

S	g_1
(0,0,0)	(0,1,0)
(1,0,0)	(1,1,0)
(1,1,0)	(1,0,0)
(1,1,1)	(1,0,0)
(0,1,0)	(0,0,0)
(1,0,1)	(1,1,0)
(0,0,1)	(0,1,0)

b)

Figure 5.3.1: Tables of Functions of Example 5.3.1.

Figure 5.3.1. It can be readily verified that the reduced structure of g_1 is indeed equal to that of $\bar{f}_1$ and $g_1 : S \to S$.

We find the reduced structure of $\bar{f}_2$ to be $L_2 = \{\{\alpha_1\},\{\alpha_2\},\{\alpha_3\}\}$. There is just one possibility of extending $\bar{f}_2$ to S, while preserving the structure: namely to map (0,0,1) into (0,1,1). But (0,1,1) is not in S and so there is no structure preserving extension of $\bar{f}_2$ to S. Thus $\mathrm{CONF}^{\bar{f}_1(X_1)}_{(S,S')}(L_1) > 0$ while $\mathrm{CONF}^{\bar{f}_2(X_2)}_{(S,S')}(L_2) = 0$. □

For the remaining part of this section we assume that S' Cartesian. This assumption is a very important one, for we show that in this case we may carry over our results from the nonstructured case. To do so we first prove the following: given any subset X of S and any function $\bar{f}$, $\bar{f}: X \to S'$, the number of extensions of $\bar{f}$ to S with a given structure $\{L_\beta\}$ is the product of the number of extensions to S of $P_\beta \cdot \bar{f}$ with location L_β, taken over all β in D'.

Proposition 5.3.2

Let $S \subseteq \bigtimes_{\alpha \in D} S_\alpha$ be irredundant, $S' = \bigtimes_{\beta \in D'} S'_\beta$ and let $\bar{f}$ be an arbitrary function, $\bar{f}: X \to S'$, where $X \subseteq S$. Then for any $\{L_\beta\}_{\beta \in D'}$, s.t. $L_\beta \subseteq D$, $\forall\ \beta$, and any $\{ub_\beta\}_{\beta \in D'}$

$$E^{\bar{f}(X)}_{(S,S')}(\{L_\beta\},\{ub_\beta\}) = \bigtimes_{\beta \in D'} E^{(P_\beta \cdot \bar{f})(X)}_{(S,S'_\beta)}(L_\beta, ub_\beta).$$

Proof

1) Let $f \in E^{\bar{f}(X)}_{(S,S')}(\{L_\beta\},\{ub_\beta\})$. Then clearly $P_\beta \cdot \bar{f}$ is a function from X to $P_\beta(S') = S'_\beta$. Also $L_\beta \in L(P_\beta \cdot f)$. $f = \bigtimes_{\beta \in D'} P_\beta \cdot f$ and obviously $P_\beta \cdot f$ is an extension of $P_\beta \cdot \bar{f}$. Clearly $\exists\ \hat{L}_\beta \in L(P_\beta \cdot f)$ with $|\hat{L}_\beta| \le ub_\beta$.

So for $\forall\ \beta \in D'$, $P_\beta \cdot f \in E^{P_\beta \cdot \bar{f}(X)}_{(S,S'_\beta)}(L_\beta, ub_\beta)$, which implies that

$f \varepsilon \bigtimes_{\beta \varepsilon D'} E_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}(X)} (L_\beta, ub_\beta)$.

2) We will now show that for every $f \varepsilon \bigtimes_{\beta \varepsilon D'} E_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}(X)} (L_\beta, ub_\beta)$, $f \varepsilon E_{(S,S')}^{\bar{f}(X)} (\{L_\beta\},\{ub_\beta\})$.

$f = \bigtimes_{\beta \varepsilon D'} f_\beta$, where $f_\beta \varepsilon E_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}(X)} (L_\beta, ub_\beta)$. So clearly $P_\beta \cdot f = f_\beta$ and $L_\beta = L(P_\beta \cdot f)$ holds. $f_\beta | X = P_\beta \cdot \bar{f} \Rightarrow f|X = \left(\bigtimes_{\beta \varepsilon D'} f_\beta\right)|X = \bigtimes_{\beta \varepsilon D'} (f_\beta | X) = \bigtimes_{\beta \varepsilon D'} (P_\beta \cdot \bar{f}) = \bar{f}$. Also since $S' = \bigtimes_{\beta \varepsilon D'} S'_\beta$, $f:S \to S'$. So $f \varepsilon E_{(S,S')}^{\bar{f}(X)} (\{L_\beta\},\{ub_\beta\})$, which completes the proof. □

With the result of Proposition 5.3.2 in hand we will now show that confidence in a structured case can be expressed as a product of "component" confidences.

<u>Corollary 5.3.1</u>

Let S be irredundant, $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ and let $S' = \bigtimes_{\beta \varepsilon D'} S'_\beta$. Let $X \subseteq S$ and let $\bar{f}:X \to S'$, where $\{L_\beta\}_{\beta \varepsilon D'}$ is a reduced structure of $\bar{f}$.

Then for any $\{ub_\beta\}_{\beta \varepsilon D'}$

(1) $CONF_{(S,S')}^{\bar{f}(X)} (\{L_\beta\},\{ub_\beta\}) = \prod_{\beta \varepsilon D'} CONF_{(S,S'_\beta)}^{(P_\beta \cdot \bar{f})(X)} (L_\beta, ub_\beta)$.

<u>Proof</u>

(See Proposition 5.3.1.)

a) $E_{(S,S')}^{\bar{f}(X)} (\cdot,\{ub_\beta\}) = 0 \Leftrightarrow \nexists f:S \to S'$ s.t. $f|X = \bar{f}$ and for $\forall \beta \varepsilon D'$ $\exists$ an $\hat{L}_\beta \varepsilon \mathcal{L}(P_\beta \cdot f)$ with $|\hat{L}_\beta| \leq ub_\beta$. It follows from Proposition 5.3.2 then, that $\exists \beta \varepsilon D'$ s.t. $\nexists f_\beta \varepsilon E_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}(X)} (\cdot, ub_\beta)$. But this implies that $\exists$ a $\beta \varepsilon D'$ s.t. $CONF_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}(X)} (L_\beta, ub_\beta) = 0$. Thus (1) holds in this case.

b) For $E^{\bar{f}(X)}_{(S,S')}(\cdot,\{ub_\beta\}) \neq \phi$, $E^{P_\beta\cdot\bar{f}(X)}_{(S,S'_\beta)}(\cdot,ub_\beta) \neq \phi$ holds and

$$CONF^{\bar{f}(X)}_{(S,S')}(\{L_\beta\},\{ub_\beta\}) = \frac{|E^{\bar{f}(X)}_{(S,S')}(\{L_\beta\},\{ub_\beta\})|}{|E^{\bar{f}(X)}_{(S,S')}(\cdot,\{ub_\beta\})|} =$$

$$\frac{\prod_{\beta\varepsilon D'} |E^{P_\beta\cdot\bar{f}(X)}_{(S,S'_\beta)}(L_\beta,ub_\beta)|}{\prod_{\beta\varepsilon D'} |E^{P_\beta\cdot\bar{f}(X)}_{(S,S'_\beta)}(\cdot,ub_\beta)|} = \prod_{\beta\varepsilon D'} CONF^{P_\beta\cdot\bar{f}(X)}_{(S,S'_\beta)}(L_\beta,ub_\beta).$$

This follows again from Proposition 5.3.2. (Note that

$$E^{\bar{f}(X)}_{(S,S')}(\cdot,\{ub_\beta\}) = \{f|f|X = \bar{f} \text{ and } |L(P_\beta\cdot f)| \leq ub_\beta, \forall \beta\} =$$

$$\bigcup_{\substack{\{L_\beta\}_{\beta\varepsilon D'} \\ |L_\beta|\leq ub_\beta}} E^{\bar{f}}_{(S,S')}(\{L_\beta\},\cdot) = \bigcup_{\substack{\{L_\beta\}_{\beta\varepsilon D'} \\ |L_\beta|\leq ub_\beta}} \left(\bigtimes_{\beta\varepsilon D'} E^{P_\beta\cdot\bar{f}}_{(S,S'_\beta)}(L_\beta,\cdot)\right) =$$

$$\bigtimes_{\beta\varepsilon D'} \left(\bigcup_{\substack{L_\beta \\ |L_\beta|\leq ub_\beta}} E^{P_\beta\cdot\bar{f}}_{(S,S'_\beta)}(L_\beta,\cdot)\right) = \bigtimes_{\beta\varepsilon D'} E^{P_\beta\cdot\bar{f}}_{(S,S'_\beta)}(\cdot,ub_\beta).)$$ □

Remark

In case $|ub_\beta| = |D| = n$ for all $\beta\varepsilon D'$, and $\{L_\beta\}$ is a structure of $\bar{f}$

$$CONF^{\bar{f}(X)}_{(S,S')}(\{L_\beta\}) = \prod_{\beta\varepsilon D'} |S'_\beta|^{-\{(|S|-|P_{L_\beta}(S)|) - (|X|-|P_{L_\beta}(X)|)\}}.$$ □

Theorem 5.3.1

Let $S \subseteq \bigtimes_{\alpha\varepsilon D} S_\alpha$ be irredundant and let $S' = \bigtimes_{\beta\varepsilon D'} S'_\beta$. Then the following hold.

a) Let $X \subseteq S$ and let $\bar{f}:X \to S'$. Then for any $\{\bar{L}_\beta\}_{\beta\varepsilon D'}$ s.t. $\bar{L}_\beta\varepsilon L(P_\beta\cdot\bar{f})$, $\forall\ \beta$, if $\{ub^1_\beta\}$ and $\{ub^2_\beta\}$ are such that

$|\bar{L}_\beta| \le ub_\beta^2 \le ub_\beta^1$ holds for $\forall\ \beta$

$$CONF^{\bar{f}(X)}_{(S,S')}(\{\bar{L}_\beta\},\{ub_\beta^1\}) \le CONF^{\bar{f}(X)}_{(S,S')}(\{\bar{L}_\beta\},\{ub_\beta^2\}).$$

b) Let X_1,X_2 be irredundant subsets of S, where $\phi \ne X_1 \subseteq X_2$ holds. Let $\bar{f}_2:X_2 \to S'$ and let $\bar{f}_1 = \bar{f}_2|X_1$. Then with $\mathcal{L}_1 = \{L(P_\beta \cdot \bar{f}_1)\}_{\beta \in D'}$ and $\mathcal{L}_2 = \{L(P_\beta \cdot \bar{f}_2)\}_{\beta \in D'}$

$CONF^{\bar{f}_1(X_1)}_{(S,S')}(\mathcal{L}_1,UB) \le CONF^{\bar{f}_2(X_2)}_{(S,S')}(\mathcal{L}_2,UB)$ holds for any family of integers $UB = \{ub_\beta\}_{\beta \in D'}$, s.t. $0 \le ub_\beta \le n$, provided that if $|L(P_\beta \cdot \bar{f}_1)| \le ub_\beta$ holds then $|L(P_\beta \cdot \bar{f}_2)| \le ub_\beta$ holds.

c) Let X be an irredundant subset of S. Let $\bar{f}:X \to S'$ and let $\mathcal{L} = \{L(P_\beta \cdot \bar{f})\}_{\beta \in D'}$. Then for any $UB = \{ub_\beta\}$

$$CONF_{(S,S')}(\mathcal{L},ub) \le CONF^{\bar{f}(X)}_{(S,S')}(\mathcal{L},UB).$$

d) Let X be an irredundant subset of S. Let $\bar{f}$ and $\bar{g}$ be functions from X to S', where $L(P_\beta \cdot \bar{f}) \subseteq L(P_\beta \cdot \bar{g})$ holds for all $\beta \in D'$. Let $\mathcal{L}_{\bar{f}}$ and $\mathcal{L}_{\bar{g}}$ denote the reduced structures of $\bar{f}$ and $\bar{g}$ respectively. Then

$CONF^{\bar{f}(X)}_{(S,S')}(\mathcal{L}_{\bar{f}},UB) \le CONF^{\bar{g}(X)}_{(S,S')}(\mathcal{L}_{\bar{g}},UB)$ holds for any family $UB = \{ub_\beta\}$, provided that if $|L(P_\beta \cdot \bar{f})| \le ub_\beta$ then $|L(P_\beta \cdot \bar{g})| \le ub_\beta$, holds for $\forall\ \beta \in D'$.

<u>Proof</u>

a) Follows directly from Propositions 4.2.2 and 5.3.2.

b) Follows directly from Theorem 4.2.1 and Proposition 5.3.2.

c) Follows directly from Proposition 4.2.3 by applying Proposition 5.3.2.

d) Follows from Propositions 4.2.4 and 5.3.2. □

In case of structured functions we define an average confidence for an irredundant subset X of S in an analogous way to that of Chapter 4. Thus for $UB = \{ub_\beta\}_{\beta \in D'}$

Z_{UB} is a random variable, $Z_{UB}: F_{(S,S')} \to \mathbb{R}$, defined by

$Z_{UB}(f) = CONF^{f|X}_{(S,S')}(STR(f|X), UB)$, where $STR(f|X)$ denotes the reduced structure of $f|X$. (This structure is unique, since X is irredundant.)

$$\text{Then } ACONF^{X}_{(S,S')}(UB) = E(Z_{UB}|UB) = \sum_{f \in F_{(S,S')}} P(f|UB) \cdot Z_{UB}(f).$$

It turns out that average confidence can also be expressed as a product of "component" average confidences.

Proposition 5.3.3

Let $S \subseteq \bigtimes_{\alpha \in D} S_\alpha$ be irredundant and let X be an irredundant subset of S. Further let $S' = \bigtimes_{\beta \in D'} S'_\beta$. Then for any $UB = \{ub_\beta\}_{\beta \in D'}$

$$ACONF^{X}_{(S,S')}(UB) = \prod_{\beta \in D'} ACONF^{X}_{(S,S'_\beta)}(ub_\beta).$$

Proof

It follows from the definition that

$$ACONF^{X}_{(S,S')}(UB) =$$

$$\frac{1}{|F_{(S,S')}(\cdot,UB)|} \sum_{f \in F_{(S,S')}(\cdot,UB)} CONF^{f|X}_{(S,S')}(\{L(P_\beta \cdot f|X)\}, UB) =$$

$$\frac{1}{\prod_{\beta \in D'} |F_{(S,S'_\beta)}(\cdot,ub_\beta)|} \left(\sum_{f \in F_{(S,S')}(\cdot,UB)} \prod_{\beta \in D'} CONF^{P_\beta \cdot f|X}_{(S,S'_\beta)}(L(P_\beta \cdot f|X), ub_\beta) \right) =$$

$$\frac{1}{\prod_{\beta \in D'} |F_{(S,S'_\beta)}(ub_\beta)|} \left(\sum_{\substack{f_\beta \in F_{(S,S'_\beta)}(ub_\beta) \\ \forall \beta \in D'}} \left(\prod_{\beta \in D'} CONF^{f_\beta|X}_{(S,S'_\beta)}(L(f_\beta|X), ub_\beta) \right) \right) =$$

$$\frac{1}{\prod_{\beta\varepsilon D'} |F_{(S,S'_\beta)}(ub_\beta)|}\Big(\prod_{\beta\varepsilon D'} \sum_{f_\beta \varepsilon F_{(S,S'_\beta)}(ub_\beta)} CONF^{f_\beta|X}_{(S,S'_\beta)}(L(f_\beta|X),ub_\beta)\Big) =$$

$$\prod_{\beta\varepsilon D}\Big(\frac{1}{|F_{(S,S'_\beta)}(ub_\beta)|}\Big(\sum_{f_\beta \varepsilon F_{(S,S'_\beta)}(ub_\beta)} CONF^{f_\beta|X}_{(S,S'_\beta)}(L(f_\beta|X),ub_\beta)\Big)\Big) =$$

$$\prod_{\beta\varepsilon D'} ACONF^{X}_{(S,S'_\beta)}(ub_\beta) \ . \qquad \square$$

We are now ready to show that results of Propositions 4.2.5 and 4.2.6 carry over to the structured case with obvious modifications.

Theorem 5.3.2

Let $S \subseteq \bigtimes_{\alpha\varepsilon D} S_\alpha$ be irredundant and let $S' = \bigtimes_{\beta\varepsilon D'} S'_\beta$. Then the following hold.

a) Let X be an irredundant subset of S and let $UB = \{ub_\beta\}_{\beta\varepsilon D'}$ be given. Then

$$ACONF^{X}_{(S,S')}(UB) =$$

$$\Big(\sum_{\bar{f}\varepsilon F_{(X,S')}(\cdot,UB)} |E^{\bar{f}}_{(S,S')}(STR(\bar{f}))|\Big) \Big/ \Big(\sum_{\bar{f}\varepsilon F_{(X,S')}(\cdot,UB)} |E^{\bar{f}}_{(S,S')}(\cdot,UB)|\Big),$$

where $STR(\bar{f})$ is the reduced structure of $\bar{f}$.

b) Let X_1, X_2 be irredundant subsets of S, where $X_1 \subseteq X_2$ holds. Then for any $UB = \{ub_\beta\}_{\beta\varepsilon D'}$

$$ACONF^{X_1}_{(S,S')}(UB) \le ACONF^{X_2}_{(S,S')}(UB).$$

Proof

a) The proof of this is analogous to the proof of Proposition 4.2.5, when Corollary 5.3.1 is applied.

b) Follows directly from Propositions 4.2.6 and 5.3.3. $\square$

The bounds for average confidence function can be readily computed on the basis of Propositions 4.2.7 and 5.3.3.

The important property of average confidence, which we were able to prove is its monotonicity with increasing data set size.

Thus as we enlarge our experimental domain subsets, the trend will be to increase our average confidence.

We now turn to the topic of prediction making in structured case. Suppose that we know the actual function f on X, a proper subset of S. We would like to predict as much of f on S-X as possible. We do not want to do it randomly, but use the information we already have about f as a basis for our predictions. We will proceed in the following way.

If $\bar{f}$ is the restriction of f to X and $L = \{\bar{L}_\beta\}$ is a reduced structure of $\bar{f}$, we will define a function $\hat{f}^L$ from $\bigcap_{\beta\varepsilon D'} \mathrm{COMPL}_S^{\bar{L}_\beta}(X)$ to S' in the following way: $\hat{f}^L = \bigtimes_{\beta\varepsilon D'} \hat{f}^L_\beta$, where $\hat{f}^L_\beta$ is the unique extension of $P_\beta \cdot \bar{f}$ to $\bigcap_{\beta\varepsilon D'} \mathrm{COMPL}_X^{\bar{L}_\beta}(X)$ with location $\bar{L}_\beta$. (See Lemma 3.3.1 and Corollary 3.3.3.) Since S' is Cartesian $\hat{f}^L$ is well defined. $\hat{f}^L$ is our prediction of f to $\bigcap_{\beta\varepsilon D'} \mathrm{COMPL}_S^{\bar{L}_\beta}(X)$ and it satisfies the following heuristic:

1) Any guess at f ought to agree with the observed portion of f ($\hat{f}^L|X = \bar{f}$).

2) Any guess at f ought to have a structure so far estimated for f ($\bar{L}_\beta \varepsilon L(P_\beta \cdot \hat{f}^L)$).

3) We are justified in guessing at point in S just when our guess is <u>uniquely</u> determined by imposing requirements 1) and 2), i.e. when our guess is constrained by so-far-acquired data.

We note that $\bigcap_{\beta\varepsilon D'} \mathrm{COMPL}_S^{\bar{L}_\beta}(X)$ is the largest subset of S with property 3. This follows from the fact that $COMPL_S^{\bar{L}_\beta}(X)$ is the largest subset Z_β of S s.t. for every $f_\beta \varepsilon E_{(S,S'_\beta)}^{P_\beta \cdot \bar{f}}(\bar{L}_\beta)$, $f_\beta | Z_\beta$ is unique. (See section 4 of Chapter 4.) We will refer to $\bigcap_{\beta \varepsilon D'} COMPL^{\bar{L}_\beta}(X)$ as an explanatory range of $\bar{f}$, denoted by $EXPL_{(S,S')}^{\bar{f}(X)}(L)$. $EXPL_{(S,S')}^{\bar{f}(X)}(L)-X$ is denoted by $PRED_{(S,S')}^{\bar{f}(X)}(L)$. $PCONF_{(S,S')}^{\bar{f}(X)}(\{L_\beta\},\{ub_\beta\})$ is defined analogously as in Chapter 4.

We remark that our prediction making approach is quite different from the usual one. The main difference being the concept of predictive range. In usual view of predictions no distinction is possible between points at which prediction is systematically determined by acquired data as opposed to points, at which prediction is not truly constrained by the data.

Proposition 5.3.4

Let $S \subseteq \bigtimes_{\alpha \varepsilon D} S_\alpha$ and $S' = \bigtimes_{\beta \varepsilon D'} S'_\beta$. Further let X be a subset of S and let $\bar{f}: X \rightarrow S'$, where $\bar{f} \varepsilon F_{(X,S')}(\cdot, UB)$.

Then for any structure $\{\bar{L}_\beta\}_{\beta \varepsilon D'}$ of $\bar{f}$

a) $PCONF_{(S,S')}^{\bar{f}(X)}(\{\bar{L}_\beta\}, UB) = \dfrac{|E_{(S,S')}^{\hat{f}(Y)}(\cdot, UB)|}{|E_{(S,S')}^{\bar{f}(X)}(\cdot, UB)|}$, where

$Y = EXPL_{(S,S')}^{\bar{f}}(\{\bar{L}_\beta\})$ and $\hat{f}$ is a unique extension of $\bar{f}$ to Y with a structure $\{\bar{L}_\beta\}$.

b) If $ub_\beta = n$, for all $\beta \varepsilon D'$

$$PCONF_{(S,S')}^{\bar{f}(X)}(\{\bar{L}_\beta\}) = |S'|^{-|PRED_{(S,S')}^{\bar{f}(X)}(\{\bar{L}_\beta\})|}.$$

Proof

Identical to that of Proposition 4.3.1. □

We thus showed that the greater the predictive range the (exponentially) smaller the probability of being correct. But note that every misprediction on the predictive range is informative--it invalidates the hypothesis that the actual f has structure $\{\bar{L}_\beta\}$, and so forces us to increase at least one component location (in case X is irredundant).

5.4 Strategies for Experimentation

In this section we will apply our results to finite autonomous discrete time systems. Those systems are time invariant systems of the form $\langle S, \delta \rangle$, where S is a state space of the system and δ its state transition function (a map on S). A system $\langle S, \delta \rangle$ evolves in discrete time, so that for any state s and any time t, $\delta(s)$ represents the state of the system at a next time step. Such a system is a special case of a system of section 5.3 (because the domain and codomain sets are same, $S = S'$).

The interpretation of a partial model of $\langle S, \delta \rangle$ and structured partial model of $\langle S, \delta \rangle$ are clear.

We will propose here several strategies, which a modeller might follow during the experimental process. They are divided into non-adaptive and adaptive classes. In non-adaptive strategies the sequence of test sets is fixed and experimentation consists of transition acquisition for the successive sets until a given structural confidence level is achieved. For this purpose we employ Y_k^y (or X_L^y) sets because of their desirable properties, in particular the computational ones. The disadvantage of the Y_k^y sets is their exponential growth, which limits the feasibility of their use to relatively small k. The Y_k^y sets may still be useful however, since models with relatively small interaction are sought in applications.

In adaptive strategies, the sequence of test sets is determined on the basis of prior experimentation. Here the problem of generating a minimal irredundant set, which includes a given set arises. Although no fully satisfactory solution has been obtained, we shall assume that given a subset X, it is feasible to generate a "reasonably" small irredundant set containing X (actually an irredundant subset of X would also work in the following strategies).

We assume now that any data point can be generated for the system being modelled. This for example can be achieved, when many identical copies of the system are available. This also will be the case, when an expression or a formula for transition function generation is available, but actual generation is done on demand by a computer program. Or simply, when all data has been collected and stored in memory in a form suitable for table look up.

The existence of feasible algorithms for location determination is also assumed. Some initial thought has revealed that these algorithms are strongly dependent on the order of coordinate testing. It will be important to investigate the computational aspects of this process before implementation of the suggested strategies is attempted.

We assume that the state set of our system is Cartesian.

The following strategy is non-adaptive. It employs once-and-for all computation.

Strategy 1

We assume the desired structural confidence level C is given. For $k = 1,2,\ldots,n-1$ we compute a minimum confidence on Y_k^y, m_k. We then find a minimum k, for which $m_k \geq C$ holds.

We note that the $\{m_k\}$ can be computed once-and-for-all, and a change in C will simply result in a different miminum k. All the computations are done prior to any experimentation. If a k as above exists, we generate the Y_k^y data and whatever the reduced structure of our partial model, the confidence in it will be high enough.

In another variant, upper bound information on location sizes is assumed given. This further increases feasibility (since m_k will be higher for every k) and makes system identification with confidence 1 possible.

The following strategy is non-adaptive. It employs a stopping rule based on achieved confidence.

Strategy 2

Again we are given a structural confidence level-C to be achieved. We start with Y_1^y, find the reduced structure of the partial model and compute the confidence in it. If this is high enough (that is at least as large as C) we stop. At this point we know the system structure with given confidence. If the confidence is not high enough we go to Y_2^y and repeat the process.

In case Y_{n-1}^y does not give us high enough confidence we may use any superset of it for structure and confidence computation. This is possible because every superset of Y_{n-1}^y is irredundant (see Remark following Corollary 3.4.2). Hopefully the required level of confidence is achieved before the entire state space has been covered.

We note that strategy 2 may be applicable even when strategy 1 is not. Of course, if $Y^y_{k_2}$ data need to be generated for strategy 2 and $Y^y_{k_1}$ for strategy 1, then $Y^y_{k_2} \subseteq Y^y_{k_1}$ will hold (assuming equal specified C values). A variant of this strategy employs average structural confidence to guide us in an initial choice of k (here we always start with k = 1). Although this might lead to unnecessarily large Y^y_k in our particular case, we may avoid several iteration steps (recomputation of partial models and confidences in their reduced structure).

The following strategy will lead to finding the system structure, when an upper bound on its complexity is given (smaller than the cardinality of the index set).

Strategy 3

We assume that all locations are smaller than or equal to k in size, where $1 \leq k \leq n-1$, and we want to find them.

Version 1

We generate Y^y_k data and find a partial model structure. This is a structure of our model. Also the state transition function of our system can be identified if necessary.

Version 2

In this version, rather than generate the entire Y^y_k set we proceed in the following way.

Let $L_1, L_2, \ldots, L_j$ be some enumeration of $\binom{n}{k}$ subsets of D with cardinality k.

We set $L_\alpha = \phi$, for all α in D. For $i = 1, \ldots, \binom{n}{k}$ we do the following.

For every α in D we compute $L(P_\alpha \cdot \bar{\delta}_i)$, where $\bar{\delta}_i$ is the transition function on $X^y_{L_i}$. We set $L_\alpha = L_\alpha \cup L(P_\alpha \cdot \bar{\delta}_i)$. If $|L_\alpha| = k$, we set $D = D-\alpha$ and go to the next α in D. If a) $D = \phi$ or b) $i = \binom{n}{k}$ we are done with $\{L_\alpha\}$ the desired model structure. If not we go to the next i and repeat the process.

<u>Version 3</u>

We assume an enumeration $L_1, L_2, \ldots, L_j$ as in version 2. Let $Z_0 = \phi$. For $i = 1, \ldots, \binom{n}{k}$ we do the following.

$$Z_i = Z_{i-1} \cup X^y_{L_i}$$

For every $\alpha \varepsilon D$ we compute $L_\alpha = L(P_\alpha \cdot \bar{\delta}_i)$, where $\bar{\delta}_i$ is a transition function on Z_i.

If $|L_\alpha| = k$, set $D = D-\alpha$ and go to the next α in D. If a) $D = \phi$ or b) $i = \binom{n}{k}$ we are done with $\{L_\alpha\}$ the reduced structure of δ. Else we go to the next i and repeat the process.

What enables us to find locations in this manner is the fact, that

$$L(\bar{\delta}|X^y_{L_1} \cup \ldots \cup X^y_{L_j}) = \bigcup_{i=1}^{j} L(\bar{\delta}|X^y_{L_i}). \quad \text{(Proposition 3.2.3.)}$$

While all the versions of strategy 3 lead to finding the reduced model structure, in the last two versions we might not have to generate the entire Y^y_k set. This might for example happen when $L(P_\alpha \cdot \delta)$ are all same and have cardinality k.

The essential difference between versions 2 and 3 is, that while in the latter we are finding locations on the union of $X^y_{L_i}$ sets, in the former we are taking the unions of corresponding locations. Efficiency of version 3 is strongly dependent on enumeration assumed. In version 2 a number of points to be compared at every step is step independent, which is not true in version 3.

Version 2 seems to be more efficient than version 3.

The next strategy is adaptive in the sense that we do not proceed in a fixed way with data generation (as was the case till now).

Strategy 4

A desired structural confidence level-C is given. We start with an arbitrary subset X_1 of S. We construct a $Y_1 = IR(X_1)$ - a minimal irredundant superset of X_1 and generate data for it. We then perform structure and confidence computations on Y_1. If the confidence is high enough we are done. If not we pick an arbitrary subset Z_1 of points of S (outside of Y_1). We set $X_2 = Y_1 \cup Z_1$ and repeat the process.

At every stage we may use increment sets-Z_i with same cardinality, say p, or vary the size at every set generation.

In a variant of the above strategy we do not attempt to construct irredundant sets at every stage. In this case however, all possible minimal structures may have to be computed. Moreover if c_i denotes a maximal confidence on X_i, we are not assured that $c_{i+1} \geq c_i$ holds.

The following strategy is adaptive and is analogous to the usual cycle of testing and modification often used in scientific modelling. We will employ here our predictive concept.

Strategy 5

We start with a small irredundant subset X_1 of S (this could be a minimal set of Proposition 2.3.2). At cycle i, we find the reduced structure, STR_i, of $\bar{\delta}$ on X_i. ($\bar{\delta}$ is the transition function observed on X_i.)

Starting at an arbitrary state in X_i we generate a trajectory employing $\bar{\delta}$ until either a) a predicted state along the trajectory does not match the corresponding experimental state or b) we reach a state in which $\hat{\delta}$ is undefined ($\hat{\delta}$ is the prediction of our transition function) or which has been previously visited (indicating a cycle has been entered into). In case a), we add the states generated along the trajectory until mismatch, to X_i and set the new X_{i+1} to a minimal irredundant set containing X_i and the added data, then we start cycle i + 1. In case b) we select a new point in X_i to initiate trajectory generation and start cycle i again.

As long as we remain in cycle i the structural confidence is increasing. This follows from Remark following Proposition 4.2.3. If we wish, we can stop when a prespecified confidence level has been achieved. If we exhaust $PRED_{(S,S)}^{\bar{\delta}(X_i)}(STR_i)$ before attaining this confidence, we set X_{i+1} to a minimal irredundant set properly containing $X_i \cup PRED_{(S,S)}^{\bar{\delta}(X_i)}(STR_i)$. We then start cycle i + 1.

Now note that the sets $X_1, X_2, \ldots$ form an increasing nested sequence so structural confidence is nondecreasing in this process.

Since the structural confidence cannot stabilize at a "false peak", it must eventually increase to any preset level. Alternatively, we can employ the heuristic rule: stop when the structural confidence computed does not change for a "long enough time".

A variant of this strategy does not employ $\bar{\delta}$ for a trajectory generation. Rather we pick subsets of $PRED_{(S,S)}^{\bar{\delta}(X_i)}(STR_i)$ for comparison of our predictions and experimental values.

Notice that in the above strategy we employ our predictive concept to orient experimentation towards tests of the hypothesis: "the actual system has the same influencer set as our partial model". Every misprediction is informative in the sense of requiring an extension of at least one influence set.

Finally, suppose that a modeller is interested in determining whether there exists a structured model of the system in j-class, i.e. such that every influencer set is smaller than or equal to j in cardinality ($j \leq n$). The necessary and sufficient conditions for excluding such a class on the basis of partial experimentation are not available. This results from the fact, that adding even one data point to a set may considerably enlarge location sizes.

We will explain next a strategy for excluding j-class of models (the sufficient conditions for such an exclusion).

<u>Strategy 6</u>

We start with an arbitrary irredundant subset X of S. We find a partial reduced model of the system. If this model is not in j-class there is no system model in this class. Otherwise for every α in D do the following. With $\{L_\alpha\}$ a reduced structure of a partial model, make the prediction $\hat{\delta}_\alpha$ of $P_\alpha \cdot \delta$ to $COMPL_S^{L_\alpha}(X) = PRED_{(S,S_\alpha)}^{P_\alpha \cdot \bar{\delta}(X)}(L_\alpha) \cup X$. Let $\mathcal{L}_\alpha = \{L | L \subseteq D, |L| = j, \text{ and } L \supseteq L_\alpha\}$. For every L in $\mathcal{L}_\alpha$, we form $COMPL_S^L(X)$. If for every $L \varepsilon \mathcal{L}_\alpha$ there exists a point of disagreement in $COMPL_S^L(X)$ between the predictions and experimental data, $|L(P_\alpha \cdot \delta)| > j$ and we stop. The model in j-class has been excluded. (This follows directly from Proposition 4.3.2.) If not set $D = D-\alpha$ and go to next α.

If $D = \phi$ and the j-class model has not been excluded yet, we may try the same method on a larger subset of S. (In the process though, our confidence in existence of a model in j-class has increased.)

The above strategies do not exhaust all the possibilities. There are many variants of them involving minor modifications. However, we feel they illustrate the spirit, in which the theory developed can be used to aid the experimental process.

CHAPTER VI

CONCLUSIONS

6.1 Summary

In this study we considered the problem of modelling autonomous discrete time systems with structured state spaces, on the basis of partial data; special emphasis was placed on structure inference and identification.

We first developed the theory of coordinatizations of abstract sets. We pointed out the importance of irredundant coordinatizations and their ramifications for modelling enterprise. Ways of irredundant set generation and criteria for irredundance were set forth.

We then studied properties of functions with structured domains. In particular the relation between locations of function restrictions to a sequence of nested subsets of a function domain has been explored. Also the construction of extensions with given locations has been studied: a method for their enumeration was given.

Ways of constructing special Cartesian domain subsets have been proposed to be used for structure identification. The computational properties of those subsets and their relative sizes have been discussed.

A notion of structured partial models has been formalized and several measures of model performance introduced. Structural confidence, predictive range and predictive confidence for a partial model were defined and their properties and dependence on parameters analyzed. Furthermore, we showed how to compute the above measures. A methodology for predicting state-transitions not yet observed was proposed, whereby predictions are constrained by so-far-acquired data.

Finally, based on the theory developed, a number of strategies for experimentation was proposed. Their advantages and drawbacks were discussed at some length.

6.2 Suggestions for Further Research

Several topics for further research emerge from this study. Considerable amount of work remains to be done in the area of algorithms for location determination. It is of interest to investigate their complexity as well as various computational trade-offs between multiple location determination and irredundant set generation.

To this end one has to seek an algorithm for a minimal irredundant superset generation. Since the solution of the above will most certainly depend on the type of coordinatization involved, the hierarchy of coordinatizations should be further investigated. More specifically, new sufficient conditions for moving up the hierarchy should be sought.

Also more research is needed to elucidate the relationships between properties of a coordinatized set X and corresponding properties of its graph-theoretic representation.

Once all of the above is accomplished,the development of an interactive computer package to aid modelling efforts would become feasible.

Finally, we suggest that the theory and methodology developed can be extended to I/O, nondeterministic and stochastic systems.

APPENDIX A

IRREDUNDANCE OF CERTAIN $\mathbb{R}^n$ SUBSETS

In general the result of Example 2.2.4 cannot be extended to hold for a convex subset with a nonempty interior. This is illustrated by the following example.

Example A.1

Consider S a subset of $\mathbb{R}^2$ as in Figure A.1 a). Then S is clearly convex and $Int(S) \neq \phi$. But S is not irredundant. This follows easily, since for the point p of S as indicated in Figure A.1 a), $[p]_{\Pi^S_{\alpha_1}} = [p]_{\Pi^S_{\alpha_2}} = \{p\}$. Clearly then, $\Pi^S_{\alpha_1} \cup \Pi^S_{\alpha_2} \neq I$. □

We will show however that every subset S_n of $\mathbb{R}^n$, where $S_n = \{(p_1,\ldots,p_n) | p_i \geq 0, \forall i, \sum_{i=1}^{n} p_i \leq 1\}$ is irredundant.

Lemma A.1

Let $S_n = \{(p_1,\ldots,p_n) | p_i \geq 0, \forall i, \text{ and } \sum_{i=1}^{n} p_i \leq 1, p_i \varepsilon \mathbb{R}\}$.

Then $Bnd(S_n) = \{(p_1,\ldots,p_n) | \exists i \text{ s.t. } p_i = 0 \text{ or } \sum_{i=1}^{n} p_i = 1\}$.

Proof

We will identify the $Int(S_n)$ and $Bnd(S_n)$.

Let $F = \{(p_1,\ldots,p_n) | \exists i \text{ s.t. } p_i = 0 \text{ or } \sum_{i=1}^{n} p_i = 1\}$.

We will show that $Bnd(S_n) = F$.

First we show that $F \subseteq Bnd(S_n)$. We will use here the Euclidean metric on $\mathbb{R}^n$, namely $d(x,y) = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}}$, where

$x = (x_i)_{i=1}^{n}$ and $y = (y_i)_{i=1}^{n}$.

Let $x \varepsilon F$ be given. To prove that $x \varepsilon \text{Bnd}(S_n)$ we need to show that for every $\varepsilon > 0$, $\exists$ a $p \varepsilon \mathbb{R}^n$ s.t. $d(x,p) < \varepsilon$, but $p \notin S_n$.

Choose any $\varepsilon > 0$. Let $\hat{\varepsilon}$ be any positive number s.t. $\hat{\varepsilon} < \varepsilon$.

a) $\exists$ an i, s.t. $x_i = 0$, say i_0. Let p be a point defined by

$$p_i = \begin{cases} x_i, & i \neq i_0 \\ -\hat{\varepsilon} & i = i_0 \end{cases}$$

. Then $p_{i_0} < 0$ and so clearly $p \notin S_n$.

Also $d(p,x) = ((-\hat{\varepsilon})^2)^{\frac{1}{2}} = \hat{\varepsilon} < \varepsilon$. So $x \varepsilon \text{Bnd}(S_n)$.

b) $\nexists$ i s.t. $x_i = 0$, but $\sum_{i=1}^{n} x_i = 1$ holds. Pick any i_0, $1 \leq i_0 \leq n$.

Let p be a point defined by $p_i = \begin{cases} x_i & i \neq i_0 \\ x_i + \hat{\varepsilon} & i = i_0 \end{cases}$.

Then $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} x_i + \hat{\varepsilon} = 1 + \hat{\varepsilon} > 1$. So $p \notin S$. But $d(p,x) = \hat{\varepsilon} < \varepsilon$.

Thus $x \varepsilon \text{Bnd}(S_n)$.

So $F \subseteq \text{Bnd}(S_n)$. To show that $\text{Bnd}(S_n) \subseteq F$, it suffices to show that every point $x \varepsilon S_n$ s.t. $x_i \neq 0$ for all i and $\sum_{i=1}^{n} x_i < 1$ is an interior point of S_n.

Let x as above be given. To show $x \varepsilon \text{Int}(S_n)$ we need to show that $\exists\ \varepsilon > 0$ s.t. $\{p \mid d(p,x) < \varepsilon\} \subseteq S_n$.

Let $\varepsilon_i > 0$ be such that $x_i - \varepsilon_i > 0$ and $x_i + \varepsilon_i < 1$. For every i, such ε_i clearly exists. Let $\hat{\varepsilon} > 0$ be such that $\sum_{i=1}^{n} x_i + n\hat{\varepsilon} < 1$. Since $\Sigma x_i < 1$, such an $\hat{\varepsilon}$ exists.

Finally, let $\varepsilon = \min(\min_i \varepsilon_i, \hat{\varepsilon})$. Then $x_i - \varepsilon > 0$ and $x_i + \varepsilon < 1$ holds for all i and $\sum_{i=1}^{n} x_i + n\varepsilon < 1$ holds. With ε so chosen we show that $\{p \mid d(p,\mathbf{x}) < \varepsilon\} \subseteq S_n$. Let p be such that $d(p,x) < \varepsilon$. This clearly implies that $|x_i - p_i| < \varepsilon$, $\forall$ i, i.e. that $x_i - \varepsilon < p_i < x_i + \varepsilon$ and thus that $p_i > 0$, $\forall$ i. Also $\sum_{i=1}^{n} p_i < \sum_{i=1}^{n} (x_i + \varepsilon) < 1$. So clearly $p \varepsilon S_n$. We showed then that if $\mathbf{x} \notin F$, $x \varepsilon \mathrm{Int}(S_n)$ and so $F = \mathrm{Bnd}(S_n)$. We also showed here that $\mathrm{Int}(S_n) \neq \phi$. □

We are now ready to prove that S_n is irredundant.

Proposition A.1

Let $S_n = \{(p_1, \ldots, p_n) \mid p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i \leq 1\}$. Then S_n is an irredundant subset of $\mathbb{R}^n$.

Proof

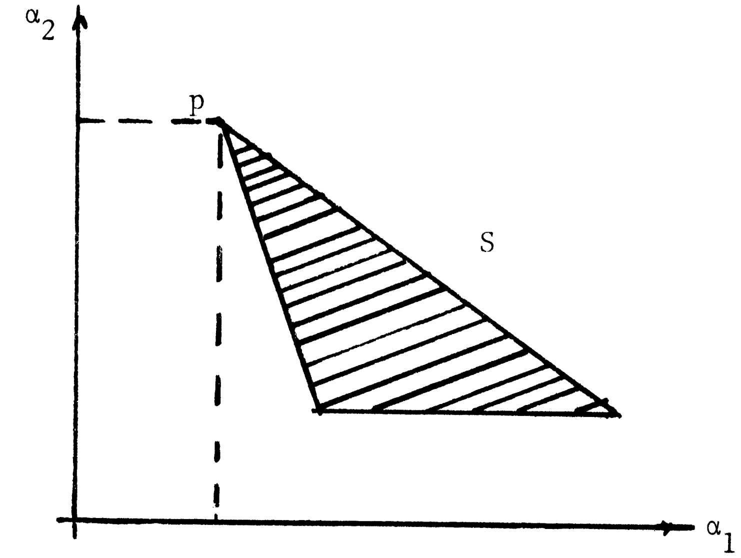
We note that S_n is convex and $\mathrm{Int}(S_n) \neq \phi$ is also convex. Thus $\mathrm{Int}(S_n)$ is irredundant. (This was proved for any open convex set in $\mathbb{R}^n$.)

1) Induction base

n=2. We show that $S_2 = \{(p_1,p_2) \mid p_1 \geq 0,\ p_2 \geq 0 \text{ and } p_1 + p_2 \leq 1\}$ is irredundant. This is clear from Figure A.1.b), since for every point p of $S - \{p^1, p^2, p^3\}$ $\exists$ an $x \varepsilon \mathrm{Int}(S_2)$ s.t. $p \Pi_{\alpha_1} x$ or $p \Pi_{\alpha_2} x$ holds and $\mathrm{Int}(S_2)$ is irredundant.

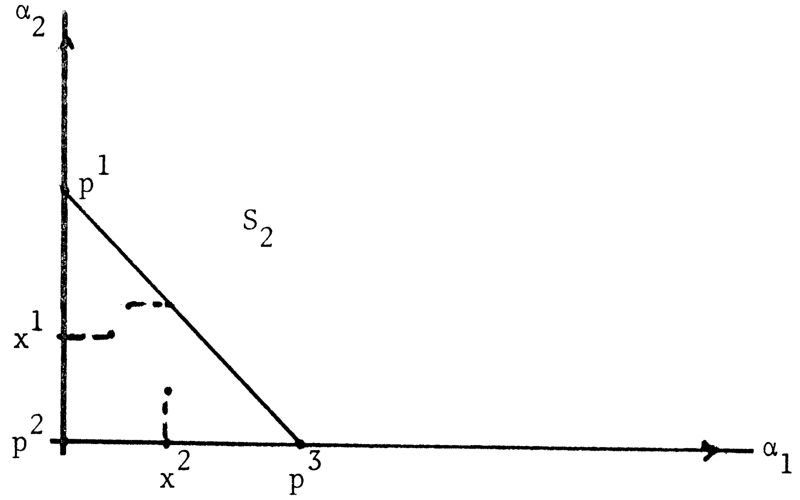
2) Induction hypothesis

We assume that S_{n-1} is irredundant ($n \geq 3$) and show that this implies irredundance of S_n.



S is convex, Int(S) $\neq \phi$, but S is not irredundant

a)



$p^1 \Pi_{\alpha_1} x^1$, $p^2 \Pi_{\alpha_2} x^2$, $p^3 \Pi_{\alpha_2} x^2$

A convex set S_2 is irredundant

b)

Figure A.1: Irredundance of Convex Subsets of $\mathbb{R}^2$.

Let D_1, D_2 be arbitrary subsets of D s.t. $D_1 \cap D_2 \neq D$ and $D_1 \cup D_2 = D$. We have to show that $\Pi_{D_1 \cap D_2} \leq \Pi_{D_1} \cup \Pi_{D_2}$.

Let $x, y \in S_n$ be such that $x\Pi_{D_1 \cap D_2} y$. We want to show, that $x\Pi_{D_1} \cup \Pi_{D_2} y$.

a) $\exists$ an $i_0 \in D_1 \cap D_2$ s.t. $x_{i_0} = y_{i_0} = 0$. Then $x, y \in \hat{S}_n$, $\hat{S}_n \subseteq S_n$, where $P_{D-i_0}(\hat{S}_n) = S_{n-1}$ and $P_{i_0}(\hat{S}_n) = 0$. But S_{n-1} is irredundant by induction hypothesis, which in turn implies irredundance of $\hat{S}_n$. So $x\Pi_{D_1} \cup \Pi_{D_2} y$ holds.

b) $\nexists$ $i \in D_1 \cap D_2$ s.t. $x_i = y_i = 0$, i.e. $x_i = y_i > 0$, for $\forall\, i \in D_1 \cap D_2$. If $\sum_{i \in D_1 \cap D_2} x_i = \sum_{i \in D_1 \cap D_2} y_i = 1$ holds then $y_i = x_i = 0$ for all $i \in D - D_1 \cap D_2$ and $x = y$, in which case we are done.

So we need to consider only x,y with $x\Pi_{D_1 \cap D_2} y$ such that

$$\sum_{i \in D_1 \cap D_2} x_i = \sum_{i \in D_1 \cap D_2} y_i < 1 \text{ and } x_i = y_i > 0,\ \forall\, i \in D_1 \cap D_2.$$

For such x and y however, $\exists$ a $p, w \in Int(S_n)$ s.t. $x\Pi_{D_1} \cup \Pi_{D_2} p$ and $w\Pi_{D_1} \cup \Pi_{D_2} y$. This then implies that $p\Pi_{D_1 \cap D_2} w$, and so that $p\Pi_{D_1} \cup \Pi_{D_2} w$.

We will show that p and w as described exist.

For x as above, $\exists$ a set of numbers ε_i such that $\varepsilon_i = 0$ for $\forall\, i \in D_1 \cap D_2$, $x_i + \varepsilon_i > 0$, $\forall$ i, and $\sum_{i=1}^{n} (x_i + \varepsilon_i) < 1$. That ε_i's as above exist can be verified in more detail by the reader. With

$$Z_i = \begin{cases} x_i & i \in D_1 \\ x_i + \varepsilon_i & i \in D - D_1 \end{cases} \quad \text{and} \quad p_i = \begin{cases} z_i + \varepsilon_i & i \in D - D_2 \\ z_i & i \in D_2 \end{cases},$$

$x\Pi_{D_1} z$ and $z\Pi_{D_2} p$ hold. Further, $p \in Int(S_n)$. Similarly we show that a w as above exists. (We note that the case $D_1 \cap D_2 = \phi$ falls into the last category.)

So S_n is irredundant, which completes the proof. □

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