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Technical Report

A NOTE ON SERIES PARALLEL IRREDUCIBILITY

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ABSTRACT

A new criterion for series-parallel irreducibility is given which makes no reference to underlying semigroups but involves only series-parallel connection operations.

A semi-automaton or transition system is a triple-~~tuple~~ $\langle X,Q,M\rangle$ where S,Q are finite sets (of input symbols and internal states respectively), and M: $Q \times X \to Q$ is the transition function. (In the usual abuse of notation we write M for $\langle X,Q,M\rangle$). In this note we shall characterize the semi-automata which are irreducible with respect to series-parallel decomposition. This augments the definition of Krohn and Rhodes [1] (see also Arbib's formulation in [2]), which is an essential way required the specification of output maps and thus held only for full automata i.e., machines of the form $\langle S,Q,O,M,N\rangle$ where O is the output set and N: $Q \to O$ the output function. Moreover, their definition of irreducibility for machines made direct reference to semigroups while the definition we shall give makes reference only to series-parallel connection operations. Except for changes in notation the presentation follows that of [2] Chapters 3 and 5.

Let S(M) denote the semigroup of M i.e.,

$$S(M) = \{\tilde{M}(\ ,x): Q \to Q \mid x \in X^*\}$$

where $\tilde{M}$ is M extended to X^*. Given a semigroup S let M_S denote the semigroup transition system i.e., $M_S: S^1 \times S \to S^1$ with $M_S(1,s) = s$ and $M_S(s,s') = ss'$ for all s, s' $\in$ S. Note that $S(M_S) = S$.[1]

In the following we consider as usual only connected machines with specified starting state.

Given transition functions $M_i: Q_i \times X_i \to Q_i$, $i = 1,2$, we say that M_2 divides M_1 written $M_2 | M_1$ if there exist $Q_1' \subseteq Q_1$ and maps $g: X_2 \to X_1^*$, $h: Q_1' \to Q_2$ (onto) such that

1) Q_1' is closed under $g(X_2)^*$ and

2) for all $q_1 \in Q_1'$, $s \in X_2$, $h(\tilde{M}_1(q_1,g(s))) = M_2(h(q_1),s)$.

Given $\langle X,Q,M\rangle$ and a positive integer n define $\Pi^n M = \langle X,Q^n,\Pi^n M\rangle$ by $\Pi^n M(q_1,\ldots,q_n,s) = (M(q_1,s),\ldots M(q_n,s))$ for all $(q_1,\ldots,q_n,s) \in Q^n \times X$. $\Pi^n M$ represents n copies of machine M (possibly in different states) which are run in parallel and are fed the same input symbol.

<u>Definition:</u> <u>M_2 π-divides M_1</u>, $M_2|_\pi M_1$ if there is a positive integer n such that $M_2|\Pi^n M$. We remark that division, and π-division are transitive relations.

<u>M_2 mutually π-divides M_1</u>, $M_2 \underset{\pi}{\equiv} M_1$ if $M_2|_\pi M_1$ and $M_1|_\pi M_2$.

We require the following statements:

1. $M_2|M_1$ implies $S(M_2)|S(M_1)$.[2]

2. $S(M_2)|S(M_1)$ implies $M_2|M_{S(M_1)}$

3. $M_{S(M)}|_\pi M$

4. $S(\Pi^n M) = S(M)$

Proofs may be found in Chapter 1 of [4]. Suffice to say that (1) and (2) are well-known; (3) is a slight extension of Fact 2.14b, Chapter 5 of [3]. For (4) we note that

$$\widetilde{\Pi^n M}(q_1,\ldots,q_n,x) = (\tilde{M}(q_1,x),\ldots,\tilde{M}(q_n,x))$$

so examining the Myhill equivalences relations:

$$x \underset{\Pi^n M}{\equiv} y \iff \text{for all } (q_1,q_2,\ldots,q_n) \in Q^n,\ \widetilde{\Pi^n M}(q_1,\ldots,q_n,x) = \widetilde{\Pi^n M}(q_1,\ldots,q_n,y)$$

$$\iff \text{for all } q \in Q,\ \tilde{M}(q,x) = \tilde{M}(q,y)$$

$$\iff x \underset{M}{\equiv} y.$$

So $S(\Pi^n M) = X^*|_{\underset{\Pi^n M}{\equiv}} = X^*|_{\underset{M}{\equiv}} = S(M)$.

Proposition 1:

$S(M_2)|S(M_1)$ if, and only if, $M_2|_\pi M_1$

Proof: Assume $S(M_2)|S(M_1)$. Then from (2), $M_2|M_{S(M_1)}$. Also from (3) $M_{S(M_1)}|_\pi M_1$ so by transitivity $M_2|_\pi M_1$.

Conversely assume $M_2|_\pi M_1$. Then for some n, $M_2|\Pi^n M_1$ so by (1) $S(M_2)|S(\Pi^n M_1)$. Recognizing that $S(\Pi^n M_1) = S(M_1)$ from (4) completes the proof.

We see that Proposition 1.allows re-interpretation of semigroup division in terms of π-division. This is not true for ordinary division-to make the converse of (1) hold, output maps have to be added to the semigroups as in Theorem 7.3.10 of [2]. The best that we can get from (1) and (2) is

5. $S(M_2) \not| S(M_1)$ if, and only if, $M_2 \not| M_{S(M_1)}$.

An interesting consequence of Proposition 1 is

Corollary 2: $M_1 \underset{\pi}{\equiv} M_2$ if, and only if, $S(M_1) = S(M_2)$.

Proof: Apply Proposition 1 twice.

The standard definitions of irreducibility are:

a) A semigroup S is irreducible if whenever $S|S_2 \, x_z \, S_1$ then $S|S_2$ or $S|S_1$. (Here $S_2 \, x_z \, S_1$ is a semidirect product of S_1 by S_2 with connecting map Z)

b) A machine M is irreducible if whenever $M|M_2 \, x_z \, M_1$ then $M|M_2$ or $M|M_1$. (Here $M_2 \, x_z \, M_1$ is the series-parallel cascade of M_1 followed by M_2 with connecting map Z.)

c) A machine M is s-irreducible if whenever $M|M_2 \, x_z \, M_1$ then $M|M_{S(M_2)}$ or $M|M_{S(M_2)}$

We add the definition:

d) A machine M is <u>π-irreducible</u> if whenever $M|M_2 \times_z M_1$ then $M|_\pi M_2$ or $M|_\pi M_1$.

Theorems 8.3.6, 8.3.7 page 4 of [2] state that M is s-irreducible if, and only if, S(M) is irreducible. On the other hand, while M is irreducible implies S(M) is irreducible, the converse does not hold.[3]
Based on Proposition 1 we can now show that the equivalence does hold for π-irreducibility.

<u>Theorem 3</u>: M is π-irreducible, if and only if, M is s-irreducible.

<u>Proof</u>: M is π-irreducible $\Longleftrightarrow$ if $M|M_2 \times_z M_1$ then $M|_\pi M_2$ or $M|_\pi M_1 \Longleftrightarrow$ if $M|M_2 \times_z M_1$ then $S(M)|S(M_2)$ or $S(M)|S(M_1)$ (from Proposition 1) $\Longleftrightarrow$ if $M|M_2 \times_z M_1$ then $M|M_{S(M_2)}$ or $M|M_{S(M_1)}$ (from (5)) $\Longleftrightarrow$ M is s-irreducible.

In conclusion, we have seen that the irreducibles are strictly included in the s-irreducibles which are co-extensive with the π-irreducibles. What this says is that although a machine M which is s-irreducible but not irreducible has a series-parallel decomposition into machines M_1, M_2 such that <u>neither</u> M_1 <u>nor</u> M_2 can simulate M, still it must be that by taking a suitable number of copies of either M_1 or M_2 we <u>can</u> simulate M, i.e., $M|_\pi M_1$ of $M|_\pi M_2$. Finally we note that Theorem 3 enables us to relate the s-irreducible machines given by the Krohn-Rhodes Theory (the simple group and unit actions) entirely to machine decomposition operations without reference to semigroup concepts.

FOOTNOTES

[1] S^1 is the smallest monoid containing S.

[2] For semigroups S_i, i = 1,2, $S_1 | S_2$ if S_1 is a homomorphic image of a sub-semigroup of S_2.

[3] Actually, there are proved for full machines but can easily be shown to be true for semiautomata.

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