THE UNIVERSITY OF MICHIGAN
COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Computer and Communication Sciences Department

Technical Report

A LEAST UPPER BOUND ON THE FEEDBACK INDEGREE FOR HOMOMORPHIC REALIZATION OF SEQUENTIAL MACHINES

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with assistance from:

Department of Health, Education, and Welfare
National Institutes of Health
Grant No. GM-12236
Bethesda, Maryland

and

Department of the Navy
Office of Naval Research
Contract No. N00014-67-A-0181-0011
Washington, D.C.

and

U. S. Army Research Office (Durham)
Grant No. DA-31-124-ARO-D-483
Durham, North Carolina

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

October 1969

ABSTRACT

It is known that for every integer d, there are transition functions not isomorphically realizable by any net having feedback indegree (the largest number of wires that any delay receives from other delays in its feedback loop) less than d. Here we show that, in contrast to the isomorphic case, every transition function can be homomorphically realized by nets of feedback indegree not exceeding 2. This is a least upper bound, since simple nets (i.e., those having feedback indegrees not exceeding 1) are shown not to be universal in this sense.

A conjecture concerning feedback complexity of logical nets (sequential machine realizations employing delay elements) was made by Holland [2] as follows: for any logical net define the feedback indegree as the largest number of input wires that any delay receives from other delays in its feedback loop; for each integer d, there is a transition function which cannot be isomorphically realized by any net of indegree less than d. This conjecture was shown to be valid by Zeigler [4,5] and the question then arose as to whether it held as well for homomorphic realization (i.e., allowing state splitting memory expansion). In this paper we show that a complexity hierarchy does not hold in this case. Specifically we show that every transition function can be homomorphically realized by nets of feedback indegree not exceeding 2 and this is the least upper bound in the sense that there are transition functions which cannot be homomorphically realized by nets of feedback indegree 1.

Logical nets and their representing digraphs were formally defined in [4]. Essentially the digraph D(A) of a logical net A considers the delay elements as points and there is a (directed) line from point α to point β just in case delay β receives an input from the output of delay α. See for example Figure 1.

For any point α of D(A) let S_α denote the strong component containing α, ($S_\alpha = \{\beta |$ there is a path from α to β and back in D(A)$\}$. I_α denotes the set of points preceeding α in D(A) i.e., the set of delays feeding delay α in the logical net and $FI_\alpha = I_\alpha \cap S_\alpha$ is the set of wires coming into α from points in its strong component.

As usual, $|I_\alpha|$ is the indegree of α and we call $|FI_\alpha|$ the feedback indegree of α.

Based on a result of Arden [1], Weiner and Hopcraft [3] show that a finite set of modules can homomorphically realize any transition function if, and only if, the set is complete (i.e., can be used to realize with some delay all finite memory span functions). As they note that there are complete modules having just two binary input wires (for example Figure 2) we can conclude that every transition function can be homomorphically realized by logical nets in which no point has indegree greater than 2.

Also in order to satisfy the completeness requirement at least some points must have indegree 2, so that 2 is the least upper bound on the indegrees of the nets which are universal in this sense. It does not follow however that this is the case for feedback indegree. Since for each point α, $|FI_\alpha| \leq |I_\alpha|$ we can conclude that 2 is an upper bound on the feedback indegree required for universality. But 2 is not necessarily the least upper bound since it is possible that every point in a net has feedback indegree 1 but also some have indegree greater than 1. (Figure 1 is an example.)

Thus it is still possible that simple nets, as defined below, are universal in the sense that every transition function can be homomorphically realised by some simple net.

Definition A logical net A is simple if for every point $\alpha \varepsilon D(A)$, $|FI_\alpha| \leq 1$.

Thus simple nets consist of cycles (in the graph theoretic sense) connected together in series-parallel fashion by feedback free circuits (Fig. 1). We now proceed to demonstrate the limitations on such nets.

First we establish a general theorem which relates the cycle characteristics of transition functions one of which can simulate the other.

<u>Definition</u> For transition functions $M_i:Q_i \times S_i \to Q_i$, i=1,2, we say that M_2 <u>divides (is simulated by)</u> M_1 if there exists $Q' \subseteq Q_1$ and maps $G:S_2 \to S_1^*$ (the free semigroup generated by S_1), $h:Q' \to Q_2$ (onto), such that Q' is closed under $g(S_2)^*$ and for all $q\varepsilon Q'$, $s\varepsilon S_2$

$$h(M_1(q,g(s)) = M_2(h(q),s)$$

($M_1:Q_1 \times S_1 \to Q_1$ is the usual extension to S_1^* of M_1, we write $qx = \check{M}(q,x)$.)

M_2 is <u>homomorphically realizable</u> by M_1 if g maps S_2 into S_1 in the above definition.

<u>Definition:</u> $M: Q \times S \to Q$ contains a <u>cycle</u> if there is a $q \varepsilon Q$ such that

$$q = qx^m = \underbrace{qx\ x \ldots x}_{m \text{ times}} \qquad \ldots 1)$$

for some $x \varepsilon S^*$ and positive integer m. Let k be the least positive integer for which (1) is true. Let the sequence $Z_1, Z_2, Z_3, \ldots Z_{k\ell(x)}$ be the se- be the sequence of initial substrings of x^k, where Z_1 is the first symbol of x^k and $Z_{k\ell(x)} = x^k$.

The sequence of states $qZ_1, qZ_2, qZ_3, \ldots, qZ_{k\ell(x)}$ is called the <u>cycle of x</u> and clearly consists of the states encountered in journey from q back to q in the order of encounter. The <u>x-period</u> of this cycle is the number of states in the subsequence $qx^1, qx^2, qx^3, \ldots, qx^k$.

We remark that the cycle of x need not form a cycle in the state digram of M in the graph theoretic sense i.e., not all qZ_i need be distinct (although all qx^i are distinct).

We say that M contains a <u>string cycle</u> of <u>string period</u>, p if it contains a cycle of x for some $x \varepsilon S^*$ which has x-period p.

<u>Theorem:</u> Let $M_i : Q_i \times S_i \to Q_i$, i = 1,2 be <u>finite</u> transition functions such such that M_2 divides M_1 with maps $h: Q_1' \to Q_2$, and $g: S_2 \to S_1^*$. If for some $x \varepsilon S_2^*$,

$$q_1', q_2', \ldots, q'_{m\ell(g(x))} = q' \in Q_1'$$

is a g(x)-cycle of M_1 of g(x)-period m, then $h(q_1), h(q_2), \ldots, h(q')$ is an x-cycle of M_2 with x-period k dividing m.

Conversely, if $q_1, q_2, \ldots, q_{k\ell(x)} = q$ is a x-cycle of M_2 with x-period k then there exists a $\tilde{g}(x)$-cycle in

$$h^{-1}(q_1) \quad h^{-1}(q_2) \ldots \quad h^{-1}(q) \text{ in } M_1$$

with $\tilde{g}(x)$-period $m > 0$ a multiple of k.

<u>Proof:</u> $\rightarrow$ Consider the subsequence of the given $\tilde{g}(x)$-cycle of M_1:

$$q'\tilde{g}(x),\ q'[\tilde{g}(x)]^2,\ \ldots,\ q'[\tilde{g}(x)]^m = q'$$

Let $H(q') = q$. Noting that

$$\begin{aligned} h(\tilde{M},(q',[g(x))]^i &= h(\tilde{M}_1(q', \tilde{g}(x^i)) \\ &= M_2(h(q'),x^i) \\ &= M_2(q,x^i) \end{aligned}$$

We see that the given subsequence maps under h to a sequence

$$qx^1,\ qx^2,\ \ldots,\ qx^m = q$$

in M_2. Not all states in this sequence need be distinct. Let k the least integer for which $qx^k = q$. Then we readily establish that $qx^m = q$ iff $m=k\ell$, for some integer $\ell \geq 0$. The reverse direction is immediate. In the forward direction, we can always write $m = k\ell + n$ where ℓ, n are integers, $\ell \geq 0$, $0 \leq n \leq k$. Then $q = qx^m = qx^{k\ell+n} = qx^n$, but k is the smallest integer with the property $qx^k = q$, $n = 0$, and hence $m = k\ell$.

Thus

$$qx,\ qx^2, \ldots,\ qx^k = q$$

is a subsequence of an x-cycle which thus has x-period k dividing m. Consider the subsequence of the x-cycle of M_2: $qx,\ qx^2,\ \ldots,\ qx^k = q$. Then the blocks $h^{-1}(qx)$, $h^{-1}(qx^2), \ldots, h^{-1}(q)$ of π_h are all distinct (since $qx,\ qx^2,\ \ldots,\ qx^k$ are all distinct). (π_h is the partition induced by h.)

Let $Z = \tilde{g}(x)$. We note first that for all $i \geq 0$,

$$q' \in h^{-1}(qx^i) \rightarrow q'Z \in h^{-1}(qx^{i+1}).$$

This is so since

$$\begin{aligned} q' \in h^{-1}(qx^i) &\text{ implies } h(q') = qx^i \\ &\text{ implies } h(M_1(q', g(x)) = M_2(qx^i, x) \\ &\text{ implies } q'g(x) \in h^{-1}(qx^{i+1}). \end{aligned}$$

Now let q_0 be a fixed state in $h^{-1}(q)$. From the preceding facts we can construct a sequence

$$q_0Z,\ q_0Z^2,\ \ldots,\ q_0Z^i\ \ldots$$

in M_1, such that for all $j \geq 0$

$$q_0Z^{jk+1} \in h^{-1}(qx),\ q_0Z^{jk+2} \in h^{-1}(qx^2),\ \ldots,$$
$$q_0Z^{jk} \in h^{-1}(qx^k = q).$$

Since $h^{-1}(q)$ is finite, not all q_0Z^{jk} can denote distinct states. Let $n_1 > 0$ be the least integer such that

$$q_0Z^{n_1k} = q_0Z^{xk}$$

for some integer $x > n_1$. Let n_2 be the least such integer x, i.e.,

$$q_0 Z^{n_1 k} = q_0 Z^{n_2 k}.$$

Then

$$q_0 Z^{n_1 k}, q_0 Z^{n_1 k+1}, \ldots, q_0 Z^{n_2 k} = q_0 Z^{n_1 k}$$

is a subsequence of a $\tilde{g}(x) = Z$-cycle in M_1 having $\tilde{g}(x)$-period $(n_2 - n_1)k$, a non-zero multiple of k.

To show that all states in this sequence are distinct (hence establishing the claim) note that

$$q_0 Z^{jk+i} \neq q_0 Z^{j'k+i'}$$

for any $i \neq i'$, $0 \leq i, i' < k$, as these elements belong to distinct blocks of π_h i.e.,

$$q_0 Z^{jk+i} \in h^{-1}(qx^i)$$

and

$$q_0 Z^{j'k+i'} \in h^{-1}(qx^{i'}).$$

Thus set $i = i'$ and $n_1 \leq j < j' < n_2$. If

$$q_0 Z^{jk+i} = q_0 Z^{j'k+i}$$

then

$$q_0 Z^{n_2 1} = q_0 Z^{(j'-j+n_2)k}$$

and hence that

$$q_0 Z^{n_1 k} = q_0 Z^{[n_2-(j-j')]k}.$$

But n_2 is the least integer for which this is true so $j-j' = 0$ and $j = j'$, a contradiction.

Since homomorphism is a special case of division we can state:

Corollary 2:

For finite transition functions, M_1, M_2, if M_2 is a homomorphic image of M_1 then the string period of any string cycle in M_1 is a non-zero multiple of the string period of its homomorphic image. Every string cycle in M_2 is the homomorphic image of a string cycle in M_1.

We apply this result to simple nets by extending a result of Holland [2].

Theorem 3 (Holland)

Let $M: Q \times S \to Q$ be isomorphically realized by a logical net A whose representing digraph D(A) is simple. For every $x \in S^*$ the period of any cycle of x in M divides 2^a l.c.m.$(\ell(x),b)$ where a, b are integers characteristic of A. Equivalently, the x-period of any x-cycle must divide

$$2^a \frac{\text{l.c.m}(\ell(x),b)}{\ell(x)} = 2^a \frac{b}{\text{g.c.d}(\ell(x),b)} .$$

(ℓ.c.m = least common multiple, g.c.d = greatest common divisor.)

Using Corollary 2 we extend this result to homomorphic realization:

Theorem 4: Let $M: Q \times S \to Q$ be homomorphically realized by a logical net A whose representing digraph is simple. For every $x \in S^*$ the x-period of any x-cycle in M must divide $2^a \dfrac{b}{\text{g.c.d}(\ell(x),b)}$.

Proof: Since A is finite, by Corollary 2, given an x-cycle in M there is an x-cycle in the transition function M_A of A. Also the x-period of the x-cycle in M divides the x-period of the x-cycle in M_A which in turn divides $2^a \frac{b}{g.c.d.(lcx),b)}$ by Theorem 3.

Corollary 5

Let $M:Q \times S \to Q$, $|S| \leq 2$, be such that there exists $q \varepsilon Q$ and $s \varepsilon S$ such that for all $x \varepsilon S^*$ $\tilde{M}(q,x) = q$, if, and only if, the number of occurences of s in x is a non-zero multiple of j, a positive integer, (M is a modulo j counter). If M is homomorphically realizable by a logical net whose representing digraph is simple, j is a power of 2.

Proof: Pick x = sy where $y \varepsilon S^*$ contains no occurances of s and $\ell(x)$ is a non-zero multiple of b (in Theorem 3). Then there is an x-cycle in M with x-period j. But by Theorem 3 this x-period must divide

$$2^a \frac{b}{g.c.d(\ell(x),b)} = 2^a ,$$

hence j divides 2^a.

Corollary6: There are transition functions, M which cannot be homomorphically realized by any logical net whose representing digraph is simple.

Proof: The modulo three counter is an example of such a finite transition function.

In sum, we have shown that the least upper bound on the feedback indegree is 2 for nets which can homomorphically realize any transition function. This involved showing that simple nets are not universal in this sense. The question of whether simple nets are universal in the sense that they can simulate (allowing rate slow dow) every transition function is still open (unfortunately, Theorem 1 cannot be applied in this case).

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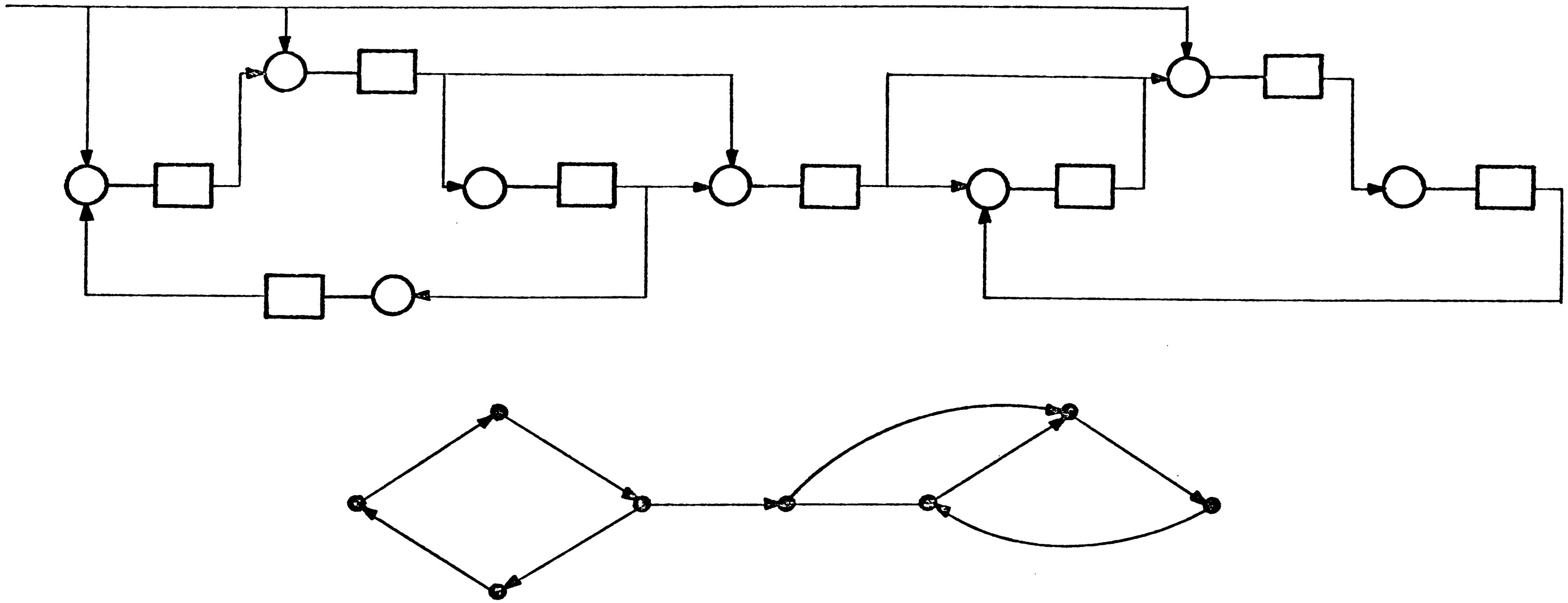

Fig. 1. A Simple Logical Net and Its Representing Digraph.

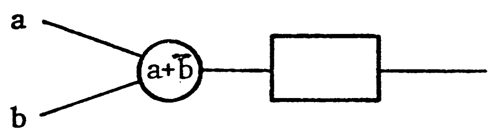

Fig. 2. A Complete Module.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* LOGIC OF COMPUTERS GROUP The University of Michigan Ann Arbor, Michigan	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A LEAST UPPER BOUND ON THE FEEDBACK INDEGREE FOR HOMOMORPHIC REALIZATION OF SEQUENTIAL MACHINES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Bernard Phillip Zeigler

6. REPORT DATE October 1969	7a. TOTAL NO. OF PAGES 14	7b. NO. OF REFS 5

8a. CONTRACT OR GRANT NO. DA-31-124-ARO-D-483 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

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