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Technical Report

MODELING AND SIMULATION: STRUCTURE PRESERVING RELATIONS FOR CONTINUOUS AND DISCRETE TIME SYSTEMS

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ABSTRACT

This paper has established a basis for the formal treatment of modeling and simulation when continuous time system occur as elements of the basic simulation triad (i.e., either the system to be simulated, the model or the computer doing the simulating is a continuous time system). The key idea is that of constructively specifying a system in a way analogous to the use of the one step transition and output functions of the usual sequential machine formulation. This enables one to develop useful criteria for determining when one constructively specified system simulates or models another.

I. INTRODUCTION

Recently, there have been a number of efforts toward a framework in which system models commonly distinguished along a digital-analog axis could be given unified treatment. Mesarovic [1] and Windeknecht [2], Zodeh and Desoer [3] have considered abstract formulations of systems theory. Kalman, Falb and Arbib [4] work with a more structured formulation which specializes more readily into the sequential machine and optimal control formalisms commonly employed. Wymore [5] has taken the first steps in considering the new applications areas made possible by a unified systems theory. These areas relate to systems containing components of both the digital and analog variety in which an adequate understanding of overall system operation can be better gained by treating such components and their interaction in a unified way.

The utility of such a systems approach is most readily apparent in the consideration of hybrid computing systems but there are other important areas of applications such as in the digital simulation of continuous state models and the design of asynchronous processors.

In this paper, I shall be concerned with the formalization of a relation existing between two systems whereby the first might be said to be a "simulation" or "realization" of the second or the second might be said to be a "model" of the first. The basis for such a development has been laid in Zeigler and Weinberg [6] and Zeigler [7]. The former paper showed how the system theoretic concepts of homomorphism and coordinate aggregation could be usefully applied in actual computer simulation of biological systems. Zeigler [7] showed how a simulation could be regarded as a triple *(system to be simulated, model, computer)* in which a preservation relation holds between model and system (determining what properties of the system are to be preserved in the model) at the same time that another preservation holds between model and computer (which governs the manner in which the model is implemented on the computer). The relations in question were formalized as *structure preserving morphisms* (with varying degrees of strength) and shown to be inclusive of existent notions of modeling and simulation. Complexity measures relevant to usage of time and space resources of a simulation were defined and their behavior under structure preserving morphisms studied with a view toward the construction of models with reduced complexity.

The entities in a simulation (simulated system, model, computer) however, were taken to be automata - i.e., essentially discrete time systems, though continuous state spaces were allowed. The theory developed in this way presupposes a discretization of time which would be unnatural when applications to continuous time models and hybrid or analog computers are considered. Thus by extending the notions of "system" and the class of structure preserving morphisms which can relate systems, a more adequate and applicable simulation theory can be developed.

The plan of this paper is as follows:

1) A concept of system is developed which allows both continuous and discrete time operation. This is essentially based on Arbib's definition [8] which I feel to be much less clumsy than Wymore's [5].

2) Behavior and function preserving morphisms are defined and studied for these systems. Behavior and function preserving morphisms are weaker than the structure preserving morphisms of [7] but are necessary prior elements in dealing with these stronger preservation concepts. There is not space in this paper to complete the connection.

3) A notion of *constructive specification* of systems is developed.

This is a generalization of the usual sequential machine *one step transition structure* formulation for discrete time systems. The great advantage afforded by such a specification is that it makes possible a practical procedure for constructing a system which will simulate or model another. This is analogous to the case of sequential machines where judgements concerning the *behavioral* relations of machines (involving extended operation) can be determined by examination of the single step transition and output functions.

Constructive specification of continuous time systems involves difficulties which do not arise in the discrete time case. Much of the paper is devoted to providing reasonable solutions to these problems.

4) Examples are provided which give evidence for the applicability of the theoretical development. It is shown, for example, how Wymore's [5] results on the realization of discrete time systems by continuous time systems and simulation of continuous time systems by discrete time systems are more easily formulated in this framework.

II. SYSTEMS

2.0 System Definition

A *time invariant mathematical system* (hereafter called a system when the context is clear) is a structure

$$S = \langle \Omega, Q, Y, \delta, \lambda \rangle$$

where Ω is a set of *input segments*, Q a nonempty set of *states*, Y a set of *outputs* and

$$\delta : Q \times \Omega \rightarrow Q$$
$$\lambda : Q \times \Omega \rightarrow Y$$

are the state *transition function* and *output function* respectively.

The objects above must satisfy the following axioms:

A.1 Structure of Ω

T denotes the underlying *time* set where T is either the reals R or the integers I.

X denotes the *input value* set.

A.1.1 *(ω is a function on a finite closed interval.)*

$\omega \in \Omega \Rightarrow$ there are $t_0, t_1 \in T$, $t_0 \leq t_1$ and ω is a map $\omega : [t_0, t_1] \rightarrow X$.

A.1.2 *(Closure under Translation)*

$\omega \in \Omega \Rightarrow$ for every $\tau \in T$ there is an $\omega' \in \Omega$ such that

$$\omega' : [t_0+\tau, t_1+\tau] \rightarrow X$$

and

$$\omega'(t) = \omega(t-\tau)$$

for all $t \in [t_0+\tau, t_1+\tau]$.

ω' is said to be a *translate* of ω.

Remark: From here on we do not distinguish between an input segment and any of its translates. More formally we define the equivalence relation $\equiv$ on Ω where

$$\omega \equiv \omega' \Leftrightarrow \omega' \text{ is a translate of } \omega.$$

An equivalence class $[\omega]$ will be represented by any one of its members as is appropriate. For example, if we are interested in segments starting at some time t_0 and write ω we mean that segment in $[\omega]$ defined on an interval $[t_0, -]$.

A.1.3 (Closure under concatenation)

$\omega \varepsilon \Omega$ and $\omega' \varepsilon \Omega$ => there is a segment $\omega\omega' \varepsilon \Omega$, where if

$$\omega:[t_0,t_1] \to X$$

and

$$\omega':[t_1,t_2] \to X$$

then

$$\omega\omega':[t_0,t_2] \to X$$

and is given by

$$\omega\omega'(t) = \omega(t) \text{ for } t \in [t_0,t_1]$$
$$= \omega'(t) \text{ for } t \in [t_1,t_2]$$

and T = R. For T = I the usual concatenation operation is assumed.

Remark: With this condition Ω becomes a semigroup since as can readily be verified concatenation is associative.

A.2 Time Invariance

For all $\omega,\omega' \varepsilon \Omega$, $q \varepsilon Q$

$\omega \equiv \omega'$ => $\delta(q,\omega) = \delta(q,\omega')$ and $\lambda(q,\omega) = \lambda(q,\omega')$.

Remark: This means that $\delta(q,\omega)$ and $\lambda(q,\omega)$ are uniquely determined once the state q and any representative ω of the equivalence class $[\omega]$ are given. As a consequence we adopt the same convention as in A.1.2, i.e., we let the context determine the particular translate of ω which is of interest.

This convention makes possible an enormous gain in notational manipulation; c.f., Wymore [5] and Kalman, et al. [4] who do not make use of this convention. Of course, it relies on a time invariant system formulation, but this is always possible to achieve by considering T as a state space component.

A.3 Composition or Semigroup Property

For all $\omega,\omega' \varepsilon \Omega$, $q \varepsilon Q$

$$\delta(q,\omega\omega') = \delta(\delta(q,\omega),\omega')$$

and

$$\lambda(q,\omega\omega') = \lambda(\delta(q,\omega),\omega').$$

With $\omega \in \Omega$ represented by $\omega:[0,\tau] \to X$ we associate two segments $\omega_{t]}$ and $\omega_{[t}$ for every $t \in [0,\tau]$.

The *left segment* $\omega_{t]} = \omega|_{[0,t]}$, i.e., the function ω restricted to the subdomain $[0,t]$.

The *right segment* $\omega_{[t} = \omega|_{[t,\tau]}$.

A system $S = \langle\Omega,Q,Y,\delta,\lambda\rangle$ is called *input decomposable* if in addition to A.1-A.3 it satisfies:

A.4 Restriction Closure

$\omega \in \Omega$ and $\omega:[0,\tau] \to X \Rightarrow \omega|_{[t,t']} \in \Omega$ for all $0 \le t \le t' < \tau$.

A.5 Consistency

For all $q \in Q$, $\delta(q,\Lambda) = q$, where Λ is any translate of $\omega_{[0,0]}$, the zero length input segment.

2.1 Time Discrete Systems

A *discrete time sequential system* is a structure $M = \langle X,Q,Y,\delta_M,\lambda_M\rangle$ where X,Q,Y are nonempty sets of *inputs, states* and *outputs* respectively and $\delta_M:Q \times X \to Q$, $\lambda:Q \to Y$ are the *one step transition* and *output functions* respectively.

This is the usual Moore Machine except that no finiteness restriction is placed on X,Q,Y.

We may associate with a discrete time system M a general system S_M as follows:

$$S_M = <\Omega, Q, Y, \tilde{\delta}, \tilde{\lambda}>$$

where Ω is the translation closure of X^+,

$$\tilde{\delta}: Q \times X^+ \rightarrow Q \text{ and } \tilde{\lambda}: Q \times X^+ \rightarrow Y$$

are the *extended* transition and output function respectively defined by

$$\tilde{\delta}(q,s) = \delta(q,s) \text{ for } s \in X$$

$$\tilde{\delta}(q,xs) = \delta(\tilde{\delta}(q,x),s) \text{ for } x \in X^+,$$

and

$$\tilde{\lambda}(q,x) = \lambda_M(\tilde{\delta}(q,x)).$$

Here $X^+ = \{\omega|\omega:\{1,2,\ldots,n\} \rightarrow X, n=1,2,\ldots\}$, i.e., the set of all finite sequences of elements of X. The underlying time set has been assumed to be the integers I. Note that by the translation closure assumption each sequence $x_1x_2\ldots x_n \in X^+$ represents any of its possible translates in Ω.

The reader may verify that S_M satisfies the axioms for a time invariant system.

2.2 Time Continuous Systems

Before exemplifying a class of continuous time systems we need the following concepts:

A system $S = (\Omega, Q, Y, \delta, \lambda)$ defines *trajectories* through the state and output sets as follows:

With each $q \in Q$, $\omega \in \Omega$ there are associated segments $STRAJ_{(q,\omega)}$, $OTRAJ_{(q,\omega)}$. Choosing the representative $\omega: [0,\tau] \to X$ we have

$$STRAJ_{(q,\omega)}: [0,\tau] \to Q$$

given by $$STRAJ_{(q,\omega)}(t) = \delta(q, \omega_{t]}) \text{ for } t \in [0,\tau];$$

$STRAJ_{(q,\omega)}$ is *state trajectory* associated with initial state q and input segment ω.

Similarly $$OTRAJ_{(q,\omega)}: [0,\tau] \to Y$$

is the *output trajectory* associated with initial state q and input segment ω,

where $$OTRAJ_{(q,\omega)}(t) = \lambda(q, \omega_{t]}) \text{ for } t \in [0,\tau].$$

Remark: It is easy to see that in the same way that ω represents its translates in $[\omega]$, $STRAJ_{(q,\omega)}$ and $OTRAJ_{(q,\omega)}$ also represents translation equivalence classes.

A *Differential Equation Specified System, DESS* is a structure

$$D = (\Omega, Q, Y, f, N)$$

where Ω is a set of input segments satisfying A.1 with $T = R$ and $X \subseteq R^m$ for some $m \geq 1$

$Q \subseteq R^n$ for some $n \geq 1$

$Y \subseteq R^p$ for some $p \geq 1$

and f,N are maps

$f: Q \times X \to Q$

$N: Q \to Y$

Remark: Often D is presented in the form

$$\frac{dq}{dt} = f(q,x)$$

$$y(t) = N(q(t)).$$

A time segment $\Phi:[0,\tau] \to Q$ is a *state trajectory* of D if there is an input segment $\omega \in \Omega$ and a (starting) state $q \in Q$ such that

$$\text{i)} \quad \Phi(0) = q$$

and

$$\text{ii)} \quad \frac{d\Phi}{dt}(t) = f(\Phi(t), \omega(t))$$

for all $t \in [0,\tau]$.

A segment $\Psi:[0,\tau] \to Y$ is an *output trajectory* of D if there is a state trajectory $\Phi:[0,\tau] \to Q$ such that

$$\Psi(t) = N(\Phi(t)) \text{ for } t \in [0,\tau].$$

A DESS will be said to have *unique solutions* of given any $q \in Q$, $\omega \in \Omega$ there is a *unique* state trajectory satisfying i) and ii), call it $\Phi_{(q,\omega)}$. Uniqueness of $\Phi_{(q,\omega)}$ clearly implies uniqueness of output trajectory $\Psi_{(q,\omega)}$.

The following is well known from other approaches:

2.2 Theorem 1:

Given a DESS with unique solutions $D = (\Omega,Q,Y,f,N)$ we may associate with D an input decomposable system S_D as follows:

$$S_D = (\Omega,Q,Y,\delta,\lambda)$$

where for $q \in Q$, $\omega \in \Omega$, $t \in T$

$$\delta(q,\omega_{t]}) = \Phi_{(q,\omega)}(t)$$

$$\lambda(q,\omega_{t]}) = \Psi_{(q,\omega)}(t).$$

Moreover S_D represents D in the sense that it generates exactly the same set of trajectories, i.e., for all $q \in Q$, $\omega \in \Omega$

$$STRAJ_{(q,\omega)} = \Phi_{(q,\omega)}$$

and

$$OTRAJ_{(q,\omega)} = \Psi_{(q,\omega)}.$$

An interesting subclass of the differential equation systems are the linear (time invariant) systems. A DESS $D = (\Omega,Q,Y,f,N)$ is *linear* if X,Q,Y are vector spaces and f,N are linear transformations given by

$$f(q,x) = Aq + Bx$$

$$N(q) = Cq$$

The underlying differential equation takes the usual form

$$\overset{\circ}{q} = Aq + Bx.$$

The corresponding system $S_D = (\Omega,Q,Y,\delta,\lambda)$ has

$$\delta(q,\omega_{t]}) = e^{At}q + \int_0^t e^{A(t-t')}B\omega(t')dt'$$

and

$$\lambda(q,\omega_{t]}) = C\delta(q,\omega_{t]}).$$

III. MODELING AND SIMULATION RELATIONS

3.0 The Essential Difference Between Discrete and Continuous Time Systems

It is important to note the essential differences between the two general classes of systems just mentioned.

In the sequential machine case, the transition function $\tilde{\delta}$ is determined by extending the given one-step transition function δ. Thus given an input string $s_1 s_2 \ldots s_n$ and a state q we can compute the trajectory $q_1, q_2, \ldots, q_n$ where $q_1 = q$ and for $i = 1,2,\ldots,n-1$, $q_{i+1} = \delta(q_i, s_i)$.

This step by step iteration is precisely the idea underlying digital computer simulation of systems.

More explicitly, given two sequential machine derived systems S_M and $S_{M'}$ there are a number of criteria available for determining whether S_M simulates $S_{M'}$ or whether $S_{M'}$ is a model of S_M. This is done essentially by examining the one-step transition and output functions (δ, λ) given in by the M,M' specifications.

In the continuous time case very few such criteria are unknown. We are reduced to having to compare the total system descriptions S and S' to determine whether S can simulate S'.

To see the import of this, consider the case where we have two DESS's D and D' and wish to determine whether S_D can simulate $S_{D'}$. Since D,D' are known we can program an analog computer to simulate S_D and $S_{D'}$ just as knowing M and M' we can program a digital computer to simulate each. In contrast to the discrete time case however, it is *generally unknown how to use the structure of these programs (or equivalently the functions f,N) to determine whether* S_D *can simulate* $S_{D'}$. *This must be done by an* *exhaustive generation and comparison of their trajectories.* (The only exception to this I know of, concerns linear systems where one can determine behavioral equivalence from the similarity of the matrices A,B,C.)

The following sections present an approach to this problem within the system theoretic framework given above. The essential idea is consider continuous time systems which have enough of the properties of discrete time systems to enable the application structure preserving criteria already known in the latter area.

3.1 Basic Preservation Relations

We begin by reviewing and generalizing known behavior and function preserving relations for systems.

The input-output behavior of a system $S = \langle \Omega, Q, Y, \delta, \lambda \rangle$ is the collection of functions $\{\beta_q | q \varepsilon Q\}$ where for $q \varepsilon Q$,

$$\beta_q : \Omega \to Y$$

and is given by

$$\beta_q(\omega) = \lambda(q,\omega)$$

for all $\omega \ \varepsilon \ \Omega$.

A *behavior preserving morphism* from S to S' is a triple (h_1, h_2, h_3) where

$$h_1 : \Omega' \to \Omega$$
$$h_2 : Q' \to Q$$
$$h_3 : Y \to Y'$$

such that for all $q \varepsilon Q'$,

$$\beta'_{q'} = h_3 \circ \beta_{h_2(q')} \circ h_1.$$

Remark: More expansively, let us say that $\beta'_{q'} | \beta_q$ ($\beta'_{q'}$ *divides* β_q) using an input encoding map h_1 and an output [illegible]ng map h_3 if $\beta'_{q'} = h_3 \circ \beta_q \circ h_1$. Then the existence of a behavior morphism (h_1, h_2, h_3) means that for each state $q'\varepsilon Q'$ there is a state $q\varepsilon Q$ (namely $h_2(q')$) such that $\beta'_{q'}|\beta_q$ using (h_1,h_2).

A *function preserving morphism* (homomorphism) from S to S' is a triple (g,h,k) where

$$g:\Omega' \to \Omega$$

$$h:Q_1 \to Q' \quad \text{(onto)}$$

$$k:Y \to Y'$$

where i) $Q_1 \subseteq Q$ is closed under $g(\Omega')$:

$$\omega'\varepsilon\Omega', q\varepsilon Q_1 \Rightarrow \delta(q,g(\omega'))\varepsilon Q_1$$

ii) for all $q\varepsilon Q_1, \omega'\varepsilon\Omega'$

$$h(\delta(q,g(\omega'))) = \delta'(h(q),\omega')$$

and

$$\beta'_{h(q)} = k\circ\beta_q\circ g$$

A behavior morphism will be said to *time local* if $h_1:\Omega' \to \Omega$ is a semigroup homomorphism, i.e., for all $\omega_1,\omega_2\varepsilon\Omega'$, $h_1(\omega_1\omega_2) = h_1(\omega_1)h_1(\omega_2)$. A function morphism is similarly designated if $g:\Omega' \to \Omega$ is a semigroup homomorphism.

Remark: Time local morphisms are desirable since they enable input segments to be encoded by concatenating encoded subsegments. Their existence comes much easier for discrete time systems than for continuous time systems, as we shall see.

3.11 Theorem: The existence of a function morphism from S to S' implies the existence of a behavior morphism from S to S'.

Proof: Let (g,h,k) be a function morphism from S to S'. Construct a behavior morphism (h_1,h_2,h_3) as follows:

Set

$$h_1 = g$$

$$h_3 = k,$$

and define $h_2:Q' \to Q$ by $h_2(q')$ is a designated representative of the inverse image $h^{-1}(q') = \{q|h(q) = q'\}$.

Then $h(h_2(q')) = q'$ implies

$$\beta'_{q'} = k \circ \beta_{h_2(q')} \circ g \quad \text{[ii) of function morphism condition]}$$
$$= h_3 \circ \beta_{h_2(q')} \circ h_1.$$

A system $S = \langle \Omega, Q, Y, \delta, \lambda \rangle$ is *reduced* if for all $q_1, q_2 \varepsilon Q$, $\beta_{q_1} = \beta_{q_2} \Rightarrow q_1 = q_2$.

3.1.2 Theorem: The existence of a time local behavior morphism from S to S' implies the existence of a time local function morphism from S to S' if S' is reduced.

Proof: Let (h_1, h_2, h_3) be a time local behavior morphism from S to S'. Construct a function morphism (g,h,k) as follows:

Set
$$g = h_1$$
$$k = h_3$$

and define $h: Q_1 \to Q'$ by

a) $Q_1 = \{q \varepsilon Q \mid \text{there is a } q' \varepsilon Q' \text{ such that } \beta'_{q'} | \beta_q \text{ using } h_1, h_3\}$

b) For $q \varepsilon Q_1$, $h(q) = q' \Leftrightarrow \beta'_{q'} | \beta_q$.

Lemma: $\beta'_{q'} | \beta_q \Rightarrow \beta'_{\delta'(q',\omega')} | \beta_{\delta(q, g(\omega'))}$.

Proof: For all $\omega_1, \omega_2, \varepsilon \Omega$

$$\beta'_{q'} | \beta_q \Rightarrow \beta'_{q'}(\omega_1 \omega_2) = h_3 \circ \beta_q \circ h_1(\omega_1 \omega_2)$$
$$\Rightarrow \beta'_{q'}(\omega_1 \omega_2) = h_3 \circ \beta_q (h_1(\omega_1) h_1(\omega_2)) \quad [h_1 \text{ is a homomorphism}]$$
$$\Rightarrow \lambda'(\delta'(q', \omega_1 \omega_2) = h_3 \circ \lambda(q, h_1(\omega_1) h_1(\omega_2))$$
$$\Rightarrow \lambda'(\delta'(q', \omega_1 \omega_2) = h_3 \circ \lambda(\delta(q, h_1(\omega_1)), h_1(\omega_2)) \quad \text{[composition property]}$$
$$\Rightarrow \beta'_{\delta'(q',\omega_1)}(\omega_2) = h_3 \circ \beta_{\delta(q, h_1(\omega_1))} \circ h_1(\omega_2)$$
$$\Rightarrow \beta'_{\delta'(q',\omega_1)} | \beta_{\delta(q, h_1(\omega_1))}.$$

Using the Lemma we see readily that Q_1 is closed under $g(\Omega')$ as required.

To show that h is well defined by b), note that if $\beta_{q'} | \beta_q$ and $\beta_{q''} | \beta_q$ we have $\beta_{q'} = h_3 \circ \beta_q \circ h_1 = \beta_{q''}$. Since S' is reduced $q' = q''$ as is required.

Next we show that (g,h,k) as defined above have the requisite commutative properties. From definition b) of h, $h(q) = q' \Rightarrow \beta'_{q'} | \beta_q \Rightarrow \beta'_{h(q)} | \beta_q$ as required.

Also by the Lemma,

$$\beta'_{\delta'(q',\omega')} | \beta_{\delta(q,g(\omega')}$$

So from definition b) of h

$$h(\delta(q,g(\omega'))) = \delta'(q',\omega')$$

$$= \delta'(h(q),\omega') \qquad \text{as required.}$$

Remark: That the homomorphisms of topological dynamics (e.g., Ura [9]) are special cases of the behavior morphism can be seen as follows:

Let S be an autonomous input decomposable continuous time system, i.e., X is a singleton set. Since Ω now consists of all constant value segments, these segments are uniquely identified by the length of their associated interval. Thus we may set $\Omega = R$; $\omega + \omega'$ then represents the segment $\omega\omega'$. $\delta: Q \times \Omega \to Q$ now satisfies the composition law: $\delta(q,\omega + \omega') = \delta(\delta(q,\omega),\omega')$ and the consistency law $\delta(q,0) = q$. Given two such systems, for time local function morphism (g,h,k) to exist, g must be a homomorphism of the real additive group R into itself. This result is the usual homomorphism of topological dynamics except that the topological structure of the state space may also be preserved.

3.2 Constructive Nature of Discrete Simulation

For time discrete systems the ability to go back and forth between transition function descriptions and behavior descriptions can be significantly strengthened.

Consider time discrete systems $M = \langle X,Q,Y,\delta,\lambda \rangle$. A *function preserving morphism* from M to M' is a triple (g,h,k) where

$$g: X' \to X^+$$

$$h: Q_1 \to Q' \quad \text{(onto)}$$

$$k: Y \to Y'$$

where $Q_1 \subseteq Q$ is closed under $g(X')$ and for all $s \in X', q \in Q_1$, (g,h,k) satisfy

$$h(\tilde{\delta}(q,g(s))) = \delta'(h(q),s))$$

and

$$k(\lambda(q)) = \lambda'(h(q)).$$

3.2.1 Theorem: Let $S_M, S_{M'}$ be systems derived from time discrete systems M and M'.

There is a function preserving morphism from M to M' if, and only if, there is a time local function morphism from S_M to $S_{M'}$.

Proof: => Let (g_M, h_M, k_M) be a function morphism from M to M'. We construct a function morphism (g,h,k) from S_M to $S_{M'}$ as follows:

Set
$$h = h_M$$
$$k = k_M$$

and define $g: (X')^+ \to X^+$

by

$$g(s_1' s_2' \ldots s_n') = g_M(s_1') g_M(s_2') \ldots g_M(s_n')$$

for all strings $s_1' s_2' \ldots s_n'$ in $(X')^+$. g is the extension to homomorphism of semigroups of the function g_M defined on the generators. Since $(X')^+$ is a free semigroup this extension is well-defined [$s_1' s_2' \ldots s_n' = s_1'' s_2'' \ldots s_n''$ implies n=m and $s_i' = s_i''$ for all i=1,2,...n].

Since g is a homomorphism as required (g,h,k) will be a time local function morphism if

$$h(\tilde{\delta}(q, g(x'))) = \tilde{\delta}'(h(q), x')$$

and

$$\beta'_{h(q)} | \beta_q \text{ using } g,k.$$

for all $q \varepsilon Q_1 . x' \varepsilon (X')^+$.

Using induction on $\ell(x')$ (the length of x') we can readily show that $h(\delta(q, g_M(s))) = \delta'(h(q), s)$ for all $s \varepsilon X'$, $q \varepsilon Q_1$ implies

$$h(\tilde{\delta}(q, g(x')) = \tilde{\delta}'(h(q), x')$$

for all $x' \varepsilon (X')^+, q \varepsilon Q_1$ as required.

Also the closure of Q_1 with respect to g(X') readily implies the closure of Q_1 with respect to $g(X')^+$.

Finally, for all $x' \varepsilon (X')^+, q \varepsilon Q_1$,

$$
\begin{aligned}
k \circ \beta_q \circ g(x') &= k(\tilde{\lambda}(q, g(x'))) && \text{[definition of } \beta_q\text{]} \\
&= k(\lambda(\tilde{\delta}(q, g(x')))) && \text{[definition of } \tilde{\lambda}\text{]} \\
&= \lambda'(h(\tilde{\delta}(q, g(x')))) && \text{[definition of function morphism]} \\
&= \lambda'(\tilde{\delta}'(h(q), x')) && \text{[definition of function morphism]} \\
&= \tilde{\lambda}'(h(q), x') && \text{[definition of } \tilde{\lambda}'\text{]} \\
&= \beta'_{h(q)}(x') && \text{as required.}
\end{aligned}
$$

<= Let (g,h,k) be a function morphism from S_M to $S_{M'}$. We construct a function morphism (g_M, h_M, k_M) from M to M' as follows:

Set

$$h = h_M$$

$$k = k_M$$

and let $g_M : X' \to X^+$ be the restriction of $g:(X')^+ \to X^+$ to X', i.e.,

$$g(s') = g_M(s') \text{ for all } s' \varepsilon X'.$$

The requisite properties now follow immediately.

IV. CONSTRUCTIVE SPECIFICATION OF CONTINUOUS SYSTEMS

4.0 Introduction

We see from the Theorems of Section 3 that the ability to determine whether one system simulates or models another by examining their one step transition structures depends crucially upon the existence of certain properties of the input encoding map $g:\Omega' \to \Omega$. Namely - 1) g is a semigroup homomorphism and 2) Ω' is generated by a smaller subset such that g need only be defined on this subset. In the case of discrete time systems these conditions are readily satisfiable. However, corresponding constructs for continuous time systems have been lacking. Thus the task now is to give a constructive specification for classes of continuous time systems analogous to that available for discrete time systems.

From here on we restrict attention to state transition system $S = \langle\Omega,Q,\delta\rangle$ *since we shall be interested in function rather than behavior morphisms as appropriate for modeling and simulation.*

4.1 Maximal Length Segmentation

Analogous to the case of discrete time systems we seek a generating set Ω_G for Ω with certain useful properties.

Given a subset $T \subseteq \Omega$ where $\Lambda \notin T$ we designate by T^+ the least set of segments containing T and closed under concatenation. It is easy to see that $T^+ = \bigcup_{i=1}^{\infty} T^i$ where $T^i = \{\omega_1\omega_2\ldots\omega_i \mid \omega_j \varepsilon T, j=1,2,\ldots i\}$.

T *generates* Ω (T is a generating set for Ω) if $T^+ = \Omega$. Members of T are called generators or generating elements.

We shall say that $\omega_1,\omega_2,\ldots,\omega_n$ is a *decomposition* of $\omega \varepsilon \Omega$ by T if $\omega = \omega_1\omega_2\ldots\omega_n$.

A set Ω_G is an *admissible* set of generators if for all $\omega \in \Omega_G^+$, if $t^* = \max \{t | \omega_{t]} \in \Omega_G\}$ then $\omega_{[t^*} \in \Omega_G^+$.

Explanation: Refer to Figure 4.1.1 where $\omega_{t^*]}$ is the longest generator which is also a left segment of ω. The condition for admissibility is that the remaining right segment $\omega_{[t^*}$ must be generated by Ω_G.

Examples of admissible sets are given in Section 4.2.

Suppose now that $\Omega_G^+ = \Omega$. This means that for every element in Ω can be obtained by concatenating a finite number of generators. However, there may be more than one way to do this. That is, in contrast to the discrete time case Ω_G^+ is not necessarily a free semigroup. To remedy this situation we do not try to force uniqueness of compositions but seek conditions guaranteeing a unique *cannonical* decomposition of elements of Ω in terms of generators Ω_G. The requirement of admissibility is just such a condition as we now show:

A decomposition $\omega_1, \omega_2, \ldots, \omega_n$ of ω by Ω_G is said to be a *maximal length segment (m.ℓ.s.)* decomposition if for each $i \in \{1,2,\ldots n\}$ whenever there is an $\omega' \in \Omega_G$ such that ω' is a left segment of $\omega_i \omega_{i+1} \cdots \omega_n$ then ω' is a left segment of ω_i.

In other words for each i, ω_i is the longest generator which is also a left segment of $\omega_i \omega_{i+1} \cdots \omega_n$.

4.1.1 Assertion: If ω has a m.ℓ.s. decomposition by Ω_G it is unique.

Proof: Let $\omega_1, \omega_2, \ldots, \omega_n$ and $\omega_1', \omega_2', \ldots \omega_n'$ be m.ℓ.s. decompositions of ω. Then ω_1 is a left segment of ω_1' (since ω_1' is the longest generator which is also a left segment of ω) and vice versa ω_1' is a left segment of ω_1. So $\omega_1 = \omega_1'$. This establishes the basis for an induction hypothesis: $\omega_i = \omega_i'$ for $i \leq j \Rightarrow \omega_{j+1} = \omega_{j+1}'$, the proof of which is straightforward.

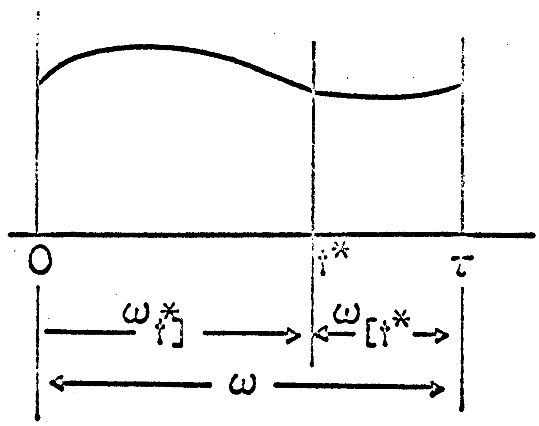

Figure 4.1.1 Maximal Length Segmentation.

<u>4.1.2 Theorem</u>: If Ω_G is an admissible set of generators for Ω then every $\omega \in \Omega$ has a unique m.ℓ.s. decomposition by Ω_G.

<u>Proof</u>: Refer to Figure 4.1.2.

Let $\omega \in \Omega$. Since Ω_G generates Ω, $\omega = \omega_1\omega_2 \ldots \omega_n$ for some $\omega_i \in \Omega_G$, $i=1,2,\ldots n$.

We wish to resegment ω so as to obtain a m.ℓ.s. decomposition $\omega_1',\omega_2',\ldots,\omega_m'$. Let us call the points $t_0,t_1,t_2,\ldots,t_n$ the *break points* of the decomposition $\omega_1,\omega_2,\ldots,\omega_n$; i.e., $\omega_1 = \omega_{t_1]}, \omega_2 = \omega|_{[t_1,t_2]}$, etc.

Let t_1' be the greatest value of t such that $\omega_{t]} \in \Omega_G$. (t_1' exists since all segments are defined on *closed* intervals.)

Then t_1' is the first break point of the m.ℓ.s. decomposition and $\omega_1' = \omega_{t_1']}$. Since Ω_G is admissible, $\omega_{[t_1'} \in \Omega_G^+$.

Note that $t_1' \geq t_1 > 0$ since $\omega_{t_1]} = \omega_1$ and $\omega_1 \neq \Lambda$.

Let k be the integer such that $t_k \leq t_1' < t_{k+1}$. (k=2 in Figure 4.1.2.)

Now $\omega_{[t_1'}$ is a proper right segment of ω and we proceed to find the break point t_2' and a maximal length segment ω_2' of $\omega_{[t_1'}$. Continuing in this way we generate a sequence $t_1',t_2',\ldots$ which must terminate since if $t_\ell \leq t_2' \leq t_{\ell+1}$ then $t_\ell > t_k$, etc.

Thus, there is an integer $m \leq n$ such that $\omega_1',\omega_2',\ldots,\omega_m'$ is a m.ℓ.s. decomposition of ω. By the Assertion 4.1.1, it is unique.

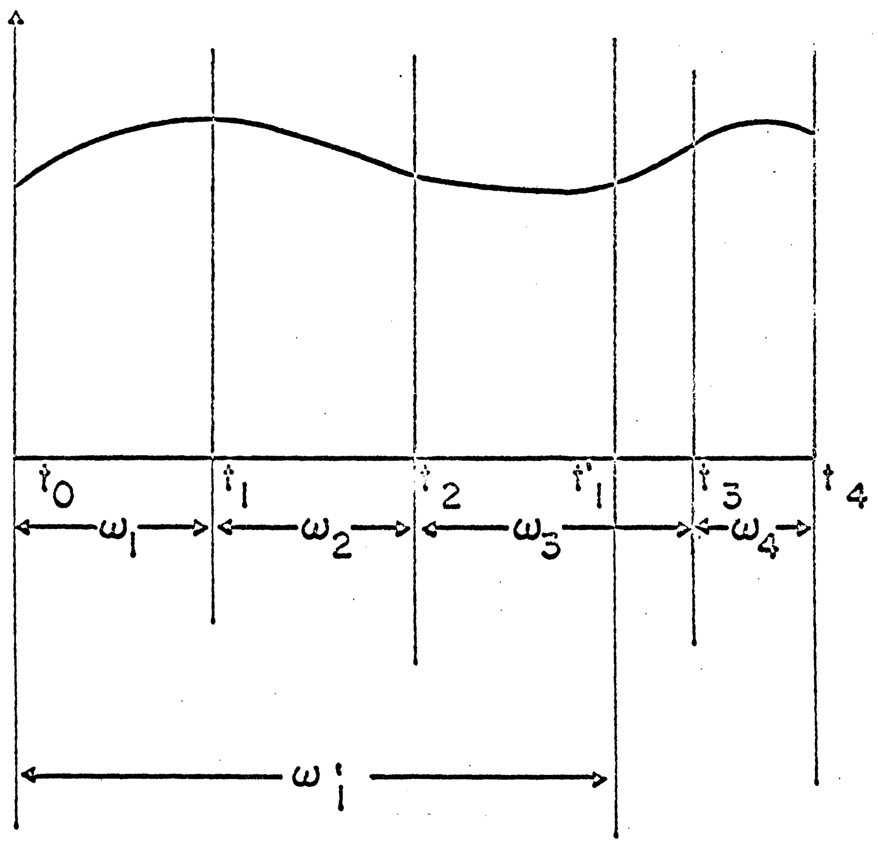

Figure 4.1.2 Initial segment of m.ℓ.s. decomposition.

4.2 Admissible Generators

There are more transparent but also more stringent conditions that guarantee admissibility.

4.2.1 Proposition: Consider the following conditions:

') *(Prefix property)*

$\omega_1 \varepsilon \Omega_G$ and $\omega_1\omega_2 \varepsilon \Omega_G \Rightarrow \omega_2 = \Lambda$ (no generator is a proper left segment of another generator).

") No generator is a proper left *or* right segment of any ther.

"') Ω_G is closed under right segmentation.

"") Ω_G is closed under *both* left and right segmentation.

Then ") => ') => Ω_G is admissible and "") => "') => Ω_G is admissible.

Proof: Left as an exercise for the reader.

4.2.1 Corollary: For discrete time systems $M = \langle X,Q,\delta\rangle$ the input symbols S are admissible generators.

Proof: $\{\omega|\omega:\{1\}\to X\}$ satisfies Proposition 4.2.1'.

4.2.2 Corollary: If Proposition 4.2.1' (the prefix property) holds then if $\omega_1,\omega_2,\ldots,\omega_n$ is a decomposition of ω by Ω_G it is the m.ℓ.s. decomposition.

We now give some examples of admissible sets.

Example 1: *Step Function Inputs*

Let a_t denote a *constant real valued function on a segment of length t,* i.e.,

$$a_t:[0,t] \to R$$

where $a_t(t') = a$ for all $t' \varepsilon [0,t]$.

Then $\Omega_G = \{a_t|t\varepsilon R, a\varepsilon R\}$ is an admissible set of generators and generates the set

$$\Omega_G^+ = \{a_{t_1} a'_{t_2-t_1} a''_{t_3-t_1} \ldots a^{(n)}_{t_n-t_{n-1}} | n = 1,2,3\ldots\}$$

Note that the segments a_t can have arbitrary length hence that essential use is being made of the continuous time base.

That Ω_G is admissible follows from Proposition 4.2.1".

Figure 4.2.1 shows three distinct decompositions of the same segment. The decomposition in a) is the m.ℓ.s. decomposition. Note that there are infinitely many decompositions but only one m.ℓ.s. decomposition. For this case of step functions the break points of the m.ℓ.s. decomposition are just the points at which step changes occur.

Example 2: *Pulse inputs*

The set $\Omega_G = \{a_{t_1} 0_{t_2-t_1} | a \neq 0, t_1 \neq 0\}$ is admissible. This can be determined directly from the definition and does not follow from Proposition 4.2.1. See Figure 4.2.2.

Counterexample 3:

The set $\Omega_G = \{a_{t_1} 0_{t_2-t_1} | a \neq 0, t_1 \neq 0\} \cup \{a_{t_1} 0_{t_2-t_1} b_{t_3-t_2} | a,b \neq 0\}$ is not admissible since $a_{t_1} 0_{t_2-t_1} b_{t_3-t_2}$ is the m.ℓ.s. of $a_{t_1} 0_{t_2-t_1} b_{t_3-t_2} 0_{t_4-t_3}$ but $0_{t_4-t_3} \notin \Omega_G$. See Figure 4.2.3.

Example 3: *Fixed Interval Inputs*

Let τ be any interval and let $\Omega_G \subseteq R^{\tau}$; i.e., Ω_G is any subset of the set of all functions with domain τ.

Then Ω_G is an admissible generating set since Proposition 4.2.1' is satisfied.

By Corollary 4.2.2, every decomposition is an m.ℓ.s. decomposition.

A well known special case is where Ω_G consists of a single segment ω; then Ω_G^+ consists of finite length periodic functions. See Figure 4.2.4.

Example 4: *Piecewise Continuous Functions*

Let $\Omega_G = \{\omega | \omega : [0,\tau] \rightarrow R$ is a continuous function$\}$. Then $\Omega_G^+ = \{\omega | \omega$ is a piecewise continuous function with a finite number of discontinuities$\}$.

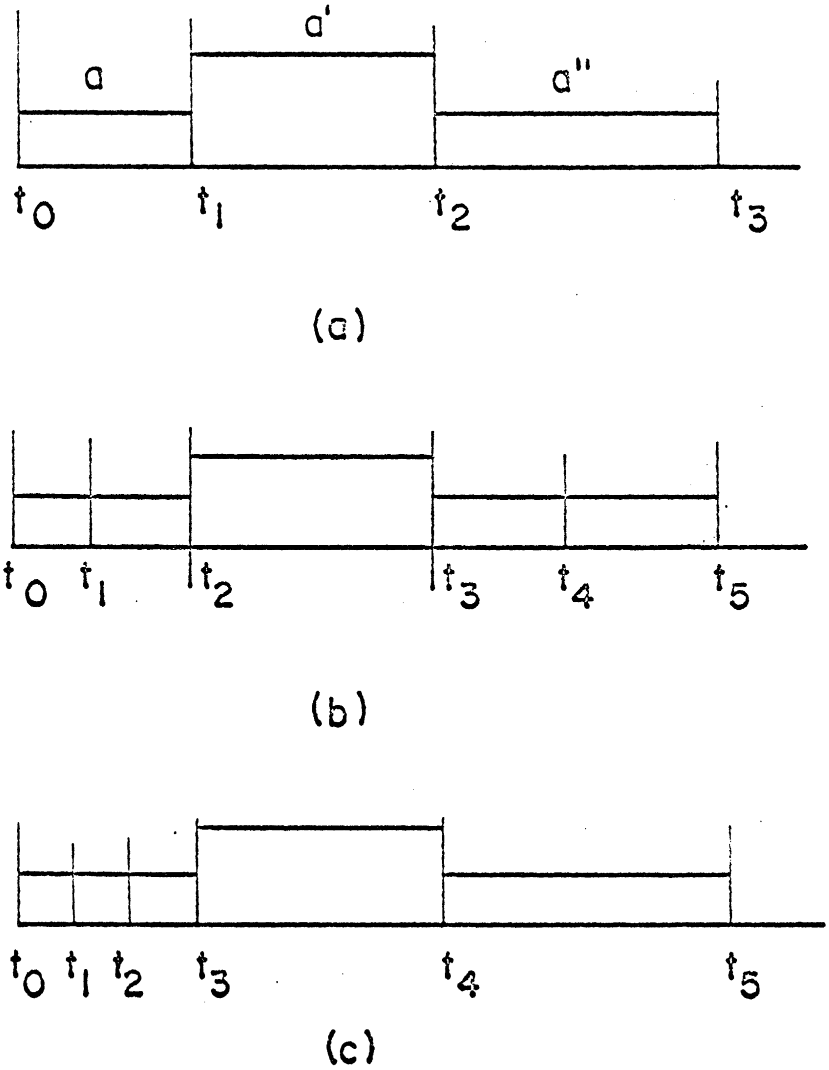

Figure 4.2.1 Three distinct decompositions of the same segment. The m.ℓ.s. decomposition is shown in a).

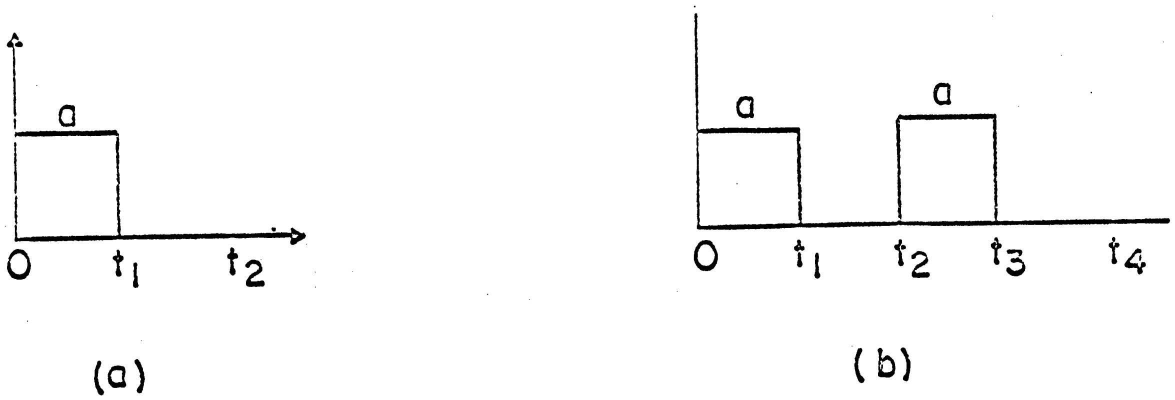

Figure 4.2.2 Generator shown in a) and generated segment shown in b).

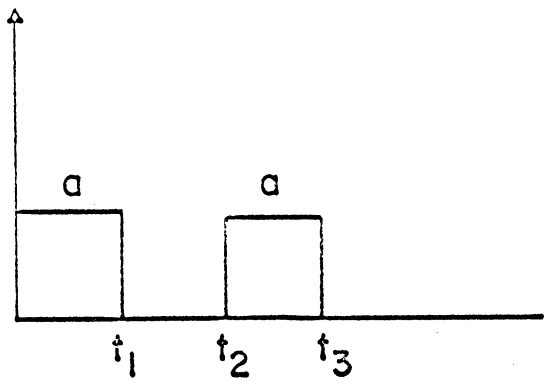

Figure 4.2.3 Adding generator shown in 4.2.3 destroys admissibility of generating set.

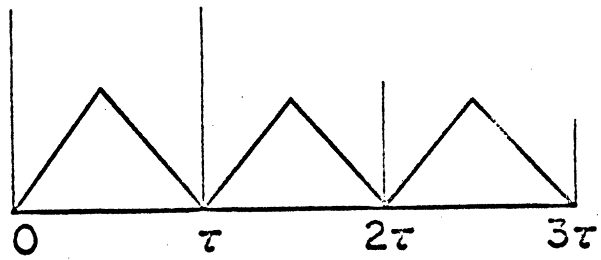

Figure 4.2.4 Periodic input segments.

That Ω_G is admissible follows from Proposition 4.2.1'''.

Note Example 1 is a special case of Example 4. Many other subsets of the continuous functions lead to admissible generators.

4.3 Some Properties of Admissible Sets

We shall need the following results later on.

For any decomposition $\omega_1,\omega_2,\ldots,\omega_n$ of ω by Ω_G let $\#(\omega) = n$, the number of generators used.

Let *size*$(\omega) = n$, where n is the number of generators used in the *m.ℓ.s. decomposition* of ω by Ω_G.

4.3.1 Assertion: size$(\omega) = \min\{\#(\omega)\}$ where the minimization is over all decompositions of ω by Ω_G.

Proof: Follows from the proof of Theorem 4.1.2.

The following is a consequence of the fact that segmentation by maximal length sequences is a "causal" operation.

4.3.2 Lemma: Let Ω_G be an admissible set.

a) If $\omega'_1,\omega'_2,\ldots,\omega'_n$ is a m.ℓ.s. decomposition for $\omega' \in \Omega_G^+$ then for any $\omega'' \in \Omega_G^+$, $\omega'_1,\omega'_2,\ldots,\omega'_{n-1}$ is the first part of a m.ℓ.s. decomposition for $\omega'\omega''$.

b) Conversely, let $\omega_1,\omega_2,\ldots,\omega_n$ be a m.ℓ.s. decomposition for $\omega \in \Omega_G^+$. For any left segment $\omega_{t]} \in \Omega_G^+$ the m.ℓ.s. decomposition of $\omega_{t]}$ is $\omega_1,\omega_2,\ldots,\omega'_m$. where $t_m \le t < t_{m+1}$ and $\omega'_m = \omega|_{[t_m,t]}$.

Proof: a) Refer to Figure 4.3.1

Clearly the maximal length segmentation of $\omega_{t_{n-1}]}$ does not depend on ω''. [n=3 in the figure.]

The segment $\omega_n = \omega_{[t_{n-1},t_n]}$ however may be subsumed by a new m.ℓ.s. segment when adjoining a left segment of ω''.

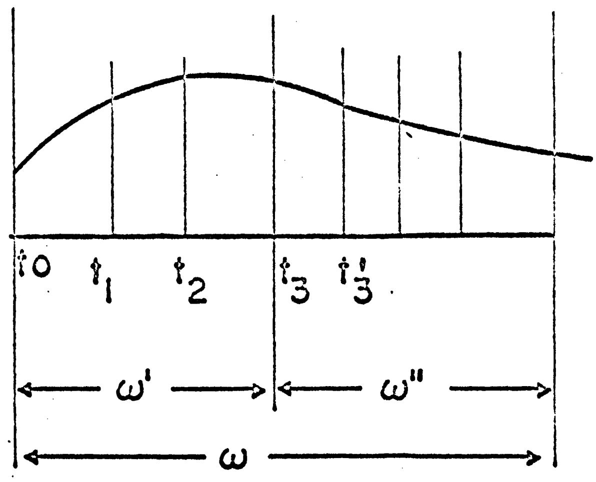

Figure 4.3.1 t_3' is the new break point after concatenating ω' with ω".

b) Refer to Figure 4.3.2. Let m be the integer for which $t_m \le t < t_{m+1}$.
The segmentation of $\omega_{t_{m-1}]}$ does not depend on $\omega_{[t}$.
This gives $\omega_{t]} = \omega_1\omega_2\ldots\omega_{m-1}\omega'_m$ where $\omega'_m = \omega|_{[t_m,t]}$.

4.3.3 Proposition:

Let Ω_G be admissible and $\omega \in \Omega_G^+$.

a) $\text{size}(\omega) \ge 1$

b) $\text{size}(\omega) = 1 \iff \omega \in \Omega_G$

c) $\omega = \omega'\omega''$ for $\omega',\omega'' \in \Omega_G^+ \Rightarrow \text{size}(\omega) \le \text{size}(\omega') + \text{size}(\omega'')$

d) $\omega = \omega'\omega''$ for $\omega',\omega'' \in \Omega_G^+ \Rightarrow \text{size}(\omega') \le \text{size}(\omega)$

Proof: a) $\Lambda \notin \Omega_G \Rightarrow \text{size}(\omega) \ge 1$

b) For $\omega \in \Omega_G$ such that $\omega:[0,\tau] \to X$, $\max_t\{\omega_t | \omega_t \in \Omega_G\} = \tau$.

c) Refer to Figure 4.3.3.

Let $\omega'_1,\omega'_2,\ldots,\omega'_p$ and $\omega''_1,\omega''_2,\ldots,\omega''_q$ be m.ℓ.s. decompositions of ω' and ω'' respectively; then $\omega = \omega'_1,\ldots,\omega'_p\ \omega''_1,\ldots,\omega''_q$; (p=4,q=3 in the Figure). Lemma 4.3.2 a) tells us that $\omega'_1,\omega'_2,\ldots,\omega'_{p-1}$ begins the m.ℓ.s. decomposition of ω. Since $\omega'_p \in \Omega_G$ and $\omega'' \in \Omega_G^+$, $\omega'_p\omega''_1\ldots\omega''_q \in \Omega_G^+$. Let $\bar{\bar{\omega}}_1,\bar{\bar{\omega}}_2,\ldots,\bar{\bar{\omega}}_r$ be the m.ℓ.s. decomposition for $\omega'_p\omega''_1\ldots\omega''_q$. Then $\omega'_1,\omega'_2,\ldots,\omega'_{p-1}\ \bar{\bar{\omega}}_1,\ldots,\bar{\bar{\omega}}_r$ is the m.ℓ.s. decomposition of ω.

Thus $\text{size}(\omega) = p-1+r$. But by Assertion 4.3.1, $r = \text{size}(\omega'_p\omega''_1\ldots\omega''_q) \le 1+q$. So $\text{size}(\omega) \le p-1 + 1+q = p+q = \text{size}(\omega')+\text{size}(\omega'')$. Thus part c) is proved

d) Note that $\text{size}(\omega') = p = \text{size}(\omega)+1-r$. Now by b) $r = \text{size}(\omega'_p\ \omega'') \geqq 1$.

4.3.2 Proposition:

Suppose Proposition 4.2.1', the prefix property holds. Then if $\omega'_1,\omega'_2,\ldots,\omega'_n$ is a m.ℓ.s. decomposition for ω' and $\omega''_1,\omega''_2,\ldots,\omega''_n$ is a m.ℓ.s. decomposition for ω'' then $\omega'_1,\omega'_2,\ldots,\omega'_n,\omega''_1,\ldots,\omega''_m$ is a m.ℓ.s. decomposition

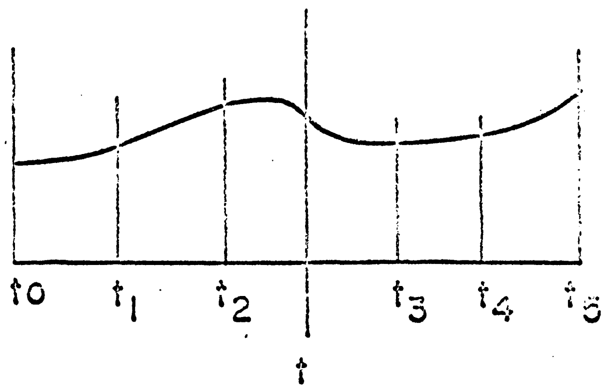

Figure 4.3.2 Since $t_2<t<t_3$, the m.ℓ.s. decomposition process does not alter the break points t_1 and t_2.

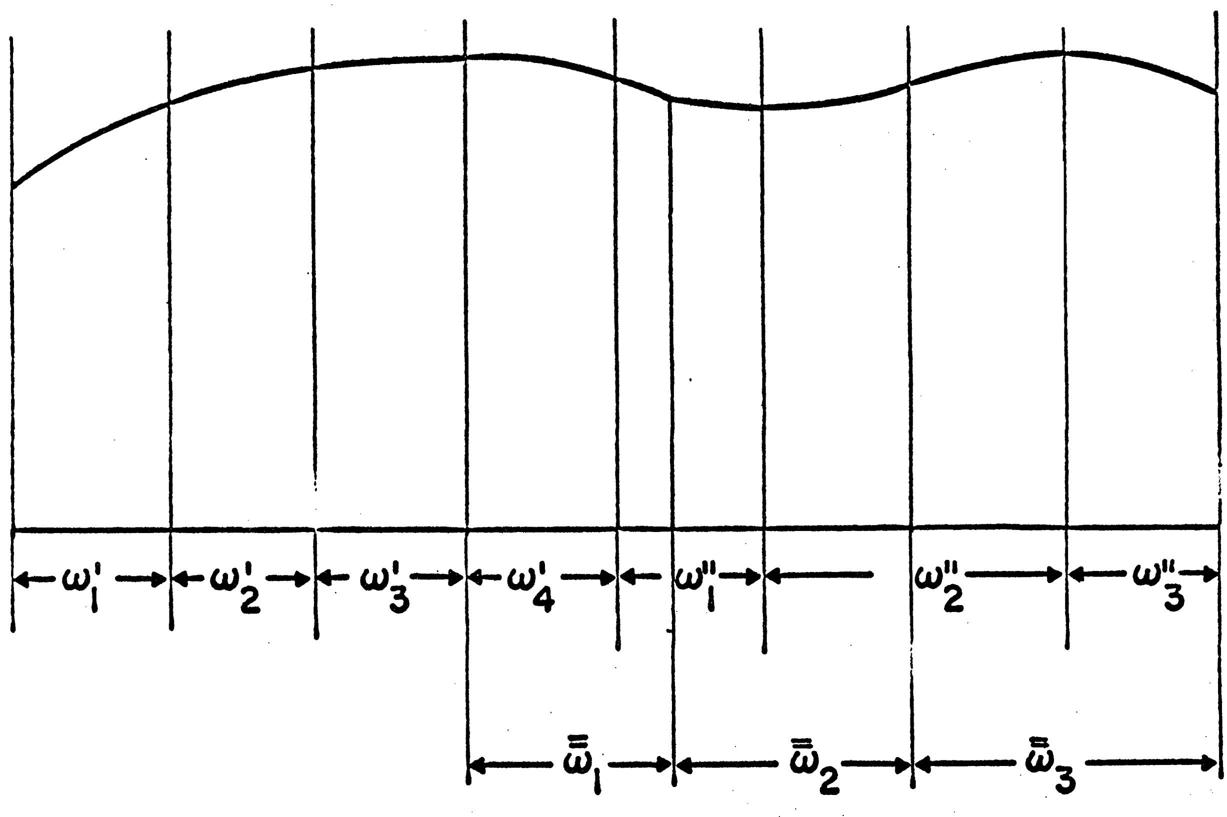

Figure 4.3.3 Size'(ω) ≤ Size(ω') + Size(ω'')

for $\omega'\omega''$.

Proof: Immediate from the discussion of Proposition 4.3.3.

4.4 Constructive Specification

We shall want to define a *one-segment transition function* $\delta_G: Q \times \Omega_G \to Q$ in such a way that it can consistently be extended to Ω_G^+. We proceed as follows:

Given a set Ω of segments, a segment μ (not necessarily in Ω) is a *right remnant* if for some $\omega \in \Omega$, $\omega\mu \in \Omega$.

A *constructive specification* is a structure $G = \langle \Omega_G, Q, \delta_G \rangle$

where Ω_G is a set of generators

Q is a nonempty set (of states)

and $\delta_G: Q \times \Omega_G \to Q$ is the *one-segment transition function.*

Ω_G must satisfy the axioms:

C.1 *Admissibility*

Ω_G is an admissible set of generators

C.2 *Closure with respect to right remnants*

μ a right remnant w.r.t. $\Omega_G \Rightarrow \mu \in \Omega_G$

δ_G must satisfy the axiom:

C.3 *Internal composition*

For all $\omega, \omega' \in \Omega_G$

$$\omega\omega' \in \Omega_G \Rightarrow \delta_G(q, \omega\omega') = \delta_G(\delta_G(q,\omega),\omega') \text{ for all } q \in Q.$$

Remark: If Ω_G has the prefix property then C.2 is vacuously satisfied since there are no right remnants.

On the other hand, if Ω_G is closed with respect to right segmentation than C.2 is (more than) satisfied.

Remark: Although they look similar, C.1 and C.2 are independent axioms. In fact one can show:

4.4.1 Proposition:

Let Ω_G be closed under right remnants. Ω_G is admissible if, and only if, Ω_G^+ is closed under right remnants.

Proof: Left for the reader.

Remark: Given a system $S = <\Omega,Q,\delta>$ we can generate constructive specifications as follows. Let Ω_G be any subset of Ω which satisfies C.1 and C.2. Let δ_G be δ restricted to Ω_G, i.e., set $\delta_G(q,\omega) = \delta(q,\omega)$ for all $q \in Q$, $\omega \in \Omega_G$. Since δ satisfies the composition axiom A.3, δ_G will automatically satisfy C.3. Actually, if S is derived from a DESS with unique solutions we need only solve the differential equation for the segments in Ω_G to obtain δ_G satisfying C.3.

Example: Step Responses of Linear First Order Lag System

Consider $G = <\Omega_G,Q,\delta_G>$
where

$$\Omega_G = \{a_t | t\varepsilon R, a\varepsilon R\}$$

$$Q = R$$

and

$$\delta_G : Q \times \Omega_G \rightarrow Q$$

is given by

$$\delta_G(q,a_t) = qe^{-t/T} + a - ae^{-t/T}.$$

Since Ω_G is closed under right segmentation it is both admissible and closed under right remnants. Since δ_G is derived from a DESS with unique solutions it is internally decomposable. Thus G is a constructive specification.

4.4.1 Theorem:

Given a constructive specification $G = <\Omega_G,Q,\delta_G>$ we can associate with it a system $S_G = <\Omega_G^+,Q,\delta>$ where $\delta : Q \times \Omega_G^+ \rightarrow Q$ is defined as follows:

For $\omega \in \Omega_G$, $q \in Q$ $\qquad \delta(q,\omega) = \delta_G(q,\omega)$;

for $\omega \in \Omega_G^+$, $q \in Q$ $\qquad \delta(q,\omega) = \delta(\delta_G(q,\omega_1),\omega_2 \ldots \omega_n))$

where $\omega_1,\omega_2,\ldots,\omega_n$ is the m.ℓ.s. decomposition of ω by Ω_G.

Note that $\omega_2,\omega_3,\ldots,\omega_n$ is the m.ℓ.s. decomposition for $\omega_2 \ldots \omega_n$.

Proof: Since δ_G is assumed to be time invariant, δ will be time invariant. That δ_G is well-defined follows from the uniqueness of the m.ℓ.s. decomposition. Thus we need only show that the composition property obtains, i.e., we must show that $\delta(q,\omega'\omega'') = \delta(\delta(q,\omega')\omega'')$ for all $\omega',\omega'' \in \Omega_G^+$. We proceed by induction on $\max\{\text{size}(\omega'), \text{size}(\omega'')\}$.

Basis: $\text{Max}\{\text{size}(\omega'), \text{size}(\omega'')\} = 1$.

By Proposition 4.3.2 a), b) $\text{size}(\omega') = \text{size}(\omega'') = 1$ and $\omega',\omega'' \in \Omega_G$.

Now by 4.3.2 c)
$$\text{size}(\omega'\omega'') \le \text{size}(\omega') + \text{size}(\omega'') \le 2.$$

If $\text{size}(\omega'\omega'') = 1$ then $\omega'\omega'' \in \Omega_G$ (by 4.3.2 a)) and by the internal composition of δ_G we have

$$\begin{aligned} \delta(q,\omega'\omega'') &= \delta_G(q,\omega'\omega'') \\ &= \delta_G(\delta_G(q,\omega'),\omega'') \\ &= \delta(\delta(q,\omega'),\omega''). \end{aligned}$$

If $\text{size}(\omega'\omega'') = 2$, refer to Figure 4.4.1.

Let $\overline{\omega},\bar{\bar{\omega}}$ be the m.ℓ.s. decomposition for $\omega'\omega''$ and let $\overline{\omega} = \omega'\mu$, so that $\mu\bar{\bar{\omega}} = \omega''$. Since $\omega'\mu \in \Omega_G$ and $\omega' \in \Omega_G$, μ is a right remnant and by C.1, $\mu \in \Omega_G$.

Now,

$$\begin{aligned} \delta(\delta(q,\omega'),\omega'') &= \delta_G(\delta_G(q,\omega'),\omega'') && [\omega',\omega'' \in \Omega_G] \\ &= \delta_G(\delta_G(q,\omega'),\mu\bar{\bar{\omega}}) && [\omega'' = \mu\bar{\bar{\omega}}] \\ &= \delta_G(\delta_G(\delta_G(q,\omega'),\mu),\bar{\bar{\omega}}) && [\mu\bar{\bar{\omega}} \in \Omega_G] \\ &= \delta_G(\delta_G(q,\omega'\mu),\bar{\bar{\omega}})) && [\omega'\mu \in \Omega_G] \\ &= \delta_G(\delta_G(q,\overline{\omega}),\bar{\bar{\omega}}) && [\omega'\mu = \overline{\omega}] \\ &= \delta_G(q,\overline{\omega\bar{\bar{\omega}}}) && [\overline{\omega},\bar{\bar{\omega}} \in \Omega_G] \\ &= \delta_G(q,\omega'\omega''). \end{aligned}$$

For the induction step, assume $\delta(q,\omega'\omega'') = \delta(\delta(q,\omega'),\omega'')$ for $\max\{size(\omega'), size(\omega'')\} \leq n$. Let $\omega = \omega'\omega''$ where $\max\{size(\omega'),size(\omega'')\} = n+1$. Refer to Figure 4.4.2.

Let $\omega_1,\omega_2,\ldots,\omega_p$ be the m.ℓ.s. decomposition for ω' where $p \leq n+1$. Let $\overline{\omega}_p$ begin the m.ℓ.s. decomposition for $\omega_p\ \omega''$ and let $\bar{\bar{\omega}}$ be the associated right segment, i.e., $\omega_p\omega'' = \overline{\omega}_p\bar{\bar{\omega}}$. By Prop. 4.3.3c, $1+size(\bar{\bar{\omega}})$ ε $size(\omega'')$ $size(\bar{\bar{\omega}}) \leq size(\omega'') \leq n+1$.

Let $\overline{\omega}_p = \omega_p\mu$; then since μ is a right remnant $\mu \in \Omega_G$.

$$
\begin{aligned}
\delta(\delta(q,\omega'),\omega'') &= \delta(\delta(q,\omega_1\omega_2\cdots\omega_{p-1}\omega_p),\omega'')) && [\omega' = \omega_1\omega_2\cdots\omega_{p-1}\omega_p] \\
&= \delta(\delta(\delta(q,\omega_1\cdots\omega_{p-1}),\omega_p),\omega'')) && [size(\omega_1\cdots\omega_{p-1}) \leq n \text{ and } size(\omega_p) = 1] \\
&= \delta(\delta(\delta(q,\omega_1\cdots\omega_{p-1}),\omega_p),\mu\bar{\bar{\omega}})) && [\mu\bar{\bar{\omega}} = \omega''] \\
&= \delta(\delta(\delta(q,\omega_1\cdots\omega_{p-1}),\omega_p\mu),\bar{\bar{\omega}}) && [\omega_p\mu \in \Omega_G] \\
&= \delta(\delta(\delta(q,\omega_1\cdots\omega_{p-1}),\overline{\omega}_p),\bar{\bar{\omega}}) && [\overline{\omega}_p = \omega_p\mu] \\
&= \delta(\delta(q,\omega_1\cdots\omega_{p-1}\overline{\omega}_p),\bar{\bar{\omega}}) && [size(\omega_1\cdots\omega_{p-1}) \leq n \text{ and } size(\overline{\omega}_p) = 1] \\
&= \delta(q,\omega_1\cdots\omega_{p-1}\overline{\omega}_p\bar{\bar{\omega}}) \\
&= \delta(q,\omega'\omega'').
\end{aligned}
$$

The next to the last line relies on the easily proved Lemma: If $\omega_1,\omega_2,\ldots,\omega_n$ is the m.ℓ.s. decomposition for ω then for $i \in \{1,2,\ldots,n-1\}$

$$\delta(\delta(q,\omega_1\omega_2\cdots\omega_i)\omega_{i+1}\cdots\omega_n) = \delta(q,\omega)$$

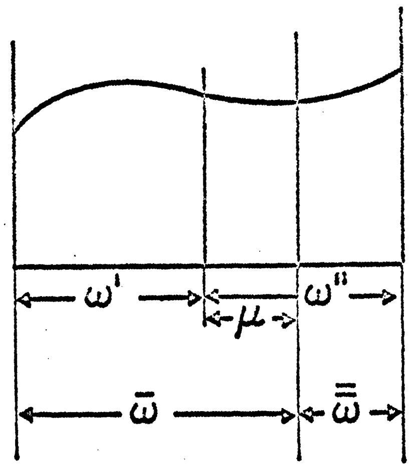

Figure 4.4.1 The case, size (ω'ω'') = 2.

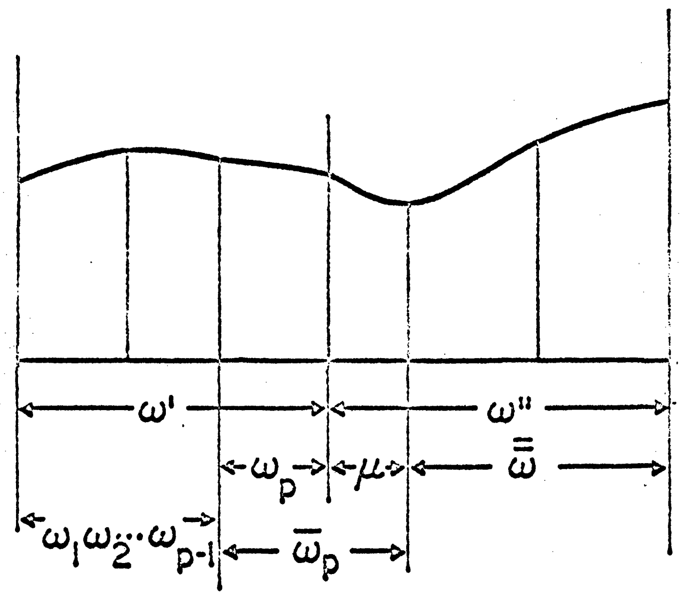

Figure 4.4.2 The case, max{size(ω'), size(ω'')} = n+1.

V. MORPHISMS FOR CONSTRUCTIVELY SPECIFIED SYSTEMS

5.0 Definitions

With the foregoing definition of constructively specified systems, we are ready to formulate a notion of simulation which retains the essential character of the discrete time idea.

A *function morphism* from $G = \langle \Omega_G, Q, \delta_G \rangle$ to $G' = \langle \Omega'_G, Q', \delta'_G \rangle$ is a pair (g,h) where

$$g : \Omega'_G \to \Omega^+_G$$

$$h : Q_1 \to Q' \quad \text{(onto)}$$

and i) $Q_1 \subsetneq Q$ is closed under $g(\Omega'_G)$

ii) $h(\tilde{\delta}_G(q, g(\omega'))) = \delta'_G(h(q), \omega'))$

for all $\omega' \in \Omega'_G$, $q \in Q_1$.

Remark: $\tilde{\delta}_G : Q \times \Omega^+_G \to Q$ is δ_G extended to Ω^+_G by the m.ℓ.s. definition of Theorem 4.4.2. Actually, to determine whether ii) holds we need only extend δ_G to segments in $g(\Omega'_G)$. If g is "size preserving", i.e., $g(\Omega'_G) \subseteq \Omega_G$ then δ_G need not be extended at all.

A *function morphism* from S_G to S'_G is a pair $(\bar{g}, \bar{h})$ such that

$$\bar{g} : (\Omega'_G)^+ \to \Omega^+_G$$

$$\bar{h} : Q_1 \to Q' \qquad \text{(onto)}$$

where i) $Q_1 \subseteq Q$ is closed under $\bar{g}((\Omega'_G)^+)$

and ii) $\bar{h}(\delta(q, \bar{g}(\omega')) = \delta'(\bar{h}(q), \omega')$.

for all $q \in Q_1$, $\omega' \in (\Omega'_G)^+$.

Generalizing Theorem 3.2.1 we have:

5.0.1 Theorem: There is a function morphism from G to G' if, and only if, there is a function morphism from S_G to $S_{G'}$.

Proof: => Let (g,h) be a function morphism from G to G'. We construct a function morphism $(\overline{g},\overline{h})$ from S_G to S'_G as follows:

$$\text{Set } \overline{h} = h$$

and define $\overline{g}:(\Omega'_G)^+ \to \Omega_G^+$ by $\overline{g}(\omega) = g(\omega_1)g(\omega_2)\ \ldots\ g(\omega_n)$

where $\omega_1,\omega_2,\ldots,\omega_n$ is the m.ℓ.s. decomposition of $\omega \in \Omega_G^+$.

First note that $\overline{g}$ is well defined because of the uniqueness of the m.ℓ.s. decomposition.

We use induction on size(ω') to show that for all $q \in Q_1$

$$\overline{h}(\delta(q,\overline{g}(\omega'))) = \delta'(\overline{h}(q),\omega').$$

Basis: size(ω') = 1 => $\omega' \in \Omega'_G$. Thus $\overline{g}(\omega') = g(\omega)$ and the equality is just that given.

Assume the equality holds for all ω' such that size(ω') = n.

Let size(ω') = n+1, and let $\omega'_1,\omega'_2,\ldots,\omega'_{n+1}$ be the m.ℓ.s. decomposition for ω'.

$$\begin{aligned}
\overline{h}(\delta(q,\overline{g}(\omega'_1\omega'_2\ldots\omega'_{n+1}))) &= \overline{h}(\delta(q,\overline{g}(\omega'_1\ldots\omega'_n)g(\omega'_{n+1}))) && [\text{definition of } \overline{g}]\\
&= \overline{h}(\delta(\delta(q,\overline{g}(\omega'_1\ldots\omega'_n),g(\omega'_{n+1})) && [\text{composition of } \delta]\\
&= \delta'(\overline{h}(\delta(q,\overline{g}(\omega'_1\ldots\omega'_n)),\omega'_{n+1}) && [\text{size}(\omega'_{n+1})=1]\\
&= \delta'(\delta(\overline{h}(q),\omega'_1\ldots\omega'_n),\omega_{n+1}) && [\text{size}(\omega'_1\ldots\omega'_n)=n]\\
&= \delta'(\overline{h}(q),\omega'_1\ldots\omega'_n\omega'_{n+1}) && [\text{composition of } \delta']
\end{aligned}$$

That Q_1 is closed under $g(\Omega')^+$ now follows readily from its assumed closure under $g(\Omega'_G)$.

This establishes the forward direction. The reverse direction is immediate.

Remark: In the foregoing theorem $\overline{g}$ is not necessarily a semigroup homomorphism of $(\Omega'_G)^+$ to Ω_G^+. In general, we cannot expect $\overline{g}(\omega\omega') = \overline{g}(\omega)\overline{g}(\omega')$ since the m.ℓ.s. decomposition for $\omega\omega'$ is not simply the concatenation of those of ω and ω'.

For example, let $\Omega_G = \{a_t | t \varepsilon R, a \varepsilon R\}$ and let $g: \Omega_G \rightarrow \Omega_G$ be defined for some $a,b,c \varepsilon R$ where $b \neq c$ by

$$\begin{aligned} g(a_t) &= b_t \quad \text{for } t \in [0,1] \\ &= c_t \quad \text{for } t > 1 \end{aligned}$$

Then $\overline{g}(a_{.5}, a_{.6}) = \overline{g}(a_{1.1}) = g(a_{1.1}) = c_{1.1}$

while

$$\overline{g}(a_{.5})\overline{g}(a_{.6}) = g(a_{.5})g(a_{.6}) = b_{.5}b_{.6} = b_{1.1}$$

5.0.1 Proposition: $\overline{g}: (\Omega'_G)^+ \rightarrow \Omega_G^+$ is a homomorphism <=> $\overline{g}(\omega\omega') = \overline{g}(\omega)\overline{g}(\omega')$ for all $\omega, \omega' \varepsilon (\Omega'_G)^+$ such that $\omega\omega' \varepsilon \Omega'_G$.

Proof: Entirely analagous to that of Theorem 4.4.2.

5.0.1 Corollary: If Prefix Property holds for Ω'_G, $\overline{g}$ is a homomorphism.

5.0.2 Corollary: If the right and left segmentation closure holds then $\overline{g}$ is a homomorphism <=> $g(\omega\omega') = g(\omega)g(\omega')$ for all $\omega, \omega' \varepsilon \Omega'_G$ such that $\omega\omega' \varepsilon \Omega'_G$.

We shall now give evidence of the applicability of these concepts by considering some illustrative problems arising in modeling and simulation.

5.1 Time and Amplitude Scaling for Continuous Time Systems

a) Time scale change

Let $G = \langle \Omega_G, Q, \delta_G \rangle$ be specified as follows:

$\Omega_G = \{\omega | \omega$ is a real valued continuous function defined on a finite closed interval$\}$

$Q = R$

$\delta_G : Q \times \Omega_G \to Q$ is given by

$$\delta_G(q, \omega_{\tau]}) = \Phi_{(q,\omega)}(\tau)$$

for all $\omega_{\tau]} \in \Omega_G$.

Here
$$\Phi_{(q,\omega)}(0) = q$$

and
$$\frac{d}{dt}\Phi_{(q,\omega)}(t) = f(\Phi_{(q,\omega)}(t), \omega(t)) \text{ for } t \in \text{dom } \omega.$$

The underlying DESS, $D = (\Omega_G, Q, f)$ is assumed to have unique solutions.

Since Ω_G is closed under right segmentation C.1 and C.2 are satisfied. Also, since the underlying DESS has unique solutions we have C.3 satisfied (see the Remark following Proposition 4.4.1) thus G is a constructive specification.

Let $G' = \langle \Omega'_G, Q, \delta'_G \rangle$ be specified by

$$\Omega'_G = \Omega_G$$

$$Q' = R$$

$$\delta'_G : Q' \times \Omega'_G \to Q'$$

is given by

$$\delta'_G(q, \omega_{\tau]}) = \Phi'_{(q,\omega)}(\tau)$$

for all $\omega_{\tau]} \in \Omega_G$.

Here $$\Phi'_{(q,\omega)}(0) = q$$

and

$$\frac{1}{\sigma}\frac{d}{dt}\Phi'_{(q,\omega)}(t) = f(\Phi'_{(q,\omega)}(t),\omega(t)) \quad \text{for } t \in [0,\tau].$$

By the same argument, $S_{G'}$ is a system. Its rate of operation, however, is σ times that of S_G where σ is a positive real number.

5.1.1 Theorem: (g,h) is a time local function morphism from G to G' where

$$h: Q \to Q' \text{ is the identity map}$$

and

$$g: \Omega'_G \to \Omega_G$$

is given by

$$g(\omega_{[0,\tau]}) = \overline{\omega}_{[0,\sigma\tau]}$$

where $$\overline{\omega}_{[0,\sigma\tau]}(t) = \omega_{[0,\tau]}(t/\sigma) \quad \text{for } t \in [0,\sigma\tau]$$

Proof: Since in this case $Q_1 = Q$ it is closed under $g(\Omega'_G)$. And since h is the identity map we need only show that for all $q \in Q$, $\omega_{[0,\tau]} \in \Omega'_G$.

$$\delta(q,g(\omega_{[0,\tau]})) = \delta'(q,\omega_{[0,\tau]})$$

Equivalently for all $\tau \in R$,

$$\delta(q,\overline{\omega}_{[0,\sigma\tau]}) = \delta'(q,\omega_{[0,\tau]})$$

i.e.,

$$\Phi_{(q,\overline{\omega})}(\sigma\tau) = \Phi'_{(q,\omega)}(\tau)$$

i.e.,

$$\Phi \cdot T_\sigma = \Phi'$$

where $T_\sigma(t) = \sigma t$ for all $t \in R$,

and $\Phi_{(q,\overline{\omega})} = \Phi$, $\Phi'_{(q,\omega)} = \Phi'$.

To show this we prove that $\Phi \cdot T_\sigma$ satisfies the differential equation for Φ'.

$$\frac{1}{\sigma}\frac{d}{dt}(\Phi\cdot T_\sigma)(t) = \frac{1}{\sigma}\frac{d}{d(\sigma t)}\Phi(\sigma t)\cdot\frac{d(\sigma t)}{dt}$$

Integrating the differential equation satisfied by Φ yields

$$\Phi(t) = q + \int_0^t f(\Phi(t'),\overline{\omega}(t'))dt'$$

Thus
$$\frac{d\Phi(\sigma t)}{d(\sigma t)} = \frac{d}{d(\sigma t)}\left(q + \int_0^{\sigma t} f(\Phi(t'),\overline{\omega}(t'))dt'\right)$$

$$= f(\Phi(\sigma t),\overline{\omega}(\sigma t))$$

$$= f(\Phi\cdot T_\sigma(t),\omega(t))$$

so

$$\frac{1}{\sigma}\frac{d}{dt}(\Phi\cdot T_\sigma)(t) = f(\Phi\cdot T_\sigma(t),\omega(t)).$$

Also,
$$\Phi\cdot T_\sigma(0) = \Phi(0) = q$$

so
$$\Phi\cdot T_\sigma = \Phi'$$

by uniqueness of solutions.

Finally, since Ω_G is closed under right and left segmentation it is enough to show that $g(\omega\omega') = g(\omega)g(\omega')$ for all $\omega,\omega' \in \Omega_G$ such that $\omega\omega' \in \Omega_G$. But it is easily seen that $\overline{\omega\omega'} = \overline{\omega}\,\overline{\omega'}$ as required.

b) Amplitude Scale Change

5.1.2 Theorem: Let G, G' above be such that $f'(q,x) = af_2(q/a,b)$ for fixed $a,b \in R$. Then (g,h) is a time local function morphism from G to G' where $h:Q \to Q'$ and $g:\Omega_G' \to \Omega_G$ are given by

$$h(q) = aq$$

$$g(\omega) = \omega/b$$

where $\omega/b(t) = \omega(t)/b$ for all $t \in \text{dom}\ \omega$.

Proof: An exercize for the reader.

5.2 Simulation of Continuous Time Systems by Discrete Time Systems

Let $G = \langle\Omega_G,Q,\delta_G\rangle$, $G' = \langle\Omega_G',Q',\delta_G'\rangle$ be discrete and continuous time constructive specifications respectively. Let (g,h) be a function morphism from G to G'. Then $g:\Omega'_G \to \Omega^+_G$.

Thus g is a mapping from functions with *real* domains into functions with *integer* domains; i.e., g must encode a continuous wave into a discrete one. Let $\Omega'_G = \{\omega' | \omega' : [0,\tau] \to R\}$; then in general Ω_G must have the cardinalitility of Ω'_G if a function preserving relation is to hold.

Clearly this is impractical and it is interesting to examine cases where Ω_G need be no more numerous than the real numbers.

<u>Example 1</u>: Sampled Equispaced Step Functions (Figure 5.3.1)

$\Omega'_G = \{a_h | a \in R\}$

$\Omega_G = \{a | a \in R\}$

$g:\Omega'_G \to \Omega_G$ given by $g(a_h) = a$

<u>Example 2</u>: Sampled Arbitrarily Spaced Step Functions (Figure 5.3.2)

$\Omega'_G = \{a_t | t \in R, a \in R\}$

$\Omega_G = \{(a,t) | t \in R, a \in R\}$

$g:\Omega'_a \to \Omega_G$ given by $g(a_t) = (a,t)$

<u>Example 3</u>: Band and Time Limited Waves

Let $\{e_i | i=1,2,\ldots,n\}$ be a finite set of functions $e_i:T \to R$. For example $\{e_i\}$ might be a truncated Fourier series, Tchebychev polynomials, etc.

Then,

$$\Omega'_G = \{\omega | \omega = \sum_{i=1}^{n} c_i e_i \quad \text{for some } c_i \in R,\ i=1,2,\ldots,n\}$$

$$\Omega_G = R$$

and $g:\Omega'_G \to \Omega^+_G$ is given by

$$g(\omega) = c_1 c_2 \ldots c_n \text{ if } \omega = \sum_{i=1}^{n} c_i e_i$$

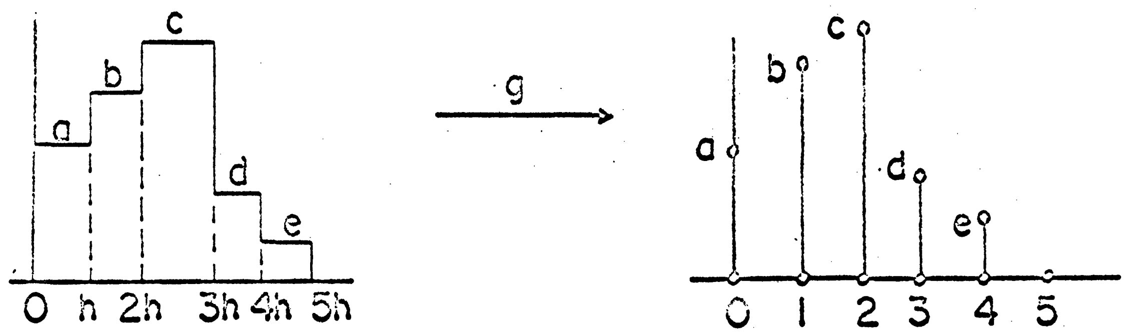

Figure 5.3.1 Sampled Equispaced Step Functions

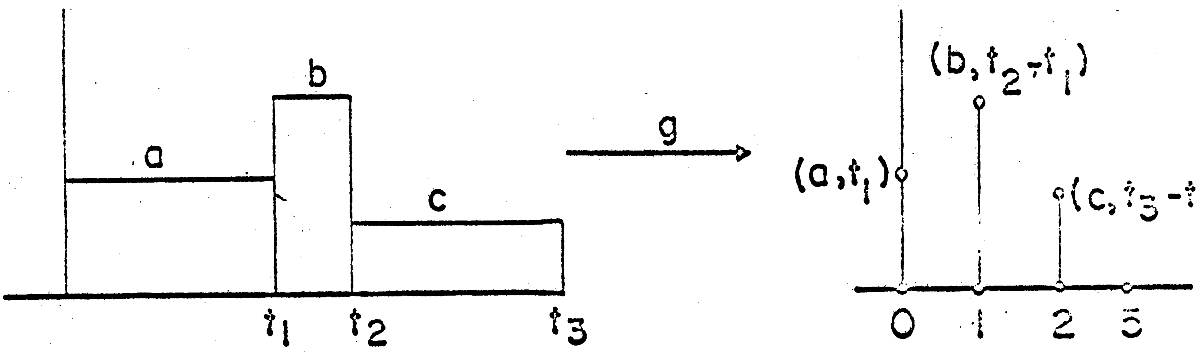

Figure 5.3.2 Sampled Variable Spaced Step Functions

This describes a common situation in speech generation where discrete sequences of parameters characterizing a speech wave form are fed to a speech production device.

5.4 Realization of Discrete Time Systems by Continuous Time Systems

With the formalism developed it is possible to characterize explicitly and simply the realization of discrete time systems by continuous time systems. The following shows how a two-state, two-input sequential machine may be realized by a set-reset flipflop with instantaneous feedback. The example may be readily extended to arbitrary finite state sequential machines and is presented as only one possible model for such realizations.

The flipflop will be modelled as a DESS $D = (\Omega, Q, f)$ where

$$\Omega = \{a_t | a \in R, t \in R\}^+$$

$$Q = R \times R$$

and

$$f: Q \times R \to Q$$

is given by

$$f\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, x\right) = \frac{1}{\tau} \begin{bmatrix} -q_1 - \mathrm{sgn}(q_2) + Z(\mathrm{sgn}(q_1), x) \\ -q_2 - \mathrm{sgn}(q_1) - Z(\mathrm{sgn}(q_1), x) \end{bmatrix} .$$

Here $\mathrm{sgn}(q) = 1$ if $q \geq 0$

$= -1$ if $q < 0$

and $Z: \{1,-1\} \times R \to \{1,-1,0\}$ is such that $Z(x,0) = 0$ for $x \in \{1,-1\}$. Figure 5.4.1 displays an analog model of this network.

It can be readily shown that D has unique solutions. Moreover, there are positive numbers σ and ρ (related to the system time constant τ) such that for any input of the form $x_\sigma 0_\rho$,

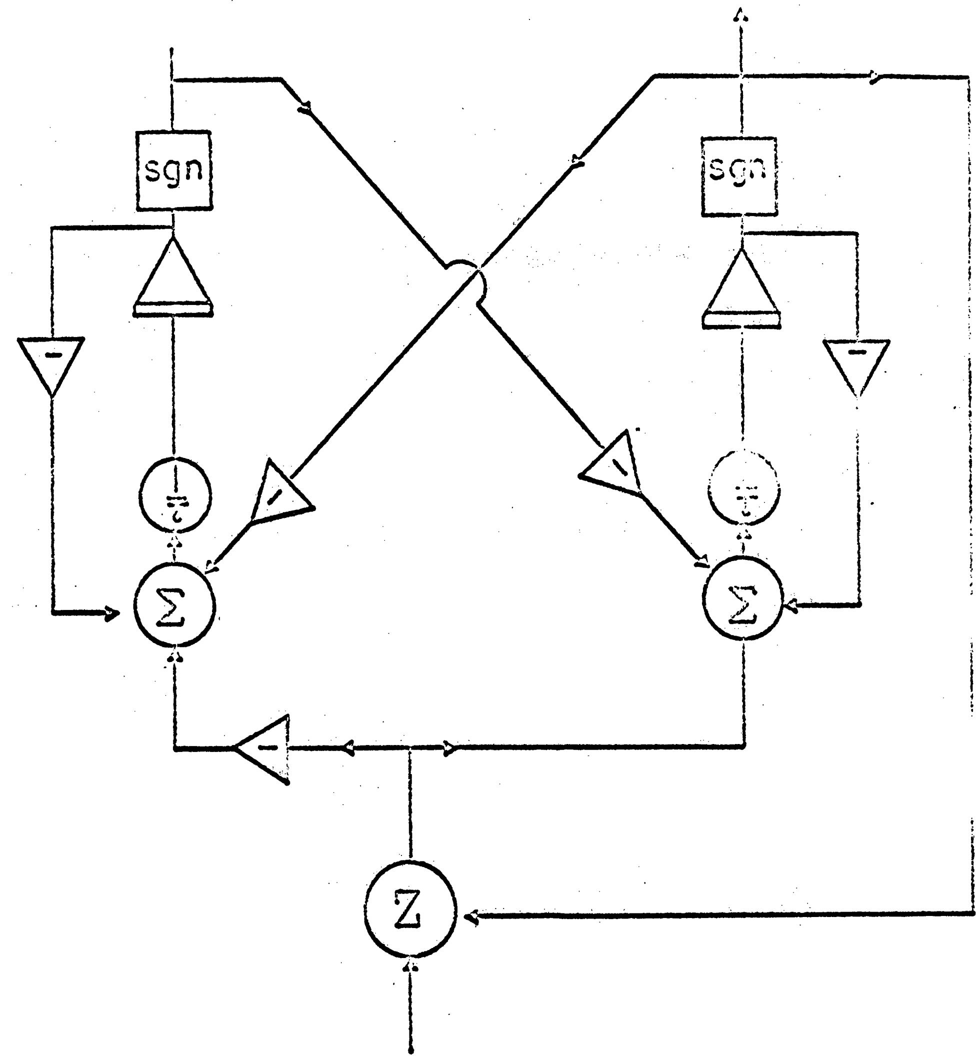

Figure 5.4.1 A set-reset flip flop realization of a two-state sequential machine.

$$\delta(q_1,q_2,x_\sigma 0_\rho) = \Phi_{(q_1,q_2,x_\sigma 0_\rho)}(\sigma+\rho)$$

$$= (Z(\mathrm{sgn}(q_1),x),-Z(\mathrm{sgn}(q_1),x)$$

Here $x \in \{1,-1\}$. In other words, the final state of the flipflop after a pulse of width σ followed by a ρ-length quiescence is completely determined by the polarities of the initial state and pulse.

5.4.1 Theorem: Let $G' = <\{1,-1\},\{1,-1\},\delta'>$ be a discrete time sequential machine. There is a continuous time constructive specification G such that G' is a time local function morphic image of G (i.e., G realizes G').

Proof: Let $G = <\Omega_G,Q,\delta_G>$ be obtained by extraction from the flipflop DESS above. Specifically, let

$$\Omega_G = \{1_\sigma 0_\rho,-1_\sigma 0_\rho\}$$

$$Q = R \times R$$

and let

$$\delta_G : Q \times \Omega_G \to Q$$

be defined by

$$\delta_G((q_1,q_2),x_\sigma 0_\rho) = (Z(\mathrm{sgn}(q_1),x),-Z(\mathrm{sgn}(q_1),x))$$

where

$$Z(y,x) = \delta'(y,x)$$

for all $y,x \in \{1,-1\}$.

Since Ω_G has the prefix property C.1 and C.2 are satisfied and C.3 is vacuously satisfied.

For the function morphism (g,h) from G to G' let

$g:\{1,-1\} \to \{1_\sigma 0_\rho,-1_\sigma 0_\rho\}$ be given by $g(x) = x_\sigma 0_\rho$.

Let $Q_1 = \{(1,-1),(-1,1)\} \subseteq R \times R$ and define $h:Q_1 \to Q'$ by

$$h(y,-y) = y \quad \text{for } y \in \{1,-1\}.$$

The reader may verify that Q_1 is closed under $\{1_\sigma 0_\rho,-1_\sigma 0_\rho\}$.

Next for $(y,-y) \varepsilon Q_1$

$$
\begin{aligned}
h(\delta_G((y,-y),g(x))) &= h(\delta_G((y,-y),x_\sigma 0_\rho)) \\
&= h(Z(\mathrm{sgn}(y),x),-Z(\mathrm{sgn}(y),x)) \\
&= Z(y,x) \\
&= \delta'(y,x) \\
&= \delta'(h(y,-y),x).
\end{aligned}
$$

Finally, since G' is a discrete time system g is extendable to a semigroup homomorphism of $\{1,-1\}^+$ to $\{1_\sigma 0_\rho,-1_\sigma 0_\rho\}^+$.

Remark: By expanding Ω_G to the set $\Omega_G = \{x_{t_1} 0_{t_2-t_1} | x \varepsilon \{1,-1\}, t_1, t_2 \varepsilon R\}$ one can study the degradation of the sequential machine realization when badly timed pulses are used.

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